Consensus Analysis of Networked Multi-Agent Systems with Second-Order Dynamics and Euler-Lagrange Dynamics

by

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B. Eng., Northwestern Polytechnical University, 2009

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of

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in the Department of Mechanical Engineering

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Supervisory Committee

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Consensus is a central issue in designing multi-agent systems (MASs). How to design control protocols under certain communication topologies is the key for solving consensus problems. This thesis is focusing on investigating the consensus protocols under different scenarios: (1) The second-order system dynamics with Markov time delays; (2) The Euler-Lagrange dynamics with uniform and nonuniform sampling strategies and the event-based control strategy.

Chapter 2 is focused on the consensus problem of the multi-agent systems with random delays governed by a Markov chain. For second-order dynamics under the sampled-data setting, we first convert the consensus problem to the stability analysis of the equivalent error system dynamics. By designing a suitable Lyapunov function and deriving a set of linear matrix inequalities (LMIs), we analyze the mean square stability of the error system dynamics with fixed communication topology. Since the transition probabilities in a Markov chain are sometimes partially unknown, we propose a method of estimating the delay for the next sampling time instant. We
explicitly give a lower bound of the probability for the delay estimation which can ensure the stability of the error system dynamics. Finally, by applying an augmentation technique, we convert the error system dynamics to a delay-free stochastic system. A sufficient condition is established to guarantee the consensus of the networked multi-agent systems with switching topologies. Simulation studies for a fleet of unmanned vehicles verify the theoretical results.

In Chapter 3, we propose the consensus control protocols involving both position and velocity information of the MASs with the linearized Euler-Lagrange dynamics, under uniform sampling and nonuniform sampling schemes, respectively. Then we extend the results to the case of applying the centralized event-triggered strategy, and accordingly analyze the consensus property. Simulation examples and comparisons verify the effectiveness of the proposed methods.
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To my parents.
Acronyms

AUV  autonomous underwater vehicle
UAV  unmanned aerial vehicle
MAS  multi-agent system
NCS  networked control system
MJLS Markov jump linear system
LMI  linear matrix inequality
LTI  linear time-invariant
ISS  input-to-state stability
Chapter 1

Introduction

1.1 An Overview of the Multi-Agent Cooperative Control

During the past decades, an enormous amount of research efforts has been devoted to the multi-agent cooperative control. The “agent” here represents a generalized individual system dynamic. It can be a single mobile robot, an unmanned air vehicle (UAV), an autonomous underwater vehicles (AUV), a helicopter or a satellite. If the agents are equipped with actuators and sensors, and their operations are coordinated through the control protocols, this kind of systems are multi-agent cooperative control systems. The implementation of multi-agent cooperative systems are of significance to accomplish complex tasks which are difficult or impossible for an individual agent; e.g., in applications such as mine-sweeping, unmanned aerial vehicles surveillance and deep sea exploration. Some leading international journals publish Special Issues on the related topics. IEEE Transactions on Automatica Control Special Issue on Networked Control Systems (Volume 49, No. 9, 2004) includes the study of the consensus problems for networked dynamic agents with fixed and switching topologies, informa-
tion flow and stability of distributed control in autonomous vehicle formations. *SIAM Journal on Control and Optimization* Special Issue on Control and Optimization in Cooperative Networks (Volume 48, No. 1, 2009) discusses protocols of multi-agent systems, distributed motion coordination, and cooperative control, and so on.

A traditional way of tackling the multi-agent cooperative control task is to have a centralized computer to collect the information of all agents. After calculation and planning, the centralized computer allocates the instructions to each agent. Unsurprisingly when the scale of the system is very large, the centralized computer needs to take a heavy load of computation and communication. If the systems or the environment change with unanticipated situations, it may lead to failure of the cooperative control. A more efficient strategy, distributed control is then widely used in multi-agent cooperative control. Each agent is equipped with an embedded microprocessor which not only collects information from the other agents but also actuates the action of the agent.

Among the research topics on multi-agent cooperative control, consensus is very critical, aiming to force a group of agents’ states to reach an agreement on certain quantities of interest. Consensus can be applied to solve many problems, such as vehicle formations [1], [2], [3], [4]; flocking [5], [6]; rendezvous problems [7]; robot position synchronization [8]; attitude alignment [9], and so on.

In order to achieve consensus, two vital issues have been intensively investigated: One is the mathematical description of the networked communication topologies, and the other is the design of the control protocols. In the next section, recent progress and the main approaches for solving the consensus problems will be reviewed.
Figure 1.1: Consensus application in cooperative control: Formation control.

Figure 1.2: Consensus application in cooperative control: Rendezvous problem.
Figure 1.3: Consensus application in cooperative control: Attitude alignment.

Figure 1.4: Consensus application in cooperative control: Robot position synchronization.
1.2 What Is A Consensus Problem?

Consensus has been extensively studied in automata theory and distributed computation in past decades [10]. Meanwhile, consensus problems have received increasing research interest in distributed cooperative control of multi-agent systems. The main goal of this section is to review the relevant research results of consensus problems for distributed cooperative multi-agent systems.

Consensus means that all the states of a multi-agent system can dynamically reach certain agreement. The states in the agreement could be some physical variables such as position, velocity, attitude, angle, temperature, and so on. With the development of digital control technology, the agents are commonly equipped with embedded sensors, microprocessor and actuators. Based on the information acquired by the sensors, designing proper control protocols for agents is the key to fulfill the collaborative tasks. In the following robot position synchronization example, we show what a consensus problem is.

In Figure 1.5, we consider the system dynamics with five one-link revolute joint arms. A consensus problem is casted that how should the joint angles $\theta_i, i = 1, 2, 3, 4, 5$, evolve under some control protocols such that the angles achieve a certain value after some time. As we mentioned above, two issues for ensuring consensus should be discussed in sequel. The first issue to be tackled is the description of the communication networks. Without information flows among the agents, it is apparently that no agent knows “where to move”, which means consensus is failed to be reached. The arrows in Figure 1.5 denote the communication links. It is shown that the angle information of agent 2 can be obtained by agent 4 and agent 5, but agent 2 does not receive the information from agent 4 and agent 5. Agent 1 and agent 4 transmit their angles information with each other since the information flow between the two agents is bidirectional. We call the agents as nodes, and describe the informa-
tion flows as edges with the graph theory. $G = (V, E, A)$, where $V = (v_1, v_2, v_3, v_4, v_5)$ denotes the node set, $E = ((v_1, v_4), (v_4, v_1), (v_2, v_5), (v_2, v_4), (v_3, v_2), (v_3, v_5))$ is the edge set indicating all existing information flows among the nodes, and $A = [a_{ij}] \in \mathbb{R}^{5 \times 5}, i, j = 1, 2, 3, 4, 5,$ is the adjacency matrix. If there exists information flow from $v_i$ to $v_j$, we say $(v_i, v_j) \in E$ and $a_{ji} \neq 0$. Suppose there is no information transmitted from the agent to itself, thus $a_{ii} = 0, i = 1, 2, \ldots$. The communication topology can be categorized as fixed and time-varying cases in the reality; see more details in Section 1.3. After mathematically describing the communication links, designing a protocol is then the second issue to be considered.

![Multi-agent system diagram](image)

**Figure 1.5**: A multi-agent system.
The dynamics of the agents could be different types. Differential equations such as first-order dynamics, second-order dynamics and Euler-Lagrange dynamics are employed to describe the MASs under different circumstances. Taking Euler-Lagrange systems in this example, the agents model in Figure 1.5 can be written as

\[ M_i(\theta_i)\ddot{\theta}_i + C_i(\theta_i, \dot{\theta}_i)\dot{\theta}_i = \tau_i, \quad i = 1, \ldots, n, \]

where \( M_i(\theta_i) \in \mathbb{R}^{n \times n} \) is the inertia matrix, \( C_i(\theta_i, \dot{\theta}_i) \dot{\theta}_i \in \mathbb{R}^p \) is the vector of Coriolis and centrifugal torques. \( \tau_i \), to be designed, is the vector of the torques produced by the actuators associated with the \( i \)th agent. The next step is to find a proper control protocol \( \tau_i = u(\theta_i, \dot{\theta}_i) \) such that the \( \lim_{t \to \infty} \| \theta_i - \theta_j \| = 0, \quad i, j = 1, 2, \ldots \). The consensus problem is solved if the states of all agents in the system converge to a common value.

1.3 Literature Review

In this subsection, we will review the theoretical progress of the consensus problem for the MASs.

Consensus problem has been receiving increasing attention over the past years. In the early literature [10], [11], consensus problems are studied in the fields of computer science and computation algorithms. Later, in [12], the consensus protocol is investigated for the headings of the moving particles with the same velocity. Jadbabaie et al. [13] provide theoretical explanation to the results in [12], and then extend the application into heading consensus for mobile autonomous agents. In [13], the average heading consensus problem for the MASs is studied with the undirected communication topology. The sufficient condition to ensure consensus requires that each agent is jointly connected to all other agents across contiguous time intervals.
In [14], Olfati-Saber and Murray study the eigenvalues and the rank of the Laplacian matrix, and then build the connectivity between matrix theory and the communication graph. Due to the communication constraints in the reality, old information flows may disappear and new ones may be set up. Thus the interaction topology among agents sometimes dynamically change. In [14], based on the algebraic graph theory, matrix theory and control theory, the authors provide the sufficient and necessary condition for an MAS to achieve consensus with switching topologies. Ren and Beard [15] further extend the study of [13] to the consensus result with directed information flows. They provide the concept “spanning tree” with the knowledge of matrix theory to explain the joint connection for the agents in a union of the directed interaction graphs. Having a spanning tree frequently enough for switching communication topologies is the sufficient condition for the MASs to achieve asymptotically consensus. In the literature we list above, the authors have developed basic concepts and protocols for consensus problem on the basis of graph theory.

In light of these elegant papers, the consensus problem has received much attention in the field of cooperative control. Here, we will review the development the consensus problem from two perspectives. One is from different views of the problem formulation, and the other is from the theoretical approaches for solving the consensus problem.

1.3.1 Consensus Problems from Different View Points

- **Different system dynamics.** Generally the dynamics is categorized as two types: Linear and nonlinear system. For linear systems, the dynamics and the control protocols are both linear. Since the linear system has the merits of simplicity and convenience for describing and solving mathematical problems, it can be used to model the dynamics in many applications, such as unmanned
flying vehicles [16], the moving particles [13], and so on. The consensus problems for linear system dynamics has been widely studied since the consensus problem began to attract the attention of researchers [14], [15]. The consensus protocols for the first-order dynamics involving the position information of the agent and its neighbors have been studied in [13], [14], [17], [18], [19]. It is shown that the joint connection in directed topologies plays a key role for the first-order dynamics to reach the asymptotical consensus. The systems in reality are always complex, thus the study of the first-order consensus problem has been extended to consensus problems for double-integrator dynamics [20], [21], [22]. Based on Lyapunov stability analysis, in [18] the authors propose the consensus protocol by converting the study of consensus problem for the second-order dynamics to the investigation of the first-order error dynamics. In [23], based on the graph theory and the matrix theory, Ren et al. show the necessary and sufficient conditions for the MASs with double-integrator dynamics achieving consensus. Some higher order protocols in consensus problem of the MASs are investigated in [23], [24], [25]. In [25], the sufficient and necessary condition is established such that consensus for general higher order system can be reached if all subsystems are asymptotically stable. The stability region is introduced and derived for the higher order system. Considering that most agent models are nonlinear, it is restrictive to study the consensus problem for linear agent dynamics. Thus it is advantageous to directly study the consensus problem for MASs with nonlinear dynamics, e.g., the synchronization of multipendulums [26], the consensus problem of robot position synchronization with Euler-Lagrange dynamics [27], [28].

- **Different time domains.** If we employ differential equations to describe the system dynamics, we say the consensus problem is studied with continuous-time
dynamics. Intuitively, the systems modeled as the continuous-time dynamics may better characterize the real dynamics in nature, such as the trajectory of flying vehicles, the angle evolutions of the robot mechanism, the temperature changing in industry control, and so on. As discussed in [29], [13], [14], [15], [30], [25], continuous-time consensus protocols are summarized as follow: The state of each agent is driven toward the states of its neighbors as time evolves. It may be the case that one agent has no information interaction with some other agents in some time intervals. Correspondingly, If we employ difference equations to describe the dynamics and propose discrete-time control protocols, we say the consensus problem is studied with discrete-time dynamics. In [13], [31] and [32], the states of the agents update at each time instant by averaging the states of the agents and their neighbors. The sampled-data system dynamics also attracts much interest in the field of consensus problem. The sampled and quantized signal of the continuous-time system is converted to a digital signal; the digital signal is processed and the yielded result is converted to the continuous-time signal and is then applied to the continuous-time dynamics; e.g., [33] and the references therein. The consensus problem is studied under the sampled-data framework in [34], [35], [36]. Usually the sampling periods are uniform [34], [36]. In [35], consensus protocols with large sampling periods and nonuniform sampling periods are investigated.

- **Communication topologies.** The time-invariant interaction topologies are fixed topologies. The fixed communication topology is relatively simple. In the early literature, more attention has been paid to it [14], [15]. In [15], eigenvalues of the *Laplacian matrix* are studied regarding the connectivity of the agents. Since the communication environments are always complicated in reality, for example, the different data transmission rates among the agents may cause
time delays; data loss may occur at the unreliable channels; the disturbance or communication range limitation may lead to the change of the communication topologies. Thus the study of the MASs cooperative control under switching topologies is very important. A stochastic matrix is called indecomposable and aperiodic (SIA), if the rows of its infinite self-products are the same. In [15], consensus protocols for the MASs under switching topologies are studied by using the property of infinite products of stochastic matrices. Another approach dealing with the consensus problem under switching topologies is based on the Markov jump linear system (MJLS) method. In [34], the first-order MJLS is converted to an equivalent error system dynamics under switching topologies, and by applying Lyapunov stability method, mean square stability of the system is analyzed.

- **Communication constraints** There exist many types of communication constraints in the study of consensus problems that may degrade or even destroy the consensus results. Time delays exist ubiquitously in the real environment. There are considerable research efforts on the consensus problem with time delays. In [14], a constant time delay is analyzed. [34] and [37] investigate consensus problems with time-varying delays for the first-order system dynamics. In [38], a sufficient condition is provided for the MASs achieving consensus under dynamically changing topologies with bounded time-varying communication delays. In [34], the authors consider the consensus problem with time delays governed by a Markov process. The switching topologies are determined by the time delays. Markov process model can characterize the stochastic property of the system, which may help reduce the conservativeness. In [39], under the sampled-data framework, by employing the graph theory and the matrix theory, the authors develop a sufficient condition for a second-order MAS with
time delays to reach consensus. Besides delays, the data loss is another type of communication constraint. The data loss occurs mainly due to the long delay or the malfunction of the communication channels. In [40], the consensus problem is studied with the data loss modeled by a Bernoulli process. Based on the stochastic stability analysis, in [41] the authors propose a maximum allowable loss probability bound for systems over random lossy networks; if the data loss probabilities are within the given bound and the communication topology has a spanning tree, the proposed control protocol solves the consensus problem.

- **Control protocols.** Besides the approaches we have mentioned in the above review, there exist a variety of control protocols in the study of the consensus problem. For example, An event-triggered consensus protocol is studied in [42]. The control action is triggered when the norm of the state error reaches certain criteria. In [43], the authors investigate event-triggered consensus protocols for the MASs with both single- and double-integrator dynamics. With this method, neighboring agents do not have to exchange information continuously, but only at specific time instants. Self-triggered consensus protocol has been considered in [44]. Based on the local information, each agent determines when to send a new measurement over the network. Moreover, the event-triggered model predictive control for the cooperation of distributed agents is studied in [45]. Recently in [46], the consensus problem is tackled using the event-triggered scheme for the first-order dynamics with time delays and second-order dynamics without time delays, respectively.

- **Newly developed approaches.** Now we introduce some newly developed approaches for solving consensus problems. In [47], it is assumed that the information is only transmitted at the sampling time instants. By using the property of stochastic matrices and algebraic graph theory, sufficient conditions
are established to ensure consensus of the MAS with double-integrator dynamics. In [48], the authors proposed a protocol by only using the information of the neighbors which are “close enough” to the agent. If the neighbors are outside the specified scope of the agent, their information will be discarded. The protocol involving the constrained physically-meaningful states is suitable for the physical limited cases. In [49], the consensus problems are studied addressing the following aspects simultaneously: Each agent only communicates with the agents in its communication range; consensus can be achieved in a finite time interval. Consensus problems are studied based on cooperative game theory in [50]. The agents here are viewed as individual “players” with partners working in a team. Each agent tends to achieve the target with the minimum cost. A cooperative working strategy may minimize the global cost function of the team. Based on the formulated linear-quadratic regulator (LQR) problem, a set of LMIs can be constructed. By solving the LMIs, a sequence of control inputs are found to optimize the team cost function while ensuring consensus.

Besides the above literature we reviewed, the past years also witnessed the increasing interest in the study of consensus problems. For example, in [51], the consensus problem is investigated for heterogeneous systems. The authors of [52] propose a control protocol involving the current information flows and former information flows.

1.3.2 Theoretical Approaches for Solving the Consensus Problems

A. Graph Theory

For describing the agents, we denote a graph with n nodes by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$. The nodes set $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ represents n agents. An edge $(v_j, v_i) \in \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the information flow from $v_j$ to $v_i$. The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, $(a_{ij} \geq$
0, ∀i, j = 1, 2, . . . , n), represents the weights of the communication channels in the graph. An undirected graph implies a graph of which the link from i to j as well as the link from j to i exist and disappear synchronously. Otherwise the graph is directed. A path from vertex i to a vertex j is a sequence of distinct vertices starting with i and ending with j. A directed graph is strongly connected if there exist a path from each agent to any other agent. Mathematically, the neighbor set \( \mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\} \) indicates the agents from which agent i receives signals. Assume that an agent can not transmit signals to itself, we have \( a_{ii} = 0 \) and \( a_{ij} \geq 0 \) for all i, j, \( i \neq j \). The graph Laplacian \( L = [l_{ij}] \in \mathbb{R}^{n \times n} \) is defined as: \( l_{ij} = -a_{ij}, \forall i \neq j; l_{ii} = \sum_{j \in \mathcal{N}_i} a_{ij}, i, j = 1, 2, . . . , n \). There is a unique \( L \) corresponding to each \( A \). More details for the graph theory can be found in [13], [14], [15] and references therein.

**B. Matrix Theory**

Matrix theory has been widely used in the study of consensus problems. It remarkably builds the bridge between the communication topology and the stability analysis of the consensus problem. Stochastic matrices are ones whose row summations all equal to 1. By using the property of infinite products of stochastic matrices, consensus protocols are studied in [13], [15]. Positive definite matrices are involved in the LMIs for solving the consensus problem [40], [34] and in the Lyapunov stability analysis in [43]. Spectral radius study to the Laplacian matrix is related to the strong connectivity of a communication topology [14]. Kronecker products are used to augment the states of the MASs to facilitate the stability analysis of consensus problems [35].

**C. Control Theory**

Many control methods have been applied in the consensus problem studies. One of the most popular approaches is the Lyapunov stability analysis [40], [53]. By using the input-to-state stability (ISS), consensus problems with nonlinear system
dynamics are solved in [54]. Adaptive control is applied in [55], [56]. Moreover, model predictive control [16], LQR control [50], passivity control [57], $H_\infty$ control [58], Nyquist sampling theorem [59], [60], Lipschitz stability analysis [49] are used in tackling the consensus problems.

1.4 Motivations and Contributions

As mentioned above, researchers have paid great attention to the investigation of consensus problems. Designing control protocols is very important for solving the consensus problems. Here, we summarize the motivations and contributions of this thesis.

1.4.1 Motivations

Consensus protocols for MASs subject to network-induced constraints, e.g., delays, have attracted much attention [14], [38], [34], [37]. In [34], the consensus problem for the MAS with first-order system dynamics subject to time delays governed by a Markov process was investigated. The motivations of Chapter 2 are as follows.

- Incorporating the probabilistic distribution of the time delays into the analysis can effectively reduce the conservativeness [34]. However, it is assumed in [34] that the transition probabilities are completely accessible, which is questionable in the realistic applications.

- Zhang and Boukas [61] study the stochastic stability of the MJLS with partially unknown transition probabilities, but time delays are not involved in their work. To the authors’ best knowledge, the consensus problem for the second-order system dynamics with Markov delays has not been fully investigated, which motivates the work in Chapter 2.
The consensus problems are widely investigated under sampled-data control strategies. The motivations of Chapter 3 are as follows.

- With the protocols proposed in [35], large uniform sampling periods and nonuniform sampling periods for consensus problems are studied. However, due to the only use of position information in [35], the convergence rate of the consensus may be slow. Thus, if both position and velocity information of the agents are measurable and simultaneously incorporated into the protocol design, the convergence rate of the consensus is intuitively expected faster.

- Event-triggered control strategy is a kind of "nonuniform" sampled-data control method, and few literature deal with event-triggered control protocols by involving both position and velocity information of the agents. Also less attention has been paid to the consensus problems with Euler-Lagrange systems under sampled-data control protocols. Above analysis motivate the work in Chapter 3.

1.4.2 Contributions

The contributions of Chapter 2 lie in three aspects:

- We propose a control protocol which solves the second-order consensus problem under the sampled-data settings. The consensus results are analyzed by studying the stability of the equivalent error system dynamics.

- Based on current time delay, we design a scheme of estimating the delay for the next sampling time instant and give a lower bound of the probability for the estimation. If the probability of estimation is within this lower bound, consensus can be achieved.
• By using the augmentation technique, a delay-free stochastic system is obtained. We show that if the delays are governed by Markov chain with partially unknown transition probabilities, the delay-free stochastic system is mean square stable and the protocols solve the consensus problem.

The contributions of Chapter 3 lie in followings:

• We linearize the Euler-Lagrange system dynamics and design a control protocol involving both position and velocity information of the agents, then accordingly analyze the consensus property. Simulation results show that the consensus rate is faster than applying control protocol with only position information of the agents.

• We propose an centralized event-triggered control protocol for the MASs with the linearized Euler-Lagrange systems by using the position and velocity information of the agents.
Chapter 2

Consensus in Second-Order Multi-Agent Systems with Random Delays Governed by a Markov Chain

2.1 Introduction

During the past years, an enormous amount of research efforts have been paid to the cooperative control of MASs [62] [63] [64] [15]. The consensus problem is one of the critical issue on MASs requiring that a group of agents’ states can reach an agreement on certain quantities of interest. Consensus can find many applications including coordinated control of vehicles, synchronization of dynamical networks, rendezvous problems, attitude alignments, and so on [22], [7], [9] [28]. Designing suitable control protocols for MASs under certain communication topologies is crucial for solving consensus problems. In [13] [14] [15], the authors develop the theoretical frameworks
for consensus problems based on the algebraic theory, which paves the way for subsequent research progresses on MASs. Many research results have been reported for consensus problems addressing different aspects.

The consensus problems are studied under the sampled-data framework in [34], [35], [36]. Generally, the sampling periods are uniform and small [34], [36]. In [35], consensus protocols with large sampling periods and nonuniform sampling periods are investigated. The consensus protocols for first-order system dynamics involve the position information of the agent and its neighbors [13], [14], [17]. The systems in reality are always complex with higher order, and the study of first-order consensus problem has been extended to the consensus problem for double-integrator dynamics [20], [21], [22].

Consensus protocols with time delays have attracted much attention. Time delays exist ubiquitously and can degrade the system performance. The delay could be constant or time varying, uniform or nonuniform. There are considerable research efforts on the consensus problem with time delays. In [14], the authors consider the constant delay and give the upper bound of the time delay to ensure consensus with the proposed control protocol. In [34], the consensus protocol for MASs with first-order discrete-time dynamics subject to random delays governed by a Markov chain is studied.

In applications, it may not be the case that all transition rates of a Markov process are available. Via LMIs formulation, Zhang and Boukas [61] study the stochastic stability of the MJLSs with partially unknown elements in the Markov transition probability matrix. In the literature, the consensus problem for MASs with the second-order dynamics subject to Markov delays has not been fully studied. The main objectives of this Chapter are three-fold:

- To propose a protocol which solves the consensus problem for the second-order
system dynamics with Markov delays under sampled-data setting.

- Supposing the Markov transition probabilities are partially unknown, based on the current time delay, to estimate the delay for the next sampling instant and to give a lower bound of the probability for the correct estimation of the delays which ensures consensus.

- To obtain the delay-free stochastic system by using the augmentation technique. Lyapunov stability analysis and LMIs are applied to study the stability of the delay-free stochastic system.

The remainder of this chapter is organized as follows. Section 2.2 introduces some backgrounds and necessary definitions. The problem formulation is presented in Section 2.3. In Section 2.4, the main results are presented: In Subsection 2.4.1 we analyze the stochastic stability of the error system dynamics with a fixed communication topology; Subsection 2.4.2 presents the stability analysis by employing the delay estimation for the next sampling instant. In Subsection 2.4.3, we study the stability of the delay-free stochastic system based on Lyapunov stability analysis.

**Notation:** The superscript ‘T’ represents the matrix transpose. A matrix $P > 0$ if and only if $P$ is symmetric and positive definite. ‘*’ in a matrix stands for a term of block that is induced by symmetry. ‘×’ represents the multiplication of matrices. $\mathbf{1}$ denotes vector $[1, 1, \ldots, 1]^T$, $\mathbf{0}$ denotes vector $[0, 0, \ldots, 0]^T$ and $I$ is the identity matrix. Matrices are assumed to be compatible with algebraic operations. $\| . \|$ denotes the Euclidean norm. For an $n \times n$ matrix $V_i$, if we define $V = [V_1, \ldots, V_N]$, it follows that $\|V\|_1 = \sum_{i=0}^{N} \|V_i\|$. $\mathbb{P}$ is the probability operator and $\mathbb{E}$ denotes the mathematical expectation.
2.2 Preliminaries

We denote a graph with $n$ nodes by $G = (\mathcal{V}, \mathcal{E}, A)$. $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ represents the vertex set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. If the nonnegative adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, $(a_{ij} \geq 0, \forall i, j = 1, 2, \ldots, n)$, is symmetric, it models the undirected communication topology among the agents. In a directed communication graph, if there is a direct link from agent $j$ to agent $i$, meaning that agent $i$ receives information from agent $j$, $a_{ij} \neq 0$; otherwise $a_{ij} = 0$. A path from agent $i$ to agent $j$ is a sequence of distinct vertices starting with $i$ and ending with $j$, such that consecutive vertices are adjacent [65]. The neighbors of agent $i$ indicate the agents from which agent $i$ receives information. We use $\mathcal{N}_i$ to denote the neighbor set of agent $i$. Assume that an agent does not receive information from itself, then $a_{ii} = 0$ and $a_{ij} \geq 0$ for all $i, j, i \neq j$. The graph Laplacian $L = [l_{ij}] \in \mathbb{R}^{n \times n}$ is defined as:

$$l_{ij} = -a_{ij}, \forall i \neq j; l_{ii} = \sum_{j \in \mathcal{N}_i} a_{ij}, i, j = 1, 2, \ldots, n. \quad (2.1)$$

By definition, there is a unique corresponding $L$ to any $A$. Next, the definition of a Markov process is given.

**Definition 1** ([66]). Let $\{X_m, m = 0, 1, 2, \ldots\}$ be a stochastic process that takes on a finite or countable number of possible values from the state space $S$, where $S = \{1, 2, \ldots, s\}$. $X_m = i$ denotes that the process is in state $i$ at time instant $m$. If $\mathbb{P}\{X_{m+1} = j | X_m = i, \} = p_{ij}$ for any time instant $m$, this stochastic process is known as a Markov process.

The transition probability matrix $P = [p_{ij}]$, for all $i, j = 1, 2, \ldots, s$ satisfies

$$\sum_{j=1}^{s} p_{ij} = 1, i = 1, 2, \ldots, s. \quad (2.2)$$
The following assumption indicates that there exists an upper bound for the time delays.

**Assumption 1.** The time delays \( \{d_k\} \) are integer multiple of the sampling period and are from a finite integer set \( \Gamma = \{\tau_1, \tau_2, \ldots, \tau_q\} \) with \( 0 \leq \tau_1 < \tau_2 < \cdots < \tau_q \). The data is sent and used with a time delay at discrete time instants.

## 2.3 Problem Formulation

Consider a group of \( n \) agents with each agent being modeled as the following second-order dynamics:

\[
x_{ci}((k+1)h) = x_{ci}(kh) + \int_{kh}^{(k+1)h} v_{ci}(t) \, dt, \quad (2.3)
\]

\[
v_{ci}(t) = v_{ci}(kh) + (t - kh)u_{ci}(kh), \quad (2.4)
\]

\[
u_{ci}(kh) = -k_c v_{ci}(kh) + \sum_{j \in N_i} a_{ij}(\tau_{ij}(t))[x_{cj}(t - \tau_{ij}(t)) - x_{ci}(t - \tau_{ij}(t))], \quad (2.5)
\]

where \( t \geq 0, \) \( h \) is the sampling period, \( x_{ci}(t), v_{ci}(t), \) and \( u_{ci}(t) \in \mathbb{R}^m \) are position, velocity and control input of the \( i \)th agent at time \( t \) in the above system, and \( k_c \) is the control gain. \( \tau_{ij}(t) \in \Gamma \) represents a random uniform delay in the system. \( a_{ij}(\tau_{ij}(t)) \) stands for the element of the adjacency matrix at time \( t = kh \) with delay \( \tau_{ij}(t) \).

With a zero-order hold, we utilize sampled-data setup to discretize the system dynamics in (2.3), (2.4) and (2.5):

\[
x_i(k+1) = x_i(k) + hv_i(k) + \frac{h^2}{2} u_i(k), \quad (2.6)
\]

\[
v_i(k+1) = v_i(k) + hu_i(k), \quad (2.7)
\]
\[ u_i(k) = -k_c v_i(k) + \sum_{j \in \mathcal{N}_i} a_{ij}(d_k)[x_j(k - d_k) - x_i(k - d_k)], \tag{2.8} \]

where \( x_i(k), v_i(k) \) and \( u_i(k) \in \mathbb{R}^m \) are position, velocity and control information of agent \( i \) at time instant \( k \). \( a_{ij}(d_k) \) is the element of adjacency matrix at time instant \( k \) with delay \( d_k \).

We define \( x(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^T \), \( v(k) = [v_1(k), v_2(k), \ldots, v_n(k)]^T \) and \( u_i(k) = [u_1(k), u_2(k), \ldots, u_n(k)]^T \). For an initial state \( x(0) = [x_1(0), x_2(0), \ldots, x_n(0)]^T \), consensus is achieved if and only if all agents' states asymptotically converge to a common value \( \alpha(x(0)) \). Equations (2.6), (2.7) and (2.8) can be further written as:

\[
\begin{bmatrix}
  x(k+1) \\
  v(k+1)
\end{bmatrix}
= \begin{bmatrix}
  I & hI \\
  0 & I
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  v(k)
\end{bmatrix}
+ \begin{bmatrix}
  \frac{h^2}{2}I \\
  hI
\end{bmatrix}
\begin{bmatrix}
  u(k)
\end{bmatrix},
\]

\[ u(k) = -k_c v(k) - L(d_k)x(k - d_k), \]

where \( L(d_k) \) is Laplacian matrix at time instant \( k \). After some algebraic manipulation, we get

\[
\begin{bmatrix}
  x(k+1) \\
  v(k+1)
\end{bmatrix}
= \begin{bmatrix}
  I & (h - \frac{h^2}{2}k_c)I \\
  0 & (1 - hk_c)I
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  v(k)
\end{bmatrix}
+ \begin{bmatrix}
  -\frac{h^2}{2}L(d_k) & 0 \\
  -hL(d_k) & 0
\end{bmatrix}
\begin{bmatrix}
  x(k - d_k) \\
  v(k - d_k)
\end{bmatrix}. \]

Define the error of position as \( \bar{x}(k) = [x_2(k) - x_1(k), x_3(k) - x_1(k), \ldots, x_n(k) - x_1(k)]^T \) and the error of velocity as \( \bar{v}(k) = [v_2(k) - v_1(k), v_3(k) - v_1(k), \ldots, v_n(k) - v_1(k)]^T \). \([1, 1, \ldots, 1]^T = 1 \in \mathbb{R}^{(n-1)\times 1} \) and \([0, 0, \ldots, 0]^T = 0 \in \mathbb{R}^{(n-1)\times 1} \). \( I \in \mathbb{R}^{(n-1)\times(n-1)} \) is the identity matrix. Let \( E = [-1, I] \) and \( F = [0, I]^T \). It is rea-
ly to see that \( \bar{x}(k) = E \bar{x}(k) \). On the other hand, noting that \( L(d_k)1 = 0 \), we have

\[
L(d_k)x(k) = L(d_k)F \bar{x}(k) + L(d_k)\begin{bmatrix} x_1(k) \\ x_1(k) \\ \vdots \\ x_1(k) \end{bmatrix} = L(d_k)F \bar{x}(k).
\]

Thus,

\[
\begin{bmatrix} \bar{x}(k+1) \\ \bar{v}(k+1) \end{bmatrix} = \begin{bmatrix} I & (h - \frac{h^2}{2}k_c) \bar{I} \\ 0 & (1 - h k_c) \bar{I} \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\ \bar{v}(k) \end{bmatrix} + \begin{bmatrix} -\frac{h^2}{2} E L(d_k)F & 0 \\ -h E L(d_k)F & 0 \end{bmatrix} \begin{bmatrix} \bar{x}(k-d_k) \\ \bar{v}(k-d_k) \end{bmatrix}.
\]

By defining the system error \( \xi(k) = \begin{bmatrix} \bar{x}(k) \\ \bar{v}(k) \end{bmatrix} \), \( A = \begin{bmatrix} I & (h - \frac{h^2}{2}k_c) \bar{I} \\ 0 & (1 - h k_c) \bar{I} \end{bmatrix} \), and \( \hat{B}(d_k) = \begin{bmatrix} -\frac{h^2}{2} E L(d_k)F & 0 \\ -h E L(d_k)F & 0 \end{bmatrix} \), we obtain

\[
\xi(k+1) = A \xi(k) + \hat{B}(d_k) \xi(k-d_k).
\]

(2.9)

It is shown that achieving consensus in (2.3), (2.4) and (2.5) is equivalent to ensuring the stability of the error system in (2.9).

**Lemma 1.** The mean square consensus of system dynamics in (2.3), (2.4) and (2.5) is achieved if and only if the error system dynamics in (2.9) is mean square stable, i.e.,

\[
\lim_{k \to \infty} E\{\|\xi(k)\|^2\} = 0.
\]

(2.10)

**Proof.** It can be proved by following the similar lines in [67] and [34]. When the MAS
in (2.3), (2.4) and (2.5) achieves consensus, all the positions of the agents reach a common value. The error dynamics in (2.9) will be stable.

Lemma 2. [68] For an MJLS as defined below:

\[ x(k + 1) = C(k)x(k), \]  

(2.11)

where \( \{C(1), C(2), \ldots \} \) is a Markov process. \( C(k) \) is determined by the time instants. The system in (2.11) is mean square stable if

\[ \exists \beta \geq 1, 0 < \zeta < 1, \text{with any } x(0), \]

\[ \mathbb{E}\{\|x(k)\|^2\} \leq \beta \zeta^k \|x(0)\|^2. \]

Proof. It can be proved by following the similar lines in Theorem 3.9, [68].

\[ \square \]

2.4 Consensus Analysis of Second-Order System Dynamics with Random Time Delays Governed by a Markov Chain

2.4.1 Case I: Fixed Communication Topology

In this subsection, the transition matrix of the Markov chain is supposed to be fully accessible. A sufficient condition of guaranteeing the stability of the error system dynamics under directed and fixed topology will be established.

Theorem 1. For the system in (2.3), (2.4) and (2.5) with random delays governed by a Markov chain, under Assumptions 1, mean square consensus is achieved if there exist matrices \( P > 0, Q_j > 0, Z_j > 0, M_j \) and \( \hat{B}(\tau_j) \), \( j = 1, 2, \ldots, q \), such that the
following matrix inequalities

\[
\begin{bmatrix}
Y_{11}(r) & Y_{12} \\
* & Y_{22}
\end{bmatrix} < 0
\]  \hspace{1cm} (2.12)

hold \( \forall r = 1, 2, \ldots, q \), where

\[
Y_{11}(r) = \begin{bmatrix}
\Phi_0 & \Psi_1(r) & \cdots & \Psi_q(r) \\
* & \Phi_1(r) & \cdots & 0 \\
* & * & \ddots & \vdots \\
* & * & * & \Phi_q(r)
\end{bmatrix}
\]

\[
\begin{bmatrix}
\sum_{j=1}^{q} (M_{0j} + M_{0j}^T) & -M_{01} + \sum_{j=1}^{q} M_{1j}^T & \cdots & -M_{0q} + \sum_{j=1}^{q} M_{qj}^T \\
* & -M_{11} - M_{11}^T & \cdots & -M_{1q} - M_{q1}^T \\
* & * & \ddots & \vdots \\
* & * & * & -M_{qq} - M_{qq}^T
\end{bmatrix}
\]

\[
Y_{12} = \begin{bmatrix}
\sqrt{\tau_1} M_1 & \sqrt{\tau_2} M_2 & \cdots & \sqrt{\tau_q} M_q
\end{bmatrix}
\]

\[
Y_{22} = -\text{diag}\{Z_1, Z_2, \cdots, Z_q\},
\]

\[
\Phi_0 = \sum_{j=1}^{q} Q_j + A^T P A - P + \sum_{s=1}^{q} \pi_{rs}(A - I)^T \left( \sum_{j=1}^{q} Z_j \right)(A - I),
\]

\[
\Phi_i(r) = \pi_{ri} \hat{B}^T(\tau_i) P \hat{B}(\tau_i) - Q_i + \pi_{ri} \hat{B}^T(\tau_i) \left( \sum_{j=1}^{q} \tau_j Z_j \right) \hat{B}(\tau_i),
\]

\[
\Psi_i(r) = \pi_{ri} A^T P \hat{B}(\tau_i) + \pi_{ri}(A - I)^T \left( \sum_{j=1}^{q} \tau_j Z_j \right) \hat{B}(\tau_i), \ i = 1, 2, \ldots, q.
\]
Proof. Consider the following Lyapunov function candidate,

\[ V(k) = V_1(k) + V_2(k) + V_3(k) \]  \hspace{1cm} (2.13)\]

where

\[ V_1(k) = \xi^T(k)P\xi(k), \]
\[ V_2(k) = \sum_{j=1}^{q} \sum_{i=k-\tau_j}^{k-1} \xi^T(i)Q_j\xi(i), \]
\[ V_3(k) = \sum_{j=1}^{q} \sum_{i=-\tau_j}^{-1} \sum_{m=k+i}^{k-1} \eta^T(m)Z_j\eta(m), \]
\[ \eta(m) = \xi(m+1) - \xi(m). \]

For \( V_3(k) \), considering (2.9), we have

\[ V_3(k) = \sum_{j=1}^{q} \sum_{i=-\tau_j}^{-1} \sum_{m=k+i}^{k-1} \eta^T(m)Z_j\eta(m), \]
\[ \eta(m) = \xi(m+1) - \xi(m). \]

Define \( d_{k-1} = \tau_r, d_k = \tau_s, r, s \in \{1, 2, \ldots, q\} \). Then the transition probability from \( d_{k-1} \) to \( d_k \) is

\[ P(d_k = \tau_s|d_{k-1} = \tau_r) = \pi_{rs}. \]  \hspace{1cm} (2.14)\]

In the sequel, when considering the difference (conditional expectation) for each term of the Lyapunov function, we define

\[ \Omega(k) = \left[ \begin{array}{cccc} \xi^T(k) & \xi^T(k-\tau_1) & \xi^T(k-\tau_2) & \cdots & \xi^T(k-\tau_q) \end{array} \right]^T \]  \hspace{1cm} (2.15)\]
Thus, we have

\[
\mathbb{E}\{\Delta V_1(k)\} = \mathbb{E}\{V_1(k+1) - V_1(k)\} \\
= \mathbb{E}\{[\xi^T(k)A^T + \xi^T(k-d_k)\hat{B}^T(d_k)]P[A\xi(k) + \hat{B}(d_k)\xi(k-d_k)] \\
- \xi^T(k)P\xi(k)\} \\
= 2\sum_{s=1}^{q} \pi_{rs}[2\xi^T(k-\tau_s)\hat{B}(\tau_s)PA\xi(k) + \xi^T(k-\tau_s)\hat{B}^T(\tau_s)PB^T(\tau_s) \\
x \xi(k-\tau_s)] + \xi^T(k)(A^TPA - P)\xi(k),
\]

\[
\mathbb{E}\{\Delta V_2(k)\} = \mathbb{E}\{V_2(k+1) - V_2(k)\} = \sum_{j=1}^{q} [\xi^T(k)Q_j\xi(k) - \xi^T(k-\tau_j)Q_j\xi(k-\tau_j)],
\]

\[
\mathbb{E}\{\Delta V_3(k)\} = \mathbb{E}\{V_3(k+1) - V_3(k)\} \\
= \mathbb{E}\left\{\sum_{j=1}^{q} \sum_{i=-\tau_j}^{q-1} [\xi^T(k)(A-I)^T + \xi^T(k-d_k)\hat{B}^T(d_k)] Z_j \\
\times [(A-I)\xi(k) + \hat{B}(d_k)\xi(k-d_k)] - [\xi^T(k+i)(A-I)^T \\
+ \xi^T(k+i-d_{k+i})\hat{B}^T(d_{k+i})] Z_j [(A-I)\xi(k+i) + \hat{B}(d_{k+i}) \\
x \xi(k+i-d_{k+i})]\right\} \\
= \sum_{s=1}^{q} \sum_{j=1}^{q} \pi_{rs} \left[\xi^T(k)(A-I)^T + \xi^T(k-\tau_s)\hat{B}(\tau_s)\right] \tau_j Z_j [(A-I)\xi(k) \\
+ \hat{B}(\tau_s)\xi(k-\tau_s)] - \sum_{j=1}^{q} \sum_{l=k-\tau_j}^{q-1} \left[\xi^T(l)(A-I)^T + \xi^T(l-d_l)\hat{B}(d_l)\right] Z_j \\
\times [(A-I)\xi(l) + \hat{B}(d_l)\xi(l-d_l)].
\]

For any matrices

\[
M_j = [M_{0j}^T \ M_{1j}^T \ M_{2j}^T \ \cdots \ M_{qj}^T]^T, \ j = 1, 2, \ldots, q, \quad (2.16)
\]

where \(M_{ij}, i = 0, 1, \ldots q,\) is the \(i\)th row vector of \(M_j.\) With appropriate dimensions,
we have the following identities

\[
\Omega^T(k)M_j \left[ \xi(k) - \xi(k - \tau_j) - \sum_{l=k-\tau_j}^{k-1} \eta(l) \right] = 0. \tag{2.17}
\]

Then,

\[
\mathbb{E}\{\Delta V(k)\} = \mathbb{E}\{\Delta V_1(k)\} + \mathbb{E}\{\Delta V_2(k)\} + \mathbb{E}\{\Delta V_3(k)\}
\]

\[
\leq \sum_{s=1}^{q} \pi_{rs} \xi^T(k - \tau_s) \hat{B}^T(\tau_s) P \left[ 2A \xi(k) + \hat{B}(\tau_s) \xi(k - \tau_s) \right]
\]

\[
+ \xi^T(k)(A^T PA - P) \xi(k) + \sum_{j=1}^{q} \left[ \xi^T(k)Q_j \xi(k) - \xi^T(k - \tau_j)Q_j \xi(k - \tau_j) \right]
\]

\[
+ \sum_{s=1}^{q} \pi_{rs} \left[ \xi^T(k)(A - I)^T + \xi^T(k - \tau_s) \hat{B}(\tau_s) \right] \left( \sum_{j=1}^{q} \tau_j Z_j \right)
\]

\[
\times \left[ (A - I) \xi(k) + \hat{B}(\tau_s) \xi(k - \tau_s) \right] - \sum_{j=1}^{q} \sum_{l=k-\tau_j}^{k-1} \left[ \xi^T(l)(A - I)^T \right.
\]

\[
+ \xi^T(l - d_l) \hat{B}(d_l) \big] Z_j \left[ (A - I) \xi(l) + \hat{B}(d_l) \xi(l - d_l) \right]
\]

\[
+ 2 \sum_{j=1}^{q} \Omega^T(k)M_j \left[ \xi(k) - \xi(k - \tau_j) - \sum_{l=k-\tau_j}^{k-1} \eta(l) \right]
\]

\[
+ \sum_{j=1}^{q} \sum_{l=k-\tau_j}^{k-1} \Omega^T(k)M_j + \eta^T(l)Z_j \big] Z_j^{-1} \left[ M_j^T \Omega(k) + Z_j \eta(l) \right]. \tag{2.18}
\]

The last term of (2.18) is nonnegative, which forms the inequality. If we let the Inequality (2.18) be less than 0, we obtain:

\[
\mathbb{E}\{V(k + 1) - V(k)\} \leq \Omega^T(k)[Y_{11}(r) - Y_{12}Y_{22}^{-1}Y_{12}^T]\Omega(k) < 0. \tag{2.19}
\]

Here, we assume that \([Y_{11}(r) - Y_{12}Y_{22}^{-1}Y_{12}^T]\) is negative definite, then the inequality in
(2.19) holds. If follows that

$$\mathbb{E}\{V(k + 1) - V(k)\} \leq -\beta \|\Omega(k)\|^2 < 0,$$

where $\beta > 0$ is the smallest eigenvalue of $[Y_{12}Y_{22}^{-1}Y_{12}^T - Y_{11}(r)]$, with $r = 1, 2, \ldots, q$. Summing (2.20) in terms of $k$ from 0 to $\infty$, it can be obtained that

$$\mathbb{E}\{V(\infty) - V(0)\} \leq -\beta \sum_{k=0}^{\infty} \|\Omega(k)\|^2,$$

$$\sum_{k=0}^{\infty} \|\Omega(k)\|^2 \leq \frac{1}{\beta} \mathbb{E}\{V(0) - V(\infty)\} \leq \frac{1}{\beta} \mathbb{E}\{V(0)\} < \infty. \tag{2.22}$$

From $\sum_{k=0}^{\infty} \|\Omega(k)\|^2 < \infty$, it is readily shown that $\lim_{k \to \infty} \mathbb{E}\{\|\xi(k)\|^2\} = 0$ which indicates that the error system dynamics in (2.9) is mean square stable. From Lemma 1, we prove that the consensus is achieved.

Now we look back into the assumption for $[Y_{11}(r) - Y_{12}Y_{22}^{-1}Y_{12}^T]$ being negative definite. By applying Schur complement to the inequality (2.19), we obtain the inequality in (2.12).

**Remark 1.** The above Lyapunov function is constructed in light of the work in [69]. If $P > 0, Q_j > 0, Z_j > 0, M_j$ and $\hat{B}(\tau_j)$ can be found, such that the inequality in (2.12) holds. But the terms $\hat{B}(\tau_j) \times P$ and $\hat{B}(\tau_j) \times Z_i$ in the Inequality (2.18) are of nonlinearity. We can take congruence transformation and apply Schur complement to derive equivalent LMIs without involving nonlinear products of matrices. However, there is no feasible solution to the LMIs due to the restricted structure of $\hat{B}(\tau_j)$. Thus, the suitable method here is to employ the fixed communication topology in the stability analysis. By applying the augmentation technique and the Lyapunov stability analysis, in Section 2.4.3, we will study the consensus problem for second-order system dynamics with Markov time delays under switching topologies.
2.4.2 Case II: Switching Topologies with Estimated Delays

The Laplacian matrix at time instant $k$ is actually determined by the delay $d_k$. In Subsection 2.4.1, it is assumed that the time delay $d_{k+1}$ can be obtained based on $d_k$ using the Markov transition matrix. In reality, the Markov transition probabilities may not be fully known. It is possible that only an estimated delay $\hat{d}_{k+1}$ of $d_{k+1}$ is accessible. In [68], a study for the mean square stability of the MJLS using estimated system dynamics is given. However, time delays are not considered in [68]. In this subsection, we investigate the mean square stability of the MAS under switching topologies with delays.

First, we define a new state variable:

$$X^T(k) = [\xi^T(k), \xi^T(k-1), \ldots, \xi^T(k-\tau_q)].$$  \hspace{1cm} (2.23)

The error system dynamics in (2.9) can be rewritten as

$$X(k+1) = (A_0 + F(k))X(k),$$ \hspace{1cm} (2.24)

where

$$A_0 = \begin{bmatrix}
A & 0 & \cdots & 0 & 0 \\
I & 0 & \cdots & 0 & 0 \\
0 & I & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & I & 0
\end{bmatrix}, \quad \text{and} \quad F(k) = \begin{bmatrix}
0 & 0 & \cdots & \hat{B}(\hat{d}_k) & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0
\end{bmatrix}.$$  \hspace{1cm} (2.25)

$F(k)$ is determined by $d_k \in \Gamma (\Gamma = \{\tau_1, \tau_2, \ldots, \tau_q\})$. Define the set $\mathcal{F} = \{F_{\tau_1}, F_{\tau_2} \ldots F_{\tau_q}\}$,
where

\[
F_{\tau_i} = \begin{bmatrix}
0 & 0 & \cdots & \underbrace{0}_{(\tau_i+1)th \ block} & \hat{B}(\tau_i) & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{bmatrix}.
\]

It is assumed that each \(F_{\tau_i}, (i = 1, 2, \ldots, q)\), can stabilize the system in (2.24). The closed-loop system in (2.24) is a stochastic system due to the existence of the stochastic variable \(d_k\). To study the stability of the augmented system in (2.24) with unknown parameters in the transition matrix of the Markov chain, we take the estimation \(\tilde{F}(k) \in \mathcal{F}\) replacing \(F(k)\). The system in (2.24) is then rewritten as

\[X(k + 1) = (A_0 + \tilde{F}(k))X(k).\]

Given \(F(k)\), the probability for \(\tilde{F}(k)\) to occur is \(\rho_{F(k)\tilde{F}(k)}\), i.e.,

\[
P(\tilde{F}(k) \mid F(k)) = \rho_{F(k)\tilde{F}(k)},
\]

with

\[
\rho_{F_{\tau_i}F_{\tau_j}} \geq 0 \quad \text{and} \quad \sum_{j=1}^{q} \rho_{F_{\tau_i}F_{\tau_j}} = 1.
\]

Denote \(Q_i(k) = \mathbb{E}[X(k)X^T(k)1_{\{F(k)=F_{\tau_i}\}}]\), \(Q(k) = [Q_1(k), \ldots, Q_q(k)]\), where \(1_{\{F(k)=F_{\tau_i}\}} = 1\) if \(F(k) = F_{\tau_i}\), otherwise 0. We further define

\[
R_j(Q(k)) = \sum_{i=1}^{q} p_{ij} (A_0 + F_{\tau_i})Q_i(k)(A_0 + F_{\tau_i})^T,
\]

\[
S_j(Q(k)) = \sum_{i=1}^{q} p_{ij} \sum_{s=1, s \neq i}^{q} \rho_{F_{\tau_i}F_{\tau_s}} (A_0 + F_{\tau_s})Q_i(k)(A_0 + F_{\tau_s})^T, \quad \text{and}
\]
\[ R(Q(k)) = [R_1(Q(k)), \ldots, R_q(Q(k))] , \]
\[ S(Q(k)) = [S_1(Q(k)), \ldots, S_q(Q(k))] . \]

Also, the norms are defined as:
\[ \| R^k \|_1 = \frac{\| R^k(Q(0)) \|_1}{\| Q(0) \|_1} , \]
\[ \| S \|_1 = \frac{\| S(Q(k)) \|_1}{\| Q(k) \|_1} . \]

Considering the assumption that each \( F(k) \in \mathcal{F} \) can stabilize the system in (2.24) and Lemma 2, we have that there exist \( \beta \geq 1 \) and \( 0 < \zeta < 1 \) such that \( \| R^k \|_1 \leq \beta \zeta^k \).

**Theorem 2.** With \( \rho = \min \rho_{F_i,F_j}, c_0 = \max \{ \| A_0 + F_i \|_2 \} \), for all \( i, j = 1, 2, \ldots, q \), \( \beta \) and \( \zeta \) defined as above, if \( \rho > \frac{\beta c_0 - 1 + \zeta}{\beta c_0} \), the stochastic system (2.24) is mean square stable.

**Proof.** By noting that \( Q_i(k) = \mathbb{E} \left[ X(k)X^T(k)1_{ \{ F(k) = F_i \} } \right] \), we have
\[
Q_j(k+1) = \mathbb{E} \left[ (A_0 + \tilde{F}(k))x(k)x^T(k)(A_0 + \tilde{F}(k))1_{ \{ F(k+1) = F_j \} } \right] \\
= \sum_{i=1}^q p_{ij} \sum_{s=1}^q \rho_{F_i,F_s} \left[ (A_0 + F_s)x(k)x^T(k)1_{ \{ F(k) = F_s \} } 1_{ \{ \tilde{F}(k) = F_j \} } (A_0 + F_s)^T \right] \\
\leq \sum_{i=1}^q p_{ij} (A_0 + F_t_i)Q_i(k)(A_0 + F_t_i)^T \\
+ \sum_{i=1}^q p_{ij} \sum_{s=1 \neq i}^q \rho_{F_i,F_s} \left[ (A_0 + F_s)x(k)x^T(k)1_{ \{ F(k) = F_s \} } \times 1_{ \{ \tilde{F}(k) = F_j \} } (A_0 + F_s)^T \right] \\
= R_j(Q(k)) + S_j(Q(k)). \quad (2.25)
\]
With \( R(Q(k)) \), \( S(Q(k)) \) and \( \| S \|_1 \) defined above, it is shown that

\[
\| S \|_1 = \frac{\| S(Q(k)) \|_1}{\| Q(k) \|_1}.
\]

\[
\leq \frac{1}{\| Q(k) \|_1} \left\{ \sum_{j=1}^{q} \sum_{i=1}^{q} P_{ij} \left\{ \sum_{s=1,s \neq i}^{q} \rho_{Fr_i,Fr_s} \| A_0 + F_{rs} \|^2 \right\} \| Q_i(k) \| \right\}
\]

\[
\leq c_0 (1 - \rho) \sum_{i=1}^{q} \| Q_i(k) \|_1 = c_0 (1 - \rho).
\]

\[
\| Q(k + 1) \|_1 \leq \| R(Q(k)) + S(Q(k)) \|_1,
\]

\[
\| Q(k) \|_1 \leq \| R^k(Q(0)) + \sum_{\kappa=0}^{k-1} R^{k-1-\kappa}(S(Q(\kappa))) \|_1
\]

\[
\leq \| R^k \|_1 \| Q(0) \|_1 + \sum_{\kappa=0}^{k-1} \| R^{k-1-\kappa} \|_1 \| S \|_1 \| Q(\kappa) \|_1.
\]

\[
\leq \beta \left\{ \zeta^k \| Q(0) \|_1 + \sum_{\kappa=0}^{k-1} \zeta^{k-1-\kappa} c_0 (1 - \rho) \| Q(\kappa) \| \right\}. \tag{2.26}
\]

Set \( \varphi(0) = \beta \| Q(0) \|_1 \), \( \varphi(k) = \zeta^k \varphi(0) + \sum_{l=0}^{k-1} \zeta^{k-1-l} \beta c_0 (1 - \rho) \| Q(l) \|_1 \). We have

\[
\varphi(k + 1) \leq (\zeta + \beta c_0 (1 - \rho)) \varphi(k) = (\zeta + \beta c_0 (1 - \rho))^k \varphi(0). \tag{2.27}
\]

Using the inequalities in (2.26) and (2.27), we can show that

\[
\| Q(k) \|_1 \leq \beta (\zeta + \beta c_0 (1 - \rho))^k \| Q(0) \|_1. \tag{2.28}
\]

According to Lemma 2 and the fact that \( \mathbb{E}(\| X(k) \|^2) = \text{tr}(\sum_{i=1}^{q} Q_i(k)) \), it is readily to verify that when \( \rho > \frac{\beta \zeta - 1}{\beta c_0} \), The system in (2.24) is mean square stable, meaning that \( \lim_{k \to \infty} \| \xi(k) \|^2 = 0 \). From Lemma 1, we can conclude that the system in (2.3), (2.4) and (2.5) can reach consensus. \( \square \)
2.4.3 Case III: Switching Topologies with Delays Governed by a Partially Unknown Markov Chain.

In this subsection, we will analyze the stability of the underlying system in (2.3), (2.4) and (2.5) with Markov delays. Some elements in the transition probability matrix are unknown:

\[
\begin{bmatrix}
  p_{11} & ? & ? & p_{14} \\
  ? & p_{22} & ? & p_{24} \\
  p_{31} & p_{32} & ? & ? \\
  ? & ? & p_{43} & ?
\end{bmatrix},
\]

where “?” represents the unknown elements of the transition probability matrix. \( p_{ij} \) denotes the transition probability from the state \( i \) to the state \( j \) in a Markov process. In [61], Zhang and Boukas investigate the stability and stabilization problem of the discrete-time MJLS with partially unknown transition rates, but the delays are not considered therein. Here, we study the stability of the MAS subject to delays governed by a Markov chain with partially unknown transition rates. By using the augmentation technique, we obtain a delay-free stochastic system in (2.24). Define the states set \( l^i_K = \{ j : p_{ij} \text{ is known} \} \), \( l^i_{UK} = \{ j : p_{ij} \text{ is unknown} \} \), \( l = l^i_K \cup l^i_{UK} \), \( \pi^i_K = \sum_{j \in l^i_K} p_{ij} \) and \( A_i = A_0 + F_{r_i}, i = 1, 2, \ldots, q \). Now we are ready to present the main result as follows.

**Theorem 3.** Consider the system in (2.3), (2.4) and (2.5) with random delays governed by a Markov chain with partially unknown transition rates. Mean square consensus of the system in (2.3), (2.4) and (2.5) is reached if there exist \( P_i > 0, i \in l \),
such that

\[
\begin{bmatrix}
-P_{K_1} & 0 & \cdots & 0 & \sqrt{p_{iK_1}}P_{K_1}A_i \\
* & -P_{K_2} & \vdots & \sqrt{p_{iK_2}}P_{K_2}A_i \\
* & * & \ddots & 0 \\
* & * & * & -P_{K_m} & \sqrt{p_{iK_m}}P_{K_m}A_i \\
* & * & * & * & -\pi_i^K P_i \\
\end{bmatrix}
< 0, \quad (2.29)
\]

\[
\begin{bmatrix}
-P_j & P_jA_i \\
* & -P_i \\
\end{bmatrix}
< 0 \quad (2.30)
\]

hold \( \forall j \in l_{UK}^i \), where \( m \) is the number of the states whose transition rates are known from state \( i \) in the Markov chain. \( K_s^i, s = 1, 2, \ldots, m, \) is the \( s \)th state whose transition probability is known from the state \( i \).

**Proof.** The stochastic system in (2.24) is an MJLS without time delays. Following the similar lines as Theorem 3 in [61], we know that the delay-free stochastic system is mean square stable if

\[
A_i^T P_{K_1}^i A_i - \pi_i^K P_i < 0, \quad (2.31)
\]

\[
A_i^T P_j A_i - P_i < 0, \forall i \in l \text{ and } j \in l_{UK}^i \quad (2.32)
\]

hold, where \( P_{K}^i = \sum_{j \in l_{K}^i} p_{ij} P_j \). By Schur complement, the inequalities in (2.31) and (2.32) are equivalent to (2.29) and (2.30), respectively. The proof is completed. \( \square \)

**Remark 2.** We analyze the stability of the stochastic system subject to delays governed by a Markov chain with partially unknown transition rates. It is observed that the system matrix \( A_0 + F(k) \) in (2.24) and the Laplacian matrices are determined by time delays. With time delays switching at different time instants, the communication
topology also changes.

### 2.5 Illustrative Examples

In this section, two sets of numerical examples will be given to verify the effectiveness of the proposed control protocols for a team of unmanned flying vehicles with the following dynamics [16]:

\[
\begin{align*}
\dot{x} &= v_x, \\
\dot{v}_x &= -v_x + u_x, \\
\dot{y} &= v_y, \\
\dot{v}_y &= -v_y + u_y,
\end{align*}
\]  

(2.33)

where \( x \) and \( y \) represent the component of position vectors in the \( x-y \) coordinate. \( v_x \) and \( v_y \) are the velocity vectors, and \( u_x \) and \( u_y \) are the control inputs.

#### 2.5.1 Consensus of the MAS under Fixed Communication Topology

Figure 2.1 shows a fixed communication topology with a spanning tree of six agents. \( L_0 \) is the corresponding Laplacian matrix. Then the vehicles start with random initial positions and velocities and evolve under the control protocol according to (2.5).
Figure 2.1: Communication topology with a directed spanning tree.

\[
L_0 = 0.4 \begin{bmatrix}
0.5 & 0 & 0 & 0 & 0 & -0.5 \\
-0.4 & 0.4 & 0 & 0 & 0 & 0 \\
0 & -0.5 & 0.5 & 0 & 0 & 0 \\
0 & -0.6 & 0.6 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.5 & 0.5 & 0 \\
0 & 0 & -0.7 & 0 & -0.2 & 0.9
\end{bmatrix}.
\] (2.34)

Here, the sampling period is \( h = 0.01 \) sec. The control gain \( k_c = 1 \) and the delay set is \( \Gamma_1 = \{40, 80, 140, 200\} \) steps. Their transition probability matrix is

\[
\Pi = \begin{bmatrix}
0.5 & 0.2 & 0.1 & 0.2 \\
0.4 & 0.3 & 0.3 & 0 \\
0.1 & 0.5 & 0.1 & 0.3 \\
0.1 & 0.2 & 0.3 & 0.4
\end{bmatrix}.
\] (2.35)

After solving LMIs by Matlab\textsuperscript{®} LMI Toolbox, it is verified that the sufficient conditions in Theorem 1 are satisfied.

The evolution of \( x \) position of agents is shown in Figure 2.2, from which it can be seen that \( x \) position evolution of the agents converges under the proposed control
protocol. Figure 2.3 demonstrates the trajectories of agents.

Figure 2.2: $x$ position evolution of agents under a fixed communication topology.

Figure 2.3: Trajectories of agents under a fixed communication topology.
2.5.2 Consensus of the MAS with Switching Communication Topologies

The second set of simulation is given to illustrate the effectiveness of the proposed control protocol applied to MAS under switching communication topologies. The delay set is \( \Gamma_2 = \{5, 7, 9, 15\} \) steps. Consider a fleet of unmanned flying vehicles in (2.33) with switching topologies as shown in Figure 2.4. As described in Section 2.4.3, the communication topology at time instant \( k \) depends on the time delay at \( k \) and switch with partially unknown transition rates. The transition probability matrix of the Markov chain is in (2.36) and the corresponding Laplacian matrices to the switching topologies are in (2.37).

\[
\Pi = \begin{bmatrix}
0.3 & ? & 0.1 & ? \\
? & ? & 0.3 & 0.2 \\
? & 0.1 & ? & 0.3 \\
0.2 & ? & ? & ?
\end{bmatrix}, \quad (2.36)
\]

\[
L_1 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
-0.7 & 0.7 & 0 & 0 \\
-0.8 & 0 & 0.8 & 0 \\
-0.5 & 0 & 0 & 0.5
\end{bmatrix}, \quad L_2 = \begin{bmatrix}
0.6 & -0.6 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -0.7 & 0.7 & 0 \\
-0.8 & 0 & 0 & 0.8
\end{bmatrix},
\]

\[
L_3 = \begin{bmatrix}
0.7 & 0 & -0.7 & 0 \\
-0.7 & 1.2 & -0.5 & 0 \\
0 & -0.6 & 0.6 & 0 \\
0 & 0 & -0.8 & 0.8
\end{bmatrix}, \quad L_4 = \begin{bmatrix}
0.6 & -0.6 & 0 & 0 \\
0 & 0.6 & -0.4 & 0 \\
-0.5 & 0 & 1.2 & -0.7 \\
0 & -0.7 & 0 & 0.7
\end{bmatrix}. \quad (2.37)
\]

The control gain \( k_c = 1 \). LMIs are built based on Theorem 3 and verified to be
feasible. The evolution of $x$ position and the trajectories of the agents are shown in Figure 2.5 and Figure 2.6, respectively. From which it can be seen that consensus is reached with our proposed control protocol for the MAS with second-order dynamics subject to random delays governed by a Markov chain with partially unknown transition rates.

Figure 2.5: $x$ position evolution of agents under switching communication topologies.

2.6 Conclusions

This Chapter investigates the consensus problem for the second-order MASs with time delays governed by a Markov chain. The consensus problem is converted into the stability analysis of the equivalent error system dynamics. Based on Lyapunov
stability analysis, sufficient conditions are established for the system dynamics with fixed communication topology to reach the mean square consensus. When the transition probabilities in the Markov chain are not fully known, we obtain the lower bound for the estimation of the time delay for ensuring the stability of the error system dynamics. Moreover, by adjusting the adjacency matrix according to the measured delay, we provide sufficient conditions for the MAS subject to random delays governed by a Markov chain with partially unknown transition rates to reach consensus. Finally, the efficacy is verified by two sets of simulations.
Chapter 3

Consensus for Multiple Euler-Lagrange Dynamics with Arbitrary Sampling Periods and Event-Triggered Strategy

3.1 Introduction

In applications, the networked control systems and multi-agent systems are implemented under a sampled-data framework [33]. In the literature, under the sampled-data framework, consensus protocols are investigated by only using the position information of the agent and its neighbors [70], [71], [72] and [34]. When tackling the consensus problem, the main methods include the property analysis of infinite products of stochastic matrices, linear matrix inequalities (LMIs) and Lyapunov stability analysis. There are two main schemes according to the different sampling pattern: Uniform sampling and nonuniform sampling. In [35], the authors propose a consensus
protocol based on detectable position information of the agent and its neighbors under sampled-data settings; the proposed protocol can increase the uniform sampling period such that the consensus is guaranteed. The nonuniform sampling mode is also studied in [35]. However, due to the use of only position information, the convergence rate of the consensus may not be fast. In practice, if both position and velocity information are measurable, then intuitively and naturally, the convergence rate of the consensus would be expected to be faster if they are simultaneously incorporated into the protocol design.

The “event-triggered” strategy essentially enables the nonuniform sampling [53]. The control action is enabled and implemented only when the state error norm satisfies a pre-scribed criteria [73] [43] [74]. A theoretical framework for event-triggered consensus protocol is studied in [43]. The sufficient conditions are derived to guarantee coordination by using the Input-to-State Stability (ISS) theory. It is worth noting that less attention has been paid to tackling the consensus problem for Euler-Lagrange dynamics based on the event-triggered strategy.

Euler-Lagrange equations can be employed to model many practical systems such as rigid body dynamics and linked robotic manipulators [75]. The consensus problem of Euler-Lagrange systems is studied in [27] [28]. In [27], the authors propose a distributed, leaderless consensus algorithm for Euler-Lagrange systems; the convergence analysis is conducted based on the algebraic graph theory, Lyapunov theory and Matrosov’s Theorem.

In light of the above analysis, we are motivated to develop event-triggered consensus protocols for the linearized Euler-Lagrange dynamics, by incorporating both position and velocity information of the agents. The remainder of this chapter is organized as follows. Section 3.2 reviews some background and necessary definitions, and presents the problem formulation. Section 3.3 reports the main results on en-
suring consensus under the arbitrary uniform and nonuniform sampling strategies, respectively. In Section 3.4, the centralized event-triggered control protocol for the linearized Euler-Lagrange dynamics is proposed. In Section 3.5, simulation results based on the Euler-Lagrange model for a revolute joint arm are provided to verify the effectiveness of the proposed algorithms. We also present the comparison of the convergence speed between the proposed algorithms and those only using position information. Finally, conclusions are offered in Section 3.6.

**Notation:** The superscript ‘T’ represents the matrix transpose. \( \mathbb{E} \) denotes the mathematical expectation. A matrix \( P > 0 \) if and only if \( P \) is symmetric and positive definite. ‘\(*\)’ in a matrix stands for a term of block that is induced by symmetry. \( \mathbf{1} \) denotes vector \([1, 1, \ldots, 1]^T\), \( \mathbf{0} \) denotes zero matrix with proper dimension and \( I \) is the identity matrix. \( \| \cdot \| \) denotes the Euclidean norm. \( \mathbb{C} \) denotes the complex numbers and \( \mathbb{R}^{m \times n} \) stands \( m \)-by-\( n \) matrices. Matrices are assumed to be compatible with algebraic operations.

### 3.2 Preliminaries

We denote a weighed graph with \( n \) nodes by \( G = (\mathcal{V}, \mathcal{E}, A) \). \( \mathcal{V} = \{v_1, v_2, \ldots, v_n\} \) represents the nodes set and \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) is the edge set. An edge \((v_j, v_i) \in \mathcal{E}\) is the information flow from \( v_j \) to \( v_i \). The nonnegative adjacency matrix \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \), \((a_{ij} \geq 0, \forall i, j = 1, 2, \ldots, n)\), models the communication topology among the agents. If there is a direct link from agent \( j \) to agent \( i \), which means agent \( i \) receives information from agent \( j \), then \( a_{ij} \neq 0 \); or else \( a_{ij} = 0 \). An undirected graph implies a graph of which the link from \( i \) to \( j \) as well as the link from \( j \) to \( i \) exist and disappear synchronously in a graph. Otherwise the graph is directed. A path from vertex \( i \) to a vertex \( j \) is a sequence of distinct vertices starting with \( i \) and ending with \( j \), such that
consecutive vertices are adjacent [65]. We use $\mathcal{N}_i$ to denote the neighbor set of agent $i$. The neighbors of $i$ indicate the agents from which $i$ receives information. Thus, $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$. Assume that an agent does not receive information from itself. With this assumption, we have $a_{ii} = 0$ and $a_{ij} \geq 0$ for all $i, j, i \neq j$. The graph Laplacian $L \in \mathbb{R}^{n \times n}$ is defined as:

$$l_{ij} = -a_{ij}, \forall i \neq j; l_{ii} = \sum_{j \in \mathcal{N}_i} a_{ij}, i, j = 1, 2, \ldots, n.$$ 

By definition, there is a unique corresponding $L$ to any $A$.

Euler-Lagrange systems are represented by

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i = \tau_i, i = 1, \ldots, n. \tag{3.1}$$

where $q_i \in \mathbb{R}^p$ is the vector of generalised coordinates, $M_i(q_i) \in \mathbb{R}^{p \times p}$ is the inertia matrix which is symmetric and positive definite. $C_i(q_i, \dot{q}_i)\dot{q}_i \in \mathbb{R}^p$ is the vector of Coriolis and centrifugal torques. $\tau_i$ is the vector of the torques exerted to $i$th agent.

We have the following assumptions: (1) There exist $k_{m}, k_{\dot{m}}$ and $k_c > 0$, for any agent, $0 \leq k_{\dot{m}} < \|M_i(q_i)\| \leq k_m$; (2) $\|C_i(q_i, \dot{q}_i)\| \leq k_c\|\dot{q}_i\|$; (3) $\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)$ is skew symmetric.

In this chapter, firstly we propose a control protocol by incorporating both position and velocity information. Assume that the system measures and obtains the position and velocity information of each agent and its neighbors simultaneously at a sequence of time instants $0, t_1, t_2, \ldots$, with $t_k < t_{k+1}, k = 1, 2, \ldots$

Considering a group of $n$ agents with Euler-Lagrange dynamics (3.1), we have the following protocol:

$$\tau_i = C_i(q_i, \dot{q}_i)\dot{q}_i + M_i(q_i)^{-1}(-k_1(\dot{q}_i(t) - \dot{q}_i(t_k)) - k_2(q_i(t) - q_i(t_k))), \tag{3.2}$$
where
\[
\hat{q}_i(t_k) = q_i(t_k) + \rho \sum_{j \in \mathcal{N}_i} a_{ij}(q_j(t_k) - q_i(t_k)),
\]
\[
\hat{\dot{q}}_i(t_k) = \dot{q}_i(t_k) + \rho \sum_{j \in \mathcal{N}_i} a_{ij}(\dot{q}_j(t_k) - \dot{q}_i(t_k)).
\]

Here, \(k_1, k_2\) are positive control gains with respect to velocity and position information of the agent and its neighbors, and \(0 < \rho < \frac{1}{2\max\{\sum_{j \in \mathcal{N}_i} a_{ij} : i = 1, 2, \ldots, n\}}\). By substituting (3.2) into (3.1), the overall system dynamics can be written in the following form:
\[
\begin{bmatrix}
\dot{q}_i(t) \\
\ddot{q}_i(t)
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-k_2 & -k_1
\end{bmatrix} \begin{bmatrix}
q_i(t) \\
\dot{q}_i(t)
\end{bmatrix} - \begin{bmatrix}
0 & 0 \\
-k_2 & -k_1
\end{bmatrix} \begin{bmatrix}
\hat{q}_i(t) \\
\hat{\dot{q}}_i(t)
\end{bmatrix}, \quad t \in [t_k, t_{k+1}).
\]

(3.3)

Next, we define the matrix \(W = [w_{ij}]\):

\[
w_{ij} = \begin{cases}
\rho a_{ij}, & i \neq j \\
1 - \rho \sum_{j \in \mathcal{N}_i} a_{ij}, & i = j.
\end{cases}
\]

**Remark 3.** Compared with the algorithm in [35], the proposed protocol has incorporated the velocity feedback in (3.3). Under the proposed protocol, the solvability of consensus problem will be studied in the sequel. We will study both the uniform and the nonuniform sampling schemes. Note that the properties of \(W\) will be applied in the following analysis and discussion. Lemma 1 in [35] shows that if \(A\) is symmetric, so is \(W\), and \(W\) is a double stochastic matrix and diagonally dominant. If the interaction topology \(G\) has a spanning tree, 1 is the algebraically simple eigenvalue of \(W\). Besides, there exists an orthogonal matrix \(P\) such that \(PWP^T = \text{diag}(\mu_1, \mu_2, \ldots, \mu_n)\), where \(\mu_1 = 1, \mu_2, \ldots, \mu_n\) are eigenvalues of \(W\) in decreasing order. \(\text{diag}(\mu_1, \mu_2, \ldots, \mu_n)\) rep-
resents the diagonal matrix with \( \mu_i \) as the \( i \)th entry on the diagonal.

### 3.3 Consensus Analysis of the MASs with Arbitrary Sampling Periods

In this section, we will study the consensus of the multiple Euler-Lagrange dynamics in (3.1) with the proposed protocol (3.2) under uniform sampling and nonuniform sampling cases, respectively.

Define \( x_i(t) = [q_i(t), \dot{q}_i(t)]^T \). After solving the differential equation in (3.3), we obtain

\[
x_i(t) = e^{K(t-t_k)}x_i(t_k) + (I - e^{K(t-t_k)}) \begin{bmatrix} 0 & 1 \\ -k_2 & -k_1 \end{bmatrix} \begin{bmatrix} \hat{q}_i(t_k) \\ \hat{\dot{q}}_i(t_k) \end{bmatrix},
\] (3.4)

where \( K = \begin{bmatrix} 0 & 1 \\ -k_2 & -k_1 \end{bmatrix} \) is Hurwitz stable.

Define \( x(t) = [x_1(t)^T, x_2(t)^T, \ldots, x_n(t)^T]^T \), we have

\[
x(t_{k+1}) = \left( I \otimes e^{K(t_{k+1}-t_k)} + W \otimes (I - e^{K(t_{k+1}-t_k)}) \begin{bmatrix} 1 & k_1 \\ k_2 & 0 \end{bmatrix} \right) x(t_k).
\] (3.5)

We now present the consensus results in the following Theorem.

**Theorem 4.** If the communication topology \( G \) has a spanning tree, by applying the protocol in (3.2), the consensus of the multiple Euler-Lagrange dynamics system in (3.1) can be achieved with any given sampling period \( h \) when \( k_1^2 \geq 4k_2 \).

**Proof.** We consider the cases when \( k_1^2 > 4k_2 \) and \( k_1^2 = 4k_2 \), respectively.
Case 1: \( k_1^2 > 4k_2 \). The Hurwitz matrix \( K \) has two negative eigenvalues \( \lambda_1 \) and \( \lambda_2 \). We define \( T_1 = \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \) and \( y_1(t) = (P \otimes T_1)x(t) \), where \( P \) is the orthogonal matrix mentioned in Remark 3. Thus, we have \( T_1 K T_1^{-1} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \) and

\[
y_1(t_{k+1}) = (P \otimes T_1) \left( I \otimes e^{K(t_{k+1}-t_k)} + W \otimes (I - e^{K(t_{k+1}-t_k)}) \begin{bmatrix} 1 & k_1 \\ k_2 & 0 \end{bmatrix} \right) \times (P \otimes T_1)^{-1} y_1(t_k).
\]

It can be further written as

\[
y_1(t_k) = \left( I \otimes \begin{bmatrix} e^{\lambda_1 h} & 0 \\ 0 & e^{\lambda_2 h} \end{bmatrix} + PW P^T \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 - e^{\lambda_2 h} \end{bmatrix} \right) T_1 \begin{bmatrix} 1 & k_1 \\ k_2 & 0 \end{bmatrix} T_1^{-1} y_1(t_k),
\]

where \( h = t_{k+1} - t_k, k = 0, 1, \ldots \) is the uniform sampling period. The above state matrix in the brackets is a diagonal block matrix with \( D_i \) on the diagonal line.

\[
D_i = \begin{bmatrix} e^{\lambda_1 h} & 0 \\ 0 & e^{\lambda_2 h} \end{bmatrix} + \frac{\mu_i}{\lambda_2 - \lambda_1} \begin{bmatrix} -\lambda_1 (1 - e^{\lambda_1 h}) & -\frac{\lambda_2}{\lambda_1} (1 - e^{\lambda_1 h}) \\ \frac{\lambda_2}{\lambda_1} (1 - e^{\lambda_2 h}) & \lambda_2 (1 - e^{\lambda_1 h}) \end{bmatrix}
\]

\[
= \begin{bmatrix} e^{\lambda_1 h} - \frac{\mu_i \lambda_1 (1 - e^{\lambda_1 h})}{\lambda_2 - \lambda_1} & -\frac{\mu_i \lambda_2 (1 - e^{\lambda_1 h})}{\lambda_1 (\lambda_2 - \lambda_1)} \\ -\frac{\mu_i \lambda_2 (1 - e^{\lambda_2 h})}{\lambda_2 (\lambda_2 - \lambda_1)} & e^{\lambda_2 h} + \frac{\mu_i \lambda_2 (1 - e^{\lambda_2 h})}{\lambda_2 - \lambda_1} \end{bmatrix}.
\]
To investigate the eigenvalues of $D_i$, we have

$$\det(sI - D_i) = (1 - \mu_i) (s - e^{\lambda_1 h}) (s - e^{\lambda_2 h}) + \mu_i (s - 1) \left( s - \frac{\lambda_2 e^{\lambda_1 h} - \lambda_1 e^{\lambda_2 h}}{\lambda_2 - \lambda_1} \right).$$

(3.6)

Next, we need to show

$$\max\{e^{\lambda_1 h}, e^{\lambda_2 h}\} < \frac{\lambda_2 e^{\lambda_1 h} - \lambda_1 e^{\lambda_2 h}}{\lambda_2 - \lambda_1} < 1.$$  

(3.7)

First, consider the case $\lambda_1 < \lambda_2$. It is readily obtained that $\frac{\lambda_2 e^{\lambda_1 h} - \lambda_1 e^{\lambda_2 h}}{\lambda_2 - \lambda_1} > e^{\lambda_2} > e^{\lambda_1}$. To prove the right-hand side of the inequality in (3.7), we have

$$\frac{\lambda_2 e^{\lambda_1 h} - \lambda_1 e^{\lambda_2 h}}{\lambda_2 - \lambda_1} - 1 = \frac{\lambda_2(e^{\lambda_1 h} - 1) - \lambda_1(e^{\lambda_2 h} - 1)}{\lambda_2 - \lambda_1}.$$

The inequality in (3.7) holds if and only if $\frac{e^{\lambda_1 h} - 1}{\lambda_1} < \frac{e^{\lambda_2 h} - 1}{\lambda_2}$. Let $f(x) = \frac{e^{xh} - 1}{x}$. Next we show $f(x)$ is monotonically decreasing when $x < 0$.

$$f'(x) = \frac{1 + (xh - 1)e^{xh}}{x^2}.$$  

Let $z(x) = (xh - 1)e^{xh}$, and $z'(x) = xh^2e^{xh}$. It is shown that $z(x)$ is monotonically decreasing when $x < 0$. We have $\min z(x) = z(0) = -1$. Thus, $f'(x) > 0$. With the same argument, for the case $\lambda_1 > \lambda_2$, the inequality in (3.7) holds.

From the above analysis, it is shown that $D_1$ has two eigenvalues, i.e., 1 and $\frac{\lambda_2 e^{\lambda_1} - \lambda_1 e^{\lambda_2}}{\lambda_2 - \lambda_1}$. From (3.6) we conclude that $D_i, i = 2, 3, \ldots, n$, has two eigenvalues in the intervals $(\min(e^{\lambda_1 h}, e^{\lambda_2 h}), \max(e^{\lambda_1 h}, e^{\lambda_2 h})) \cup \left( \frac{\lambda_2 e^{\lambda_1} - \lambda_1 e^{\lambda_2}}{\lambda_2 - \lambda_1}, 1 \right)$. Therefore, $D_i$ is Schur stable.
\( P \) is an orthogonal matrix with the first row \((1/\sqrt{n})1^T\). We have

\[
y_1(t_k) = (P \otimes T)x(t_k) = \begin{bmatrix} D_1 & \cdots & \cdots \end{bmatrix}^k y_1(0) = \begin{bmatrix} D_1 & \cdots & \cdots \end{bmatrix}^k (P \otimes T)x(0).
\]

Therefore,

\[
x(t_k) = (P \otimes T)^{-1} \begin{bmatrix} D_1 & \cdots & \cdots \end{bmatrix}^k (P \otimes T)x(0). \tag{3.8}
\]

Since \( D_1 \) has a 1 eigenvalue and the other eigenvalue in \((\max\{e^{{k_1}h}, e^{{k_2}h}\}, 1)\) and \( D_i, i = 2, 3, \ldots, n \) are Schur stable. From (3.8), we conclude that \( \lim_{k \to +\infty} x(t_k) \) asymptotically converges to some steady state.

**Case 2:** \( k_1^2 = 4k_2 \).

Let \( \lambda = -\frac{k_1}{2}, T_2 = \begin{bmatrix} 0 & -1 \\ \frac{k_1}{2} & 1 \end{bmatrix} \) and \( y_2(t) = (P \otimes T_2) \). Similarly, we obtain the evolution of \( y_2(t) \) as follows

\[
y_2(t_{k+1}) = \Xi y(t_k),
\]

where \( \Xi \) is also the block diagonal system matrix. We have

\[
\Xi = \begin{bmatrix} D_1' & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & D_n' \end{bmatrix}, \tag{3.9}
\]
and
\[
D'_i = \begin{bmatrix}
  e^{\lambda h} - \mu_i \lambda h e^{\lambda h} & -\lambda h e^{\lambda h} + \mu_i \lambda h e^{\lambda h} \\
  \mu_i (e^{\lambda h} - 1) & e^{\lambda h} + \mu_i (1 - e^{\lambda h})
\end{bmatrix}.
\]

Then
\[
\det(sI - D'_i) = (1 - \mu_i)(s - e^{\lambda h})^2 + \mu_i(s - 1)(s - e^{\lambda h} + \lambda h e^{\lambda h}). \quad (3.10)
\]

Next, we need to show \(0 < e^{\lambda h} - \lambda h e^{\lambda h} < 1\). Let \(f(x) = e^{xh} - xhe^{xh}\), we have \(f'(x) = -xh^2 e^{xh} < 0\), when \(x < 0\). Thus \(f(x)\) is monotonically decreasing. \(\lim_{x \to -\infty} f(x) = 0\) and \(\lim_{x \to 0} f(x) = 1\). \(D'_i\), has two eigenvalues: 1 and \(e^{\lambda h} - \lambda h e^{\lambda h}\). \(D'_i, i = 2, 3, \ldots, n\) has two eigenvalues in the interval \((e^{\lambda h} - \lambda h e^{\lambda h}, 1)\). \(D'_i\) is Schur stable. Then with the similar analysis, when \(k_1^2 = 4k_2\), the proposed protocol in (3.2) can ensure the consensus.

**Remark 4.** In light of the elegant results in [35], we analyze the consensus conditions for the linearized multiple Euler-Lagrange dynamics system under the proposed protocol that incorporates not only the position but also the velocity information, whereas the result in [35] considers double-integrator systems and only uses the position information. Along a similar line, we study the case when control gains \(k_1\) and \(k_2\) satisfy some condition such that the proposed control protocol in (3.2) can ensure the consensus. Since the additional velocity information has been incorporated into the protocol design, unsurprisingly, the convergence speed will be faster than the results in [35]. The comparison study is provided in the simulations.

**Remark 5.** For the nonuniform sampling scheme, Theorem 2 in [35] proves that the control protocol without velocity feedback can guarantee the consensus if any sampling period \(h_k\) is larger than a given lower bound. The main technique of the proof lies
in the following: One block diagonal matrix of the evolution equations for \( y(t_k) \) is with eigenvalues 1 and another real value in the interval \((-1, 1)\); other block diagonal matrices are all Schur stable. Then it can be proved that infinite products of such system matrices asymptotically converge to certain values. Under the proposed protocol with additional velocity information, we have proved that the system matrices of \( y_1(t_k) \) and \( y_2(t_k) \) are both block diagonal matrices with one eigenvalue 1 and other eigenvalues being in the interval \((-1, 1)\). Therefore, we can show that, with arbitrary sampling periods, the proposed protocol in (3.2) can ensure the consensus.

In the next section, we will propose a new centralized event-triggered protocol.

### 3.4 Consensus Based on an Centralized Event-triggered Strategy

In practical networked dynamic systems or networked multi-agent systems, the network bandwidth and resources are limited. To make full use of the limited resources, if the involved variables or signals do not vary significantly, one may choose not to sample the signals periodically with a fast frequency; instead, one can sample or update the signals only if some prescribed measure will be satisfied. Here, we propose a new protocol based on the centralized event-triggered strategy incorporating both position and velocity information of the agent and its neighbors.

For each agent \( i \) (\( i = 1, 2, \ldots, n \)), we consider the second-order dynamics described by (3.3). Define the new variable

\[
X(t) = [q_1(t), q_2(t), \ldots, q_n(t), \dot{q}_1(t), \dot{q}_2(t), \ldots, \dot{q}_n(t)]^T.
\]
Then we have

\[ \dot{X}(t) = \begin{bmatrix} 0 & I \\ k_2 \bar{W} & k_1 \bar{W} \end{bmatrix} X(t), \]  

(3.11)

where

\[
\bar{W} = \begin{bmatrix}
- \sum_{j \in N_1} w_{1j} & w_{12} & \cdots & w_{1n} \\
& - \sum_{j \in N_2} w_{2j} & \cdots & w_{2n} \\
& & \ddots & \vdots \\
w_{n1} & w_{n2} & \cdots & - \sum_{j \in N_n} w_{nj}
\end{bmatrix}.
\]

**Lemma 3.** [76] (Geršgorin Theorem) Let \( G = [g_{ij}] \in \mathbb{R}^{n \times n} \), and let \( R_i(G) \equiv \sum_{j=1, j \neq i}^n |g_{ij}|, 1 \leq i \leq n \). Then all the eigenvalues of \( G \) are located in the union of \( n \) discs: 

\[
\bigcup_{i=1}^n \{ z \in \mathbb{C}; |z - g_{ii}| \leq R_i(G) \}.
\]

**Lemma 4.** [76] If \( M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \), where \( A, B, C, D \) are \( n \times n \) matrices and \( CD = DC \), then \( \det(M) = \det(AD - BC) \).

Now we study the eigenvalues of \( R = \begin{bmatrix} 0 & I \\ k_2 \bar{W} & k_1 \bar{W} \end{bmatrix} \). From Lemma 4, we obtain

\[
\det(sI - R) = \det \begin{bmatrix} sI & -I \\ -k_2 \bar{W} & sI - k_1 \bar{W} \end{bmatrix} = \det(s^2 - k_1 s \bar{W} - k_2 \bar{W}).
\]

Using Lemma 3, we obtain that \( \bar{W} \) is negative definite (see Figure 3.1). Let \( u_i < 0, i = 1, 2, \ldots, n \) be the eigenvalues of \( \bar{W} \). Thus we have \( s = \frac{k_1 u_i \pm \sqrt{k_1^2 u_i^2 - 4 u_i k_2}}{2} \). The real part of \( s \) is negative. We claim that \( R \) is a negative definite matrix.

Define an error term \( e(t) = X(t_k) - X(t), t \in [t_k, t_{k+1}) \). For the event-triggered strategy, the control input will be kept as a constant between any two triggering time
instants \( t_0, t_1, \ldots \). If let \( \tilde{R} = -R \), then \( \tilde{R} \) is positive definite. The overall system can be described as

\[
\dot{X}(t) = -\tilde{R}X(t_k), \ t \in [t_k, t_{k+1}).
\]

(3.12)

Figure 3.1: A demonstration of Geršgorin Theorem applied to \( \tilde{W} \).

Consider the following Lyapunov function candidate

\[
V(t) = \frac{1}{2} X^T(t)\tilde{R}X(t).
\]

Then we have

\[
\dot{X}(t) = -\tilde{R}X(t_k) = -\tilde{R}(X(t) + e(t)), \ t \in [t_k, t_{k+1}),
\]

\[
\dot{V} = -X^T(t)\tilde{R}\tilde{R}(X(t) + e(t)) = -\|\tilde{R}X(t)\|^2 - X^T(t)\tilde{R}\tilde{R}e(t)
\]

\[
\leq -\|\tilde{R}X(t)\|^2 + \|\tilde{R}X(t)\|\|\tilde{R}\|\|e(t)\|.
\]

(3.13)

Therefore, if we set

\[
\|e(t)\| \leq \frac{\|\tilde{R}X(t)\|}{\|\tilde{R}\|},
\]

(3.14)

we can enforce the Lyapunov function to stay negative. However, it could be sensitive
and risky by setting the triggering time directly based on \( \| e(t) \| = \frac{\| \tilde{R} X(t) \|}{\| R \|} \). Sometimes, when \( \| e(t) \| \) satisfies (3.14), but gets very close to \( \frac{\| \tilde{R} X(t) \|}{\| R \|} \), the next checking of \( \| e(t) \| \) may fail such that the derivative of the Lyapunov function may not be negative. Thus, we set the triggering time when \( \| e(t) \| \) satisfies

\[
\| e(t) \| \geq \sigma \frac{\| \tilde{R} X(t) \|}{\| R \|},
\]

where \( 0 < \sigma < 1 \) and \( \dot{V} \leq (\sigma - 1) \| \tilde{R} X(t) \|^2 \). Once the triggering is activated, the error is set to 0 and the control input remains constant as proposed in (3.12) until the next triggering is activated. Now we study the consensus property under the proposed protocol and the triggering mechanism.

**Theorem 5.** If the communication topology is connected, by applying the proposed protocol and assuming \( 0 < \sigma < 1 \), all the agents can asymptotically reach the weighted average initial position and the velocities converge to 0.

**Proof.** Since \( \dot{V} \leq (\sigma - 1) \| \tilde{R} X(t) \|^2 \), we can see that \( \lim_{k \to \infty} \tilde{R} X(t_k) = 0 \). From the structure of \( \tilde{R} \) and (3.12) we can conclude that the velocities of the agents converge to 0. Further, the positions are equal to a steady state with the weighted average with respect to \( \tilde{W} \).

\[\Box\]

### 3.5 Illustrative Examples

In this section, two numerical examples will be given to verify the proposed control protocols for a group of revolute joint arms. In the first set of simulation, we show that by applying the protocol in (3.2), consensus can be reached and the convergence rate is improved compared to the algorithms given in [35]. The second set of simulation is presented to demonstrate the event-triggered strategy: Consensus can be achieved
with faster speed than the method by only using the position information [43].

It is assumed that for every joint arm the dynamics parameters are the same: The mass of the link is 0.6kg, the length of the link is 0.5m, the distances from the joint to the center of mass of the link is 0.25m, the moment of inertia of the link is 0.2kgm$^2$. Six joint arms start from random initial conditions $q_i(0)$ and $\dot{q}_i(0), \ i = 1, 2, \cdots, 6$. Then the states of agents evolve under the control protocol in (3.2) in the following cases. Figure 3.2 shows the fixed system topology with a spanning tree.

When $k_1^2 > 4k_2$, Figures 3.3–3.8 show the comparison results regarding the convergence speed between control protocols with and without the velocity information [35] under uniform sampling schemes. It can be observed that: When the uniform sampling periods are small ($h \leq 1$), the control protocol in (3.2) demonstrates better performance; when the sampling periods are large, there is no obvious difference on the convergence speed between these two protocols.

Figures 3.9 and 3.10 illustrate the performance of using these two protocols when the nonuniform sampling schemes are applied. At each sampling time, we randomly choose $h_k \in [0.01, 5]\sec$. It can be seen that the convergence speed has been improved.

![Communication topology with a directed spanning tree.](image)

The similar trends can be observed for the case $k_1^2 = 4k_2$. The results are shown
Figure 3.3: The angles evolution without velocity feedback control when $k_1 > 4k_2$, $k_1 = 3$, $k_2 = 1$ and the sampling period $h = 0.1$.

in Figures 3.11–3.18.
Figure 3.4: The angles evolution with velocity feedback control when $k_1 > 4k_2$, $k_1 = 3, k_2 = 1$ and the sampling period $h = 0.1$.

Figure 3.5: The angles evolution without velocity feedback control when $k_1 > 4k_2$, $k_1 = 3, k_2 = 1$ and the sampling period $h = 1$. 
Figure 3.6: The angles evolution with velocity feedback control when $k_1 > 4k_2$, $k_1 = 3$, $k_2 = 1$ and the sampling period $h = 1$.

Figure 3.7: The angles evolution without velocity feedback control when $k_1 > 4k_2$, $k_1 = 3$, $k_2 = 1$ and the sampling period $h$ is large, $h = 20$. 
Figure 3.8: The angles evolution with velocity feedback control when \( k_1 > 4k_2 \), \( k_1 = 3, k_2 = 1 \) and the sampling period \( h \) is large \( h = 20 \).

Figure 3.9: The angles evolution without velocity feedback control when \( k_1 > 4k_2 \), \( k_1 = 3, k_2 = 1 \) and the sampling period \( h \) is nonuniform, \( h_k \in [0.01, 5] \).
Figure 3.10: The angles evolution with velocity feedback control when $k_1 > 4k_2$, $k_1 = 3, k_2 = 1$ and the sampling period $h$ is nonuniform, $h_k \in [0.01, 5]$.

Figure 3.11: The angles evolution without velocity feedback control when $k_1 = 4k_2$, $k_1 = 2, k_2 = 1$ and the sampling period $h = 0.1$. 
Figure 3.12: The angles evolution with velocity feedback control when $k_1 = 4k_2$, $k_1 = 2$, $k_2 = 1$ and the sampling period $h = 0.1$.

Figure 3.13: The angles evolution without velocity feedback control when $k_1 = 4k_2$, $k_1 = 2$, $k_2 = 1$ and the sampling period $h = 1$. 
Figure 3.14: The angles evolution with velocity feedback control when $k_1 = 4k_2$, $k_1 = 2, k_2 = 1$ and the sampling period $h = 1$.

Figure 3.15: The angles evolution without velocity feedback control when $k_1 = 4k_2$, $k_1 = 2, k_2 = 1$ and the sampling period $h$ is large, $h = 20$. 
Figure 3.16: The angles evolution with velocity feedback control when $k_1 = 4k_2$, $k_1 = 2$, $k_2 = 1$ and the sampling period $h$ is large, $h = 20$. 
Figure 3.17: The angles evolution without velocity feedback control when $k_1 = 4k_2$, $k_1 = 2, k_2 = 1$ and the sampling period $h$ is nonuniform, $h_k \in [0.1, 5]$.

Figure 3.18: The angles evolution with velocity feedback control when $k_1 = 4k_2$, $k_1 = 2, k_2 = 1$ and the sampling period $h$ is nonuniform, $h_k \in [0.1, 5]$. 
The second set of simulation is to verify the centralized event-triggered strategy as shown in (3.12). We choose $\sigma = 0.6$. Figures 3.19 and 3.20 show that the consensus can be reached by using the proposed protocol in (3.12) and by using the control protocol without velocity information, respectively. It is also seen that the convergence speed has been improved. Figures 3.21 and 3.22 show the $\|e(t)\|$ and the bound of triggering.

Figure 3.19: The angles evolution using centralized event-triggered control protocol without velocity information.
Figure 3.20: The angles evolution using centralized event-triggered control protocol with velocity information.

Figure 3.21: $\|e(t)\|$ and the trigger bound using centralized event-triggered control protocol without velocity information.
Figure 3.22: $\|e(t)\|$ and the trigger bound using centralized event-triggered control protocol with velocity information.
3.6 Conclusion

In this Chapter, we have investigated the consensus problem for multi-agent systems with the linearized Euler-Lagrange dynamics, under the sampled-data setting. We proposed a control protocol by incorporating the position and velocity information, under uniform and nonuniform sampling schemes, respectively. Then, we proposed a new centralized event-triggered strategy for the linearized Euler-Lagrange dynamics. Simulation results verified the effectiveness of the proposed methods.
Chapter 4

Conclusions

4.1 Summary of the Thesis

This thesis firstly studies the consensus problem of the second-order MAS with random delays governed by a Markov process. Then, the uniform, nonuniform sampled-data control protocols and the centralized event-triggered control protocol for the linearized Euler-Lagrange system dynamics are investigated.

Under the sampled-data framework, Chapter 2 investigates the consensus problem of the MAS with second-order system dynamics. We convert the consensus problem into the stability analysis of the equivalent error system dynamics. The consensus problem is dealt with in the presence of uniform delays governed by a Markov chain. All the communication channels are subject to the same delay at each time instant. Then based on Lyapunov stability analysis and LMIs, we provide sufficient conditions for the MAS to reach consensus under fixed communication topology. Since not all transition probabilities in a Markov chain are always accessible in reality, we estimate the delay for the next time instant and derive a lower bound of the probability for the estimation in order to ensure the stability of the error system dynamics. For the MAS
under switching topologies, control protocols are designed according to the measured delays. We establish the sufficient condition for the MAS with delays governed by a Markov chain with partially unknown transition rates to reach the mean square consensus.

Chapter 3 investigates the consensus problem of the MAS with the linearized Euler-Lagrange dynamics under the uniform sampling schemes and the arbitrary nonuniform sampling scheme, respectively. Firstly, we assume that the position and velocity information of agents are measurable and can be used in the control protocols. Based on the graph theory and the matrix theory, we analyze the stability of the system. Further, a centralized event-triggered control strategy is proposed for the MAS with the linearized Euler-Lagrange system dynamics in Chapter 3. The results of the consensus protocols in Chapter 3 lead to an improvement in the convergence rate compared with existing protocols by only involving the position information of the agents.

4.2 Future Work

In the previous Chapters, we have made several assumptions for formulating and solving the consensus problem. However, these assumptions sometimes limit the application of the proposed control protocols in the reality. Thus, further studies are required to relax the assumptions in designing control protocols.

4.2.1 Extension of the Results in Chapter 2

In Chapter 2, the first assumption is that the uniform time delays are from a finite integer set $\Gamma = \{\tau_1, \tau_2, \ldots, \tau_q\}$ steps. The study of the problem needs to be extended in the following aspects: (1) By the nature of the agent, each agent does not have time
delays on itself. If only the signals from neighbors are with time delays, intuitively, the consensus result can be preserved and the convergence rate will be improved. Thus, the study of the nonuniform time delays governed by a Markov chain needs more attention. (2) The time delays are from a finite integer set, for example, \( \Gamma = \{10, 20, 30, 40\} \) steps. This assumption limits the application. Suppose the real delay is 21 steps which is not in \( \Gamma \). It is a minor difference between the real delay and some element in \( \Gamma \). Naturally the system states with time delay 21 steps and the system states with time delay 20 steps are almost same (If the sampling period is small enough). But the effectiveness of the proposed control protocol is questionable and needs further study.

4.2.2 Extension of the Results in Chapter 3

Chapter 3 firstly studies the consensus problem of the MAS with the linearized Euler-Lagrange dynamics under uniform sampling scheme. However, the consensus is analyzed based on the condition \( k_1 \geq 4k_2 \). For the case \( k_1 < 4k_2 \), the consensus result is still under investigation. Further study will be given to complement the protocols for general scenarios. Moreover, the faster convergence rate is shown by the results of simulations. Theoretical analysis of the convergence speed will be focused in future.

The experimental validation by applying the protocols to real systems is also a part of future works.
Bibliography


Appendix A

Publications

• Refereed journal papers that have been accepted


• Refereed journal paper that is under preparation


• Refereed conference papers that have appeared or been accepted


Winnipeg, Manitoba, Canada, June 4-6, 2012.

- **Refereed conference paper that is under review**