Fractional Calculus Operators and Their Applications Involving Power Functions and Summation of Series

M-P. Chen & H.M. Srivastava

DMS-717-IR September 1995

FRACTIONAL CALCULUS OPERATORS AND THEIR APPLICATIONS INVOLVING POWER FUNCTIONS AND SUMMATION OF SERIES

MING-PO CHEN AND H.M. SRIVASTAVA

Abstract

Many earlier works on the subject of fractional calculus contain interesting accounts of the theory and applications of fractional calculus operators in a number of areas of mathematical analysis (such as ordinary and partial differential equations, integral equations, summation of series, *et cetera*). The main object of the present paper is to examine rather systematically (and extensively) some of the most recent contributions on the applications of fractional calculus operators involving power functions and in finding the sums of several interesting families of infinite series. Various other classes of infinite sums found in the mathematical literature by these (or other) means, and their validity or hitherto unnoticed connections with some known results, are also considered.

Key words: Fractional calculus, power functions, summation of series, ordinary and partial differential equations, integral equations, differintegral operator, Riemann-Liouville operator, Weyl operator, hypergeometric function, Leibniz rule, Pochhammer symbol, Psi (or Digamma) function, confluent hypergeometric function, hypergeometric sums and transformations, analytic continuation formula, Legendre's duplication formula, reflection formula, Kummer's summation theorem.

§1. Introduction and Definitions

One of the most frequently encountered tools in the theory of fractional calculus (that is, differentiation and integration of an *arbitrary* real or complex order) is furnished by the

1991 *Mathematics Subject Classification.* Primary 26A33; Secondary 33B15, 33C05.

familiar differintegral operator ${}_cD_z^\mu$ defined by

$$
{}_cD_z^\mu\{f(z)\} :=
\begin{cases}
\frac{1}{\Gamma(-\mu)}\int_c^z (z-\zeta)^{-\mu-1} f(\zeta)\,d\zeta \quad (c\in\mathbb{R};\ \Re(\mu)<0) \\
\frac{d^m}{dz^m}\,{}_cD_z^{\mu-m}\{f(z)\} \quad (m-1\leq\Re(\mu)<m;\ m\in\mathbb{N}:=\{1,2,3,\cdots\}),
\end{cases}
\tag{1.1}
$$

provided that the integral exists. For $c=0$, the operator D_z^μ given by [*cf.* Equation (1.1)]

$$
D_z^\mu\{f(z)\} := {}_0D_z^\mu\{f(z)\} \qquad (\mu\in\mathbb{C}) \tag{1.2}
$$

corresponds essentially to the classical *Riemann-Liouville fractional derivative* (or *integral*) of order μ (or $-\mu$). Moreover, when $c\to\infty$, Equation (1.1) may be identified with the definition of the familiar *Weyl fractional derivative* (or *integral*) of order μ (or $-\mu$).

In recent years there has appeared a great deal of literature discussing the application of the aforementioned fractional calculus operators in a number of areas of mathematical analysis (*cf.*, *e.g.*, [8], [10], [11], [12], [15], [20], [23], and [25]; see also a recent paper by Ross *et al.* [19] dealing with functions with no first-order derivative that might have *fractional* derivatives of all orders less than 1). In the present paper we aim at examining rather systematically (and extensively) some of the most recent works on the applications of fractional calculus operators involving power functions and in finding the sums of several interesting families of infinite series. We also consider various other classes of infinite sums [which have been found in the mathematical literature by these (or other) means] and investigate their validity or hitherto unnoticed connections with some known results.

It should be remarked in passing that the familiar Leibniz rule for *ordinary* derivatives admits itself of the following extension in terms of the Riemann-Liouville operator D_z^μ given by (1.2):

$$
D_z^\mu\{f(z)g(z)\} = \sum_{n=0}^{\infty}\binom{\mu}{n} D_z^{\mu-n}\{f(z)\}\,D_z^n\{g(z)\} \tag{1.3}
$$

$$
\left(\mu\in\mathbb{C};\ \binom{\mu}{\kappa} := \frac{\Gamma(\mu+1)}{\Gamma(\mu-\kappa+1)\,\Gamma(\kappa+1)} = \binom{\mu}{\mu-\kappa}\ (\mu,\kappa\in\mathbb{C})\right),
$$

which will be required in our present investigation.

§2. Application Involving Power Functions

Making use of a certain Eulerian integral representing the Beta function, Vyas and Banerji [29] proved the following fractional integral formula for the power function $(\alpha z + \beta)^{\lambda}$:

$$ {}_cD_z^{-\nu}\left\{(\alpha z+\beta)^{\lambda}\right\} = \alpha^{-\nu}\,\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\nu+1)}\,(\alpha z+\beta)^{\lambda+\nu} \tag{2.1} $$

$$ (c=-\beta/\alpha;\ \ \Re(\nu)>0;\ \ \Re(\lambda)>-1;\ \ \alpha\neq 0;\ \ z\neq-\beta/\alpha). $$

Nishimoto [13, p. 13], on the other hand, *similarly* proved the *equivalent* case $\alpha = 1$ of this same fractional integral formula (2.1), which essentially is one of the main results in a subsequent work by Tu and Nishimoto [28, p. 37, Theorem 1]. In fact, as already observed by Srivastava and Nishimoto [24], the fractional integral formula (2.1) follows readily from, and is no more general than, the following well-known (rather classical) result (*cf.*, *e.g.*, Erdélyi *et al.* [5, p. 185, Equation 13.1(7)]):

$$ D_z^{-\nu}\{z^{\lambda}\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\nu+1)}\,z^{\lambda+\nu}\ \ (\Re(\nu)>0;\ \ \Re(\lambda)>-1), \tag{2.2} $$

which incidentally was used by, among others, S.F. Lacroix in 1819 and P.M.H. Laurent in 1884.

In terms of the Riemann-Liouville operator D_z^{μ} given by (1.2), Ross [18, p. 87, Theorem 1] made use of the familiar binomial expansion and term-by-term integration in order to prove the fractional integral formula:

$$ D_z^{-\nu}\left\{(\alpha z+\beta)^{\lambda}\right\} = \beta^{\lambda}\sum_{n=0}^{\infty}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)}\left(\frac{\alpha}{\beta}\right)^{n}\frac{z^{\nu+n}}{\Gamma(\nu+n+1)} \tag{2.3} $$

$$ (\Re(\nu)>0;\ \ \Re(\lambda)>-1;\ \ |\alpha z/\beta|<1). $$

He also gave a fractional derivative formula by merely replacing ν in (2.3) by $-\nu$ (*cf.* [18, p. 88, Theorem 2]).

Since

$$ \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)} = (-1)^n\,(-\lambda)_n \qquad (\lambda\in\mathbb{C};\ n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}), \tag{2.4} $$

where $(\lambda)_n$ denotes the Pochhammer symbol defined by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}), \end{cases} \tag{2.5}$$

the fractional integral formula (2.3) can be rewritten at once in the hypergeometric form:

$$D_z^{-\nu}\left\{(\alpha z+\beta)^{\lambda}\right\} = \frac{\beta^{\nu} z^{\nu}}{\Gamma(\nu+1)}\, {}_2F_1\left(1,-\lambda;\ \nu+1;\ -\frac{\alpha z}{\beta}\right) \tag{2.6}$$

$$(\Re(\nu)>0;\ \Re(\lambda)>-1;\ |\alpha z/\beta|<1),$$

where, as usual, ${}_2F_1$ denotes the Gauss hypergeometric function.

Much more general fractional integral formulas than (2.3) or (2.6) are already available in the mathematical literature. For example, we have (*cf.* Erdélyi *et al.* [5, p. 186, Equation 13.1(9)])

$$D_z^{-\nu}\left\{z^{\rho}(z+\gamma)^{\lambda}\right\} = \frac{\Gamma(\rho+1)}{\Gamma(\rho+\nu+1)}\gamma^{\lambda} z^{\rho+\nu}\, {}_2F_1\left(\rho+1,-\lambda;\ \nu+1;\ -\frac{z}{\gamma}\right) \tag{2.7}$$

$$(\Re(\nu)>0;\ \Re(\lambda)>-1;\ |\arg(z/\gamma)| \leq \pi-\epsilon \qquad (0<\epsilon<\pi)),$$

which, in its *special* case when $\rho = 0$ *and* $\gamma = \beta/\alpha$, immediately yields Ross's formula (2.6). More generally, in terms of the generalized hypergeometric ${}_pF_q$ function, it is known that [5, p. 200, Equation 13.1(95)]

$$D_z^{\nu}\left\{z^{\rho}\, {}_pF_q(\alpha_1,\cdots,\alpha_p;\ \beta_1,\cdots,\beta_q;\ z)\right\} = \frac{\Gamma(\rho+1)}{\Gamma(\rho+\nu+1)} z^{\rho+\nu} \tag{2.8}$$

$$\cdot\, {}_{p+1}F_{q+1}(\rho+1,\alpha_1,\cdots,\alpha_p;\ \rho+\nu+1,\beta_1,\cdots,\beta_q;\ z)$$

$$(\Re(\nu)>0;\ \Re(\lambda)>-1;\ |z|<\infty \text{ when } p \leq q;\ |z|<1 \text{ when } p=q+1),$$

which does yield (2.7) in the special case when

$$p-1=q=0, \quad \alpha_1=-\lambda, \quad \text{and} \quad z \to -z/\gamma.$$

With a view to presenting a *further* generalization of the fractional integral formula (2.8), we recall the operator $I_{0,z}^{\mu,\delta,\eta}$ defined by (*cf.* Srivastava *et al.* [26, p. 413, Equation (1.4)])

$$I_{0,z}^{\mu,\delta,\eta}\{f(z)\} := \frac{z^{-\mu-\delta}}{\Gamma(\mu)}\int_0^z (z-\zeta)^{\mu-1}\, {}_2F_1\left(\mu+\delta,-\eta;\ \mu;\ 1-\frac{\zeta}{z}\right) f(\zeta)\, d\zeta, \tag{2.9}$$

so that, obviously,

$$I_{0,z}^{\mu,-\mu,\eta}\{f(z)\} = D_z^{-\mu}\{f(z)\} \qquad (\Re(\mu) > 0), \tag{2.10}$$

provided, of course, that the integral in (2.9) exists. For this general fractional integral operator $I_{0,z}^{\mu,\delta,\eta}$, it is also known that (*cf.* [26, p. 415, Equation (2.3)])

$$I_{0,z}^{\nu,\delta,\eta}\{z^{\lambda}\} = \frac{\Gamma(\lambda+1)\Gamma(\lambda-\delta+\eta+1)}{\Gamma(\lambda-\delta+1)\Gamma(\lambda+\nu+\eta+1)}\, z^{\lambda-\delta} \tag{2.11}$$

$$(\Re(\nu) > 0;\ \ \Re(\lambda) > \max\{0,\ \Re(\delta-\eta)\} - 1),$$

which, for $\delta = -\nu$, immediately yields the classical result (2.2).

By comparing the definitions (1.1), (1.2), and (2.9), it is readily seen that

$$I_{0,z}^{\mu,\delta,0}\{f(z)\} = z^{-\mu-\delta}\ D_z^{-\mu}\{f(z)\} \tag{2.12}$$

$$(\Re(\mu) > 0;\ \ \delta \in \mathbb{C})$$

and, more interestingly, that

$$I_{0,z}^{\mu,\delta,\eta}\{f(z)\} = \sum_{n=0}^{\infty} \frac{(\mu+\delta)_n(-\eta)_n}{n!}\, z^{-\mu-\delta-n}\ D_z^{-\mu-n}\{f(z)\}, \tag{2.13}$$

provided that this last series converges. It may be worthwhile to note that, in view of the relationship (2.12), the fractional integral formula (2.11) would reduce to the classical result (2.2) also when $\eta = 0$.

Making use of the fractional integral formula (2.11), it is not difficult to show that

$$I_{0,z}^{\nu,\delta,\eta}\{z^{\rho}\ {}_pF_q(\alpha_1,\cdots,\alpha_p;\ \beta_1,\cdots,\beta_q;\ z)\} = \frac{\Gamma(\rho+1)\Gamma(\rho-\delta+\eta+1)}{\Gamma(\rho-\delta+1)\Gamma(\rho+\nu+\eta+1)}\, z^{\rho-\delta}$$

$$\cdot\ {}_{p+2}F_{q+2}\left[\begin{matrix}\rho+1,\ \rho-\delta+\eta+1,\ \alpha_1,\cdots,\alpha_p; \\ \rho-\delta+1,\ \rho+\nu+\eta+1,\ \beta_1,\cdots,\beta_q;\end{matrix}\ z\right] \tag{2.14}$$

$$\Big(\Re(\nu) > 0;\ \ \Re(\lambda) > \max\{-1,\ \Re(\delta-\eta-1)\};$$

$$|z| < \infty \text{ when } p \leq q;\ \ |z| < 1 \text{ when } p = q+1\Big)$$

and, more generally, that

$$I_{0,z}^{\nu,\delta,\eta}\left\{z^{\rho}\,\Phi(z)\right\}=\sum_{n=0}^{\infty}\frac{\Gamma(\rho+n+1)\Gamma(\rho-\delta+\eta+n+1)}{\Gamma(\rho-\delta+n+1)\Gamma(\rho+\nu+\eta+n+1)}\,\Omega_n\,z^{\rho-\delta+n} \tag{2.15}$$

$$\left(\Phi(z):=\sum_{n=0}^{\infty}\Omega_n\,z^n;\ \ \Re(\nu)>0;\ \ \Re(\lambda)>\max\{-1,\,\Re(\delta-\eta-1)\}\right),$$

provided that each member of this last result (2.15) exists.

§3. The Leibniz Rule (1.3) and Its Consequences

The application of the Riemann-Liouville operator D_z^{μ} in evaluating sums of infinite series is based largely upon the Leibniz rule (1.3). Many of the workers on this subject did indeed revive, as illustrations emphasizing the usefulness of the fractional calculus techniques, various special cases and consequences of the following *well-known* (rather *classical*) result in the theory of the Psi (or Digamma) function $\psi(z)$:

$$\sum_{n=1}^{\infty}\frac{(\nu)_n}{n(\lambda)_n}=\psi(\lambda)-\psi(\lambda-\nu) \tag{3.1}$$

$$(\Re(\lambda-\nu)>0;\ \ \lambda\neq 0,-1,-2,\cdots),$$

where

$$\psi(z):=\frac{d}{dz}\{\log\Gamma(z)\}=\frac{\Gamma'(z)}{\Gamma(z)}. \tag{3.2}$$

For a detailed historical account of the summation formula (3.1), and of its numerous consequences and generalizations, the reader may be referred to one of the recent works on the subject by Nishimoto and Srivastava [14], who also provided many relevant *earlier* references on summation of infinite series by means of fractional calculus. See also Srivastava [22], Al-Saqabi *et al.* [1], Aular de Durán *et al.* [3], and Tu and Chyan [27]. These last authors (Tu and Chyan [27]) consider many obvious variations of the summation formula (3.1).

Now we turn to the work of Galué *et al.* [7] who derived the following interesting consequence of the familiar Leibniz rule (1.3):

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(\mu)_n}{n!}\sum_{k=0}^{\infty}\binom{\mu}{k}D_z^{-n-k}\{f(z)\}\,D_z^{n+k}\{g(z)\} = \sum_{n=1}^{\infty}\binom{\mu}{n}D_z^{-n}\{f(z)\}\,D_z^{n}\{g(z)\}, \tag{3.3}$$

provided that each side of (3.3) exists. Subsequently, as further applications of (3.3), Galué [6] set

(i) $f(z)=e^{az}$ and $g(z)=z^b$ $(a\neq 0)$;

(ii) $f(z)=z^b$ and $g(z)=e^{az}$ $(a\neq 0)$;

(iii) $f(z)=e^{az}$ and $g(z)=\log(bz)$ $(a\neq 0;\ b\neq 0)$,

and deduced from (3.3) the following summation formulas:

$$\sum_{n=0}^{\infty}\binom{m+n-1}{n}\frac{(-b)_n}{(az)^n}\,{}_2F_0\left(-m,\,n-b;\,\text{—};\,\frac{1}{az}\right)=1 \quad (m\in\mathbb{N}); \tag{3.4}$$

$$\sum_{n=0}^{\infty}(-1)^n\binom{m+n-1}{n}\frac{(az)^n}{(b+1)_n}\,{}_1F_1(-m;\,n+b+1;\,-az)=1 \qquad (m\in\mathbb{N}); \tag{3.5}$$

$$\sum_{n=1}^{\infty}\binom{m+n-1}{n}\frac{\Gamma(n)}{(az)^n}\,{}_2F_0\left(-m,\,n;\,\text{—};\,\frac{1}{az}\right) = -\sum_{n=1}^{m}\frac{(-m)_n}{n!}\,\frac{\Gamma(n)}{(az)^n} \qquad (m\in\mathbb{N}). \tag{3.6}$$

The summation formula (3.5) occurs erroneously in Galué's paper [6, p. 64, Equation (9)], where the negative sign in the argument of the confluent hypergeometric ${}_1F_1$ function is missing. More importantly, the summation formula (3.6) can be shown to follow readily from (3.4). Observe that, since [*cf.* Definition (2.5)]

$$(-b)_n=\frac{\Gamma(n-b)}{\Gamma(-b)} \qquad (n\in\mathbb{N}_0),$$

the summation formula (3.4) can be rewritten at once in its *equivalent* form:

$$\sum_{n=1}^{\infty}\binom{m+n-1}{n}\frac{\Gamma(n-b)}{(az)^n}\,{}_2F_0\left(-m,\,n-b;\,—;\,\frac{1}{az}\right)$$
$$=\Gamma(-b)\left\{1-{}_2F_0\left(-m,\,-b;\,—;\,\frac{1}{az}\right)\right\}\qquad(m\in\mathbb{N}),$$

which, when expressed as a (finite) series on the right-hand side, becomes

$$\sum_{n=1}^{\infty}\binom{m+n-1}{n}\frac{\Gamma(n-b)}{(az)^n}\,{}_2F_0\left(-m,\,n-b;\,—;\,\frac{1}{az}\right)$$
$$=-\sum_{n=1}^{m}\frac{(-m)_n}{n!}\,\frac{\Gamma(n-b)}{(az)^n}\qquad(m\in\mathbb{N}).\tag{3.7}$$

The summation formula (3.6) is an *obvious* special case of this last result (3.7) when $b=0$.

In view of the above observation that the summation formula (3.6) is contained in (3.4), we need only examine the summation formulas (3.4) and (3.5). As a matter of fact, a closer examination of these summation formulas would suggest the existence of their unification (and generalization) in the form:

$$\sum_{n=0}^{\infty}\binom{\mu+n-1}{n}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\,{}_{p+1}F_q\left[\begin{matrix}-\mu,\alpha_1+n,\cdots,\alpha_p+n;\\ \beta_1+n,\cdots,\beta_q+n;\end{matrix}\ \zeta\right]\zeta^n=1\tag{3.8}$$

$$(\mu\in\mathbb{C};\ |\zeta|<\infty\ \text{ when }\ p\leq q-1;\ |\zeta|<1\ \text{ when }\ p=q),$$

where the constraints upon ζ, p, and q may be waived when $\mu=m$ $(m\in\mathbb{N})$.

The summation formula (3.4) follows from the general result (3.8) when

$$\mu=m\ (m\in\mathbb{N}),\quad p-1=q=0,\quad \alpha_1=-b,\quad\text{and}\quad \zeta=\frac{1}{az},$$

while (3.8), in its special case when

$$\mu=m\ (m\in\mathbb{N}),\quad p=q-1=0,\quad \beta_1=b+1,\quad\text{and}\quad \zeta=-az,$$

would readily yield the summation formula (3.5).

Our proof of the general summation formula (3.8) is direct (that is, *without* using fractional calculus techniques). If, for convenience, we denote the first member of (3.8) by $\Lambda(\zeta)$, we readily find that

$$\begin{aligned}\Lambda(\zeta) &= \sum_{n,k=0}^{\infty} \frac{(\mu)_n(-\mu)_k(\alpha_1)_{n+k}\cdots(\alpha_p)_{n+k}}{(\beta_1)_{n+k}\cdots(\beta_q)_{n+k}} \frac{\zeta^{n+k}}{n!\ k!} \\ &= \sum_{k=0}^{\infty} \frac{(-\mu)_k(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k} \frac{\zeta^k}{k!}\ {}_2F_1(-k,\ \mu;\ \mu-k+1;\ 1),\end{aligned} \tag{3.9}$$

where the change of the order of summation is justifiable, by absolute convergence of the series involved, under the conditions stated with (3.8). Now, by the Vandermonde summation theorem, we have

$${}_2F_1(-k,\ \mu;\ \mu-k+1;\ 1) = \frac{(1-k)_k}{(\mu-k+1)_k} = \begin{cases} 1 & (k=0) \\ 0 & (k\in\mathbb{N}), \end{cases} \tag{3.10}$$

which, when substituted into the last member of (3.9), yields the desired result:

$$\Lambda(\zeta) = 1 \tag{3.11}$$

under the various constraints stated already with (3.8).

It is not difficult to apply the above proof *mutatis mutandis* in order to derive the following *further* generalization of the summation formula (3.8):

$$\sum_{n=0}^{\infty} \binom{\mu+n-1}{n} \sum_{k=0}^{\infty} (-\mu)_k\ \Delta_{n+k}\ \frac{\zeta^{n+k}}{k!} = 1, \tag{3.12}$$

provided that the series involved converge absolutely, $\{\Delta_n\}_{n=0}^{\infty}$ being a suitably bounded sequence of complex numbers.

§4. A Symmetrical Generalization of the Leibniz Rule (1.3)

The Leibniz rule (1.3), which was applied by Galué *et al.* [7] in order to derive the summation identity (3.3), suffers from an apparent drawback in the sense that the interchange of the functions $f(z)$ and $g(z)$ on the right-hand side is ***not*** obvious. A further (*symmetrical*) generalization of (1.3), considered by Watanabe [30] and Osler [16], ***without*** such a drawback, is given by (*cf.*, *e.g.*, Samko *et al.* [20, p. 316, Equation (17.12)]

$$D_z^\mu \{f(z)g(z)\} = \sum_{n=-\infty}^{\infty} \binom{\mu}{\sigma+n} D_z^{\mu-\sigma-n}\{f(z)\}\, D_z^{\sigma+n}\{g(z)\} \tag{4.1}$$

$$(\mu, \sigma \in \mathbb{C}),$$

which, in the special case when $\sigma = 0$, yields the Leibniz rule (1.3).

By applying the generalized Leibniz rule (4.1), Aular de Durán *et al.* [3] derived the following summation identity [3, p. 752, Equation (3.9)]:

$$\begin{aligned}\binom{-\mu}{\alpha} \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \binom{\mu}{\beta+n} D_z^{-\alpha-\beta-n}\{f(z)\}\, D_z^{\alpha+\beta+n}\{g(z)\} \\ = f(z)g(z) - \binom{-\mu}{\alpha}\binom{\mu}{\beta} D_z^{-\alpha-\beta}\{f(z)\}\, D_z^{\alpha+\beta}\{g(z)\} \\ - \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \binom{-\mu}{\alpha+n} \sum_{k=-\infty}^{\infty} \binom{\mu}{\gamma+k} D_z^{-\gamma-\alpha-n-k}\{f(z)\} \\ \cdot D_z^{\gamma+\alpha+n+k}\{g(z)\} \qquad (\alpha,\beta,\gamma,\mu \in \mathbb{C}),\end{aligned} \tag{4.2}$$

which, in the special case when $\alpha = \beta = \rho = 0$, would reduce to the simpler summation identity (3.3).

An obvious special case of the summation identity (4.2) when $\alpha = \beta = \rho$ happens to be the *main* result of Al-Zamel and Kalla [2, p. 30, Equation (5)], who also presented several examples illustrating the usefulness of this particular case of (4.2) in deriving various relationships involving infinite sums. The general summation identity (4.2), obtained earlier by Aular de Durán *et al.* [3], is potentially more advantageous in this respect than the special case used by Al-Zamel and Kalla [2].

§5. Further Applications of the Generalized Leibniz Rules

By setting

$$f(z) = 1 \quad \text{and} \quad g(z) = z^N \qquad (N \in \mathbb{N}_0)$$

in the generalized Leibniz rule (1.3), Owa [17] proved the hypergeometric summation formula:

$$ {}_2F_1(-\mu,\ -N;\ 1-\mu;\ 1) = \frac{N!}{(1-\mu)_N} \qquad (N \in \mathbb{N}_0;\ \mu \notin \mathbb{N}). \tag{5.1}$$

Formula (5.1), as observed also by Owa [17], is a special case of the Gauss summation theorem:

$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{5.2}$$

$$(\Re(c-a-b) > 0; \quad c \neq 0, -1, -2, \cdots)$$

when $a = -\mu$, $b = -N$, and $c = 1-\mu$ ($N \in \mathbb{N}_0$; $\mu \notin \mathbb{N}$). In fact, the Gauss summation theorem (5.2) would also emerge from the generalized Leibniz rule (1.3) upon setting

$$\mu = -a, \quad f(z) = z^{c-a-1}, \quad \text{and} \quad g(z) = z^{-b},$$

and applying the formula (2.2) on each side. More interestingly, the generalized Leibniz rule (4.1) can be shown to yield the well-known *Dougall's formula* [4, p. 7, Equation 1.4(1)]:

$$\sum_{n=-\infty}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(d+n)} = \frac{\pi^2}{\sin(\pi a)\,\sin(\pi b)}\ \frac{\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)} \tag{5.3}$$

$$(\Re(c+d-a-b) > 1;\ a, b \notin \mathbb{Z} := \{0, \pm1, \pm2, \cdots\}),$$

when we set

$$\mu = c-a-1, \quad \sigma = c-1, \quad f(z) = z^{d-a-1}, \quad \text{and} \quad g(z) = z^{c-b-1}, \tag{5.4}$$

and make use of the formula (2.2) on both sides of (4.1).

In view of the definition (2.5), a special case of Dougall's formula (5.3) when $d = 1$ (or, equivalently, when $c = 1$) immediately yields the Gauss summation theorem (5.2).

Now we turn to a *mild* extension of the generalized Leibniz rule (4.1) in the form (*cf.*, *e.g.*, Samko *et al.* [20, p. 317]):

$$D_z^\mu \{f(z)g(z)\} = \sum_{n=-\infty}^{\infty} \kappa \binom{\mu}{\sigma+\kappa n} D_z^{\mu-\sigma-\kappa n}\{f(z)\}\, D_z^{\sigma+\kappa n}\{g(z)\} \tag{5.5}$$

$$(\mu, \sigma \in \mathbb{C}; \quad 0 < \kappa \leq 1),$$

which, for $\kappa = 1$, reduces at once to (4.1). Indeed, if we choose the parameters μ and σ (and the functions f and g) just as indicated in (5.4), and apply the formula (2.2) on both sides of (5.5), we shall obtain the summation formula:

$$\sum_{n=-\infty}^{\infty} \frac{\Gamma(a+\kappa n)\Gamma(b+\kappa n)}{\Gamma(c+\kappa n)\Gamma(d+\kappa n)} \sin[\pi(a+\kappa n)] \sin[\pi(b+\kappa n)] \tag{5.6}$$

$$= \frac{\pi^2\, \Gamma(c+d-a-b-1)}{\kappa\, \Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)}$$

$$(\Re(c+d-a-b) > 1; \quad a, b \notin \mathbb{Z}; \quad 0 < \kappa \leq 1),$$

which, in the special case when $\kappa = 1$, would readily yield Dougall's formula (5.3).

Since [5, p. 188, Entry (24)]

$$D_z^\mu \{z^\lambda \log z\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} z^{\lambda-\mu} [\log z + \psi(\lambda+1) - \psi(\lambda-\mu+1)] \tag{5.7}$$

$$(\Re(\lambda) > -1; \quad \Re(\mu) < 0),$$

which yields the following special case when $\lambda = 0$:

$$D_z^\mu \{\log z\} = \frac{z^{-\mu}}{\Gamma(1-\mu)} [\log z - \gamma - \psi(1-\mu)] \qquad (\Re(\mu) < 0), \tag{5.8}$$

and since

$$D_z^n \{\log z\} = \begin{cases} \log z & (n = 0) \\ (-1)^{n-1}(n-1)!\, z^{-n} & (n \in \mathbb{N}), \end{cases} \tag{5.9}$$

the classical result (3.1) would follow from the generalized Leibniz rule (1.3) when we set

$$\mu = -\nu, \quad f(z) = z^{\lambda-\nu-1}, \quad \text{and} \quad g(z) = \log z.$$

More generally, by setting

$$f(z) = z^{\lambda-1} \quad \text{and} \quad g(z) = \log z,$$

and applying the formulas (5.7) and (5.8), in conjunction with (2.2), we find from (5.5) that

$$\begin{aligned} &\log z + \psi(\lambda) - \psi(\lambda - \mu) \\ &= \sum_{n=-\infty}^{\infty} \kappa \binom{\mu}{\sigma + \kappa n} \frac{[\log z - \gamma - \psi(1 - \sigma - \kappa n)]\Gamma(\lambda - \mu)}{\Gamma(1 - \sigma - \kappa n)\Gamma(\lambda - \mu + \sigma + \kappa n)} \\ &\qquad (\Re(\lambda) > 0; \quad \mu, \sigma \in \mathbb{C}; \quad 0 < \kappa \leq 1), \end{aligned} \tag{5.10}$$

provided further that *each* member of (5.10) exists.

Finally, we rewrite the generalized Leibniz rule (5.5) in the form:

$$\begin{aligned} D_z^{-\mu}\{f(z)g(z)\} &= \xi \binom{-\mu}{\alpha} D_z^{-\mu-\alpha}\{f(z)\}\, D_z^{\alpha}\{g(z)\} \\ &+ \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \xi \binom{-\mu}{\alpha + \xi n} D_z^{-\mu-\alpha-\xi n}\{f(z)\}\, D_z^{\alpha+\xi n}\{g(z)\} \\ &\qquad (\alpha, \mu \in \mathbb{C}; \quad 0 < \xi \leq 1), \end{aligned} \tag{5.11}$$

which, in view of (5.5) itself, yields the following (presumably new) summation identity:

$$\begin{aligned} &\xi \binom{-\mu}{\alpha} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \eta \binom{\mu}{\beta + \eta n} D_z^{-\alpha-\beta-\eta n}\{f(z)\}\, D_z^{\alpha+\beta+\eta n}\{g(z)\} \\ &= f(z)g(z) - \xi\eta \binom{-\mu}{\alpha}\binom{\mu}{\beta} D_z^{-\alpha-\beta}\{f(z)\}\, D_z^{\alpha+\beta}\{g(z)\} \\ &- \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \xi \binom{-\mu}{\alpha + \xi n} \sum_{k=-\infty}^{\infty} \zeta \binom{\mu}{\gamma + \zeta k} D_z^{-\gamma-\alpha-\xi n-\zeta k}\{f(z)\} \\ &\cdot D_z^{\gamma+\alpha+\xi n+\zeta k}\{g(z)\} \qquad (\alpha, \beta, \gamma, \mu \in \mathbb{C}; \; 0 < \xi, \eta, \zeta \leq 1). \end{aligned} \tag{5.12}$$

The summation identity (4.2), given earlier by Aular de Durán *et al.* [3], is an obvious special case of this last result (5.12) when

$$\xi = \eta = \zeta = 1.$$

§6. Validity of Some Hypergeometric Sums and Transformations

For the Gauss hypergeometric function, the following integral representation is fairly well-known:

$$ {}_2F_1(a,b;\ c;\ z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-t}\, t^{a-1}\ {}_1F_1(b;\ c;\ zt)\, dt \tag{6.1} $$

$$ (\Re(a) > 0; \quad |z| < 1). $$

And, for the confluent hypergeometric function involved in the integrand of (6.1), there exist several multiplication theorems expressing this function in series of similar functions (*cf.*, *e.g.*, Erdélyi *et al.* [4, p. 283, Equations 6.14(1),(2), and (3)]). Making use of these multiplication theorems in (6.1), and applying the Gauss summation theorem (5.2), Samtani and Bhatt [21] proved three transformation formulas for the Gauss hypergeometric function. We choose to recall their main results in the following slightly modified (and, where necessary, corrected) forms (*cf.* [21, p. 65, Equation (1.1); p. 66, Equations (3.1) and (3.2)]):

$$ {}_2F_1(a,b;\ c;\ z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\, z^{-a}\ {}_2F_1\left(a, a+1-c;\ a+b+1-c;\ 1-\frac{1}{z}\right); \tag{6.2} $$

$$ \begin{aligned} {}_2F_1(a,b;\ c;\ z) = {} & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\, z^{1-c} \\ & \cdot {}_2F_1\left(a+1-c, b+1-c;\ a+b+1-c;\ 1-z\right); \end{aligned} \tag{6.3} $$

$$ {}_2F_1(a,b;\ c;\ z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\ {}_2F_1\left(a,b;\ a+b+1-c;\ 1-z\right), \tag{6.4} $$

in *each* of which the parameter a is assumed to be a *negative integer.*

In view of Euler's transformation [4, p. 64, Equation 2.1.4(23)]:

$$ {}_2F_1(a,b;\ c;\ z) = (1-z)^{c-a-b}\ {}_2F_1(c-a, c-b;\ c;\ z) \tag{6.5} $$

$$ (|\arg(1-z)| \leq \pi - \epsilon \qquad (0 < \epsilon < \pi)), $$

the transformation (6.3) is an *immediate* consequence of (6.4). As a matter of fact, ***each*** of the transformations (6.2) and (6.4) is itself implied by a ***known*** analytic continuation formula for the Gauss hypergeometric function. For example, since

$$\frac{1}{\Gamma(a)}=0 \qquad (a=0,-1,-2,\cdots), \tag{6.6}$$

the transformations (6.2) and (6.4) would follow immediately from the known results (*cf.*, *e.g.*, Erdélyi *et al.* [4, p. 109, Equation 2.10(4); p. 108, Equation 2.10(1)]):

$$\begin{aligned}{}_2F_1(a,b;\,c;\,z)&=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,z^{-a}\;{}_2F_1\left(a,a+1-c;\,a+b+1-c;\,1-\frac{1}{z}\right)\\&+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\,z^{a-c}(1-z)^{c-a-b}\;{}_2F_1\left(c-a,1-a;\,c-a-b+1;\,1-\frac{1}{z}\right)\\&\qquad(|\arg(z)|\leq\pi-\epsilon\quad(0<\epsilon<\pi))\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}{}_2F_1(a,b;\,c;\,z)&=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\;{}_2F_1\left(a,b;\,a+b+1-c;\,1-z\right)\\&+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\,(1-z)^{c-a-b}\;{}_2F_1\left(c-a,c-b;\,c-a-b+1;\,1-z\right)\\&\qquad(|\arg(1-z)|\leq\pi-\epsilon\quad(0<\epsilon<\pi)),\end{aligned} \tag{6.8}$$

respectively, whenever the parameter a is restricted to take on negative integer values only. It should be remarked in passing that, since $\Re(c-a-b)>0$, the Gauss summation theorem (5.2) can be deduced readily from (6.7) and (6.8) by letting $z\to 1$.

In the same paper, by setting

$$z=\frac{1}{2}\quad\text{and}\quad c=\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}$$

in the transformation formulas (6.2), (6.3), and (6.4), but seemingly ignoring the various parametric constraints emerging from their derivations, Samtani and Bhatt [21, Section 4] obtained three strange (and overly involved) sums for the hypergeometric series

$${}_2F_1\left(a,b;\,\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2};\,\frac{1}{2}\right),$$

each of which is significantly different from the well-known sum:

$$ {}_2F_1\left(a, b; \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}; \frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2})\Gamma(\frac{1}{2}b + \frac{1}{2})} \tag{6.9} $$

$$ \left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2} \neq 0, -1, -2, \cdots\right), $$

which is due, in fact, to Kummer [9, p. 134].

The right-hand side of Kummer's formula (6.9) vanishes whenever

$$ a \quad \text{or} \quad b = -2m - 1 \qquad (m \in \mathbb{N}_0). $$

Furthermore, since

$$ \frac{(\frac{1}{2})^{-b}\,\Gamma(\frac{1}{2}b + 1)}{\Gamma(b+1)} = \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}b + \frac{1}{2})}, \tag{6.10} $$

by Legendre's duplication formula, and since (by the familiar reflection formula for the Γ-function)

$$ \frac{\Gamma(\frac{1}{2} - \frac{1}{2}a - \frac{1}{2}b)\,\Gamma(\frac{1}{2} + \frac{1}{2}a + \frac{1}{2}b)}{\Gamma(\frac{1}{2} - \frac{1}{2}a + \frac{1}{2}b)\,\Gamma(\frac{1}{2} + \frac{1}{2}a - \frac{1}{2}b)} = 1 \tag{6.11} $$

whenever

$$ a \quad \text{or} \quad b = -2m \qquad (m \in \mathbb{N}), $$

each of the aforementioned hypergeometric sums (derived by Samtani and Bhatt [21, Section 4]) is actually an unnecessarily involved *special* case of Kummer's formula (6.9) when the parameter b is a negative integer. As a matter of fact, Kummer's formula in its *general* form (6.9) would follow from the known analytic continuation formula (6.7) when we set

$$ z = \frac{1}{2} \quad \text{and} \quad c = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}, $$

and apply Kummer's summation theorem [9, p. 134]:

$$ {}_2F_1(a, b;\, a - b + 1;\, -1) = \frac{\Gamma(a - b + 1)\Gamma(\frac{1}{2}a + 1)}{\Gamma(\frac{1}{2}a - b + 1)\Gamma(a + 1)} \tag{6.12} $$

$$ (\Re(b) < 1; \quad a - b + 1 \neq 0, -1, -2, \cdots) $$

in order to sum each of the resulting hypergeometric ${}_2F_1(-1)$ series on the right-hand side of (6.7). The details involved in this derivation of the general result (6.9) may be left as an exercise for the interested reader.

Acknowledgments

The present investigation was initiated during the second-named author's visits to the Institute of Mathematics (Academia Sinica) at Taipei, the National Chang-Hua University of Education at Chang-Hua, and National Tsing Hua University at Hsin-Chu in July and August 1995. This work was supported, in part, by the *National Science Council of the Republic of China* under Grant NSC-85-2121-M-001-013 and, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

References

[1] B.N. Al-Saqabi, S.L. Kalla, and H.M. Srivastava, A certain family of infinite series associated with Digamma functions, *J. Math. Anal. Appl.* **159**(1991), 361–372.

[2] A. Al-Zamel and S. Kalla, An application of generalized Leibniz rule to infinite sums, *J. Fractional Calculus* **7**(1995), 29–33.

[3] J. Aular de Durán, S.L. Kalla, and H.M. Srivastava, Fractional calculus and the sums of certain families of infinite series, *J. Math. Anal. Appl.* **190**(1995), 738–754.

[4] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Higher Transcendental Functions*, Vol. **I**, McGraw-Hill Book Company, New York, Toronto, and London, 1953.

[5] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Tables of Integral Transforms*, Vol. **II**, McGraw-Hill Book Company, New York, Toronto, and London, 1954.

[6] L. Galué, Application of fractional calculus to infinite sums (II), *J. Fractional Calculus*, **7**(1995), 61–67.

[7] L. Galué, S.L. Kalla, and K. Nishimoto, Application of fractional calculus to infinite sums, *J. Fractional Calculus* **1**(1992), 17–21.

[8] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Pitman Research Notes in Mathematics Series **301**, Longman Scientific and Technical, Harlow, Essex (John Wiley and Sons, New York), 1994.

[9] E.E. Kummer, Über die hypergeometrische Reihe

$$1+\frac{\alpha\cdot\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3+\cdots,$$

J. Reine Angew. Math. **15**(1836), 39–83 and 127–172.

[10] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, New York, Chichester, Brisbane, Toronto, and Singapore, 1993.

[11] K. Nishimoto, *Fractional Calculus*, Vols. **I**, **II**, **III**, and **IV**, Descartes Press, Koriyama, 1984, 1987, 1989, and 1991.

[12] K. Nishimoto, *An Essence of Nishimoto's Fractional Calculus (Calculus in the 21st Century): Integrations and Differentiations of Arbitrary Order*, Descartes Press, Koriyama, 1991.

[13] K. Nishimoto, Power functions in fractional calculus of Nishimoto and that of Lacroix and Riemann-Liouville, *J. Fractional Calculus* **2**(1992), 11–25.

[14] K. Nishimoto and H.M. Srivastava, Certain classes of infinite series summable by means of fractional calculus, *J. College Engrg. Nihon Univ. Ser. B* **30**(1989), 97–106.

[15] K.B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Academic Press, New York and London, 1974.

[16] T.J. Osler, Leibniz rule for fractional derivatives generalized and an application to infinite series, *SIAM J. Appl. Math.* **18**(1970), 658–674.

[17] S. Owa, An application of the fractional derivative to hypergeometric series (in Japanese), *Sugaku* **38**(1986), 360–362.

[18] B. Ross, A formula for the fractional integration and differentiation of $(a + bx)^c$, *J. Fractional Calculus* **5**(1994), 87–89.

[19] B. Ross, S.G. Samko, and E.R. Love, Functions that have no first order derivative might have fractional derivatives of all orders less than 1, *Real Anal. Exchange* **20**(1994/1995), 140–157.

[20] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Reading, Tokyo, Paris, Berlin, and Langhorne (Pennsylvania), 1993.

[21] R.K. Samtani and R.C. Bhatt, A useful hypergeometric transformation, *Gaṇita Sandesh* **8**(1994), 65–67.

[22] H.M. Srivastava, A simple algorithm for the evaluation of a class of generalized hypergeometric series, *Stud. Appl. Math.* **86**(1992), 79–86.

[23] H.M. Srivastava and R.G. Buschman, *Theory and Applications of Convolution Integral Equations*, Mathematics and Its Applications, Vol. **79**, Kluwer Academic Publishers, Dordrecht, Boston, and London, 1992.

[24] H.M. Srivastava and K. Nishimoto, A note on a certain fractional integral formula, *J. Fractional Calculus* **3**(1993), 87–89.

[25] H.M. Srivastava and S. Owa (Editors), *Univalent Functions, Fractional Calculus, and Their Applications*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1989.

[26] H.M. Srivastava, M. Saigo, and S. Owa, A class of distortion theorems involving certain operators of fractional calculus, *J. Math. Anal. Appl.* **131**(1988), 412–420.

[27] S.-T. Tu and D.-K. Chyan, A certain family of infinite series, differintegrable functions and Psi functions, *J. Fractional Calculus* **7**(1995), 41–46.

[28] S.-T. Tu and K. Nishimoto, On the fractional calculus of functions $(cz - a)^{\beta}$ and $\log(cz - a)$, *J. Fractional Calculus* **5**(1994), 35–43.

[29] D.N. Vyas and P.K. Banerji, Fractional integral formula of the function $(\alpha z + \beta)^{a}$, *J. Fractional Calculus* **2**(1992), 83–86.

[30] Y. Watanabe, Notes on the generalized derivative of Riemann-Liouville and its applications to Leibniz's formula. I and II, *Tôhoku Math. J.* **34**(1931), 8–27 and 28–41.

MING-PO CHEN:
Institute of Mathematics
Academia Sinica
Nankang
Taipei 11529
Taiwan
Republic of China
E-Mail: MAAPO@CCVAX.SINICA.EDU.TW

H.M. SRIVASTAVA:
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4
Canada
E-Mail: HMSRI@UVVM.UVIC.CA