IRREDUNDANCE IN THE QUEENS’ GRAPH

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Abstract

The vertices of the queens’ graph $Q_n$ are the squares of an $n \times n$ chessboard and two squares are adjacent if a queen placed on one covers the other. Informally, a set $I$ of queens on the board is irredundant if each queen in $I$ covers a square (perhaps its own) which is not covered by any other queen in $I$. It is shown that the cardinality of any irredundant set of vertices of $Q_n$ is at most $\lfloor 6n + 6 - 8 \sqrt{n + 3} \rfloor$ for $n \geq 6$. We also show that the bound is not exact since $IR(Q_8) \leq 23$.

1. Introduction

The lower (upper) domination numbers $\gamma(G), (\Gamma(G))$, independence numbers $i(G)$ ($\beta(G)$) and irredundance numbers $ir(G)$ ($IR(G)$) of a graph $G$ are respectively the smallest (largest) cardinalities of minimal dominating, maximal independent and maximal irredundant vertex sets of $G$.

These six parameters are well-studied in the literature (see [3]) and satisfy the following chain of inequalities:

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G).$$

In particular there has been considerable recent interest in the evaluation of these parameters for graphs defined from $n \times n$ chessboards ([2]). This is perhaps due to the fact that two of these problems, namely the determination of $\gamma$ and $i$ for the Queens’ graph
$Q_n$ (defined in the following paragraph), have defied all efforts at solution for at least a hundred years (see [2]).

The Queens’ graph $Q_n$ has the $n^2$ squares of the chessboard as its vertex set and two vertices are adjacent if a queen placed on one covers the other, i.e. if the two squares are on the same line (row, column or diagonal) of the board.

The survey paper ([2]) gives an excellent account of recent results on the six parameters for $Q_n$. Since that paper was written, Weakley ([5]) and Burger, Cockayne and Mynhardt ([1]) have established new values of $\gamma(Q_n)$.

This paper is concerned with the upper irredundance number $IR(Q_n)$. Informally, a set $I$ of queens on the board is irredundant if each queen in $I$ covers a square (perhaps its own) which is not covered by any other queen in $I$. Weakley ([5]) has shown $\Gamma(Q_n)$ (and hence $IR(Q_n)$) $\geq 2n-5$ and McCrae ([4]) has used computer techniques to generate examples which show this lower bound is not exact. In the present work we show that $IR(Q_n) \leq \lfloor 6n+6-8\sqrt{n+3} \rfloor$ and show that our bound is also not exact since $IR(Q_8) \leq 23$.

2. The Upper Bound for $IR(Q_n)$

The following notation and terminology will be required. The rows and columns are numbered in obvious matrix fashion. The sum-diagonal numbered $k$ contains the squares $(i, j)$ such that $(i + j) - (n + 1) = k$. The difference-diagonal numbered $k$ contains the squares $(i, j)$ satisfying $i - j = k$. There are $(2n-1)$ sum-diagonals (difference-diagonals) which are numbered $0, \pm 1, \pm 2, \ldots, \pm (n-1)$.

For a vertex $v$ of $Q_n$, $r(v)$, $c(v)$, $d(v)$, $s(v)$ denote respectively the row, column, difference-diagonal and sum-diagonal which contain $v$. A set $I$ of vertices of a graph $G$ is irredundant if for each $v \in I$

$$N[v] - \bigcup_{u \in I \setminus \{v\}} N[u] \neq \emptyset.$$ 

Let $I$ be an irredundant vertex set of $Q_n$, $A$ be the set of isolated vertices of $G[I]$ where $|A| = \alpha \leq n$ (since $\beta(Q_n) = n$) and $X = \{x_1, \ldots, x_t\} = I - A$. Since $I$ is irredundant, for each $i = 1, \ldots, t$, $x_i$ is adjacent to $y_i \in V - I$ (a private neighbour of $x_i$) which is not
adjacent to any vertex of \( I - \{x_i\} \). Vertices \( x_i \) and \( y_i \) are on a line \( \ell_i \). Let \( \{y_1, \ldots, y_t\} = Y \), and \( Z = V - (I \cup Y) \). The private neighbour property implies that \( \ell_1, \ldots, \ell_t \) are distinct.
Define \( U = \{\ell_1, \ldots, \ell_t\} \).

We begin with a few simple propositions.

**Proposition 1.** If \( x_i, x_j \) (or \( y_i, y_j \)) are adjacent on a line \( \ell \), then \( \ell \notin U \).

*Proof.* If \( \{x_i, x_j\} \subseteq \ell \) and \( \{x_k, y_k\} \subseteq \ell \in U \), then one of \( i, j \) (say \( i \)) is distinct from \( k \).
Then \( \{x_i, x_k\} \subseteq \ell \) contradicting the private neighbour property of \( y_k \). (Similar proof for \( \{y_i, y_j\} \subseteq \ell \).)

**Proposition 2.** Let \( \ell \) be any line. If \( \{x_i, x_j\} \subseteq \ell \), then \( \{y_k, y_{\ell}\} \notin \ell \) and conversely.

*Proof.* Suppose \( \{x_i, x_j, y_k, y_{\ell}\} \subseteq \ell \). Clearly the private neighbour property is contradicted.

**Proposition 3.** Let \( \ell_i \) be the line defined by \( x_i, y_i \). Then none of the other six lines which contain \( x_i, y_i \) are in \( U \).

*Proof.* Suppose \( m \neq \ell_i \) contains \( x_i \) and \( m \in U \). Then for some \( j \), \( \{x_j, y_j, x_i\} \subseteq m \), contrary to the private neighbour property of \( y_j \). (Similar proof for \( y_i \in m \).)

**Proposition 4.** If \( v \in A \), then \( \{r(v), c(v), d(v), s(v)\} \cap U = \emptyset \).

*Proof.* Suppose \( v \in A \) is on line \( \ell \in U \). Then for some \( i \), \( \{x_i, y_i, v\} \subseteq \ell \), contrary to the private neighbour property.

**Proposition 5.** If \( v \in A \), then \( N(v) \subseteq Z \).

*Proof.* Vertex \( v \) is isolated in \( G[I] \) and is not adjacent to \( y \in Y \) (for otherwise \( y \) is not a private neighbour).

**Proposition 6.** For each \( i = 1, \ldots, t \), \( \ell_i - \{x_i, y_i\} \subseteq Z \).

*Proof.* Let \( w_i \in \ell_i - \{x_i, y_i\} \). If \( w_i = x \in I \), then both \( x \) and \( x_i \) are adjacent to \( y_i \).
If \( w_i = y_j \in Y \), then \( x_i \) is adjacent to both \( y_i \) and \( y_j \). In each case the private neighbour property is contradicted.
Now suppose that $U$ contains $r$, $c$, $s$, $d$ rows, columns, sum-diagonals and difference-diagonals respectively.

**Proposition 7.** If $r + \alpha \geq n - 4$ (or $c + \alpha \geq n - 4$), then $|I| \leq 3n$.

**Proof.** Since $A$ is independent, the rows occupied by vertices of $A$ are distinct and the $r$ rows of $U$ are distinct. Further by Proposition 4, these two sets of rows are disjoint. By Propositions 5 and 6, $Z$ contains $(n - 2)$ elements of $r$ rows of $U$ and $(n - 1)$ elements from $\alpha$ additional rows. Since $|Z| = n^2 - 2t - \alpha$,

$$r(n - 2) + \alpha(n - 1) \leq n^2 - 2t - \alpha.$$ 

Therefore

$$2t \leq n^2 - (r + \alpha)n + 2r$$

$$\leq n^2 - (n - 4)n + 2r$$

$$= 4n + 2r.$$ 

Hence $|I| = t + \alpha \leq 2n + (r + \alpha) \leq 3n$. 


We now establish the upper bound for $IR(Q_n)$.

**Theorem 8.** For $n \geq 6$, $IR(Q_n) \leq \lfloor 6n + 6 - 8\sqrt{n + 3} \rfloor$.

**Proof.** If $r + \alpha$ (or $c + \alpha$) $\geq n - 4$, then by Proposition 7, $|I| \leq 3n \leq 6n + 6 - 8\sqrt{n + 3}$ for $n \geq 6$. Hence assume $r + \alpha \leq n - 5$ and $c + \alpha \leq n - 5$. Assume, without loss of generality, that $d \leq s$ and re-label $X, Y$ so that $\ell_1, \ldots, \ell_s$ are sum-diagonals. Let $r_1, \ldots, r_s$ (respectively $r'_1, \ldots, r'_s$) be the rows occupied by $x_1, \ldots, x_s$ ($y_1, \ldots, y_s$). Note that there may be repetitions among $r_1, \ldots, r_s$ and among $r'_1, \ldots, r'_s$, but no $r_i$ is equal to an $r'_j$ ($r_i \neq r'_j$ since $x_i, y_i$ are on a sum-diagonal and $r_i = r'_j$ $(j \neq i)$ contradicts the private neighbour property).

Suppose $L$ is the set of lines which are neither in $U$ nor pass through any vertex of $A$. Let $\lambda$ be the largest multiplicity of a row in the sequence $r_1, \ldots, r_s$. Then there are at least $\lceil s/\lambda \rceil$ distinct rows in the sequence. These rows are in $L$ (by Proposition 3 and the fact that no vertex of $A$ is adjacent to vertices of $X \cup Y$). Further the $\lambda$ vertices of $X$ which are
on the same row, occupy distinct columns and distinct difference-diagonals. These $2\lambda$ lines are also in $L$. Hence we have a set of lines $L_1 \subseteq L$ satisfying (using elementary calculus)

$$|L_1| = \left\lceil \frac{s}{\lambda} \right\rceil + 2\lambda \geq \frac{s}{\lambda} + 2\lambda \geq 2\sqrt{2s}.$$ 

Applying the same argument to the sequence $r'_1, \ldots, r'_s$, we obtain a set $L_2 \subseteq L$ with $|L_2| \geq 2\sqrt{2s}$ and $L_1 \cap L_2 = \emptyset$ (otherwise $x_i, y_j$ are on same row, column or difference-diagonal which contradicts the private neighbour property). We conclude $|L| \geq 4\sqrt{2s}$.

The total number of lines is $6n - 2$. Hence

$$t + 4\alpha + 4\sqrt{2s} \leq 6n - 2$$

and

$$|I| = t + \alpha \leq 6n - 2 - 4\sqrt{2s} - 3\alpha.$$ 

Therefore

$$|I| \leq f_1(s) = 6n - 2 - 4\sqrt{2s}. \quad (1)$$ 

Moreover

$$|I| = (r + \alpha) + c + s + d \leq (n - 5) + (n - 5) + 2s.$$ 

Therefore

$$|I| \leq f_2(s) = (2n - 10) + 2s. \quad (2)$$ 

Hence

$$|I| \leq \max_{1 \leq s \leq 2n-3} \left( \min(f_1(s), f_2(s)) \right).$$

The maximum occurs where $f_1(s) = f_2(s)$.

Solving the quadratic for $\sqrt{2s}$, we find that the maximum occurs when $\sqrt{2s} = 2\sqrt{n + 3} - 2$ and so from (1)

$$|I| \leq 6n - 2 - 4(2\sqrt{n + 3} - 2)$$

$$= 6n + 6 - 8\sqrt{n + 3}.$$
3. An upper bound for $IR(Q_8)$

In this section we show that the bound of Section 2 is not exact. The bound for $IR(Q_8)$ is 27. However we prove

**Theorem 9.** $IR(Q_8) \leq 23$.

*Proof.* Suppose $I$ is an irredundant set of 24 vertices of $Q_8$. Then $t = 24 - \alpha$ and $|Z| = 64 - 2t - \alpha = 16 + \alpha$. Suppose $\alpha \geq 2$ and $a_1, a_2 \in A$. By Proposition 5, $N(a_1) \cup N(a_2) \subseteq Z$ and the minimum degree of $Q_8$ is 21. Hence

$$|Z| \geq |N(a_1)| + |N(a_2)| - |N(a_1) \cap N(a_2)|$$

$$= 42 - |N(a_1) \cap N(a_2)|.$$

But for any $n$ and non-adjacent $v_1, v_2 \in V(Q_n)$, $|N(v_1) \cap N(v_2)| \leq 12$, hence $|Z| = 16 + \alpha \geq 30$, which is impossible since $\alpha \leq \beta(Q_n) = 8$. We have shown that $\alpha = 0$ or 1.

Suppose $U$ contains 4 or more lines which contain 8 squares (i.e. 4 or more rows, columns or major diagonals). Let four of these lines be $\ell_1, \ldots, \ell_4$ and $Z_i = \ell_i - \{x_i, y_i\}$. By Proposition 6, $\bigcup_{i=1}^{4} Z_i \subseteq Z$. Therefore

$$|Z| \geq \left| \bigcup_{i=1}^{4} Z_i \right| \geq \sum_{i=1}^{4} |Z_i| - \sum_{1 \leq i < j \leq 4} |Z_i \cap Z_j|$$

$$\geq 24 - 6 = 18.$$

Hence $16 + \alpha \geq 18$, a contradiction.

It follows (using $\alpha \in \{0, 1\}$) that $U$ contains at least 20 lines from the set of sum-diagonals and difference-diagonals numbered $\pm 1, \ldots, \pm 6$.

Without losing generality, $U$ contains at least 10 sum-diagonals from this list say $s_1, \ldots, s_{10}$ and these are disjoint. By Proposition 6

$$|Z| \geq \sum_{i=1}^{10} |s_i - \{x_i, y_i\}| \geq 2(0 + 1 + 2 + 3 + 4) = 20.$$

Therefore $16 + \alpha \geq 20$, a contradiction which shows that there is no 24-vertex irredundant set. $\blacksquare$
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