Gray code numbers of complete multipartite graphs

by

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B.Sc., University of Victoria, 2012

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics

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Let $G$ be a graph and $k \geq \chi(G)$ be an integer. The $k$-colouring graph of $G$ is the graph whose vertices are $k$-colourings of $G$, with two colourings adjacent if they colour exactly one vertex differently. We explore the Hamiltonicity and connectivity of such graphs, with particular focus on the $k$-colouring graphs of complete multipartite graphs. We determine the connectivity of the $k$-colouring graph of the complete graph $K_n$ for all $n$, and show that the $k$-colouring graph of a complete multipartite graph $K$ is 2-connected whenever $k \geq \chi(K) + 1$. Additionally, we examine a conjecture that every connected $k$-colouring graph is 2-connected, and give counterexamples for $k \geq 4$. As our main result, we show that for all $k \geq 2t$, the $k$-colouring graph of a complete $t$-partite graph is Hamiltonian. Finally, we characterize the complete multipartite graphs $K$ whose $(\chi(K) + 1)$-colouring graphs are Hamiltonian.
## Contents

Supervisory Committee ii  
Abstract iii  
Table of Contents iv  
List of Figures v  

1 Introduction 1  

2 Background 4  
   2.1 Definitions and Notation . . . . . . . . . . . . . . . . . . . . . . . . . 4  
   2.2 Useful Theorems . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5  
   2.3 The Modified Existence Theorem . . . . . . . . . . . . . . . . . . . . . 7  

3 Connectivity of Colour Graphs 10  
   3.1 Connectivity of \( C_k(K) \) . . . . . . . . . . . . . . . . . . . . . . . 10  
   3.2 2-Connectedness of colour graphs . . . . . . . . . . . . . . . . . . . . . 14  

4 Hamilton Paths and Cycles in SDR Graphs 16  
   4.1 Preliminaries . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 16  
   4.2 Hamilton path and cycle constructions . . . . . . . . . . . . . . . . . . 18  

5 Hamiltonicity of \( C_k(K) \) 44  
   5.1 Construction Theorems . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 44  
   5.2 Upper and lower bounds for \( k_0(K) \) . . . . . . . . . . . . . . . . . . . 48  
   5.3 Hamiltonicity of \( C_{t+1}(K) \) . . . . . . . . . . . . . . . . . . . . . . . 50  

6 Open Problems 56  

Bibliography 57
List of Figures

3.1 $P_{x,1}$ and $P_{x,2}$ do not intersect $P_{y,1}$ or $P_{y,2}$. .................. 13
3.2 One of $P_{x,1}$ or $P_{x,2}$ intersects $P_{y,1}$ or $P_{y,2}$. .................. 13
3.3 Both $P_{x,1}$ and $P_{x,2}$ intersect one of $P_{y,1}$ or $P_{y,2}$. ................. 13
3.4 The graph $H_4$ with colouring $f$ ........................................ 14

4.1 $l = 0, a_1 = 3, a_2 = 1$. ........................................ 21
4.2 $l = 0, a_1 = 2, a_2 = 2$. ........................................ 21
4.3 $l = 1, a_1 = 3, a_2 = 1$. ........................................ 22
4.4 $l = 1, a_1 = 2, a_2 = 2$. ........................................ 22
4.5 $l = 2, a_1 = 1, a_2 = 1$. ........................................ 22
4.6 $l = 2, a_1 = 2, a_2 = 1$. ........................................ 23
4.7 $l = 2, a_1 = 2, a_2 = 2$. ........................................ 23
4.8 $l = 3, a_1 = 1, a_2 = 1$. ........................................ 23
4.9 $l = 3, a_1 = 2, a_2 = 1$. ........................................ 24
4.10 $l = 3, a_1 = 2, a_2 = 2$. ....................................... 24
4.11 $l$ odd, $n$th coordinate $x_i$. ...................................... 27
4.12 $l$ odd, $n$th coordinate $y_i$. ...................................... 27
4.13 $l$ even, $n$th coordinate $x_i$. ..................................... 27
4.14 $l$ even, $n$th coordinate $y_i$. ..................................... 27
4.15 Hamilton cycle $C_1$ in $G_S$. ...................................... 32
4.16 Hamilton cycle $C_2$ in $G_S$. ...................................... 32
4.17 A Hamilton cycle in $G_Y$. ........................................ 34
4.18 Stitching cycles together. ........................................ 38
4.19 A Hamilton Cycle in $G_S$, when $A_t = \{x_1, x_2, y_1\}$. ................. 40

5.1 $C_4(K_3)$ and $C_4(K_{3,3,1})$ ......................................... 51
Let $G$ be a graph and let $k$ be a positive integer. The focus of our work is on the $k$-colouring graph of $G$, denoted $C_k(G)$, which is the graph whose vertices are proper $k$-colourings of $G$, with two colourings adjacent if and only if they differ in the colour of exactly one vertex. For a graph $G$, we consider the Hamiltonicity and connectivity of $C_k(G)$, for various values of $k$. Primarily, we will give results on the Hamiltonicity and connectivity of $k$-colouring graphs of complete multipartite graphs.

The problem of determining the Hamiltonicity of $C_k(G)$ was first considered by Choo [8] in 2003 (also see [9]). Choo has shown that, given a graph $G$, there is a number $k_0(G)$ such that for all $k \geq k_0(G)$, $C_k(G)$ is Hamiltonian. The number $k_0(G)$ is referred to as the Gray code number of $G$, as a Hamilton cycle in $C_k(G)$ is a combinatorial Gray code.

The existence of $k_0(G)$ for any graph $G$ suggests the obvious question: Given $G$, what is $k_0(G)$? Choo [8] answers this question for complete graphs, trees and cycles. Further work on this problem has been done by Celaya et al. [3], who determine Gray code numbers of complete bipartite graphs. The results of Celaya et al. [3] are a basis for the results of this thesis.

Connectivity of the $k$-colouring graph has been explored more thoroughly than Hamiltonicity of the $k$-colouring graph. This is in no small part due to its relevance to the Glauber dynamics Markov chain of $k$-colourings. This is the Markov chain whose states are $k$-colourings, and a transition between states occurs by selecting a colour $c$ uniformly at random, and a vertex uniformly at random to be coloured with $c$. Algorithms for random sampling of $k$-colourings and approximating the number of $k$-colourings arise from these Markov chains, and connectivity of the $k$-colouring graph plays a pivotal role. Jerrum [18] gives a fully polynomial randomized approximation
scheme for estimating the number of $k$-colourings of a graph when $k \geq 2\Delta(G) + 1$. Dyer et al. [11] give an algorithm for almost uniformly randomly generating a $k$-colouring of a random graph $G$ with constant average degree, when $k$ is sufficiently small compared to $\Delta(G)$. Lucier and Molloy [19] give results on Glauber dynamics Markov chains of bounded degree trees.

The problem of determining connectivity of the $k$-colouring graph of a graph $G$ in general is considered by Cereceda et al. [5] in 2008. This work includes a proof that $C_k(G)$ is connected whenever $k \geq 1 + Col(G)$, which follows from a result of Dyer et al. [11]. In addition, it is shown that in general there is no function $\phi(\chi(G))$ such that the $\phi(\chi(G))$-colouring graph of $G$ is connected. If $\chi(G) = 2$ or 3, then the $\chi(G)$-colouring graph of $G$ is not connected, and when $\chi(G) \geq 4$ there exist graphs for which the $\chi(G)$-colouring graph of $G$ is connected. Connectivity of the 3-colouring graph of a bipartite graph is examined by Cereceda et al. [6]. Given a bipartite graph $G$, it is shown that $C_3(G)$ is connected if and only if $G$ is pinchable to $C_6$, where pinching refers to indentifying two vertices at distance two, and a graph $G$ is pinchable to $H$ when there is a series of pinches that transforms $G$ into $H$. Some complexity results are also given. The problem of deciding whether or not the 3-colour graph of a bipartite graph is connected is shown to be coNP-Complete. In contrast, the problem of deciding whether or not two $k$-colourings are in the same component of $C_k(G)$ is PSPACE-Complete when $k \geq 4$ [2], and in P when $k = 3$ [4].

Some alternate colour graphs have also been considered. Finbow and MacGillivray [12] consider variations of the $k$-colouring graph, the $k$-Bell colour graph and the $k$-Stirling colour graph. The $k$-Bell colour graph of $G$ is the graph whose vertices are the partitions of the vertices of $G$ into at most $k$ independent sets. The $k$-Stirling colour graph of $G$ is the graph whose vertices are the partitions of the vertices of $G$ into exactly $k$ independent sets. Various results on the Hamiltonicity and connectivity of such graphs are given.

Two colorings are referred to as non-isomorphic if they admit different partitions of $V(G)$. In 2012, Haas [16] examined the canonical $k$-colouring graph of $G$, whose vertices are non-isomorphic $k$-colourings which are lexicographically least under some enumeration $\pi$ of the vertices of $G$. Two vertices are adjacent if and only if they differ in the colour of exactly one vertex. It is shown that every graph has a canonical $k$-colouring graph which is not connected for some $\pi$ and $k$. Additionally, it is shown that every tree $T$ has an ordering $\pi$ of its vertices such that the canonical $k$-colouring graph of $T$ under $\pi$ is Hamiltonian for every $k \geq 3$. Finally, it is shown that the
canonical $k$-colouring graph of a cycle $C$, with $k \geq 4$, will always be connected under some $\pi$.

This thesis continues the work of finding Gray code numbers for classes of graphs. In particular, the class of complete multipartite graphs is examined. We also give results on the connectivity of colour graphs of complete multipartite graphs. Chapter 2 gives formal definitions and notation which will be used throughout this thesis, as well as an overview of some theorems that we will commonly reference. In Chapter 3, we discuss the connectivity of the colour graph of complete multipartite graphs. We find the connectivity of the $k$-colouring graph of a complete graph $K_t$, for all $k \geq t+1$. We show that the $k$-colouring graph of a complete multipartite graph has connectivity at least 2 whenever it is connected. We address whether or not a connected $k$-colouring graph is in general necessarily 2-connected, and show that this is false for $k \geq 4$. In Chapter 4, we examine a class of graphs, a subclass of what we call SDR graphs, which appear as subgraphs of $k$-colouring graphs of complete multipartite graphs. We show that these graphs will have always have Hamilton paths, and give results on the structure of such paths. In Chapter 5, for complete multipartite graphs $K$, we give our results regarding the Gray code number $k_0(K)$ of $K$. We establish an upper-bound on $k_0(K)$, and characterize the graphs $K$ whose $(\chi(K) + 1)$-colouring graphs are Hamiltonian. In Chapter 6, we close with a brief discussion of open problems.
Chapter 2

Background

In this chapter, we introduce the definitions and notation which will be used throughout the rest of this thesis. In addition, we present a selection of useful theorems on Hamiltonicity and connectivity of colour graphs.

2.1 Definitions and Notation

Let $G$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. For $u, v \in V(G)$, we use the notation $u \sim v$ to denote $uv \in E(G)$. A proper $k$-colouring of $G$ is a function $f: V(G) \to \{1, 2, \ldots, k\}$ such that if $v_i \sim v_j$, then $f(v_i) \neq f(v_j)$. We say a proper $k$-colouring uses the colour $c \in \{1, 2, \ldots, k\}$ if for some $v \in V(G)$, $f(v) = c$. The proper $k$-colouring graph of $G$, $C_k(G)$ is the graph whose vertex set is the set of proper $k$-colourings of $G$, with two colourings being adjacent if and only if they differ in the colour of exactly one vertex of $G$. As we restrict our attention to only proper $k$-colourings, we will refer to the proper $k$-colouring graph and proper $k$-colourings as simply the $k$-colour graph and $k$-colourings respectively.

A complete $t$-partite graph $K_{a_1,a_2,\ldots,a_t}$ is the graph whose vertex set is partitioned by sets $V_1, V_2, \ldots, V_t$, with $|V_i| = a_i$, and for $v \in V_i$ and $u \in V_j$, $u \sim v$ if and only if $i \neq j$. Notice that the complete $t$-partite graph $K_{1,1,\ldots,1}$ is isomorphic to $K_t$. Unless otherwise stated, we assume without loss of generality that $a_1 \geq a_2 \geq \cdots \geq a_t$.

The Gray code number $k_0(G)$ of $G$ is the smallest number such that $C_k(G)$ is Hamiltonian for all $k \geq k_0(G)$. The existence of $k_0(G)$ for any graph $G$ was shown by Choo [8], and a proof will be given at the end of this chapter. A Hamilton cycle in $C_k(G)$ corresponds to a cyclic list of the $k$-colourings of $G$ such that consecutive
colourings in the list differ in the colour of exactly one vertex.

Let $G$ be a graph where $C_k(G)$ is Hamiltonian, and let $C = f_0, f_1, \ldots, f_{N-1}, f_0$ be a Hamilton cycle in $C_k(G)$. We say that $C$ has Property $A$ if for all $c \in \{1, 2, \ldots, k\}$, there is an integer $s$ such that, interpreting indices modulo $N$, neither $f_s$ nor $f_{s+1}$ color any vertex with $c$. If the integer $s$ is assigned to colours $c_1$ and $c_2$, one of $c_1$ or $c_2$ can be reassigned the integer $s+1$, as adjacent colourings differ in the colour of exactly one vertex. Therefore, if a Hamilton cycle has Property $A$, then each colour $c$ can be assigned a unique integer $s$ such that neither $f_s$ nor $f_{s+1}$ color any vertex with $c$. If $C_k(G)$ has a Hamilton cycle with Property $A$, we say $C_k(G)$ is $A$-Hamiltonian. This property is introduced in this thesis, and is used extensively as a construction tool throughout.

The notation $v_1, v_2, \ldots, v_i$ will be used to denote a path from $v_1$ to $v_i$, and the notation $v_1, v_2, \ldots, v_i, v_1$ will be used to denote a cycle. If $P_1 = v_1, v_2, \ldots, v_i$, $P_2 = u_1, u_2, \ldots, u_j$, and $v_i \sim u_1$, then $P_1P_2$ is used to denote the path $v_1, v_2, \ldots, v_i, u_1, u_2, \ldots, u_j$. Similar notation is used to concatenate the path $P_1$ with the single vertex $u_1$. That is, $P_1u_1$ denotes the path $v_1, v_2, \ldots, v_i, u_1$.

Let $G$ be a graph, and let $\pi = v_1, v_2, \ldots, v_n$ be an enumeration of the vertices of $G$. Let $G_i$ denote the subgraph of $G$ induced by the vertices $\{v_1, v_2, \ldots, v_i\}$, and let $d_{G_i}(v)$ denote the degree of $v$ in $G_i$. Let $D_\pi = \max_{1 \leq i \leq n} d_{G_i}(v_i)$. The colouring number of $G$, denoted $Col(G)$, is the value $\min_{\pi} D_\pi + 1$.

Let $\mathcal{G}$ be a group and $X \subset \mathcal{G}$. The Cayley graph $Cay(X : \mathcal{G})$ is defined as the graph with vertex set $V(Cay(X : \mathcal{G})) = \mathcal{G}$ and with vertices $g$ and $g'$ adjacent if and only if $g' = gx$ for some $x \in X$. Results on Hamiltonicity of Cayley graphs can be found in [20] and [10].

Any further terminology and notation will be consistent with Bondy and Murty [1].

### 2.2 Useful Theorems

Among the results of Cereceda et al. [5] regarding connectivity of $k$-colouring graphs is the following theorem, a slight modification of a theorem by Dyer et al. [11], which shows $C_k(G)$ is connected for a sufficiently large $k$.

**Theorem 2.2.1** (Cereceda et al. [5]). Let $G$ be a graph. If $k \geq 1 + Col(G)$, then $C_k(G)$ is connected.
An analogous result for Hamiltonicity was given by Choo [8].

**Theorem 2.2.2** (Choo [8]). Let $G$ be a graph. If $k \geq 2 + \text{Col}(G)$, then $C_k(G)$ is Hamiltonian.

This theorem proves the existence of $k_0(G)$ for any graph $G$. In the next section of this chapter, we will give the proof of this theorem, modified such that the construction produces a Hamilton cycle with Property $\mathcal{A}$. Along with this existence result, Choo [8] establishes Gray code numbers for complete graphs, trees and cycles.

**Theorem 2.2.3** (Choo [8]). $k_0(K_1) = 3$, and $k_0(K_n) = n + 1$ for $n \geq 2$.

The proof of this theorem shows that $C_{t+1}(K_t) \cong \text{Cay}(X : S_{t+1})$, where $X$ is the generating set of transpositions $X = \{(1, t+1), (2, t+1), \ldots, (t, t+1)\}$. We will see in Chapter 5 that the structure of $C_{t+1}(K_{a_1,a_2,\ldots,a_t})$ closely depends on the structure of $C_{t+1}(K_t)$.

**Theorem 2.2.4** (Choo [8]). Let $T$ be a star with $n + 1 \geq 2$ vertices. Then $C_3(T)$ is Hamiltonian if and only if $n$ is odd.

Given that a star $T$ with $n + 1$ vertices is isomorphic to $K_{n,1}$, this result also has particular relevance to our problem.

**Theorem 2.2.5** (Choo [8]). Let $T$ be a tree. If $T$ is a star with $2k + 1 \geq 3$ vertices, then $k_0(T) = 4$. Otherwise, $k_0(T) = 3$.

**Theorem 2.2.6** (Choo [8]). For all $n \geq 3$, we have $k_0(C_n) = 4$.

Further work has been done by Celaya et al. [3], who gave Gray code numbers for complete bipartite graphs. The ideas presented in [3] are a basis for the work done in this thesis. We attempt to generalize these results on complete 2-partite graphs to results on complete $t$-partite graphs.

**Theorem 2.2.7** (Celaya et al. [3]). For positive integers $l$ and $r$, $C_2(K_{l,r})$ is not Hamiltonian, and $C_3(K_{l,r})$ is Hamiltonian if and only if $l$, $r$ are both odd.

In Chapter 5, we generalize this theorem to characterize the complete $t$-partite graphs $K = K_{a_1,a_2,\ldots,a_t}$ for which $C_{t+1}(K)$ is Hamiltonian.

**Theorem 2.2.8** (Celaya et al. [3]). Let $1 \leq l \leq r$ and let $k \geq 4$. Then $C_k(K_{l,r})$ is Hamiltonian.
CHAPTER 2. BACKGROUND

The main result of this thesis is the following generalization of this theorem, which we will prove in Chapter 5.

**Theorem 2.2.9.** Let $a_1, a_2, \ldots, a_t$ be positive integers such that $a_1 \geq a_2 \geq \cdots \geq a_t$. Then, $C_k(K_{a_1,a_2,\ldots,a_t})$ is Hamiltonian for all $k \geq 2t$.

### 2.3 The Modified Existence Theorem

As a final preliminary, we give a proof of Theorem 2.2.2, modified such that it constructs Hamilton cycles with Property $\mathcal{A}$. To begin, we introduce a useful class of graphs known as $C$-Graphs. In this section, we consider subscripts to be modulo $N$.

A $C$-Graph is a graph $G$ whose vertices may be partitioned into sets $F_0, F_1, \ldots, F_{N-1}$ such that for $i \in \{0, 1, \ldots, N - 1\}$, $|F_i| \geq 3$ and $F_i$ induces a Hamilton connected subgraph of $G$. We will now give some conditions under which a $C$-Graph is Hamiltonian, proofs of which can be found in [9] (Choo and MacGillivray). Let $[F_j, F_{j+1}]$ denote the set of edges with one vertex in $F_j$ and one vertex in $F_{j+1}$.

**Lemma 2.3.1** (Choo and MacGillivray [9]). Let $G$ be a $C$-Graph with vertex partition $F_0, F_1, \ldots, F_{N-1}$. If, for each $i \in \{0, 1, \ldots, N - 1\}$, there exist vertex disjoint edges $x_iy_{i+1}$ where $x_i \in F_i$ and $y_{i+1} \in F_{i+1}$, then $G$ is Hamiltonian.

[Choo and MacGillivray [9]]

**Corollary 2.3.2** (Choo and MacGillivray [9]). Let $G$ be a $C$-Graph with vertex partition $F_0, F_1, \ldots, F_{N-1}$. Suppose for each $j \in \{0, 1, \ldots, N - 1\}$ that $[F_j, F_{j+1}]$ contains at least 2 vertex disjoint edges. If there exists $i \in \{0, 1, \ldots, N - 1\}$ such that some vertex $x \in F_i$ has a neighbour in $F_{i+1}$, and $[F_{i-1}, F_i - \{x\}]$ contains at least two vertex disjoint edges, then $G$ is Hamiltonian.

**Corollary 2.3.3** (Choo and MacGillivray [9]). Let $G$ be a $C$-Graph with vertex partition $F_0, F_1, \ldots, F_{N-1}$. Suppose for each $j \in \{0, 1, \ldots, N - 1\}$ that $[F_j, F_{j+1}]$ contains at least 2 vertex disjoint edges. If there exists $i \in \{0, 1, \ldots, N - 1\}$ such that $[F_i, F_{i+1}]$ contains at least three vertex disjoint edges, then $G$ is Hamiltonian.

In light of these results, we have all the tools we need to prove the modified existence theorem. The construction used in this proof is identical to the construction used by Choo [8]. This proof merely notes that the Hamilton cycle constructed does in fact have Property $\mathcal{A}$.
We will show that \( C_k \) colouring in that 

Proof. Let \( C_k \) be a graph. If \( k \geq \text{Col}(G) + 2 \), then \( C_k(G) \) has a Hamilton cycle with Property \( A \).

Let \( \sigma = v_1v_2 \ldots v_n \) be an ordering of \( V(G) \) such that \( D_\sigma = \min_\pi D_\pi \). Let \( k \geq 3 + D_\sigma = 2 + \text{Col}(G) \). Let \( G_i \) denote the subgraph of \( G \) induced by \( v_1, v_2, \ldots, v_i \). We will show that \( C_k(G_i) \) has a Hamilton cycle with Property \( A \) by induction on \( i \).

Let \( \{1, 2, \ldots, k\} \) be our set of colours. Then, \( C_k(G_i) = C_k(K_k) \cong K_k \). This graph clearly has a Hamilton cycle, and since \( k \geq 3 \), for each \( j \in \{1, 2, \ldots, k\} \) any such Hamilton cycle must have consecutive colourings which do not use \( j \). Thus, Property \( A \) is present.

For some \( i \in \{2, 3, \ldots, n - 1\} \), let \( f_0, f_1, \ldots, f_{N-1}, f_0 \) be a Hamilton cycle in \( C_k(G_{i-1}) \) which has Property \( A \). Let \( F_j \) be the set of colourings in \( C_k(G_i) \) which agree with \( f_j \) on \( V(G_{i-1}) \), for \( 0 \leq j \leq N - 1 \). Now, since \( k \geq 3 + D_\sigma \geq 3 + d_{G_i}(v_i) \), we have \( |F_j| \geq 3 \). We also have that \( F_j \) induces a complete subgraph of \( C_k(G_i) \). Therefore, since complete graphs are Hamilton connected, \( C_k(G_i) \) is a \( C \)-Graph with vertex partition \( F_0, F_1, \ldots, F_{N-1} \).

Now, consider some \( j \in \{0, 1, \ldots, N - 1\} \). For colourings \( c_j \in F_j \) and \( c_{j+1} \in F_{j+1} \), \( c_j \sim c_{j+1} \) if and only if \( c_j \) and \( c_{j+1} \) colour \( v_i \) the same colour. As a result, edges in \([F_j, F_{j+1}]\) are vertex disjoint. Let \( w_j \) denote the unique vertex such that \( f_j(w_j) \neq f_{j+1}(w_j) \). If \( v_i \sim w_j \), then each vertex in \( F_j \) has a neighbour in \( F_{j+1} \). In this case, \([F_j, F_{j+1}]\) contains at least three vertex disjoint edges. If \( v_i \sim w_j \), a vertex in \( F_j \) which colours \( v_i \) with the colour \( f_{j+1}(w_j) \) will not have a neighbour in \( F_{j+1} \). Therefore, in this case we may only guarantee that \([F_j, F_{j+1}]\) has at least two vertex disjoint edges. Therefore, if for some \( j \), \( w_j \sim v_i \), then \( C_k(G_i) \) is Hamiltonian by Corollary 2.3.3.

Suppose that \( w_j \sim v_i \) for each \( j \in \{0, 1, \ldots, N - 1\} \). We have already shown that \([F_j, F_{j+1}]\) contains at least two vertex disjoint edges for each \( j \). Let \( c_{N-1} \) be a colouring in \( F_{N-1} \) which has a neighbour in \( F_0 \). Let \( r \) be the largest integer such that \( f_{r-1} \) uses the colour \( c_{N-1}(v_i) \), but \( f_r \) does not. Let \( c_r \) be the colouring in \( F_r \) which assigns \( v_i \) the colour \( c_{N-1}(v_i) \). By definition of \( r \), \( f_{r+1} \) does not use \( c_{N-1}(v_i) \). Then, \( c_r \) has a neighbour in \( F_{r+1} \), and does not have a neighbour in \( F_{r-1} \). Therefore, \([F_{r-1}, F_r - \{c_r\}]\) has at least two vertex disjoint edges. Then, \( C_k(G_i) \) is Hamiltonian by Corollary 2.3.2.

All that is left is to verify the Property \( A \) holds for these Hamilton cycles. First, it is important to notice that the Hamilton cycle constructed by Corollary 2.3.2, and similarly by Corollary 2.3.3, is the concatenation of Hamilton paths of the \( F_i \)'s. Therefore, our constructed Hamilton cycle visits the vertices of \( F_j \) and \( F_{j+1} \) consecutively,
for each $j \in \{0, 1, \ldots, N - 1\}$. By induction, for $l \in \{1, 2, \ldots, k\}$ there exists $j_l$ such that neither $f_{j_l}$ nor $f_{j_l+1}$ use the colour $l$. Then, $F_{j_l}$ and $F_{j_l+1}$ each contain a single vertex which uses colour $l$. Since $|F_{j_l}| + |F_{j_l+1}| \geq 6$, and exactly two of these vertices use $l$, there must be consecutive vertices which do not use $l$. Thus, our Hamilton cycles maintains Property $\mathcal{A}$. □
Chapter 3

Connectivity of Colour Graphs

In an effort to improve our overall understanding of colour graphs, in particular colour graphs of complete multipartite graphs, we considered the connectivity of these graphs. Chartrand and Kapoor [7] show that the cube of a connected graph is Hamiltonian. In the context of colour graphs, for a graph $G$, a Hamilton cycle in the cube of $C_k(G)$ corresponds to a cyclic list of the $k$-colourings of $G$ such that consecutive colourings in the list differ in the colour of at most three vertices. In this chapter, we will give a few basic results on the connectivity of colour graphs. To begin, we examine the connectivity of the colour graph of $K_t$, the simplest complete $t$-partite graph. We show that the connectivity of $C_k(K_t)$ is equal to its minimum degree. We then turn our attention toward $C_k(K_{a_1,a_2,...,a_t})$, proving that we have connectivity at least 2 whenever $k \geq t + 1$, an obvious necessary condition for Hamiltonicity. To finish the chapter, we take a brief look at connectivity of colour graphs in general.

3.1 Connectivity of $C_k(K)$

The first section of this chapter considers the connectivity of colour graphs of complete multipartite graphs, starting with $K_t$. In the case of $K_t$, we are able to establish the connectivity of $C_k(K_t)$ by proving the following theorem.

**Theorem 3.1.1.** $C_k(K_t)$ has connectivity $\delta(C_k(K_t)) = t(k - t)$, whenever $k \geq t + 1$.

**Proof.** We will prove the result by induction on the number of colours, $k$. As we have previously noted, $C_{t+1}(K_t) \cong Cay(X: S_{t+1})$, where $X$ is the minimal generating set of transpositions $X = \{(1, t+1), (2, t+1), \ldots, (t, t+1)\}$. It follows from a theorem of
Godsil [14] that $Cay(X : S_{t+1})$ has connectivity $t$, and thus $C_{t+1}(K_t)$ has connectivity $t = t((t + 1) - t)$ as well.

Suppose for some $k - 1 \geq t + 1$ the result holds, and consider $C_k(K_t)$. Let $x$ and $y$ be any two non-adjacent vertices in $C_k(K_t)$. We will prove our result in two cases, based on the number of colours used by $x$ and $y$.

Case 1: There is a colour $c$ not used by $x$ or $y$.

In this case, we will show that any set which disconnects $x$ from $y$ must have size at least $t(k - t)$ by describing $t(k - t)$ internally vertex disjoint paths between $x$ and $y$. Let $c$ be any colour not used by $x$ or $y$. Let $G$ be the subgraph of $C_k(K_t)$ induced by the vertices which do not use $c$. Note that $x, y \in V(G)$. Then, $G \cong C_{k-1}(K_t)$, and therefore by induction $G$ contains $t((k-1)-t)$ internally vertex disjoint $xy$-paths. We will utilize the fact that none of these paths contain a vertex which uses the colour $c$ to construct $t$ additional internally vertex disjoint $xy$-paths. Let $P_0 = x, f_1, f_2, \ldots, f_\alpha, y$ denote any one of our $xy$-paths contained in $G$. Let $x^i$ denote the vertex obtained by recolouring $v_i$ in $x$ with colour $c$, for $1 \leq i \leq t$. Define $y^i$ similarly. Let $f_j^i$ denote the vertex obtained by recolouring $v_i$ in $f_j$ to $c$, for $1 \leq i \leq t$ and $1 \leq j \leq \alpha$. Then, $x, x^i, f_1^i, f_2^i, \ldots, f_\alpha^i, y^i, y$ is a walk from $x$ to $y$. Though it may not itself be a path due to the possibility of repeated vertices, it contains a path $P_0$ from $x$ to $y$.

Then $P_1, P_2, \ldots, P_t$ are our $t$ additional internally vertex disjoint $xy$-paths, and we have a total of $t(k - t)$ paths, as desired.

Case 2: All $k$ colours are used by $x$ or $y$.

The proof is by contradiction. Let $S$ be a minimal set that separates $x$ from $y$ in $C_k(K_t)$, and suppose $|S| < t(k - t)$. Let $G_x$ and $G_y$ denote the components of $C_k(K_t) - S$ which contain $x$ and $y$ respectively. Let $S_x$ and $S_y$ denote the sets of colours not used by $x$ and $y$ respectively. Note that we have $S_x \cap S_y = \emptyset$. By Case 1, any vertex which does not use a colour $c_x \in S_x$ is either in $G_x$ or in our cut-set $S$. Similarly, any vertex which does not use a colour $c_y \in S_y$ is either in $G_y$ or in $S$. Therefore, any colouring which uses neither $c_x$ nor $c_y$ must be in $S$.

The number of such colourings is $N = (k - 2)(k - 3) \cdots (k - (t + 1))$. Since $(k - 2) \geq t$, we have $N \geq t(k - t)$ when $t \geq 3$, contradicting $|S| < t(k - t)$. When $t = 2$, we must have $k = 4$, as at most four colours may be used by $x$ and $y$. Without loss of generality, we may assume $S_x = \{1, 2\}$ and $S_y = \{3, 4\}$, and there are 8 vertices, $(2, 4), (4, 2), (2, 3), (3, 2), (1, 4), (4, 1), (1, 3)$ and $(3, 1)$ which must be in $S$. Again, a contradiction is reached as $|S| < 4$, and the result is proven.

\[\square\]
In light of the previous theorem, one might wonder if in general the connectivity of $C_k(K_{a_1,a_2,\ldots,a_t})$ is equal to its minimum degree. This is, however, not the case. Consider the graph $C_{t+1}(K_{a_1,1,1,\ldots,a_t=1})$, with $a_1 \geq 3$. This graph has minimum degree $a_1$, but connectivity at most 2, as any two $t$-colourings which differ only in the colour of the vertices in $V_1$ form a cut set. Therefore, there are colour graphs of complete multipartite graphs with arbitrarily large minimum degree, but with connectivity at most 2. What then, can we say about the connectivity of these colour graphs in general? The following theorem gives a simple lower bound on the connectivity of such graphs.

**Theorem 3.1.2.** For $K = K_{a_1,a_2,\ldots,a_t}$, the graph $C_k(K)$ has connectivity at least 2 whenever $k \geq t + 1$.

**Proof.** By Menger’s theorem, it is sufficient to show that between any two vertices in $C_k(K)$ there are two vertex-disjoint paths. To do this we will show that any two vertices lie on a common cycle.

Let $V_1, V_2, \ldots, V_t$ be the $t$-partition of $K$. Consider $K_t$, the complete graph on $t$ vertices, with vertex set $V(K_t) = \{v_1, v_2, \ldots, v_t\}$. By Theorem 2.2.3, $C_k(K_t)$ is Hamiltonian whenever $k \geq t + 1$. Let $N$ denote the number of vertices in $C_k(K_t)$. For the remainder of this proof, we interpret subscripts modulo $N$. Let $C = f_0, f_1, \ldots, f_{N-1}, f_0$ be a Hamilton cycle in $C_k(K_t)$. Let $F_i$ denote the vertex in $C_k(K)$ where for each $j \in \{1, 2, \ldots, t\}$ and each $u \in V_j$, $F_i(u) = f(v_j)$. That is, $F_i$ is the colouring of $K$ which colours the vertices of $V_j$ with the colour used by $f_i$ to colour $v_j$. Since $f_i$ and $f_{i+1}$ are adjacent colourings, $F_i$ and $F_{i+1}$ differ only in the colour of vertices in $V_j$, for some $j$. Let $P_i$ denote some path in $C_k(K)$ from $F_i$ to a neighbour of $F_{i+1}$ obtained by successively changing the colour of vertices in $V_j$ from $f_i(v_j)$ to $f_{i+1}(v_j)$. Then, $C' = P_0 P_1 \cdots P_{N-1} F_0$ is a cycle in $C_k(K)$ which contains every $t$ colouring of $K$.

We claim that for every vertex $x$ not contained in $V(C')$, there are at least two internally vertex-disjoint paths from $x$ to distinct vertices of $C'$. Since $x$ uses at least $t + 1$ colours, for some $j$, $V_j$ uses at least two distinct colours, $c_{x,1}$ and $c_{x,2}$, to colour its vertices. Let $P_{x,i}$, for $i = 1$ or $2$, be the path obtained by recolouring the vertices of $V_j$ which are not already coloured with $c_{x,i}$ to $c_{x,i}$ one by one, and then recolouring the vertices of each $V_h$ which uses more than one colour until $V_h$ is monocoloured. The paths $P_{x,1}$ and $P_{x,2}$ are internally vertex disjoint, as, aside from $x$, no vertex of $P_{x,1}$ colours $V_j$ the same as a vertex of $P_{x,2}$. Each path ends in a $t$-colouring, and the claim is proven.
It is now straightforward to see that any two distinct vertices $x, y \in V(C_k(K))$ lie on a common cycle. If both $x$ and $y$ lie on $C'$, this is trivial. If exactly one of $x$ or $y$ lies on $C'$, using our two internally vertex disjoint paths, a common cycle is again found immediately. If neither $x$ nor $y$ lie on $C'$, there are three possible cases.

Figure 3.1: $P_{x,1}$ and $P_{x,2}$ do not intersect $P_{y,1}$ or $P_{y,2}$.

Figure 3.2: One of $P_{x,1}$ or $P_{x,2}$ intersects $P_{y,1}$ or $P_{y,2}$.

Figure 3.3: Both $P_{x,1}$ and $P_{x,2}$ intersect one of $P_{y,1}$ or $P_{y,2}$.
Figures 3.1-3.3 show examples of these three cases, and how to find our cycle in each. Cycles can be found in each case using methods similar to those shown by our figures.

![Graph](image)

Figure 3.4: The graph $H_4$ with colouring $f$, a leaf of the connected colour graph $C_4(H_4)$.

### 3.2 2-Connectedness of colour graphs

In this section, we turn our attention to a problem which is only tangentially related to our main focus, but is still worth consideration. Although never published, Horak [17] conjectured that every colour graph which is connected must also be 2-connected. A theorem of Fleischner [13] states that the square of every 2-connected graph is Hamiltonian. For a graph $G$, a Hamilton cycle in the square of $C_k(G)$ corresponds to a cyclic list of the $k$-colourings of $G$, such that consecutive colourings in the list differ in the colour of at most two vertices. If Horak’s conjecture is true, the square of every connected colour graph is Hamiltonian. Indeed, this is the case for colour graphs of complete multipartite graphs. However, we will show that for each $k \geq 4$, there is at least one graph $G$ such that $C_k(G)$ is connected, but not 2-connected.

Let $H_4$ be the graph displayed in Figure 3.4. The colour of a vertex in Figure 3.4 is represented by its shape. For $i \geq 5$, let $H_i = H_{i-1} + \{u_i\}$, where $H_{i-1} + \{u_i\}$ is the graph obtained by adding a dominating vertex $u_i$ to $H_{i-1}$. Given an ordering $\pi =$
$x_1, x_2, \ldots, x_{i+4}$ of the vertices of $H_i$, we have $Col(H_i) \leq D_\pi + 1 = \max_{1 \leq j \leq i+4} d_{H_i}(x_i) + 1$. Let $\sigma$ be the ordering $x_1 = u_1, x_2 = u_2, \ldots, x_{i-4} = u_{i-4}, x_{i-3} = v_1$, $x_{i-2} = v_2, \ldots, x_{i+4} = v_8$. Then, $D_\sigma + 1 = i - 1 \geq Col(H_i)$. By Theorem 2.2.1, we know that $C_k(H_i)$ is connected whenever $k \geq Col(H_i) + 1$. Therefore, $C_i(H_i)$ is connected. Furthermore, the colouring $f$ in $C_4(H_4)$ shown in Figure 3.4 has only a single vertex which can change colour: the vertex $v_8$ may change from square to diamond. Let $f'$ denote the colouring obtained by recolouring $v_8$ to diamond. Then, $f'$ is a cut vertex in $C_4(H_4)$, and $C_4(H_4)$ is therefore not 2-connected. By extending $f$ to an $i$-colouring of $H_i$ by using the additional $i - 4$ colours to colour $u_1$ through $u_{i-4}$, we may use a similar argument to show that $H_i$ is connected, but not 2-connected. Therefore, we arrive at the following conclusion:

**Theorem 3.2.1.** For $k \geq 4$, there is a graph $G$ such that $C_k(G)$ is connected, but not 2-connected.

The question still remains whether or not Horak’s conjecture is true when $k = 3$. When $k = 1$, the conjecture holds vacuously. Let $Q_n$ denote the $n$–cube, the graph whose vertex set is the set of binary strings of length $n$, where two strings are adjacent if they differ in exactly one position. When $k = 2$, if $C_k(G)$ is connected, then $G$ can contain no edges, and $C_k(G) \cong Q_n$, where $n = |V(G)|$. It is well known that $Q_n$ is Hamiltonian whenever $n \geq 2$, and is therefore 2-connected.
Chapter 4

Hamilton Paths and Cycles in SDR Graphs

In this chapter, we will examine some properties of the following class of graphs. For a collection of sets $S = A_1, A_2, \ldots, A_t$, where $A_i = \{x_{i,1}, x_{i,2}, \ldots, x_{i,a_i}\}$, we define a graph $G_S$ corresponding to reconfigurations of SDRs of $S$. Let $V(G_S) = \{(v_1, v_2, \ldots, v_t) | v_i \in A_i$ and $v_i \neq v_j$ if $i \neq j\}$, and $E(G_S) = \{((v_1, v_2, \ldots, v_t),$ $(u_1, u_2, \ldots, u_t)) | \exists i$ such that $v_j = u_j \iff j \neq i\}$. In other words, if $u$ and $v$ are vertices of $G_S$, then $u$ and $v$ are SDRs of $S$, where the $i$th coordinate corresponds to the representative of $A_i$. We have $u \sim v \iff u$ and $v$ differ in exactly one coordinate. Our study of this class of graphs is motivated by their relation to colour graphs. For example, consider the complete graph $K_t$ with vertex set $\{v_1, v_2, \ldots, v_t\}$. Then, $G_S$ is isomorphic to the graph of vertex colourings of $K_t$ where the colour of $v_i$ is restricted to elements of $A_i$.

4.1 Preliminaries

Not all collections $S$ produce graphs $G_S$ which are relevant to our study of colour graphs. We define the set $\mathcal{S}_t$, which contains the $t$-collections of sets we will examine in this chapter.

For $t \geq 2$, let $\mathcal{S}_t$ denote the set of collections $S = A_1, A_2, \ldots, A_t$ for which the following properties hold:

- $A_i = \{x_1, x_2, \ldots, x_t, y_{i,1}^1, y_{i,2}^2, \ldots, y_{i,a_i}^t\}$,
- $a_1 \geq a_2 \geq \cdots \geq a_t \geq 1$. 
• $A_i \cap A_j = \{x_1, x_2, \ldots, x_l\}$ \forall i, j such that $i \neq j$,

• $|\bigcup_{i=1}^t A_i| \geq 2t$.

Let $S \in \mathcal{S}_t$. Then, $S$ is a collection of sets where an element $z$ of $\bigcup_{i=1}^t A_i$ is either in every set, or exactly one set. Specifically, each $y^i_j$ is distinct. In the Chapter 5, we will see that the colour graph $C_k(K_{b_1,b_2,\ldots,b_t})$, where $b_i \geq 2$ for every $i$, can be partitioned into some number of subgraphs, each of which is isomorphic to $G_S$, for some $S \in \mathcal{S}_t$. The lemmas in this chapter show in a variety of ways that we may always find a Hamilton path in $G_S$ which suits our needs. This is a very difficult task. In order to prove the results of the section, we must consider a property of $G_S$ analogous to Property $\mathcal{A}$ in colour graphs. In the context of an SDR graph $G_S$, we will say a Hamilton cycle $C$ in $G_S$ has Property $\mathcal{A}$ if for each $i \in \{1, 2, \ldots, l\}$, there exist consecutive vertices in $C$ which do not use $x_i$. We say $G_S$ is $\mathcal{A}$-Hamiltonian if it contains a Hamilton cycle with Property $\mathcal{A}$. For the remainder of this chapter, when discussing a collection of sets $S$, it is assumed $S \in \mathcal{S}_t$ unless otherwise stated.

In our examination of $S$, it is extremely useful to utilize the automorphisms of $G_S$.

Let $X = \bigcup_{i=1}^t A_i$. Let $\pi^x : X \rightarrow X$ be any bijection where for every $i$ and $j$, $\pi^x(y^i_j) = y^j_i$. In other words, $\pi^x$ is some function which permutes the $x_i$s. For $v = (v_1, v_2, \ldots, v_t) \in V(G_S)$, let $\pi^x(v) = (\pi^x(v_1), \pi^x(v_2), \ldots, \pi^x(v_t))$.

**Automorphism Property I**: $\pi^x$ is an automorphism of $G_S$.

Let $i \in \{1, 2, \ldots, t\}$, and let $\pi^{y^i} : X \rightarrow X$ be any bijection where for every $k$, $\pi^{y^i}(y^k_j) = y^k_j$ when $j \neq i$, and $\pi^{y^i}(x_j) = x_j$. Then, $\pi^{y^i}$ is a function which permutes the $y^i_j$s for some fixed $i$. As before, for $v = (v_1, v_2, \ldots, v_t) \in V(G_S)$, let $\pi^{y^i}(v) = (\pi^{y^i}(v_1), \pi^{y^i}(v_2), \ldots, \pi^{y^i}(v_t))$.

**Automorphism Property II**: $\pi^{y^i}$ is an automorphism of $G_S$.

Automorphism Properties I and II utilize the fact that if $a$ and $b$ are elements of the same sets in some collection of sets $S$, swapping the labels of $a$ and $b$ in any SDR of $S$ will give you an SDR of $S$. The automorphism of Automorphism Property I permutes the labels of elements which are in every set of a collection of sets $S$, while the automorphism of Automorphism Property II permutes the labels of elements
which are in exactly one set of such a collection. We now give a third automorphism of $G_S$.

Suppose $|A_i| = |A_j|$ for some $i < j$. We define the function $\phi_{ij} : V(G_S) \rightarrow V(G_S)$ by the following rule:

$$\phi_{ij}((v_1, v_2, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_{j-1}, v_j, v_{j+1}, \ldots, v_t)) = (v_1, v_2, \ldots, v_{i-1}, v'_j, v_{i+1}, \ldots, v_{j-1}, v'_i, v_{j+1}, \ldots, v_t)$$

$$v'_i = \begin{cases} v_i & \text{if } v_i \in \{x_1, x_2, \ldots, x_l\} \\ y'_k & \text{if } v_i = y'_k \end{cases}$$

$$v'_j = \begin{cases} v_j & \text{if } v_j \in \{x_1, x_2, \ldots, x_l\} \\ y'_k & \text{if } v_j = y'_k \end{cases}$$

The function $\phi_{ij}$ captures the notion of swapping the $i$th and $j$th coordinates of every vertex in $G_S$.

**Automorphism Property III:** $\phi_{ij}$ is an automorphism of $G_S$.

Let $X^t$ be the unique collection in $S_t$ which satisfies the additional properties $l = 1$, $a_1 = t$, and $a_i = 1$ for $i \in \{2, 3, \ldots, t\}$. Then, $X^t$ is the collection where $A_1 = \{x_1, y_1^1, y_2^1, \ldots, y_l^1\}$, and $A_i = \{x_1, y_i^1\}$. The collection $X^t$ is a special case, and must be separately addressed.

### 4.2 Hamilton path and cycle constructions

Our first result restricts attention to the case $t = 2$. This result will be used as a base case for induction to prove results for larger values of $t$. Consider the following example to demonstrate the use of Automorphism Property I and Automorphism Property II.

Suppose $S = A_1, A_2$, with $A_1 = \{x_1, x_2, x_3, y_1^1, y_2^1, y_3^1, y_4^1\}$ and $A_2 = \{x_1, x_2, x_3, y_2^2\}$. Say a Hamilton cycle in $G_S$ which contains the edge $e = (u, v)$, where $u = (y_1^1, x_1)$ and $v = (y_2^1, x_2)$ is required. Consider any edge of the form $e' = (u', v')$ with $u' = (y_1^1, x_j)$ and $v' = (y_2^1, x_k)$ for some $i \in \{1, 2, 3, 4\}$ and $j, k \in \{1, 2, 3\}, j \neq k$. By Automorphism Properties I and II, some automorphism of $G_S$ maps $e$ to $e'$. Then, a Hamilton cycle
in $G_S$ which contains the edge $e'$ can be mapped to a Hamilton cycle which contains the edge $e$ by some automorphism.

We refer to the orbit of an edge $e$ under the automorphisms of Automorphism Properties I and II as its edge type. Suppose we want to show that for every edge $e$ of $G_S$, there is a Hamilton cycle in $G_S$ which contains $e$. In light of the above, we only need to show that an edge from every edge type appears in a Hamilton cycle in $G_S$.

With this fact in mind, we are now ready to prove our result.

**Lemma 4.2.1.** Let $S \in S_2 - \{X^2\}$. For any edge $e$ in $G_S$, there is a Hamilton cycle with Property A which contains $e$.

**Proof.** To prove this result in a reasonably efficient manner, we appeal to Automorphism Properties I and II. Consider an edge $e$ of $G_S$. Instead of finding a Hamilton cycle which contains $e$, one may find a Hamilton cycle $C'$ which contains any edge $e'$ which shares an edge type with $e$. Using an automorphism which maps $e'$ to $e$, we may transform $C'$ into a Hamilton cycle which contains $e$.

We use $(x, y) \rightarrow (x', y)$ to denote the set of edges $((v_1, v_2), (v'_1, v_2))$, where $v_1, v_2 \in \{x_1, x_2, \ldots, x_l\}$, $v_1 \neq v'_1$ and $v_2 \in \{y_1^2, y_2^2, \ldots, y_{a_2}^2\}$. The other edge types listed define sets of edges in a similar manner. The following are the 12 possible edge types, which we label as 1 through 12.

1. $(x, x') \leftrightarrow (x'', x')$
2. $(x, x') \leftrightarrow (x, x'')$
3. $(x, x') \leftrightarrow (y, x')$
4. $(x, x') \leftrightarrow (x, y)$
5. $(y, y') \leftrightarrow (x, y')$
6. $(y, y') \leftrightarrow (y, x)$
7. $(y, y') \leftrightarrow (y'', y')$
8. $(y, y') \leftrightarrow (y, y'')$
9. $(x, y) \leftrightarrow (x', y)$
10. $(x, y) \leftrightarrow (x, y')$
11. \((y, x) \leftrightarrow (y', x)\)

12. \((y, x) \leftrightarrow (y, x')\)

The following is a list of possible values for \(l, a_1\) and \(a_2\), together with when each edge type will occur.

\(l = 0:\)

- \(a_1 \leq 2, a_2 = 1\) : No graphs.
- \(a_1 \geq 3, a_2 = 1\) : 7.
- \(a_1 \geq 2, a_2 \geq 2\) : 7, 8.

\(l = 1:\)

- \(a_1 = 1, a_2 = 1\) : No graphs.
- \(a_1 = 2, a_2 = 1\) : \(G_{X^2}\).
- \(a_1 \geq 3, a_2 = 1\) : 5, 6, 7, 11.
- \(a_1 \geq 2, a_2 \geq 2\) : 5, 6, 7, 8, 10, 11.

\(l = 2:\)

- \(a_1 = 1, a_2 = 1\) : 3, 4, 5, 6, 9, 12.
- \(a_1 \geq 2, a_2 = 1\) : 3, 4, 5, 6, 7, 9, 11, 12.
- \(a_1 \geq 2, a_2 \geq 2\) : 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

\(l \geq 3:\)

- \(a_1 = 1, a_2 = 1\) : 1, 2, 3, 4, 5, 6, 9, 12.
- \(a_1 \geq 2, a_2 = 1\) : 1, 2, 3, 4, 5, 6, 7, 9, 11, 12.
- \(a_1 \geq 2, a_2 \geq 2\) : 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.
Figures 4.1-4.10 show the smallest graphs of the possibilities listed above, and Hamilton cycles in those graphs which have the edge types we desire, as well as Property $\mathcal{A}$. The method of generalizing these cycles to larger graphs should be apparent, as additional rows and columns of vertices are easily included. Figures which contain two cycles in two copies of the graph are those whose edge types are not all covered by a single cycle. In each figure, the dashed edges represent the set of edges used to verify Property $\mathcal{A}$ on the cycle. Note that some edges are not shown in these figures. Vertices which share either a row or a column are adjacent. The SDR which a vertex represents is indicated by row and column. For example, if a vertex is in row $x_2$ and column $y_1^2$, it represents the SDR $(x_2, y_1^2)$. The rows are labelled with elements of $A_1$, and the columns are labelled with elements of $A_2$.

Figure 4.1: $l = 0, a_1 = 3, a_2 = 1$.

Figure 4.2: $l = 0, a_1 = 2, a_2 = 2$. 
Figure 4.3: $l = 1, a_1 = 3, a_2 = 1.$

Figure 4.4: $l = 1, a_1 = 2, a_2 = 2.$

Figure 4.5: $l = 2, a_1 = 1, a_2 = 1.$
Figure 4.6: \( l = 2, a_1 = 2, a_2 = 1 \).

Figure 4.7: \( l = 2, a_1 = 2, a_2 = 2 \).

Figure 4.8: \( l = 3, a_1 = 1, a_2 = 1 \).
Our work in the next chapter has more specific requirements for a few particular collections in $S_t$. We will now resolve one such case. Let $J_n$ denote the graph $K_n \square K_n - \{(1, 1), (2, 2), \ldots, (n-1, n-1)\}$. Celaya et al. [3] prove the following result regarding $J_n$.

**Lemma 4.2.2** (Celaya et al. [3]). For $n \geq 3$, $J_n$ has a Hamilton path from $(n, n)$ to every other vertex of $J_n - \{(n, n)\}$.

We use this result as the basis for induction to prove the following lemma.
Lemma 4.2.3. Given a collection of sets $S = A_1, A_2, \ldots, A_n$, where $n \geq 2$, and $A_i = \{x_1, x_2, \ldots, x_l, y_l\}$, with $l \geq n$, the graph $G_S$ has a Hamilton path from $y = (y_1, y_2, \ldots, y_n)$ to any other vertex in $V(G_S)$.

Proof. First, note that $S \in S_r$, as $|\bigcup_{i=1}^{n} A_i| = l + n \geq 2n$. Hence, we turn to the automorphisms of $G_S$ to simplify the problem. By Automorphism Properties I and III, for any vertex $x \in V(G_S)$, there exists an automorphism $\phi$ such that $\phi(y) = y$ and $\phi(x) = v$, where $v \in X_n = \{(x_1, y_2, y_3, \ldots, y_n), (x_1, x_2, y_3, \ldots, y_n), \ldots, (x_1, x_2, x_3, \ldots, x_n)\}$. Therefore, to prove this lemma it is sufficient to find a Hamilton path beginning at $v$ and ending at $y$ for each $v \in X_n$.

We prove this result by induction on $n$. Notice that when $n = 2$, $G_S \cong J_{l+1}$. Therefore, by Lemma 4.2.2, the result holds when $n = 2$. Now, suppose for some $i$, $i \geq 2$, the result holds. Let $n = i + 1$. For $j \in \{1, 2, \ldots, l\}$, let $H_j$ denote the subgraph of $G_S$ induced by vertices in which the $n$th coordinate is $x_j$. Let $H_0$ denote the subgraph of $G_S$ induced by vertices in which the $n$th coordinate is $y_3$. For $j \in \{1, 2, \ldots, l\}$, $H_j \cong G_{S_j}$, where $S_j = A_1 - \{x_j\}, A_2 - \{x_j\}, \ldots, A_{n-1} - \{x_j\}$, and by induction $G_{S_j}$ has a Hamilton path from $(y_1, y_2, \ldots, y_{n-1})$ to any other vertex in $V(G_{S_j})$. Similarly, $H_0 \cong G_{S_0}$, where $S_0 = A_1, A_2, \ldots, A_{n-1}$, and $G_{S_0}$ has a Hamilton path from $(y_1, y_2, \ldots, y_{n-1})$ to any other vertex in $V(G_{S_0})$.

For $0 \leq j \leq l$, let $u_j$ denote the vertex in $H_j$ which the $k$th coordinate is $y_k$, for each $1 \leq k \leq i$. Notice that $u_0 = y$, and that $u_j \sim u_k$ for each $j, k \in \{0, 1, 2, \ldots, l\}$, with $j \neq k$. For some $Y \subset \{1, 2, \ldots, l\}$, let $H_Y$ denote the subgraph of $G_S$ induced by vertices in which the $n$th coordinate is an element of $\{x_i | i \in Y\}$. We will now show that for any $j, k \in \{1, 2, \ldots, l\}$, where $j \neq k$, there exists a Hamilton path in $H_{\{j, k\}}$ beginning at $u_j$ and ending at $u_k$. First, let $P_j$ be a Hamilton path in $H_j$ from $u_j$ to $v_j$, where $v_j$ is any vertex for which no coordinate is $x_j$. Let $P_k$ be a Hamilton path in $H_k$ from $v_k$ to $u_k$, where $v_k$ is the vertex obtained by switching the $(i + 1)$st coordinate of $v_j$ from $x_j$ to $x_k$. Then, $P_j P_k$ is the desired path. By combining several such paths, it follows that for any even subset $I \subset \{1, 2, \ldots, l\}$, the graph $H_I$ has a Hamilton path from $u_j$ to $u_k$ for any $j, k \in I$, with $j \neq k$.

Let $v \in X_n$. We are now ready to construct a Hamilton path from $v$ to $y = u_0$. We consider cases, based on the parity of $l$, and the $n$th coordinate of $v$.

Case 1.1: $l$ odd, $n$th coordinate $x_n$.

Let $P_0$ be a Hamilton path in $H_0$ from $u_0$ to $v_0$, where $v_0$ is the vertex $(x_n, y_2, y_3, \ldots, y_n)$. Let $w_0$ be the vertex which follows $u_0$ in $P_0$. Let $P_0'$ denote the path from $w_0$ to $v_0$ obtained by removing $u_0$ from $P_0$. Since $w_0$ is adjacent to $u_0$, there is exactly one
CHAPTER 4. HAMILTON PATHS AND CYCLES IN SDR GRAPHS

26

Let $P_t$ be a Hamilton path in $H_t$ from $u_t$ to $v_t$, where $v_t$ is the vertex obtained by switching the $n$th coordinate of $w_0$ from $y_n$ to $x_t$. Let $r \in \{1, 2, \ldots, l\} - \{t, n\}$. Let $P_r$ denote a Hamilton path in $H_r$ from $v_r$ to $u_r$, where $v_r$ is the vertex obtained by switching the $n$th coordinate of $v_0$ from $y_n$ to $x_r$. Let $P_n$ be a Hamilton path in $H_n$ from $u_n$ to $v_n = v$. Now, since $l$ is odd, $I = \{1, 2, \ldots, l\} - \{n, r, t\}$ is even. Let $P_I$ be a Hamilton path in $H_I$ from $u_j$ to $u_k$ for some $j, k \in I$. Now, a Hamilton path in $G_S$ from $y = u_0$ to $v = v_n$ is $u_0P_I P_0' P_r P_I P_n$ (See Figure 4.11.)

Case 1.2: $l$ odd, $n$th coordinate $y_n$.

Let $P_0$ be a Hamilton path in $H_0$ from $u_0$ to $v_0 = v$. Define $w_0$, $P_0'$ and $P_t$ analogously to the previous case. Again, $I = \{1, 2, \ldots, l\} - \{t\}$ is even, so we may define $P_I$ in a similar fashion to the previous case as well. Then, $u_0P_I P_t P_0'$ is the desired Hamilton path. (See Figure 4.12.)

Case 2.1: $l$ even, $n$th coordinate $x_n$.

Define $P_0$ and $P_n$ as in Case 1.1. Let $t \in \{1, 2, \ldots, l\} - \{n\}$. Let $P_t$ be a Hamilton path in $H_t$ from $v_t$ to $u_t$, where $v_t$ is the vertex obtained by switching the $n$th coordinate of $v_0$ from $y_n$ to $x_t$. Then, $I = \{1, 2, \ldots, l\} - \{n, t\}$ is even, and we may define $P_I$ similar to the previous cases. Then, $P_0P_t P_I P_n$ is the desired Hamilton path. (See Figure 4.13.)

Case 2.2: $l$ even, $n$th coordinate $y_n$.

Define $P_0'$ and $P_t$ as in Case 1.2. Let $w_t$ be the vertex which follows $u_t$ in $P_t$, and let $P_t'$ be the path from $w_t$ to $v_t$ obtained by removing $u_t$ from $P_t$. Again, since $w_t$ is adjacent to $u_t$, there is exactly one $s \in \{1, 2, \ldots, l\}$ such that some coordinate of $w_t$ is $x_s$. Let $r \in \{1, 2, \ldots, l\} - \{s, t\}$. Let $P_r$ be a Hamilton path in $H_r$ from $u_r$ to $v_r$, where $v_r$ is the vertex obtained by switching the $n$th coordinate of $w_t$ from $x_t$ to $x_r$. Once again, $I = \{1, 2, \ldots, l\} - \{r, t\}$ is even, and we may define $P_I$ analogously to the previous cases. Then, $u_0u_t P_t P_I P_t' P_0'$ is the desired Hamilton path. (See Figure 4.14.)

The result follows by induction.\[\Box\]
We now address another special case, the collection of sets $X_t$. Recall that this is the collection where $A_1 = \{x_1, y_1^1, y_2^1, \ldots, y_t^1\}$ and $A_i = \{x_1, y_i^i\}$ for $i \in \{2, 3, \ldots, t\}$. 

Figure 4.11: $l$ odd, $n$th coordinate $x_i$.

Figure 4.12: $l$ odd, $n$th coordinate $y_i$.

Figure 4.13: $l$ even, $n$th coordinate $x_i$.

Figure 4.14: $l$ even, $n$th coordinate $y_i$. 

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Lemma 4.2.4. For each \( t \geq 3 \), the graph \( G_{X^i} \) contains a Hamilton path. Furthermore, any pair of vertices \( u = (u_1, u_2, \ldots, u_t) \) and \( v = (v_1, v_2, \ldots, v_t) \) of \( V(G_{X^i}) \) may be chosen as the endpoints of such a path as long as \( u_i = x_1 \) or \( v_i = x_1 \) for some \( i \). If, in addition, \( u \sim v \), then the path has an edge neither of whose endpoints use \( x_1 \).

Proof. Let \( H_i \) be the subgraph of \( G_{X^i} \) induced by vertices in which the \( i \)th coordinate is \( x_1 \). For \( i \in \{2, 3, \ldots, t\} \), \( H_i \) is isomorphic to \( K_t \), with the \( t \) vertices corresponding to the \( t \) possible choices for the first coordinate. The choices for the first coordinate are \( \{y_1^1, y_2^1, \ldots, y_t^1\} \), with each other coordinate being fixed. Since \( H_i \) is isomorphic to \( K_t \), it must contain a Hamilton path from any vertex to any other vertex. The graph \( H_1 \) is simply the single vertex \((x_1, y_1^1, y_2^1, \ldots, y_t^1)\).

Let \( H_0 \) be the subgraph of \( G_{X^i} \) induced by vertices in which no coordinate is \( x_1 \). Notice that \( \bigcup_{i=0}^{t} V(H_i) = V(G_{X^i}) \), and \( V(H_i) \cap V(H_j) = \emptyset \) whenever \( i \neq j \). \( H_0 \) is also isomorphic to \( K_t \). Each vertex in \( H_0 \) is adjacent to a vertex in \( H_i \), \( i \in \{1, 2, \ldots, t\} \), by switching the \( i \)th coordinate to \( x_1 \). As such, for every edge \((v_1, v_2)\) in \( H_0 \), there exists a path beginning at \( v_1 \) and ending at \( v_2 \) whose internal vertices are exactly the vertices of \( H_i \).

Let \( u \) and \( v \) denote the vertices we wish to be the endpoints of our Hamilton path. We now consider two cases, based on whether or not \( u \) and \( v \) are adjacent.

Case 1: \( u \sim v \).

Consider a Hamilton cycle in \( H_0 \) with edges \( e_1, e_2, \ldots, e_t \), and replace each of \( e_i \) with a path to \( H_i \) as described above. The result is a Hamilton cycle \( C \) of \( G_{X^i} \).

We must now confirm that without loss of generality this Hamilton cycle contains the edge \( e = (u, v) \). We know at least one of \( u \) or \( v \) uses \( x_1 \) on some coordinate; therefore, \( e \) is not an edge of \( H_0 \). For each \( i \in \{2, 3, \ldots, t\} \), the cycle we described contains some edge which switches the \( i \)th coordinate from \( x_1 \) to \( y_i^1 \), as well as an edge which fixes the \( i \)th coordinate at \( x_1 \), changing the first coordinate from \( y_i^1 \) to \( y_j^1 \) for some \( i \neq j \). For \( H_1 \), our cycle contains an edge which switches the first coordinate from \( x_1 \) to \( y_j^1 \) for some \( j \). These edges cover all possible edge types outside of edges within \( H_0 \). By Automorphism Property II, we can take our cycle \( C \) and permute the \( y_j^1 \)’s to get the edge we want.

Case 2: \( u \sim v \).

Suppose \( u \in V(H_1) \). Then \( u \) is adjacent to every vertex of \( H_0 \). Therefore, we must have \( v \in V(H_i) \) for some \( i \in \{2, 3, \ldots, t\} \). Starting from \( v \), it is simple to construct a path which first visits every vertex of \( H_i \), then visits every vertex in \( H_0 \), and finally moves to \( u \). The path we have constructed contains \( t - 1 \) edges of \( H_0 \). Then, since
$|\{2, 3, \ldots, i-1, i+1, \ldots, t\}| = t - 2$, for each $j \in \{2, 3, \ldots, i-1, i+1, \ldots, t\}$ we may assign an edge $(v_{j-1}, v_j)$ of our path to be replaced with a path from $v_{j-1}$ to $v_j$ whose internal vertices are the vertices of $H_j$, completing our Hamilton path from $u$ to $v$.

Suppose $u \in V(H_0)$. We again must have $v \in V(H_i)$ for some $i \in \{2, 3, \ldots, t\}$, as $u$ is adjacent to the lone vertex of $H_1$, and also to every other vertex in $H_0$. Since $t \geq 3$, there exists a Hamilton path in $H_i$ starting at $v$ and ending at a vertex $w$ not adjacent to $u$. From $w$, we can take a Hamilton path in $H_0$ ending at $u$. Again, this path uses $t - 1$ edges from $H_0$, so we may connect our remaining $t - 2$ subgraphs to form a Hamilton path as we have done previously.

Suppose $u \in V(H_i)$, for $i \in \{2, 3, \ldots, t\}$. The last remaining case to check is $v \in H_j$, for $j \in \{2, 3, \ldots, t\}$ and $j \neq i$. Start by taking any Hamilton path in $H_i$ starting at $u$. Move into $H_0$, and take a Hamilton path ending at any vertex not adjacent to $v$, which is always possible as $t \geq 3$. Now, move into $H_j$, and take a Hamilton path ending at $v$. The result is a path starting at $u$ and ending at $v$, which visits every vertex in each of $H_i$, $H_j$ and $H_0$. Additionally, this path uses $t - 1$ edges within $H_0$. As such, we can connect the remaining $t - 2$ subgraphs in the manner described above.

In each of these three cases, the Hamilton path described contains an edge in $H_0$, which is an edge that does not use $x_1$ on either of its end points, and we are done.

Our final goal for this chapter is to generalize Lemma 4.2.1 for larger values of $t$. In other words, we want to prove the following theorem.

**Theorem 4.2.5.** Let $S \in S_t - \{X^t\}$. For any edge $e$ in $G_S$, there is a Hamilton cycle with Property $A$ in $G_S$ which contains $e$.

The proof for this theorem is quite long and involved. We will present the proof as a series of lemmas. The general tactic for the proof is by induction on $t$, using Lemma 4.2.1 to verify the base case $t = 2$. For induction we assume that the result is true for any $S \in S_k$, when $2 \leq k \leq t - 1$.

Recall that for $S = A_1, A_2, \ldots, A_t \in S_t$, we define $A_i = \{x_1, x_2, \ldots, x_i, y_1^i, y_2^i, \ldots, y_{a_i}^i\}$, with $a_1 \geq a_2 \geq \cdots \geq a_t$. For $z \in A_t$, let $H_z$ denote the subgraph of $G_S$ induced by vertices in which the $t$-th coordinate is $z$. If $z$ is used on the $t$-th coordinate it cannot be used on any other coordinate, and so we have $H_z \cong G_{S'}$, where $S' = A_1 - \{z\}, A_2 - \{z\}, \ldots, A_t - \{z\} = A_1', A_2', \ldots, A_t'$. In order to use induction, we must verify that $S' \in S_{t-1}$, and we must address that possibility that $S' = X^{t-1}$.
It is not difficult to see that the only property of sets in $S_{t-1}$ which $S'$ might not satisfy is the requirement $\alpha = |\bigcup_{i=1}^{t-1} A'_i| \geq 2(t-1)$. Since $S \in S_t$, we know $|\bigcup_{i=1}^{t} A_i| \geq 2t$. Therefore, if $z \in \{y_1, y_2, \ldots, y_{l}^{t-1}\}$ then $A_i \setminus \{z\} = A_i$, and $\alpha \geq 2t - a_t$. If $a_t \leq 2$, we are done. Suppose $a_t \geq 3$. Since $a_1 \geq a_2 \geq \cdots \geq a_t$, in this case $|\bigcup_{i=1}^{t} A_i| \geq a_t t \geq a_t(t-1) \geq 2(t-1)$, and we are done. If, on the other hand, $z \in \{x_1, x_2, \ldots, x_l\}$, then $\alpha \geq 2t - a_t - 1$. If $a_t \leq 1$, we are done. Suppose $a_t \geq 2$. Here we have $|\bigcup_{i=1}^{t} A_i| \geq a_t + 1$, as the element $z$ must also be accounted for. Therefore, we have $\alpha \geq a_t + 1 - a_t - 1 = a_l(t-1) \geq 2(t-1)$, and we are done. Therefore, $S' \in S_{t-1}$.

Since $S' \in S_{t-1}$, we may assume by the induction hypothesis that for any edge $e$ in $G_{S'}$, there is a Hamilton cycle in $G_{S'}$ with Property $\mathcal{A}$ which contains $e$, provided $S'$ is not $X^{t-1}$. We resolve the case $S' = X^{t-1}$ separately.

Suppose $S' = X^{t-1}$. In this case, we have $a_t = 1$. If $z = y_1$, then we must have $A_1 = \{x_1, y_1, y_2, \ldots, y_{l}^{t-1}\}$, and $A_i = \{x_1, y_i\}$ for $i \in \{2, 3, \ldots, t\}$. However, this implies $|\bigcup_{i=1}^{t} A_i| = 2t - 1$, a contradiction. Therefore, we must have $l = 2$, and $z = x_1$ or $z = x_2$. In this case, $A_1 = \{x_1, x_2, y_1, y_2, \ldots, y_{l}^{t-1}\}$ and $A_i = \{x_1, x_2, y_i\}$. Let $Y^t \in S_t$ denote this collection of sets. We resolve this special case with the following lemma.

**Lemma 4.2.6.** Let $S \in S_k$, $S \neq X^k$, for some $k$, $2 \leq k \leq t - 1$. If, for any edge $e$ of $G_S$, there is a Hamilton cycle with Property $\mathcal{A}$ in $G_S$ which contains $e$ then, for any edge $e$ in $Y^t$, there is a Hamilton cycle with Property $\mathcal{A}$ in $G_{Y^t}$ which contains $e$.

**Proof.** The case $t = 3$ will be handled via diagrams.

When $t = 3$, consider the following ten edge types, with notation similar to the notation used in the proof of Lemma 4.2.1. Similarly to the proof of Lemma 4.2.1, we use Automorphism Properties I, II and III in order to restrict our search to Hamilton cycles that contain each of these edge types.

1. $(x, x', y) \rightarrow (y, x', y)$
2. $(x, x', y) \rightarrow (x, y, y')$
3. $(y, x, x') \rightarrow (y', x, x')$
4. $(y, x, x') \rightarrow (y, y', x')$
5. $(x, y, y') \rightarrow (y'', y, y')$
6. \((x, y, y') \rightarrow (x', y, y')\)
7. \((y, x, y') \rightarrow (y, y'', y')\)
8. \((y, x, y') \rightarrow (y'', x, y')\)
9. \((y, x, y') \rightarrow (y, x', y')\)
10. \((y, y', y'') \rightarrow (y''', y', y'')\)

For \(\alpha \in \{x_1, x_2, y_t^1\}\), let \(H_\alpha\) denote the subgraph of \(G_{Y^t}\) induced by the vertices in which the third coordinate is \(\alpha\). See Figure 4.15 and Figure 4.16 for Hamilton cycles \(C_1\) and \(C_2\) with Property A in \(G_{Y^3}\), which cover all ten possible edge types.

Now, suppose \(t \geq 4\). The graph \(G_{Y^t}\) may be partitioned into the three disjoint subgraphs \(H_{x_1}, H_{x_2},\) and \(H_{y_t^1}\). Since \(H_{x_1}\) and \(H_{x_2}\) are both isomorphic to \(G_{X^t}\), we may apply Lemma 4.2.4. The graph \(H_{y_t^1}\) is isomorphic to \(G_{S_{y_t^1}}\), where \(S_{y_t^1} = A_1, A_2, \ldots, A_{t-1}\). It can be easily checked that \(S_{y_t^1} \in S_{t-1}\). By assumption, we may find a Hamilton cycle in \(H_{y_t^1}\) which contains a prescribed edge, and has Property A. Let \(e = (u, v) = ((u_1, u_2, \ldots, u_{k+1}), (v_1, v_2, \ldots, v_{k+1}))\) be any edge in \(G_{Y^t}\). We will construct a Hamilton cycle \(C\) in \(G_{Y^t}\) which contains the edge \(e\), and then we will verify that \(C\) has consecutive vertices not containing \(x_i\), for \(i \in \{1, 2\}\). To do this, we will utilize the symmetries of \(G_{Y^t}\).

By Automorphism Properties I, II and III, as before, it suffices to find enough Hamilton cycles with Property A so that an edge of each edge type (listed below) appears in one of them. We can then apply the appropriate automorphism to find the Hamilton cycle we desire. Note that by Automorphism Property III, since \(a_2 = a_3 = \cdots = a_t = 1\), we only need to consider the edges in which the first or the second coordinate changes. The following are the edge categories of \(G_{Y^t}\), grouped by which coordinate changes.

**Case 1:** First coordinate changes:

a. from \(y_1^1\) to \(y_j^1\), with \(x_1\) and \(x_2\) not used.

b. from \(y_i^1\) to \(y_j^1\), with \(x_1\) used, \(x_2\) not used.

c. from \(y_1^1\) to \(y_j^1\), with \(x_1\) and \(x_2\) used.

d. from \(y_1^1\) to \(x_1\), with \(x_2\) not used.
Figure 4.15: Hamilton cycle $C_1$ in $G_S$.

Figure 4.16: Hamilton cycle $C_2$ in $G_S$. 
e. from $y^1_t$ to $x_1$, with $x_2$ used.

f. from $x_1$ to $x_2$.

**Case 2:** Second coordinate changes:

a. from $y^2_t$ to $x_1$, with $x_2$ not used.

b. from $y^2_t$ to $x_1$, with $x_2$ used on the first coordinate.

c. from $y^2_t$ to $x_1$, with $x_2$ used on the $i$th coordinate, $i \in \{3, 4, \ldots, t\}$.

d. from $x_1$ to $x_2$.

We will now describe a set of Hamilton cycles with Property $\mathcal{A}$ in $G_{Y'}$, which contain an edge of each of these ten edge categories. First, we will partition $H_{y^1_t}$ into three subgraphs. Let $H_{x_1,y^1_t}$, $H_{x_2,y^1_t}$, and $H_{y^{t-1}_t,y^1_t}$ be the subgraphs of $H_{y^1_t}$ induced by vertices with $(t-1)$th coordinate $x_1$, $x_2$, and $y^{t-1}_t$, respectively. We will denote these by $H_{x_1,y}$, $H_{x_2,y}$ and $H_{y,y}$. We have $t \geq 4$, and each of these subgraphs is isomorphic to the graph $G_{S'}$ for a collection of $t-2$ sets $S' \in S_{t-2}$. Therefore, by assumption, $H_{x_1,y}$, $H_{x_2,y}$ and $H_{y,y}$ all have Hamilton cycles with Property $\mathcal{A}$.

Consider a Hamilton cycle $C_{y,y}$ in $H_{y,y}$. Clearly, $C_{y,y}$ must contain a pair of consecutive vertices $u_1$ and $u_2$, where $u_1$ uses neither $x_1$ nor $x_2$ on any coordinate, and $u_2$ uses exactly one of $x_1$ or $x_2$ on its coordinates. Without loss of generality, assume some coordinate of $u_2$ is $x_1$. Let $P_{y,y}$ be the Hamilton path in $H_{y,y}$ from $u_2$ to $u_1$. Let $u'_1$ be the vertex obtained by changing the last coordinate of $u_1$ to $x_1$. Let $u'_2$ be the vertex obtained by changing the last coordinate of $u_2$ to $x_2$. Notice, $u'_1 \in H_{x_1}$ and $u'_2 \in H_{x_2}$. Now, by Lemma 4.2.4, let $P_{x_1}$ be a Hamilton path in $H_{x_1}$ starting at $u'_1$ and ending at some vertex $v'_1$, which uses $x_2$ in the $(t-1)$st coordinate and is not adjacent to $u'_1$. Let $P_{x_2}$ be a Hamilton path in $H_{x_2}$ starting at the vertex $w'_1$, which uses $x_1$ in the $(t-1)$st coordinate, and is equal to $v'_1$ on coordinates one through $t-2$, and ending at $u'_2$. Since $u'_2$ uses $x_1$ on some coordinate other than the $(t-1)$st, we have that $w'_1$ and $u'_2$ are not adjacent. In addition, by Lemma 4.2.4, $P_{x_1}$ and $P_{x_2}$ can be constructed to contain a single edge which does not use $x_2$ or $x_1$, respectively.

Let $v_1$ be the vertex obtained by changing the last coordinate of $v'_1$ to $y^1_t$, and let $w_1$ be the vertex obtained by changing the last coordinate of $w'_1$ to $y^1_t$. Notice $v_1 \in H_{x_2,y}$ and $w_1 \in H_{x_1,y}$, and that $v_1 \sim w_1$. Let $v_2$ be the vertex obtained by switching the first coordinate of $v_1$ to $y^1_t$ for some $i \in \{1, 2, \ldots, t-1\}$, and let $w_2$
be the vertex obtained by switching the first coordinate of $w_1$ to $y_1^t$ as well. Then, $v_2 \sim w_2$. Now, by induction, there is a Hamilton path $P_{x_1,y}$ in $H_{x_1,y}$ from $w_2$ to $w_1$, and there is a Hamilton path $P_{x_2,y}$ in $H_{x_2,y}$ from $v_1$ to $v_2$. Now, $P_{y,y}P_{x_1}P_{x_2,y}P_{x_1,y}P_{x_2}u_2$ is a Hamilton cycle $C_{Y^t}$ in $G_{Y^t}$. (See Figure 4.17.)

This construction does not depend on which Hamilton cycle we choose for $H_{y,y}$. By induction, there exists a Hamilton cycle $C_{y,y}$ in $H_{y,y}$ which uses any edge of $H_{y,y}$, and therefore the cycle we constructed may contain any such edge. Looking at the list of the ten edge types, this covers almost all of them. The only possible edges not covered are 1$c$ and 2$c$, and only when $t$ is exactly four (as the $(t-1)$st and $t$th coordinates are fixed at $y_1^{t-1}$ and $y_1^t$ respectively within $H_{y,y}$). However, both edge types 1$c$ and 2$c$ are contained in our Hamilton paths $P_{x_1}$ in $H_{x_1}$, and $P_{x_2}$ in $H_{x_2}$. As such, $C_{Y^t}$ may contain any edge in $G_{Y^t}$. We must now verify that $C_{Y^t}$ has Property $\mathcal{A}$. By Lemma 4.2.4, and since $u_1' \sim v_1'$ and $u_2' \sim w_1'$, $P_{x_1}$ and $P_{x_2}$ will contain an edge that fits our needs

![Figure 4.17: A Hamilton cycle in $G_{Y^t}$.

In light of this result, we now consider $S \in S_t$, $S \neq X^t$ and $S \neq Y^t$. As previously discussed, for $z \in A_t$, we may now assume that for any edge $e$ of $H_z$, there is a Hamilton cycle with Property $\mathcal{A}$ in $H_z$ which contains $e$. The problem of proving $G_S$ has the Hamilton cycle we want is still far from easy. We will split the remainder of
our proof into three parts, based on the value of \( l \). The first case considers only the collections \( S \) for which \( l = 0 \).

**Lemma 4.2.7.** Let \( S \in S_l - \{X^t, Y^t\} \), with \( l = 0 \) and \( t \geq 3 \). For any edge \( e \) in \( G_S \), there is a Hamilton cycle with Property \( A \) in \( G_S \) which contains \( e \).

*Proof.* In this case, \( G_S \cong K_{a_1} \square K_{a_2} \square \cdots \square K_{a_t} \). This graph has a Hamilton cycle whenever \( a_1 + a_2 + \cdots + a_t > t + 1 \). Since we have \( a_1 + a_2 + \cdots + a_t \geq 2t \), we have a Hamilton cycle. By Automorphism Property II, we may assume that this cycle contains the edge \( e \) we desire. When \( l = 0 \), any Hamilton cycle vacuously satisfies the requirements of Property \( A \). \( \square \)

**Lemma 4.2.8.** Let \( S \in S_l - \{X^t, Y^t\} \), with \( l = 1 \) and \( t \geq 3 \). For any edge \( e \) in \( G_S \), there is a Hamilton cycle with Property \( A \) in \( G_S \) which contains \( e \).

*Proof.* Let \( H_0 \) denote the subgraph of \( G_S \) induced by the vertices which do not use \( x_1 \) on any coordinate, and let \( H_i, i \in \{1, 2, \ldots, t\} \), denote the subgraph of \( G_S \) induced by the vertices which use \( x_1 \) on coordinate \( i \). Notice that \( H_0 \cong K_{a_1} \square K_{a_2} \square \cdots \square K_{a_t} \), and \( H_i \cong K_{a_i} \square K_{a_{i+1}} \square \cdots \square K_{a_{i+1+1}} \square \cdots \square K_{a_t} \). Recalling that \( a_1 \geq a_2 \geq \cdots \geq a_t \), we consider two cases based on the value \( a_2 \).

**Case 1:** \( a_2 = 1 \).

Here, we have \( A_1 = \{x_1, y_{1}^1, y_1^2, \ldots, y_{a_1}^1\} \), and \( A_i = \{x_1, y_i^1\} \) when \( i \in \{2, 3, \ldots, t\} \). We may assume \( a_1 \geq t + 1 \), as \( |\bigcup_{i=1}^t A_i| < 2t \) if \( a_1 < t \), and \( S = X^t \) if \( a_1 = t \). We can use the same technique as in Case 1 of the proof of Lemma 4.2.4 to construct a Hamilton cycle \( C_S \) in \( G_S \). However, in this case \( H_0 \cong K_{a_1} \), which contains \( a_1 \geq t + 1 \) vertices. As such, \( E(C_S) \) will contain some edge of \( H_0 \). This edge does not use \( x_1 \) on any coordinate, and by the construction our cycle may contain any edge not contained within \( H_0 \). By Automorphism Property II, we may assume \( C_S \) uses a particular edge contained within \( H_0 \). Therefore, there is a cycle that contains any edge we want, and will always have some edge contained in \( H_0 \). This gives consecutive vertices in \( C_S \) which do not use \( x_1 \) on any coordinate. Hence, Property \( A \) holds.

**Case 2:** \( a_2 \geq 2 \).

First, we show that \( H_i, i \in \{1, 2, \ldots, t\} \), contains a Hamilton path from \( u \) to \( v \) whenever \( u \sim v \). Recall that \( H_i \cong K_{a_1} \square K_{a_2} \square \cdots \square K_{a_{i-1}} \square K_{a_{i+1}} \square \cdots \square K_{a_t} \). In this case, we have \( a_1 + a_2 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_t \geq t \). If \( a_1 + a_2 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_t > t \), for any edge \( e \) in \( H_i \), \( H_i \) contains a Hamilton cycle which contains \( e \), for the reasons described in the proof of Lemma 4.2.7. If \( a_1 + a_2 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_t = t \), 

then $H_i \cong K_2$. In either case, it is clear that $H_i$ contains a Hamilton path from $u$ to $v$ whenever $u \sim v$.

Now, we also have $a_1 + a_2 + \cdots + a_t \geq 2t - 1$, and therefore for any edge $e$ with endpoints in $H_0$, there is a Hamilton cycle $C_0$ in $H_0$ which contains $e$, for reasons stated in the proof of Lemma 4.2.7. For some edge $e = (u, v)$ in $C_0$, there are distinct vertices $u_i$ and $v_i$ in $H_i$ such that $u \sim u_i$, $v \sim v_i$, and $u_i \sim v_i$ if and only if $u$ and $v$ do not differ on the $i$th coordinate. If that is the case for the edge $e$, we may replace $e$ with a Hamilton path in $H_i$ from $u_i$ to $v_i$. Therefore, if for each $i \in \{1, 2, \ldots, t\}$, there is an edge $e_i$ with endpoints that do not differ in the $i$th coordinate, and $e_i \neq e_j$ whenever $i \neq j$, we may construct a Hamilton cycle in $G_S$ by replacing $e_i$ with a Hamilton path in $H_i$ for each $i$. We now show that such a set of edges exists.

Let $s = \max\{i \mid a_i \geq 2\}$. In this case, we have $s \geq 2$. Notice that if $i \in \{s + 1, s + 2, \ldots, t\}$, then there are no edges in $H_0$ which change coordinate $i$, so any edge in $C_0$ may be chosen as $e_i$. For each $i \in \{1, 2, \ldots, s\}$, there are at least $a_i \geq 2$ edges in $C_0$ which change the $i$th coordinate. For $i \in \{1, 2, \ldots, s - 1\}$, let $e_i$ be any edge in $C_0$ which changes the $(i + 1)$st coordinate. Let $e_s$ be any edge in $C_0$ which changes the first coordinate. We will now use a counting argument to show that there are enough “unclaimed” edges of $C_0$ to assign to the remaining $t - s$ subgraphs $H_i$. Let $m$ denote the number of edges in $C_0$. There are then $m - s$ unclaimed edges of $C_0$. Noting that $m \geq a_1 + a_2 + \cdots + a_s$, consider the following:

\[
2t - 1 \leq a_1 + a_2 + \cdots + a_t \\
2t - 1 \leq a_1 + a_2 + \cdots + a_s + (t - s) \\
(t - 1) + s \leq a_1 + a_2 + \cdots + a_s
\]

Thus, we have $m - s \geq a_1 + a_2 + \cdots + a_s \geq (t - 1) + s - s = (t - 1)$. We have only $t - s \leq t - 2 = t - 2$ edges left to choose, and at least $(t - 1)$ edges from which to choose. Choose any set of $t - s$ of these edges to be $e_i$ for $i \in \{s + 1, s + 2, \ldots, t\}$. Notice that we have at least one edge $e_0$ in $C_0$ where $e_0 \neq e_i$ for each $i \in \{1, 2, \ldots, t\}$. Furthermore, we can choose $e_i$ for $i \in \{s + 1, s + 2, \ldots, t\}$ so that we can assume $e_0$ changes the $j$th coordinate, for some $j \in \{1, 2, \ldots, s\}$. Our Hamilton cycle $C_S$ of $G_S$ is constructed by replacing each $e_i$ in $C_0$ with a Hamilton path in $H_i$. The resulting cycle $C_S$ must contain at least one edge of $H_0$, which gives us consecutive vertices which do not use $x_1$ on any coordinate. We must now show that for any edge
Let $e = (u, v)$ in $G_S$, our construction can produce a Hamilton cycle which contains $e$. We will do so in three cases, based on the three possible edge types in $G_S$.

Suppose $u \in V(H_0), v \in V(H_i), i \in \{1, 2, \ldots, t\}$. In this case, the edge $e$ switches the $i$th coordinate from $y_j^i$ to $x_1$, for some $j \in \{1, 2, \ldots, a_i\}$. The Hamilton cycle in $G_S$ will always contain some edge $e'$ with the same edge type as $e$, and by Automorphism Property II, we can permute the $y_j^i$s so that $e'$ maps to $e$, and we are done.

Suppose $u, v \in V(H_0)$. Let $w$ denote the coordinate in which $u$ and $v$ differ. Note that we must have $w \in \{1, 2, \ldots, s\}$. As described above, we may construct our Hamilton cycle such that the edge $e_0$ changes coordinate $w$. Again, by Automorphism Property II, we can permute the $y_j^i$s so that the edge $e_0$ maps to our desired edge $e$, and we are done.

Suppose $u, v \in V(H_i)$ for some $i \in \{1, 2, \ldots, t\}$. Let $w$ denote the coordinate in which $u$ and $v$ differ. Again, we have $w \in \{1, 2, \ldots, s\}$. If $H_i$ has only two vertices, then we are done. So, assume $H_i$ has at least three vertices. If the Hamilton path in $H_i$ we used to construct our Hamilton cycle contains some edge $e'$ which changes coordinate $w$, we may permute the $y_j^i$s such that $e'$ maps to $e$, and we are done. The only way such an $e'$ will not exist is if the endpoints of the Hamilton path in $H_i$ differ in the $w$th coordinate. This occurs precisely when $e_i$ changes the $w$th coordinate. If $i \in \{s+1, s+2, \ldots, t\}$, we can alter the choice of $e_i$ to an edge which does not change the $w$th coordinate. If $i \in \{1, 2, \ldots, s\}$, we alter the choice of $e_i$ easily unless $s = 2$. However, if $s = 2$, each edge in $H_i$ will be an edge which changes the $w$th coordinate. Therefore we may construct a Hamilton cycle such that it contains an $e'$, which we can map to $e$ under some permutation of the $y_j^i$s by Automorphism Property II.

These three cases cover all the possible edge types in $G_S$, and therefore, for any $e \in E(G_S)$, we may construct a Hamilton cycle in $G_S$ which contains $e$, and has an edge $e_0$ which does not use $x_1$ on any coordinate, so we are done.

\[ \boxempty \]

Lemma 4.2.9. Let $S \in S_t - \{X^t, Y^t\}$, with $l \geq 2$ and $t \geq 3$. For any edge $e$ in $G_S$, there is a Hamilton cycle with Property A in $G_S$ which contains $e$.

Proof. Let $e = (v, v')$ be any edge in $G_S$. We will construct a Hamilton cycle in $G_S$ which contains the edge $e$ in two cases, based on whether or not $v$ and $v'$ differ on the $t$-th coordinate, or some other coordinate.

Case 1: $v = (v_1, v_2, \ldots, v_{l-1}, v_l), v' = (v_1, v_2, \ldots, v_{l-1}, v'_l), v_l \neq v'_l$.

Our first step is to construct a cycle which contains exactly the vertices of $H_{v_l}$ and $H_{v'_l}$, and which contains the edge $e$. By induction, $H_{v_l}$ and $H_{v'_l}$ will each contain
Hamilton cycles which use prescribed edges. Let $C_{vt}$ be a Hamilton cycle in $H_{vt}$ using the edge $(v, u) = ((v_1, v_2, \ldots, v_{t-1}, v_t), (v_1, v_2, \ldots, v_{t-1}, v'_t, v_{t+1}, \ldots, v_{2t-1}, v_t))$ and let $C'_{vt}$ be a Hamilton cycle in $H_{vt}'$ using the edge $(v', u') = ((v_1, v_2, \ldots, v_{t-1}, v'_t), (v_1, v_2, \ldots, v_{t-1}, v'_t, v_{t+1}, \ldots, v_{2t-1}, v_t))$.

We must justify that such a $v'_t$ exists. If $a_i > 1$ for some $i$, one of $y'_1$ or $y'_2$ is a valid candidate. Otherwise, if $v_j \neq y'_1$ for some $j$, $y'_1$ is a valid candidate. Now, if $a_i = 1$ and $v_i = y'_i$ for all $i$, we must have $l \geq 3$, as $|\bigcup_{i=1}^{t} A_i| \geq 2t$ and $t - 1 \geq 2$. Then, one of $x_1, x_2$ or $x_3$ is a valid candidate. Therefore, we may always find the desired $v'_t$, for some $2 \leq i \leq t - 1$.

Now, as $v \sim v'$ and $u \sim u'$, we may simply delete the edges $(v, u)$ and $(v', u')$ and add the edges $(v, v')$ and $(u, u')$ to create the desired cycle. Figure 4.18 displays this process. We will informally refer to this process as stitching the cycles $C_{vt}$ and $C'_{vt}$ together.

![Figure 4.18: Stitching cycles together.](image)

In several steps we will extend our cycle to include the vertices of $H_a$, for each $a \in A_t - \{v_t, v'_t\}$.

First, we will extend our cycle to include the vertices of $H_{x_i}$, for $x_i \in \{x_1, x_2, \ldots, x_t\} - \{v_t, v'_t\}$. By induction, the cycle $C_{vt}$ we have chosen for $H_{vt}$ will have consecutive vertices which do not use $x_i$ on any coordinate (it is possible that these vertices are the vertices $u$ and $v$, which is a problem that is addressed later). Let $(v_i, u_i) = ((w_1, w_2, \ldots, w_{t-1}, v_t), (w_1, w_2, \ldots, w_{t-1}, w'_i, w_{i+1}, \ldots, w_{2t-1}, v_t))$ be an edge of $H_{vt}$ which does not use $x_i$. By induction, we may choose a Hamilton cycle $C_{x_i}$ in $H_{x_i}$ which contains the edge $(v'_i, u'_i) = ((w_1, w_2, \ldots, w_{t-1}, x_i), (w_1, w_2, \ldots, w_{t-1}, w'_i, w_{i+1}, \ldots, w_{2t-1}, x_i))$. We may now stitch $C_{x_i}$ onto our existing cycle by deleting the edges $(v_i, u_i)$ and
(v'_t, u'_t), and adding the edges (v_t, v'_t) and (u_t, u'_t).

We must now extend our cycle to include the vertices of $H_{y'_t}$, for $y'_t \in \{y'_1, y'_2, \ldots, y'_{a_t}\} - \{v_t, v'_t\}$. In this case, any choice of an edge in $H_{y_t}$ which has not been stitched onto can be chosen to attach $H_{y'_t}$, in a similar manner. Note that the number of edges used by a Hamilton cycle in $H_{v_t}$ is large enough that running out of edges to attach cycles onto is not a concern.

After this process is complete, we are left with a Hamilton cycle $C_S$ in $G_S$, which contains the edge $e$. We must verify that $C_S$ has Property A. By induction, $C_{v'_t}$ must contain consecutive vertices which do not use $x_t$ on any coordinate, for each $x_t \in \{x_1, x_2, \ldots, x_l\} - \{v_t, v'_t\}$. If $u'$ and $v'$ are the only such vertices for some $x_t$, then note that the edge $e$ must not use $x_t$.

Suppose $v_t \in \{x_1, x_2, \ldots, x_l\}$. If $u'$ and $v'$ are not the only consecutive vertices in $C_{v'_t}$ which do not use $v_t$ on any coordinate, then we have consecutive vertices in $C_S$ which do not use $v_t$ on any coordinate. Suppose they are. Then, for each $x_t \in \{x_1, x_2, \ldots, x_l\} - \{v_t\}$, there must be consecutive vertices in $C_{v'_t}$ which do not use $x_t$ on any coordinate, and which are not both $u'$ and $v'$. Then, instead of stitching a Hamilton cycle $C_a$ in $H_a$, for some $a \in A_t - \{v_t, v'_t\}$, to an edge of $C_{v_t}$, we may stitch it to an edge $e' \neq (u', v')$ of $C_{v'_t}$. By assumption, $e'$ will use $v_t$, and by induction $C_a$ must contain an edge which does not use $v_t$, our final cycle $C_S$ will contain an edge which does not use $v_t$, and is contained within $H_a$. The edge in $C_{v_t}$ we used previously to stitch $C_a$ onto will be in our final cycle, and hence we will still have an edge which does not use $a$.

Suppose $v'_t \in \{x_1, x_2, \ldots, x_l\}$. We may assume that $v_t \in \{x_1, x_2, \ldots, x_l\}$, otherwise we may switch the roles of $v_t$ and $v'_t$ in the construction and use the previous argument. Since $a_t \geq 1$, we must have $y'_1 \in A_t - \{v_t, v'_t\}$. We may stitch a Hamilton cycle $C_{y'_1}$ in $H_{y'_1}$ onto some edge of $C_{v_t}$ which uses $v'_t$ on one of its end vertices, and our cycle $C_S$ is then guaranteed to have consecutive vertices contained within $H_{y'_1}$ which do not use $v'_t$ for similar reasons as those discussed in the previous case.

In the case that $A_t = \{x_1, x_2, y_{t,1}\}$, with $v_t = x_1$ and $v'_t = x_2$, there is only one subgraph, $H_{y'_1}$, to attach. Therefore, we cannot use both of the previous arguments, as each requires the attachment of such a subgraph. An alternate construction is given for this case. Here, we have $v = (y_{v_1}^1, y_{v_2}^2, \ldots, y_{v_{t-2}}^{t-2}, y_{v_{t-1}}^{t-1}, x_1)$ and $v' = (y_{v_1}^1, y_{v_2}^2, \ldots, y_{v_{t-2}}^{t-2}, y_{v_{t-1}}^{t-1}, x_2)$. Let:

- $u = (y_{v_1}^1, y_{v_2}^2, \ldots, y_{v_{t-2}}^{t-2}, x_2, x_1)$,
Figure 4.19: A Hamilton Cycle in $G_S$, when $A_t = \{x_1, x_2, y_1^t\}$.

- $u' = (y_{v_1}^1, y_{v_2}^2, \ldots, y_{v_{t-1}}^{t-2}, x_2, y_1^t)$,
- $w = (y_{v_1}^1, y_{v_2}^2, \ldots, y_{v_{t-1}}^{t-2}, x_1, x_2)$, and
- $w' = (y_{v_1}^1, y_{v_2}^2, \ldots, y_{v_{t-1}}^{t-2}, x_1, y_1^t)$.

Then, we have $u \sim v$, $u \sim u'$, $w \sim v'$, $w \sim w'$, and $u' \sim w'$. Figure 4.19 illustrates how to connect Hamilton cycles in $H_{x_1}$, $H_{x_2}$, and $H_{y_1^t}$ into a Hamilton cycle $C_S$ in $G_S$. Since some coordinate of $u'$ is $x_2$, and some coordinate of $w'$ is $x_1$, by induction, there must be consecutive vertices in $C_S$ which do not use $x_1$ on any coordinate, and consecutive vertices which do not use $x_2$ on any coordinate, each pair contained within $H_{y_1^t}$. Therefore, we can guarantee that our constructed cycle $C_S$ will have Property $A$.

We must now address the possibility that, for some $x_i$, the only consecutive vertices in $C_{v_t}$ which do not use $x_i$ on any coordinate are the vertices $v$ and $u$. Suppose this is the case. If the vertices $v'$ and $u'$ are not the only consecutive vertices in $C_{v_t'}$ which do not contain $x_i$, $C_{x_i}$ may simply be stitched onto $C_{v_t'}$ instead of $C_{v_t}$. Suppose that the vertices $v'$ and $u'$ are the only consecutive vertices in $C_{v_t'}$ which do not contain $x_i$.

We may resolve this case easily when $H_{x_i}$ is not the only subgraph whose vertices we need to include with our cycle in $H_{v_t}$ and $H_{v_t'}$; that is, when $A_t \neq \{v_t, v'_t, x_i\}$. In this case, take some $a \in A_t - \{v_t, v'_t, x_i\}$, and stitch a cycle $C_a$ in $H_a$ to $C_{v_t}$ in the manner described above. Now, by induction, the cycle $C_a$ will have two consecutive vertices which do not use $x_i$ on any coordinate. Furthermore, those vertices cannot be the vertices used to stitch $C_a$ to $C_{v_t}$, as then we must have another pair of
consecutive vertices in $C_{v_l}$ which do not use $x_i$ on any coordinate, which contradicts the assumption that $v$ and $u$ are the only consecutive vertices in $C_{v_l}$ which do not use $x_i$ on any coordinate. Therefore, by induction we may choose a cycle $C_{x_i}$ in $H_{x_i}$ which will allow us to stitch together $C_{x_i}$ and $C_u$. Stitching the remaining $C_{v_1}$s to our existing cycle can proceed in the manner described above. Note that the edge $e$ must not use $x_i$ because $v_t \neq x_i$ and $v$ does not use $x_i$. Hence, we may construct our cycle such that Property $A$ is satisfied.

Now, suppose that $A_t = \{v_t, v'_t, x_i\}$. Since we know $a_t \geq 1$, and in this case $l \geq 2$, we must have $A_t = \{x_1, x_2, y'_t\}$. Without loss of generality, we may assume $v_t = y'_t, v'_t = x_1$, and $x_i = x_2$. Recall that when we stitched $C_{v_t}$ and $C_{v'_t}$ together using the edges $(v, u)$ in $C_{v_t}$ and $(v', u')$ in $C_{v'_t}$, the choice of $u$ was made arbitrarily. We chose any $u$ that was adjacent to both $v$, and some vertex $u'$ in $H_{v'_t}$. Knowing that $v$ and $v'$ do not use $x_i$ on any coordinate, we may choose $u$ to be a vertex that does use $x_i$ on some coordinate. Simply take any coordinate of $v$ other than the last and change it to $x_i$ to get such a $u$. The choice for $u'$ follows. As before, we can choose $C_{v'_t}$ to contain our new $(v', u')$, and $C_{v_t}$ to contain our new $(v, u)$, and stitch them together with our prescribed edge $e = (v, v')$ and the edge $(u, u')$. By induction, both $C_{v_t}$ and $C_{v'_t}$ will have consecutive vertices which do not use $x_i$ on any coordinate, and neither of these vertices can be $u$ or $u'$ respectively, as each use $x_i$ on some coordinate. We can then choose a cycle $C_{x_i}$ in $H_{x_i}$ using the appropriate edge, and stitch it to $C_{v_t}$ as before. We know that $C_{v'_t}$ will have some edge that does not use $x_i$ on any coordinate, and this edge will occur in our final cycle $C_S$. Now, we must show that our cycle has consecutive vertices which do not use $v'_t = x_1$ on any coordinate. Let $(w, w')$ denote consecutive vertices in $C_{v_t}$ which do not use $x_2$ on any coordinate. If either $w$ or $w'$ use $x_1$ on some coordinate, then by stitching $C_{x_2}$ onto $(w, w')$ and appealing to induction, we know that our final cycle $C_S$ will have consecutive vertices which do not use $x_1$ on any coordinate and which lie in $H_{x_i}$. Suppose neither $w$ nor $w'$ use $x_1$ on any coordinate. Consider the vertex $w''$ following $w'$ on the cycle $C_{v_t}$. Since $w'$ uses neither $x_2$ nor $x_1$ on any coordinate, $w''$ can use at most one of $x_2$ and $x_1$ its coordinates. Thus, a careful selection of one of $(w, w')$ or $(w', w'')$ as the edge to attach $C_{x_2}$ onto will yield a Hamilton cycle $C_S$ in $G_S$ which contains consecutive vertices which do not use $x_1$ on any coordinate.

We now consider the case where $v$ and $v'$ differ in some coordinate other than the $t$-th coordinate.

**Case 2:** $v = (v_1, v_2, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_{i-1}, v_i)$, $v' = (v_1, v_2, \ldots, v_{i-1}, v'_i, v_{i+1},$
\[
\ldots, v_{t-1}, v_t, v_i \neq v_i', i \neq t.
\]

We will split this case into sub-cases, based on the \( t \)-th coordinate of our edge \( e \).

**Case 2.1:** \( v_t = x_i \), for some \( i \in \{1, 2, \ldots, l\} \).

By induction, let \( C_{x_i} \) be a Hamilton cycle in \( H_{x_i} \) which contains the edge \( e \). Let \( (u, w) \) be any other edge of \( C_{x_i} \). Let \( u' \) and \( w' \) be the vertices obtained by switching the \( t \)-th coordinate of \( u \) and \( w \) respectively to \( y_i' \). By induction, let \( C_{y_i'} \) be a Hamilton cycle with Property \( A \) in \( H_{y_i} \) which contains the edge \( (u', w') \). As before, we may remove the edges \( (u, w) \) from \( C_{x_i} \) and \( (u', w') \) from \( C_{y_i'} \), and add in the edges \( (u, w') \) and \( (w, u') \) to create a cycle which uses the vertices of \( H_{x_i} \) and \( H_{y_i} \). By induction, \( C_{y_i'} \) has a distinct edge \( e_j \) for every \( j \in \{1, 2, \ldots, l\} \) \( \setminus \{i\} \), such that neither of the endpoints of \( e_j \) use \( x_j \) on any coordinate. Let \( e_j' \) be the edge of \( H_{x_j} \) obtained by switching the \( t \)-th coordinate of the vertices of \( e_j \) to \( x_j \). By induction, let \( C_{x_j} \) be a Hamilton cycle with Property \( A \) in \( H_{x_j} \) which contains the edge \( e_j' \). Then, as before, we may stitch each such cycle onto our existing cycle with the appropriate pair of edge deletions and additions.

It is possible that for some \( j \), the only candidate for the edge \( e_j \) is the edge \( (u', w') \). If \( l = 2 \), we may simply choose the edge \( (u, w) \) to be an edge which uses \( x_j \), which prevents the possibility that \( e_j = (u', w') \). Otherwise, for each \( h \in \{1, 2, \ldots, l\} \) \( \setminus \{i, j\} \), \( e_h \) must use \( x_j \), and therefore the edge \( e_h' \) will too. Then, by induction, \( C_{x_h} \) will have an edge \( e_{j,h} \) which does not use \( x_j \), and that edge cannot be the edge \( e_h' \). We may then switch the \( t \)-th coordinates of the vertices of \( e_{j,h} \) to obtain the edge \( e_{j,h}' \) in \( H_{x_j} \), use induction to create a Hamilton cycle \( C_{x_j} \) in \( H_{x_j} \) which contains the edge \( e_{j,h}' \), and stitch \( C_{x_j} \) onto \( C_{x_h} \).

For any \( y \in \{y_{i2}', y_{i3}', \ldots, y_{il}'\} \), we may pick any unused edge \( e_y \) of \( C_{y_i'} \). Let \( e_y' \) be the edge obtained by switching the \( t \)-th coordinates of the vertices of \( e_y \) to \( y \). Let \( C_y \) be a Hamilton cycle in \( H_y \) which contains the edge \( e_y' \), and stitch \( C_y \) to \( C_{y_i'} \).

After this process is complete, we have our Hamilton cycle \( C_S \) in \( G_S \) which contains the edge \( e \). We must now show that this cycle has Property \( A \). By induction, we know \( C_{x_i} \) has an edge which does not use \( x_j \) for \( j \in \{1, 2, \ldots, l\} \) \( \setminus \{i\} \). If \( (u, w) \) is such an edge for some \( x_j \), then the edges \( (u, w') \) and \( (w, w') \) are both edges in \( C_S \) which do not use \( x_j \). It remains to show that \( x_i \) is not used on an edge of \( C_S \).

Consider any of the edges \( e_j \), for \( j \in \{1, 2, \ldots, l\} \) \( \setminus \{i\} \). If \( e_j \) uses \( x_i \), then \( e_j' \) uses \( x_i \) as well. Therefore, \( C_S \) will have an edge contained within \( H_{x_j} \) which does not use \( x_i \), and we are done. Suppose then that \( e_j \) does not use \( x_i \). Then, \( e_j' \) does not use \( x_i \) as well. Therefore, each of the edges used to stitch \( C_{x_j} \) to \( C_{y_i'} \) will not use \( x_i \), and we
are done.

Case 2.2: $v_i = y_i^1$, for some $i \in \{1, 2, \ldots, a_t\}$.

By induction, let $C_y^1$ be a Hamilton cycle in $H_y^1$, which uses the edge $e$. By induction, $C_y^1$ has an edge $e_j$ for every $j \in \{1, 2, \ldots, l\}$, such $e_j$ does not use $x_j$. Let $e_j'$ be the edge of $H_{x_j}$ obtained by switching the $t$-th coordinate of the vertices of $e_j$ to $x_j$. By induction, let $C_{x_j}$ be a Hamilton cycle in $H_{x_j}$ which contains the edge $e_j'$. We may stitch $C_{x_j}$ to $C_y^1$ on the edge $e_j$. This possibility that $e_j = e$ for some $j$ is resolved in a similar manner as the possibility that $e_j = (u', w')$ for some $j$ in the previous case.

For any $y \in \{y_1^1, y_2^1, \ldots, y_{at}^1\} - \{y_i^1\}$, we may pick any unused edge $e_y$ of $C_y^1$. Let $e_y'$ be the edge obtained by switching the $t$-th coordinates of the vertices of $e_y$ to $y$. Let $C_y$ be a Hamilton cycle in $H_y$ which contains the edge $e_y'$, and stitch $C_y$ to $C_y^1$ as before.

We are left with a Hamilton cycle $C_S$ in $G_S$, which contains our edge $e$. As always, we must confirm $C_S$ has Property $A$.

First, suppose that $e_j \neq e$, for all $j \in \{1, 2, \ldots, l\}$. If this is the case, then either $C_{x_2}$ or the set of edges used to stitch $C_{x_1}$ and $C_y^1$ together will contain edges which do not use $x_j$ for $j \in \{2, 3, \ldots, l\}$. Additionally, either $C_{x_2}$ or the set of edges used to stitch $C_{x_2}$ and $C_y^1$ together will contain an edge which does not use $x_1$.

Suppose that for some $j \in \{1, 2, \ldots, l\}$, $e$ is the only edge of $C_y^1$ which does not use $x_j$. In this case, we will have stitched $H_{x_j}$ onto $H_{x_h}$ for some $h \in \{1, 2, \ldots, l\} - \{j\}$. Then, by induction, $C_{x_j}$ will have consecutive vertices which do not use $x_m$ on any coordinate, for $m \in \{1, 2, \ldots, l\} - \{j, h\}$. Since by assumption $e$ does not use $x_j$, we just need to find consecutive vertices in $C_S$ which do not use $x_h$ on any coordinate. This can be done in the same manner as finding an edge which does not use $x_i$ from the previous case. However, in this case, there is the possibility that when $l = 2$ this cannot be done. If this is the case, without loss of generality we may say that $C_{x_1}$ is stitched to $C_y^1$, $C_{x_2}$ is stitched to $C_{x_1}$, and the edge $e$ is the only edge in $C_y^1$ which does not use $x_2$. Then, consider any vertex $u$ of $C_y^1$ which uses neither $x_1$ nor $x_2$ on any coordinate. Since $e$ is the only edge which does not use $x_2$, the neighbours $w$ and $w'$ of $u$ in $C_y^1$ will not use $x_2$. Therefore, neither $(u, w)$ nor $(u, w')$ will use $x_1$ on any coordinate. Since we only use one edge of $C_y^1$ to stitch $C_{x_1}$ onto, our Hamilton cycle $C_S$ will use at least one of $(u, w)$ and $(u, w')$, and we are done. 

This completes the proof of Theorem 4.2.5.
Chapter 5

Hamiltonicity of $C_k(K)$

Recall that we denote a complete multipartite graph $K_{a_1,a_2,...,a_t}$ by $K$. In this chapter, we address our main problem: determining whether or not $C_k(K)$ is Hamiltonian for various values of $k$. In the first section, we present two “construction” theorems, which given a graph $G$ and subgraph $H$ of $G$, under particular circumstances allow us to construct a Hamilton cycle in the colour graph of $G$ from a Hamilton cycle in the colour graph of $H$. We then use these theorems to give our main result, an upper bound on $k_0(K)$, the Gray code number of a complete multipartite graph $K$. We give a result on the lower bound on $k_0(K)$ with respect to the number of parts of $K$ with size two, and we fully characterize the graphs $K$ for which $C_{t+1}(K)$ is Hamiltonian.

5.1 Construction Theorems

The first of our construction theorems examines the symmetries between the colour graph $C_k(G)$, and the colour graph $C_{k+1}(G + \{v\})$, where $G + \{v\}$ is the graph obtained by adding a dominating vertex to $G$. This is of particular use for the purpose of constructing Hamilton cycles in complete multipartite graphs, as $K_{a_1,a_2,...,a_t} + \{v\} \cong K_{a_1,a_2,...,a_t,1}$.

**Theorem 5.1.1.** For any graph $G$, if $C_k(G)$ is $A$-Hamiltonian and $k \geq \chi(G) + 2$, then $C_{k+1}(G + \{v\})$ is $A$-Hamiltonian.

**Proof.** Let $H_i$ denote the subgraph of $C_{k+1}(G + \{v\})$ induced by the vertices which colour $v$ with $i$. Notice that each $H_i \cong C_k(G)$, and is therefore $A$-Hamiltonian by our hypothesis. Let $C_1 = f_0^1, f_1^1, \ldots, f_{N-1}^1, f_0^1$ be a Hamilton cycle in $H_1$ with Property $A$. For $i \in \{2,3,\ldots,k+1\}$, let $s_i$ be an integer such that the colour $i$ is not used
in both of $f_{s_i}^1$ and $f_{s_{i+1}}^1$. By Property $A$, such an $s_i$ must exist, and we may assume $s_i = s_j \iff i = j$. We may also assume $s_i < s_j$ whenever $i < j$, by relabeling our colours if necessary.

For $i \in \{2, 3, \ldots, k + 1\}$, let $C_i = \pi_i(C_1) = \pi_i(f_0^1), \pi_i(f_1^1), \ldots, \pi_i(f_{N-1}^1), \pi_i(f_0^1) = f_0^i, f_1^i, \ldots, f_{N-1}^i, f_0^i$, where $\pi_i = (1 \ i)$. It is not difficult to see that if $f_j^i \sim f_k^i$, then $f_j^i \sim f_k^i$. Therefore, $C_i$ is a Hamilton cycle in $H_i$. Furthermore, $f_{s_i}^i \sim f_{s_{i+1}}^i$ as $f_{s_i}^i$ colours no vertex with $i$, and therefore the two colourings differ only in the colour of $v$. Similarly, $f_{s_{i+1}}^i \sim f_{s_i}^i$.

Let $P_i = f_{s_i}^i, f_{s_{i+1}}^i, \ldots, f_{s_i}^i$. $P_i$ is a Hamilton path in $H_i$ from $f_{s_i}^i$ to $f_{s_{i+1}}^i$. For $i \in \{2, 3, \ldots, k\}$, let $P'_i = f_{s_i}^1, f_{s_{i+1}}^1, \ldots, f_{s_i}^1$, and let $P'_{k+1} = f_{s_k}^1, f_{s_{k+1}}^1, \ldots, f_{s_2}^1$. Then, the following is a Hamilton cycle in $C_{k+1}(G \cup \{v\}): C = P_2 P'_2 P_3 P'_3 \cdots P_{k+1} P'_{k+1} f_{s_2}^1$.

To complete the proof, we must show that $C$ has Property $A$. We know that $H_1 \cong C_k(G)$, and therefore $C_1$ contains consecutive vertices which do not use the colour $i$, for $i \in \{2, 3, \ldots, k + 1\}$. Therefore, since $C_i = \pi_i(C_1)$, $C_i$ must contain consecutive vertices which do not use the colour $j$, for $j \in \{1, 2, \ldots, j - 1, j, j + 1, \ldots, k + 1\}$. However, $P_i$ is $C_i$ with the edge $e_i = (f_{s_i}^i, f_{s_{i+1}}^i)$ removed. By construction, $e_i$ does not use colour 1, and therefore $P_i$ may not contain consecutive vertices which do not use 1. At this point, we know that $P_i \cup P_j$, with $j \neq i$ and $i, j \in \{2, 3, \ldots, k + 1\}$, must contain consecutive vertices which do not use colour $h$, for $h \in \{2, 3, \ldots, k + 1\}$. Therefore, all that remains to verify Property $A$ is to find consecutive vertices on $C$ which do not use colour 1.

To find such vertices, we appeal to our assumption that $k \geq \chi(G) + 2$. For some $i \in \{2, 3, \ldots, k + 1\}$, consider integers $p$ and $q$ such that $f_p^1$ and $f_q^1$ use exactly $\chi(G)$ colours, and do not use the colour $i$. Let $c_p$ be another colour not used by $f_p^1$, and let $c_q$ be another colour not used by $f_q^1$. For some $c \in \{i, c_p, c_q\}$, there exist $s, t \in \mathbb{Z}$, $s \neq t$, such that $f_s^i, f_{s+1}^i, f_t^i, f_{t+1}^i$ all do not use the colour $c$. Therefore, as $H_c = \pi_c(H_1)$, $f_s^c, f_{s+1}^c, f_t^c, f_{t+1}^c$ all do not use the colour 1, and so $C_c$ has at least two edges which do not use colour 1. Therefore, $P_c$ must contain at least one edge which does not use colour 1. Therefore, $C$ has Property $A$, and we have shown $C_{k+1}(G \cup \{v\})$ is $A$-Hamiltonian, as desired.

**Corollary 5.1.2.** If $C_k(K_{a_1, a_2, \ldots, a_t})$ is $A$-Hamiltonian, and $k \geq t + 2$, then $C_{k+1}(K_{a_1, a_2, \ldots, a_t, 1})$ is $A$-Hamiltonian.

The second of our two construction theorems is the one which necessitated our work in Chapter 4. Unlike our first construction theorem, this theorem is particular
to complete multipartite graphs. We show that, given a sufficiently large number of colours, \( k \), adding a vertex to each part of a complete multipartite graph whose \( k \)-colour graph is \( \mathcal{A} \)-Hamiltonian will result in a graph whose \( k \)-colour graph is also \( \mathcal{A} \)-Hamiltonian. The result is a generalization of work done by Celaya et al. [3] on colour graphs of complete bipartite graphs.

**Lemma 5.1.3.** If \( C_k(K_{a_1,a_2,\ldots,a_t}) \) is \( \mathcal{A} \)-Hamiltonian and \( k \geq 2t \), \( C_k(K_{a_1+1,a_2+1,\ldots,a_t+1}) \) is \( \mathcal{A} \)-Hamiltonian.

**Proof.** Let \( K' = K_{a_1,a_2,\ldots,a_t} \) and \( K = K_{a_1+1,a_2+1,\ldots,a_t+1} \). Let \( \{V_1,V_2,\ldots,V_t\} \) be the \( t \)-partition of the vertices of \( K' \), and let \( v_1,v_2,\ldots,v_t \) be vertices such that \( \{V_1\cup\{v_1\},V_2\cup\{v_2\},\ldots,V_t\cup\{v_t\}\} \) is the \( t \)-partition of \( K \). Let \( f_0,f_1,\ldots,f_{N-1},f_0 \) be a Hamilton cycle with Property \( \mathcal{A} \) in \( C_k(K') \). For \( j \in \{0,1,\ldots,N-1\} \), let \( F_j \) denote the set of colourings in \( C_k(K) \) which agree with \( f_j \) on the colours of \( V(K') \). Let \( A_{j,i} \) denote the set of colours that could be assigned to \( v_i \) to extend \( f_j \) to a colouring of \( K \). Since the \( v_i \)'s induce a clique, each must be assigned a different colour. Therefore, colourings in \( F_j \) correspond to SDRs of the collection \( S_j = A_{j,1},A_{j,2},\ldots,A_{j,t} \). Additionally, two colourings in \( F_j \) are adjacent if and only if their corresponding SDRs are adjacent in the SDR graph of \( S_j \). Therefore, the subgraph induced by \( F_j \) is isomorphic to the SDR graph of \( S_j \). We will now show that \( S_j \) must be isomorphic to one of the SDR graphs we examined in Chapter 4.

Consider some colour \( c \in \{1,2,\ldots,k\} \). If \( c \) is used in \( f_j \), it must be used on exactly one part, say \( V_n \). Then, \( c \in A_{j,i} \) if and only if \( i = n \). If \( c \) is not used in \( f_j \), then \( c \in A_{j,i} \) for \( i \in \{1,2,\ldots,t\} \). In other words \( c \) is either available to exactly one of the \( v_i \)'s, or it is available to every \( v_i \). Let \( X_j = \{x_1,x_2,\ldots,x_l\} \) be the set of colours such that \( x_n \in A_{j,i} \) for \( 1 \leq n \leq l_j \) and \( 1 \leq i \leq t \). Let \( Y_{j,i} = \{y_{i,1},y_{i,2},\ldots,y_{i,b_i}\} \) for \( i \in \{1,2,\ldots,t\} \) be the set of colours such that, for \( 1 \leq n \leq b_{j,i} \), \( y_{i,n} \in A_{j,m} \) if and only if \( m = i \). Then, \( A_{j,i} = X_j \cup Y_{j,i} \).

Clearly, we must have \( b_{j,i} \geq 1 \), for each \( i \), as at least one colour must be used in \( f_j \) to colour the vertices of \( V_i \). Additionally, we have \( A_{j,h} \cap A_{j,i} = X_j \) for \( h,i \in \{1,2,\ldots,t\} \). Since \( k \geq 2t \), we must have \( |\bigcup_{i=1}^t A_{j,i}| \geq 2t \). Since \( t \geq 2 \), we have all the conditions we need to guarantee \( S_j \in S_t \), and we may therefore apply either Lemma 4.2.4 or Theorem 4.2.5.

For any \( j \in \{1,2,\ldots,t\} \), we call a vertex \( w \in F_j \) a sink if it is adjacent to a vertex in \( F_{j+1} \). If a vertex \( w \in F_j \) is not a sink, this means that for some \( i \), the colour used by \( w \) to colour \( v_i \) is not available to \( v_i \) to extend \( f_{j+1} \). That is, \( w(v_i) \in A_{j,i} \), but
CHAPTER 5. HAMILTONICITY OF \(C_K(K)\)

Let \(w(v_i) \notin A_{j+1,i}\). Since \(f_j\) and \(f_{j+1}\) differ in the colour of only one vertex, if \(w\) is not a sink, \(f_j\) colours no vertex with \(w(v_i)\), and \(f_{j+1}\) uses the colour \(w(v_i)\) on some vertex in \(V_1 \cup V_2 \cup \ldots \cup V_{i-1} \cup V_{i+1} \cup \ldots \cup V_t\).

We will find a Hamilton cycle in \(C_k(K)\) using a similar idea as in the proof of Lemma 2.3.1 (See [9]). For \(i \in \{0, 1, \ldots, N - 1\}\), we will define vertices \(t_i\) and \(s_i\) such that there exists a Hamilton path in \(F\) from \(s_i\) to \(t_i\), and \(t_i \sim s_{i+1}\). Without loss of generality, we may assume \(f_0\) is the colouring which assigns each vertex in \(V_j\) colour \(j\), for \(1 \leq j \leq t\). Let \(t_0 \in F_0\) be the vertex such that \(t_0(v_j) = j\). Since \(f_0\) uses colour \(j\) on \(V_j\) for each \(j \in \{1, 2, \ldots, t\}\), \(A_{1,j}\) must contain the colour \(j\). Therefore, \(t_0\) is a sink. We define \(t_j\) and \(s_j\) for \(1 \leq j \leq N - 1\) inductively. Suppose \(t_0, s_1, t_1, s_2, t_2, \ldots, s_{j-1}, t_{j-1}\) have been defined. Let \(s_j\) be the neighbour of \(t_{j-1}\) in \(F_j\).

If \(S_j \in \mathcal{S} - \{X^i\}\), then by Theorem 4.2.5 we know that \(F_j\) contains a Hamilton cycle which contains some prescribed edge in \(F_j\). As a result, \(F_j\) contains a Hamilton path from \(s_j\) to any vertex adjacent to \(s_j\). We now show that some neighbour of \(s_j\) in \(F_j\) must be a sink. If a neighbour of \(s_j\) is not a sink, then there exists a colour \(c\) which is not used by \(f_j\) but is used by \(f_{j+1}\). Say \(c\) is used by \(f_{j+1}\) to colour a vertex in \(V_i\). Then, any vertex \(w \in F_j\) which either does not use \(c\), or uses \(c\) to colour \(v_i\) is a sink. Since \(|A_{j,h}| \geq 2\) for each \(h \in \{1, 2, \ldots, t\}\), clearly some neighbour of \(s_j\) satisfies one of those conditions, and is therefore a sink. Let \(t_j\) be such a sink. By Theorem 4.2.5, there is a Hamilton path in \(F_j\) from \(s_j\) to \(t_j\).

If \(S_j = X^i\), then \(A_{j,i} = \{x_1, y_1, 1, y_1, \ldots, y_i, 1\}\) for some \(i \in \{1, 2, \ldots, t\}\), and \(A_{j,h} = \{x_1, y_{h, 1}\}\) for all \(h \in \{1, 2, \ldots, t\} - \{i\}\). In this case, we know by Lemma 4.2.4 that \(F_j\) contains a Hamilton path between any two vertices, so long as at least one of them uses the colour \(x_1\). Since \(x_1\) is the only colour not used by \(f_j\), there exists \(\alpha \in \{1, 2, \ldots, t\}\) such that every vertex in \(F_j\) which uses \(x_1\) to colour \(v_{\alpha}\) is a sink. Let \(t_j\) be any such vertex, with \(t_j \neq s_j\). Since, \(t_j\) uses \(x_1\), by Lemma 4.2.4 there is a Hamilton path in \(F_j\) from \(s_j\) to \(t_j\).

Now, by our choice of \(f_0\), the vertex \(f_0\) cannot use any colour which \(f_{N-1}\) does not use. Therefore, every vertex of \(F_{N-1}\) is a sink. As such, we have enough freedom to ensure that our choice of \(t_{N-1}\) is not adjacent to \(t_0\). Finally, let \(s_0\) be the neighbour of \(t_{N-1}\) in \(F_0\). We have \(S_0 = A_0, 1, A_0, 2, \ldots, A_0, t\), with \(A_0, i = \{x_1, x_2, \ldots, x_{l}, y_{i, 1}\}\) for each \(i \in \{1, 2, \ldots, t\}\). Then, by Lemma 4.2.3, there is a Hamilton path in \(F_0\) from \(t_0\) to any other vertex in \(F_0\). In particular, there is a Hamilton path in \(F_0\) from \(t_0\) to \(s_0\).

Now that we have defined \(s_i\) and \(t_i\) for each \(i \in \{1, 2, \ldots, N - 1\}\), we may describe a Hamilton cycle in \(C_k(K)\).
For $i \in \{1, 2, \ldots, N - 1\}$, let $P_i$ be a Hamilton path in $F_i$ from $s_i$ to $t_i$. Then, $P_0 P_1 \cdots P_{N-1} s_0$ is a Hamilton cycle in $C_k(K)$. That this cycle has Property $A$ follows from Theorem 4.2.5, and the fact that $f_0, f_1, \ldots, f_{N-1}, f_0$ has Property $A$. \hfill \Box

\section{Upper and lower bounds for $k_0(K)$}

In this section we give our main result, which uses our constructions theorems Theorem 5.1.1 and Theorem 5.1.3 to give an upper bound on $k_0(K)$. Additionally, we will show that there exist $C_k(K)$ which are not Hamiltonian for $k$ as large as $t + \lceil \frac{t}{2} \rceil$.

The following simple lemma, which will be presented without proof, illustrates the construction used to prove our upper bound theorem.

**Lemma 5.2.1.** Every non-increasing sequence of natural numbers $a_1, a_2, \ldots, a_t$ can be reduced to a sequence $b, 1$ using the following two operations:

\begin{align*}
O_1(b_1, b_2, \ldots, b_s) &= b_1 - 1, b_2 - 1, \ldots, b_s - 1, \\
O_2(b_1, b_2, \ldots, b_{s-1}, 1) &= b_1, b_2, \ldots, b_{s-1}.
\end{align*}

For example, consider the sequence $s_0 = 7, 5, 4, 2, 1, 1$. The series of moves which reduces $s_0$ to the length two sequence $3, 1$ is as follows:

- $s_1 = O_2(s_0) = 7, 5, 4, 2, 1$
- $s_2 = O_2(s_1) = 7, 5, 4, 2$
- $s_3 = O_1(s_2) = 6, 4, 3, 1$
- $s_4 = O_2(s_3) = 6, 4, 3$
- $s_5 = O_1(s_4) = 5, 3, 2$
- $s_6 = O_1(s_5) = 4, 2, 1$
- $s_7 = O_2(s_6) = 4, 2$
- $s_8 = O_2(s_7) = 3, 1$

We are now ready to prove the main result.

**Theorem 5.2.2.** If $k \geq 2t$, $C_k(K_{a_1, a_2, \ldots, a_t})$ is $A$-Hamiltonian.
**Proof.** By Lemma 5.2.1, there exists a sequence $G_0, G_1, \ldots, G_n$ of complete multipartite graphs such that:

- $G_0 = K_{b,1}$ for some $b$,
- $G_n = K_{a_1,a_2,\ldots,a_t}$,
- if $G_i = K_{b_1,b_2,\ldots,b_{s-1},b_s}$, then either $b_s > 1$ and $G_{i-1} = K_{b_1-1,b_2-1,\ldots,b_{s-1}1,b_s-1}$, or $b_s = 1$ and $G_{i-1} = K_{b_1,b_2,\ldots,b_{s-1}}$, for $1 \leq i \leq n$.

Since $G_0 = K_{b,1}$ is a star, we have that $Col(G_0) = 2$. Therefore, by Theorem 2.3.4, $C_l(G_0)$ is $A$-Hamiltonian whenever $l \geq 4$. Since $k \geq 2t$, we must have $k - (t - 2) \geq t + 2 \geq 4$, and so $C_{k-(t-2)}(G_0)$ is $A$-Hamiltonian. Suppose that for some $G_j$ that $G_j$ is $s$-partite, and $C_{k-(t-s)}(G_j)$ is $A$-Hamiltonian. Since $k \geq 2t$, we have $k - (t - s) \geq t + s \geq 2s$. If $G_{j+1}$ is obtained from $G_j$ by adding a vertex to each part of $G_j$, then by Lemma 5.1.3, $C_{k-(t-s)}(G_{j+1})$ is $A$-Hamiltonian. If $G_{j+1}$ is obtained from $G_j$ by adding a single dominating vertex, then by Theorem 5.1.1, $C_{k-(t-(s+1))}(G_{j+1})$ is $A$-Hamiltonian. By induction, for each $j \in \{0, 1, \ldots, n\}$, if $G_j$ is $s$-partite, then $C_{k-(t-s)}(G_j)$ is $A$-Hamiltonian. In particular, $C_{k-(t-t)}(G_n) = C_k(K_{a_1,a_2,\ldots,a_t})$ is $A$-Hamiltonian.

**Corollary 5.2.3.** $k_0(K_{a_1,a_2,\ldots,a_t}) \leq 2t$.

An improvement to this upper bound can be made when $K_{a_1,a_2,\ldots,a_t}$ has some number of parts of size one by applying Theorem 5.1.1.

**Corollary 5.2.4.** For $K = K_{a_1,a_2,\ldots,a_t}$ with $a_1 \geq a_2 \geq \cdots \geq a_t$, if $a_i = a_{i+1} = \cdots = a_t = 1$ where $t > i \geq 3$, then $C_{t+i-1}(K)$ is Hamiltonian.

**Proof.** $C_{2i-2}(K_{a_1,a_2,\ldots,a_{i-1}})$ is $A$-Hamiltonian. The result follows by applying Theorem 5.1.1 $t - (i - 1)$ times.

We now turn our attention towards a lower bound for $k_0(K)$. It is clear that $C_t(K)$ is not connected, much less Hamiltonian. Considering our upper bound, we are only left to concern ourselves with the Hamiltonicity of $C_k(K)$ when $t + 1 \leq k \leq 2t - 1$. As will be shown in the next section, $C_{t+1}(K)$ is Hamiltonian for some choices of $K$, and not Hamiltonian for others. As a result, it is likely the case that $k_0(K) \geq t + 1$ is the best general lower bound that exists. It is however worth noting that it may
be possible for $\mathcal{C}_{k'}(K)$ to be Hamiltonian, but $\mathcal{C}_k(K)$ be non-Hamiltonian for some $2t \geq k \geq k'$. The result that follows is an attempt to gain insight on why some complete multipartite graphs fail to have Hamiltonian colour graphs for a particular number of colours while others succeed. Furthermore, this result shows that for $k$ as large as $t + \lceil \frac{s}{2} \rceil$, there are complete $t$-partite graphs whose $k$-colour graphs are not Hamiltonian.

For a graph $G$, let $\omega(G)$ denote the number of connected components of $G$. Before we continue, recall a well known necessary condition for Hamiltonicity of $G$ (see [1], page 53).

**Lemma 5.2.5.** If $G$ is Hamiltonian, and $S \subseteq V(G)$, then $\omega(G - S) \leq |S|$.

We use this condition to prove the following result.

**Lemma 5.2.6.** For $K = K_{a_1, a_2, \ldots, a_t}$, if $K$ has $s$ parts of size two, then $\mathcal{C}_k(K)$ is not Hamiltonian for $k \leq t + \lceil \frac{s}{2} \rceil$

**Proof.** Suppose $k = t + c$, where $c \leq s$. Let $T \subseteq V(\mathcal{C}_k(K))$ denote the set of colourings of $K$ in which $c$ parts of size two are coloured with two colours, and the remaining $t - c$ parts are coloured with a single colour. Let $S \subseteq V(\mathcal{C}_k(K))$ denote the set of colourings of $K$ in which $c - 1$ parts of size two are coloured with two colours, the remaining $t - c + 1$ parts are coloured with a single colour, and one colour is unused. Notice that the subgraph induced by $T$ contains no edges, as the only vertices which can change colour are those in the parts of size two which are coloured with two colours, and they can only be changed to the colour of the other vertex in their part. As such, vertices in $T$ are adjacent only to vertices in $S$. Then, $\omega(\mathcal{C}_k(K) - S) \geq |T| + 1$. Therefore, if $|S| \leq |T|$, $\mathcal{C}_k(K)$ is not Hamiltonian by Lemma 5.2.5. It is a simple counting exercise to show that $|T| = \binom{s}{c}(t + c)!$, and $|S| = \binom{s}{c - 1}(t + c)!$. So, $|S| \leq |T|$ whenever $\binom{s}{c} \geq \binom{s}{c - 1}$. This inequality holds whenever $c \leq \lceil \frac{s}{2} \rceil$. \hfill $\square$

A computer search for Hamilton cycles in colour graphs of some small complete 3-partite graphs suggests that $K_{2,2,2}$ is the only complete 3-partite graph whose 5-colour graph is not Hamiltonian.

### 5.3 Hamiltonicity of $\mathcal{C}_{t+1}(K)$

Let $V_1, V_2, \ldots, V_t$ be the $t$-partition of $K$, where $|V_i| = a_i$. It is clear that $\mathcal{C}_t(K)$ is not connected for $t \geq 2$, and of course cannot be Hamiltonian. This is not the case
with $C_{t+1}(K)$. In this section we characterize the complete $t$-partite graphs whose $(t+1)$-colour graphs are Hamiltonian. Once again our results are a generalization of the work done by Celaya et al. [3] on colour graphs of complete bipartite graphs. To begin, we will examine some basic necessary conditions.

Consider $C_{t+1}(K_{1,1,...,1}) = C_{t+1}(K_t)$. Let $V(K_t) = \{v_1, v_2, \ldots, v_t\}$. There is an important relationship between Hamilton cycles in $C_{t+1}(K_t)$ and Hamilton cycles in $C_{t+1}(K)$. Let $e = (f_1, f_2) \in E(C_{t+1}(K_t))$. Let $H_e$ be the subgraph of $C_{t+1}(K)$ induced by the colourings where, for each $i$, $V_i$ is coloured using colours from $\{f_1(v_i), f_2(v_i)\}$. Since $f_1(v_i) = f_2(v_i)$ for all but one vertex, this subgraph simply contains colourings in which one part $V_j$ is coloured using two colours, and each other part is monocoloured.

Recall that $Q_n$ denotes the $n$-cube, the graph whose vertex set is the set of binary strings of length $n$, where two strings are adjacent if they differ in exactly one position. Notice that $H_e \cong Q_{a_j}$, as there is a clear correspondence between binary strings of length $a_j$ and colourings of $V_j$ using two colours. Figure 5.1 gives an example of how the edges of $C_4(K_3)$ correspond to subgraphs of $C_4(K_{3,3,1})$.

It should be clear that every vertex in $C_{t+1}(K)$ is contained in $H_e$ for some $e$. More specifically, if $f \in C_{t+1}(K)$ is a colouring which uses all $t+1$ colours, it will appear
in exactly one subgraph $H_e$, and if $f$ uses $t$ colours, it will appear in $t$ subgraphs $H_e$, corresponding to the $t$ edges incident to the vertex in $C_{t+1}(K_t)$ corresponding to $f$.

Consider $H_e$ for some edge $e \in E(C_{t+1}(K_t))$. Let $f_1$ and $f_2$ denote the two colourings in $V(H_e)$ which use $t$ colours. Since no colour used on $V_i$ can be used on $V_j$ for any $j \neq i$ and any vertex $y \in V(H_e) - \{f_1, f_2\}$ uses all $t+1$ colours, any path from a vertex $x \in V(C_{t+1}(K)) - V(H_e)$ to $y$ must contain either $f_1$ or $f_2$. In other words, $\{f_1, f_2\}$ is a disconnecting set of $C_{t+1}(K)$. As a result, a Hamilton cycle in $C_{t+1}(K)$ must be composed of a sequence of Hamilton paths in the $H_e$s which begin and end at the vertices of $H_e$ which use $t$ colours. Each such path corresponds to an edge $e \in C_{t+1}(K_t)$, and therefore a Hamilton cycle in $C_{t+1}(K)$ will correspond to a Hamilton cycle in $C_{t+1}(K_t)$. Our first condition comes from the fact that we need $H_e$ to contain a Hamilton path beginning and ending at its $t$ colourings. Since $H_e \cong Q_{a_j}$ for some $j \in \{1, 2, \ldots, t\}$, this is equivalent to finding a Hamilton path in $Q_{a_j}$ from $00\ldots0$ to $11\ldots1$. The following lemma, a proof of which may be found in [9], illustrates when this can be done.

**Lemma 5.3.1.** For $n \geq 1$, there is a Hamilton path in $Q_n$ from $00\ldots0$ to $11\ldots1$ if and only if $n$ is odd.

As a result, we immediately get the following corollary.

**Corollary 5.3.2.** If $C_{t+1}(K_{a_1, a_2, \ldots, a_t})$ is Hamiltonian, $a_i$ is odd for each $i$.

We now complete the argument for complete bipartite graphs, reaffirming the result of Celaya et al. [3]. The proof we give is similar to the proof given by Celaya et al.

**Theorem 5.3.3 (Celaya et al. [3]).** $C_3(K_{a_1, a_2})$ is Hamiltonian if and only if $a_1$ and $a_2$ are both odd.

**Proof.** We know that $C_3(K_2)$ is Hamiltonian, and since each vertex in $C_3(K_2)$ has degree two, every edge of $C_3(K_2)$ is used in this cycle. Note that $C_3(K_2)$ has $3! = 6$ vertices. Let $f_0, f_1, \ldots, f_5, f_0$ denote the Hamilton cycle in $C_3(K_2)$, and let $e_i = (f_i, f_{i+1})$, interpreting indices modulo 6. Let $f'_i$ and $f'_{i+1}$ denote the $t$ colourings in $H_{e_i}$ which use the same colours on the same parts as $f_i$ and $f_{i+1}$ respectively. If $a_1$ and $a_2$ are both odd, by Lemma 5.3.1 there will always be a Hamilton path $P_{e_i}$ in $H_{e_i}$ from $f'_i$ to $f'_{i+1}$. Then, $P_{e_0}P_{e_1}\cdots P_{e_5}$ is a Hamilton cycle in $C_3(K_{a_1, a_2})$. If either $a_1$ or $a_2$ is even, then some for some edge $e_i$, by Lemma 5.3.1 $H_{e_i}$ will not contain a Hamilton path from $f'_i$ to $f'_{i+1}$, and a Hamilton cycle in $C_3(K_{a_1, a_2})$ cannot exist. □
CHAPTER 5. HAMILTONICITY OF $C_K(K)$

Since $C_{t+1}(K_t)$ is also Hamiltonian for $t \geq 3$, one might imagine a similar idea would work to find Hamilton cycles in $C_{t+1}(K_{a_1,a_2,\ldots,a_i})$ when $t \geq 3$. This however is not necessarily the case. Since the degree of a vertex in $C_{t+1}(K_t)$ is $t$, if $t \geq 3$, there must be some edges which are not used in a Hamilton cycle $C$ in $C_{t+1}(K_t)$. Consider such an edge $e$, and say $e$ changes the colour of $v_i$ in $K_t$. Then, since $H_e \cong Q_{a_i}$, if $a_i \geq 2$, there will be vertices in $H_e$ which are not visited by the cycle in $C_{t+1}(K)$ which corresponds to $C$. Thus, $C$ does not correspond to a Hamilton cycle in $C_{t+1}(K)$. Furthermore, if $a_i > 1$, any Hamilton cycle $C'$ in $C_{t+1}(K_t)$ which does correspond to a Hamilton cycle in $C_{t+1}(K)$ must use every edge $e$ which changes the colour of $v_i$. If $a_i = 1$ then $H_e$ contains only two vertices, the two possible $t$ colourings, and these vertices will appear in $H_{e'}$ for some $e'$ which is used by $C$. Every vertex in $C_{t+1}(K_t)$ is incident with exactly one edge which changes the colour of $v_i$, and indeed the set $E_i$ of all edges in $C_{t+1}(K_t)$ which change the colour of $v_i$ is a perfect matching.

To reiterate, each vertex $v_i$ has a corresponding value $a_i$, which is the size of $V_i$. Additionally, if $a_i > 1$, $v_i$ has a corresponding set of edges $E_i$ which must be used in any Hamilton cycle in $C_{t+1}(K_t)$ which corresponds with a Hamilton cycle in $C_{t+1}(K) = C_{t+1}(K_{a_1,a_2,\ldots,a_i})$. Note that $E(C_{t+1}(K_t)) = E_1 \cup E_2 \cup \cdots \cup E_t$, and that $E_i \cap E_j = \emptyset$ whenever $i \neq j$. If there exist $a_i, a_j$ and $a_k$, with $i \neq j \neq k, a_i \geq a_j \geq a_k \geq 2$, then there clearly cannot be a Hamilton cycle in $C_{t+1}(K_t)$ which uses every edge in $E_i \cup E_j \cup E_k$. Therefore, no Hamilton cycle in $C_{t+1}(K_t)$ can correspond to a Hamilton cycle in $C_{t+1}(K)$. Since a Hamilton cycle in $C_{t+1}(K)$ necessarily has a corresponding Hamilton cycle in $C_{t+1}(K_t)$, we conclude that $C_{t+1}(K)$ is not Hamiltonian. Suppose then, that only $a_1$ and $a_2$ are greater than one. A corresponding Hamilton cycle in $C_{t+1}(K_t)$ must use every edge in $E_1 \cup E_2$, and since $E_1$ and $E_2$ are disjoint perfect matchings, a corresponding Hamilton cycle in $C_{t+1}(K_t)$ cannot use any edge that is not in $E_1 \cup E_2$. Since these edges correspond to changing the colour of vertices $v_1$ and $v_2$ respectively, clearly we cannot have a Hamilton cycle, as the colour of $v_3$ is never changed using only these edges. This proves the following lemma.

**Lemma 5.3.4.** If $C_{t+1}(K_{a_1,a_2,\ldots,a_i})$ is Hamiltonian and $t \geq 3$, then $a_2 = a_3 = \cdots = a_t = 1$.

At this point, we are left to consider $C_{t+1}(K)$ for $K = K_{a_1,1,1,\ldots,1}$, with $a_1$ odd. Furthermore, a Hamilton cycle in $C_{t+1}(K)$ must correspond to a Hamilton cycle in $C_{t+1}(K_t)$ which uses every edge in the perfect matching $E_1$, the set of edges which change the colour of $v_1$. Notice that the edges of such a cycle must alternate between
edges in $E_1$ and edges not in $E_1$, as using two edges in $E_1$ consecutively contradicts that $E_1$ is a perfect matching, and using two edges not in $E_1$ consecutively would result in the cycle missing some edge in $E_1$. We will turn our efforts towards discerning when such a cycle in $C_{t+1}(K_t)$ will exist. To do so, we turn our attention to a particular Cayley graph.

Choo [9] noticed that $C_{t+1}(K_t) \cong \text{Cay}(X : S_{t+1})$, where $X = \{(1, t+1), (2, t+1), \ldots, (t, t+1)\}$. We allow the permutation $\pi$ of $\{1, 2, \ldots, t\}$ to correspond with a colouring $f$ of $\{v_1, v_2, \ldots, v_{t+1}\}$ in the obvious manner, where $\pi(i) = f(v_i)$ for $i \in \{1, 2, \ldots, t\}$. In addition, $\pi(t+1)$ corresponds to the single colour not used by $f$. Thus, the transposition $(i, t+1)$ is equivalent to switching the colour of $v_i$ to the single unused colour. A Hamilton cycle $\pi_1, \pi_2, \ldots, \pi_{t+1}, \pi_1$ in $\text{Cay}(X : S_{t+1})$ can be represented by a sequence $t_1, t_2, \ldots, t_{(t+1)!}$ of transpositions in $X$ such that $t_i \circ \pi_i = \pi_{i+1}$, interpreted modulo $(t+1)!$. Recall that we are looking for a Hamilton cycle in $C_{t+1}(K_t)$ in which every other edge changes the colour of $v_1$. In order to find this Hamilton cycle, we can equivalently find a Hamilton cycle in $C_{t+1}(K_t)$ which is represented by $(1, t+1), t_2, (1, t+1), t_3, \ldots, (1, t+1), t_{t+1}$, where $t_i \in X - \{(1, t+1)\}$.

This is equivalent to finding a Hamilton cycle in the directed Cayley graph $D_{t+1} = \text{Cay}(X' : A_{t+1})$, where $X' = \{(1, t+1)(i, t+1) : 2 \leq i \leq t\} = \{(1, i, t+1) : 2 \leq i \leq t\}$, and $A_{t+1} \subseteq S_{t+1}$ is the set of even permutations of a set of size $t+1$. The group $A_{t+1}$ is known as the alternating group. We then have the following lemma.

**Lemma 5.3.5.** $C_{t+1}(K_{a_1, a_2, \ldots, a_t})$, with $t \geq 3$, is Hamiltonian if and only if

- $a_1$ is odd,
- $a_i = 1$, for $2 \leq i \leq t$,
- $D_{t+1}$ is Hamiltonian.

Gould and Roth [15] proved the following theorem.

**Theorem 5.3.6.** $D_n$ is Hamiltonian if $n = 3$ or $n \geq 5$, and $D_n$ is not Hamiltonian if $n = 4$.

Therefore, we have fully characterized the complete $t$-partite graphs with Hamiltonian $(t+1)$-colour graphs with the following three theorems.

**Theorem 5.3.7.** $C_3(K_{a_1, a_2})$ is Hamiltonian if and only if $a_1$ and $a_2$ are odd.
Theorem 5.3.8. $C_4(K_{a_1,a_2,a_3})$ is Hamiltonian if and only if $a_1 = a_2 = a_3 = 1$.

Theorem 5.3.9. $C_{t+1}(K_{a_1,a_2,...,a_t})$ is Hamiltonian if and only if $a_1$ is odd, and $a_i = 1$ for $2 \leq i \leq t$. 
Chapter 6

Open Problems

Our analysis leaves the following open problems regarding colouring graphs of complete multipartite graphs.

- For \( k \leq t + \left\lceil \frac{t}{2} \right\rceil \), for which complete \( t \)-partite graphs is \( C_k(K) \) Hamiltonian?

- Given a complete \( t \)-partite graph \( K \), is \( C_k(K) \) Hamiltonian whenever \( k > t + \left\lceil \frac{t}{2} \right\rceil \)? Our results, supported by a limited computer search, suggest that this may be the case.

- More specifically, is \( K_{2,2,2} \) the only complete 3-partite graph whose 5 colour graph is non-Hamiltonian?

Concerning connectivity of colouring graphs, the following problem remains unsolved.

- Does there exist a 3-colouring graph which is connected, but not 2-connected?

On the Hamiltonicity of colouring graphs:

- Determine Gray code numbers of further classes of graphs.

- If \( C_k(G) \) is Hamiltonian, is \( C_{k+1}(G) \) always Hamiltonian?

To study the Gray code numbers of unexplored classes of graphs, the best candidates seem to be classes of highly structured graphs, such as outerplanar graphs, \( k \)-trees, and perhaps chordal graphs. With regards to whether or not Hamiltonicity of \( C_k(G) \) implies Hamiltonicity of \( C_{k+1}(G) \), a similar result by Cereceda et al. [5] suggests that this is not the case. It is shown that there exist graphs \( G \) and integers \( k \) such that \( C_k(G) \) is connected and \( C_{k+1}(G) \) is not connected. The graphs used for these examples are good candidates for an analagous result regarding Hamiltonicity.
Bibliography


