On the Depression of Graphs

by

Mark Schurch
B.Sc., Simon Fraser University, 1998
M.Sc., University of Victoria, 2005

A Dissertation Submitted in Partial Fulfillment of the
Requirements for the Degree of

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ABSTRACT

An edge ordering of a graph $G = (V, E)$ is an injection $f : E \to \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. A path in $G$ for which the edge ordering $f$ increases along its edge sequence is called an $f$-ascent; an $f$-ascent is maximal if it is not contained in a longer $f$-ascent. The depression of $G$ is the smallest integer $k$ such that any edge ordering $f$ has a maximal $f$-ascent of length at most $k$. In this dissertation we discuss various results relating to the depression of a graph. We determine a formula for the depression of the class of trees known as double spiders. A $k$-kernel of a graph $G$ is a set of vertices $U \subseteq V(G)$ such that for any edge ordering $f$ of $G$ there exists a maximal $f$-ascent of length at most $k$ which neither starts nor ends in $U$. We study the concept of $k$-kernels and discuss related depression results, including an improved upper bound for the depression of trees. We include a characterization of the class of graphs with depression three and without adjacent vertices of degree three or higher, and also construct a large class of graphs with depression three which contains graphs with adjacent vertices of high degree. Lastly, we apply the concept of ascents to edge colourings using possibly fewer than $|E(G)|$ colours (integers). We consider the problem of determining the minimum number of colours for which there exists an edge colouring such that the length of a shortest maximal path of edges with increasing colors has a given length.
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Chapter 1

Preliminaries

1.1 Introduction

Suppose the edges of a finite graph are totally ordered. The flatness of such an edge ordering is the length of a shortest maximal path along which the edges increase with respect to the ordering. The depression of a graph is the maximum value of the flatness over all edge orderings. In this dissertation we discuss various topics relating to the depression of a graph.

In Chapter 2 we provide a formula for the depression of the class of trees known as double spiders. In Chapter 3 we discuss a method of constructing graphs for which the depression is bounded from above. We establish various results relating to this method, which include an improvement on an existing upper bound for the depression of trees. In Chapter 4 we characterize the class of graphs with depression three and no adjacent vertices of degree three or more. We also construct a large class of graphs with depression three, which includes graphs with adjacent vertices of high degree. In Chapter 5 we allow partial edge orderings and consider the problem of determining the minimum number of labels such that there exists a partial edge
ordering with given flatness. Our results include general bounds, classes of graphs which attain some of these bounds, an upper bound for a particular class of trees, and an upper bound for complete graphs. In Chapter 6 we summarize our results and discuss some open problems and further directions of research. In Appendix A we outline an algorithm which was used to verify the depression of many of the example graphs in this dissertation.

In the rest of this chapter we define concepts and introduce notation which is relevant to the main body of work. In general we follow the notation and terminology of [5], which the reader may refer to for background on graph theory topics not defined here. We also discuss some of the motivating results for this research.

1.2 Edge orderings and ascents

An edge ordering of a graph \( G = (V, E) \) is an injection \( f : E \to \mathbb{R} \), where \( \mathbb{R} \) denotes the set of real numbers. Denote the set of all edge orderings of \( G \) by \( \mathcal{F}(G) \). A path \( \lambda \) in \( G \) for which \( f \in \mathcal{F}(G) \) increases along its edge sequence is called an \( f \)-ascent (or simply ascent if the ordering is clear), and if the path \( \lambda \) has length \( k \), it is also called a \((k, f)\)-ascent. If the path \( \lambda \) with vertex sequence \( v_0, v_1, ..., v_k \) or edge sequence \( e_1, e_2, ..., e_k \) forms an \( f \)-ascent, we denote this fact by writing \( \lambda \) as \( v_0v_1 \cdots v_k \) or \( e_1e_2 \cdots e_k \). An \( f \)-ascent is maximal if it is not contained in a longer \( f \)-ascent. The height of an edge ordering \( f \), denoted by \( H(f) \), is the length of a longest \( f \)-ascent of \( G \). The flatness of an edge ordering \( f \), denoted by \( h(f) \), is the length of a shortest maximal \( f \)-ascent of \( G \).
1.3 The altitude of a graph

In [3], the altitude of $G$ was defined as

$$\alpha(G) = \min_{f \in \mathcal{F}(G)} \{H(f)\}.$$ 

The interpretation of the altitude of a graph $G$ is that any edge ordering $f \in \mathcal{F}(G)$ has an $f$-ascent of length at least $\alpha(G)$, and $\alpha(G)$ is the largest integer for which this statement is true. Note that $\alpha(G) \geq 2$ if $G$ has a vertex of degree at least two, and if $H$ is a subgraph of $G$, then $\alpha(H) \leq \alpha(G)$.

The study of lengths of increasing paths was initiated by Chvátal and Komlós [6], who posed the problem of determining the altitude of the complete graph. This is a difficult problem and $\alpha(K_n)$ is known only for $1 \leq n \leq 8$ (see [3, 6]). In particular, $\alpha(K_6) = 4$ and $\alpha(K_7) = \alpha(K_8) = 5$. Some bounds for the altitude of complete and complete bipartite graphs are given in [3] and some general bounds of $\alpha(G)$ were obtained by Graham and Kleitman in [13]. In [9], Cockayne and Mynhardt gave exact values for $\alpha(K_{3,n})$ and a lower bound for $\alpha(K_{m,n})$. The problem of determining the height of a given edge ordering was shown to be NP-hard by Katrenič and Semanišin [15]. Other work on altitude can be found in e.g. [2, 4, 7, 17, 20, 23].

1.4 The depression of a graph

In [8], the depression of $G$ was defined as

$$\varepsilon(G) = \max_{f \in \mathcal{F}(G)} \{h(f)\}.$$ 

The interpretation of the depression of a graph $G$ is that any edge ordering $f \in \mathcal{F}(G)$ has a maximal $f$-ascent of length at most $\varepsilon(G)$, and $\varepsilon(G)$ is the smallest integer for
which this statement is true.

Clearly, \( \varepsilon(G) = 1 \) if and only if \( K_2 \) is a component of \( G \). Let \( \tau(G) \) denote the length of a longest path in \( G \), called the *detour length* of \( G \). If we assume that \( G \) is connected and of size at least two, then

\[ 2 \leq \varepsilon(G), \alpha(G) \leq \tau(G). \]

By taking the edge ordering \( f \) for the path \( P_n \), where \( n \geq 3 \), to increase along its edge sequence we see that \( \varepsilon(P_n) = \tau(P_n) = n - 1 \). On the other hand, by taking the edge ordering \( g \) for the path \( P_n \), \( n \geq 3 \), as \( 1, n - 1, 2, n - 2, \ldots, \lceil \frac{n}{2} \rceil \) along its edge sequence, we see that \( \alpha(P_n) = 2 \).

### 1.5 Known results on depression

#### 1.5.1 Graphs with depression two

Clearly, if a connected graph \( G \) has a vertex which is adjacent to two pendant vertices or to two adjacent vertices of degree two, then \( \varepsilon(G) = 2 \). In [8] it was shown that the converse of this statement is also true, which gives the following characterization of graphs with depression two.

**Theorem 1.5.1.** [8] *If \( G \) is connected, then \( \varepsilon(G) = 2 \) if and only if \( G \) has a vertex adjacent to two pendant vertices or to two adjacent vertices of degree two.*

The graph shown in Figure 1.1(a) is an example of a graph with a vertex adjacent to two pendant vertices while Figure 1.1(b) is an example of a graph with a vertex adjacent to two adjacent vertices of degree two. In each figure \( v_1v_2v_3 \) is a maximal ascent for the depicted edge ordering.
Theorem 1.5.1 shows that there is no forbidden subgraph characterisation of graphs with depression two, because if any vertex of an arbitrary graph $G$ is joined to two new vertices, the resulting graph has depression two. It also follows from Theorem 1.5.1 that if $G$ is a 2-connected graph of order at least four, then $\varepsilon(G) \geq 3$.

### 1.5.2 Diameter of line graphs

When considering the problem of determining the depression of a graph it is natural to consider the line graph of $G$, which we denote by $L(G)$, and specifically, how $\varepsilon(G)$ is related to $\text{diam}(L(G))$, the diameter of the line graph of $G$. The following proposition from [8] shows that there exists an infinite class of graphs $G$ with $\varepsilon(G) \leq \text{diam}(L(G)) + 1$.

**Proposition 1.5.2.** [8]

(a) If $\text{diam}(L(G)) = 1$, then $\varepsilon(G) = 2$.

(b) If $\text{diam}(L(G)) = 2$, then $\varepsilon(G) \leq 3$.

On the other hand, it is not an easy task to find graphs $H$ with $\varepsilon(H) > \text{diam}(L(H)) + 1$. An example of one such graph was given in [8] and is shown in Figure 1.2. In [11], the graph shown in Figure 1.2 was used to construct an infinite class of graphs for which $\varepsilon(H) > \text{diam}(L(H)) + 1$. Furthermore, in [11], this class of graphs is also used.
to show that the difference $\varepsilon(G) - \text{diam}(L(G))$ may arbitrarily large, as stated in the following result.

**Theorem 1.5.3.** [11] For every integer $n \geq 1$ there is a graph $G$ for which $\varepsilon(G) - \text{diam}(L(G)) > n$.

1.5.3 Complete graphs and multipartite complete graphs

The depressions of complete graphs and multipartite graphs are corollaries of Theorem 1.5.1 and Proposition 1.5.2.

**Corollary 1.5.4.** [8] $\varepsilon(K_n) = 3$ for all $n \geq 4$.

See Figure 1.3 for an edge ordering of $K_4$ with flatness three.
Since $\alpha(K_{n-1}) \leq \alpha(K_n)$ and $\alpha(K_6) = 4$, it follows that the complete graphs $K_n$ for $n \geq 6$ constitute an infinite class of graphs $G$ for which $\alpha(G) > \varepsilon(G)$.

**Corollary 1.5.5.** [8] $\varepsilon(K_{m,n}) = 3$ for all $2 \leq m \leq n$.

**Corollary 1.5.6.** [8] $\varepsilon(K_{m_1,m_2,\ldots,m_k}) = 3$ for all $m_1 \leq \cdots \leq m_k$ and $k \geq 3$, except $k = 3$ and $m_3 = 1$.

### 1.5.4 Paths and cycles

The depressions of paths and cycles were determined in [8].

**Proposition 1.5.7.** [8] $\varepsilon(P_n) = n - 1$ for all $n \geq 2$.

**Proposition 1.5.8.** [8] $\varepsilon(C_n) = \left\lceil \frac{n+1}{2} \right\rceil$ for all $n \geq 3$.

It is a simple matter to verify that the altitude of a cycle of length $n$ is given by

$$\alpha(C_n) = \begin{cases} 2, & \text{if } n = 3 \text{ or } n \text{ is even} \\ 3, & \text{otherwise.} \end{cases}$$

Thus, the cycles $C_n$ for $n \geq 6$ constitute an infinite class of graphs $G$ for which $\alpha(G) < \varepsilon(G)$.

### 1.5.5 Trees

Theorem 1.5.1 gives the following characterization of trees with depression 2.

**Corollary 1.5.9.** [8] If $T$ is a tree, then $\varepsilon(T) = 2$ if and only if some vertex of $T$ is adjacent to at least two leaves.

A *branch vertex* of a tree is a vertex with degree at least three and a *support vertex* is a vertex adjacent to a leaf. Let $B(T)$ and $L(T)$ respectively denote the sets
Figure 1.4: An edge ordering of $S(3,3,3)$ for which the maximal ascents are $1\ 2\ 3\ 4\ 5\ 6,\ 1\ 2\ 3\ 7\ 8\ 9$, and $4\ 7\ 8\ 9$.

of all branch vertices and all leaves of the tree, and $\ell(T)$ the minimum length of a path $P$ between two leaves of $T$ such that no two consecutive vertices of $P$ are branch vertices.

**Theorem 1.5.10.** [8] For any tree $T$, $\varepsilon(T) \leq \ell(T)$.

For any tree $T$ and $v \in V(T)$, a $v-L$ path is a path from $v$ to some leaf of $T$. For $v \in V(T)$ and $l \in L(T)$, a $(v,l)$-endpath, or $v$-endpath if $l$ is unimportant, or endpath if neither $v$ nor $l$ is important, is a path $P$ from $v$ to $l$ such that each internal vertex of $P$ has degree two in $T$. A spider $S(a_1,a_2,\ldots,a_r)$ is a tree with exactly one branch vertex $v$ and $v$-endpaths of lengths $a_1,a_2,\cdots,a_r$, where $1 \leq a_1 \leq a_2 \leq \cdots \leq a_r$ and $r = \deg v$. The depression of spiders is given in [8].

**Proposition 1.5.11.** [8] $\varepsilon(S(a_1,a_2,\ldots,a_r)) = \min\{a_1 + a_2, a_3 + 1\}$.

See Figure 1.4 for an edge ordering $f$ of $S(3,3,3)$ with $h(f) = 4 = \varepsilon(S(3,3,3))$.

An upper bound for the depression of trees related to the above result for spiders was determined in [8]. Those spiders obtained by removing all edges of the tree that are not edges of endpaths are called hanging spiders of $T$. Let $\mathcal{H}(T)$ denote the set of all hanging spiders $H = S(a_1,\ldots,a_r)$, where $r \geq 3$, of $T$ and define

$$s(T) = \min_{H \in \mathcal{H}(T)} \{a_3 + 1\}.$$
Theorem 1.5.12. [8] For any tree $T$, $\varepsilon(T) \leq \min\{\ell(T), s(T)\}$.

The bound does not necessarily give the exact value of the depression of trees, even in the case where $B(T)$ is independent. An improvement on this bound is given in Theorem 3.3.2, which does give the exact value of $\varepsilon(T)$ in the case where $B(T)$ is independent.

A lower bound for the depression of trees was given in [10] and it was shown that this bound gives the exact value of $\varepsilon(T)$ in the case where $B(T)$ is independent. The bound requires the following definition. For $\alpha \in B(T)$ with $\deg \alpha = r$, let $e_1(\alpha), e_2(\alpha), \ldots, e_r(\alpha)$ be an arrangement of the edges incident with $\alpha$ and $\ell_i(\alpha)$ the length of a shortest $\alpha - L$ path that contains $e_i(\alpha)$. We abbreviate $e_i(\alpha)$ and $\ell_i(\alpha)$ to $e_i$ and $\ell_i$, if the vertex $\alpha$ is clear from the context. An arrangement $e_1, \ldots, e_r$ is called suitable if $\ell_i \leq \ell_j$ whenever $i < j$. From a suitable arrangement $e_1, \ldots, e_r$ of the edges incident with $\alpha$, define

$$\rho(\alpha) = \min\{\ell_1(\alpha) + \ell_2(\alpha), \ell_3(\alpha) + 1\}.$$ 

Theorem 1.5.13. [10] For any tree $T$, $\varepsilon(T) \geq \min_{\alpha \in B(T)}\{\rho(\alpha)\}$. Moreover, if $B(T)$ is independent, then $\varepsilon(T) = \min_{\alpha \in B(T)}\{\rho(\alpha)\}$.

1.5.6 Trees with depression three

The following characterization of trees with depression three was given in [16].

Let $S_k$ be the class of trees $S_k$, $k \geq 1$, that can be constructed recursively as follows. Let $S_0 = K_2$ with $V(S_0) = \{x_0, y_0\}$. Define $U_0 = \emptyset$ and $Y_0 = \{y_0\}$. Once $S_i$ has been constructed, construct $S_{i+1}$ by performing one of the following two operations.
\textbf{O1}: For any $y \in Y_i$, join $y$ to the vertex $u$ of a new edge $ux$; let $U_{i+1} = U_i \cup \{u\}$ and $Y_{i+1} = Y_i$.

\textbf{O2}: For any $y \in Y_i$, join $y$ to the central vertex $w$ of a new $P_5 : s, r, w, t, z$; let $U_{i+1} = U_i \cup \{w\}$ and $Y_{i+1} = Y_i \cup \{r, t\}$.

Let $S = \bigcup_{k \geq 1} S_k$. Note that $S_0 = K_2$ is not in $S$. For a tree $S \in S$, define $U_S = U_k$. Let $\mathcal{G}$ be the class of all graphs $G_S$ constructed as follows.

\textbf{O3}: Add any set $A = A(G_S)$ of new vertices to a tree $S \in S$ and arbitrary edges between vertices in $A \cup U_S$.

Note that $G_S \in \mathcal{G}$ is a tree if and only if $U_S$ is independent, $\langle A \rangle$ is acyclic, and there is exactly one edge between each component of $\langle A \rangle$ and $U_s$.

Let $\mathcal{T} = \{T \in \mathcal{G} : T \text{ is a tree}\}$.

\textbf{Theorem 1.5.14}. [16] \textit{For any tree $T$, $\varepsilon(T) = 3$ if and only if $T \in \mathcal{T}$ and no vertex of $T$ is adjacent to two leaves.}

In the proof of Theorem 1.5.14 the author shows that for each $G \in \mathcal{G}$, $\varepsilon(G) \leq 3$. Therefore,

$$\mathcal{H} = \{H \in \mathcal{G} : H \text{ is not a tree}\}$$

defines a family of graphs which are not trees and have depression at most three.

This result is generalized in Section 4.2.
Chapter 2

Double Spiders

The lower bound in Theorem 1.5.13 for the depression of a tree is not exact if the tree contains adjacent branch vertices. In this chapter we determine the depression of a class of trees, called double spiders, which contain adjacent branch vertices.

A double spider is a tree with exactly two branch vertices, these two vertices being adjacent. More precisely, the double spider \( S(a_1, \ldots, a_k : b_1, \ldots, b_{k'}) \) consists of two adjacent vertices \( u \) and \( v \) with \( \deg u = k + 1, \deg v = k' + 1 \), together with \( k \geq 2 \) \( u \)-endpaths \( R_i \) of lengths \( a_i, \ i = 1, \ldots, k \), and \( k' \geq k \) \( v \)-endpaths \( R'_j \) of lengths \( b_j, \ j = 1, \ldots, k' \), where \( a_1 \leq \cdots \leq a_k \) and \( b_1 \leq \cdots \leq b_{k'} \). Figure 2.1 depicts the double spider \( S(2, 3 : 2, 3, 4) \).

2.1 A lower bound

The following lower bound for the depression of double spiders was determined in [10].

Proposition 2.1.1. [10] Let \( T = S(a_1, \ldots, a_k : b_1, \ldots, b_{k'}) \). Then

\[
\varepsilon(T) \geq \min\{a_1 + a_2, a_3 + 1, b_1 + b_2, b_3 + 1, a_1 + b_2 + 1, a_2 + b_1 + 1\},
\]

where we ignore the term $a_3 + 1$ if $k = 2$, and the term $b_3 + 1$ if $k' = 2$.

Note that the bound in Proposition \ref{prop:double_spider} improves the bound in Theorem 1.5.13 for double spiders. For example, let $T = S(1, 6 : 3, 4)$ with branch vertices $u$ (the support vertex) and $v$. Using the notation defined in Section 1.5.5, we see that $\ell_1(u) = 1, \ell_2(u) = 4, \ell_3(u) = 6, \ell_1(v) = 2, \ell_2(v) = 3$ and $\ell_3(v) = 4$. Therefore $\min_{x \in B(T)} \{\rho(x)\} = 5$. However, by Proposition \ref{prop:double_spider}, $\varepsilon(T) \geq \min\{7, 7, 6, 10\} = 6$.

The bound in Proposition \ref{prop:double_spider} is not best possible in all cases. A formula for the depression of double spiders may not be of particular interest in itself, but it may provide ideas for determining the depression of trees in general, or perhaps for special classes of trees with adjacent branch vertices. In Section 2.2 we prove that the depression of double spiders is given by the following formula.

**Theorem 2.1.2.** Let $T = S(a_1, \ldots, a_k : b_1, \ldots, b_{k'})$. Then

$$
\varepsilon(T) = \begin{cases} 
\min\{a_1 + a_2, a_1 + b_2 + 1, a_2 + b_1 + 1, b_1 + b_2\} & \text{if } k = k' = 2 \\
\min\{a_1 + a_2, b_1 + b_2, b_3 + 1, \max\{a_2 + 2, a_1 + b_2 + 1\}\} & \text{if } k = 2 < k' \\
\min\{a_1 + a_2, a_3 + 1, b_1 + b_2, b_3 + 1\} & \text{if } k, k' \geq 3.
\end{cases}
$$

(2.1)
The statement of the theorem and the relative complexity of its proof illustrate some of problems one encounters when attempting to determine the depression of graphs with adjacent branch vertices, even for trees with exactly two branch vertices, these vertices being adjacent.

2.2 Proof of the formula for the depression of double spiders

We begin with a lemma.

Lemma 2.2.1. For any edge ordering $f'$ of a tree $T$ there is an edge ordering $f$ of $T$ with $h(f) \geq h(f')$ such that for each $\alpha \in B(T)$, each $\alpha$-endpath is a proper subpath of a maximal $f$-ascent.

Proof. Let $v$ be a branch vertex of $T$ and $Q_1, \ldots, Q_r$ the $v$-endpaths of lengths $q_1, \ldots, q_r$, respectively. We first construct an edge ordering $f_v$ that satisfies the requirements for $v$. Let $Q_1$ be the endpath $v = v_0, v_1, \ldots, v_{q_1}$. The result is obvious for any endpath of length 1, so we assume $q_1 \geq 2$.

Without loss of generality let $f'(v_1v_2) < f'(v_1v_2)$. If $Q_1$ is not an $f'$-ascent, then there is some smallest index $i \geq 2$ such that $f'(v_{i-2}v_{i-1}) < f'(v_{i-1}v_i) > f'(v_iv_{i+1})$. Relabel the edges $v_iv_{i+1}, \ldots, v_{q_1-1}v_{q_1}$ so that $Q_1$ is an ascent increasing from $v$ to $v_{q_1}$ (using large enough labels to ensure a total ordering). Relabel the edges of $Q_2, \ldots, Q_r$ similarly if necessary; if $f'$ decreases along the first two edges of $Q_i$ and hence small labels are required to form an ascent from the leaf to $v$, increase the labels of the other edges by some constant where necessary to ensure a total ordering. The resulting edge ordering $g$ has height no less than that of $f'$.

If no $Q_i$ is a maximal $g$-ascent, let $f_v = g$ and we are done. Hence assume without loss of generality that $Q_1$ is a maximal $g$-ascent. Since $Q_1$ is an ascent from $v$ to the
without loss of generality that \( Q \) the maximal \( f \) \( g \) \( g \) is a \( g \)-ascent which is contained in at least one maximal \( g \)-ascent \( R_u \) that starts at \( v_1 \). For \( u \) fixed, denote the set of all these maximal \( g \)-ascents \( R_u \) by \( \mathcal{R}_u \). Define \( g' \) by \[
g'(e) = \begin{cases} g(e) + q_1 & \text{if } e \in E(T) - E(Q_1) \\ i & \text{if } e = v_{q_1+i+1}v_{q_1-i}, \; i = 1, \ldots, q_1. \end{cases}
\]
Then \( g'(vv_1) = \min_{u \in N(v)} \{g'(uv)\} \). The maximal \( g' \)-ascents in \( T - E(Q_1) \) are exactly the same as the maximal \( g \)-ascents contained in \( T - E(Q_1) \). The other maximal \( g' \)-ascents are \( Q_1 \cup R_u \) for each \( R_u \in \mathcal{R}_u \) and each \( u \in N(v) - \{v_1\} \). Therefore \( g'(f) \geq h(f') \).

If no \( Q_i \) is a maximal \( g' \)-ascent, let \( f_v = g' \) and we are done. Hence assume without loss of generality that \( Q_r \) is a maximal \( g' \)-ascent; say \( Q_r \) is the path \( v = w_0, w_1, \ldots, w_{q_r} \). The maximality of \( Q_r \) and the fact \( g'(vv_1) = \min_{u \in N(v)} \{g'(uv)\} \) imply that \( Q_r \) is an ascent from the leaf \( w_{q_r} \) to \( v \), i.e. \( g(vw_1) = \max_{u \in N(v)} \{g'(uv)\} \) and \( g'(vw_1) > g'(w_1w_2) \). It follows that for each \( u \in N(v) - \{w_1\} \), \( uwv_1 \) is a \( g' \)-ascent which is contained in a maximal \( g' \)-ascent \( R'_u \) that starts at \( w_1 \). In particular, each \( Q_i, i \neq r \), is a proper subpath of a maximal \( g' \)-ascent. For \( u \) fixed, denote the set of all these maximal \( g' \)-ascents \( R'_u \) by \( \mathcal{R}'_u \). By definition of \( g' \), each \( R'_u \) is also a maximal \( g \)-ascent. Let \( m = \max_{e \in E(T) - E(Q_r)} \{g'(e)\} \) and define \( f_v \) by \[
f_v(e) = \begin{cases} g'(e) & \text{if } e \in E(T) - E(Q_r) \\ m + i & \text{if } e = w_{i-1}w_i, \; i = 1, \ldots, q_r. \end{cases}
\]
The maximal \( f_v \)-ascents in \( T - E(Q_r) \) are exactly the same as the maximal \( g' \)-ascents contained in \( T - E(Q_r) \). The other maximal \( f_v \)-ascents are \( Q_r \cup R'_u \) for each \( R'_u \in \mathcal{R}'_u \) and each \( u \in N(v) - \{w_1\} \neq \emptyset \). Therefore each \( Q_i \) is a proper subpath of a maximal \( f_v \)-ascent and \( h(f_v) \geq h(g') \geq h(f') \).
By the definition of $f_v$, each $f'$-ascent contained entirely in $T - \bigcup_{i=1}^{r} E(Q_i)$ is a subpath of a maximal $f_v$-ascent, i.e. the relabelling of the paths $Q_i$ did not change the $f'$-ascents in $T - \bigcup_{i=1}^{r} E(Q_i)$. Hence we may repeat the above process for each branch vertex separately until we obtain the required edge ordering $f$. \hfill \Box

Proof of Theorem 2.1.2. Let $u_i$ be the neighbour of $u$ on $R_i$ and $v_j$ the neighbour of $v$ on $R'_j$. Define the edge ordering $f$ as follows.

- From the leaf to $u$, label the edges of $R_1$ with the integers $1, \ldots, a_1$.
- From the leaf to $v$, label the edges of $R'_1$ with $a_1 + 1, \ldots, a_1 + b_1$.
- For $i = 2, \ldots, k - 1$, and in this order, label the edges of $R_i$ with consecutive integers $a_1 + b_1 + 1, \ldots, \sum_{i=1}^{k-1} a_i + b_1$ to form ascents from $u$ to the leaf.
- For $j = 2, \ldots, k' - 1$, and in this order, label the edges of $R'_j$ with consecutive integers $\sum_{i=1}^{k-1} a_i + b_1 + 1, \ldots, \sum_{i=1}^{k-1} a_i + \sum_{j=1}^{k'} b_j$ to form ascents from $v$ to the leaf.
- Let $f(uv) = \sum_{i=1}^{k-1} a_i + \sum_{j=1}^{k'} b_j + 1$.
- From $u$ to the leaf, label the edges of $R_k$ with $\sum_{i=1}^{k-1} a_i + \sum_{j=1}^{k'} b_j + 2, \ldots, \sum_{i=1}^{k} a_i + \sum_{j=1}^{k'} b_j + 1$.
- From from $v$ to the leaf, label the edges of $R'_k$ with the remaining integers.

See Figures 2.2 and 2.3(a) for the edge ordering $f$ of the double spiders $S(2, 3, 4 : 2, 3, 4)$ and $S(2, 3 : 2, 3, 4)$ respectively.

The maximal $f$-ascents are

- $R_1 \cup R_i$ for $i = 2, \ldots, k$ and $R'_1 \cup R'_j$ for $j = 2, \ldots, k'$,
- $u_iu \cup R_j$ for $2 \leq i < j \leq k$ and $v_iv \cup R'_j$ for $2 \leq i < j \leq k'$,
- $R_1 \cup uv \cup R'_{k'}$ and $R'_1 \cup vu \cup R_k$. 
Figure 2.2: The edge ordering $f$ of the double spider $S(2, 3, 4 : 2, 3, 4)$.

- $u_iuv \cup R'_k$ for $i = 2, \ldots, k - 1$ and $v_jvu \cup R_k$ for $j = 2, \ldots, k' - 1$.

Therefore

$$
\varepsilon(T) \geq h(f) = \min\{a_1 + a_2, b_1 + b_2, a_3 + 1, b_3 + 1, a_1 + 1 + b_{k'}, b_1 + 1 + a_k, b_{k'} + 2, a_k + 2\}
\begin{cases}
    \min\{a_1 + a_2, a_1 + b_2 + 1, a_2 + b_1 + 1, b_1 + b_2\} & \text{if } k = k' = 2 \\
    \min\{a_1 + a_2, a_2 + 2, b_1 + b_2, b_3 + 1\} & \text{if } k = 2 < k' \\
    \min\{a_1 + a_2, a_3 + 1, b_1 + b_2, b_3 + 1\} & \text{if } k, k' \geq 3.
\end{cases}
$$

(2.2)

If $k = 2$, we also define another edge ordering $g$ as follows.

- From the leaf to $u$, label the edges of $R_1$ with the integers $1, \ldots, a_1$.
- From the leaf to $v$, label the edges of $R'_1$ with $a_1 + 1, \ldots, a_1 + b_1$.
- Let $g(uv) = a_1 + b_1 + 1$.
- Label the edges of $R_2$ with consecutive integers $a_1 + b_1 + 2, \ldots, a_1 + a_2 + b_1 + 1$ to form an ascent from $u$ to the leaf.
Figure 2.3: (a) The edge ordering $f$ of the double spider $S(2, 3 : 2, 3, 4)$. (b) The edge ordering $g$ of the double spider $S(2, 3 : 2, 3, 4)$.

- For $j = 2, \ldots, k' - 1$, and in this order, label the edges of $R'_j$ with consecutive integers $a_1 + a_2 + b_1 + 2, \ldots, a_1 + a_2 + \sum_{j=1}^{k'} b_j + 1$ to form ascents from $v$ to the leaf.

See Figure 2.3(b) for the edge ordering $g$ of the double spider $S(2, 3 : 2, 3, 4)$.

The maximal $g$-ascents are

- $R_1 \cup R_2$ and $R'_1 \cup R'_j$ for $j = 2, \ldots, k'$,
- $v_i v \cup R'_j$ for $2 \leq i < j \leq k'$,
- $R_1 \cup uv \cup R'_j$ for $2 \leq j \leq k'$ and $R'_1 \cup vu \cup R_2$.

Therefore

$$
\varepsilon(T) \geq h(g) = \min\{a_1 + a_2, b_1 + b_2, b_3 + 1, a_1 + b_2 + 1, a_2 + b_1 + 1\}
= \begin{cases} 
\min\{a_1 + a_2, a_1 + b_2 + 1, a_2 + b_1 + 1, b_1 + b_2\} & \text{if } k = k' = 2 \\
\min\{a_1 + a_2, b_1 + b_2, b_3 + 1, a_1 + b_2 + 1, a_2 + b_1 + 1\} & \text{if } k = 2 < k'.
\end{cases}
$$

(2.3)
Combining (2.2) and (2.3) for the case $k = 2 < k'$, we get

$$\varepsilon(T) \geq \min\{a_1 + a_2, b_1 + b_2, b_3 + 1, \max\{a_2 + 2, a_1 + b_2 + 1\}\}$$

as required.

For the upper bound we prove that each edge ordering of $T$ has height at most the stated minimum. By Theorem 1.5.12, for any edge ordering $f$ of $T$,

$$h(f) \leq \min\{a_1 + a_2, a_3 + 1, b_1 + b_2, b_3 + 1\}.$$ 

Thus (2.1) holds for $k, k' \geq 3$. Therefore we assume without loss of generality that $k = 2$. We must show that

$$\varepsilon(T) \leq \begin{cases} 
\min\{a_1 + b_2 + 1, a_2 + b_1 + 1\} & \text{if } k = k' = 2 \\
\max\{a_2 + 2, a_1 + b_2 + 1\} & \text{if } k = 2 < k' 
\end{cases}.$$ (2.4)

If $a_2 = 1$ or $b_2 = 1$, then $u$ or $v$ is adjacent to two leaves, $\varepsilon(T) = 2$ by Corollary 1.5.9 and we are done, hence we assume that $a_2, b_2 \geq 2$ so that $\varepsilon(T) \geq 3$. Since the values in (2.4) are at least four, we may assume that $\varepsilon(T) \geq 5$, otherwise we are done. By Lemma 2.2.1 we only need to consider edge orderings $f$ with $h(f) = \varepsilon(T)$ such that all endpaths $R_i$ and $R'_j$ are proper subpaths of maximal $f$-ascents.

Let $f$ be any edge ordering with this property. If $\lambda$ is a maximal $f$-ascent that contains any edge of $R_i$ (respectively $R'_j$) other than $uu_i$ (respectively $vv_j$), then by the choice of $f$, $R_i \subseteq \lambda$ (respectively $R'_j \subseteq \lambda$). Therefore, if no endpath of length greater than one is entirely contained in $\lambda$, then $\lambda$ contains at most one edge of any endpath, and edges from at most two endpaths, hence $h(f) \leq 3$, contrary to the
choice of $f$. We therefore assume that

any maximal $f$-ascent contains at least

one endpath of length greater than one. \hfill (2.5)

Without loss of generality say $R_1$ is an $f$-ascent from the leaf to $u$. Since (by the choice of $f$) $R_1$ is not a maximal $f$-ascent,

$$f(u_1u) < \max\{f(uu_2), f(uv)\}. \hfill (2.6)$$

If $R_2$ is an ascent from the leaf to $u$ and $f(uu_2) > f(uu_1)$, then $R_1 \cup uu_2$ is a maximal $f$-ascent of length $a_1 + 1$ and (2.4) holds, so we assume this is not the case. If $R_2$ is an ascent from $u$ to the leaf and $f(uu_2) < f(uu_1)$, then $u_2uu_1$ is a maximal $f$-ascent, contradicting (2.5). Hence either

A1. $R_2$ is an ascent from $u$ to the leaf and $f(uu_2) > f(uu_1)$, so that $R_1 \cup R_2$ is a maximal $f$-ascent of length $a_1 + a_2$, or

A2. $R_2$ is an ascent from the leaf to $u$ and $f(uu_2) < f(uu_1)$, so that $R_2 \cup uu_1$ is a maximal $f$-ascent of length $a_2 + 1$. In this case (2.4) holds if $k = 2 < k'$ and we only have to show that it also holds if $k = k' = 2$.

Suppose $f(uu_1) > f(uv)$. By (2.6), $f(uu_1) < f(uu_2)$, hence A1 holds and $R_1 \cup R_2$ is an $f$-ascent. If $f(vv_1) < f(uv)$, then by (2.5), $R'_1$ is an ascent (of length greater than one) from the leaf to $v$, $R'_1 \cup vv_1u_1$ is a maximal $f$-ascent of length $b_1 + 2$ and (2.4) is satisfied. On the other hand, if $f(vv_1) > f(uv)$, then by (2.5), $R'_1$ is an ascent from $v$ to the leaf, so that $uv \cup R'_1$ is a maximal $f$-ascent of length $b_1 + 1$ and we are done.
Therefore we assume that $f(uv_1) < f(uv)$. If $f(uv) = \max_{v' \in N(v)} \{f(vv')\}$, then $R_1 \cup uv$ is a maximal $f$-ascent of length $a_1 + 1$ and (2.4) holds, so we assume this is not the case.

Suppose $f(uv) < f(vv_1)$. If $R'_1$ is an ascent from the leaf to $v$, then $R_1 \cup uvv_1$ is a maximal $f$-ascent of length $a_1 + 2$, satisfying (2.4), and if $R'_1$ is an ascent from $v$ to the leaf, then $R_1 \cup uv \cup R'_1$ is a maximal $f$-ascent of length $a_1 + b_1 + 1$, which also satisfies (2.4).

Assume therefore that $f(uv) > f(vv_1)$. If $f(uv) > f(uu_2)$, then $R'_1 \cup vu$ is a maximal $f$-ascent of length $b_1 + 1$ (because by the choice of $f$, $v_1 vu$ is not a maximal $f$-ascent), and we are done. Hence we assume that $f(uv) < f(uu_2)$. Then $f(uu_1) < f(uv) < f(uv_2)$ and A1 holds. By (2.5), $v_1 vu u_2$ is not a maximal $f$-ascent, so either $v_1 vu \cup R_2$ is a maximal $f$-ascent of length $a_2 + 2$, or $R'_1 \cup vu \cup R_2$ is a maximal $f$-ascent of length $a_2 + b_1 + 1$.

**Case 1:** $v_1 vu \cup R_2$ is a maximal $f$-ascent. Then $R'_1$ is an ascent from $v$ to the leaf and (2.4) is satisfied for $k = 2 < k'$, so we assume $k = k' = 2$. If $R'_2$ is an ascent from $v$ to the leaf, then $R'_1 \cup vv_2$ or $v_1 v \cup R'_2$ is a maximal $f$-ascent of length at most $b_2 + 1$, so (2.4) holds. If $R'_2$ is an ascent from the leaf to $v$, then (2.5) implies that $f(v_2 v) < f(vv_1)$ and so $f(uv) = \max_{v' \in N(v)} \{f(vv')\}$, contrary to assumption.

**Case 2:** $R'_1 \cup vu \cup R_2$ is a maximal $f$-ascent. Then $R'_1$ is an ascent from the leaf to $v$. If $f(uv) < f(vv_2)$, then either $R_1 \cup uvv_2$ or $R_1 \cup vv_2 \cup R'_2$ is a maximal $f$-ascent of length $a_1 + 2$ or $a_1 + b_1 + 1$. This together with the maximal ascent $R'_1 \cup vu \cup R_2$ satisfies (2.4). Assume that $f(uv) > f(vv_2)$. If $R'_2$ is an ascent from the leaf to $v$, then $R'_1 \cup R'_2$ contains a maximal $f$-ascent of length at most $b_2 + 1$, and this together with the maximal ascent $R'_1 \cup vu \cup R_2$ ensures that (2.4) holds. Thus assume $R'_2$ is an ascent from $v$ to the leaf. By (2.5), $R'_1 \cup R'_2$ is an ascent. Also, $v_2 vu \cup R_2$ is a maximal
$f$-ascent of length $a_2 + 2$, hence (2.4) holds if $k = 2 < k'$. Moreover, if $k = k' = 2$, then $R_1 \cup uv$ is a maximal $f$-ascent of length $a_1 + 1$ and (2.4) holds in this case too. □
Chapter 3

Kernels

Consider two disjoint graphs $G_1$ and $G_2$, and vertices $v_i \in V(G_i)$. The *vertex-coalescence of $G_1$ and $G_2$ via $v_1$ and $v_2$* is the graph obtained by identifying $v_1$ and $v_2$ to form a new vertex $v$, and is denoted $(G_1 \cdot G_2)(v_1, v_2 : v)$. In forming $G = (G_1 \cdot G_2)(v_1, v_2 : v)$, if $v_2$ is unimportant we also say we *attach $G_1$ to $G_2$ at $v_1$*, and if $G$ is the resulting graph, we say that $G$ contains $G_1$ as an *attachment at $v_1$*. From Theorem 1.5.1 we see that if $v$ is the central vertex of $P_3$ or any vertex of $K_3$, and $G$ is any connected graph containing $P_3$ or $K_3$ as an attachment at $v$, then $\varepsilon(G) = 2$.

From this result an interesting question arises.

**Question 1.** What properties should $H$ and $v \in V(H)$ satisfy so that if we attach $H$ to an arbitrary graph at $v$, the resulting graph has depression at most $k$?

In this chapter we answer this question and extend our consideration to subsets of $V(H)$. That is, for a graph $H$, we determine a necessary condition for a set $U \subset V(H)$, called a *$k$-kernel of $H$*, so that if we add a set of vertices $A$ to $H$ and arbitrary edges between $A \cup U$, the resulting graph has depression at most $k$. This investigation leads us to an improved upper bound for the depression of trees. The
work in this chapter has been accepted for publication in [19].

In Section 3.1 we define $k$-kernels and $\varepsilon$-kernels of graphs. In Section 3.2 we characterize $\varepsilon$-kernels of paths and $k$-kernels of cycles. We provide a sufficient condition in Section 3.3 for a set of vertices to be an $\varepsilon$-kernel of a spider and use this result to improve the upper bound for the depression of trees given in Theorem 1.5.12. Lastly, in Section 3.4 we describe a sufficient condition for a vertex of a graph $G$ with $\text{diam}(L(G)) = 2$ to be a $k$-kernel of $G$ for $k \in \{2, 3\}$. The work in this chapter has been submitted for publication.

### 3.1 Definition

To help answer Question 1 and to aid our study of graphs with depression three in Chapter 4, we include the following definition which is similar to the one introduced in [16].

We define a kernel of a graph $G$ as a set $U \subseteq V(G)$ such that for any edge ordering $f$ of $G$ there exists a maximal $f$-ascent that neither starts nor ends at a vertex in $U$. Specifically, we say $U$ is a $k$-kernel if for any edge ordering $f$ of $G$ there exists a maximal $(l, f)$-ascent, where $l \leq k$, that neither starts nor ends at a vertex in $U$, and $k$ is smallest integer for which this is true. If $k = \varepsilon(G)$ we say that $U$ is an $\varepsilon$-kernel of $G$. For an illustration of this concept see Figure 3.1, which shows that a vertex $v$ of $P_4$ with degree two is a 3-kernel of $P_4$. Furthermore, since $\varepsilon(P_4) = 3$, we also say that $v$ forms an $\varepsilon$-kernel of $P_4$.

Note that the definition of a kernel given in [16] is equivalent to an $\varepsilon$-kernel as defined here. We have expanded this definition to include cases where $k > \varepsilon(G)$. As illustration of a case where we may be interested in a value of $k > \varepsilon(G)$, consider the graph $G$ shown in Figure 3.2. By Theorem 1.5.1, $\varepsilon(G) = 2$ and the labelling $f$ in the
Figure 3.1: The vertex $v$ forms an $\varepsilon$-kernel of $P_4$.

Figure 3.2: The vertex $u$ is not an $\varepsilon$-kernel but is a 3-kernel.

The following theorem relates the concept of kernels to Question 1.

Remark 3.1.1. Any vertex $v \in V(G)$ with $\deg(v) \geq 2$ forms a $k$-kernel of $G$ for some $\varepsilon(G) \leq k \leq \tau(G)$.
Theorem 3.1.2. [16] Let $U$ be a $k$-kernel of a graph $H$. Form a new graph $G$ by adding any set $A$ of new vertices and arbitrary edges joining vertices in $U \cup A$. Then $\varepsilon(G) \leq k$.

Therefore, if we are able to identify a $k$-kernel of a graph $G$, Theorem 3.1.2 provides us with a method of forming a family of graphs with depression at most $k$ for some $k \geq \varepsilon(G)$. For example, if $v$ is a vertex of $P_4$ with degree 2 and $G$ is a graph that contains $P_4$ as an attachment at $v$, then by Theorem 3.1.2, $\varepsilon(G) \leq \varepsilon(P_4) = 3$.

To aid us in our discussion of kernels we introduce the following terminology.

Let $f$ be an edge ordering of a graph $G$. If an $f$-ascent $\lambda$ neither starts nor ends in a set $A \subset V(G)$, we say that $\lambda$ is an $A$-avoiding (maximal) $f$-ascent or an $a$-avoiding (maximal) $f$-ascent if $A$ contains a single vertex $a$ (and $\lambda$ is not contained in a longer $f$-ascent).

In order to identify a set $U \subseteq V(G)$ as a $k$-kernel of a graph $G$, we must show that for every edge ordering $f \in \mathcal{F}(G)$ there exists a $U$-avoiding maximal ascent of length at most $k$.

In the following sections we identify $k$-kernels for various classes of graphs.

3.2 Paths and cycles

In this section we identify $k$-kernels of paths and cycles. Since $\varepsilon(P_n) = \tau(P_n)$, it follows that any $k$-kernel of $P_n$ is necessarily an $\varepsilon$-kernel.

Proposition 3.2.1. Let $U \subseteq V(P_n)$, where $n \geq 3$. Then $U$ forms an $\varepsilon$-kernel of $P_n$ if and only if $U$ is an independent set and for each $u \in U$, $\deg(u) = 2$.

Proof. Suppose to the contrary that $U$ is an independent set, for each $u \in U$, $\deg(u) = 2$, and $U$ is not an $\varepsilon$-kernel of $P_n = v_1, v_2, ..., v_n$. Then there exists an edge ordering of $P_n$, say $f$, such that every maximal $f$-ascent of length at most $\varepsilon(P_n)$ either starts...
or ends in \( U \). Since \( \varepsilon(P_n) = n - 1 \), this means that every maximal \( f \)-ascent of \( P_n \) starts or ends at a vertex in \( U \). Necessarily, for some \( 3 \leq k \leq n \), either \( v_1v_2 \cdots v_k \) or \( v_kv_{k-1} \cdots v_1 \) is a maximal \( f \)-ascent. Without loss of generality we assume the former. Since \( v_1 \not\in U \), it follows that \( v_k \in U \) and \( k < n \). Since \( v_1v_2 \cdots v_k \) is a maximal \( f \)-ascent, \( f(v_{k-1}v_k) > f(v_kv_{k+1}) \), which means that \( v_{k+1}v_kv_{k-1} \) is an \( f \)-ascent that ends at \( v_{k-1} \). Since \( U \) is an independent set, \( v_{k-1} \) and \( v_{k+1} \) are not in \( U \). This implies \( v_{k+1}v_kv_{k-1} \) is contained in a longer \( f \)-ascent \( \lambda \). Since \( \lambda \) starts or ends in \( U \), the initial vertex, say \( v_{k'} \), is in \( U \) and \( k' > k + 1 \). By a similar argument, \( v_{k'-1}v_kv_{k'+1} \) is an \( f \)-ascent contained in a longer \( f \)-ascent, say \( \lambda' \), and the end vertex \( k'' \) of \( \lambda' \) is in \( U \), where \( k'' > k' - 1 \). Since \( P_n \) is of finite length, eventually we obtain a maximal \( f \)-ascent which neither starts nor ends in \( U \), a contradiction.

Conversely, suppose \( U \) forms an \( \varepsilon \)-kernel of \( P_n \). Then every edge ordering \( f \) contains a \( U \)-avoiding maximal \( f \)-ascent. Suppose that \( U \) is not an independent set. Let \( v_i, v_{i+1} \in U \) for some \( 2 \leq i \leq n - 1 \). Let \( f \) be the edge ordering defined as \( f(v_iv_{i+1}) = 1 \), \( f(v_jv_{j+1}) = j + 1 - i \) for each \( j > i \), and \( f(v_jv_{j+1}) = i - j + f(v_{n-1}v_n) \) for each \( j < i \). Thus any maximal \( f \)-ascent of \( P_n \) starts at either \( v_i \) or \( v_{i+1} \), a contradiction. Suppose that \( U \) contains an end vertex of \( P_n \). Consider the edge ordering \( f \) defined by \( f(v_iv_{i+1}) = i \) for all \( 1 \leq i \leq n - 1 \). Clearly, \( v_1v_2 \cdots v_n \) is the only maximal \( f \)-ascent and by our assumption it starts or ends in \( U \), a contradiction.

\( \square \)

**Proposition 3.2.2.** Let \( U \subseteq V(C_n) \), where \( n \geq 3 \). If \( U \) forms a \( k \)-kernel of \( C_n \), then \( k = n - 1 \). Furthermore, \( U \) is an \((n - 1)\)-kernel of \( C_n \) if and only if \( |U| = 1 \).

**Proof.** By Remark 3.1.1 any single vertex \( v \) forms a \( k \)-kernel of \( C_n \) for some \( \varepsilon(C_n) \leq k \leq n - 1 \). Consider a cycle \( C_n = v_1, v_2, \ldots, v_n \) and the edge labelling \( f \) given by \( f(v_iv_{i+1}) = i \) for \( 1 \leq i \leq n - 1 \) and \( f(v_nv_1) = n \). Then for \( v = v_2 \) the only \( v \)-avoiding maximal \( f \)-ascent has length \( n - 1 \). Hence \( v \) is an \((n - 1)\)-kernel of \( C_n \). Since \( C_n \) is
Suppose that \( U \subseteq V(G) \) and \(|U| \geq 2\). Let \( u, v \in U \), and say \( u = v_1 \) and \( v = v_k \) where \( 2 \leq k \leq n \). Let \( f : E(C_n) \to \{1, \ldots, n\} \) such that \( f(v_1v_2) = 1 \), \( f(v_kv_{k+1}) = n \) (or \( f(v_kv_1) = n \) if \( k = n \)), and the remaining edges are labelled so that there are exactly two maximal \( f \)-ascents in \( G \), both of which start with the edge labelled 1 and end with the edge labelled \( n \) (one in each direction around the cycle). One of the ascents starts at \( u \) and the other ends at \( v \), which implies that \( U \) is not a kernel of \( C_n \). Therefore, if \( U \) forms a \( k \)-kernel of \( C_n \), then \(|U| = 1\) and \( k = n - 1 \).

### 3.3 Spiders

In this section we identify sets which are \( \varepsilon \)-kernels of a spider \( S(a_1, a_2, \ldots, a_r) \) and use this result to determine a new upper bound for the depression of trees. Recall that for a tree \( T \) and a vertex \( \alpha \in B(T) \) with \( \deg \alpha = r \), an arrangement of the edges \( e_1, \ldots, e_r \) incident with \( \alpha \) is called suitable if \( \ell_i(\alpha) \leq \ell_j(\alpha) \) whenever \( i < j \), where \( \ell_i(\alpha) \) is the length of shortest \( \alpha \)-endpath containing \( e_i \).

**Proposition 3.3.1.** Let \( T = S(a_1, a_2, \ldots, a_r) \). If \( U \subseteq V(T) - L(T) \) and \( U \cup B(T) \) is independent, then \( U \) forms an \( \varepsilon \)-kernel of \( T \).

**Proof.** Let \( B(T) = \{v\} \) and \( U \subseteq V(T) - L(T) \) such that \( U \cup \{v\} \) is independent. By Theorem 1.5.11, \( \varepsilon(S(a_1, a_2, \ldots, a_r)) = \min\{a_1 + a_2, a_3 + 1\} \). Hence, to prove the result we must show that for any edge ordering \( f \) of \( T \) there exists a \( U \)-avoiding maximal \( f \)-ascent of length at most \( \min\{a_1 + a_2, a_3 + 1\} \).

Let \( e_1, e_2, \ldots, e_r \) be a suitable arrangement of the edges incident with the branch vertex \( v \). For \( 1 \leq i < j \leq 3 \), let \( P_{i,j} \) be the path of length \( a_i + a_j \) which contains \( e_i \) and \( e_j \). From Proposition 3.2.1, for \( G = P_{1,2} \), any independent set of \( V(G) \) forms an \( \varepsilon \)-kernel of \( G \), where \( \varepsilon(G) = a_1 + a_2 \). This implies that for any edge ordering \( f \) of \( T \),
there exists a \((U \cup \{v\})\)-avoiding maximal \(f\)-ascent of length at most \(a_1 + a_2\) which is contained in \(P_{1,2}\). Similarly, there exist \((U \cup \{v\})\)-avoiding maximal \(f\)-ascents of lengths at most \(a_1 + a_3\) and \(a_2 + a_3\) contained in \(P_{1,3}\) and \(P_{2,3}\) respectively. Let \(\lambda_{i,j}\) be a \((U \cup \{v\})\)-avoiding maximal \(f\)-ascent contained in the path \(P_{i,j}\) where \(1 \leq i < j \leq 3\).

Let \(f\) be an edge ordering of \(T\). If \(a_1 + a_2 \leq a_3 + 1\), then we are done. Hence we assume \(a_1 + a_2 > a_3 + 1\). Suppose to the contrary that there does not exist a \(U\)-avoiding maximal \(f\)-ascent of length at most \(a_3 + 1\) in \(T\). Then each \(\lambda_{i,j}\) has length at least \(a_3 + 2 \geq a_2 + 2 \geq a_1 + 2\), which implies the edges \(e_i\) and \(e_j\) are contained in \(\lambda_{i,j}\).

Without loss of generality assume that \(f(e_1) < f(e_2)\). For \(1 \leq i \leq 3\), let \(e_i'\) be the edge adjacent to \(e_i\) and not incident with \(v\). Then, since the length of \(\lambda_{1,2}\) is at least \(a_2 + 2\), \(f(e_1') < f(e_1)\) and \(f(e_2) < f(e_2')\). This implies that \(f(e_3') < f(e_3) < f(e_2)\) or else the length of \(\lambda_{2,3}\) is at most \(a_3 + 1\), which is a contradiction. But then either \(\lambda_{1,3}\) has length at most \(a_1 + 1\) (if \(f(e_3) > f(e_1')\)), or \(\lambda_{1,3}\) has length at most \(a_3 + 1\) (if \(f(e_3) < f(e_1')\)), which again is a contradiction.

We use Proposition 3.3.1 to establish an upper bound for the depression of a tree. The bound requires the following definition.

An \textit{embedded spider} of a tree \(T\) is a subgraph \(H = S(a_1, a_2, \ldots, a_r)\) of \(T\) such that

(i) \(H\) is a spider,

(ii) no endpath of \(H\) contains consecutive vertices in \(B(T)\), and

(iii) leaves of \(H\) are also leaves of \(T\).

Let \(\mathcal{H}_{es}(T)\) denote the set of all embedded spiders \(H = S(a_1, a_2, \ldots, a_r)\) of \(T\), where \(r \geq 3\). If \(\mathcal{H}_{es}(T) \neq \emptyset\), define

\[
\sigma(T) = \min_{H \in \mathcal{H}_{es}(T)} \{a_3 + 1\}.
\]
Figure 3.3: A tree $T$ with $\varepsilon(T) = 4$.

If $H_{es}(T) = \emptyset$, then define $\sigma(T) = \infty$.

Note that $\sigma(T) \leq s(T)$, where $s(T) = \min_{H \in \mathcal{H}(T)} \{a_3 + 1\}$ and $\mathcal{H}(T)$ is the set of all hanging spiders of $T$ with at least three leaves.

Recall that $\ell(T)$ is the minimum length of a path $P$ between two leaves of $T$ such that $P$ contains no two consecutive branch vertices.

**Theorem 3.3.2.** For any tree $T$, $\varepsilon(T) \leq \min\{\ell(T), \sigma(T)\}$.

**Proof.** If $\min\{\ell(T), \sigma(T)\} = \ell(T)$, then the result follows from Theorem 1.5.10. Suppose then that $\ell(T) > \sigma(T)$. Let $H = S(a_1, a_2, ..., a_r)$ be an embedded spider of $T$ such that $a_3 + 1 = \sigma(T)$. Let $U$ be the set of vertices of $H$ that are adjacent to vertices of $T - H$. Since $H$ is an embedded spider, $U \cup B(H)$ is indepdt. By Proposition 3.3.1, $U$ forms an $\varepsilon$-kernel of $H$. By Theorem 3.1.2, $\varepsilon(T) \leq \min\{a_1 + a_2, a_3 + 1\}$, and since $\ell(T) > \sigma(T)$, $a_1 + a_2 > a_3 + 1$ and the bound is established.

The bound in Theorem 3.3.2 is an improvement on the bound in Theorem 1.5.12. For example, consider the tree $T$ shown in Figure 3.3. We note that $\ell(T) = 5$ and that $T$ does not contain any hanging spiders with at least three leaves, thus from Theorem 1.5.12 it follows that $\varepsilon(T) \leq 5$. On the other hand, for the embedded spider $S(3, 3, 3)$ indicated by the emphasized edges, $a_3 + 1 = 4$, which implies $\sigma(T) \leq 4$. Hence by Theorem 3.3.2, $\varepsilon(T) \leq 4$. 
For the tree shown in Figure 3.3 the lower bound in Theorem 1.5.13 gives \( \varepsilon(T) \geq 4 \) and it was shown in [10] that this bound is tight for trees with no adjacent branch vertices. Hence, for this example, the bound from Theorem 3.3.2 is best possible.

Next we show that in general the bound in Theorem 3.3.2 gives the exact value of \( \varepsilon(T) \) whenever \( B(T) \) is independent.

Recall that for \( \alpha \in B(T) \) with \( \deg(\alpha) = r \), from a suitable arrangement \( e_1, \ldots, e_r \) of the edges incident with \( \alpha \),

\[
\rho(\alpha) = \min\{\ell_1(\alpha) + \ell_2(\alpha), \ell_3(\alpha) + 1\}.
\]

**Theorem 3.3.3.** If \( B(T) \) is independent, then \( \varepsilon(T) = \min\{\ell(T), \sigma(T)\} \).

*Proof.* If \( T \) is a path, then the result is obvious. We consider then only trees for which \( B(T) \neq \emptyset \). To prove the result we show that if \( B(T) \) is independent, then the lower bound in Theorem 1.5.13 is equivalent to the upper bound in Theorem 3.3.2, that is, \( \min\{\ell(T), \sigma(T)\} = \min_{\alpha \in B(T)}\{\rho(\alpha)\} \). Since for any tree \( T \), \( \min_{\alpha \in B(T)}\{\rho(\alpha)\} \leq \varepsilon(T) \leq \min\{\ell(T), \sigma(T)\} \), it is enough to show that \( \min\{\ell(T), \sigma(T)\} \leq \min_{\alpha \in B(T)}\{\rho(\alpha)\} \).

Let \( T \) be a tree with \( B(T) \) independent, and \( v \) a vertex in \( B(T) \) such that \( \rho(v) = \min_{\alpha \in B(T)}\{\rho(\alpha)\} = k \). Necessarily, \( v \) is the branch vertex of an embedded spider of \( T \), say \( S(a_1, a_2, \ldots, a_r) \) where \( r \geq 3 \). By definition \( \rho(v) = \min\{a_1 + a_2, a_3 + 1\} \). Moreover, \( \ell(T) \leq a_1 + a_2 \), and \( \sigma(T) \leq a_3 + 1 \). Hence, \( \min\{\ell(T), \sigma(T)\} \leq \rho(v) \) and the result follows. \( \square \)

The bound in Theorem 3.3.2 is not always exact for trees with adjacent branch vertices. For example, consider the double spider \( S(1,5 : 3,3) \) shown in Figure 3.4. From Theorem 3.3.2 we have \( \varepsilon(S(1,5 : 3,3)) \leq 6 \), yet \( \varepsilon(S(1,5 : 3,3)) = 5 \) by Theorem 2.1.2.
3.4 Graphs whose line graph has diameter two

Recall that if diam($L(G)$) = 2, then $\varepsilon(G) \leq 3$. In this section we describe a sufficient condition for a vertex of a graph $G$ with diam($L(G)$) = 2 to be a $k$-kernel of $G$ for $k \in \{2, 3\}$.

We introduce the following notation which we utilize in this section. For a graph $G$ and sets $A, B \subseteq V(G)$, define $E(A, B)$ as the set of all edges $\{a, b\} \in E(G)$ such that $a \in A$ and $b \in B$.

**Theorem 3.4.1.** Let $G$ be a graph with diam($L(G)$) = 2. If $v$ is a vertex such that $N[v]$ is a vertex cover of $G$, then $v$ forms a $k$-kernel of $G$, where $k \in \{2, 3\}$.

**Proof.** Let $v \in V(G)$ be a vertex such that $N[v]$ is a vertex cover of $G$. It suffices to show that for any edge ordering $f$ there exists a $v$-avoiding maximal $f$-ascent of length at most three. Suppose $|E(G)| = n$ and let $f : E(G) \to \{1, \ldots, n\}$ be an edge ordering of $G$. Let $uw$ and $xy$ be the edges with $f(uw) = 1$ and $f(xy) = n$. Since diam($L(G)$) = 2, $uw$ and $xy$ lie on a common $P_4$. If $v \in \{u, w\} \cap \{x, y\}$, say $v = w = y$, then $ux$ is a $v$-avoiding maximal $f$-ascent of length at most three. Similarly, if (say) $w = y$ and $v \notin \{u, w, x, y\}$, then $ux$ is a $v$-avoiding maximal
ascent. If \( \{u, w\} \cap \{x, y\} = \emptyset \) and \( v \notin \{u, w, x, y\} \), then \( E(\{u, w\}, \{x, y\}) \neq \emptyset \) since \( uw \) and \( xy \) lie on a common \( P_4 \). Any \( e \in E(\{u, w\}, \{x, y\}) \) has label \( k \) with \( 1 < k < n \), thus one of \( uwxy, uwyx, wuxy \) and \( wuyx \) is a maximal \( v \)-avoiding ascent. We may therefore assume that \( v \in \{u, w, x, y\} \) and \( v \notin \{u, w\} \cap \{x, y\} \).

Without loss of generality suppose \( v = y \). We consider two cases.

**Case 1** \( \{u, w\} \cap \{x, v\} = \emptyset \). Since \( N[v] \) is a vertex cover, \( v \) is joined to \( u \) or \( w \) with an edge labelled \( k \), where \( 1 < k < n \). In the former case \( wuvx \) is a maximal \( v \)-avoiding \( f \)-ascent, and in the latter case \( uwvx \) is such an ascent.

**Case 2** \( \{u, w\} \cap \{x, v\} \neq \emptyset \). By our assumption \( v \notin \{u, w\} \) and we may assume without loss of generality that \( x = w \). Let \( f(z) = n - 1 \) and suppose \( v \notin \{z, r\} \). If \( r \in \{u, w\} \), then \( uwz \) or \( wuz \) is a maximal \( v \)-avoiding ascent. Hence we may assume \( r \notin \{u, w\} \) and similarly \( z \notin \{u, w\} \). But \( zr \) and \( uw \) lie on a common \( P_4 \), hence there exists an edge \( e \in E(\{z, r\}, \{u, w\}) \) and this edge has label \( k \) with \( 1 < k < n - 1 \), thus forming a \( v \)-avoiding maximal \( (3, f) \)-ascent. Therefore \( v \in \{z, r\} \); say \( v = r \).

(Note that possibly \( z = u \).)

Let \( f(u_1, w_1) = 2 \) and suppose \( \{u_1, w_1\} \cap \{u, v, w, z\} = \emptyset \). Since \( N[v] \) is a vertex cover, \( v \) is adjacent to \( u_1 \) or \( w_1 \) and this edge has label \( k \) with \( 2 < k < n - 1 \). Since \( u_1 \notin \{u, w\} \), \( u_1w_1 \) is not adjacent to an edge with a smaller label. Thus \( u_1w_1vw \) or \( w_1u_1vw \) is a \( v \)-avoiding maximal ascent. It follows that \( \{u_1, w_1\} \cap \{u, v, w, z\} \neq \emptyset \).

We show that the edge labelled 2 is incident with \( w \).

Now suppose that \( |\{u_1, w_1\} \cap \{u, v, w, z\}| = 1 \) and without loss of generality \( w_1 \in \{u, v, w, z\} \). If \( w_1 = z \) then \( u_1zvw \) is a maximal ascent and if \( w_1 = v \) then \( u_1vw \) is a maximal ascent. Suppose then that \( w_1 = u \). Then \( u_1 \) is not incident with an edge with a smaller label. Since \( N[v] \) is a vertex cover, \( v \) is joined to \( u_1 \) or \( u \) by an edge with label \( k \), \( 2 < k < n \) (possibly \( k = n - 1 \) if \( u = z \)), so \( u_1uvw \) or \( uu_1vw \) is a \( v \)-avoiding maximal ascent. Hence \( w_1 = w \) (see Figure 3.5(a) and 3.5(b)). If
$|E(G)| = 4$, then $zvw$ is a $v$-avoiding maximal $f$-ascent.

Suppose next that $|\{u_1, w_1\} \cap \{u, v, w, z\}| = 2$. If $u_1w_1 = zu$, then $z \neq u$ and $uzvw$ is a $v$-avoiding maximal $f$-ascent. If $u_1w_1 = vu$, then $z \neq u$ and $uvw$ is such an ascent. Hence $u_1w_1 = zw$; say $u_1 = z$ and $w_1 = w$ and note that $z \neq u$. Thus we see that in each case the edge labelled 2 is incident with $w$ (see Figure 3.5(c)). If $|E(G)| = 4$, then $zvw$ is a $v$-avoiding maximal $f$-ascent.

Assume $|E(G)| \geq 5$ and let $f(u_2w_2) = 3$. Suppose $u_2w_2 = uu_1$. If $z \notin \{u, u_1\}$, then since $N[v]$ is a vertex cover, $v$ is joined to $u$ or $u_1$ by an edge with label $k$, $3 \leq k \leq n - 1$. Suppose $uw \in E(G)$ and consider the ascent $u_1uvw$. Since $u_1$ is not incident with the edge labelled 1, and the addition of the edge $u_1w$ with label 2 forms a 4-cycle, $u_1uvw$ is a $v$-avoiding maximal $f$-ascent. Similarly, if $u_1v \in E(H)$, then $uu_1vw$ is a $v$-avoiding maximal $f$-ascent. If $z = u$, $u_1uvw$ is a $v$-avoiding maximal ascent, and if $z = u_1$, then $uu_1vw$ is such an ascent. For all other possibilities similar arguments as for $u_1w_1$ show that $u_2w_2 = zw$ (if $f(zw) \notin \{1, 2\}$) or, without loss of generality, $w_2 = w$ and no edge incident with $u_2$ has label 1, 2, $n - 1, n$. Let $u_0 = u$.

By repeating the above argument we see that each edge $u_iw_i$ with $f(u_iw_i) = i + 1$, $i = 0, 1, \ldots, n - 3$, is incident with $w$, say $w_i = w$, and possibly $u_i = z$ for one $i = 0, 1, \ldots, n - 3$. Therefore, the graph $H$ is either the graph $H_1$ or $H_2$ in Figure 3.6.

But in either graph the ascent $zvw$ is a $v$-avoiding maximal $f$-ascent and the proof of Case 2 is complete. \qed
Corollary 3.4.2. Let $G$ be a graph with $\text{diam}(L(G)) = 2$ and $\varepsilon(G) = 3$. If $v$ is a vertex such that $N[v]$ is a vertex cover of $G$, then $v$ forms an $\varepsilon$-kernel of $G$.

To illustrate the above corollary, note that for $n \geq 4$, $\text{diam}(L(K_n)) = 2$, $\varepsilon(K_n) = 3$, and for any vertex $v \in V(K_n)$, $N[v]$ is a vertex cover of $K_n$. Therefore, by Corollary 3.4.2 we see that for any $v \in V(K_n)$, $v$ forms an $\varepsilon$-kernel of $K_n$, $n \geq 4$.

Theorem 3.1.2 and Theorem 3.4.1 allow us to identify a large class of graphs with depression at most three. We state this result in the following corollary.

Corollary 3.4.3. Let $G$ be a graph with an end-block $B$ such that $\text{diam}(L(B)) \leq 2$, and $v$ the cut vertex of $G$ contained in $B$. If $N[v]$ is a vertex cover of $B$, then $\varepsilon(G) \leq 3$.

Next we show that the converse of Theorem 3.4.1 is false. As a counterexample, consider the vertex $v$ of the graph $G$ shown in Figure 3.7. Clearly, $\text{diam}(L(G)) = 2$ and $N[v]$ is not a vertex cover of $G$. In order to show that $v$ forms a $k$-kernel, where $k \in \{2, 3\}$, we must show that for every edge ordering $f$ of $G$ there exists a $v$-avoiding maximal $f$-ascent of length at most three.

Suppose to the contrary that $f : E(G) \to \{1, 2, \ldots, 8\}$ is an edge ordering of $G$ for which there does not exist a $v$-avoiding maximal $f$-ascent of length at most
three. If \( \{ f^{-1}(1), f^{-1}(8) \} \subseteq \{ e_3, e_4, e_5, e_6, e_7, e_8 \} \), then there clearly exists a \( v \)-avoiding maximal ascent of length at most three. Thus for some \( e \in \{ e_1, e_2 \} \), \( f(e) \in \{ 1, 8 \} \), say \( f(e_1) = 1 \). Then \( e_1e_2 \) is contained in a \( v \)-avoiding maximal \( f \)-ascent \( \lambda \), and by our assumption, \( \lambda \) has length four. Thus either \( \lambda = e_1e_2e_3e_4 \) or \( \lambda = e_1e_2e_7e_4 \) and without loss of generality we assume the former.

Let \( k = \min(\{ f(e_3), f(e_5), f(e_6), f(e_7) \}) \). If \( f(e_3) = k \), then either \( e_3e_7 \) or \( e_3e_7e_5 \) is a \( v \)-avoiding maximal \( f \)-ascent. Similarly, if \( f(e_7) = k \), then \( e_7e_3 \) or \( e_7e_3e_6 \) is a \( v \)-avoiding maximal \( f \)-ascent, and if \( f(e_5) = k \), then \( e_5e_6 \) or \( e_5e_6e_3 \) is a \( v \)-avoiding maximal \( f \)-ascent.

We assume then that \( f(e_6) = \min(\{ f(e_3), f(e_5), f(e_6), f(e_7) \}) \). Then \( e_6e_5 \) are the first two edges of a maximal ascent, and since \( G \) does not contain a \( v \)-avoiding maximal \( f \)-ascent of length at most three, it follows that \( f(e_5) < f(e_7) < f(e_2) \). Now if \( f(e_8) > f(e_2) \), then \( e_6e_8 \) is a \( v \)-avoiding maximal \( f \)-ascent. Assume then that \( f(e_8) < f(e_2) \). Since \( e_1e_2e_3e_4 \) is an \( f \)-ascent, it follows that \( f(e_8) < f(e_3) \). If \( f(e_8) < f(e_5) \), then \( e_8e_5e_4 \) is a \( v \)-avoiding maximal \( f \)-ascent. Finally, if \( f(e_8) > f(e_5) \), then \( e_5e_8e_3 \) is a \( v \)-avoiding maximal \( f \)-ascent.

This covers all cases and establishes the proof of the counterexample to the converse of Theorem 3.4.1.
Chapter 4

Graphs With Depression Three

Theorem 1.5.1 provides a simple characterization of graphs with depression two, which leads us to consider the problem of characterizing graphs with depression three. Although this remains an unsolved problem, a characterization of trees with depression three is given in Theorem 1.5.14 of [16]. In Section 4.1 of this chapter we characterize graphs with depression three and the added property that the graph contains no two adjacent vertices of degree three or more. The work in this section has been accepted for publication in [18]. In Section 4.2 we define a large class of graphs with depression three that do contain adjacent vertices of degree three or more. The work in this section has been submitted for publication.

4.1 Graphs with depression three and no adjacent vertices of degree three or more

Let $\mathcal{H}$ be the set of graphs consisting of $K_{2,m}$ for $m \geq 1$, $K_3$, $P_4$, and the spider $S(2,2,2)$ – see Figure 4.1. An $\varepsilon$-kernel of a graph $H \in \mathcal{H}$ is not necessarily unique, but if an $\varepsilon$-kernel consists of a single vertex of $H$, then we call such a vertex an
The set of graphs $\mathcal{H}$. 

Our aim in this section is to prove the following theorem.

**Theorem 4.1.1.** Let $G$ be a connected graph with $\text{diam}(L(G)) \geq 3$ and no adjacent vertices of degree three or more. Then $\varepsilon(G) \leq 3$ if and only if $G = S(2,2,2)$, or for some $H \in \mathcal{H}$, $G$ contains $H$ as an attachment at an $\varepsilon$-kernel vertex of $H$.

A vertex $v \in V(G)$ is called an end-vertex if $\deg v = 1$, a link vertex if $\deg v = 2$ and a hub vertex if $\deg v \geq 3$. A $u$-$v$ path in $G$ in which the internal vertices are link vertices is called a $u$-$v$ direct path or simply a direct path if $u$ and $v$ are unimportant. For $U \subseteq V(G)$ and $u \in U$, a $v$-$u$ direct path is also called a $v$-$U$ direct path. As for trees, a $u$-$v$ path $P$ in which $v$ is an end-vertex and the internal vertices are link vertices is called a $u$-$v$ endpath or a $u$-endpath if $v$ is unimportant.

We first prove three lemmas.

**Lemma 4.1.2.** Let $H$ be a graph in $\mathcal{H}$ and $w$ a vertex in $V(H)$ such that $\deg(w) = \Delta(H)$. Then $w$ is an $\varepsilon$-kernel vertex of $H$.

**Proof.** Suppose firstly that $H = K_3$. Then $\varepsilon(H) = 2$ and $\tau(H) = 2$. Hence the result follows from Remark 3.1.1. If $H = K_{2,1}$ or $H = P_4$, then the result follows from Proposition 3.2.1. Suppose then that $H = K_{2,m}$ where $m \geq 2$. Then by Theorem
1.5.1 and Proposition 1.5.2, $\varepsilon(H) = 3$. Moreover, for any vertex $v \in V(K_{2,m})$ where $m \geq 2$, $N[v]$ is a vertex cover of $K_{2,m}$. Thus the result follows from Corollary 3.4.2. Lastly, suppose $H = S(2,2,2)$. Then the result follows from Proposition 3.3.1.

From Lemma 4.1.2 we see that for each $H \in \mathcal{H}$, the vertex labelled $w$ in Figure 4.1 is an $\varepsilon$-kernel vertex of $H$. Furthermore, if a graph $G$ has no adjacent vertices of degree three or more, $\text{diam}(L(G)) > 3$, and $G$ contains a graph $H \in \mathcal{H}$ as an attachment at an $\varepsilon$-kernel vertex of $H$, then $G$ contains $H$ as an attachment at $w$.

**Lemma 4.1.3.** For any graph $G$, if an edge of $G$ incident with a link vertex is subdivided to form a graph $H$, then $\varepsilon(H) \geq \varepsilon(G)$.

**Proof.** Let $G$ be a graph with a link vertex $v$ and let $e_1 = uv$ and $e_2$ be the edges incident with $v$. For any edge ordering $f : E(G) \to \mathbb{Z}$ of $G$, either $e_1e_2$ or $e_2e_1$ is an $f$-ascent. Without loss of generality assume $e_1e_2$ is an $f$-ascent and let $H$ be the graph formed from $G$ by removing $e_1$ and adding the vertex $w$ and the edges $uw$ and $wv$. Define the edge ordering $f'$ of $H$ by $f'(uw) = f(e_1)$, $f'(wv) = f(e_1) + 0.5$, and $f'(e) = f(e)$ whenever $e \in E(G)$. Thus for any maximal $f$-ascent $\lambda$ of $G$ containing $e_1$ and $e_2$ there exists a corresponding $f'$-ascent $\lambda'$ in $H$ which contains $uw, wv, e_2$, thus containing one more edge than $\lambda$. Furthermore, for any maximal $f$-ascent $\lambda$ of $G$ containing $e_1$ and not $e_2$, there exists a corresponding $f'$-ascent of the same length in $H$ which contains the edges of $\lambda$ except with $e_1$ replaced by $uw$. Lastly, any maximal $f$-ascent $\lambda$ in $G$ not containing $e_1$ is an $f'$-ascent in $H$. Therefore for each edge ordering $f$ of $G$, there exists an edge ordering $f'$ of $H$ such that $h(f') \geq h(f)$.

**Lemma 4.1.4.** Let $G$ be a graph with $\varepsilon(G) = k \geq 4$. If $H$ is formed by attaching $P_n$ to $G$ at an end-vertex of $P_n$, where $n \geq k$, then $\varepsilon(H) \geq k$.

**Proof.** Let $f : E(G) \to \{1,2,...,|E(G)|\}$ such that $h(f) = k$. Also, let $G' = v_2, v_3, ..., v_{n+1}$ be a path of order $n \geq k$. Let $v_1$ be an arbitrary vertex of $G$ and
define $H = (G \cdot G')(v_1, v_2 : v)$. Define $f' : E(H) \to \{1, 2, \ldots, |E(H)|\}$ such that $f'(e) = f(e)$ whenever $e \in E(G)$ or $e = uv$ where $u \in V(G)$, and label the edges on the $v-v_{n+1}$ direct path $\lambda$ so that its edges form a $v-v_{n+1}$ $f'$-ascent. Then, for any edge $a$ on $\lambda$ and $b \in E(G)$, $f'(a) > f'(b)$. Thus for all edges $e$ incident with $v$ that are not on $\lambda$, $f'(e) < f(vv_3)$. This implies that any maximal $f'$-ascent that contains edges of $\lambda$ has length at least $k$. Therefore $h(f') \geq k$ and $\varepsilon(H) \geq k$. \hfill \square

We define the following properties for a graph $G$.

**P1:** A graph has property $\mathbf{P1}$ if and only if it contains a graph $H \in \mathcal{H}$ as an attachment at an $\varepsilon$-kernel vertex of $H$.

**P2:** A graph has property $\mathbf{P2}$ if and only if it contains adjacent vertices of degree three or more.

**Proof of Theorem 4.1.1.** Suppose that $G = S(2,2,2)$ or $G$ contains $H \in \mathcal{H}$ as an attachment at an $\varepsilon$-kernel vertex of $H$. By Proposition 1.5.2 or Proposition 1.5.11, $\varepsilon(H) \leq 3$, and by Theorem 3.1.2, $\varepsilon(G) \leq \varepsilon(H) \leq 3$.

Conversely, suppose that $G \neq S(2,2,2)$ and for all $H \in \mathcal{H}$, $G$ does not contain $H$ as an attachment at an $\varepsilon$-kernel vertex of $H$. That is, $G$ has neither property $\mathbf{P1}$ nor property $\mathbf{P2}$. It follows from Lemma 4.1.2 that $G$ does not contain a graph $H \in \mathcal{H}$ as an attachment at $w$, where $w$ is the vertex labelled as such for each graph in Figure 4.1.

We show that $\varepsilon(G) \geq 4$.

First suppose $\Delta(G) = 2$. Since $\text{diam}(L(G)) \geq 3$, $G = C_n$, where $n \geq 6$, or $G = P_n$, where $n \geq 5$. In the former case, $\varepsilon(G) = \left\lceil \frac{n+1}{2} \right\rceil \geq 4$ by Proposition 1.5.11, and in the latter case $\varepsilon(G) = \varepsilon(P_n) = n - 1 \geq 4$. 


Therefore we assume henceforth that $\Delta(G) \geq 3$. Denote the set of all end-vertices of $G$ by $Z$, the set of all link vertices by $Y$ and the set of all hub vertices by $X'$. By Lemma 4.1.4 we may limit our consideration to graphs with endpaths of length at most two, thus, we may assume $d(z, X') \leq 2$ for each $z \in Z$. Define the partition $Z_1, Z_2$ of $Z$ by $z \in Z_i$ if and only if $z \in Z$ and $d(z, X') = i$.

Let $B_1, B_2, \ldots, B_k$ be the end-blocks of $G$ that are cycles. By Lemma 4.1.3 and since $G$ does not have property $\textbf{P1}$, we may assume that $B_i \cong C_5$ for each $i$. For each $B_i$, $1 \leq i \leq k$, choose a vertex $v_i$ such that $d(v_i, X') = 2$, and let $X'' = \{v_1, \ldots, v_k\}$ and $X = X' \cup X''$.

Suppose $|X| = 1$. Since $\Delta(G) \geq 3$, $|X'| \geq 1$. Hence $X'' = \emptyset$ and $G$ has exactly one vertex of degree three or more. Thus $G$ is a spider. Since $G \text{diam}(L(G)) \geq 3$ and $G \neq S(2,2,2)$, Proposition 1.5.11 shows that $\varepsilon(G) \geq 4$. Hence we assume that $|X| \geq 2$.

**Lemma 4.1.1.1** If $|X \cup Z| = 2$, then $\varepsilon(G) \geq 4$.

*Proof of Lemma 4.1.1.1.* As assumed above, $|X| \geq 2$. Now if $|X \cup Z| = 2$, then $Z = \emptyset$. If $|X'| = 1$, say $X' = \{x\}$, then $x$ lies on two or more cycles, none of which has a cut-vertex other than $x$, so that $|X''| \geq 2$ and $|X \cup Z| \geq 3$, which is not the case. Thus $|X'| = 2$ and $|X''| = |Z| = 0$. Let $X' = \{u, v\}$. Since $\Delta(G) \geq 3$ there exist at least three $u$-$v$ direct paths in $G$. Furthermore, since $\text{diam}(L(G)) \geq 3$, at least two $u$-$v$ direct paths contain three or more edges, or at least one $u$-$v$ direct path contains four or more edges.

Suppose $G$ contains exactly two $u$-$v$ direct paths of length three, $e_1, e_2, e_3$ and $e_1', e_2', e_3'$, where $e_1$ and $e_1'$ are incident with $u$, and $k \geq 1$ $u$-$v$ direct paths of length 2. We now describe an edge-ordering $f \in F(G)$ with flatness four. Let $f(e_2) = 1$ and $f(e_2') = m$, where $m = |E(G)|$. Also, let $f(e_1) = 2$, $f(e_3) = m - 2$, $f(e_1') = 3$, $f(e_3') = m - 1$. Label the edges of the $u$-$v$ direct paths of length two so that each
path is labelled with distinct values from the set \( \{4, 5, \ldots, m-4, m-3\} \) so that each direct path forms a \( u-v \) direct \( f \)-ascent. The paths \( e_2e_1'e_1' \) and \( e_2e_3'e_3' \) are both maximal \((4, f)\)-ascents. Any maximal \( f \)-ascent that contains a \( u-v \) direct path of length two has length at least 4. Thus \( f \) has flatness four – see Figure 4.2(a). By Lemma 4.1.3, if any of the edges on the \( u-v \) direct paths in \( G \) are subdivided, the depression does not decrease.

Suppose therefore that \( G \) contains one \( u-v \) direct path \( Q \) of length four and \( k \geq 2 \) \( u-v \) direct paths of length two. Let \( e_1, e_2, e_3, e_4 \) be the edges of the \( u-v \) direct path of length four and let \( f(e_2) = 1, \ f(e_1) = 2, \ f(e_3) = m-2 \) and \( f(e_4) = m-1 \). Label the edges of the \( u-v \) direct paths of length two with distinct values from the set \( \{3, 4, \ldots, m-3, m\} \) so that each \( u-v \) direct path is a \( u-v \) direct \( f \)-ascent. This labelling is shown in Figure 4.2(b) and also has flatness four. Again, by Lemma 4.1.3, if any of the edges on the \( u-v \) direct paths in \( G \) are subdivided, the depression does not decrease. \( \blacklozenge \)

Assume henceforth that \( |X \cup Z| \geq 3 \). Let \( G' \) be the graph with vertex set \( X \cup Z \) such that \( uv \in E(G') \) if and only if \( u, v \in X \cup Z \) and there exits a \( u-v \) direct path in \( G \). If \( Z \neq \emptyset \), let \( x_0 \in Z \), otherwise let \( x_0 \) be a vertex that is not a cut vertex of \( G' \). Let \( \mu = \text{diam} G' \) and let \( V_0, V_1, \ldots, V_{\mu} \) be a partition of \( V(G') \) such that \( V_0 = \{x_0\} \)
and for each \( i \geq 1, v \in V_i \) whenever \( d_{G'}(x_0, v) = i \). Denote the set of end-vertices of \( G' \) that are not in \( Z \) by \( \Omega \); note that \( X'' \subseteq \Omega \) and that possibly \( X' \cap \Omega \neq \emptyset \).

We use \( G' \) to aid us in defining a vertex ordering on \( X \cup Z \) which we then use to define an edge ordering of \( E(G) \) with flatness at least 4. This will show that \( \varepsilon(G) \geq 4 \). By Lemma 4.1.3 we need only consider graphs in which all direct paths between vertices \( u, v \in X \) have length 2, unless \( u \) or \( v \) is in \( \Omega \), in which case, since \( G \) does not have property P1, one of the \( u \)-\( v \) direct paths has length 3 and all others have length 2.

We define a vertex ordering \( g_k : V(G') \rightarrow \mathbb{R} \) of \( G' \) recursively as described below.

G1. Define the ordering \( g_0 : X \cup \{x_0\} \rightarrow \{1, 2, ..., |X \cup \{x_0\}|\} \) so that for \( u \in V_i \) and \( v \in V_j, g_0(u) < g_0(v) \) whenever \( 0 \leq i < j \leq k \), and within each \( V_i \), the vertices in \( \Omega \) receive the largest labels, while the vertices in \( X - \Omega \) that are not adjacent to a vertex in \( V_{i+1} \) receive the smallest labels.

Let \( X_0 \) be the set of vertices \( v \in X - \Omega \) such that \( v \) is not adjacent to a vertex in \( Z \), \( g_0^{-1}(g_0(v) - 1) \notin N_{G'}(v) \), and for each vertex \( u \in N_{G'}(v) \), \( g_0(v) > g_0(u) \). From an ordering \( g_i \) on \( V(G') - Z \) and its associated set \( X_i \subseteq X - \Omega \) we now define the ordering \( g_{i+1} : X \rightarrow \{1, 2, ..., |X \cup \{x_0\}|\} \) and the set \( X_{i+1} \).

G2. Let \( v \) be the vertex such that \( g_i(v) = \max_{x \in X_i} \{g_i(x)\} \). Let \( u \) be the vertex with the maximum value assigned under \( g_i \) over all vertices in \( N_{G'}(v) \). Define \( g_{i+1}(u) = g_i(v) - 1 \) and \( g_{i+1}(v) = g_i(v) \).

Now let \( N_u \) be the set of vertices \( t \) such that \( t \in N_{G'}(u) \) and \( g_i(u) < g_i(t) < g_i(v) \).

G3. Under \( g_{i+1} \), label the vertices in \( N_u \) with the values from the set \( \{g_i(v) - 2, g_i(v) - 3, ..., g_i(v) - 1 - |N_u|\} \) such that \( g_{i+1}(t) < g_{i+1}(t') \) whenever \( g_i(t) < g_i(t') \) for all \( t, t' \in N_u \). For all vertices \( x \in X - N_u - \{u, v\} \) let \( g_{i+1}(x) \in \{1, 2, ..., |X \cup \{x_0\}|\} \) and the set \( X_{i+1} \).
\{x_0\} - \{g_i(v), g_i(v) - 1, \ldots, g_i(v) - 1 - |N_u|\} \text{ such for all } x, x' \in X - N_u - \{u, v\},
g_{i+1}(x) < g_{i+1}(x') \text{ whenever } g_i(x) < g_i(x').

Let \(X_{i+1} \subseteq X - \Omega\) be the set of vertices \(v\) such that \(g_{i+1}^{-1}(g_{i+1}(v) - 1) \notin N_{G'}(v)\) and for each vertex \(u \in N_{G'}(v)\), \(g_{i+1}(v) > g_{i+1}(u)\).

Figure 4.3(b) shows a labelling \(g_0\) of \(X \cup \{x_0\} \subseteq V(G')\) as defined in G1 where \(G'\) is derived from the graph \(G\) shown in Figure 4.3(a). Note that \(z, z' \in Z, y \in \Omega, X_0 = \{v\}\), and \(u\) is the neighbour of \(v\) assigned the largest label under \(g_0\). The labelling \(g_1\) shown in Figure 4.3(c) is obtained by applying G2 and G3. Note that \(X_1 = \emptyset\).

We make two remarks concerning the labels of \(x_0\) and its neighbours for future reference. Recall that \(x_0\) is not a cut vertex. Thus, in G2, if \(v \in V_1\), then \(v\) is adjacent to some vertex \(u' \in V_1 \cup V_2\) such that \(g_0(u') > 1 = g_0(x_0)\).

R1. Hence \(u \neq x_0\) and \(g_{i+1}(x_0) = 1\) for each \(i\).

Also, since \(X_i \subseteq X - \Omega\), \(v\) is adjacent to at least two vertices of \(G'\); hence if \(v \in V_j\) for \(j \geq 2\), then \(g_i(u) > 2\) for each \(i\).

R2. Therefore the vertex \(w\) such that \(g_{i+1}(w) = 2\) is adjacent to \(x_0\) for all \(i\).

**Lemma 4.4.1.2** \(X_{i+1} \not\subseteq X_i\).

**Proof of Lemma 4.4.1.2.** Let \(x \in X - \Omega - X_i\). Then (i) \(x\) is adjacent to a vertex \(y\) with a larger label under \(g_i\) or (ii) \(x\) is adjacent to a vertex \(y'\) such that \(g_i(y') = g_i(x) - 1\). In either case, if \(x \in N_u \cup \{u\}\), then by G2 or G3, \(x\) is adjacent to a vertex with a larger label under \(g_{i+1}\), namely \(v\) or \(u\); hence \(x \notin X_{i+1}\). Assume therefore that \(x \notin N_u \cup \{u\}\) and suppose first that (i) holds. If \(y \in N_u \cup \{u\}\), then \(g_i(x) < g_i(y) < g_i(v)\) and by G3, \(g_{i+1}(y) \in \{g_i(v) - 1, g_i(v) - 2, \ldots, g_i(v) - 1 - |N_u|\}\) while \(g_{i+1}(x) < g_i(v) - 1 - |N_u|\),
Figure 4.3: Examples of orderings defined by G1-G5 and F1-F3.
hence \( g_{i+1}(x) < g_{i+1}(y) \). If \( y \notin N_u \cup \{u\} \) then \( g_{i+1}(x) < g_{i+1}(y) \) by G3. Hence \( x \notin X_{i+1} \).

Suppose that (ii) holds. If \( y' \in N_u \cup \{u\} \), then \( g_i(y') < g_i(v) \) and thus \( g_i(x) < g(v) \). Therefore \( y' \neq u \) and by G3, \( g_{i+1}(y') \in \{g_i(v) - 2, g_i(v) - 3, \ldots, g_i(v) - 1 - |N_u|\} \) while \( g_{i+1}(x) < g_i(v) - 1 - |N_u| \), so that \( g_{i+1}(x) < g_{i+1}(y') \). If \( y \notin N_u \cup \{u\} \) then \( g_{i+1}(y') = g_{i+1}(x) - 1 \) by G3. In either case \( x \notin X_{i+1} \).

Therefore \( X_{i+1} \subseteq X_i \) and the result follows from the fact that \( v \notin X_{i+1} \). \( \diamond \)

Beginning with \( i = 0 \), repeatedly refine \( g_i \) to \( g_{i+1} \) using the procedure described above until an ordering \( g_k \) such that \( X_k = \emptyset \) is obtained.

Next extend the labelling \( g_k \) to the vertices in \( Z \) as follows. For each \( x \in X' \), let \( Z(x) = Z \cap N_{G'}(x) - \{x_0\} \). Since \( G \) does not have property \( P1 \), \( |Z(x)| \leq 2 \) for all \( x \in X' \). Furthermore, if \( |Z(x)| = 2 \), then the two vertices in \( Z(x) \) are at distance 2 from \( x \) in \( G \).

G4. If \( Z(x) \) contains a single vertex \( z \), let \( g_k(z) = g_k(x) + 0.25 \). If \( Z(x) \) contains two vertices, say \( z \) and \( z' \), let \( g_k(z) = g_k(x) - 0.25 \) and \( g_k(z') = g_k(x) + 0.25 \).

Figure 4.3(d) contains an example of an extension of a labelling \( g_k \) to \( Z - \{x_0\} \) as defined in G4.

Now we make one last set of refinements to form the labelling \( g'_k \) from \( g_k \). Let \( W \subseteq X - \Omega \) be the set of vertices \( v \) such that \( g_k(v) = \max\{g_k(u) : u \in N_{G'}[v]\} \) and for \( u \in N_{G'}(v) \) with \( g_k(u) = g(v) - 1 \), \( Z(u) \neq \emptyset \). Note that if \( v \in W \), then \( Z(v) = \emptyset \).

G5. For each \( v \in W \), let \( g'_k(v) = g_k(u) - 0.5 \). For all other vertices \( v \in V(G') \), let \( g'_k(v) = g_k(v) \). This ensures that the label of \( v \) is less than the labels of \( u \) and the vertices in \( Z(u) \), but still greater than those of the vertices in \( N_{G'}(v) - \{u\} \).

In Figure 4.3(d) we see that \( v \in W \) since \( v \) is assigned the largest value over all its neighbours in \( G' \), and for the vertex \( u \) which is assigned the label \( g_1(v) - 1 \),
\[ Z(u) \neq \emptyset. \] Furthermore, \( v \) is the only vertex in \( W \), thus, the ordering \( g'_1 \) as shown in Figure 4.3(e) is obtained by the refinement defined in G5.

Let \( S = \{0, 1, 2, \ldots, |V(G')| - 1\} \) and \( \rho : V(G') \to S \) such that \( \rho(u) < \rho(v) \) if and only if \( g'_k(u) < g'_k(v) \). By R1, \( \rho(x_0) = 0 \). Let \( V(G') = \{x_i : i \in S\} \), where \( \rho(x_i) = i \) for each \( i \in S \).

We next define the edge ordering \( f' : E(G) \to \{1, 2, \ldots, m\} \), where \( m = |E(G)| \), as follows.

F1. For each pair of vertices \( x_i, x_j \in V(G') \), if \( i < j \) then each \( x_i-x_j \) direct path is an \( x_i-x_j \) direct \( f' \)-ascent.

F2. For \( x_i, x_j, x_r, x_s \in V(G') \), where \( i < j \) and \( r < s \), if \( (i, j) \) precedes \( (r, s) \) in a lexicographic ordering of \( S \times S \), then each label defined by \( f' \) on the edges of the \( x_i-x_j \) direct paths is less than each label defined by \( f' \) on the edges of the \( x_r-x_s \) direct paths.

F3. For \( x_i, x_j \in V(G') \), if there exist multiple \( x_i-x_j \) direct paths, then a longest \( x_i-x_j \) direct path contains the smallest and largest labels under \( f' \) over all edges contained on \( x_i-x_j \) direct paths.

Figure 4.3(f) and Figure 4.4(a) each show an edge ordering which satisfies the constraints outlined in F1-F3. Note that the edge ordering in Figure 4.4(a) has flatness three as indicated by the emphasized maximal ascents.

We now make a set of refinements to \( f' \) to form the edge ordering \( f : E(G) \to \mathbb{R} \). Let \( \text{deg}_{G'} x_0 = r \) and \( N_{G'}(x_0) = \{x_{a_1}, x_{a_2}, \ldots, x_{a_r}\} \), where \( a_1 < a_2 < \cdots < a_r \). By R2 and the definition of \( \rho \), \( a_1 = 1 \).

F4. If \( r > 1 \), let \( e \) be the edge with the smallest label under \( f' \) over all edges on \( x_0-x_{a_2} \) direct paths and let \( f(e) = 1.5 \).
F5. Let $I$ be the set of indices such that $i \in I$ if and only if $\deg_{G'} x_i > 1$ and $i > j$ for all $x_j \in N_{G'}(x_i)$. For each $i \in I$, let $e_i$ be the edge incident with $x_i$ with the largest label under $f'$ and let $x_{i'}$ be the vertex with the second largest index over all vertices in $N_{G'}(x_i)$. Let $e_{i'}$ be the edge with the largest label over all $x_{i'}$-$x_i$-direct paths and define $f(e_i) = f'(e_i) - 0.5$.

F6. For all edges $e \in E(G)$ not yet labelled under $f$, let $f(e) = f'(e)$.

Figure 4.4(b) shows the edge ordering $f$ obtained by applying F4-F6 to the edge ordering $f'$ shown in Figure 4.4(a). Note that the ordering shown in Figure 4.3(f) will remain unchanged after the refinements F4-F6.

**Lemma 4.1.1.3** $h(f) \geq 4$.

**Proof of Lemma 4.1.1.3.** For each vertex $v \in V(G')$, let $E(v)$ be the set of edges of $G$ incident with $v$, $E^-(v)$ the set of edges of $G$ incident with $v$ that are on a $u$-$v$ direct $f$-ascent of length 2 or more for some $u \in V(G)$, and $E^+(v) = E(v) - E^-(v)$. By F1 and F2, for each vertex $v \in V(G')$ and edges $e_1 \in E^-(v)$ and $e_2 \in E^+(v)$,
\[
f(e_2) \geq f(e_1). \text{ Also, for all } z \in Z_1, \ E^-(z) = \emptyset \text{ (since } z \text{ does not lie on a } u-z \text{ direct path of length at least two), and for all } v \in X - \{x_0\}, \ |E^-(v)| \geq 1.
\]

Let \( \lambda \) be any maximal \( f \)-ascent in \( G \). Then \( \lambda \) has length at least 2. Let \( v_0, v_1 \) and \( v_2 \) be the first three vertices of \( \lambda \), and let \( e_1 = v_0v_1 \) and \( e_2 = v_1v_2 \). There are two possibilities for \( v_1 \): \( v_1 \in Y \) or \( v_1 \in X' \).

**Case 1** \( v_1 \in Y \). Since \( \lambda \) is a maximal \( f \)-ascent and \( |E^-(v)| \geq 1 \) for all \( v \in X - \{x_0\} \), \( v_0 \in Z \) or \( v_0 = x_0 \).

First assume \( v_0 \in Z - \{x_0\} \). Since \( v_1 \in Y \), it follows that \( v_0 \in Z_2 \) and \( v_2 \in X' \). By \( G4 \) and the definition of \( \rho \), \( \rho(v_2) = \rho(v_0) + 1 \). Then by \( F2 \), \( f(e_2) = \max_{e \in E^-(v_2)} \{f(e)\} \) and thus the ascent \( e_1e_2 \) cannot be extended along an edge of \( E^-(v_2) \). Furthermore, since \( \rho(v_2) = \rho(v_0) + 1 \), \( G4 \) also implies that there exists a vertex \( x_i \in Z_2 \) that is adjacent to \( v_2 \) in \( G' \) and \( i = \rho(v_2) + 1 \). Thus \( E^+(v_2) \neq \emptyset \), and since \( G \) does not have properties \( P1 \) or \( P2 \), each edge in \( E^+(v_2) \) is contained in a \( v_2-x_j \) direct \( f \)-ascent of length at least 2 for some \( j \). Therefore \( \lambda \) has length at least 4.

Now we consider the case where \( v_0 = x_0 \). Since \( v_1 \in Y, v_0 \in X \cup Z_2 \). Recall that \( a_1 = 1 \) (see \( R2 \)). Let \( P \) be the \( x_0-x_1 \) direct path whose initial edge is the edge \( e' \) such that \( f(e') = 1 \) and let \( P' \) be any other \( x_0-x_1 \) direct path (if it exists). Let \( e'' \) be the terminal edge of \( P \). By \( F3 \), \( P' \) followed by \( e'' \) is an \( f \)-ascent; call it \( \ell \). However, if \( x_0 \in \Omega \), then \( e' \) followed by \( \ell \) is an \( f \)-ascent of length 4 (by \( F3 \) and since \( G \) does not have property \( P1 \)), and if \( x_0 \in X - \Omega \), then by \( F4 \), the edge \( e \) with the smallest label under \( f \) over all edges on \( x_0-x_{a_2} \) direct paths followed by \( \ell \) is an \( f \)-ascent of length 4. Thus \( F1 \) – \( F4 \) and the maximality of \( \lambda \) imply that \( f(e_1) = 1 \) and that \( v_1 \) is on an \( x_0-x_1 \) direct \( f \)-ascent \( \lambda' \) that is a subpath of \( \lambda \), and the edge \( e^* \) of \( \lambda' \) incident with \( x_1 \) is in \( E^-(x_1) \).

- First suppose \( x_0 \in \Omega \). Since \( G \) does not have property \( P1 \), \( \lambda' \) has length 3. Since \( |V(G')| \geq 3 \), \( x_1 \in X' \) and \( x_j \in N_{G'}(x_1) \) for some \( j > 1 \). Thus \( E^+(x_1) \neq \emptyset \),
which implies that $\lambda$ has length at least 4.

- Suppose next that $x_0 \in X - \Omega$. Then by the choice of $x_0$, $Z = \emptyset$. Since $x_0$ is not a cut vertex, $x_1 \in X - \Omega$ and again $E^+(x_1) \neq \emptyset$. Since $Z = \emptyset$, each edge in $E^+(x_1)$ is contained in a $x_1$-$x_i$ direct $f$-ascent of length at least 2 for some $i > 1$. Hence any extension of $\lambda'$ along an edge in $E^+(x_1)$ has length at least 4. By F3, $f(e^*) = \max_{e \in E^{-}(x_1)} \{f(e)\}$. Therefore $\lambda'$ cannot be extended along an edge in $E^-(x_1)$ and thus $\lambda$ has length at least 4.

- Lastly, suppose $x_0 \in Z_2$. Then $\deg_G x_1 \geq 3$, thus $E^+(x_1) \neq \emptyset$ while $E^-(x_1) = \{e_2\}$. Since $G$ does not have property P1 (in particular, $G$ does not have an end-vertex at distance 1 from $x_1$), each edge in $E^+(x_1)$ is contained in a $x_1$-$x_i$ direct $f$-ascent of length at least 2 for some $i > 1$. Again $\lambda$ has length at least 4.

Case 2 $v_1 \in X'$. By the definition of the edge ordering $f$ and since $G$ does not have property P2, $v_0 \in Z_1$ or $v_0 \in Y$.

Subcase 2.1 $v_0 \in Z_1$. Then by G4 and F2, $e_2$ is on a $v_1$-$x_i$ direct path for some $i > \rho(v_0) = \rho(v_1) + 1$. Since $v_0$ is an end-vertex and $G$ does not have property P1 (in particular, $G$ does not have an end-vertex at distance 1 or 2 from $v_1$), $x_i \in X$.

- If $x_i \in \Omega$, then by F3 the edge incident with $x_i$ with the largest label under $f$ is on the $v_1$-$x_i$ direct path $Q$ of length 3. Regardless of whether $e_2$ is on $Q$ or not, each extension of $e_1e_2$ to a $(3, f)$-ascent can be extended to a $(4, f)$-ascent. Hence $\lambda$ has length at least 4.

- Suppose that $x_i \in X - \Omega$. Then the $v_1$-$x_i$ direct path containing $e_2$ has length 2. Let $e_3 = v_2x_i$. Since $\rho(v_1) < \rho(x_i)$, the definition of $g'_k$ in G5 implies that either $g_k(x_i) \neq \max\{g_k(u) : u \in N_{G'}[x_i]\}$, or $g_k(x_i) = \max\{g_k(u) : u \in N_{G'}[x_i]\}$ and
$g_k(v_1) \neq \max\{g_k(u) : u \in N_{G'}(x_i)\}$. In the former case it is clear that $f(e_3) \neq \max_{e \in E(x_i)}\{f(e)\}$. In the latter case there exists a vertex $u' \in N_{G'}(x_i) - \{v_1\}$ such that $\rho(u') > \rho(v_1)$, and again it follows that $f(e_3) \neq \max_{e \in E(x_i)}\{f(e)\}$. Therefore $v_0 v_1 v_2 x_i$ can be extended to a $(4, f)$-ascent and $\lambda$ has length at least 4.

**Subcase 2.2** $v_0 \in Y$. Since $\lambda$ is a maximal $f$-ascent, $e_1$ is not on a $w$-$v_1$ direct $(2, f)$-ascent for any $w \in V(G)$; that is, $v_0 v_1$ cannot be extended to an $f$-ascent $wv_0 v_1$. Since all direct paths between vertices in $X$ have length 2 or 3, either $v_1 \in X$, or $v_1 \in Y$ and $v_2 \in X$. But by F1 and the maximality of $\lambda$ the latter case is impossible. Therefore $v_1 \in X$ and by definition of $E^{-}(v_1)$, $e_1 \notin E^{-}(v_1)$; hence $e_1 \in E^{+}(v_1)$. Thus $e_1$ is on a $v_1$-$x_i$ direct $f$-ascent $\lambda_1$ (containing $v_0$) of length at least 2 for some $i > \rho(v_1)$. Since $f(e_2) > f(e_1)$, F1 and F2 now imply that $e_2$ is on a $v_1$-$x_j$ direct $f$-ascent $\lambda_2$ for some $j \geq i > \rho(v_1)$. If $j = i$, then clearly $x_j \notin Z$, and if $j > i$, then $j > \rho(v_1) + 1$ and thus, by G4, $x_j \notin Z$. Hence $x_j \in X$ and $\lambda_2$ has length at least 2. However, we may assume that $\lambda_2$ has length 2, otherwise $\lambda$ has length at least 4. Let $e_3 = v_2 x_j$.

- Suppose $i < j$. We assume that $e_3$ is the edge incident with $x_j$ with the largest label under $f$, otherwise $\lambda$ has length at least 4. Then F3 and the facts that $\lambda_2$ has length 2, $G$ does not have property P1 and $x_j \notin Z$ imply that $x_j \in X - \Omega$. The assumption that $e_3$ has the largest label amongst the edges incident with $x_j$ now implies two facts: $j = \max\{\rho(u) : u \in N_{G'}[x_j]\}$ and $\rho(v_1) = \max\{\rho(u) : u \in N_{G'}(x_j)\}$. But $\rho(v_1) < j - 1$, hence we have a contradiction to the fact that $X_k = \emptyset$.

- Hence $i = j$. Let $e_4$ be the edge on $\lambda_1$ incident with $x_i$. Let $e^* = \max\{f(e) : e$ is incident with $x_i\}$. By F3 and the fact that $f(e_1) < f(e_2)$, $e_3 \neq e^*$. If $e_4 \neq e^*$, then $\lambda = e_1 e_2 e_3 ... e^*$ has length at least 4, hence assume $e_4 = e^*$. Now if $\lambda_1$ has
length 3, then $\lambda = e_1 e_2 e_3 e_4$ is a $(4, f)$-ascent, therefore we assume that $\lambda_1$ has length 2, and so by F3 all $v_1-x_j$ direct paths have length 2. Since $G$ does not have property $P1$, $N_{G'}(x_i) \neq \{v_1\}$. If $i < \rho(u)$ for any $u \in N_{G'}(x_i)$ then $\lambda$ has length at least 4. Assume then that $i > \rho(u)$ for all $u \in N_{G'}(x_i)$. Since $X_k = \emptyset$, $\rho(u) = i - 1$ for some $u \in N_{G'}(x_i)$, that is, $x_{i-1} \in N_{G'}(x_i)$. By F2, $e^* = e_4$ is on an $x_{i-1}-x_i$ direct path, hence $v_1 = x_{i-1}$. Let $x_{i'}$ be the vertex with the second largest index over all vertices in $N_{G'}(x_i)$ and let $\alpha$ be the edge with the largest label over all $x_{i'}-x_i$-direct paths. By F5, $f(\alpha) = f(e_4) - 0.5 > f(e_3)$ and therefore $\lambda$ has length at least 4.

Since the above cases exhaust all possibilities we conclude that $h(f) \geq 4$. $\diamond$

Therefore $\varepsilon(G) \geq 4$ and the result follows. $\square$

The next corollary follows immediately from Theorems 4.1.1, 1.5.2 and 1.5.1.

**Corollary 4.1.5.** If $G$ is a connected graph of order at least three, then $\varepsilon(G) = 3$ if and only if no vertex of $G$ is adjacent to two end-vertices or to two adjacent vertices of degree two, and

(i) $\text{diam}(L(G)) = 2$, or

(ii) $\text{diam}(L(G)) \geq 3$ and

(a) $G$ has a vertex $v$ and two end-vertices $u_1$ and $u_2$ such that $d(v, u_i) = i$, or

(b) $G$ has a vertex $v$ and three end-vertices $u_1, u_2, u_3$ such that $d(v, u_i) = 2$ for each $i$, or

(c) $G$ has an end-block $B = K_{2, m}$, $m \geq 2$, whose cut vertex is a vertex of $B$ of degree $m$. 
4.2 A class of graphs with depression three

Trees with depression three were characterized in [16] and the previous section characterizes graphs with depression three and no adjacent vertices of degree three or more. Furthermore, in Section 3.4 we showed that $\varepsilon(G) \leq 3$ whenever $G$ contains a graph $H$ as an attachment at a vertex $v$ such that $v$ is a vertex cover of $H$ and $\text{diam}(L(H)) \leq 2$. In this section we construct a large class of graphs with depression at most three which contains graphs with cycles and adjacent vertices of degree three or more. The construction is a generalization of the construction used in [16] to characterize trees with depression three.

Let $S'_k$ be the class of graphs $S_k$, $k \geq 1$, that can be constructed recursively in $k$ steps as follows. Let $S_0 = K_2$ with $V(S_0) = \{x_0, y_0\}$. Define $U_0 = \emptyset$ and $Y_0 = \{y_0\}$. Once $S_i$ has been constructed, construct $S_{i+1}$ by performing one of the following five operations.

O1: For any $y \in Y_i$, join $y$ to the vertex $u_1$ of a new edge $u_1x_1$; let $U_{i+1} = U_i \cup \{u_1\}$ and $Y_{i+1} = Y_i$.

O2: For any $y \in Y_i$, join $y$ to the central vertex $u_2$ of a new $P_5 : x_2, y_2, u_2, y'_2, x'_2$; let $U_{i+1} = U_i \cup \{u_2\}$ and $Y_{i+1} = Y_i \cup \{y_2, y'_2\}$.

O3: For any $y \in Y_i$, join $y$ to the vertices $u_3$ and $v_3$ of a new edge $u_3v_3$; let $U_{i+1} = U_i \cup \{u_3\}$ and $Y_{i+1} = Y_i$.

O4: For any $y \in Y_i$, join $y$ to the central vertex $y_4$ and an end vertex $u_4$ of a new $P_3 : u_4, y_4, x_4$; let $U_{i+1} = U_i \cup \{u_4\}$ and $Y_{i+1} = Y_i$.

O5: For any $y \in Y_i$, join $y$ to the vertex $v_5$ of the graph $G_5 = (\{x_5, x'_5, v_5, v'_5, v''_5, u_5, y_5\}, \{v_5y_5, y_5x_5, v'_5, v''_5, v'_5v''_5, v'_5u_5, u_5x'_5\})$; let $U_{i+1} = U_i \cup \{u_5\}$ and $Y_{i+1} = Y_i \cup \{y_5\}$. 


The operations **O1-O5** performed on $S_0$ are illustrated in Figure 4.5.

Let $S_k$ be the family of graphs such that $S_k \in S_k$ whenever $S_k \in S'_k$ and in the construction of $S_k$, any vertex $y \in Y_k$ is involved in **O3** at most once. Define $S = \bigcup_{k \geq 1} S_k$. Note that $S_0 = K_2$ is not in $S$. For a graph $S = S_k \in S$, define $U_S = U_k$ and $Y_S = Y_k$. Let $G$ be the class of all graphs $G_S$ formed by performing the following two operations.

**O6:** Add any set $A = A(G_S)$ of new vertices to a graph $S \in S$ and arbitrary edges between vertices in $A \cup U_S$.

**O7:** Add any arbitrary edges between vertices in $Y_S$.

**Remark 4.2.1.** Let $S \in S$. The operations **O1-O5** show that if $y \in Y_S$, then $y$ is adjacent to exactly one vertex of degree one.

We define the following property for a graph $G$.

**P3:** A graph $G$ has property **P3** with respect to an edge ordering $f$ and sets $U_G, Y_G \subseteq V(G)$, if for each $y \in Y_G$ for which a $U_G$-avoiding maximal $(2, f)$- or $(3, f)$-ascent
ends (starts) at $y$, there exists a $U_G$-avoiding maximal $(2, f)$- or $(3, f)$-ascent such that its last (first) edge is assigned the largest (smallest) value under $f$ over all edges incident with $y$.

**Lemma 4.2.2.** If $S \in \mathcal{S}$ and $f$ is an edge ordering of $S$ for which there exists a $U_S$-avoiding maximal $f$-ascent of length at most three and all such ascents start or end in $Y_S$, then $S$ has property $P3$ with respect to $f$, $U_S$ and $Y_S$.

**Proof.** Let $y \in Y_S$ be a vertex for which a $U_S$-avoiding maximal $(2, f)$- or $(3, f)$-ascent ends at $y$, $A_y$ be the set of all such $f$-ascents, and $\lambda = aby$ or $\lambda = acby$, where $\lambda$ is the maximal $f$-ascent such that its last edge $by$ is assigned the largest value over all edges of ascents in $A_y$. Let $x$ be the end vertex adjacent to $y$. Clearly, $f(by) > f(yx)$.

Suppose to the contrary that $f(by) \neq \max_{v \in N(y)} \{f(vy)\}$. Then there exists an edge $wy \in E(S)$ such that $w \neq b$ and $f(wy) = \max_{v \in N(y)} \{f(vy)\}$. Since $\lambda$ is a maximal $f$-ascent, $w$ is a vertex of $\lambda$. By the construction of graphs in $\mathcal{S}$, all cycles of $S$ have length three and we may assume that $wby$ is a 3-cycle. If the cycle was introduced by $O3$, then $\lambda = wby$, $b \in U_S$, $w \notin U_S \cup Y_S$, and both $w$ and $b$ have degree 2. But since $f(yw) > f(wb)$ and $\deg(w) = 2$, $xyw$ is a $U_S \cup Y_S$-avoiding maximal $f$-ascent, a contradiction.

Suppose then that the cycle $wby$ was introduced by $O4$. Then $w \in Y_S$ and there exists an end vertex $x'$ adjacent to $w$. If $f(x'w) < f(wy)$, then $x'wy$ is a maximal $f$-ascent, which contradicts our choice of $\lambda$. Now if $f(x'w) > f(wy)$, then $xywx'$ is a maximal $f$-ascent which is also a contradiction.

A similar argument may be used to show that if a $U_S$-avoiding maximal $f$-ascent of length at most three starts at $y$, then there exists a $U_S$-avoiding maximal $(2, f)$- or $(3, f)$-ascent $\lambda$ such that for the initial edge $yb$ of $\lambda$, $f(yb) = \min_{v \in N(y)} \{f(yv)\}$. $\square$
\textbf{Theorem 4.2.3.} For each $S \in S$, $\varepsilon(S) \leq 3$ and $U_S$ is a $k$-kernel of $S$ for some $k \in \{2, 3\}$.

\textit{Proof.} The proof is by induction on $k$, the number of steps used to construct $S = S_k$ from $K_2 = S_0$. To prove the result we must show that for any edge ordering $f$ of $S$ there exists a $U_S$-avoiding maximal $(2, f)$- or $(3, f)$-ascent.

If $k = 1$, then $S$ was constructed by performing one of the operations $O1$-$O5$ on $K_2 = S_0$

\textbf{Case 1 O1} is performed. Then $S = P_4$ and $U_S = \{u_1\}$. Since diam$(L(S)) = 2$ and $N[u_1]$ is a vertex cover of $S$, the result follows from Theorem 3.4.1.

\textbf{Case 2 O2} is performed. Then $S = S(2, 2, 2)$ and $U_S = \{u_2\}$. Consider any edge ordering $f$ of $S$. Without loss of generality we may assume $f(x_0y_0) < f(y_0u_2)$. If $f(y_0u_2) > y(u_2y_2)$, then either $x_2y_2u_2y_0$ (if $f(x_2y_2) < f(y_2u_2)$) or $y_2u_2y_0$ (if $f(x_2y_2) > f(y_2u_2)$) are $u_2$-avoiding maximal $f$-ascents of $S$ with length at most three. The same argument applies if $f(y_0u_2) > f'(u'_2y'_2)$. Suppose then that $f(y_0u_2) < f(u_2y_2)$ and $f(y_0u_2) < f(u'_2y'_2)$. To avoid a $u_2$-avoiding maximal $f$-ascents of length at most three, both $x_0y_0u_2x_2y_2$ and $x_0y_0u_2x'_2y'_2$ are maximal $(4, f)$-ascent of $S$. This implies either $y_2u_2y'_2x'_2$ (if $f(y_2u_2) < f(u_2y'_2)$) or $y'_2u_2y_2x_2$ (if $f(y_2u_2) > f(u_2y'_2)$) is a $u_2$-avoiding maximal $f$-ascent of the required length.

\textbf{Case 3 O3} is performed. Then $U_S = \{u_3\}$. Since diam$(L(S)) = 2$ and $N[u_3]$ is a vertex cover of $S$, the result follows from Theorem 3.4.1.

\textbf{Case 4 O4} is performed. Then $U_S = \{u_4\}$. Since diam$(L(S)) = 2$ and $N[u_4]$ is a vertex cover of $S$, once again, the result follows from Theorem 3.4.1.

\textbf{Case 5 O5} is performed. Then $U_S = \{u_5\}$. Suppose to the contrary that $u_5$ is not a 3-kernel of $S$. Let $f$ be an edge ordering $f$ of $S$ for which all maximal $(2, f)$- and $(3, f)$-ascents either start or end at $u_5$. Necessarily, either $x_0y_0u_5y_5x_5$ or its reverse is a $(4, f)$-ascent of $S$, and without loss of generality we assume the former. Furthermore,
by our assumption, neither $v_5''v_5's$ nor its reverse is a maximal $(2, f)$-ascent of $S$, which implies either $v_5''v_5'su_5$, $v_5''v_5'su_5x_5'$, or the reverse of one of these paths is a maximal $f$-ascent. We need only consider the former two of these cases since for any $f$-ascent present in an edge ordering extended from these cases, its reverse will be present in one of the latter cases–with the roles of $x_0$ and $y_0$ switched with $x_5$ and $y_5$ respectively. These cases are shown in Figure 4.6 where the paths labelled $abcd$ and $rst$ are $f$-ascents of $S$. Moving forward we will refer to the labels in this figure to simplify notation.

Firstly, suppose $rst$ is a maximal $f$-ascent. Then $t > \pi$ and, since $u_5$ is not a 3-kernel of $S$, $\pi t \phi r$ is a $(4, f)$-ascent. But then $t < \phi < r < s < t$, which is a contradiction.

Secondly, suppose that $rst\pi$ is an $f$-ascent of $S$. If $r < b$, then since $t > r$, either $rb$ (if $\phi > r$) or $\phi rb$ (if $\phi < r$) is a maximal $f$-ascent, which in either case is a contradiction. Therefore we may assume $r > b$. We may also assume that $\phi > r$, or else $abr$ is a $u_5$-avoiding maximal $f$-ascent. Furthermore, if $c > r$, then $rcd$ is a maximal $f$-ascent, so we may assume $c < r$. Now if $\phi < s$, then $\phi s$ is a $u_5$-avoiding maximal $f$-ascent, which is a contradiction. Thus we may assume $\phi > s$. Since $r < s$ by assumption, we now have $c < r < s < \phi$, which implies that $cs\phi$ is a maximal $f$-ascent, and again we have a contradiction.
This case completes the basis step of the proof.

Assume the result to be true for graphs in $S$ constructed from $K_2$ in fewer than $k \geq 2$ steps. Consider any graph $S = S_k$ constructed from $K_2$ in $k$ steps, and any edge ordering $f$ of $S$.

Suppose that in the construction of $S$ one of O1, O2 or O5 was performed at least once. Then $S$ contains $y \in Y_S$ such that $y$ was joined to a new vertex in step $i \geq 2$ and such that $y$ is incident with at least two bridges. Let $y \in Y_S$ be incident to at least two bridges, and $x$ be the vertex of degree one adjacent to $y$. Note that one of the bridges incident with $y$ is $xy$. Let $G_1, G_2, ..., G_m$ be the components of $S - y$ which consist of at least two vertices. For each $1 \leq i \leq m$, let $G'_i$ be the subgraph induced by $\{x, y\} \cup V(G_i)$. Then each $G'_i \in S_j$ for some $1 \leq j < k$. If $G'_i \cong S_j \in S_j$, then let $U_{G'_i} = U_j$ and $f'_i$ be the edge ordering of $G'_i$ induced by $f$.

Since $y$ is incident with a bridge other than $xy$, there exists an $i$, say $i = 1$, such that $\deg_{G'_1}(y) = 2$. Let $H = S - G_1$ and $f_H$ be the edge ordering of $H$ induced by $f$. Then $H \cong S_j \in S_j$ for some $1 \leq j < k$. Let $U_H = U_j$. By the induction hypothesis there exists at least one $U_H$-avoiding maximal $(2, f_H)$- or $(3, f_H)$-ascent and we may assume that all such maximal $f_H$-ascents start or end at $y$, or else there exists a $U_S$-avoiding maximal $f$-ascent of length at most three in $S$ and we are done. Without loss of generality assume that there exists a $U_H$-avoiding maximal $f_H$-ascent of length at most three which ends at $y$. Then by Lemma 4.2.2 there exists a maximal $f_H$-ascent $\lambda = aby$ or $\lambda = acby$ such that $f_H(by) = \max_{v \in N(y)} \{f_H(vy)\}$ and $a \in V(H) - U_H$.

Let $b_1$ be the neighbour of $y$ in $G_1$. By the induction hypothesis, there exists at least one $U_{G'_1}$-avoiding maximal $(2, f'_1)$- or $(3, f'_1)$-ascent and we may assume that all such maximal $f'_1$-ascents start or end at $y$, or else we are done. Thus either $b_1y$ is the initial or final edge of a $U_{G'_1}$-avoiding maximal $f'_1$-ascent $\alpha$ of length at most three. If $\alpha$ starts at $y$, then $f'_1(b_1y) < f(xy) < f(by)$ and $\lambda$ is a $U_S$-avoiding maximal $f$-ascent
Figure 4.7: $S$ is constructed from $S_{k-1}$ by joining $y$ to $y_4$ and $u_4$ of a new $P_3: u_4, y_4, x_4$ of length at most three. If $\alpha$ ends at $y$, then in $S$ either $\alpha$ (if $f'_1(b_1y) > f_H(by)$) or $\lambda$ (if $f'_1(b_1y) < f_H(by)$) is a $U_S$-avoiding maximal $f$-ascent of length at most three.

Suppose then that only $O3$ and $O4$ are used in the construction of $S$.

Firstly, suppose that $S$ is constructed from $S_{k-1}$ by joining $y$ to $y_4$ and $u_4$ of a new $P_3: u_4, y_4, x_4$ (see Figure 4.7). Then $U_S = U_{k-1} \cup \{u_4\}$. Let $f'$ be the edge ordering of $S_{k-1}$ induced by $f$, and $x$ the end vertex adjacent to $y$. By the induction hypothesis, in $S_{k-1}$ there exists a $U_{k-1}$-avoiding maximal $f'$-ascent of length at most three. We may assume that all such $f'$-ascents start or end at $y$ or else we are done. Without loss of generality assume that there exists a $U_{k-1}$-avoiding maximal $f'$-ascent of length at most three which ends at $y$. By Lemma 4.2.2 there exists a maximal $f'$-ascent $\lambda = aby$ or $\lambda = acby$ such that $f'(by) = \max_{v \in N(y) \setminus y_4} \{f'(vy)\}$ and $a \in V(S_{k-1}) - U_{k-1}$.

If $\lambda$ is a maximal $f$-ascent, then we are done so we may assume that either

$$f(yu_4) > f(by) \text{ or } f(yy_4) > f(by). \quad (4.1)$$

- Suppose $f(yu_4) > f(by)$. Then $f(yu_4) = \max_{v \in N(y) \setminus y_4} \{f(vy)\}$.

  - If $f(yu_4) < f(u_4y)$, then either $y_4u_4y$ or $x_4y_4u_4y$ is a $U_S$-avoiding maximal $f$-ascent.
Suppose \( f(y_4 u_4) > f(u_4 y) \). Then \( f(x_4 y_4) > f(y_4 u_4) \), or else \( x_4 y_4 u_4 \) is a \( U_S \)-avoiding maximal \( f \)-ascent.

- If \( f(y y_4) > f(y_4 x_4) \), then \( f(y_4 y) = \max_{v \in N(y)} \{ f(v y) \} \) and \( x_4 y_4 y \) is a \( U_S \)-avoiding a maximal \( f \)-ascent.

- If \( f(y y_4) < f(y_4 x_4) \), then either \( x y y_4 x_4 \) (if \( f(x y) < f(y y_4) \)) or \( y_4 y x \) (if \( f(x y) > f(y y_4) \)) is a \( U_S \)-avoiding maximal \( f \)-ascent.

• Suppose then that \( f(y u_4) < f(b y) \). Then by (4.1), \( f(y y_4) > f(b y) \) and \( f(y y_4) = \max_{v \in N(y)} \{ f(v y) \} \). This implies either \( x y y_4 x_4 \) (if \( f(y y_4) < f(y_4 x_4) \)) or \( x_4 y_4 y \) (if \( f(y y_4) > f(y_4 x_4) \)) is a maximal \( f \)-ascent, neither of which starts or ends in \( U_S \).

Secondly, suppose that \( S \) is constructed from \( S_{k-1} \) by joining \( y \in Y_{k-1} \) to the vertices \( v_3 \) and \( u_3 \) of a new edge \( u_3 v_3 \). Then \( U_S = U_{k-1} \cup \{ u_3 \} \). Let \( S' \) be the subgraph of \( S \) induced by \( \{ x, y, v_3, u_3 \} \), \( f' \) the edge ordering of \( S' \) induced by \( f \), and \( f'' \) the edge ordering of \( S_{k-1} \) induced by \( f \). Note that \( S' \cong S_1 \in S \). Let \( U_{S'} = \{ u_3 \} \). By the basis step, there exists a \( u_3 \)-avoiding maximal \( f' \)-ascent \( \alpha \) of length at most three. We may assume that \( \alpha \) either starts or ends at \( y \), or else we are done. Without loss of generality assume that \( \alpha \) starts at \( y \). Necessarily, \( \alpha = y u_3 v_3 \) and \( f'(x y) > f'(y u_3) \). Furthermore, we may assume that \( f'(y v_3) > f'(y u_3) \), or else \( f'(y v_3) < f'(y u_3) < f'(x y) \) and \( v_3 y x \) is a \( U_S \)-avoiding maximal \( f \)-ascent of length two and we are done. Thus \( f'(y u_3) = \min_{v \in N(y)} \{ f'(v y) \} \).

By the induction hypothesis, there exists a \( U_{k-1} \)-avoiding maximal \( f'' \)-ascent \( \lambda \) of length at most three in \( S_{k-1} \). We may assume that \( \lambda \) starts or ends at \( y \) or else we are done. If \( \lambda \) starts at \( y \), then by Lemma 4.2.2 there exists a maximal \( f'' \)-ascent \( \lambda' = a b y \) or \( \lambda' = a c b y \) such that \( f''(b y) = \min_{v \in N(y)} \{ f''(v y) \} \) and \( a \in V(S_{k-1}) - U_{k-1} \). This implies either \( \lambda' \) or \( \alpha \) is a \( U_S \)-avoiding maximal \( f \)-ascent of length at most three. Assume then that \( \lambda \) ends at \( y \), and furthermore, that all \( U_{k-1} \)-avoiding maximal \( f'' \)-
Figure 4.8: $S$ is constructed from $S_{k-1}$ by joining $y$ to $u_3$ and $v_3$ of a new edge $\{u_3, v_3\}$.

ascent of length at most three end at $y$. Then there exists an edge $vy \in E(S_{k-1})$ such that $f''(vy) < f'(yu_3)$ otherwise $\alpha$ is a $U_S$-avoiding maximal $f$-ascent of length two and we are done. Let $wy$ be the edge in $S_{k-1}$ such that $f''(wy) = \min_{v \in N(y)} \{f''(vy)\}$. Then $f''(wy) < f'(yu_3) < f'(vy)$ which implies $f(wy) = \min_{v \in N(y)} \{f(vy)\}$. Recall that we have assumed $S$ is constructed using only $O3$ and $O4$, and that for any graph in $S$, each vertex in $y \in Y_S$ is involved in $O3$ at most once. Thus the edge $wy$ was introduced by $O4$, which implies either $w = u' \in U_{k-1}$ and is adjacent to a vertex $y' \in Y_{k-1}$, or $w = y' \in Y_{k-1}$ and is adjacent to a vertex $u' \in U_{k-1}$. In either case, let $x'$ be the vertex of degree one adjacent to $y'$ – see Figure 4.8.

Suppose $w = y'$. If $f(x'y') < f(y'y)$, then, since $f(y'y) < f(xy)$, $x'y'yx$ is a $U_S$-avoiding maximal $f$-ascent of length three. If $f(x'y') > f(y'y)$, then, since $f(y'y) = \min_{v \in N(y)} \{f(vy)\}$, $yy'x'$ is a $U_S$-avoiding maximal $f$-ascent of length two.

Suppose then that $w = u'$. Let $G_1$ be the component of $S - y$ containing $w$, and $G_1'$ the subgraph of $S$ induced by $V(G_1) \cup \{y, x\}$. Then $G_1' \cong S_j \in S_j$ for some $1 \leq j < k$. Let $U_{G_1'} = U_{S_j}$ and $f'_1$ be the edge ordering of $G_1'$ induced by $f$. By the induction hypothesis, there exists a $U_{G_1'}$-avoiding maximal $f'_1$ ascent of length at most three in $G_1'$. Necessarily all $U_{G_1'}$-avoiding maximal $f'_1$ ascent of length at most three start or end at $y$ or else we are done. Suppose there exists such an ascent which starts
at $y$. By Lemma 4.2.2 there exists a $U_{G_1'}$-avoiding maximal $f'_1$ ascent $\lambda$ of length at most three whose initial edge is $yw = yu'$. But since $f(yu') = \min_{v \in N(y)} \{f(vy)\}$, $\lambda$ is also a $U_S$-avoiding maximal $f$-ascent which is a contradiction. Hence we may assume that there exists a $U_{G_1'}$-avoiding maximal $f'_1$-ascent $\lambda$ of length at most three which ends at $y$. Since $f'_1(u'y) = \min_{v \in N(y)} \{f'_1(vy)\}$, $f'_1(u'y) > f'_1(xy)$ and the last edge of $\lambda$ is $y'y$. This implies $f'_1(yy') > f'_1(xy)$ or equivalently, $f(yy') > f(yx)$. Necessarily, $f(x'y') < f(yy')$, or else $xyy'x'$ is a $U_S$-avoiding maximal $f$-ascent of length at most three. Now we look at three cases for the value of $f(y'u')$. In these cases we assume that $\deg_S(y') \geq 3$ or else either $xyy'$ (if $f(y'u') < f(yy')$) or $yu'y'$ (if $f(y'u') > f(yy')$) is a $U_S$-avoiding maximal $f$-ascent.

**Case 1** $f(yu') < f(y'u') < f(x'y')$. Then $yu'y'x'$ is a $U_S$-avoiding maximal $f$-ascent.

We define the following to aid us in the next two cases. Let $H_1$ be the component of $S_{k-1} - y'$ containing $w$, $H'_1$ the the subgraph of $S_{k-1}$ induced by $V(H_1) \cup \{y', x'\}$, and $H'_2$ the subgraph of $S_{k-1}$ induced by $V(S_{k-1}) - V(H_1)$. Then each $H_i \in S_{\ell}$ for some $1 \leq \ell < k$. If $H_i' \cong S_{\ell} \in S_{\ell}$, then let $U_{H_i'} = U_{\ell}$ and $f_i$ be the edge ordering of $H_i'$ induced by $f$.

**Case 2** $f(y'u') < f(x'y')$ and $f(y'u') < f(u'y)$. Then, in $H_1'$, $y'u'yx$ is a $U_{H_1'}$-avoiding maximal $f_1$-ascent starting at $y'$ and $xyy'$ is a $U_{H_1'}$-avoiding maximal $f_1$-ascent ending at $y$. By the induction hypothesis, in $H_2$, there exists a $U_{H_2'}$-avoiding maximal $f_2$-ascent of length at most three. We may assume that all such $f_2$-ascents start or end at $y'$. Without loss of generality suppose there exists a $U_{H_2'}$-avoiding maximal $f_2$-ascent of length at most three that ends at $y'$. By Lemma 4.2.2, there exists a $U_{H_2'}$-avoiding maximal $f_2$-ascent $\lambda = aby'$ or $\lambda = acby'$ such that $f_2(by') = \max_{v \in N(y')} \{f_2(vy')\}$. Thus, in $S$, either $\lambda$ or $xyy'$ is a $U_S$-avoiding maximal $f$-ascent of length at most three.

**Case 3** $f(y'u') > f(x'y')$. Then either $xyy'$ (if $f(y'u') < f(yy')$) or $yu'y'$ (if $f(y'u') >
Figure 4.9: A graph $G$ constructed from $S_0$ by performing $O_3$ twice at $y_0$, and an edge labelling $f$ of $G$ for which every maximal $f$-ascent of length at most three starts or ends in $U_G = \{u_3, u'_3\}$.

$f(yy')$ is a $U_{H'_1}$-avoiding maximal $f_1$-ascent which ends at $y'$. Again, by the induction hypothesis, in $H_2$, there exists a $U_{H'_2}$-avoiding maximal $f_2$-ascent of length at most three and we assume that all such $f_2$-ascents start or end at $y'$. Suppose there exists a $U_{H'_2}$-avoiding maximal $f_2$-ascent of length at most three that ends at $y'$. By Lemma 4.2.2, there exists a $U_{H'_2}$-avoiding maximal $f_2$-ascent $\lambda = aby'$ or $\lambda = acby'$ such that $f_2(by') = \max_{v \in N(y') \{f_2(vy')\}}$. Therefore, in $S$, either $\lambda$, $xyy'$, or $xu'y'$ is a $U_S$-avoiding maximal $f$-ascent of length at most three. Suppose then that there exists a $U_{H'_2}$-avoiding maximal $f_2$-ascent of length at most three that starts at $y'$. By Lemma 4.2.2, there exists a $U_{H'_2}$-avoiding maximal $f_2$-ascent $\lambda = aby'$ or $\lambda = acby'$ such that $f_2(by') = \min_{v \in N(y')} \{f_2(vy')\}$. Necessarily, $f(by') < f(y'x')$, and since $f(y'y) > f(y'x')$ and $f(y'u') > f(y'x')$, $\lambda$ is a $U_S$-avoiding maximal $f$-ascent of length at most three.

In the construction of $S_k \in S_k$, any vertex $y \in Y_k$ is involved in $O_3$ at most once. If not, then $U_k$ is no longer a 3-kernel of $S_k$. Consider the graph $G$ shown in Figure 4.9, which is constructed from $S_0$ by performing $O_3$ twice at $y_0$. Let $U_G = \{u_3, u'_3\}$. For the edge labelling $f$ of $G$ shown in the figure, any maximal $f$-ascent of length at most three starts or ends in $U_G$.

Recall that the graphs $G_S \in G$ are obtained from a graph $S \in S$ by performing
operations $O_6$ and $O_7$. We now show that these graphs also have depression at most three.

**Theorem 4.2.4.** For each $G_S \in \mathcal{G}$, $\varepsilon(G) \leq 3$.

**Proof.** Let $G_S'$ be constructed from $S \in \mathcal{S}$ by adding $n \geq 0$ edges between vertices in $Y_{G_S'} = Y_S$ and let $U_{G_S'} = U_S$. If $n = 0$, then $G_S' \in \mathcal{S}$ and by Theorem 4.2.3, $\varepsilon(G'(S)) \leq 3$ and $U_{G_S'}$ is a $k$-kernel of $G_S'$, where $k \in \{2, 3\}$.

Suppose that $n \geq 1$. Let $f$ be an edge ordering of $G_S'$, and $f'$ the edge ordering of $S$ induced by $f$. If there exists a $(U_S \cup Y_S)$-avoiding maximal $f'$-ascent of length at most three, then $h(f) \leq 3$. Suppose then that there does not exist a $(U_S \cup Y_S)$-avoiding $f'$-ascent of length at most three. By Theorem 4.2.3 there exists a $U_S$-avoiding maximal $f'$-ascent of length at most three in $S$, thus all maximal $U_S$-avoiding $(2, f')$- or $(3, f')$-ascents start or end in $Y_S$.

Without loss of generality we assume there exists a maximal $U_S$-avoiding ascent of length at most three which ends in $Y_S$. By Lemma 4.2.2, $S$ has property $P_3$, which implies that there exists a maximal $f'$-ascent $\lambda = aby_1$ or $\lambda = acby_1$ such that $y_1 \in Y_S$ and $f'(by_1) = \max_{v \in N_S(y_1)} \{f'(vy_1)\}$. Suppose that in $G_S'$ there exists an edge $y_1w$ such that $f(y_1w) = \max_{v \in N_{G_S'}(y_1)} \{f(vy_1)\} > f(by_1)$ and $w$ is not a vertex of $\lambda$. Necessarily, $y_1w \notin E(S)$ which implies $w \in Y_S$. Let $w = y_2$, and $x_1$ and $x_2$ be the vertices of degree one adjacent to $y_1$ and $y_2$ respectively. Since $\lambda$ is a maximal $f'$-ascent in $S$, it follows that $f(y_1x_1) < f(by_1) < f(y_1y_2)$. Therefore, either $x_1y_1y_2x_2$ (if $f(y_2x_2) > f(y_1y_2)$) or $x_2y_2y_1$ (if $f(y_2x_2) < f(y_1y_2)$) is a $U_{G_S'}$-avoiding maximal $f$-ascent. Hence $U_{G_S'}$ is a $k$-kernel of $G_S'$, where $k \in \{2, 3\}$.

Let $G_S \in \mathcal{G}$ be constructed from $G_S'$ by adding any set $A = A(G_S)$ of new vertices to $G_S'$ and arbitrary edges between vertices in $A \cup U_{G_S'}$. Then by Theorem 3.1.2, $\varepsilon(G_S) \leq 3$. 

Note that $\kappa(G_S) = 1$ for each $G_S \in \mathcal{G}_S$. We also note that for each graph $G$ in
the classes of graphs with depression three defined in Sections 1.5, 3.4, and 4.1, either \(\text{diam}(L(G)) = 2\) or \(\kappa(G) = 1\). The graph \(H\) shown in Figure 4.10 is an example of a graph with \(\kappa(H) > 1\), \(\text{diam}(L(H)) > 2\), and \(\varepsilon(H) = 3\). We provide the following argument to support the claim that \(\varepsilon(H) = 3\). Suppose to the contrary that \(\varepsilon(H) > 3\).

Let \(f : E(H) \to \{1, 2, \ldots, 8\}\) be an edge ordering of \(H\) for which every maximal \(f\)-ascent has length at least 4. Since \(e_1\) and \(e_8\) are the only edges in \(H\) which are at distance three in \(L(H)\), it follows that \(\{f(e_1), f(e_8)\} = \{1, 8\}\).

Without loss of generality we may assume that \(f(e_1) = 1\) and \(f(e_8) = 8\). Without loss of generality we may also assume that \(f(e_5) = \max\{f(e_2), f(e_3), f(e_4), f(e_5)\}\).

Then, since \(h(f) > 3\) and \(f(e_4) < f(e_5)\), it follows that \(e_7e_2e_4e_5\) is a maximal \(f\)-ascent. However, this implies \(e_1e_2\) is a maximal \(f\)-ascent, a contradiction.
Chapter 5

Edge Colourings

Up to now the edge orderings we have considered were complete orderings; no two edges had the same label. Edge orderings by definition are proper edge colourings since adjacent edges receive different labels/colours. However, in some cases it is also possible to use proper edge colourings in fewer than $|E(G)|$ colours as partial orderings with flatness equal to $\varepsilon(G)$. For example, colouring the edges of $C_4$ in sequence 1, 2, 3, 2 gives a partial ordering with flatness $3 = \varepsilon(C_4)$. In this chapter we consider the following question.

What is the smallest integer $r$ such that there exists a proper edge colouring $c: E(G) \rightarrow \{1, 2, \ldots, r\}$ for which a shortest maximal ascent has given length $k$?

Path lengths and colourings have previously been linked. There is an elementary relationship between edge colourings and the altitude of a graph, $\alpha(G)$. Upper bounds on $\alpha$ are established using edge colourings in the following method, also used in [3, 4, 7, 9, 13, 20, 23]. We colour the edges of the graph (not necessarily obtaining a proper colouring) in $t \geq 2$ colours. We then obtain an edge ordering $f$ by first labelling all the edges of one colour with consecutive integers, and then the edges of the next colour, etc. In any $f$-ascent, once we use edges of one colour, we cannot
use edges of a previous colour since all such edges have smaller labels. In [23], this method is used to show that $\alpha(G) \leq \chi_1(G)$, where $\chi_1(G)$ denotes the edge chromatic number of $G$.

Also, Gallai [12], Hasse [14], Roy [21] and Vitaver [22] independently give a relation between the longest directed path for an orientation $D$ of a graph $G$, which we denote by $\ell_D(G)$, and the chromatic number of $G$, $\chi(G)$.

**Theorem 5.0.5.** [12, 14, 21, 22] For any orientation $D$ of a graph $G$, $\ell_D(G) + 1 \geq \chi(G)$. Moreover, there exists an orientation $D$ of $G$ such that $\ell_D(G) + 1 = \chi(G)$.

Since an edge ordering of $G$ is a vertex ordering of $L(G)$, from Theorem 5.0.5 we again arrive at $\alpha(G) \leq \chi_1(G)$.

In Section 5.1 we define the $\varepsilon$-ascent chromatic index of a graph, denoted $\chi_\varepsilon(G)$. We also state some trivial bounds and characterize the class of graphs $G$ for which $\chi_\varepsilon(G) = \chi_1(G)$, the edge chromatic index of $G$. In Section 5.2 we provide a lower bound for $\chi_\varepsilon(G)$ in terms of $\varepsilon(G)$ and $\chi_1(G)$. In Section 5.3 we determine the $\varepsilon$-ascent chromatic index for paths and cycles, and show that $\chi_\varepsilon(G) = \varepsilon(G)$ if and only if $G$ is a path or an even cycle. In Section 5.4 we characterize trees for which $\chi_\varepsilon(T) = |E(T)|$. In this section we also define the $k$-ascent chromatic index of a graph, denoted $\chi_{(k)}(G)$, and determine an upper bound for $\chi_{(3)}(T)$, which is in turn used to establish a bound for the $\varepsilon$-chromatic index of trees with depression three. Finally, in Section 5.5 we provide an upper bound for $\varepsilon$-chromatic index of complete graphs and determine $\chi_\varepsilon(K_n)$ for $2 \leq n \leq 5$ and bounds for $\chi_\varepsilon(K_6)$.

## 5.1 The $\varepsilon$-ascent chromatic index of a graph

For an edge colouring we assume the colours are integers and the flatness of a colouring $c$, denoted by $h(c)$, is analogous to the flatness of an edge ordering. We define the $\varepsilon$-
ascent chromatic index of a graph, denoted \( \chi_\varepsilon(G) \), as the minimum number of colours of a proper edge colouring \( c \) with \( h(c) = \varepsilon(G) \).

Therefore, to prove that \( \chi_\varepsilon(G) = k \), we must show that

(a) there exists a proper \( k \)-edge colouring \( c \) such that \( h(c) = \varepsilon(G) \), i.e. \( \chi_\varepsilon(G) \leq k \), and

(b) for all proper \( (k - 1) \)-edge colourings \( c \), \( h(c) < \varepsilon(G) \), i.e. \( \chi_\varepsilon(G) \geq k \).

Denote the chromatic index (the minimum number of colours in a proper edge colouring) of \( G \) by \( \chi_1(G) \).

**Remark 5.1.1.** For any graph \( G \),

(a) \( \varepsilon(G) \leq \chi_\varepsilon(G) \),

(b) \( \chi_1(G) \leq \chi_\varepsilon(G) \), and

(c) \( \chi_\varepsilon(G) \leq |E(G)| \).

As mentioned in Remark 5.1.1, \( \chi_\varepsilon(G) \geq \chi_1(G) \). We next characterize graphs \( G \) for which \( \chi_\varepsilon(G) = \chi_1(G) \).

**Proposition 5.1.2.** For any graph \( G \), \( \chi_\varepsilon(G) = \chi_1(G) \) if and only if \( \varepsilon(G) \leq 2 \).

**Proof.** If \( \varepsilon(G) = 1 \), then \( G \) has exactly one edge and thus \( \chi_\varepsilon(G) = \chi_1(G) = 1 \). If \( \varepsilon(G) = 2 \), then \( G \) has at least two adjacent edges. Any proper edge colouring of \( G \) has the property that its maximal ascents have length at least two, thus \( \chi_\varepsilon(G) = \chi_1(G) \). Now suppose \( \varepsilon(G) \geq 3 \) and \( \chi_\varepsilon(G) = k \), and let \( c : E(G) \to \{1, 2, \ldots, k\} \) be a proper edge colouring of \( G \) in \( \chi_\varepsilon(G) \) colours whose maximal ascents have lengths at least three. Since \( G \) has no maximal \( (2, c) \)-ascents, the colours 1 and \( k \) do not occur at the same vertex, for if \( c(uv) = 1 \) and \( c(vw) = k \), then \( uvw \) is a maximal \( (2, c) \)-ascent. Therefore each edge \( e \) such that \( c(e) = k \) can be recoloured with colour 1 to give a proper \( (k - 1) \)-edge colouring of \( G \). Thus \( \chi_1(G) < \chi_\varepsilon(G) \). \( \square \)
5.2 A lower bound

As mentioned in Remark 5.1.1, $\chi(\varepsilon(G)) \geq \chi_1(G)$, and by Proposition 5.1.2, equality holds if and only if $\varepsilon(G) = 2$. We now improve this bound for $\varepsilon(G) \geq 3$.

**Proposition 5.2.1.** If $\varepsilon(G) \geq 2$, then $\chi(\varepsilon(G)) \geq \chi_1(G) + \varepsilon(G) - 2$.

**Proof.** If $\varepsilon(G) = 2$, the result follows from Remark 5.1.1. Let $G$ be a graph with $\varepsilon(G) = k \geq 3$. Suppose, to the contrary, that $\chi(\varepsilon(G)) = m \leq \chi_1(G) + k - 3$ and consider a proper $m$-edge colouring $c$ of $G$ for which $h(c) = \varepsilon(G)$. Suppose that an edge $e$ for which $c(e) = 1$ is adjacent to an edge $e'$ with $c(e') = \chi_1(G)$. Then any maximal ascent containing $ee'$ has length at most $k - 1$, a contradiction. In general, if an edge $e$ for which $c(e) = i$ is adjacent to an edge $e'$ for which $c(e') = \chi_1(G) + i - 1$, then $h(c) \leq i + 1 + (\chi_1(G) + k - 3) - (\chi_1(G) + i - 1) = k - 1$. Thus, for $1 \leq i \leq k - 2$, an edge assigned colour $i$ is not adjacent to an edge assigned colour $\chi_1(G) + i - 1$. Therefore we may reassign the edges coloured $i$ with colour $\chi_1(G) + i - 1$ for each $i$ such that $1 \leq i \leq k - 2$. Since $k \geq 3$, this gives us a proper edge colouring in $m - (k - 2) \leq \chi_1(G) - 1$ colours, a contradiction. \qed

**Corollary 5.2.2.** If $\varepsilon(G) \geq 2$, then $\chi(\varepsilon(G)) \geq \Delta(G) + \varepsilon(G) - 2$.

For each $k \geq 3$, there exists a graph $G$ with $\varepsilon(G) = k$ for which $\chi(\varepsilon(G))$ realizes the bound in Proposition 5.2.1. For example, consider the spider $S(1, t, t)$, where $t \geq 2$. By Theorem 1.5.11, $\varepsilon(S(1, t, t)) = t + 1$. Furthermore, $\chi_1(S(1, t, t)) = 3$, and by Proposition 5.2.1, $\chi(\varepsilon(S(1, t, t)) \geq t + 1$. To establish $\chi(\varepsilon(S(1, t, t)) \leq t + 2$ we consider the edge colouring $c$ of $S(1, t, t)$ shown in Figure 5.1, for which it is easy to verify that $h(c) = t + 1$. Thus for all $t \geq 2$, $\chi(\varepsilon(S(1, t, t)) = t + 2$.

The difference $\chi(\varepsilon(G)) - (\chi_1(G) + \varepsilon(G) - 2)$ can also be arbitrarily large. Consider the spider $S(t, t, t)$. By Theorem 1.5.11, $\varepsilon(S(t, t, t)) = t + 1$. Let $v$ be the vertex of $S(t, t, t)$ with degree three, $e_{1,1}, e_{2,1}, e_{3,1}$ the edges incident with $v$, and $\lambda = e_{i,1}e_{i,2}\cdots e_{i,t}$ the
Let $c$ be a proper $r$-edge colouring with $h(c) = t + 1$. Without loss of generality we may assume that $c(e_{1,1}) < c(e_{2,1}) < c(e_{3,1})$. Necessarily, $c(e_{3,1}) < c(e_{3,2}) < \cdots < c(e_{3,t})$, otherwise $\lambda_3$ contains a maximal $(k,c)$-ascent, where $k \leq t$, which is a contradiction. By a similar argument, $c(e_{1,1}) > c(e_{1,2}) > \cdots > c(e_{1,t})$. Hence $r \geq 2t + 1$. If we let $c(e_{2,i}) = c(e_{3,i})$ for $2 \leq i \leq t$, then $r \leq 2t + 1$ and the resulting edge colouring has the required flatness. Thus $\chi_{\varepsilon}(S(t,t,t)) = 2t + 1$, and $\chi_{\varepsilon}(S(t,t,t)) - (\chi_1(S(t,t,t)) + \varepsilon(S(t,t,t)) - 2) = t - 1$.

### 5.3 Paths and cycles

In this section we determine the $\varepsilon$-ascent chromatic index for paths and cycles. We also show that the only graphs for which $\chi_{\varepsilon}(G) = \varepsilon(G)$ are paths and even cycles.

**Proposition 5.3.1.** $\chi_{\varepsilon}(P_n) = n - 1$ for all $n \geq 2$.

*Proof.* Since $\varepsilon(P_n) = n - 1 = |E(P_n)|$, it follows from Remark 5.1.1 that $\chi_{\varepsilon}(T) = n - 1$. \hfill $\square$

**Proposition 5.3.2.** $\chi_{\varepsilon}(C_n) = \lceil \frac{n}{2} \rceil + 1$ for all $n \geq 3$.

*Proof.* By Proposition 1.5.8 and Remark 5.1.1, $\chi_{\varepsilon}(C_n) \geq \varepsilon(C_n) = \lceil \frac{n+1}{2} \rceil$. Let $C_n = e_1 e_2 \cdots e_n$. 

![Figure 5.1: An edge colouring of $S(1,t,t)$ with flatness $t + 1$.](image)
Case 1: \( n \) is even. Define the edge ordering \( c_e \) of \( C_n \) by
\[
c_e(e_i) = \begin{cases} 
  i & \text{if } 1 \leq i \leq \frac{n}{2} + 1 \\
  n - i + 2 & \text{if } \frac{n}{2} + 1 < i \leq n.
\end{cases}
\]

It is easy to verify that \( c_e \) is a proper \((\frac{n}{2} + 1)\)-edge colouring with \( h(c_e) = \varepsilon(C_n) \). Since \( n \) is even, \( \frac{n}{2} + 1 = \left\lceil \frac{n+1}{2} \right\rceil = \varepsilon(C_n) \). Hence, by Remark 5.1.1, \( \chi_{\varepsilon}(C_n) = \frac{n}{2} + 1 \), and the result holds.

Case 2: \( n \) is odd. Define the edge colouring \( c_o \) of \( C_n \) by
\[
c_o(e_i) = \begin{cases} 
  i & \text{if } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil + 1 \\
  n - i + 2 & \text{if } \left\lceil \frac{n}{2} \right\rceil + 1 < i \leq n.
\end{cases}
\]

It is easy to verify that \( c_o \) is a proper \((\left\lceil \frac{n}{2} \right\rceil + 1)\)-edge colouring with \( h(c_o) = \varepsilon(C_n) \). Since \( n \) is odd, \( \left\lceil \frac{n}{2} \right\rceil + 1 = \left\lceil \frac{n+1}{2} \right\rceil = \varepsilon(C_n) + 1 \). To prove the result we must show that any proper \((\frac{n+1}{2})\)-edge colouring of \( C_n \) has flatness at most \( \frac{n-1}{2} \). Suppose, to the contrary, that there exists a proper \((\frac{n+1}{2})\)-edge colouring \( c'_o \) of \( C_n \) with flatness \( \varepsilon(c_o) = \frac{n+1}{2} \). Let \( c'_o(e_1) = 1 \). Since any maximal \( c'_o \)-ascent in \( C_n \) has length at least \( \frac{n+1}{2} \), it follows that
\[
c'_o(e_i) = \begin{cases} 
  i & \text{if } 2 \leq i \leq \frac{n+1}{2} \\
  n - i + 2 & \text{if } \frac{n+1}{2} < i \leq n.
\end{cases}
\]

But then \( c'_o(e_k) = c'_o(e_{k+1}) \) for \( k = \frac{n+1}{2} \), and \( c'_o \) is not a proper edge colouring, a contradiction. \( \square \)

As mentioned in Remark 5.1.1, \( \chi_{\varepsilon}(G) \geq \varepsilon(G) \). We next characterize graphs \( G \) for which \( \chi_{\varepsilon}(G) = \varepsilon(G) \).

**Proposition 5.3.3.** If \( G \) is connected and \( \chi_{\varepsilon}(G) = \varepsilon(G) \), then \( G = C_{2n} \) or \( G = P_n \).
for $n \geq 2$.

**Proof.** Let $G$ be a graph such that $\chi_\varepsilon(G) = \varepsilon(G)$. By Corollary 5.2.2, $\Delta(G) \leq 2$. If $G = P_n$, then $\varepsilon(G) = n - 1$. Hence, from Proposition 5.3.1, $\chi_\varepsilon(G) = n - 1 = \varepsilon(G)$.

From Proposition 1.5.8, $\varepsilon(C_n) = \lceil \frac{n+1}{2} \rceil$. If $G = C_{2k+1}$, then $\varepsilon(G) = k + 1$ and by Proposition 5.3.2, $\chi_\varepsilon(G) = k + 2$. If $G = C_{2k}$, then $\varepsilon(G) = k + 1$ and by Proposition 5.3.2, $\chi_\varepsilon(G) = k + 1$. \qed

### 5.4 Trees

Remark 5.1.1 states that $\chi_\varepsilon(G) \leq |E(G)|$. In this section we characterize trees $T$ for which $\chi_\varepsilon(T) = |E(T)|$. We also bound $\chi_\varepsilon(T)$ for trees with $\varepsilon(T) = 3$ and in doing so introduce a variation on the parameter $\chi_\varepsilon(G)$.

**Theorem 5.4.1.** Let $T$ be a tree. Then $\chi_\varepsilon(T) = |E(T)|$ if and only if $T = P_n$ or $T = K_{1,n}$, $n \geq 2$.

**Proof.** Suppose $T = P_n$, where $n \geq 2$. Then $\varepsilon(T) = n - 1 = |E(T)|$, and by Remark 5.1.1, $\chi_\varepsilon(T) = |E(T)|$. Suppose $T = K_{1,n}$ where $n \geq 2$. Then $\chi_1(T) = n = |E(T)|$, and again by Remark 5.1.1, $\chi_\varepsilon(T) = |E(T)|$.

Conversely, suppose that $T \neq P_n$ and $T \neq K_{1,n}$ for $n \geq 2$. If $T$ contains a vertex which is adjacent to two leaves, then $\varepsilon(T) = 2$ and by Proposition 5.1.2, $\chi_\varepsilon(T) = \chi_1(T)$. Furthermore, since $T$ is not a star, $\text{diam}(L(T)) \geq 2$ which implies $\chi_1(T) < |E(T)|$.

Assume then that no vertex of $T$ is adjacent to two leaves, which by Corollary 1.5.9, implies that $\varepsilon(T) \geq 3$. Let $f$ be an edge ordering of $T$ with $h(f) = \varepsilon(T)$. Suppose also that $T$ has at least three endpaths of length two or more. Let $e_1, e_2$ and $e_3$ be pendant edges on three such endpaths. Necessarily each $e_i$ is either the initial or final edge of a maximal $f$-ascent in $T$. Without loss of generality we may
assume that $e_1$ and $e_2$ are both initial edges of a maximal $f$-ascent in $T$. Let $c$ be an edge colouring of $T$ such that $c(e_1) = c(e_2) = \min_{e \in E(T)} \{ f(e) \}$ and for all other edges $e \notin \{ e_1, e_2 \}$, $c(e) = f(e)$. Then $h(c) = h(f) = \varepsilon(T)$, which implies $\chi_\varepsilon(T) < |E(T)|$.

Suppose then that $T$ contains at most two endpaths of length two or more. Consider the case where $B(T) = \{ v \}$. Since $v$ is not adjacent to two leaves and $v$ has exactly two endpaths of length two or more, $T = S(1, k_1, k_2)$, where $2 \leq k_1 \leq k_2$. By Theorem 1.5.11, $\varepsilon(S(1, k_1, k_2)) = 1 + k_1$. Now we describe an edge colouring $c$ using fewer than $|E(T)|$ labels such that $h(c) = 1 + k_1$. Assign the three edges incident with $v$ the three smallest labels under $f$ where the pendant edge receives the smallest of these labels. For the edges not incident with $v$, the values of $f$ increase along each of the endpaths and the edges that are the same distance from $v$ are assigned the same values. Thus at least two edges will be assigned the same value under the colouring $c$ and it is easily verified that $h(c) = 1 + k_1 = \varepsilon(T)$.

Suppose then that $|B(T)| = k \geq 2$. Denote the number of leaves of $T$, which is also the number of endpaths of $T$, by $l$. Then $l \geq 2 + k$ and if any vertex in $B(T)$ has degree four or more, $l > 2 + k$ (see Theorem 3.7 in [5]). Necessarily, there exist at least two vertices in $B(T)$, say $v_1$ and $v_2$, such that each $v_i$ is incident with exactly one edge that does not lie on a $v_i$-endpath. Then each $v_i$ has at least two $v_i$-endpaths. Since no vertex in $T$ is adjacent to two leaves and $T$ has at most two endpaths of length two or more, each $v_i$ has an endpath which is a pendant edge and another which has length two or more, and the degree of $v_1$ and $v_2$ is three. Let $\lambda_i$ be the $v_i$-endpath of length two or more, $y_i$ the pendant edge of $\lambda_i$, $w_i$ the pendant edge incident with $v_i$, and $f$ an edge ordering of $T$ with $h(f) = \varepsilon(T)$. Then each $y_i$ is either the initial or final edge in a maximal $f$-ascent in $T$. If both are initial or both are final edges of a maximal $f$-ascent in $T$, then, as before, there exists an edge colouring $c$ of $G$ using fewer than $|E(G)|$ labels such that $h(c) = h(f) = \varepsilon(T)$ and we are done. We assume
then, without loss of generality, that \( y_1 \) is the initial edge of a maximal \( f \)-ascent of \( T \) and \( y_2 \) is the final edge of a maximal \( f \)-ascent of \( T \).

We focus our attention on the vertex \( v_1 \). Let the endpath \( \lambda_1 \) be denoted by the edges \( e_1 e_2 \ldots e_t \) where \( t \geq 2 \) and \( e_t = y_1 \), and \( e' \) be the edge incident with \( v_1 \) not on a \( v_1 \)-endpath. We may assume \( \lambda_1 \) is an \( f \)-ascent of \( T \) of which \( e_t \) is the initial edge; if not, then for some \( i \), \( 2 \leq i \leq t-1 \), \( f(e_{i+1}) < f(e_i) > f(e_{i-1}) \), and there exists an edge colouring \( c \) with \( c(e_{i+1}) = c(e_{i-1}) \) and \( h(c) = h(f) \). We also may assume that \( f(w_1) > f(e_1) \), otherwise \( w_1 e_1 \) is a maximal \( (2,f) \)-ascent and \( \varepsilon(T) = 2 \), a contradiction. If \( f(e') < f(e_1) \), let \( j \) be the largest index such that \( f(e_j) > f(e') \). If \( j < t \), relabel \( e_{j+1} \) with \( f(e') \), otherwise, relabel \( e_t \) with \( f(e') \). In either case this new colouring has flatness \( \varepsilon(T) \) using fewer than \( |E(G)| \) labels. If \( f(w_1) > f(e') \), then we define the edge colouring \( c \) of \( T \) by \( c(w_1) = c(y_2) = \max_{e \in E(T)} \{ f(e) \} \) and for all other edges \( e \), \( c(e) = f(e) \). Since \( h(c) = h(f) = \varepsilon(T) \), \( \chi_\varepsilon(T) < |E(T)| \).

We assume then that \( f(w_1) < f(e') \) and \( f(e_1) < f(e') \). Let \( \alpha \) be a shortest maximal \( f \)-ascent containing the path \( w_1 e' \). Necessarily, the length of \( \alpha \) is at least \( \varepsilon(T) \). Define the edge colouring \( c \) as follows: \( c(w_1) = f(e_1) \), \( c(e_1) = f(w_1) \), for \( 2 \leq i \leq t \), \( c(e_i) = f(e') + i - 2 \), and for \( e \notin \{ e_1, e_2, \ldots, e_t, w_1, e' \} \), \( c(e) = f(e) \). Note that the maximal \( f \)-ascent containing \( \lambda_1 \) and \( w_1 \) is now a maximal \( c \)-ascent with its direction reversed. Furthermore, the maximal \( f \)-ascent \( \alpha \) is also a maximal \( c \)-ascent. Also, note that any shortest maximal \( c \)-ascent containing \( e_1 e' \) has the same length as \( \alpha \), whose length is at least \( \varepsilon(T) \). Thus \( h(c) = h(f) = \varepsilon(T) \). Moreover, at least two edges receive the same label, hence \( \chi_\varepsilon(T) < |E(T)| \).

Next we discuss a bound of \( \chi_\varepsilon(T) \) for trees \( T \) with \( \varepsilon(T) = 3 \). We begin with a slightly more general result.

**Theorem 5.4.2.** Let \( T \) be a tree with \( \varepsilon(T) \geq 3 \). Then there exists a proper edge colouring \( c \) of \( T \) using at most \( \chi_1(T) + 2 \) colours such that \( h(c) \geq 3 \).
Proof. Since $\varepsilon(T) \geq 3$ it follows that $|E(T)| \geq 3$, and by Theorem 1.5.1, no vertex of $T$ is adjacent to two leaves. Note that for any tree $T$, $\chi_1(T) = \Delta(T)$. If $|E(T)| = 3$, then $T = P_4$, and since $\chi_1(P_4) = 2$ and $\varepsilon(P_4) = 3$, the result follows. Suppose then that $|E(T)| = 4$. The only tree $T$ with four edges and $\varepsilon(T) \geq 3$ is $T = P_5$ and again, since $\chi_1(P_5) = 2$ and $\varepsilon(P_5) = 4$, the result follows.

Suppose the result is true for all trees $T$ with $3 \leq |E(T)| < t$ for some $t \geq 5$ and consider a tree $T'$ with $|E(T')| = t$ and $\varepsilon(T') \geq 3$.

Suppose $T'$ is a path, say $T' = v_0v_1 \ldots v_t$. Let $T = T' - \{v_{t-1}, v_t\}$. By the induction hypothesis there exists a proper edge colouring $c$ of $T$ using at most 4 colours such that $h(c) \geq 3$. If $c(v_{t-3}v_{t-2}) \geq 3$, then let $c(v_{t-1}v_{t-2}) = 1$ and $c(v_{t-1}v_{t-2}) = 2$, otherwise let $c(v_{t-1}v_{t-2}) = 4$ and $c(v_{t-1}v_{t-2}) = 3$. In either case $v_{t-3}v_{t-2}v_{t-3}$ is a $(3, c)$-ascent in $T'$ which implies that we may extend the colouring $c$ to $T'$ so that it has flatness at least three using at most $\chi_1(T') + 2$ colours.

Suppose now that $T'$ is not a path. Then there exist at least three endpaths in $T'$. Recall that $B(T')$ is the set of branch vertices of $T'$.

**Case 1** There exists a $v$-endpath of length two for some $v \in B(T')$. Let $vxy$ be a $v$-endpath of length two and $T = T' - \{x, y\}$. Since $T'$ does not contain a vertex adjacent to two leaves, it follows that $T$ also does not contain a vertex adjacent to two leaves. Hence $\varepsilon(T) \geq 3$ and by the induction hypothesis $T$ has a proper edge colouring $c$ using at most $\chi_1(T) + 2$ colours such that $h(c) \geq 3$. Let $\deg_T(v) = k$ and $C_v$ be the set of colours assigned to edges incident with $v$. Note that $\chi_1(T) \geq k$, hence $\chi_1(T') \geq k + 1$, so there are at least $k + 3$ colours available to colour $T'$.

Suppose that either $\{1, 2\} \cap C_v = \emptyset$ or $\{k + 2, k + 3\} \cap C_v = \emptyset$. In the former case we extend the edge colouring $c$ to $T'$ by $c(vx) = 2$ and $c(xy) = 1$ and in the latter case by $c(vx) = k + 2$ and $c(xy) = k + 3$. In either case, in $T'$, $h(c) \geq 3$, and since $\chi_1(T') \geq k + 1$, the result holds.
Suppose then that \(\{1, 2\} \cap C_v \neq \emptyset\) and \(\{k + 2, k + 3\} \cap C_v \neq \emptyset\). Then there exists \(j \notin C_v\) such that \(3 \leq j \leq k + 1\). Suppose there exists \(i \in C_v\) such that \(i < j\) and for the edge \(uv\) assigned \(i\), any edge \(e\) adjacent to \(uv\) such that \(c(e) < c(uv)\) is incident with \(v\). Then, since \(h(c) \geq 3\), for any edge \(vw\) with \(c(vw) > i\), there exists an edge \(e'\) incident with \(w\) but not with \(v\) such that \(c(e') > c(vw)\). Thus, if we extend the edge colouring \(c\) to \(T'\) by \(c(vx) = j\) and \(c(xy) = j + 1\), then the resulting edge colouring of \(T'\) has flatness at least three. Therefore, we may assume that for any edge \(uv\) with \(c(uv) < j\), there exists an edge \(uw\) such that \(c(uw) < c(uv)\). Hence, if we extend the edge colouring \(c\) to \(T'\) by \(c(vx) = j\) and \(c(xy) = j - 1\), the resulting edge colouring of \(T'\) has flatness at least three.

**Case 2** For all \(v \in B(T')\), there does not exist a \(v\)-endpath of length two. Since \(\varepsilon(T') \geq 3\), there exists an endpath of length three or more. Let \(v_0\) be a branch vertex incident with at most one edge not on an endpath, \(\deg(v_0) = k\), and \(\lambda = v_0v_1, \ldots v_j, j \geq 3\), be a \(v_0\)-endpath of maximum length. Let \(T = T' - \{v_j, v_{j-1}\}\).

- Suppose \(\varepsilon(T) \geq 3\). By the induction hypothesis there exists a proper edge colouring \(c\) of \(T\) in at most \(\chi_1(T) + 2\) colours with flatness at least three. If \(c(v_{j-3}v_{j-2}) \geq 3\), then let \(c(v_jv_{j-1}) = 1\) and \(c(v_{j-1}v_{j-2}) = 2\), otherwise let \(c(v_jv_{j-1}) = 4\) and \(c(v_{j-1}v_{j-2}) = 3\). In either case \(v_jv_{j-1}v_{j-2}v_{j-3}\) is a \((3, c)\)-ascent in \(T'\), which implies that we may extend the colouring \(c\) to \(T'\) so that it has flatness at least three.

- Suppose \(\varepsilon(T) = 2\). Then in \(T'\), \(v_0\) is incident with a pendant edge \(e_0 = v_0w\). If \(T'\) is a spider, then \(T' \cong S(1, 3, \ldots, 3)\) and Figure 5.2 shows an edge colouring in \(\chi_1(T') + 2\) colours with flatness four. Hence assume \(T'\) has at least two branch vertices and there are \(k - 2 \geq 1\) \(v_0\)-endpaths of length three which we denote by \(\lambda_1, \lambda_2, \ldots, \lambda_{k-2}\). Let \(e_i\) be the edge incident with \(v_0\) which lies on endpath \(\lambda_i\) and \(e'\) the edge incident with \(v_0\) that does not lie on an endpath. Let \(T_1\)
be the component of $T - \{e_1, e_2, \ldots, e_{k-2}\}$ which contains $e_0$, and note that $\deg_{T_1}(v_0) = 2$. Also, no vertex of $T_1$ is adjacent to two leaves, hence $\varepsilon(T_1) \geq 3$.

By the induction hypothesis there exists a proper edge colouring $c$ of $T_1$ in at most $\chi_1(T_1) + 2$ colours such that $h(c) \geq 3$, and without loss of generality we may assume that $c(e') > c(e_0)$. Then we may also assume without loss of generality that $c(e_0) = 1$.

Let $c(e') = m$. We colour the edges of $\lambda_i$ with $i+1, i+2, i+3$, for $1 \leq i \leq k-2$ and $i \neq m-1$, and the edges of $\lambda_{m-1}$ (if it exists, in which case $m < k$) with $k, k+1, k+2$ to obtain a proper edge colouring $c'$ of $T'$. Since $k \leq \chi_1(T')$ and $\chi_1(T_1) \leq \chi_1(T')$, it follows that $c'$ uses at most $\chi_1(T') + 2$ colours.

Clearly, the maximal ascents contained in the union of the $v_0$-endpaths, $v_0w$, $\lambda_1, \lambda_2, \ldots, \lambda_{k-2}$ all have length at least three (four in fact). If $c(e') < c(e_i)$, then $e'$ followed by $\lambda_i$ is a $(4, c')$-ascent. Suppose $c(e') > c(e_i)$. Then since $h(c) \geq 3$, there exists an edge $e''$ adjacent to $e'$ such that $e_0e'e''$ is a $(3, c')$-ascent. Then $e_0e'e''$ is also a $(3, c')$-ascent. Hence $h(c') \geq 3$ and we are done.

We combine the lower bound from Proposition 5.2.1 with the result from Theorem 5.4.2 in the following corollary.

**Corollary 5.4.3.** For any tree $T$ with $\varepsilon(T) = 3$, $\chi_1(T) + 1 \leq \chi_e(T) \leq \chi_1(T) + 2$. 
Theorem 5.4.2 does not provide an upper bound for $\chi_\varepsilon(T)$ when $\varepsilon(T) \geq 4$ since we are only guaranteed an edge colouring with flatness at least three. This motivates a generalization of the parameter $\chi_\varepsilon(G)$.

We define the $k$-ascent chromatic index of a graph $G$, $\chi_{(k)}(G)$, as the minimum number of colours so that there exists a proper edge colouring with flatness $k$, where $2 \leq k \leq \varepsilon(G)$. Note that $\chi_{(2)}(G) = \chi_1(G)$.

We now restate Theorem 5.4.2 using the parameter $\chi_{(k)}(G)$.

**Theorem 5.4.4.** Let $T$ be a tree with $\varepsilon(T) \geq 3$. Then $\chi_{(3)}(T) \leq \chi_1(T) + 2$.

The bound in Theorem 5.4.4 does not hold for all graphs in general. For example, consider the graph $G$ shown in Figure 5.3. Since $c(e_1) = c(e_4) = 1$, $c(e_2) = c(e_6) = 2$, $c(e_3) = c(e_5) = 3$ is a proper 3-edge colouring of $G$, we conclude that $\chi_1(G) = 3$, and from Theorem 1.5.1 and Proposition 1.5.2, it follows that $\varepsilon(G) = 3$. Suppose $\chi_{(3)}(G) = \chi_\varepsilon(G) \leq \chi_1(G) + 2 = 5$. Then there exists a proper 5-edge colouring $c$ of $G$ with $h(c) = 3$. Necessarily, the edge coloured 1 is not adjacent to the edge coloured 5.

Moreover, $\{c(e_2), c(e_3), c(e_4)\} \cap \{1, 5\} = \emptyset$, otherwise at least one of $e_1e_2, e_1e_3, e_4e_5$ or their reverse is a maximal $(2, c)$-ascent. Therefore, without loss of generality we may assume $c(e_1) = 1$ and $c(e_5) = 5$. Since $|E(G)| = 6$, two edges are assigned the same colour, and since no other edge is assigned 1 or 5, $c(e_2) = c(e_6) = i$ where $i \in \{2, 3, 4\}$.

If $i = 2$, then $e_6e_5$ is a maximal $(2, c)$-ascent, and if $i = 4$, then $c(e_4) \in \{2, 3\}$ and $e_4e_6$ is a maximal $(2, c)$-ascent. Suppose then that $i = 3$. If $c(e_4) = 2$, then $e_4e_6$ is a maximal $(2, c)$-ascent, and if $c(e_4) = 4$, then $e_6e_4$ is a maximal $(2, c)$-ascent.

This completes all cases and we conclude that there does not exist a proper 5-edge colouring of $G$ with flatness three. Hence, $\chi_{(3)}(G) = \chi_\varepsilon(G) = 6 > \chi_1(G) + 2$. 
5.5 Complete graphs

In this section we consider the problem of determining \(\chi_\varepsilon(K_n)\). Trivially, we note that \(\chi_\varepsilon(K_2) = 1\), and since \(\chi_1(K_3) = |E(K_3)| = 3\), it follows that \(\chi_\varepsilon(K_3) = 3\). From Proposition 1.5.4, \(\varepsilon(K_n) = 3\) for all \(n \geq 4\), hence, for \(n \geq 4\) the problem involves determining the minimum number of colours required for there to exist a proper edge colouring of \(K_n\) with flatness three.

We first determine an upper bound for \(\chi_\varepsilon(K_n)\).

**Theorem 5.5.1.** \(\chi_\varepsilon(K_n) \leq 2n - 3\) for all \(n \geq 2\).

*Proof.* As noted previously, \(\chi_\varepsilon(K_2) = 1\) and \(\chi_\varepsilon(K_3) = 3\), thus the result holds for \(n = 2\) and \(n = 3\). Consider \(K_n\) where \(n \geq 4\). Let \(V(K_n) = \{v_0, v_1, \ldots, v_{n-1}\}\) and define the edge colouring \(c\) by \(c(v_i, v_j) = i + j\). Clearly \(c\) is a proper edge colouring of \(K_n\), and furthermore, \(c\) uses \(2n - 3\) colours. To complete the proof we need only show that every \((2, c)\)-ascent is contained in a longer \(c\)-ascent.

Suppose \(\lambda = v_a v_b v_c\) is a \((2, c)\)-ascent of \(K_n\). Note that \(a \neq n - 1\) since \(v_{n-1}v_b\) is assigned the largest value over all edges incident with \(v_b\) and hence \(v_{n-1}v_b v_c\) is not a \(c\)-ascent. We now consider the following cases.

**Case 1** \(c = n - 1\). If \(b \leq n - 3\) and \(a \leq n - 3\), or \(b \leq n - 4\) and \(a = n - 2\), then there exists an edge \(v_{n-1}v_k\) such that \(k \neq a\) and \(c(v_{n-1}v_k) > b + n - 1\), which implies \(\lambda\) is not a maximal \(c\)-ascent. If \(b = n - 3\) and \(a = n - 2\), then there exists an edge \(v_a v_k\) such that \(c(v_a v_k) < c(v_a v_b)\), and \(\lambda\) is not a maximal \(c\)-ascent. If \(b = n - 2\), then
there exists an edge $v_av_k$ such that $k < n - 2$, which implies $c(v_av_k) < c(v_av_b)$, and again, $\lambda$ is not a maximal $c$-ascent.

**Case 2** $b = n - 1$. By the definition of the edge colouring $c$, it follows that $v_av_{n-1}$ is assigned the largest value over all edges incident with $v_a$. Thus, since $n \geq 4$, there exists an edge $v_av_k$ such that $k \neq c$ and $c(v_av_k) < c(v_av_{n-1})$, which implies that $\lambda$ is not a maximal $c$-ascent.

**Case 3** $n-1 \notin \{a, b, c\}$. By the definition of the edge colouring $c$, $c(v_bv_c) < c(v_cv_{n-1})$, which implies that $\lambda$ is not a maximal $c$-ascent.

We have now considered all possible cases for a $(2, c)$-ascent $\lambda$ and shown that in each case $\lambda$ is not a maximal $c$-ascent. \hfill $\square$

Figures 5.4(a) and 5.5 depict the edge colouring $c$ for $K_4$ and $K_5$ respectively as defined in the proof of Theorem 5.5.1. Next we determine $\chi_\epsilon(K_4)$ and $\chi_\epsilon(K_5)$.

**Proposition 5.5.2.** $\chi_\epsilon(K_4) = 5$.

**Proof.** By Theorem 5.5.1, $\chi_\epsilon(K_4) \leq 5$.

To complete the proof we now show that $\chi_\epsilon(K_4) \geq 5$. Suppose, to the contrary, that there exists a colouring $c : E(K_4) \rightarrow \{1, 2, 3, 4\}$ of $K_4$ with flatness three. Necessarily, an edge assigned colour 1 is not incident with an edge coloured 4. This
Figure 5.5: A 7-edge colouring $c$ of $K_5$ with $h(c) = \varepsilon(K_5) = 3$.

implies the colours 1 and 4 are used once each and furthermore, the colours 2 and 3 are each used twice. Any proper colouring with this configuration is equivalent to the one shown in Figure 5.4(b) and both of the ascents 2 3 are maximal ascents. Thus $\chi_{\varepsilon}(K_4) \geq 5$.

**Proposition 5.5.3.** $\chi_{\varepsilon}(K_5) = 7$.

**Proof.** By Theorem 5.5.1, $\chi_{\varepsilon}(K_5) \leq 7$.

We now show that $\chi_{\varepsilon}(K_5) \geq 7$. Suppose, to the contrary, that there exists a proper 6-edge colouring $c$ of $K_5$ with $h(c) = 3$. Let $V(K_5) = \{v_0, v_1, v_2, v_3, v_4\}$. Without loss of generality let $c(v_0v_1) = 1$. If $c(v_i v_j) = 6$, then $i, j \geq 2$, otherwise $h(c) = 2$. Without loss of generality let $c(v_2v_3) = 6$. To avoid a maximal $(2, c)$-ascent, $c(e) \in \{2, 3, 4, 5\}$ for each $e \in E(K_5) - \{v_0v_1, v_2v_3\}$. Then, since the maximum size of an independent edge set of $K_5$ is two and there are 8 edges in $E(K_5) - \{v_0v_1, v_2v_3\}$, $|c^{-1}(i)| = 2$ for each $i \in \{2, 3, 4, 5\}$. Additionally, for each colour $i \in \{2, 3, 4, 5\}$, there exists $x \in \{0, 1, 2, 3\}$ such that $c(v_xv_4) = i$. Note that $c(v_0v_4) \neq 5$ or else $v_1v_0v_4$ is a maximal $(2, c)$-ascent, a contradiction. By similar arguments, $c(v_1v_4) \neq 5$, $c(v_2v_4) \neq 2$, and $c(v_3v_4) \neq 2$. Thus, either $c(v_2v_4) = 5$ or $c(v_3v_4) = 5$, and we assume without loss of generality that $c(v_2v_4) = 5$. Since another edge must also be assigned the colour 5, either $c(v_0v_3) = 5$ or $c(v_1v_3) = 5$ and without loss of generality we may
assume that \( c(v_0v_3) = 5 \). If \( c(v_1v_3) = 2 \), then \( v_1v_3v_0 \) is a maximal \((2, c)\)-ascent. Hence
\[ 3 \leq c(v_1v_3) \leq 4. \]
Since \( c(v_2v_3) = 6 \) and \( c(v_0v_3) = 5 \), if \( c(v_3v_4) = 3 \), then \( c(v_1v_3) = 4 \) and \( v_3v_4v_2 \) is a maximal \((2, c)\)-ascent. This implies \( c(v_3v_4) = 4 \) and \( c(v_1v_3) = 3 \). Now \( c(v_0v_4), c(v_1v_4) \in \{2, 3\} \) and since \( c(v_1v_3) = 3 \), \( c(v_1v_4) = 2 \) and \( c(v_0v_4) = 3 \). Moreover,
since \( c(v_1v_2), c(v_0v_2) \in \{2, 4\} \) and \( c(v_1v_4) = 2 \), \( c(v_1v_2) = 4 \) and \( c(v_0v_2) = 2 \). Then
\( v_0v_2v_1 \) is a maximal \((2, c)\)-ascent and the result follows.

For \( 2 \leq n \leq 5 \), the bound provided in Theorem 5.5.1 is best possible. However, this is not always the case as we illustrate next with \( K_6 \).

**Proposition 5.5.4.** \( 7 \leq \chi_\varepsilon(K_6) \leq 8 \).

**Proof.** Suppose \( \chi_\varepsilon(K_6) \leq 6 \). Then there exists a colouring \( c : E(K_6) \to \{1, 2, 3, 4, 5, 6\} \) with \( h(c) = 3 \). Since the maximum size of an independent edge set of \( K_6 \) is three, any colour can be used at most three times. Moreover, an edge coloured 1 is not adjacent to an edge coloured 6, otherwise \( h(c) = 2 \). Suppose the colours 1 and 6 are each used exactly once. Since \( |E(K_6)| = 15 \), there are 13 edges to be coloured with colours from the set \( \{2, 3, 4, 5\} \), and by the pigeonhole principle one of these colours is used at least four times, a contradiction. Hence, three edges are coloured 1 or 6.
Without loss of generality we assume two edges are coloured 1 and one is coloured 6; necessarily these edges form an independent edge set. Since there are 12 edges of $K_6$ to be coloured with colours from the set $\{2, 3, 4, 5\}$, each colour is used exactly three times. Consider the three edges coloured 5. Since only one edge is coloured 6, there exists at least one edge, say $e$, coloured 5 which is not adjacent to the edge coloured 6. Furthermore, the edge $e$ is adjacent to an edge $e'$ which is coloured 1. Thus $e'e$ is a maximal $(2, c)$-ascent, which is a contradiction. Therefore $\chi_\epsilon(K_6) \geq 7$.

To show that $\chi_\epsilon(K_6) \leq 8$, we show that the 8-edge colouring of $K_6$ in Figure 5.6 has flatness three. Note that in the figure the labels $k$ and $k'$ are assumed to be the same label for each $k \in \{2, 3, \ldots, 8\}$ and the notation is used to differentiate between edges which are assigned the same colour. We lexicographically consider all $(2, c)$-ascents which are the first two edges of a maximal $c$-ascent and include in brackets the colour of an edge which extends the ascent to a $(3, c)$-ascent. The $(2, c)$-ascents we need to consider are $1\ 2\ 6'$, $1\ 2'\ 8'$, $1\ 3\ 6'$, $1\ 3'\ 6'$, $1\ 4\ 5'$, $1\ 4'\ 8'$, $1\ 5\ 6'$, $1\ 6\ 7$, $2\ 3\ 6'$, $2\ 3'\ 4$, $2\ 4'\ 5'$, $2\ 6\ 8$, $2'\ 3'\ 6'$, $2'\ 4\ 7$, $2'\ 4'\ 6$, $2'\ 5\ 6'$, $3\ 4'\ 8'$, $3\ 6\ 7$, $4\ 5\ 6'$, $4'\ 6\ 7$. Hence $h(c) = 3$. \qed
Chapter 6

Summary and Further Directions of Research

In this chapter we provide a summary of the work accomplished in this dissertation. We conclude with a list of several open problems related to this body of work.

6.1 Summary of results

In this dissertation we considered various problems related to the depression of a graph. We determined a formula for the depression of the class of trees known as double spiders. We investigated the concept of $k$-kernels of a graph $G$. In particular, we identified $k$-kernels for paths, cycles, and spiders. From the $k$-kernel result for spiders we were able to determine an improved upper bound for the depression of trees. We also provided a sufficient condition for a vertex $v \in V(G)$ to be a 3-kernel (or 2-kernel) of a graph $G$ with $\text{diam}(L(G)) = 2$.

In this dissertation we also investigated graphs with depression three. We provided a characterization of graphs with depression three and the added property that no two adjacent vertices both have degree three or more. We also constructed a large
class of graphs with depression three which includes cyclic graphs and graphs with adjacent vertices of degree three or more.

We considered the problem of determining the smallest integer $r$ such that there exists a proper edge colouring $c : E(G) \rightarrow \{1, 2, \ldots, r\}$ for which a shortest maximal ascent has given length $k$, denoted by $\chi_{(k)}(G)$, or $\chi_{\varepsilon}(G)$ if $k = \varepsilon(G)$. We established various general bounds for $\chi_{\varepsilon}(G)$ and in some cases identified classes of graphs which achieve these bounds. We determined $\chi_{\varepsilon}(G)$ for paths and cycles, and showed that $\chi_{(3)}(T) \leq \chi_1(T) + 2$ for any tree $T$. We also determined $\chi_{\varepsilon}(K_n)$ for $2 \leq k \leq 5$, and showed that $\chi_{\varepsilon}(K_n) \leq 2n - 3$ for all $n \geq 2$.

We developed an algorithm, which is included in Appendix A, to determine the depression of a given graph $G$. This algorithm was used to verify the depression of many of the example graphs in this dissertation.

6.2 Open problems

A characterization of trees with depression three is given in [16] and graphs with depression three and no adjacent vertices of degree three or more are characterized in Section 4.1 (Theorem 4.1.1), however, the characterization of graphs with depression three remains an unsolved problem.

Problem 6.2.1. Characterize the class of graphs with depression three.

Problem 6.2.2. Does there exist a characterization similar to Theorem 4.1.1 for graphs with depression $k \geq 4$ and no adjacent vertices of degree three or more?

In Section 4.2 we described 7 operations, $O_1 - O_7$, which are used to construct a large class of graphs with depression three.

Problem 6.2.3. Does there exist a finite number of operations of the type $O_1 - O_7$ that would yield all graphs with depression three?
In Chapter 2 we describe a formula for the depression of the family of trees known as double spiders. As mentioned previously, a characterization of trees with depression three was established in [16]. Further research into the depression of trees with the end goal of determining a formula for the depression of trees in general may include: a characterization of trees with depression four, a formula for the depression of other families of trees, an improved algorithm for determining the depression of a given tree, and improved bounds on the depression of trees, perhaps by using the depression of double spiders.

Problem 6.2.4. Determine a formula for the depression of trees.

In Section 5.2 we proved that $\chi_\varepsilon(G) \geq \chi_1(G) + \varepsilon(G) - 2$ and we also showed that the difference $\chi_\varepsilon(G) - [\chi_1(G) + \varepsilon(G) - 2]$ can be arbitrarily large. Prove or disprove:

Problem 6.2.5. The ratio $\chi_\varepsilon(G)/[\chi_1(G) + \varepsilon(G)]$ is bounded.

Theorem 5.4.1 states that the only trees $T$ for which $\chi_\varepsilon(T) = |E(T)|$ are stars and paths. The graph shown in Figure 6.1 is an example of a cyclic graph $G$ for which $\chi_\varepsilon(G) = |E(G)|$ (we omit verification of this claim). It remains an open problem to identify all such graphs.

Problem 6.2.6. Characterize the class of graphs for which $\chi_\varepsilon(G) = |E(G)|$.

In Section 5.5 we investigated the problem of determining the $\varepsilon$-ascent chromatic index for complete graphs. We established an upper bound for $\chi_\varepsilon(K_n)$ when $n \geq 2$, and determined the exact value for $\chi_\varepsilon(K_n)$ when $2 \leq n \leq 5$. With further research
into this problem we aspire to determine a general formula of $\chi_\varepsilon(K_n)$. Failing this we seek to improve the bounds for $\chi_\varepsilon(K_n)$.

**Problem 6.2.7.** Determine $\chi_\varepsilon(K_n)$ for $n \geq 6$.

In Appendix A we outline an algorithm for determining the depression of a graph.

**Problem 6.2.8.** Develop an improved algorithm for determining the depression of a graph.

In [15], the problem of determining the height of a given edge ordering was shown to be NP-hard. The complexity of determining the flatness of a given edge ordering is not known.

**Problem 6.2.9.** Determine the complexity of the problem of evaluating the flatness of a given edge ordering.
Appendix A

Depression Algorithm

Here we outline an algorithm for determining the depression of a graph $G$.

One task in determining the depression of a graph $G$ is to determine the length of the shortest maximal $f$-ascent for a given edge ordering $f : E(G) \rightarrow \{1, 2, \ldots, |E(G)|\}$. Let $e_i$ be the edge in $G$ for which $f(e_i) = i$ for $i = 1, \ldots, m$, where $m = |E(G)|$. Now consider the line graph of $G$, denoted by $L(G)$, where $V(L(G)) = \{e_1, e_2, \ldots, e_m\}$.

We aim to construct our maximal $f$-ascents of $G$ in $L(G)$. Note that not all paths in $L(G)$ correspond to a path in $G$. Specifically, a path $P = v_1, v_2, \ldots, v_k$ in $L(G)$ corresponds to a path in $G$ if and only if, for all $0 \leq i < j \leq k$, if $\{v_i, v_j\} \in E(L(G))$, then $j = i + 1$. That is, a path $P$ in $L(G)$ corresponds to a path in $G$ if and only if $P$ is an induced path in $L(G)$. Furthermore, in order for a path $P = v_1, v_2, \ldots, v_k$ to correspond to an $f$-ascent in $G$, it follows that for each edge $\{v_i = e_j, v_{i+1} = e_k\}$ of $P$, $k > j$.

The following procedure $EXTEND(i)$ determines all paths $P$ in $L(G)$ which correspond to a maximal $f$-ascent in $G$ that begins with the vertex $e_i$. These maximal ascents are stored in the set $X$, hence, if we run the procedure $EXTEND(i)$ for each $i$ from 1 to $n - 1$, sequentially and initializing $P := e_i$ each time, then the flatness of
$f$ is the length of the shortest path in $X$.

**proc** EXTEND($i$)

$m := i + 1$;

**while** $m \leq n$ **do**

**if** \{$e_i, e_m$\} $\in E(L(G))$ and $N(P \setminus e_i) \cap \{e_m\} = \emptyset$ **then**

add $e_m$ to $P$; **EXTEND**($m$);

remove $e_m$ from $P$; $m := m + 1$;

**else**

$m := m + 1$;

**end if**

**end do**

**if** $P$ is not a subpath of a path already in $X$ **then**

$X := X \cup \{P\}$;

**end if**

**end proc**

Naively, we could repeat this procedure over all $m!$ edge orderings of $G$, but many of these orientations are equivalent. It is easy to see that a vertex ordering $f$ of a graph $G$ can be used to define an acyclic orientation $D$ of the edges of $G$ by orienting the edge \{$a, b$\} as $(a, b)$ whenever $f(a) < f(b)$. It is also easy to see that more than one vertex ordering may correspond to the same acyclic orientation. For example, consider $P_3 = v_1, v_2, v_3$. The vertex orderings $f = \{(v_1, 2), (v_2, 1), (v_3, 3)\}$ and $g = \{(v_1, 3), (v_2, 1), (v_3, 2)\}$ determine the same acyclic orientation of $P_3$ – see figure A.1. Moreover, if two edge orderings of $G$, say $h$ and $f$, correspond to the same acyclic orientation of $L(G)$, then $h(f) = h(g)$. Thus, we need only consider a minimal set of edge orderings of $G$ which determines the set of all acyclic orientations.
Figure A.1: Two vertex orderings which define equivalent orientations of $P_3$.

of $L(G)$.

To construct the set of all acyclic orientations of $L(G)$ we use the algorithm outlined in [1] and for each of these orientations we obtain a total ordering of the vertices of $L(G)$ (or an edge ordering of $G$) using a topological sort. We may further reduce our set of orderings by a factor of two by noting that two acyclic orientations $d$ and $\overline{d}$ which are complementary, that is, $(a, b) \in d$ if and only if $(b, a) \in \overline{d}$, will correspond to orderings which yield the same flatness. Thus we generate all acyclic orientations of $L(G)$ with the orientation of one edge fixed. The depression of $G$ is then the maximum flatness over all such edge orderings.
Bibliography


