Random Distances Associated with Arbitrary Triangles: A Recursive Approach with an Arbitrary Reference Point

Maryam Ahmadi and Jianping Pan

University of Victoria, BC, Canada

Abstract

In this work, we propose a decomposition and recursion approach in order to obtain the distance distributions associated with arbitrary triangles. The focus of this work is to derive the distance distributions from an arbitrary reference point to a random point within the triangle, where the reference point can be inside or outside of the triangle. Our approach is based on the distance distributions from a vertex of an arbitrary triangle to a random point inside. By decomposing the original triangle, using the probabilistic sum, and using the distance distributions from the vertex of the decomposed triangles, we obtain the desired distance distributions. We compare our analytical results with those of simulation, where a close match can be seen between them. Since any polygon can be decomposed into triangles, this approach also applies to the random distances from an arbitrary reference point to an arbitrary polygon, regardless convex or concave.

Index Terms

Random distances with a reference point; distance distribution functions; arbitrary triangles; arbitrary polygons

I. Problem Statement

The coverage area of the base stations (BSs) in cellular networks is usually approximated by hexagons or circles, in order to make the modeling and analysis of such networks more tractable.

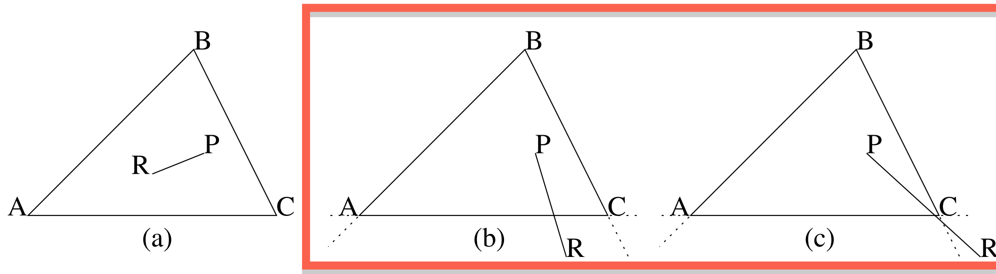

Fig. 1: Arbitrary Reference Point R.

In [1] the authors derived the distance distributions from a fixed point to a random point inside of a hexagon and utilized the results for a performance study of wireless networks. Distance distributions from an interior reference point to a random point within a regular polygon were derived in [2]. The approach proposed in [2], however, is not applicable to the scenario with an exterior reference point. In real world, the coverage area of BSs is not exactly in the form of a hexagon or circle, but more of an irregular shape, as it depends on many factors. Thus, the distance distributions associated with irregular shapes can be used to accurately model the interference (and other performance metrics related to the distance) within a cell or between neighboring cells.

In this work, we focus on the distance distributions associated with arbitrary triangles, as any polygon can be triangulated. A systematic approach is proposed to derive the distribution of the distance from an arbitrary reference point to a random point inside of an arbitrary triangle. The reference point can be located either inside or outside of the triangle. Basically, the triangle could be any arbitrary triangle including equilateral, isosceles, and right, as well as other irregular triangles.

1) An Interior Reference Point: Figure 1(a) shows the case where the reference point, R, is located inside of an arbitrary triangle $\triangle ABC$. P is a random point inside of the triangle. The problem is to find the distribution of the distance between R and any random point P inside of the triangle.

2) An Exterior Reference Point: Figure 1(b) and (c) correspond to the case where the reference point, R, is located outside of the triangle. In (c), the reference point R is located in the area formed from the extensions of the edges at vertex C as shown in the figure, while in (b), the reference point is located outside of this specific area for any of the vertices. The two cases will be separately discussed in Subsection II-B.

II. Decomposition Approach

In this section, we describe how we decompose the triangle and apply a recursive approach to find the distance distribution from an interior/exterior reference point to a random point within an arbitrary triangle.

A. The Interior Reference Point

When the reference point, R, is located inside of the triangle, connecting R to the vertices will decompose the triangle into three smaller triangles, $\triangle ABR$, $\triangle BCR$, and $\triangle ARC$, as shown in Fig. 2(a).

For now, assume that the distance distribution from a vertex of an arbitrary triangle to a random point within the triangle is known (will be explained in detail in Section III). In other words, the distance distribution from point R to a random point inside of $\triangle ABR$ can be found. Following the same approach, the distance distribution from R to a random point inside of $\triangle BCR$ and $\triangle ARC$ can be found as well. Utilizing the probabilistic sum concept, the CDF of the distance from R to a random point within $\triangle ABC$ is the probabilistic sum of the distance distributions from R to a random point within the three triangles which compose $\triangle ABC$. Denote the area of $\triangle ABC$, $\triangle ABR$, $\triangle BCR$, and $\triangle ARC$ as $||\triangle ABC||$, $||\triangle ABR||$, $||\triangle BCR||$, and $||\triangle ARC||$, respectively. Thus,

$$F_{ABC}(r) = \frac{||\triangle ABR||}{||\triangle ABC||} F_{ABR}(r) + \frac{||\triangle BCR||}{||\triangle ABC||} F_{BCR}(r) + \frac{||\triangle ARC||}{||\triangle ABC||} F_{ARC}(r), \qquad (1)$$

where F_t corresponds to the CDF of the distance from point R to a random point inside of triangle t and r is the random variable representing the distance between R and the random point inside of the triangle.

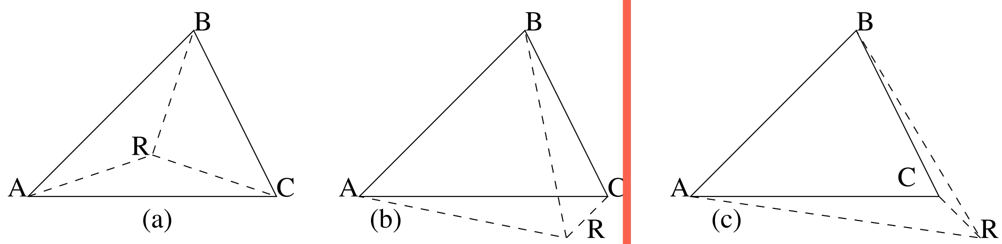

Fig. 2: Decomposition.

B. The Exterior Reference Point

When R is located outside of $\triangle ABC$, two possible cases can happen as shown in Fig. 2(b) and (c): 1) the reference point is located in the area formed from the extensions of the edges at one vertex, as shown in Fig. 2(c) and Fig. 1(c), 2) the reference point is in any other location, but not the specific areas formed from the extension of the edges at a vertex, as shown in Fig. 2(b). As demonstrated in Fig. 2(c), connecting R to the vertices does not intersect with any of the edges, while in (b), connecting R to vertex B, intersects with edge AC, thus resulting in a different decomposition pattern.

As demonstrated in Fig. 2(b), using the probabilistic sum we have

$$\frac{||\triangle ABC||}{||\square ABCR||}F_{ABC}(r)+\frac{||\triangle ACR||}{||\square ABCR||}F_{ACR}(r)=\frac{||\triangle ABR||}{||\square ABCR||}F_{ABR}(r)+\frac{||\triangle BCR||}{||\square ABCR||}F_{BCR}(r), \tag{2}$$

where $||\square ABCR||$ is the area of $\square ABCR$. As a result, $F_{ABC}(r)$ can be obtained since all other terms in (2) are known (or can be derived using the approach in Section III).

In Fig. 2(c), we have

$$F_{ABR}(r)=\frac{||\triangle ABC||}{||\triangle ABR||}F_{ABC}(r)+\frac{||\triangle BRC||}{||\triangle ABR||}F_{BRC}(r)+\frac{||\triangle ACR||}{||\triangle ABR||}F_{ACR}(r), \tag{3}$$

so, $F_{ABC}(r)$ can be found.

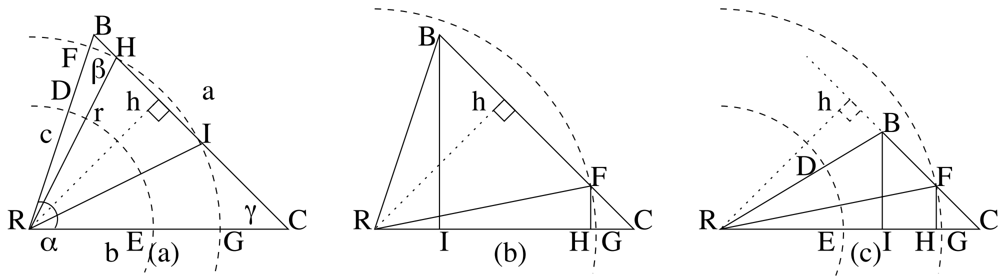

Fig. 3: Distance Distributions from Vertex R to A Point Inside.

III. Random Distances from a Vertex of The Triangle

In this section, we provide detailed explanation on how to obtain the distance distributions from a vertex R of an arbitrary triangle $\triangle RBC$ to a random point inside (without loss of generality, assume $|RB| \leq |RC|$). Two cases are separately discussed below.

A. The Inside Altitude Case

Figure 3(a) and (b) correspond to this case, where the perpendicular line from R to side BC is located inside of $\triangle RBC$. In order to find the distance distribution from R to a random point within $\triangle RBC$, we start with drawing a circle centered at R, where the radius of the circle, denoted as r, corresponds to the distance between R and the random point within $\triangle RBC$. The probability that the distance is smaller than r (the CDF function), is equal to the area of the intersection between the circle and $\triangle RBC$ divided by $||\triangle RBC||$. Several possible cases are discussed below, where h is the length of the perpendicular line from R to side BC and can be derived as

$$h = \frac{2||\triangle RBC||}{|BC|}, \tag{4}$$

where $||\triangle RBC||$ is

$$||\triangle RBC|| = \sqrt{s(s-|RB|)(s-|BC|)(s-|RC|)}, \tag{5}$$

and

$$s = \frac{|RB|+|BC|+|RC|}{2}. \tag{6}$$

1) $0 \leq r \leq h$

   As shown in Fig. 3(a), the circle with radius r cuts the triangle at two points D and E. The intersection area between the circle and the triangle can be easily calculated as $\frac{\alpha}{2}r^2$, where α is $\angle BRC$.

2) $h \leq r \leq |RB|$

   As demonstrated in Fig. 3(a), the circle with radius $h \leq r \leq |RB|$ cuts BC at two points, H and I, side RB at F, and side RC at G. The intersection area can be found as $|| \lhd RFH|| + ||\triangle RHI|| + || \lhd RIG||$. The area of $\triangle RHI$ can be expressed as $\frac{h|HI|}{2}$, where the length of HI is

$$|HI| = 2\sqrt{r^2 - h^2}. \tag{7}$$

   Let us denote the angle $\angle HRI$ as α_1. The summation of the areas of $\lhd RFH$ and $\lhd RIG$ can be calculated as the sector with radius $\alpha - \alpha_1$, where

$$\frac{\alpha_1}{2} = \arccos\left(\frac{h}{r}\right). \tag{8}$$

   Thus,

$$|| \lhd RFH|| + || \lhd RIG|| = \frac{\alpha - \alpha_1}{2}r^2. \tag{9}$$

3) $|RB| \leq r \leq |RC|$

   The intersection area can be calculated as $||\triangle RBF|| + || \lhd RFG||$, demonstrated in Fig. 3(b). $||\triangle RBF||$ can be expressed as $\frac{h|BF|}{2}$, where

$$|BF| = \sqrt{|RB|^2 - h^2} + \sqrt{r^2 - h^2}. \tag{10}$$

   $|| \lhd RFG||$, which is the area of a sector of the circle can be calculated as $\frac{\alpha_2}{2}r^2$, and

$$\alpha_2 = \alpha - \left(\arccos\left(\frac{h}{|RB|}\right) + \arccos\left(\frac{h}{r}\right)\right). \tag{11}$$

4) $r \geq |RC|$

   When $r \geq |RC|$, the triangle will be completely inside of the circle with radius r. Thus, the intersection area is equal to the area of $\triangle RBC$.

B. The Outside Altitude Case

As shown in Fig. 3(c), the perpendicular line from R to side BC falls outside of $\triangle RBC$. Several cases are discussed below.

1) $0 \leq r \leq |RB|$

   The circle with radius r and centered at R, intersects with $\triangle RBC$ at two points, D and E. The intersection area, sector $\triangleleft RDE$, can be easily calculated as $\frac{\alpha}{2}r^2$, where α is $\angle BRC$ of the triangle $\triangle RBC$ and is known.

2) $|RB| \leq r \leq |RC|$

   The intersection area consists of two parts: the area of $\triangle RBF$ and sector $\triangleleft RFG$. The area of $\triangle RBF$ is expressed as

$$||\triangle RBF|| = \frac{h|BF|}{2}, \tag{12}$$

   where, $|BF| = \sqrt{r^2 - h^2} - \sqrt{|RB|^2 - h^2}$.

   Finally, the area of sector $\triangleleft RFG$ is

$$|| \triangleleft RFG|| = \frac{\arcsin\left(\frac{h}{r}\right) - \gamma}{2} r^2, \tag{13}$$

   where γ is the angle $\angle BCR$ shown in Fig. 3.

3) $r \geq |RC|$

   When $r \geq |RC|$, the triangle will be completely inside of the circle with radius r. Thus, the intersection area is equal to the area of $\triangle RBC$.

IV. Results and Verification

In this section, we provide simple examples to derive the distance distribution from an arbitrary interior/exterior reference point to a random point within a triangle. We also compare our results with those of simulation.

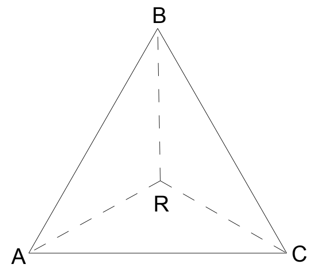

Fig. 4: Example 1, An Interior Reference Point.

A. Example 1: An Interior Reference Point

Denote the vertices of the triangle, A, B, and C with coordinates $(0,0)$, $(\frac{1}{2},\frac{\sqrt{3}}{2})$, and $(1,0)$, respectively, assuming that $(0,0)$ is the origin. Moreover, assume that R is located at the geometrical center of the triangle $(\frac{1}{2},\frac{\sqrt{3}}{6})$. As shown in Fig. 4, connecting R to the vertices of $\triangle ABC$ decomposes the triangle into three triangles: $\triangle ARC$, $\triangle ABR$, and $\triangle BCR$. As explained earlier in Section II, using the recursive approach we have

$$F_{ABC}(r) = \frac{1}{3}F_{ARC}(r) + \frac{1}{3}F_{ABR}(r) + \frac{1}{3}F_{BCR}(r), \tag{14}$$

where, the area of the three small triangles is the same and is equal to $\frac{1}{3}||\triangle ABC||$, and F represents the CDF.

Based on the approach explained in Section III, we obtain that $F_{ARC}(r) = F_{ABR}(r) = F_{BCR}(r)$, and is equal to

$$F_{ARC}(r) = F_{ABR}(r) = F_{BCR}(r) = \begin{cases} \frac{4}{3}\pi\sqrt{3}r^2 & 0 \le r \le \frac{\sqrt{3}}{6} \\ 2\sqrt{r^2 - \frac{1}{12}} - 4\sqrt{3}r^2\cos^{-1}\frac{\sqrt{3}}{6r} & \\ \quad + \frac{4}{3}\pi\sqrt{3}r^2 & \frac{\sqrt{3}}{6} \le r \le \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{6} \\ 1 & r \ge \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{6} \end{cases}. \tag{15}$$

Then, based on (14) and (15), $F_{ABC}(r)$ can be obtained, which is equal to (15).

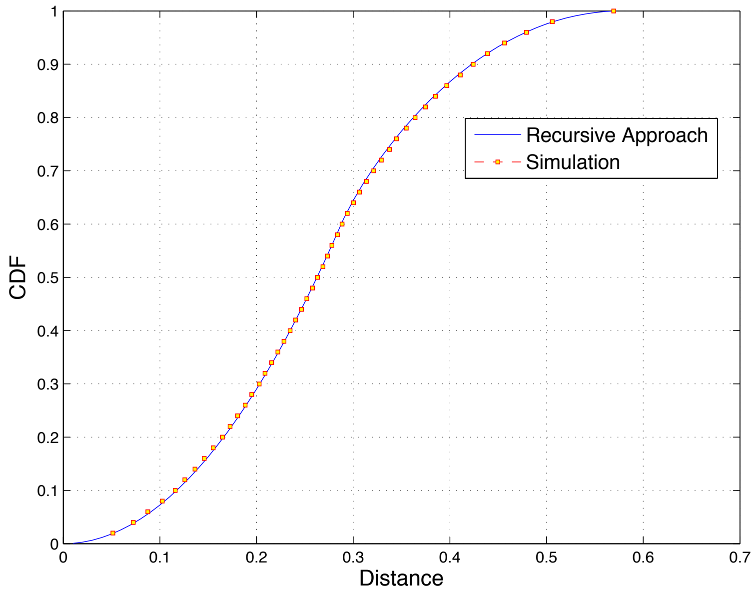

Fig. 5: Example 1, Recursive Approach vs. Simulation.

Finally, we compare the above results with the numerical results from simulation. As shown in Fig. 5, the results from our recursive approach match very closely with the simulation results, verifying our approach.

B. Example 2: An Exterior Reference Point

In this example, we investigate the case when R is located outside of the triangle. The vertices of the triangle are $A(0,0)$, $B(\frac{1}{2},\frac{\sqrt{3}}{2})$, and $C(1,0)$, as shown in Fig. 6. The reference point, R, is located at $(\frac{1}{2},-\frac{\sqrt{3}}{2})$.

Based on the probabilistic sum we have

$$\frac{||\triangle ABR||}{||\square ABCR||}f_{ABR}(r)+\frac{||\triangle BCR||}{||\square ABCR||}f_{BCR}(r)=\frac{||\triangle ABC||}{||\square ABCR||}f_{ABC}(r)+\frac{||\triangle ACR||}{||\square ABCR||}f_{ACR}(r), \tag{16}$$

where f denotes the PDF.

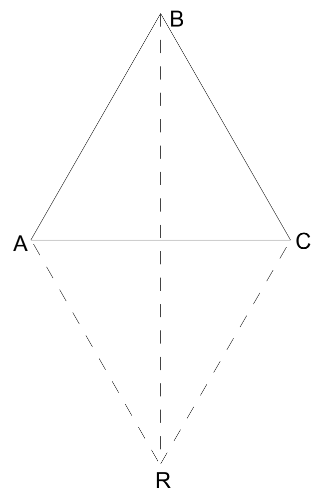

Fig. 6: Example 2, An Exterior Reference Point.

It is obvious that

$$\frac{||\triangle ABR||}{||\Box ABCR||} = \frac{||\triangle BCR||}{||\Box ABCR||} = \frac{||\triangle ABC||}{||\Box ABCR||} = \frac{||\triangle ACR||}{||\Box ABCR||} = \frac{1}{2}. \tag{17}$$

We also have

$$f_{ABR}(r) = f_{BCR}(r) = \begin{cases} \frac{2\pi\sqrt{3}}{9} r & 0 \leq r \leq 1 \\ \frac{r}{\sqrt{r^2 - \frac{3}{4}}} - \frac{1}{\sqrt{1 - \frac{3}{4r^2}}} - \frac{4\sqrt{3}}{3} r \cos^{-1}(\frac{\sqrt{3}}{2r}) + \frac{4\pi\sqrt{3}}{9} r & 1 \leq r \leq \sqrt{3} \\ 0 & r \geq \sqrt{3} \end{cases}, \tag{18}$$

and

$$f_{ACR}(r) = \begin{cases} \frac{4\pi\sqrt{3}}{9} r & 0 \leq r \leq \frac{\sqrt{3}}{2} \\ \frac{2r}{\sqrt{r^2 - \frac{3}{4}}} - \frac{2}{\sqrt{1 - \frac{3}{4r^2}}} - \frac{8\sqrt{3}}{3} r \cos^{-1}(\frac{\sqrt{3}}{2r}) + \frac{4\pi\sqrt{3}}{9} r & \frac{\sqrt{3}}{2} \leq r \leq 1 \\ 0 & r \geq 1 \end{cases}. \tag{19}$$

Thus, according to (16), (17), (18), and (19), $f_{ABC}(r)$ can be derived as

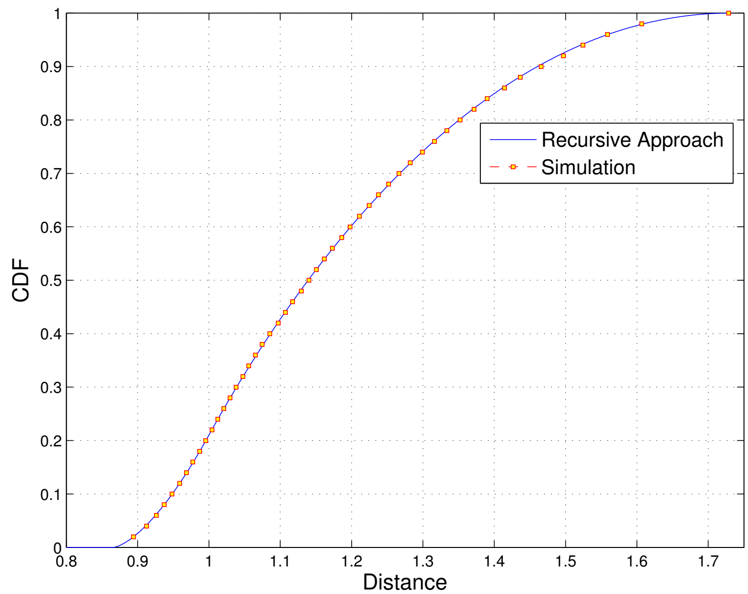

Fig. 7: Example 2, Recursive Approach vs. Simulation.

$$f_{ABC}(r)=\begin{cases}0 & 0\le r\le \frac{\sqrt{3}}{2}\\ -\frac{2r}{\sqrt{r^2-\frac{3}{4}}}+\frac{2}{\sqrt{1-\frac{3}{4r^2}}}+\frac{8\sqrt{3}}{3}r\cos^{-1}(\frac{\sqrt{3}}{2r}) & \frac{\sqrt{3}}{2}\le r\le 1\\ \frac{2r}{\sqrt{r^2-\frac{3}{4}}}-\frac{2}{\sqrt{1-\frac{3}{4r^2}}}-\frac{8\sqrt{3}}{3}r\cos^{-1}(\frac{\sqrt{3}}{2r})+\frac{8\pi\sqrt{3}}{9}r & 1\le r\le\sqrt{3}\\ 0 & r\ge\sqrt{3}\end{cases}. \quad (20)$$

Figure 7 demonstrates the comparison of the results from the simulation and the results derived above. As shown in the figure, the results match closely, verifying our approach and results.

V. Discussion and Conclusions

In this work, we proposed a systematic approach based on decomposition and recursion to find the distance distributions from an arbitrary reference point to a random point within an arbitrary triangle. The reference point can be located inside or outside of the triangle. This basically covers all cases for distance distributions from an arbitrary reference point associated with triangles.

As shown in Fig. 8, any convex or concave polygon can be triangulated and thus our approach

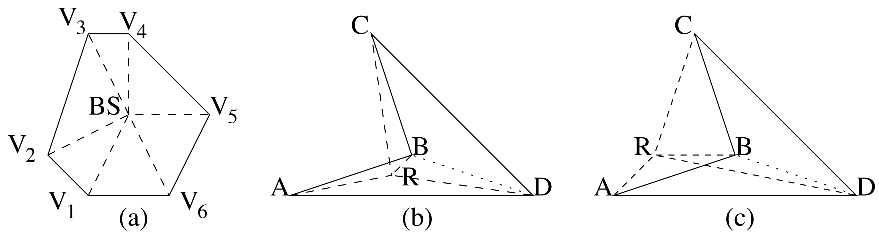

Fig. 8: Triangulation of Convex/Concave Polygons.

can be applied. In Fig. 8(a), the distance distribution from the BS to a random point within the cell can be found by using the probabilistic sum of the distance distributions between the BS and a random point in each of the triangles. In Fig. 8(b), $\square ABCD$ is decomposed into $\triangle ABD$ and $\triangle BCD$. The distance distribution from an interior R to a random point inside $\square ABCD$ can be obtained by the probabilistic sum of the distance distributions from R as an interior reference point to $\triangle ABD$ and as an exterior reference point to $\triangle BCD$ using the approach explained in this report. Finally, in Fig. 8(c), the distance distribution from an exterior R to a random point inside $\square ABCD$ can be obtained by the probabilistic sum of the distance distributions from R as an exterior reference point to $\triangle ABD$ and $\triangle BCD$. Thus, our approach can be applied to convex/concave polygons with an interior/exterior reference point.

Having the distance distributions from a fixed point to random points within a polygon can greatly help with the analysis of wireless networks where the shapes of the cells are irregular. Using these distance distributions, the distributions of the distance-related metrics, such as interference, can be derived. In the next report, we show the random distances associated with arbitrary triangles between two random points, extending our previous work on rhombuses [3], hexagons [4], equilateral triangles [5], and isosceles trapezoids [6].

ACKNOWLEDGMENT

This work is supported in part by the NSERC, CFI and BCKDF. The authors would like to thank Fei Tong for his help.

REFERENCES

[1] Y. Zhuang, Y. Luo, L. Cai, and J. Pan, "A Geometric Probability Model for Capacity Analysis and Interference Estimation in Wireless Mobile Cellular Systems," *In Proc. of* IEEE Globecom, 2011.

[2] Z. Khalid and S. Durrani, "Distance Distributions in Regular Polygons," arXiv:1207.5857v3, 2013.

[3] Y. Zhuang and J. Pan, "Random Distances Associated with Rhombuses," arXiv:1106.1257, 2011.

[4] Y. Zhuang and J. Pan, "Random Distances Associated with Hexagons," arXiv:1106.2200, 2011.

[5] Y. Zhuang and J. Pan, "Random Distances Associated with Equilateral Triangles," arXiv:1207.1511, 2012.

[6] M. Ahmadi and J. Pan, "Random Distances Associated with Trapezoids," arXiv:1307.1444, 2013.