Refined Inertias Related to Biological Systems and to the Petersen Graph

by

Garrett James Culos
B.Sc., University of British Columbia, 2013

A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics

© Garrett Culos, 2015
University of Victoria

All rights reserved. This thesis may not be reproduced in whole or in part, by photocopying or other means, without the permission of the author.
Refined Inertias Related to Biological Systems and to the Petersen Graph

by

Garrett James Culos
B.Sc., University of British Columbia, 2013

Supervisory Committee

Dr. P. van den Driessche, Co-Supervisor
(Department of Mathematics and Statistics)

Dr. D.D. Olesky, Co-Supervisor
(Department of Computer Science)
Supervisory Committee

Dr. P. van den Driessche, Co-Supervisor
(Department of Mathematics and Statistics)

Dr. D.D. Olesky, Co-Supervisor
(Department of Computer Science)

ABSTRACT

Many models in the physical and life sciences formulated as dynamical systems have a positive steady state, with the local behavior of this steady state determined by the eigenvalues of its Jacobian matrix. The first part of this thesis is concerned with analyzing the linear stability of the steady state by using sign patterns, which are matrices with entries from the set \{+, -, 0\}. The linear stability is related to the allowed refined inertias of the sign pattern of the Jacobian matrix of the system, where the refined inertia of a matrix is a 4-tuple \((n_+, n_-, n_z, 2n_p)\) with \(n_+\) \((n_-)\) equal to the number of eigenvalues with positive \(\)negative\) real part, \(n_z\) equal to the number of zero eigenvalues, and \(2n_p\) equal to the number of nonzero pure imaginary eigenvalues. This type of analysis is useful when the parameters of the model are of known sign but unknown magnitude. The usefulness of sign pattern analysis is illustrated with several biological examples, including biochemical reaction networks, predator-prey models, and an infectious disease model. The refined inertias allowed by sign patterns with specific digraph structures have been studied, for example, for tree sign patterns. In the second part of this thesis, such results on refined inertias are extended by considering sign and zero-nonzero patterns with digraphs isomorphic to strongly connected orientations of the Petersen graph.
# Contents

Supervisory Committee ............................................. ii  
Abstract ............................................................... iii  
Table of Contents ...................................................... iv  
List of Tables ......................................................... vi  
List of Figures ........................................................ vii  
Acknowledgements ..................................................... ix  
Dedication ............................................................... x  

## 1 Introduction ....................................................... 1  
1.1 Basic Results on \( \mathbb{H}_n \) ................................ 3  
1.2 Dynamical Systems ............................................. 5  
\hspace{1em} 1.2.1 Example: a Lorentz-Like System ............... 6  
1.3 Overview ......................................................... 7  

## 2 Sign Patterns and Biological Systems ......................... 8  
2.1 Biochemical Reaction Networks I ............................ 8  
2.2 Biochemical Reaction Network II ............................ 10  
2.3 The Dynamics of Metabolic-Genetic Circuits .............. 14  
\hspace{1em} 2.3.1 Generalized Core Metabolator ..................... 14  
\hspace{1em} 2.3.2 Reverse Metabolator ............................... 16  
2.4 Model of Fox Rabies ........................................... 17  
2.5 A Predator–Prey Model With a Scavenger .................. 19  
2.6 A Predator–Prey Food Chain Model ......................... 21  
2.7 Spatially Heterogeneous Predator–Prey Models ............ 22
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.7.1 Two Patches with Linear Prey Growth</td>
<td>22</td>
</tr>
<tr>
<td>2.7.2 Two Patches with Generalized Growth Rates</td>
<td>25</td>
</tr>
<tr>
<td>2.7.3 Generalized n-Patch Model with Linear Prey Growth</td>
<td>26</td>
</tr>
<tr>
<td>2.7.4 Generalized n-Patch Sign Pattern</td>
<td>27</td>
</tr>
<tr>
<td>2.8 Lotka–Volterra Systems With Patch Structures</td>
<td>30</td>
</tr>
<tr>
<td>2.8.1 Homogeneous Predator–Prey System with Diffusion</td>
<td>30</td>
</tr>
<tr>
<td>2.8.2 Three-Competitor System</td>
<td>31</td>
</tr>
<tr>
<td>2.9 Dispersion in a Host-Parasitoid Systems</td>
<td>32</td>
</tr>
<tr>
<td>3 Refined Inertias Related to the Petersen Graph</td>
<td>33</td>
</tr>
<tr>
<td>3.1 Preliminary Theorems</td>
<td>34</td>
</tr>
<tr>
<td>3.2 Nonnegative Patterns</td>
<td>36</td>
</tr>
<tr>
<td>3.3 Zero-Nonzero Patterns</td>
<td>38</td>
</tr>
<tr>
<td>3.4 A Sign Pattern</td>
<td>46</td>
</tr>
<tr>
<td>4 Discussion and Conclusions</td>
<td>48</td>
</tr>
<tr>
<td>Bibliography</td>
<td>50</td>
</tr>
<tr>
<td>A Petersen Graph</td>
<td>54</td>
</tr>
<tr>
<td>A.1 Characteristic Polynomials</td>
<td>55</td>
</tr>
<tr>
<td>A.2 Strongly Connected Orientations of the Petersen Graph</td>
<td>56</td>
</tr>
</tbody>
</table>
List of Tables

Table 2.1 Refined inertias of (2.7) with parameters $b, c, d, e, f = 1$ ........ 12
Table 2.2 Refined inertias of (2.8) with parameters $a, b, c, d, e = 1$ ........ 12
Table 2.3 Refined inertias of (2.11) with parameters $\eta_2 = 7.92$ and $\rho = 5.15$ 17
Table 2.4 Refined inertias of (2.32) with parameters $a = 1/100, b = 1, c = 1/10, \text{ and } d = 1$ ........ 32

Table 3.1 Characteristic polynomials of $C_i$ from Appendix A.1 with all parameters $a, b, c, d, e, f$ set to one, and their corresponding refined inertias .......................................................... 37
Table 3.2 Examples of parameter values for all possible refined inertias of zero-nonzero patterns $A_1$ and $A_2$. Realization $A_1$ of $A_1$ has parameters $a, c, d = 1$, while realization $A_2$ of $A_2$ has parameters $a, c, e = 1$ .................. 39
Table 3.3 Examples of parameter values for all possible refined inertias of zero-nonzero patterns $A_3$ and $A_4$. Realization $A_3$ of $A_3$ has parameter values $e = 2, b = 1/2$ while realization $A_4$ of $A_4$ has parameter values $c = 1, e = 3$ .................. 41
Table 3.4 Examples of values for the coefficients of the characteristic polynomials $C_5$ to $C_{10}$ that can be obtained by choosing nonzero values of $a, b, c, d, e, \text{ and } f$ to give all possible refined inertias of $A_5$ to $A_{10}$. For brevity we do not list the parameter values $a, ..., f$ .... 42
Table 3.5 Examples of values for the coefficients of the characteristic polynomials $C_{13}$ to $C_{18}$ that can be obtained by choosing nonzero values of $a, b, c, d, e, \text{ and } f$ to give all possible refined inertias of $A_{13}$ to $A_{18}$. For brevity we do not list the parameter values $a, ..., f$ 46
# List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Digraph $D(S')$</td>
<td>10</td>
</tr>
<tr>
<td>2.2</td>
<td>Digraph $D(S)$</td>
<td>11</td>
</tr>
<tr>
<td>2.3</td>
<td>Parameters $k_1 = 0.01$, $k_2 = 0.15$, $k_3 = 0.01$, $k_4 = 10$, $k_5 = 0.008$, $k_6 = 0.09$, $k_7 = 1$, $V = 0.0004$, $K = 100$ and $a = 10$ as found in [27, Figure 3]. Solutions approach the steady state $u^* = (0.00009499, 0.001425, 1.781, 0.0001282)$</td>
<td>13</td>
</tr>
<tr>
<td>2.4</td>
<td>Parameters as in Figure 2.3, except that $k_2 = 0.5$. Solutions oscillate about the steady state $u^* = (0.00003234, 0.001617, 2.021, 0.0001455)$</td>
<td>13</td>
</tr>
<tr>
<td>2.5</td>
<td>Digraph $D(S)$ for the tree sign pattern $S$</td>
<td>15</td>
</tr>
<tr>
<td>2.6</td>
<td>Digraph $D(P_4)$</td>
<td>23</td>
</tr>
<tr>
<td>3.1</td>
<td>$D(A_c)$, a weighted digraph of $P_3$</td>
<td>47</td>
</tr>
<tr>
<td>A.1</td>
<td>$D(A_1)$, a weighted digraph of $P_1$ with directed spanning tree (solid arcs)</td>
<td>56</td>
</tr>
<tr>
<td>A.2</td>
<td>$D(A_2)$, a weighted digraph of $P_2$ with directed spanning tree (solid arcs)</td>
<td>56</td>
</tr>
<tr>
<td>A.3</td>
<td>$D(A_3)$, a weighted digraph of $P_3$ with directed spanning tree (solid arcs)</td>
<td>57</td>
</tr>
<tr>
<td>A.4</td>
<td>$D(A_4)$, a weighted digraph of $P_4$ with directed spanning tree (solid arcs)</td>
<td>57</td>
</tr>
<tr>
<td>A.5</td>
<td>$D(A_5)$, a weighted digraph of $P_5$ with directed spanning tree (solid arcs)</td>
<td>58</td>
</tr>
<tr>
<td>A.6</td>
<td>$D(A_6)$, a weighted digraph of $P_6$ with directed spanning tree (solid arcs)</td>
<td>58</td>
</tr>
<tr>
<td>A.7</td>
<td>$D(A_7)$, a weighted digraph of $P_7$ with directed spanning tree (solid arcs)</td>
<td>59</td>
</tr>
</tbody>
</table>
Figure A.8 $D(A_8)$, a weighted digraph of $P_8$ with directed spanning tree (solid arcs) ............................................. 59
Figure A.9 $D(A_9)$, a weighted digraph of $P_9$ with directed spanning tree (solid arcs) ............................................. 60
Figure A.10 $D(A_{10})$, a weighted digraph of $P_{10}$ with directed spanning tree (solid arcs) ............................................. 60
Figure A.11 $D(A_{11})$, a weighted digraph of $P_{11}$ with directed spanning tree (solid arcs) ............................................. 61
Figure A.12 $D(A_{12})$, a weighted digraph of $P_{12}$ with directed spanning tree (solid arcs) ............................................. 61
Figure A.13 $D(A_{13})$, a weighted digraph of $P_{13}$ with directed spanning tree (solid arcs) ............................................. 62
Figure A.14 $D(A_{14})$, a weighted digraph of $P_{14}$ with directed spanning tree (solid arcs) ............................................. 62
Figure A.15 $D(A_{15})$, a weighted digraph of $P_{15}$ with directed spanning tree (solid arcs) ............................................. 63
Figure A.16 $D(A_{16})$, a weighted digraph of $P_{16}$ with directed spanning tree (solid arcs) ............................................. 63
Figure A.17 $D(A_{17})$, a weighted digraph of $P_{17}$ with directed spanning tree (solid arcs) ............................................. 64
Figure A.18 $D(A_{18})$, a weighted digraph of $P_{18}$ with directed spanning tree (solid arcs) ............................................. 64
ACKNOWLEDGEMENTS

First, let me thank my funding sources, the University of Victoria and an NSERC CGS M research grant, for keeping me sheltered and fed. Second, and possibly more important, I would like to thank:

**Alyssa Halpin**, my wonderful partner in life and scientific crime (but not actual crime). You have been a constant source of inspiration and support.

**My family**, Mom, Dad, Tony, and Brit. We may not live in the same place, or even close, but I love you all, and appreciate all that you have done.

**My friends**, not going to name names, but near and far, thanks for being a distraction when I needed one.

**Pauline and Dale**, for your mentoring, support, encouragement, patience, the enormous amount of effort you both have put into my education, and letting me into your academic family. Thank you.

*Time.*

> It erodes mountains, makes diamonds, forms stars, the universe, and us.  
> Time created us and time will take us.  
> I hope we all take advantage of the little time we are given.
DEDICATION

To my Uncle Barrie and my Nan, you were a huge part of my life, miss you both. Pop, thank you for supporting my education; I wouldn’t be here if it wasn’t for you.
Chapter 1

Introduction

We begin with some pertinent definitions. A zero-nonzero pattern \( A = [\alpha_{ij}] \) is a matrix with entries from \( \{0, \star\} \), where \( \star \) represents any nonzero real number. A matrix realization \( A = [a_{ij}] \) of a zero-nonzero pattern \( A \) has \( a_{ij} = 0 \) if \( \alpha_{ij} = 0 \), and \( a_{ij} \) nonzero (either a constant or a variable) if \( \alpha_{ij} = \star \). A sign pattern \( A = [\alpha_{ij}] \) is a matrix with entries from \( \{0, +, -\} \), and a matrix realization \( A = [a_{ij}] \) of \( A \) has \( \text{sgn}(a_{ij}) = \text{sgn}(\alpha_{ij}) \). Each nonzero entry of such a realization can be either a constant or a variable having a value of fixed sign. Similarly, if \( A = [a_{ij}] \) is a matrix with entries of fixed sign (i.e., \( A \) is sign-definite), then the sign pattern of \( A \), denoted \( \text{sgn}(A) \), has entries \( \alpha_{ij} = + \) if \( a_{ij} > 0 \), \( \alpha_{ij} = - \) if \( a_{ij} < 0 \), and \( \alpha_{ij} = 0 \) if \( a_{ij} = 0 \). The associated sign pattern class (qualitative class) is \( Q(A) = \{A = [a_{ij}] | \text{sgn}(a_{ij}) = \alpha_{ij} \text{ for all } i, j\} \).

We refer to a pattern as either a sign pattern or a zero-nonzero pattern. A pattern \( A \) allows property \( X \) if there is a realization \( A \) of \( A \) with property \( X \). A pattern \( A \) requires property \( X \) if every realization \( A \) of \( A \) has property \( X \). A pattern \( A \) is sign nonsingular if it requires a nonzero determinant, that is, \( \text{det}(A) \neq 0 \) for all realizations \( A \) of \( A \). If matrix \( A \) has algebraic relationships between its entries, then \( A \) is a restricted realization of sign pattern \( \text{sgn}(A) \). If we define a sign pattern \( A \) from a matrix with restrictions, then \( A \) is said to be a restricted sign pattern. A pattern \( B = [\beta_{ij}] \) is a superpattern of \( A = [\alpha_{ij}] \) if \( \beta_{ij} = \alpha_{ij} \) whenever \( \alpha_{ij} \neq 0 \), and \( A \) is a subpattern of \( B \).

A pattern \( A \) is equivalent to \( B \) (\( A \sim B \)) if \( A = B^T \), or \( A = P^T BP \) for some permutation pattern \( P \) (a sign pattern of 0 and +, in which a + occurs exactly once in each row and column), or \( A = S^T BS \) for some signature pattern \( S \) (a diagonal sign pattern, in which each diagonal entry is + or -), or any combination of the above. The latter two transformations are permutation and signature similarities.
The importance of equivalence will become apparent later.

The weighted digraph of an $n$-by-$n$ matrix $A = [a_{ij}]$, denoted $D(A)$, is a digraph on $n$ vertices labelled $1, ..., n$ that has an arc $i \rightarrow j$ if and only if $a_{ij} \neq 0$, where this arc is labelled with weight $a_{ij}$. An arc from $i \rightarrow i$ is called a loop and is labelled with weight $a_{ii}$. If $A$ is a matrix realization of a zero-nonzero pattern, then each weight of $D(A)$ has arbitrary sign. If $A$ is a matrix realization of a sign pattern, then each weight of $D(A)$ has a fixed sign. Since it is labelled, any weighted digraph with $n$ vertices uniquely determines an $n$-by-$n$ matrix with nonzero entries corresponding to the weights on the arcs of the digraph. If $A$ is a matrix realization of a pattern $\mathcal{A}$, then $D(\mathcal{A})$ is equal to $D(A)$ with each weight replaced by $*$ (for a zero-nonzero pattern), or a $+$ or $-$ sign (for a sign pattern). If $\alpha_{i_1i_2}\alpha_{i_2i_3}\cdots\alpha_{i_ki_1}$ is nonzero for distinct $i_1, \ldots, i_k$, $k \geq 2$, then $D(\mathcal{A})$ has a $k$-cycle $i_1 \rightarrow i_2 \rightarrow \ldots \rightarrow i_k \rightarrow i_1$, and it is signed as $\text{sgn}(\alpha_{i_1i_2}\alpha_{i_2i_3}\cdots\alpha_{i_ki_1})$ if $\mathcal{A} = [\alpha_{ij}]$ is a sign pattern. If $D(\mathcal{A})$ is strongly connected and has no $k$-cycles for $k \geq 3$, then $\mathcal{A}$ is a tree sign pattern. The equivalence operations correspond to operations on the vertex set and arc set of $D(A)$. If $\mathcal{A}$ is a pattern, then transposition reverses the arcs of $D(\mathcal{A})$, while a permutation similarity is a relabeling of the vertices. For a sign pattern $\mathcal{A}$, a signature similarity negates all arcs entering and exiting a set of vertices. Signature similarities do not change zero-nonzero patterns. We use several well-known sign patterns throughout this thesis: $I_n$ denotes the $n$-by-$n$ sign pattern $\text{diag}(+, \ldots, +)$ and $C_n$ the sign pattern with digraph equal to a negative $n$-cycle.

As introduced by Kim et al. [23], the refined inertia of an $n$-by-$n$ real matrix $A$ is an ordered 4-tuple denoted by $\text{ri}(A) = (n_+, n_-, n_z, 2n_p)$, where $n_+$ is the number of eigenvalues with positive real part, $n_-$ is the number of eigenvalues with negative real part, $n_z$ is the number of zero eigenvalues, and $2n_p$ is the number of nonzero pure imaginary eigenvalues with $n = n_+ + n_- + n_z + 2n_p$. This is a refinement of the classical inertia of a matrix, which is an ordered 3-tuple $(n_+, n_-, n_z + 2n_p)$ with entries defined as above.

The refined inertia of a pattern $\mathcal{A}$ is the set of refined inertias for all realizations $A$ of $\mathcal{A}$ and is denoted by $\text{ri}(\mathcal{A})$. Refined inertias of sign patterns have been considered, for example, in [3, 12, 36], of zero-nonzero patterns in [8, 37], and of more general patterns in [5]. If $\mathcal{A} \sim \mathcal{B}$, then $\text{ri}(\mathcal{A}) = \text{ri}(\mathcal{B})$ since the eigenvalues of matrices are preserved under transposition, signature and permutation similarity transformations. As a result, pattern analysis need only be done up to equivalence.
We are interested in studying the specific set of refined inertias
\[ \mathbb{H}_n = \{(0, n, 0, 0), (0, n - 2, 0, 2), (2, n - 2, 0, 0)\}, \]
for sign patterns. This set of refined inertias corresponds to the transition of a pair of complex nonzero eigenvalues across the imaginary axis. Here, an \( n \)-by-\( n \) matrix \( A \) is \textit{stable} if all of its eigenvalues are in the open left-half plane and an \( n \)-by-\( n \) sign pattern \( \mathcal{A} \) is called \textit{sign stable} if \( \text{ri}(\mathcal{A}) = \{(0, n, 0, 0)\} \). If a sign pattern \( \mathcal{A} \) allows \((0, n, 0, 0)\), then \( \mathcal{A} \) is \textit{potentially stable}. Note that all sign patterns that allow/require \( \mathbb{H}_n \) must be potentially stable, and if \( \mathcal{A} \) is potentially stable, then \( \mathcal{A} \) must contain a negative diagonal entry. Therefore, a sign pattern that allows/requires \( \mathbb{H}_n \) must contain a negative diagonal entry. Potentially stable sign patterns have been studied in detail in [15, 11, 22, 21], but a general characterization of potential stability is unknown.

In addition to studying sign patterns that allow (or require) \( \mathbb{H}_n \), we also study patterns that allow other interesting sets of refined inertias and explore the refined inertias of patterns with certain digraph structures. These structures (e.g., number of negative cycles or negative loops) may be critical in analyzing the refined inertias of sign and zero-nonzero patterns of large order.

1.1 Basic Results on \( \mathbb{H}_n \)

It is easy to determine the refined inertias of all 2-by-2 sign patterns; see [29] for a complete list. All of the 3-by-3 tree sign patterns and their refined inertias can also be found in [29]. Beyond tree sign patterns, [3] and [12] give all 3-by-3 sign patterns that require \( \mathbb{H}_n \). A complete list of all 3-by-3 sign patterns and their allowed refined inertias is currently unknown.

Often a sign pattern \( \mathcal{A} \) is a superpattern of a known pattern \( \mathcal{B} \) (perhaps a tree sign pattern); therefore, it is useful to study how the refined inertia of \( \mathcal{B} \) relates to the refined inertia of a superpattern \( \mathcal{A} \). Bodine et al. [3] prove the following result for superpatterns of 3-by-3 sign patterns.

\textbf{Theorem 1.} [3, Theorem 2.2] If \( \mathcal{A} \) is a 3-by-3 sign nonsingular sign pattern that allows (and hence requires) \( \mathbb{H}_3 \) and \( \mathcal{A}' \) is a sign nonsingular superpattern of \( \mathcal{A} \), then \( \mathcal{A}' \) allows (and hence requires) \( \mathbb{H}_3 \).
We are able to extend this to the following new result. Note that the spectral norm of a matrix $A$ is $||A||_2 = \sqrt{\lambda_{\text{max}}(A^*A)}$, where $\lambda_{\text{max}}$ denotes the largest eigenvalue.

**Theorem 2.** Any superpattern $B$ of a sign pattern $A$ that allows $\mathbb{H}_3$ also allows $\mathbb{H}_3$.

**Proof.** Let $A$ be any 3-by-3 sign pattern that allows $\mathbb{H}_3$. Let $A \in Q(A)$ have refined inertia $(0,1,0,2)$ with spectrum $\sigma(A) = \{-\delta, \pm \beta i\}$ and $\delta, \beta > 0$. Since $A = [a_{ij}]$ is nonsingular, $\exists \epsilon > 0$ such that every matrix $C = [c_{ij}]$ with $||A - C||_2 < \epsilon$ is also nonsingular. For any such $\epsilon$, let $C_\epsilon$ denote any open convex set of such nonsingular matrices that contains $A$.

Since $A$ allows $\mathbb{H}_3$, $\exists A_1 = [a_{ij}^{(1)}] \in Q(A) \cap C_\epsilon$ having refined inertia $(0,3,0,0)$ with $\sigma(A_1) = \{-\delta_1, -\alpha_1 \pm \beta_1 i\}$, and $\exists A_2 = [a_{ij}^{(2)}] \in Q(A) \cap C_\epsilon$ having refined inertia $(2,1,0,0)$ with $\sigma(A_2) = \{-\delta_2, \alpha_2 \pm \beta_2 i\}$, where $\delta_i, \alpha_i, \beta_i > 0$ for $i = 1, 2$.

Let $B$ be any 3-by-3 superpattern of $A$, and $B_1 = [b_{ij}^{(1)}]$ and $B_2 = [b_{ij}^{(2)}]$ be matrices in $Q(B)$ such that

$$\max_{i,j} \left\{|a_{ij}^{(1)} - b_{ij}^{(1)}|, |a_{ij}^{(2)} - b_{ij}^{(2)}| \right\} < \tilde{\epsilon}$$

is sufficiently small so that $B_1, B_2 \in C_\epsilon$ and

$$\sigma(B_1) = \{-\mu_1, -v_1 \pm \omega_1 i\}, \quad \sigma(B_2) = \{-\mu_2, v_2 \pm \omega_2 i\}$$

where $\mu_i, v_i, \omega_i > 0$ for $i = 1, 2$. Thus $\text{ri}(B_1) = (0,3,0,0)$ and $\text{ri}(B_2) = (2,1,0,0)$.

Now consider a convex combination of $B_1$ and $B_2$, namely $B(t) = (1 - t)B_1 + tB_2$, with $t \in [0,1]$. Then $B(0) = B_1$ and $B(1) = B_2$, and all $B(t) \in Q(B)$. Moreover, all $B(t) \in C_\epsilon$ since every matrix $B(t)$ is contained in the convex set $C_\epsilon$, and every matrix $B(t)$ must have at least one negative eigenvalue (since $n = 3$). By continuity, there exists a value $\hat{t} \in (0,1)$ such that $B(\hat{t})$ also has a pair of nonzero pure imaginary eigenvalues and thus $\text{ri}(B(\hat{t})) = (0,1,0,2)$. Hence $B$ allows $\mathbb{H}_3$. \hfill \Box

It follows from Theorem 2 that all superpatterns of (8), (11), and (12) in Appendix B of [29] allow $\mathbb{H}_3$.

**Example 1.** The sign pattern

$$\begin{bmatrix}
- & + & + \\
0 & 0 & + \\
- & 0 & 0
\end{bmatrix}$$

is an example of a 3-by-3 sign pattern that allows $\mathbb{H}_3$ [14, Section 3.4] but has no subpattern that allows $\mathbb{H}_3$ since every subpattern is either singular or not potentially
Bodine et al. [3] give several other useful theorems relating sign patterns to refined inertias, specifically to the set $\mathbb{H}_n$. We list a few of them here, where $\oplus$ denotes the direct sum (see, e.g., [17, p. 12]).

**Theorem 3.** [3, Theorem 2.3] If $\mathcal{A}$ is a 4-by-4 sign nonsingular pattern that requires a negative trace and allows $\mathbb{H}_4$, then $\mathcal{A}$ requires $\mathbb{H}_4$.

**Theorem 4.** [3, Theorem 3.3] If an $n$-by-$n$ sign pattern $\mathcal{A}$ allows $\mathbb{H}_n$ and has all diagonal entries negative, then any superpattern of $\mathcal{A}$ allows $\mathbb{H}_n$.

**Theorem 5.** [3, Corollary 3.4] If an $n$-by-$n$ sign pattern $\mathcal{A}$ allows $\mathbb{H}_n$ and has all diagonal entries negative, then every $(n+m)$-by-$(n+m)$ superpattern of $\mathcal{A} \oplus -I_m$ allows $\mathbb{H}_{n+m}$.

**Theorem 6.** [3, Corollary 3.6] For $n \geq 3$, if every entry of an $n$-by-$n$ sign pattern $\mathcal{A}$ is negative, then $\mathcal{A}$ allows $\mathbb{H}_n$.

Sign patterns of order 4 and $\mathbb{H}_4$ are considered in [3, 12]. It is known that a tree sign pattern $\mathcal{A}$ requires $\mathbb{H}_4$ if and only if it is potentially stable, sign-nonsingular, not sign stable and does not allow refined inertia $(4, 0, 0, 0)$; see [12, Theorem 2.9]. Moreover, in [12, Section 2], all tree sign patterns of order 4, up to equivalence, are given. However, a complete list of all 5-by-5 tree sign patterns that require $\mathbb{H}_5$ does not exist. For an $n$-by-$n$ tree sign pattern $\mathcal{A}$ to allow $\mathbb{H}_n$, $\mathcal{A}$ must contain a negative 2-cycle. This follows from the fact that if $\mathcal{A}$ has all positive 2-cycles, then $\mathcal{A}$ can be symmetrized (see Theorem 1.8 in [9]). Several examples of sign patterns of order $n$ that allow $\mathbb{H}_n$ have been found (see, for example, [3]); however, necessary and sufficient conditions for a sign pattern to allow $\mathbb{H}_n$ are unknown. It is also unknown whether or not there exists a sign pattern of order $n \geq 8$ that requires $\mathbb{H}_n$.

### 1.2 Dynamical Systems

The dynamics of many systems, physical and biological, are accurately explained using a nonlinear system of differential equations

$$\frac{dx}{dt} = F(x(t), p)$$  (1.1)
where $x$ is an $n$-dimensional vector, $F$ is a vector valued function, and $p$ is an $m$-dimensional vector of parameters. A steady state (or equilibrium) of system (1.1) defines a vector $x^*$ where $F(x^*, p) = 0$, and represents a state of system (1.1) where the dynamics are unchanging. Note that a system may have more than one steady state. If, under small perturbations of $x^*$, the system evolves back to $x^*$, this steady state is said to be linearly stable. Mathematically this occurs when the Jacobian matrix

\[
J(x^*) = \left[ \frac{\partial}{\partial x_j} F_i(x(t), p) \right]_{x^*},
\]

is (negative) stable, i.e., all of the eigenvalues of $J$ have negative real parts ($\text{ri}(J) = (0, n, 0, 0)$). This type of analysis, called the linear stability of (1.1), is valid for small regions around $x^*$ and gives insight into the dynamics of the nonlinear system. The characteristic polynomial of a matrix $J(x^*)$ is defined as $\text{det}(J(x^*) - Iz)$, with the zeros of this polynomial in $z$ giving the eigenvalues of $J(x^*)$.

System (1.1) is said to undergo a bifurcation if its stability changes when one parameter in $p$ changes. We study a specific type of bifurcation called a Hopf bifurcation. A Hopf bifurcation occurs when a pair of nonzero complex conjugate eigenvalues of the Jacobian matrix (1.2) move from the left-half side of the complex plane to the right-half side, while the real parts of all other eigenvalues remain negative. Often finding this transition is algebraically difficult and/or numerically intense. This motivates the study of sign patterns that require $\mathbb{H}_n$, as the transition of eigenvalues in a Hopf bifurcation corresponds to the transition of refined inertias in $\mathbb{H}_n$. Using sign patterns, we are able to determine whether a system has the potential to undergo a Hopf bifurcation giving rise to periodic solutions. If $\text{sgn}(J(x^*))$ allows $\mathbb{H}_n$, then we say the system may undergo a Hopf bifurcation.

1.2.1 Example: a Lorentz-Like System

To illustrate the basic concepts, consider the following Lorentz-like system (called the “T-system” in [20]) described by the set of ordinary differential equations (ODEs)

\[
\begin{align*}
\frac{dx}{dt} & = a(y - x), \\
\frac{dy}{dt} & = (c - a)x - axz, \\
\frac{dz}{dt} & = xy - bz,
\end{align*}
\]
where \( a, b, c > 0 \). The positive equilibrium \((x^*, y^*, z^*)\) for this system is

\[
x^* = y^* = \sqrt{\frac{b}{a}(c - a)}, \quad z^* = \frac{c - a}{a},
\]

assuming \( c - a > 0 \). The Jacobian for this system at \((x^*, y^*, z^*)\) is

\[
J(x^*, y^*, z^*) = \begin{bmatrix}
-a & a & 0 \\
0 & 0 & -ax^* \\
y^* & x^* & -b
\end{bmatrix},
\]

which has entries of fixed sign. Matrix \( J \) has sign pattern

\[
\mathcal{S} = \begin{bmatrix}
- & + & 0 \\
0 & 0 & - \\
+ & + & -
\end{bmatrix},
\]

which is equivalent to the first sign pattern in Appendix A.3 of Bodine et al. [3], and thus requires \( \mathbb{H}_3 \). However, \( J \) has magnitude restrictions (\( J \) is a restricted realization of \( \mathcal{S} \)). Using parameter values from [20], let \( a = 4 \) and \( b = 2 \). For \( 4 < c < 16 \), \( \text{ri}(J) = (0, 3, 0, 0) \), for \( c = 16 \), \( \text{ri}(J) = (0, 1, 0, 2) \), and for \( c > 16 \), \( \text{ri}(J) = (2, 1, 0, 0) \). Therefore the restricted sign pattern \( \text{sgn}(J) \) requires \( \mathbb{H}_3 \). It follows that this system may exhibit periodic solutions occurring through a Hopf bifurcation, as shown analytically and confirmed numerically in [20].

\section{1.3 Overview}

This thesis has two directions: the application of sign patterns to biological systems (Chapter 2), and the refined inertias of all strongly connected orientations of the Petersen graph (Chapter 3). Chapter 2 explores the usefulness of sign patterns in analyzing the local stability of differential equations. Moreover, this chapter explores how sign patterns give insight into the dynamics of a system when the parameters are unknown or varied. We analyze several different examples from the literature using a variety of techniques in hope of expanding the reader’s “toolbox”. In Chapter 3, we look at the refined inertias of patterns resulting from a very unique and well known graph. The analysis used is novel, and particularly useful when the characteristic polynomial has several zero coefficients.
Chapter 2

Sign Patterns and Biological Systems

In Chapter 2 we study several different types of biological systems. In Sections 2.1, 2.2, and 2.3 we study the dynamics of biochemical reaction networks, modeling enzymatic reactions. In Section 2.4 a disease model having susceptible, latent, and infectious populations is studied. In Sections 2.5, 2.6, 2.7, 2.8, and 2.9 a variety of different predator–prey systems are explored. In Section 2.5 a predator–prey system with scavenging is analyzed, Section 2.6 contains a food chain model, while Sections 2.7, 2.8, and 2.9 look at predator–prey systems with patch structures. Sections 2.2, 2.3, 2.7, 2.8, and 2.9 have been published in the Journal of Mathematical Biology [7].

2.1 Biochemical Reaction Networks I

The Goodwin model is used to study many biological systems, such as circadian rhythms [32] and enzymatic regulation. The Goodwin model from [26] describes the interaction between four variables in two compartments, which represent the nucleus and the cytoplasm of a cell. The dynamics of mRNA and a repressor enzyme in the two compartments are formulated as the system

\[
\begin{align*}
\dot{u}_1 &= f(u_2) - u_1 + k_1(u_3 - u_1), \\
\dot{u}_2 &= -k_2u_1 + k_3(u_4 - u_2), \\
\dot{u}_3 &= -u_3 + k_4(u_1 - u_3), \\
\dot{u}_4 &= k_5u_3 - k_6u_4 + k_7(u_2 - u_4).
\end{align*}
\]
The function \( f(u_2) = 1/(1 + k u_2^h) \), \( u_1 \) and \( u_2 \) represent the concentrations of mRNA and the repressor enzyme in the nucleus, while \( u_3 \) and \( u_4 \) are concentrations of the mRNA and repressor enzyme in the cytoplasm (see [26] for a more detailed explanation of these biological interactions). The Jacobian matrix around the positive equilibrium \((u_1^*, u_2^*, u_3^*, u_4^*)\) is

\[
J(u_1^*, u_2^*, u_3^*, u_4^*) = \begin{bmatrix}
-1 - k_1 & f'(u_2^*) & k_1 & 0 \\
0 & -k_2 - k_3 & 0 & k_3 \\
k_4 & 0 & -1 - k_4 & 0 \\
0 & k_7 & k_5 & -k_6 - k_7
\end{bmatrix} \tag{2.2}
\]

where \( f'(u_2^*) = -kh(u_2^*)^{h-1}/(1 + k(u_2^*)^h)^2 < 0, h, k > 0, \text{ and } k_i > 0 \text{ for all } i = 1, \ldots, 7. \) All entries of (2.2) have fixed sign, therefore, its sign pattern is

\[
\text{sgn}(J) = \mathcal{S} = \begin{bmatrix}
- & - & + & 0 \\
0 & - & 0 & + \\
+ & 0 & - & 0 \\
0 & + & + & -
\end{bmatrix}. \tag{2.3}
\]

By permutation and signature similarity transformations, namely switching rows and columns 3 and 4, and negating the first row and column, \( \mathcal{S} \) is equivalent to

\[
\mathcal{S}' = \begin{bmatrix}
- & + & 0 & - \\
0 & - & + & 0 \\
0 & + & - & + \\
- & 0 & 0 & -
\end{bmatrix},
\]

which has digraph given in Figure 2.1 and is a superpattern of the following sign pattern (2.4) (the boxed entries are nonzero in \( \mathcal{S}' \)):

\[
\begin{bmatrix}
- & + & 0 & \boxed{0} \\
0 & - & + & 0 \\
0 & \boxed{0} & - & + \\
- & 0 & 0 & -
\end{bmatrix}. \tag{2.4}
\]

Sign pattern (2.4) with the boxed entries zero is \(-\mathcal{I}_4 + \mathcal{C}_4\). Thus, by [3, Theorem 3.9] this sign pattern requires \( \mathbb{H}_4 \), and by [3, Theorem 3.3] the sign pattern (2.3) allows \( \mathbb{H}_4 \). We are left to show that the restricted realization \( \tilde{J} \) of \( \mathcal{S} \) allows the refined
inertias in $\mathbb{H}_4$ (the restrictions being $J_{33} = -1 - J_{31}$, $J_{11} = -1 - J_{13}$, $J_{22} = -k_2 - J_{24}$, $J_{44} = -k_6 - J_{42}$). Substituting in parameter values for $J$ in (2.2) with $k = 1000$, $k_1 = k_2 = k_3 = k_4 = k_6 = k_7 = 1/2$, $h = 7$, $k_5 = 10,11$, and approximately 10.466465, the refined inertias of the Jacobian matrix (2.2) are $(0, 4, 0, 0)$, $(2, 2, 0, 0)$ and $(0, 2, 0, 2)$, respectively. We may conclude that since the restricted sign pattern $S$ allows $\mathbb{H}_4$, system (2.1) may undergo a Hopf bifurcation. In [26] graph-theoretic methods were used to show the occurrence of Hopf bifurcation and periodic solutions, but no numerics were given.

2.2 Biochemical Reaction Network II

We consider a two-cycle Goodwin model [27, Equation (13)], similar to the one in Section 2.1, described by the set of ODEs

\[
\begin{align*}
\frac{du_1}{dt} &= -k_1 u_1 + \Phi(u_3, u_4), \\
\frac{du_2}{dt} &= k_2 u_1 - k_3 u_2, \\
\frac{du_3}{dt} &= k_4 u_2 - k_5 u_3, \\
\frac{du_4}{dt} &= k_6 u_2 - k_7 u_4.
\end{align*}
\]

(2.5)

Here $u_1$ is the concentration of mRNA, $u_2$ is the concentration of the enzyme translated from mRNA, $u_3$ and $u_4$ are the concentrations of repressors synthesized by the enzyme $u_2$, and the function $\Phi(u_3, u_4) = V/(K + u_3^a + u_4^a)$ is the only non-linearity in this system. All parameters $V, K, a$ and $k_i$ are positive. This system has a unique pos-
itive equilibrium \( u^* = (u_1^*, u_2^*, u_3^*, u_4^*) \) that depends on the parameters. The Jacobian matrix for this system evaluated at the positive equilibrium is

\[
J(u^*) = \begin{bmatrix}
-k_1 & 0 & \frac{\partial \Phi}{\partial u_3}(u_3^*, u_4^*) & \frac{\partial \Phi}{\partial u_4}(u_3^*, u_4^*) \\
- k_3 & -k_4 & 0 & 0 \\
0 & k_4 & -k_5 & 0 \\
k_6 & 0 & -k_7 & 0
\end{bmatrix}, \tag{2.6}
\]

where \( \frac{\partial \Phi}{\partial u_j} = -V a_j u_j a_j^{-1} (K + u_3 a + u_4 a)^{-2} \), for \( j = 3, 4 \), evaluated at \( u^* \) are both negative. Note that

\[
\frac{\partial \Phi}{\partial u_3} / \frac{\partial \Phi}{\partial u_4} = (u_3^*/u_4^*) a - 1 = (k_4 k_7 / k_5 k_6) a - 1
\]

is a magnitude restriction that we have to take into account.

First, consider the unrestricted case. Matrix \( J(u^*) \) in (2.6) has fixed sign and thus has the following sign pattern

\[
\text{sgn}(J(u^*)) = S = \begin{bmatrix}
- & 0 & - & - \\
+ & - & 0 & 0 \\
0 & + & - & 0 \\
0 & + & 0 & -
\end{bmatrix}
\]

with its signed digraph \( \mathcal{D}(S) \) given in Figure 2.2.

![Figure 2.2: Digraph \( \mathcal{D}(S) \)](image)

We can without loss of generality (using a positive diagonal similarity transformation) choose a realization

\[
\begin{bmatrix}
-b & 0 & -f & -g \\
1 & -c & 0 & 0 \\
0 & 1 & -d & 0 \\
0 & 1 & 0 & -e
\end{bmatrix}, \tag{2.7}
\]
of $S$ with $b, \ldots, g > 0$. This realization has the refined inertias given in Table 2.1 for the specified values of the parameters, showing that $S$ allows $\mathbb{H}_4$. In fact, since

\begin{table}[h]
\centering
\caption{Refined inertias of (2.7) with parameters $b, c, d, e, f = 1$}
\begin{tabular}{ll}
\hline
Values of $g$ & Refined Inertia \\
\hline
$0 < g < 7$ & $(0, 4, 0, 0)$ \\
$g = 7$ & $(0, 2, 0, 2)$ \\
$7 < g < \infty$ & $(2, 2, 0, 0)$ \\
\hline
\end{tabular}
\end{table}

$S$ requires a positive determinant and negative trace, application of Theorem 3 gives that $S$ requires $\mathbb{H}_4$, and so a Hopf bifurcation leading to periodicity is likely to occur.

We now consider matrices with the realization $G$ in (2.8) and positive parameters $a, \ldots, f$, where the $(1, 4)$ entry is a function of the other entries, corresponding to the magnitude restriction in the Jacobian matrix; i.e., $j_{14} = j_{13}(j_{42}j_{33}/j_{32}j_{44})^{a-1}$, which implies that $j_{14} = -g = -f(d/e)^{a-1}$. We show in Table 2.2 that this realization allows $\mathbb{H}_4$, and thus the restricted sign pattern $\text{sgn}(J(u^*))$ in (2.6) requires $\mathbb{H}_4$. Therefore, system (2.5) likely exhibits periodic solutions around $u^*$. Substituting in the magnitude restriction gives

$$
G = \begin{bmatrix}
-b & 0 & -f & -f \left(\frac{d}{e}\right)^{a-1} \\
1 & -c & 0 & 0 \\
0 & 1 & -d & 0 \\
0 & 1 & 0 & -e
\end{bmatrix}, \quad (2.8)
$$

which has the refined inertias shown in Table 2.2.

\begin{table}[h]
\centering
\caption{Refined inertias of (2.8) with parameters $a, b, c, d, e = 1$}
\begin{tabular}{ll}
\hline
Values of $f$ & Refined Inertia \\
\hline
$0 < f < 4$ & $(0, 4, 0, 0)$ \\
$f = 4$ & $(0, 2, 0, 2)$ \\
$4 < f < \infty$ & $(2, 2, 0, 0)$ \\
\hline
\end{tabular}
\end{table}

Mincheva and Roussel [27] developed and used a graph-theoretic approach based on the Hurwitz matrix to show that a Hopf bifurcation occurs in system (2.5). These authors observed numerically that periodic orbits around the steady state $u^*$ arise for
parameter values due to this Hopf bifurcation (see [27, Figure 3]). For \( k_1 = 0.01, k_2 = 0.15, k_3 = 0.01, k_4 = 10, k_5 = 0.008, k_6 = 0.09, k_7 = 1, V = 0.0004, K = 100 \) and \( a = 10 \), we find numerically that \( u^* \) is stable (see Figure 2.3); whereas if \( k_2 = 0.5 \), then periodic solutions occur (see Figure 2.4). Note that in Figures 2.3 and 2.4, \( u_1 \) and \( u_4 \) remain positive although small in magnitude. Mincheva and Roussel [27, Section 4] also consider Turing–Hopf bifurcation when the repressor represented by \( u_4 \) diffuses. Their partial differential equation system linearized about the homogeneous equilibrium gives the same sign pattern \( S \) with the same magnitude restriction as in (2.6). The above analysis suggests that Turing–Hopf bifurcation may be possible, as found numerically by Mincheva and Roussel [27, Figure 5].

Figure 2.3: Parameters \( k_1 = 0.01, k_2 = 0.15, k_3 = 0.01, k_4 = 10, k_5 = 0.008, k_6 = 0.09, k_7 = 1, V = 0.0004, K = 100 \) and \( a = 10 \) as found in [27, Figure 3]. Solutions approach the steady state \( u^* = (0.00009499, 0.001425, 1.781, 0.0001282) \)

Figure 2.4: Parameters as in Figure 2.3, except that \( k_2 = 0.5 \). Solutions oscillate about the steady state \( u^* = (0.00003234, 0.001617, 2.021, 0.0001455) \)
2.3 The Dynamics of Metabolic-Genetic Circuits

The focus of [31] is to develop a modeling framework to understand the dynamics of metabolic-genetic circuits. Reznick et al. formulated different network topologies including the core metabolator [31, Equation (1)] and the generalized core metabolator [31, Equation (12)], which model two enzymes $g_1$ and $g_2$ that interconvert two substrates $m_1$ and $m_2$; they also consider the reverse metabolator [31, Equation (18)] in which the regulation of $g_1$ and $g_2$ is reversed. For information on metabolic-genetic models, see [31] and references therein.

2.3.1 Generalized Core Metabolator

The generalized model of the core metabolator investigates the topological structure of the core metabolator model [31, Equation (1)], while allowing for more general functions. This system takes the following form [31, Equation (12)]:

\[
\begin{align*}
\frac{dg_1}{dt} &= P_1(-m_2) - D_1(g_1), \\
\frac{dg_2}{dt} &= P_2(m_2) - D_2(g_2), \\
\frac{dm_1}{dt} &= I + g_2R_2(m_2) - g_1R_1(m_1), \\
\frac{dm_2}{dt} &= g_1R_1(m_1) - g_2R_2(m_2) - R_3(m_2),
\end{align*}
\]

where $I$ is the constant inflow, $R_3$ (a function of $m_2$) is the outflow catalyzed by $g_1$ and $g_2$, and for $i = 1, 2$, $P_i$ is the production of proteins, $D_i$ is the degradation of proteins, $R_i$ corresponds to reactions. Using the total amount of substrate $c = m_1 + m_2$, this change of variables gives the following equivalent system:

\[
\begin{align*}
\frac{dg_1}{dt} &= P_1(-m_2) - D_1(g_1), \\
\frac{dg_2}{dt} &= P_2(m_2) - D_2(g_2), \\
\frac{dc}{dt} &= I - R_3(m_2), \\
\frac{dm_2}{dt} &= g_1R_1(c - m_2) - g_2R_2(m_2) - R_3(m_2),
\end{align*}
\]
where it is assumed that \( R_i, P_i, \) and \( D_i \) are differentiable and monotone increasing functions of their arguments. We assume that there is a unique positive steady state \((g^*_1, g^*_2, c^*, m_2^*)\); for the core metabolator model with specific functions, this equilibrium is given explicitly in [31, Section II]. The Jacobian matrix of system (2.9) evaluated around its positive steady state is

\[
J(g^*_1, g^*_2, c^*, m_2^*) = \begin{bmatrix}
-\partial_{g_1} D_1 & 0 & 0 & -\partial_{m_2} P_1 \\
0 & -\partial_{g_2} D_2 & 0 & \partial_{m_2} P_2 \\
0 & 0 & 0 & -\partial_{m_2} R_3 \\
R_1 & -R_2 & g_1 \partial_c R_1 & -g_1 \partial_{m_2} R_1 - g_2 \partial_{m_2} R_2 - \partial_{m_2} R_3
\end{bmatrix}.
\]

The sign pattern

\[
S = \text{sgn}(J) = \begin{bmatrix} - & 0 & 0 & - \\
0 & - & 0 & + \\
0 & 0 & 0 & - \\
+ & - & + & - \end{bmatrix}
\]

has the signed digraph \( \mathcal{D}(S) \) in Figure 2.5.

![Figure 2.5: Digraph \( \mathcal{D}(S) \) for the tree sign pattern \( S \)](image)

Digraph \( \mathcal{D}(S) \) is strongly connected and has no \( k \)-cycles for \( k \geq 3 \); thus, \( S \) is irreducible and is a tree sign pattern. Also, if \( A = [a_{ij}] \) is any realization of \( S \), then \( \det(A) > 0 \), \( a_{ii} \leq 0 \) for all \( i \), and if \( i \neq j \) then \( a_{ij} a_{ji} \leq 0 \). By [4, Theorem 10.2.2], \( S \) is sign stable (since property (v) there also holds), so \( \text{ri}(S) = (0, 4, 0, 0) \), and \( S \) does not allow \( \mathbb{H}_4 \).

Since \( S \) is sign stable, the matrix \( J \) for the general metabolator has \( \text{ri}(J) = (0, 4, 0, 0) \) and thus the positive steady state is linearly stable. No periodic solutions can arise by a Hopf bifurcation. The Routh–Hurwitz conditions and algebraic manipulations were used in [31, Appendix B] to obtain this conclusion.
2.3.2 Reverse Metabolator

The reverse metabolator [31, Equation (18)] again describes the relationship between enzymes \( g_1 \) and \( g_2 \), and substrates \( m_1 \) and \( m_2 \), using specific functions. With the regulating connections of the core metabolator reversed, the set of ODEs describing this system is

\[
\begin{align*}
\frac{dg_1}{dt} &= \frac{m_2}{1 + m_2} - g_1, \\
\frac{dg_2}{dt} &= \frac{1}{1 + m_2} - g_2, \\
\frac{dc}{dt} &= \rho(1 - m_2), \\
\frac{dm_2}{dt} &= -\rho m_2 + \eta_1 g_1(c - m_2) - \eta_2 g_2 m_2.
\end{align*}
\]

Similar to (2.9), system (2.10) uses the total substrate \( c = m_1 + m_2 \), and the parameters \( \rho, \eta_1, \eta_2 \) are assumed to be positive. The system linearized about the steady state \( g_1^* = g_2^* = 1/2, m_2^* = 1, \) and \( c^* = (\eta_1 + \eta_2 + 2\rho)/\eta_1 \) has the following Jacobian matrix

\[
J(g_1^*, g_2^*, c^*, m_2^*) = \begin{bmatrix}
-1 & 0 & 0 & 1/2 \\
0 & -1 & 0 & -1/2 \\
0 & 0 & 0 & -\rho \\
2\rho + \eta_2 & -\eta_2 & \eta_1/2 & (-\eta_1 - \eta_2 - 2\rho)/2
\end{bmatrix},
\]

with sign pattern

\[
S = \text{sgn}(J) = \begin{bmatrix}
-0 & 0 & 0 & + \\
0 & -0 & -0 & - \\
0 & 0 & 0 & - \\
+0 & 0 & 0 & -
\end{bmatrix}.
\]

The signed digraph of this tree sign pattern \( S \) in (2.12) has two positive 2-cycles (1 \( \rightarrow \) 4 \( \rightarrow \) 1 and 2 \( \rightarrow \) 4 \( \rightarrow \) 2). By first permuting the rows and columns to obtain

\[
\begin{bmatrix}
- - + + \\
- - 0 0 \\
+ 0 - 0 \\
- 0 0 0
\end{bmatrix},
\]
and then negating the second row and column, \( S \) is equivalent to

\[
S' = \begin{bmatrix}
- & + & + & + \\
+ & - & 0 & 0 \\
+ & 0 & - & 0 \\
- & 0 & 0 & 0
\end{bmatrix},
\]

which is known to require \( \mathbb{H}_4 \) [12, Section 2.2]. Thus, \( S \) requires \( \mathbb{H}_4 \), and a Hopf bifurcation leading to periodicity is possible.

However, there are magnitude restrictions in the entries of the Jacobian matrix \( J = [j_{kl}] \) of (2.11), for example, \( j_{44} = -(2j_{43} + j_{41})/2 \) and \( j_{41} = -2j_{34} - j_{42} \). Since \( S = \text{sgn}(J) \) and \( S \) requires \( \mathbb{H}_4 \), matrix \( J \) can only obtain the refined inertias in \( \mathbb{H}_4 \). In Table 2.3, we show that as \( \eta_1 \) varies, \( J \) can achieve each refined inertia in \( \mathbb{H}_4 \). Therefore, the restricted sign pattern \( \text{sgn}(J) \) requires \( \mathbb{H}_4 \), and a Hopf bifurcation leading to periodicity may be possible. This bifurcation was found by Reznik et al. [31, Figure 2] using Routh–Hurwitz criteria and numerical computations with AUTO.

<table>
<thead>
<tr>
<th>Values of ( \eta_1 )</th>
<th>Refined Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>(0, 4, 0, 0)</td>
</tr>
<tr>
<td>1.39597 (approx.)</td>
<td>(0, 2, 0, 2)</td>
</tr>
<tr>
<td>1.3</td>
<td>(2, 2, 0, 0)</td>
</tr>
</tbody>
</table>

2.4 Model of Fox Rabies

Swart [33] investigated a simple model for fox rabies describing the dynamics between the susceptible, latent (i.e., infected by not yet infectious), and infectious populations. The set of differential equations below describes this interaction.

\[
\begin{align*}
\frac{dX}{dt} &= rX - \gamma XN - \beta XY, \\
\frac{dY}{dt} &= \sigma I - (\alpha + b + \gamma N)Y, \\
\frac{dI}{dt} &= \beta XY - (\sigma + b + \gamma N)I,
\end{align*}
\] (2.13)
The susceptible population \( X \) has per capita growth rate \( a \), and death rate \( b \), where
\( r = a - b \) is the net growth rate of the susceptible population. Note that population \( X \)
is the only one able to reproduce and all other populations are subject to the natural death rate \( b \). The susceptible individuals become latently infected (population \( I \)) by a mass action transmission term involving susceptible and infectious populations. Latently infected individuals become infectious (population \( Y \)) at constant rate \( \sigma \).

The growth of populations \( X, Y, \) and \( Z \) is limited by a density-dependent term, with constant of proportionality \( \gamma \). The infectious individuals have an additional mortality rate \( \alpha \) due to infection.

Using the total population numbers \( N = X + I + Y \), system (2.13) can be replaced by

\[
\begin{align*}
\frac{dX}{dt} &= (a - b)X - \gamma X N - \beta XY, \\
\frac{dY}{dt} &= \sigma(N - X - Y) - (\alpha + \beta + \gamma N)Y, \\
\frac{dN}{dt} &= aX - (b + \gamma N)N - \alpha Y.
\end{align*}
\]

(2.14)

For a positive steady state \( (X^*, Y^*, N^*) \) to occur the conditions \( \sigma \beta - a \gamma > 0 \) and \( a - b - \gamma N^* > 0 \) must be satisfied. The Jacobian matrix of (2.14) equals

\[
J(X^*, Y^*, N^*) = \begin{bmatrix}
0 & -\beta X^* & -\gamma X^* \\
-\sigma & -(\alpha + \sigma + b + \gamma N^*) & \sigma - \gamma Y^* \\
a & -\alpha & -R
\end{bmatrix}
\]

where \( R = b + 2\gamma N^* \). The (unrestricted) sign pattern \( S \) of matrix \( J \) is

\[
S = \begin{bmatrix}
0 & - & - \\
- & - & \oplus \\
+ & - & -
\end{bmatrix},
\]

where the \( \oplus \) entry can be +, -, 0. By interchanging rows/columns 1 and 2 and by a signature similarity of the second row/column, we find

\[
S \sim \begin{bmatrix}
- & + & \oplus \\
+ & 0 & + \\
- & - & -
\end{bmatrix}.
\]
This sign pattern is a superpattern of
\[
\begin{bmatrix}
- & + & 0 \\
+ & 0 & + \\
0 & - & -
\end{bmatrix},
\]
which is known to allow $\mathbb{H}_3$ [29, Appendix B, (11a)]. Therefore, by Theorem 2, $S$ allows $\mathbb{H}_3$. However, the matrix $J$ has parameter restrictions, so it remains to show that $J$ can obtain all refined inertias in $\mathbb{H}_3$. We use a computer program to find the positive steady state of (2.14), substitute $(x^*, y^*, z^*)$ into $J$, and then numerically solve for the eigenvalues. For parameter values $a = 1, b = 1/2, \beta = 80, \sigma = 13, \alpha = 73, \gamma$ equal to 0.058, 0.0583397(approx.), and 0.06, chosen based on [33], the refined inertia of matrix $J$ equals $(0, 3, 0, 0), (0, 1, 0, 2)$ and $(2, 1, 0, 0)$, respectively. It follows that periodic solutions may arise in system (2.14) via a Hopf bifurcation, as found in [33] by analytical techniques verified by a computer search.

### 2.5 A Predator–Prey Model With a Scavenger

Previte et al. [30] considered a more complex predator–prey model, broader in scope than the original Lotka-Voltera system formulated in 1910. This system incorporates a third equation representing a scavenging population, denoted by $z$, and is described by the set of ODEs
\[
\begin{align*}
\frac{dx}{dt} &= x(1 - bx - y - z), \\
\frac{dy}{dt} &= y(-c + x), \\
\frac{dz}{dt} &= z(-e + fx + gy - hz),
\end{align*}
\]
where the prey, predator and scavenger populations ($x$, $y$, and $z$) have been non-dimensionalized. The prey population has growth limited by a density dependent term and decreases only by means of predation and scavenging. The predator population, subject to a death rate $c$, converts prey to new predators at a rate arbitrarily set to one. The scavenger population, limited by density-dependence, increases by feeding on prey (at rate $f$) and on the remains of the predator–prey interaction (i.e., scavengers eat the leftovers at rate $g$), denoted as scavenging.

Setting $z$ to zero, we obtain an invariant manifold, reducing the system to the already solved, standard, predator–prey model with linear functional response (see,
When solutions are not confined to this manifold, there is one positive equilibrium 
\[(x^*, y^*, z^*) = \left( \frac{h + e - fc - bch}{g + h}, \frac{-e + fc + g - gbc}{g + h} \right), \tag{2.16} \]
provided \(h + e > c(f + bh)\) and \(fc + g > e + gbc\). Our goal is to show, using sign patterns, that this model may exhibit in periodic solutions, arising from a Hopf bifurcation. Consider the Jacobian matrix at the equilibrium point (2.16)

\[
J(x^*, y^*, z^*) = \begin{bmatrix}
-bx^* & -x^* & -x^* \\
y^* & 0 & 0 \\
fz^* & gz^* & -hz^*
\end{bmatrix} = x^* \begin{bmatrix}
-b & -1 & -1 \\
\frac{y^*}{x^*} & 0 & 0 \\
\frac{fz^*}{x^*} & \frac{gz^*}{x^*} & -\frac{hz^*}{x^*}
\end{bmatrix}, \tag{2.17}
\]
which has sign pattern
\[
S = \begin{bmatrix}
- & - & - \\
+ & 0 & 0 \\
+ & + & -
\end{bmatrix}.
\]
Using a permutation and signature similarity transformation, \(S\) is similar to
\[
S' = \begin{bmatrix}
- & + & + \\
- & - & + \\
- & 0 & 0
\end{bmatrix}.
\]
Sign pattern \(S'\) requires \(\mathbb{H}_3\) [3, Appendix A.3] and thus \(S\) requires \(\mathbb{H}_3\). Since \(\text{sgn}(J)\) is a restricted sign pattern, it remains to show that \(J\) obtains the refined inertias of \(\mathbb{H}_3\). Letting \(b = 0.9, c = f = 0.1, g = 13\) and \(e = 2\) (chosen based on [30]), we find that for \(h = 2\) and \(h = 3\), \((x^*, y^*, z^*)\) is positive and the matrix \(J\) has refined inertias \((2, 1, 0, 0)\) and \((0, 3, 0, 0)\), respectively. For \(h \approx 2.3129\), \(J\) has refined inertia \((0, 1, 0, 2)\); therefore, the restricted sign pattern \(\text{sgn}(J)\) requires \(\mathbb{H}_3\) and a Hopf bifurcation may occur. Previte et al. [30] demonstrated that system (2.15) undergoes a Hopf bifurcation producing periodic solutions, as well as chaotic solutions that arise through period doubling bifurcations.
2.6 A Predator–Prey Food Chain Model

Fussmann et al. [10] considered a predator–prey food chain system, modeling the interactions between the reproducing and total density of planktonic rotifers *Brachionus* (populations *R* and *B*, respectively), which feed on unicellular green algae *Chlorella* (population *C*). The green algae’s growth is limited by the amount of nitrogen (population *N*) within the system. This model is described by the system of ODE’s

\[
\begin{align*}
\frac{dN}{dt} &= \delta (N_i - N) - \frac{b_C NC}{k_C + N}, \\
\frac{dC}{dt} &= \frac{b_C NC}{k_C + N} - \frac{b_B CB}{\epsilon (k_B + C)} - \delta C, \\
\frac{dR}{dt} &= \frac{b_B CR}{k_B + C} - (\delta + m + \lambda) R, \\
\frac{dB}{dt} &= \frac{b_B CR}{k_B + C} - (\delta + m) B,
\end{align*}
\]

(2.18)

where \(\delta\) is the dilution rate, \(N_i\) is the inflow of nitrogen, \(k_C, k_B, b_C, b_B, m, \lambda\) and \(\epsilon\) are positive parameters (see [10] for parameter explanation). Analyzing the linear stability of this system about the positive steady state \((N^*, C^*, R^*, B^*)\), we find the Jacobian matrix

\[
J(N^*, C^*, R^*, B^*) = \begin{bmatrix}
-\delta - \frac{b_C k_C C^*}{(k_C + N^*)^2} & -\frac{b_C N^*}{k_C + N^*} & 0 & 0 \\
\frac{b_C C^* k_C}{(k_C + N^*)^2} & \frac{b_B B^* C^*}{\epsilon (k_B + C^*)^2} & 0 & -\frac{b_B C^*}{\epsilon (k_B + C^*)} \\
0 & \frac{b_B R^* k_B}{(k_B + C^*)^2} & 0 & 0 \\
0 & \frac{b_B R^* k_B}{(k_B + C^*)^2} & \frac{b_B C^*}{k_B + C^*} & -\delta - m
\end{bmatrix}
\]

All entries of \(J\) are of fixed sign; therefore, we consider the following sign pattern \(\mathcal{S} = \text{sgn}(J)\) and without loss of generality a realization \(A\):

\[
\mathcal{S} = \begin{bmatrix} - & - & 0 & 0 \\ + & + & 0 & - \\ 0 & + & 0 & 0 \\ 0 & + & + & - \end{bmatrix}, \quad A = \begin{bmatrix} -a & -b & 0 & 0 \\ 1 & c & 0 & -d \\ 0 & 1 & 0 & 0 \\ 0 & 1 & e & -f \end{bmatrix},
\]

with \(a, ..., f > 0\). Since \(\det(A) = dea\), \(\mathcal{S}\) is sign nonsingular. For parameters \(a = f = 3, b = c = e = 1\), and for \(d = 5, 4, 3\), the refined inertias of \(A\) equal \((2, 2, 0, 0)\) \((0, 2, 0, 2)\), and \((0, 4, 0, 0)\), respectively. It follows that \(\mathcal{S}\) allows \(\mathbb{H}_4\). Since \(J\) has
restrictions (e.g., $j_{32} = j_{42}$) we are left to show that this restricted realization allows $\mathbb{H}_4$. For parameter values $b_C = 3.3, \epsilon = 0.25, b_B = 2.25, k_C = 4.3, k_B = 15, m = 0.055, \lambda = 0.4, \delta \approx 0.15098283$, and $N_i = 80$, as found in [10], the non-trivial steady state is $(6.64, 5.53, 1.44, 4.22)$ and $\text{ri}(J) = (0, 2, 0, 2)$. If $\delta$ is increased slightly, then $\text{ri}(J) = (0, 4, 0, 0)$, and if $\delta$ is decreased then $\text{ri}(J) = (2, 2, 0, 0)$. Hence, this restricted realization $J$ allows $\mathbb{H}_4$. Fussmann et al. [10] found coexistence on limit cycles both by numerical simulations of (2.18) and also biological experimentations.

2.7 Spatially Heterogeneous Predator–Prey Models

There have been many extensions and refinements to the classical predator–prey model, which is neutrally stable about its positive equilibrium, i.e., its Jacobian matrix has refined inertia $(0, 0, 0, 2)$. We look at one refinement by Holt [16] that incorporates spatial heterogeneity, resulting in the coexistence of prey species.

2.7.1 Two Patches with Linear Prey Growth

First consider the scenario where there are two separate patches. Let $P_i$ and $R_i$ be the population levels of predator species and prey species in patch $i$, respectively. The predators move between patches feeding on prey, while the prey populations grow linearly, only dying to predation. A linear Lotka–Volterra type functional response is postulated. This system is formulated as

$$\begin{align*}
\frac{dP_1}{dt} &= P_1(a_1 b_1 R_1 - C_1) - E_{11} P_1 + E_{12} P_2, \\
\frac{dP_2}{dt} &= P_2(a_2 b_2 R_2 - C_2) - E_{22} P_2 + E_{21} P_1, \\
\frac{dR_1}{dt} &= r_1 R_1 - a_1 P_1 R_1, \\
\frac{dR_2}{dt} &= r_2 R_2 - a_2 P_2 R_2.
\end{align*}$$

(2.19)

Here $r_i$ is the per capita growth rate for each prey species, $a_i$ is the rate at which predators catch prey, $a_i b_i$ is the rate of foraging for predator $i$, $C_i$ is the mortality rate for predator $i$, and the $E_{ij}$ are positive immigration and emigration rates for predators moving between patches. All parameters are assumed to be positive. Let $d_i = r_i/a_i$. 
Then the non-trivial steady state of this system is given by $P_1^* = d_1$, $P_2^* = d_2$, $R_1^* = (C_1 + E_{11} - E_{12}d_2/d_1)/(a_1b_1)$, and $R_2^* = (C_2 + E_{22} - E_{21}(d_2/d_1)^{-1})/(a_2b_2)$, where these are assumed positive, i.e.,

$$\frac{C_1 + E_{11}}{E_{12}} > \frac{d_2}{d_1} > \frac{E_{21}}{C_2 + E_{22}}.$$  

Linearizing around this positive steady state $(P_1^*, P_2^*, R_1^*, R_2^*)$ results in the following Jacobian matrix

$$J = \begin{bmatrix} a_1b_1R_1^* - C_1 - E_{11} & E_{12} & a_1b_1P_1^* & 0 \\ E_{21} & a_2b_2R_2^* - C_2 - E_{22} & 0 & a_2b_2P_2^* \\ -a_1R_1^* & 0 & r_1 - a_1P_1^* & 0 \\ 0 & -a_2R_2^* & 0 & r_2 - a_2P_2^* \end{bmatrix}. $$

After substituting for $P_1^*, P_2^*, R_1^*$, and $R_2^*$, this becomes

$$J = \begin{bmatrix} -E_{12}d_2/d_1 & E_{12} & r_1b_1 & 0 \\ E_{21} & -E_{21}d_1/d_2 & 0 & r_2b_2 \\ -a_1R_1^* & 0 & 0 & 0 \\ 0 & -a_2R_2^* & 0 & 0 \end{bmatrix}. \quad (2.20)$$

This Jacobian matrix is sign-definite with sign pattern $\mathcal{S} = \text{sgn}(J)$. By application of a permutation and signature similarity, it follows that $\mathcal{S}$ is equivalent to $\mathcal{P}_4$, a tree sign pattern in (2.21) that is known to require $\mathbb{H}_4$ [12, Section 2.1], and has the signed digraph shown in Figure 2.6.

$$\mathcal{S} = \begin{bmatrix} - & + & + & 0 \\ + & - & 0 & + \\ - & 0 & 0 & 0 \\ 0 & - & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & + & 0 & 0 \\ + & 0 & 0 & - \\ - & + & 0 & + \\ 0 & 0 & 0 & - \end{bmatrix} = \mathcal{P}_4. \quad (2.21)$$

![Figure 2.6: Digraph $\mathcal{D}(\mathcal{P}_4)$](image)
Note that the upper left 2-by-2 submatrix of $J$ in (2.20) is singular. The following result takes into account this magnitude restriction.

**Proposition 7.** Matrix $J$ in (2.20) is semi-stable, and stable if and only if

$$E_{12}E_{21}(a_1b_1r_1^* - a_2b_2r_2^*)^2 > 0.$$  

**Proof.** Let $D = \text{diag} \left( 1, \sqrt{\frac{E_{12}}{E_{21}}}, \sqrt{\frac{r_1b_1}{a_1R_1^*}}, \sqrt{\frac{r_2b_2}{a_2R_2^*}E_{12}} \right)$. Then, by similarity, $J$ has the same eigenvalues as a matrix in the form

$$DJD^{-1} = A = \begin{bmatrix} B & C \\ -C & 0 \end{bmatrix},$$

where $B = B^T$, $\det B = 0$, $b_{ii} < 0$, $b_{12} = \sqrt{E_{12}E_{21}}$, and $C$ is a positive diagonal matrix with $c_{ii} = \sqrt{a_i b_i r_i R_i^*}$. Then

$$\frac{A + A^T}{2} = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix},$$

which implies by Bendixson’s theorem [25, Chapter III, p. 140] that $\lambda$, any eigenvalue of $A$, satisfies

$$\text{Re}(\lambda) \leq \max \left\{ \text{eigenvalues of } \frac{A + A^T}{2} \right\} \leq 0,$$

since one eigenvalue of $B$ is 0 and the other eigenvalue is negative. Note that $A$ has nonzero eigenvalues since $S = \text{sgn}(A)$ is sign nonsingular. Computing the second additive compound, denoted by $A^{[2]}$, see [1, 28], gives

$$\det(A^{[2]}) = E_{12}E_{21}(a_1b_1r_1^* - a_2b_2r_2^*)^2.$$

By [1, Lemma 2.5], if $\det(A^{[2]}) > 0$, then $A$ cannot have a pair of pure imaginary eigenvalues, and so the positive steady state is linearly stable. If $\det(A^{[2]}) = 0$, then a pair of pure imaginary eigenvalues occurs.

In [16, Appendix 1], the Routh–Hurwitz stability criteria were calculated to obtain the result of Proposition 7. Thus, even though the unrestricted sign pattern $S$ requires $\mathbb{H}_4$, this model does not exhibit Hopf bifurcation because of magnitude restrictions. For almost all parameter values (except when $a_1b_1r_1^* = a_2b_2r_2^*$), the system is linearly stable, showing the stabilizing effect of predator movement.
2.7.2 Two Patches with Generalized Growth Rates

If the growth rates of the prey species are generalized to some function $\phi_i(R_i)$, then the Jacobian matrix at the steady state becomes

$$J = \begin{bmatrix}
-E_{12} d_2 / d_1 & E_{12} & r_1 b_1 & 0 \\
E_{21} & -E_{21} d_1 / d_2 & 0 & r_2 b_2 \\
-a_1 R_1^* & 0 & \phi_1'(R_1^*) R_1^* & 0 \\
0 & -a_2 R_2^* & 0 & \phi_2'(R_2^*) R_2^*
\end{bmatrix}. \quad (2.22)$$

Assume that the growth rate functions $\phi_i$ are both negative density-dependent (i.e., $\phi_i' < 0$) at the steady state. Then,

$$S = \begin{bmatrix}
- & + & + & 0 \\
+ & - & 0 & + \\
- & 0 & - & 0 \\
0 & - & 0 & -
\end{bmatrix} \sim \begin{bmatrix}
- & + & 0 & 0 \\
- & - & + & 0 \\
0 & + & - & + \\
0 & 0 & - & -
\end{bmatrix} = T.$$

The sign pattern $T$ does not require $\mathbb{H}_4$ because it is not sign nonsingular. By the results of Section 2.7.1 and continuity, $T$ allows $(0, 4, 0, 0)$ and $(2, 2, 0, 0)$. Take a realization $A$ to be a realization of $P_4$ in (2.21) with eigenvalues $\lambda_{1,2} = a \pm ib$ where $a > 0$ and $\text{Re}(\lambda_3, \lambda_4) < 0$ (i.e., $\text{ri}(A) = (2, 2, 0, 0)$). Also take $T$ to be a realization of $T$ such that $T = A - aI$. Then $T$ has eigenvalues $\lambda_{1,2} = \pm ib, \lambda_3 - a$ and $\lambda_4 - a$ so $\text{ri}(T) = (0, 2, 0, 2)$, and $T$ allows $\mathbb{H}_4$. The above ideas give the following result, which slightly generalizes Theorem 4. A sign pattern $B = [\beta_{ij}]$ is a superpattern of $A = [\alpha_{ij}]$ if $\alpha_{ij} \neq 0$ whenever $\beta_{ij} \neq 0$.

**Theorem 8.** Let $A$ be an $n$-by-$n$ sign pattern with all diagonal entries nonpositive that allows $\mathbb{H}_n$. If $B$ is a superpattern of $A$ with all diagonal entries negative, then $B$ allows $\mathbb{H}_n$.

By an argument similar to that in Section 2.7.1, it can be shown in this more general case that the restricted sign pattern $\text{sgn}(J)$ in (2.22) is semi-stable.
2.7.3 Generalized \( n \)-Patch Model with Linear Prey Growth

Let us extend system (2.19) to the \( n \)-patch predator–prey system described by the following ODEs for \( i = 1, \ldots, n \):

\[
\frac{dP_i}{dt} = P_i(a_ib_iR_i - C_i - E_{ii}) + \sum_{j \neq i} E_{ij}P_j,
\]

\[
\frac{dR_i}{dt} = R_i(r_i - a_iP_i),
\]

(2.23)

where only the predators move between patches. At equilibrium, \( P_i^* = r_i/a_i \) and \( R_i^* = (C_i + E_{ii} - (1/P_i^*) \sum_{j \neq i} E_{ij}P_j^*)/a_ib_i \), which are assumed positive. The Jacobian matrix around this equilibrium, after a positive diagonal similarity, takes the form

\[
A = \begin{bmatrix} B & C \\ -C & 0 \end{bmatrix}.
\]

(2.24)

Let \( r_i/a_i = d_i \). Then, the matrices \( B = [b_{ij}] \) and \( C = [c_{ij}] \) have entries \( b_{ii} = -\frac{1}{d_i} \sum_{j \neq i} E_{ij}d_j, b_{ij} = E_{ij} \) for \( i \neq j \), \( c_{ii} = \sqrt{a_ib_ir_iR_i^*} \), \( c_{ij} = 0 \) for \( i \neq j \), and thus

\[
B = \begin{bmatrix}
-\frac{1}{d_1} \sum_{j \neq 1} E_{1j}d_j & E_{12} & \cdots & E_{1n} \\
E_{21} & -\frac{1}{d_2} \sum_{j \neq 2} E_{2j}d_j & \cdots & E_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
E_{n1} & E_{n2} & \cdots & -\frac{1}{d_n} \sum_{j \neq n} E_{nj}d_j
\end{bmatrix}.
\]

(2.25)

With regard to the columns of \( B \), note that \( d_1c_{11} + d_2c_{21} + \cdots + d_nc_{1n} = 0 \), thus \( B \) is singular with a positive right nullvector \([d_1, d_2, \ldots, d_n]^T\). The proof of Proposition 7 relied on the fact that \( B \) is 2-by-2, so it can be symmetrized by a positive diagonal similarity transformation. For larger dimensions, we use \( M \)-matrix theory; see, e.g., [2, Chapter 6] and [17, Section 2.5].

**Lemma 9.** With \( B \) given by (2.25), the matrix \(-B\) is a singular \( M \)-matrix.

**Proof.** Matrix \(-B\) has the \( Z \)-sign pattern (i.e., all off-diagonal entries are nonpositive) and is singular. By [2, Chapter 6, Lemma 4.1] \(-B\) is a singular \( M \)-matrix if and only if \(-B + \epsilon I\) is a nonsingular \( M \)-matrix for every \( \epsilon > 0 \). It is sufficient to show that, for some \( x > 0 \), we have \(( -B + \epsilon I ) x > 0 \). If \( x = (d_1, \ldots, d_n)^T \) then \(( -B + \epsilon I ) x = \epsilon x > 0 \).
(as \(-Bx = 0\), implying that \(-B + \epsilon I\) is a nonsingular \(M\)-matrix; therefore \(-B\) is a singular \(M\)-matrix.

**Proposition 10.** The 2\(n\)-by-2\(n\) matrix \(A\) in (2.24), with \(-B\) a singular \(M\)-matrix and \(C\) a diagonal matrix with all \(c_{ii} > 0\), is semi-stable.

**Proof.** Since \(-B\) is an \(n\)-by-\(n\) singular \(M\)-matrix it, follows from [2, p. 136] and continuity that there exists an \(n\)-by-\(n\) diagonal matrix \(D\) with all \(d_{ii} > 0\) so that \(B^T D + DB\) is negative semi-definite. Consider

\[
A^T(D \oplus D) + (D \oplus D)A = \begin{bmatrix}
B^T & -C \\
C & 0
\end{bmatrix}
\begin{bmatrix}
D & 0 \\
0 & D
\end{bmatrix}
+ \begin{bmatrix}
0 & D \\
D & 0
\end{bmatrix}
\begin{bmatrix}
B & C \\
-C & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
B^T D & -CD \\
CD & 0
\end{bmatrix}
+ \begin{bmatrix}
DB & DC \\
-DC & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
B^T D + DB & -CD + DC \\
CD - DC & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
B^T D + DB & 0 \\
0 & 0
\end{bmatrix},
\]

where the last equality uses the fact that \(C\) and \(D\) are diagonal matrices. Thus, \(A^T(D \oplus D) + (D \oplus D)A\) is negative semi-definite, and so \(A\) is semi-stable [17, Lemma 2.4.5].

The above result shows that the generalized \(n\)-patch predator–prey model with predators moving between patches cannot exhibit a Hopf bifurcation, thus extending the result of [16] to \(n\) patches. Li and Shuai [24, Section 6] considered a similar \(n\)-patch predator–prey model but with only the prey dispersing. They used a Lyapunov function to prove that if matrix \(E = [E_{ij}]\) in (2.23) is irreducible, then whenever a positive equilibrium exists, it is unique and globally stable in the positive cone.

### 2.7.4 Generalized \(n\)-Patch Sign Pattern

A logical question to follow from Section 2.7.3 is whether or not the sign pattern of matrix (2.24) allows \(\mathbb{H}_{2n}\). Consider the sign pattern
where $I = \text{diag}(+, +, \ldots, +)$.

**Theorem 11.** The sign pattern $\mathcal{A}$ in (2.26) is sign nonsingular and allows $\mathbb{H}_{2n}$.

**Proof.** By a diagonal similarity, as in (2.24), let

$$
\mathcal{A} = \begin{bmatrix}
- & + & + & 0 \\
& \ddots & & \\
& & - & 0 \\
0 & & & 0
\end{bmatrix} = \begin{bmatrix}
B & I \\
-I & 0
\end{bmatrix},
$$

(2.26)

Our goal now is to show that $\mathcal{A}$ allows $\mathbb{H}_{2n}$. First, we show that $\mathcal{A}$ allows refined inertia $(0, 2n, 0, 0)$. We do this by exhibiting a realization $A$ of $\mathcal{A}$ that has a properly signed nest (see [22]), i.e.,

$$
\text{sgn} (\det A[1, \ldots, k]) = (-1)^k \text{ for } k = 1, \ldots, 2n,
$$

where $A[1, \ldots, k]$ is the principal submatrix of $A$ on rows and columns $1, \ldots, k$. Take the positive entries of $B$ to be zero and the diagonal entries to have magnitude one, so $B = -I_n$ obviously has a properly signed nest. Consider the block matrix

$$
\begin{bmatrix}
-I_n & C_{nk} \\
-C_{kn} & 0
\end{bmatrix},
$$

(2.27)

where $C_{nk}$ is the leading $n$-by-$k$ submatrix of the diagonal matrix $C = \text{diag}(c_{ii})$ with $c_{ii} = 1$ $(1 \leq k \leq n)$. Using the Schur complement of $-I_n$ in the matrix (2.27), we
obtain (see [2, p. 292] for the definition and properties of the Schur complement)

\[
\det \left( \begin{bmatrix} -I_n & C_{nk} \\ -C_{kn} & 0 \end{bmatrix} \right) = \det(-I_n) \det(C_{kn}(-I_n)^{-1}C_{nk}) = (-1)^n \det(-I_k) = (-1)^{n+k}.
\]

By continuity, when the positive entries of \( B \) are small in magnitude, \( A \) has a properly signed nest, and thus by [22, Theorem 2.1] \( A \) is potentially stable, i.e., \( A \) allows refined inertia \((0, 2n, 0, 0)\).

Next, choose a realization of \( A \) such that \( B \) is symmetric and semi-stable with a simple zero eigenvalue, and \( C = I \). By Lemma 9, and by choosing \( E_{ij} = E_{ji} \) in (2.25), such a matrix \( B \) exists. Now consider \( \tilde{B} = B + \alpha I \) with \( \alpha > 0 \) and small in magnitude. Then, \( \tilde{B} \) has \( \alpha \) as a simple eigenvalue and all other eigenvalues are negative. Take a realization of \( A \) in the form

\[
\tilde{A} = \begin{bmatrix} \tilde{B} & I \\ -I & 0 \end{bmatrix}
\]

with \( \tilde{B}x = \alpha x \). Take \( \beta \neq 0 \) and consider an eigenvector equation in the following form:

\[
\begin{bmatrix} \tilde{B} & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} x \\ -\frac{x}{\beta} \end{bmatrix} = \beta \begin{bmatrix} x \\ -\frac{x}{\beta} \end{bmatrix}.
\]

It follows that \( \alpha x - x / \beta = \beta x \). Since \( x \neq 0 \) (as it is an eigenvector of \( \tilde{B} \) corresponding to \( \alpha \)), \( \beta^2 - \alpha \beta + 1 = 0 \), which implies \( \beta = (\alpha \pm \sqrt{\alpha^2 - 4}) / 2 \). Thus, for \( \alpha \) small, the refined inertia of \( \tilde{A} \) is \((2, 2n - 2, 0, 0)\). For \( \alpha = 0 \), when zero is a simple eigenvalue of \( \tilde{B} = B \), the refined inertia of \( \tilde{A} \) is \((0, 2n - 2, 0, 2)\). Thus, the sign pattern \( A \) allows \( \mathbb{H}_{2n} \).

The general \( n \)-patch predator–prey model thus has properties similar to the 2-patch model for both its sign pattern and the restricted Jacobian matrix, namely, the sign pattern allows \( \mathbb{H}_{2n} \), but the restricted sign pattern is sign semi-stable. In Holt [16, p. 392], the dynamics of this \( n \)-patch model were reported as unknown, even for the case of \( E_{ij} = E \) (i.e., predator movement is equiprobable in all directions).
2.8 Lotka–Volterra Systems With Patch Structures

In this section, we give two Lotka–Volterra systems in 6 dimensions that lead to sign nonsingular patterns that allow $H_6$.

2.8.1 Homogeneous Predator–Prey System with Diffusion

Consider the following Lotka–Volterra system [34, p. 182] with two competitive prey species $x_i, y_i$, with predator $z_i$ in patch $i$ for $i = 1, 2$, and movement between the patches, described by the following system of ODEs:

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1(1 - x_1 - \alpha y_1 - \epsilon z_1) + D_x(x_2 - x_1), \\
\frac{dy_1}{dt} &= y_1(1 - \beta x_1 - y_1 - \mu z_1) + D_y(y_2 - y_1), \\
\frac{dz_1}{dt} &= z_1(-1 + dx_1 + d\mu y_1) + D_z(z_2 - z_1), \\
\frac{dx_2}{dt} &= x_2(1 - x_2 - \alpha y_2 - \epsilon z_2) + D_x(x_1 - x_2), \\
\frac{dy_2}{dt} &= y_2(1 - \beta x_2 - y_2 - \mu z_2) + D_y(y_1 - y_2), \\
\frac{dz_2}{dt} &= z_2(-1 + dx_2 + d\mu y_2) + D_z(z_1 - z_2).
\end{align*}
\]

Here positive parameters $\alpha$ and $\beta$ represent the competition between the prey species, $\mu$ and $\epsilon$ represent the prey consumption by predator, $d$ is a conversion rate from prey to predator, and $D_x, D_y,$ and $D_z$ represent the rate of dispersion for each species between the two patches. We assume that there is a positive steady state $(x_1^*, y_1^*, z_1^*, x_2^*, y_2^*, z_2^*)$. Then, after some manipulation, the Jacobian matrix at this steady state has the following sign pattern

\[
S = \begin{bmatrix}
F & I \\
I & F
\end{bmatrix}
\]

(2.29)

where $F = \begin{bmatrix} - & - & - \\ - & - & - \\ + & + & - \end{bmatrix}$. The sign pattern $F$ is a superpattern of

\[
\begin{bmatrix}
- & - & 0 \\
- & - & - \\
0 & + & 0
\end{bmatrix},
\]

which allows $H_3$ [29, Appendix B, (8b)]. Since all of the diagonal entries of $F$ are negative, it allows $H_3$ by Theorem 8. By Theorem 5, it follows that $S$ allows $H_6$. This indicates the possibility of periodic solutions. Takeuchi [34] numerically shows
that system (2.28) demonstrates a stable limit cycle around the positive steady state.

### 2.8.2 Three-Competitor System

We now consider a three-species competition model [34, p. 191] with each species competing in a common patch $X$ and also having its own refuge patch $Y_i$, for species $i = 1, 2, 3$. Let $x_i$ be the population size competing in patch $X$ and $y_i$ be the population size in refuge $Y_i$. The model is described by the following system of ODEs:

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1 r_1 (1 - x_1 - \alpha_2 x_2 - \beta_3 x_3) + \epsilon_1 (y_1 - x_1), \\
\frac{dx_2}{dt} &= x_2 r_2 (1 - \beta_1 x_1 - x_2 - \alpha_3 x_3) + \epsilon_2 (y_2 - x_2), \\
\frac{dx_3}{dt} &= x_3 r_3 (1 - \alpha_1 x_1 - \beta_2 x_2 - x_3) + \epsilon_3 (y_3 - x_3), \\
\frac{dy_1}{dt} &= y_1 R_1 (1 - y_1) + \epsilon_1 (x_1 - y_1), \\
\frac{dy_2}{dt} &= y_2 R_2 (1 - y_2) + \epsilon_2 (x_2 - y_2), \\
\frac{dy_3}{dt} &= y_3 R_3 (1 - y_3) + \epsilon_3 (x_3 - y_3).
\end{align*}
\]  

(2.30)

Here positive parameters $\alpha_i, \beta_i$ are competition coefficients, and $\epsilon_i$ represents the dispersal rate for species $i$ between patch $X$ and patch $Y_i$. Also, $R_i$ and $r_i$ are intrinsic growth rates of species $i$ in its refuge patch and competition patch, respectively. We assume that there exists a positive steady state $x_i^*, y_i^*$. Then the Jacobian matrix around this positive steady state has the following sign pattern

\[
S = \begin{bmatrix} N & I \\ I & -I \end{bmatrix}
\]  

(2.31)

where $N$ is the 3-by-3 sign pattern with every entry negative. Sign pattern $N$ allows $\mathbb{H}_3$ by Theorem 6. Thus, by Theorem 5, $S$ allows $\mathbb{H}_6$, since $S$ is a superpattern of $N \oplus -I$. Takeuchi [34, Figure 6.4.1] shows numerically that system (2.30) has periodic solutions for specific parameter values.

This competitor system can be generalized to an $n$-species competitor model with each pattern $N$ and $I$ being $n$-by-$n$. By a similar argument, the resulting sign pattern $S$ allows $\mathbb{H}_{2n}$.
2.9 Dispersion in a Host-Parasitoid Systems

Sign patterns arising from dynamical systems can motivate results about sign patterns that allow $\mathbb{H}_n$. An example arises from a host–parasitoid system with a Type II functional response (see [35, Equation (5)]). The Jacobian matrix for this system at the positive equilibrium has the sign pattern

$$S = \begin{bmatrix} + & - & 0 \\ + & - & + \\ 0 & + & - \end{bmatrix}.$$ 

Without loss of generality, take a realization $A$ of $S$ given by

$$A = \begin{bmatrix} a & -1 & 0 \\ b & -c & 1 \\ 0 & d & -e \end{bmatrix} \quad (2.32)$$

with $a, \ldots, e > 0$. If $ad + be = ace$, then $A$ is singular, so $S$ does not require $\mathbb{H}_3$. Numerically, $A$ allows $\mathbb{H}_3$ (see Table 2.4), thus a Hopf bifurcation occurs for some parameter values, as found by Weisser et al. [35, Figure 2]. Note that, although all 3-by-3 sign patterns that require $\mathbb{H}_3$ are known [14], there does not exist a complete list of 3-by-3 sign patterns that allow $\mathbb{H}_3$ (except for tree sign patterns, see (8), (11), and (12) in Appendix B of [29]).
Chapter 3

Refined Inertias Related to the Petersen Graph

We are interested in studying the refined inertias of patterns corresponding to strongly connected digraphs, and in particular we focus on digraphs arising from the well-known Petersen graph; see, for example, [6]. Note that the content of this chapter has been submitted for publication to Linear Algebra and its Applications.

The Petersen graph is 3-regular, has 10 vertices, and 15 edges. Using Sage it can be shown that there are 18 non-isomorphic strongly connected orientations of the Petersen graph, denoted $P_1, ..., P_{18}$. For $i = 1, ..., 18$, we label the vertices of $P_i$ with 1, ..., 10 and label the 15 arcs with real-valued variables to give weighted digraphs $D(A_i)$ having digraph $P_i$. Each digraph $D(A_i)$ determines a unique matrix $A_i$ (with exactly 15 nonzero entries). For eigenvalue determination, in matrix $A_i$ the entries corresponding to the arcs of a directed spanning tree may be set to one (see, for example, [13, Theorem 2.3]); i.e., there is a diagonal similarity that reduces the number of variables in $A_i$ to six while keeping the same refined inertia. Hence, each matrix $A_i$ with weighted digraph $D(A_i)$ can be specified in terms of nonzero parameters $a, b, c, d, e, f$. See Appendix A.2 for these matrices $A_i$ and their weighted digraphs $D(A_i)$ with their directed spanning trees. If each parameter can be either positive or negative, then each $A_i$ corresponds to one of $2^6 = 64$ realizations. Thus, there exist a total of $64 \times 18 = 1152$ different sign patterns, each one isomorphic to a strongly connected orientation of the Petersen graph. If each parameter is a nonzero real number, then $A_i$ corresponds to a realization of a zero-nonzero pattern that has a digraph isomorphic to a strongly connected orientation of the Petersen graph.
(18 different zero-nonzero patterns). In Section 3.2 we take $a, ..., f$ positive and in Section 3.3 we consider zero-nonzero patterns taking $a, ..., f$ nonzero. In Section 3.4 we illustrate a sign pattern with a particular refined inertia set, taking $a, b, c, d, f$ positive and $e$ negative.

### 3.1 Preliminary Theorems

For use in our analysis, we introduce the elementary symmetric functions (see [18, Definition 1.2.9])

$$S_\ell(a_1, ..., a_m) = \sum_{1 \leq i_1 < i_2 < ... < i_\ell \leq m} a_{i_1}a_{i_2}a_{i_3}...a_{i_\ell}$$

E.g., $S_2(a_1, ..., a_4) = a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4$. Let $S_i(A)$ be the coefficient of the $x^{n-i}$ term of the characteristic polynomial of an $n$-by-$n$ matrix $A$; then

$$S_i(A) = (-1)^iS_i(\lambda_1, ..., \lambda_n),$$

where $\lambda_1, ..., \lambda_n$ are the eigenvalues $A$ (see [18, Theorem 1.2.12]). Note that if the real parts of $a_1, ..., a_m > 0$, then

$$S_{k-1}(a_1, ..., a_m)S_1(a_1, ..., a_m) > S_k(a_1, ..., a_m), \quad (3.1)$$

where $k \geq 2$.

Several of the coefficients in all of the characteristic polynomials $C_i$ of the matrices $A_i$ are zero; see Appendix A.1. The following three theorems eliminate the possibility of certain refined inertias for arbitrary matrices having some zero coefficients in their characteristic polynomials, i.e., certain elementary symmetric functions are equal to zero.

**Theorem 12.** If $A$ is a matrix with $S_1(A) = \text{Tr}(A) = 0$, then $\text{ri}(A)$ cannot equal $(0, n_-, n_z, 2n_p)$ or $(n_+, 0, n_z, 2n_p)$, where $n_+$ and $n_-$ are positive.

**Proof.** Let $A$ be a matrix with $S_1(A) = \text{Tr}(A) = 0$, hence the sum of the eigenvalues of $A$ is equal to zero. Since pure imaginary complex conjugate eigenvalues have zero sum, it follows that if $A$ has eigenvalues with only positive or negative real parts, then $\text{Tr}(A) \neq 0$. Therefore, $\text{ri}(A)$ cannot equal $(0, n_-, n_z, 2n_p)$ or $(n_+, 0, n_z, 2n_p)$ with $n_-, n_+$ positive. \(\square\)
Theorem 13. Let $A$ be an $n$-by-$n$ matrix with $n \geq 3$. If $A$ has elementary symmetric functions $S_1(A) = S_3(A) = 0$, then $\text{ri}(A)$ cannot equal $(n_+, 1, n_z, 2n_p)$ with $n_+ \geq 2$, or $(1, n_-, n_z, 2n_p)$ with $n_- \geq 2$.

Proof. Let $A$ be an $n$-by-$n$ matrix with exactly one negative eigenvalue, $k \geq 2$ eigenvalues with positive real parts, $2m \geq 0$ nonzero pure imaginary complex conjugate eigenvalues, and $n - 2m - 1 - k$ zero eigenvalues. The characteristic polynomial of $A$ factors into the form

$$x^{n-2m-1-k}(x + a) \prod_{i=1}^{k} (x - b_i) \prod_{j=1}^{m} (x^2 + \phi_j)$$

for $a, \phi_1, \ldots, \phi_m > 0$ and the real parts of $b_1, \ldots, b_k > 0$. Since the elementary symmetric function $S_1(A) = 0$, it follows that

$$-S_1(A) = a - S_1(b_1, \ldots, b_k) = 0.$$

First assume $n \geq 4$ and $k \geq 3$. Then the function $S_3(A)$ becomes

$$-S_3(A) = S_1(\phi_1, ..., \phi_m) [a - S_1(b_1, ..., b_k)] + aS_2(b_1, ..., b_k) - S_3(b_1, ..., b_k)$$

$$= aS_2(b_1, ..., b_k) - S_3(b_1, ..., b_k) = S_1(b_1, ..., b_k)S_2(b_1, ..., b_k) - S_3(b_1, ..., b_k),$$

which is positive by (3.1), a contradiction.

If $n \geq 3$ and $k = 2$, then

$$-S_3(A) = S_1(\phi_1, ..., \phi_m) [a - S_1(b_1, b_2)] + aS_2(b_1, b_2) = ab_1b_2 > 0,$$

again a contradiction.

Therefore, if $S_1(A) = S_3(A) = 0$, then $\text{ri}(A)$ cannot equal $(k, 1, n-2m-1-k, 2m)$ for $k \geq 2$. A similar argument holds for $A$ having exactly one positive eigenvalue, and it follows that $\text{ri}(A)$ cannot equal $(1, k, n-2m-1-k, 2m)$ for $k \geq 2$.

\(\square\)

Theorem 14. Let $A$ be an $n$-by-$n$ matrix with $n \geq 4$. If $A$ has elementary symmetric functions $S_2(A) = S_4(A) = 0$, then $\text{ri}(A)$ cannot equal $(1, 1, n_z, 2n_p)$ with $n_p \geq 1$.

Proof. Let $A$ be an $n$-by-$n$ matrix with $n \geq 4$. Let $p(x)$ be the characteristic polynomial of $A$ with elementary symmetric function $S_2(A) = 0$. Assume that $A$ has exactly
one negative eigenvalue, exactly one positive eigenvalue, and \( m \geq 1 \) pairs of nonzero pure imaginary complex conjugate eigenvalues. Then \( p(x) \) can be factored into the form

\[
x^{n-2m-2}(x + a)(x - b) \prod_{j=1}^{m}(x^2 + \phi_j)
\]

for \( a, b, \phi_1, ..., \phi_m > 0 \), and thus

\[
S_2(A) = S_1(\phi_1, ..., \phi_m) - ab, \\
S_4(A) = S_2(\phi_1, ..., \phi_m) - abS_1(\phi_1, ..., \phi_m).
\]

If \( S_2(A) = 0 \), then \( S_1(\phi_1, ..., \phi_m) = ab \). Since \( S_1(\phi_1, ..., \phi_m)^2 > S_2(\phi_1, ..., \phi_m) \) by (3.1), it follows that \( S_4(A) < 0 \), a contradiction. Hence, if \( S_2(A) = S_4(A) = 0 \), then \( ri(A) \neq (1, 1, n-2m-2, 2m) \) for \( m \geq 1 \). \[\Box\]

### 3.2 Nonnegative Patterns

For \( i = 1, ..., 18 \), let \( \mathcal{A}_i \) denote the nonnegative pattern having a matrix realization \( A_i \) in Appendix A.2 with all parameters \( a, ..., f \) positive. Thus \( \mathcal{D}(A_i) \) is a strongly connected orientation of the Petersen graph having each arc weighted with a + sign. We show that these nonnegative patterns have fixed refined inertias.

**Theorem 15.** Let \( \mathcal{A} \) be a nonnegative pattern with \( \mathcal{D}(\mathcal{A}) \) isomorphic to a strongly connected orientation of the Petersen graph. Then \( \mathcal{A} \) has unique refined inertia. Specifically, the refined inertias are as follows:

i. \((3, 3, 4, 0)\) if \( \mathcal{A} \) requires exactly four zero eigenvalues,
ii. \((5, 3, 2, 0)\) if \( \mathcal{A} \) requires exactly two zero eigenvalues,
iii. \((5, 4, 1, 0)\) if \( \mathcal{A} \) requires a single zero eigenvalue,
iv. \((6, 4, 0, 0)\) if \( \mathcal{A} \) is sign nonsingular.

**Proof.** Let \( \mathcal{A} \) be a sign pattern with entries from the set \{0, +\}, where \( \mathcal{D}(\mathcal{A}) \) has digraph isomorphic to a strongly connected orientation of the Petersen graph. If \( A \) is a realization of \( \mathcal{A} \), then the characteristic polynomial of \( A \) corresponds to one of the \( C_i \) given in Appendix A.1, with all of the parameters in \( C_i \) positive. We first show that the \( C_i \) do not allow nonzero pure imaginary zeros. If \( x = ik \) for \( k > 0 \) is a zero of
any one of the polynomials $C_1$ to $C_4$, then the imaginary part gives $-\alpha ik^5 = 0$ with $\alpha > 0$, a contradiction. Similarly, for the remaining polynomials $C_5$ to $C_{18}$, if $x = ik$ with $k > 0$, then either

$$-\alpha ik^5 - D i k = 0 \quad \text{or} \quad -\alpha ik^5 - \zeta ik = 0$$

with $\alpha, D, \zeta > 0$, giving a contradiction.

Since these polynomials do not allow nonzero pure imaginary zeros and the polynomials have 0 as a zero of fixed multiplicity, it is impossible to transition from one refined inertia to another by continuity (i.e., nonzero eigenvalues of $A$ are fixed in their respective regions, the left or right half of the complex plane). With all parameter values $a, ..., f$ set to one, the refined inertias for each matrix realization $A_i$ of $\mathcal{A}_i$ are given in Table 3.1.

Table 3.1: Characteristic polynomials of $C_i$ from Appendix A.1 with all parameters $a, b, c, d, e, f$ set to one, and their corresponding refined inertias

<table>
<thead>
<tr>
<th>$i$</th>
<th>$C_i$</th>
<th>$\text{ri}(A_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x^{10} - 4x^5 - 2x^4$</td>
<td>(3, 3, 4, 0)</td>
</tr>
<tr>
<td>2</td>
<td>$x^{10} - 4x^5 - 2x^4$</td>
<td>(3, 3, 4, 0)</td>
</tr>
<tr>
<td>3</td>
<td>$x^{10} - 3x^5 - 2x^4 - x^2$</td>
<td>(5, 3, 2, 0)</td>
</tr>
<tr>
<td>4</td>
<td>$x^{10} - 3x^5 - 2x^4 - x^2$</td>
<td>(5, 3, 2, 0)</td>
</tr>
<tr>
<td>5</td>
<td>$x^{10} - 2x^5 - 2x^4 - 2x^2 - x$</td>
<td>(5, 4, 1, 0)</td>
</tr>
<tr>
<td>6</td>
<td>$x^{10} - 3x^5 - 2x^4 - 2x^2 - x$</td>
<td>(5, 4, 1, 0)</td>
</tr>
<tr>
<td>7</td>
<td>$x^{10} - 2x^5 - 2x^4 - x^2 - x$</td>
<td>(5, 4, 1, 0)</td>
</tr>
<tr>
<td>8</td>
<td>$x^{10} - 3x^5 - 2x^4 - x^2 - x$</td>
<td>(5, 4, 1, 0)</td>
</tr>
<tr>
<td>9</td>
<td>$x^{10} - 3x^5 - x^4 - 2x^2 - x$</td>
<td>(5, 4, 1, 0)</td>
</tr>
<tr>
<td>10</td>
<td>$x^{10} - 3x^5 - x^4 - x^2 - x$</td>
<td>(5, 4, 1, 0)</td>
</tr>
<tr>
<td>11</td>
<td>$x^{10} - 2x^5 - 3x^4 - 2x^2 - 2x$</td>
<td>(5, 4, 1, 0)</td>
</tr>
<tr>
<td>12</td>
<td>$x^{10} - 2x^5 - 3x^4 - 2x^2 - 2x$</td>
<td>(5, 4, 1, 0)</td>
</tr>
<tr>
<td>13</td>
<td>$x^{10} - 3x^5 - x^4 - x^2 - x - 1$</td>
<td>(6, 4, 0, 0)</td>
</tr>
<tr>
<td>14</td>
<td>$x^{10} - 3x^5 - x^4 - x^2 - x - 1$</td>
<td>(6, 4, 0, 0)</td>
</tr>
<tr>
<td>15</td>
<td>$x^{10} - 4x^5 - x^4 - 2x^2 - 2x + 1$</td>
<td>(6, 4, 0, 0)</td>
</tr>
<tr>
<td>16</td>
<td>$x^{10} - 4x^5 - x^4 - 2x^2 - 2x + 1$</td>
<td>(6, 4, 0, 0)</td>
</tr>
<tr>
<td>17</td>
<td>$x^{10} - 3x^5 - 2x^4 - 2x^2 + x + 1$</td>
<td>(6, 4, 0, 0)</td>
</tr>
<tr>
<td>18</td>
<td>$x^{10} - 3x^5 - 2x^4 - 2x^2 - x + 1$</td>
<td>(6, 4, 0, 0)</td>
</tr>
</tbody>
</table>
3.3 Zero-Nonzero Patterns

For \( i = 1, \ldots, 18 \), we now consider zero-nonzero patterns \( A_i \) with matrix realizations \( A_i \) having the weighted digraphs \( D(A_i) \) in Appendix A.2. Thus each of the parameters \( a, \ldots, f \) in \( A_i \) can have any nonzero value. The refined inertia \((n_-, n_+, n_z, 2n_p)\) is called the reversal of the refined inertia \((n_+, n_-, n_z, 2n_p)\), and the following result holds; see, for example, [8].

**Theorem 16.** A zero-nonzero pattern allows refined inertia \((n_+, n_-, n_z, 2n_p)\) if and only if it allows its reversal \((n_-, n_+, n_z, 2n_p)\).

**Theorem 17.** If \( A \) is a zero-nonzero pattern with weighted digraph \( D(A) \) isomorphic to \( P_1 \) or \( P_2 \), then

\[
\text{ri}(A) = \{(0, 0, 10, 0), (2, 3, 5, 0), (3, 2, 5, 0), (4, 2, 4, 0), (2, 4, 4, 0), (3, 3, 4, 0), (2, 2, 4, 2)\}.
\]

**Proof.** Let \( A \) be a matrix realization of either \( A_1 \) or \( A_2 \). For nonzero values of \( a, b, c, d, e, f \) in \( A_1 \) or \( A_2 \) (see Appendix A.2), the characteristic polynomial is of the form

\[
p(x) = x^4(x^6 - \alpha x - \beta),
\]

with \( \alpha, \beta \in \mathbb{R} \). If \( \alpha = \beta = 0 \), then \( p(x) = x^{10} \) and \( \text{ri}(A) = (0, 0, 10, 0) \). If \( \beta = 0 \), then \( p(x) = x^5(x^5 - \alpha) \). If in addition \( \alpha > 0 \), then the solutions to \( x^5 = \alpha \) are evenly distributed around the unit circle (with one positive root); therefore, \( \text{ri}(A) = (3, 2, 5, 0) \). By Theorem 16, \( A \) also allows \((2, 3, 5, 0)\). If \( \alpha = 0 \), then \( p(x) = x^4(x^6 - \beta) \). If in addition \( \beta > 0 \), then \( \text{ri}(A) = (3, 3, 4, 0) \), whereas if \( \beta < 0 \), then \( \text{ri}(A) = (2, 2, 4, 2) \).

Suppose \( \alpha \neq 0 \) and \( \beta \neq 0 \). Since \( A \) has four zero eigenvalues and \( S_i(A) = 0 \) for \( i = 1, \ldots, 4 \), by Theorems 12, 13, and 14 the only other possible refined inertias of \( A \) are \((4, 2, 4, 0), (2, 4, 4, 0)\), and \((0, 0, 4, 6)\). If the refined inertia \((0, 0, 4, 6)\) can occur, then the factored polynomial \( x^4(x^2 + \phi_1)(x^2 + \phi_2)(x^2 + \phi_3) \) has elementary symmetric function \( S_2(A) = \phi_1 + \phi_2 + \phi_3 > 0 \), a contradiction. Therefore, refined inertia \((0, 0, 4, 6)\) cannot occur. The remaining refined inertias, \((4, 2, 4, 0)\) and \((2, 4, 4, 0)\), are found numerically, with parameter values demonstrating the allowed refined inertias of \( A_1 \) and \( A_2 \) given in Table 3.2.

**Theorem 18.** If \( A \) is a zero-nonzero pattern with weighted digraph \( D(A) \) isomorphic
Table 3.2: Examples of parameter values for all possible refined inertias of zero-nonzero patterns $A_1$ and $A_2$. Realization $A_1$ of $A_1$ has parameters $a, c, d = 1$, while realization $A_2$ of $A_2$ has parameters $a, c, e = 1$

<table>
<thead>
<tr>
<th></th>
<th>$A_1$</th>
<th></th>
<th>$A_2$</th>
<th></th>
<th></th>
<th></th>
<th>ri($A_1$) and ri($A_2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$b$</td>
<td>$e$</td>
<td>$f$</td>
<td>$b$</td>
<td>$d$</td>
<td>$f$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>-2</td>
<td>2</td>
<td>-2</td>
<td>1</td>
<td>-1</td>
<td>-2</td>
<td>-1</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>-2</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>-2</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>2</td>
<td>1</td>
<td>-2</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
<td>2</td>
<td>1</td>
<td>-2</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>

To $P_3$ or $P_4$, then

$$\text{ri}(A) = \{(3, 5, 2, 0), (5, 3, 2, 0), (4, 4, 2, 0), (3, 3, 2, 2), (2, 2, 2, 4)\}.$$  

Proof. Let $A$ be a matrix realization of either $A_3$ or $A_4$. For nonzero values of $a, b, c, d, e, f$ in $A_3$ or $A_4$, the characteristic polynomials $C_3$ and $C_4$ are of the form

$$x^{10} - \alpha x^5 - \beta x^4 - C x^2,$$  

where $\alpha, \beta \in \mathbb{R}$ and $C \neq 0$. Since (3.3) has a factor $x^2$, by Theorems 12, 13, and 14 the only possible refined inertias of $A$ are $(6, 2, 2, 0), (5, 3, 2, 0), (4, 2, 2, 2), (4, 4, 2, 0), (3, 3, 2, 2), (2, 2, 2, 4), (0, 0, 2, 8)$, and their reversals.

If $A$ has a nonzero pure imaginary eigenvalue, then $\alpha = 0$, and (3.3) contains of only even powers of $x$. It follows that if $\lambda$ is an eigenvalue, then $-\lambda$ is also an eigenvalue; therefore, $(4, 2, 2, 2)$ and its reversal cannot occur as $n_+ \neq n_-$. If $\text{ri}(A) = (6, 2, 2, 0)$, then the characteristic polynomial of $A$ can be factored into the form

$$x^2(x + a_1)(x + a_2)\prod_{i=1}^{6}(x - b_i),$$
where the real parts of \(a_1, a_2, b_1, \ldots, b_6 > 0\). Then
\[
-S_1(A) = S_1(a_1, a_2) - S_1(b_1, \ldots, b_6) = 0,
\]
\[
S_2(A) = S_2(b_1, \ldots, b_6) - S_1(a_1, a_2)S_1(b_1, \ldots, b_6) + S_2(a_1, a_2) = 0,
\]
\[
-S_3(A) = -S_1(b_1, \ldots, b_6)S_2(a_1, a_2) + S_2(b_1, \ldots, b_6)S_1(a_1, a_2) - S_3(b_1, \ldots, b_6) = 0.
\]
A contradiction is obtained by substituting the above three conditions into
\[
S_4(A) = S_2(b_1, \ldots, b_6)S_2(a_1, a_2) - S_3(b_1, \ldots, b_6)S_1(a_1, a_2) + S_4(b_1, \ldots, b_6),
\]
giving
\[
S_4(A) = S_2(b_1, \ldots, b_6)S_2(a_1, a_2) + S_4(b_1, \ldots, b_6) - S_1(b_1, \ldots, b_6) - (S_2(b_1, \ldots, b_6)S_1(b_1, \ldots, b_6) - S_1(b_1, \ldots, b_6) \left[ S_1(b_1, \ldots, b_6)^2 - S_2(b_1, \ldots, b_6) \right]),
\]
\[
= S_2(b_1, \ldots, b_6)S_2(a_1, a_2) + S_1(b_1, \ldots, b_6)^2 \left[ S_1(b_1, \ldots, b_6)^2 - 2S_2(b_1, \ldots, b_6) \right] + S_4(b_1, \ldots, b_6),
\]
\[
= S_2(b_1, \ldots, b_6)S_2(a_1, a_2) + S_1(b_1, \ldots, b_6)^2(b_1^2 + \ldots + b_6^2) + S_4(b_1, \ldots, b_6) > 0. \quad (3.4)
\]
This contradicts the fact that \(S_4(A) = 0\) in (3.3); hence, \(A\) does not allow refined inertia \((6, 2, 2, 0)\) or its reversal.

If the refined inertia \((0, 0, 2, 8)\) occurs, then the characteristic polynomial
\[
x^2 \prod_{i=1}^{4} (x^2 + \phi_i)
\]
with all \(\phi_i > 0\) has elementary symmetric function \(S_2(A) = S_1(\phi_1, \ldots, \phi_4) > 0\), a contradiction since \(S_2(A) = 0\). Hence, refined inertia \((0, 0, 2, 8)\) cannot occur. Parameter values that demonstrate the allowed refined inertias of \(A_3\) and \(A_4\) are given in Table 3.3.

\[\square\]

**Theorem 19.** If \(A\) is a zero-nonzero pattern with weighted digraph \(D(A)\) isomorphic to one of the digraphs \(P_5, \ldots, P_{10}\), then
\[
\text{ri}(A) = \{(6, 3, 1, 0), (3, 6, 1, 0), (4, 3, 1, 2), (3, 4, 1, 2), (5, 4, 1, 0), (4, 5, 1, 0)\}.
\]

**Proof.** Let \(A\) be a matrix realization of \(A_i\) for \(i = 5, \ldots, 8\). For nonzero values of
Table 3.3: Examples of parameter values for all possible refined inertias of zero-nonzero patterns \( A_3 \) and \( A_4 \). Realization \( A_3 \) of \( A_3 \) has parameter values \( e = 2, b = 1/2 \) while realization \( A_4 \) of \( A_4 \) has parameter values \( c = 1, e = 3 \)

<table>
<thead>
<tr>
<th>( A_3 )</th>
<th>( A_4 )</th>
<th>( \text{ri}(A_3) ) and ( \text{ri}(A_4) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{-2}{3} )</td>
<td>( \frac{1}{3} )</td>
<td>( -3 )</td>
</tr>
<tr>
<td>( -2 )</td>
<td>( 1 )</td>
<td>( -1 )</td>
</tr>
<tr>
<td>( \frac{2}{3} )</td>
<td>( 1 )</td>
<td>( 3 )</td>
</tr>
<tr>
<td>( -1 )</td>
<td>( 1 )</td>
<td>( -3 )</td>
</tr>
<tr>
<td>( 1 )</td>
<td>( 2 )</td>
<td>( 2 )</td>
</tr>
</tbody>
</table>

\( a, b, c, d, e, f \) in \( A_5, \ldots, A_8 \), the characteristic polynomials are of the form

\[
p(x) = x^{10} - \alpha x^5 - \beta x^4 - C x^2 - Dx,
\]

with \( \alpha, \beta \in \mathbb{R} \) and \( C, D \neq 0 \). Since (3.5) has a factor of \( x \), by Theorems 12, 13, and 14 the only possible refined inertias of \( A \) are \((7, 2, 1, 0), (6, 3, 1, 0), (5, 4, 1, 0), (5, 2, 1, 2), (4, 3, 1, 2), (3, 2, 1, 4)\), and their reversals.

If (3.5) has a complex zero \( x = i \sqrt{\phi} \) with \( \sqrt{\phi} > 0 \), then \( p(i \sqrt{\phi}) = 0 \) implies that \( \alpha = -D/\phi^2 \) and \( C = \phi^4 + \beta \phi \). Substituting these conditions into (3.5) gives

\[
p(x) = x \left( x^9 + \frac{D}{\phi^2} x^4 - \beta x^3 - (\phi^3 + \beta) \phi x - D \right),
\]

which factors into the form

\[
p(x) = x (x^2 + \phi) \left( x^7 - \phi x^5 + \phi^2 x^3 + \frac{D}{\phi^2} x^2 - (\phi^3 + \beta) x - \frac{D}{\phi} \right).
\]

Since \( D \neq 0 \) and \( \phi > 0 \), if \( q(x) \) has a complex zero \( x = it \) with \( t > 0 \), then

\[
- \frac{D}{\phi^2} t^2 - \frac{D}{\phi} = 0,
\]
which is a contradiction. Thus, the refined inertia corresponding to the zeros of \( q(x) \) is of the form \((\hat{n}_+, \hat{n}_-, 0, 0)\) for some fixed values of \( \hat{n}_+, \hat{n}_- \) with \( \hat{n}_+ + \hat{n}_- = 7 \) for all values of \( \phi > 0, D \neq 0 \) and \( \beta \in \mathbb{R} \). It can be shown numerically that if \( D > 0 \), then \((\hat{n}_+, \hat{n}_-) = (3, 4)\), and if \( D < 0 \), then \((\hat{n}_+, \hat{n}_-) = (4, 3)\). Consequently, if \( A \) has fixed eigenvalues 0 and \( \pm i\sqrt{\phi} \) for any \( \phi > 0 \), \( D \neq 0 \) and \( \beta \in \mathbb{R} \). It can be shown numerically that if \( D > 0 \), then \((\hat{n}_+, \hat{n}_-) = (3, 4)\), and if \( D < 0 \), then \((\hat{n}_+, \hat{n}_-) = (4, 3)\). Consequently, if \( A \) has fixed eigenvalues 0 and \( \pm i\sqrt{\phi} \) for any \( \phi > 0 \), then since \( D \neq 0 \), the other seven eigenvalues are such that \( \text{ri}(A) = (4, 3, 1, 2) \) or \( (3, 4, 1, 2) \) as \( \beta \) and \( D \) vary. In particular, the refined inertias \((5, 2, 1, 2)\), \((3, 2, 1, 4)\) cannot occur.

If \( \text{ri}(A) = (7, 2, 1, 0) \), then the characteristic polynomial of \( A \) can be factored into the form

\[
x(x + a_1)(x + a_2) \prod_{i=1}^{7} (x - b_i)
\]

where the real parts of \( a_1, a_2, b_1, ..., b_7 > 0 \). The same argument as for refined inertia \((6, 2, 2, 0)\) in Theorem 18 on replacing \( b_1, ..., b_6 \) by \( b_1, ..., b_7 \) gives a contradiction, namely \( S_4(A) > 0 \) as in (3.4). Thus refined inertia \((7, 2, 1, 0)\) and its reversal are impossible.

If \( A \) is a realization of \( A_9 \) or \( A_{10} \), then the characteristic polynomials of \( A \) are of the form

\[
x^{10} - \alpha x^5 - Bx^4 - Cx^2 - Dx,
\]

with \( \alpha \in \mathbb{R} \) and \( B, C, D \neq 0 \). The above analysis holds for this polynomial as the nonzero restriction on the coefficient \( -\beta = S_6(A) \) of \( x^4 \) does not affect the analysis. Parameter values are given in Table 3.4 showing the allowed refined inertias of \( A_5 \) to \( A_{10} \).

Table 3.4: Examples of values for the coefficients of the characteristic polynomials \( C_5 \) to \( C_{10} \) that can be obtained by choosing nonzero values of \( a, b, c, d, e, \) and \( f \) to give all possible refined inertias of \( A_5 \) to \( A_{10} \). For brevity we do not list the parameter values \( a, ..., f \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \beta, B )</th>
<th>( C )</th>
<th>( D )</th>
<th>( \text{ri}(A_5) ) to ( \text{ri}(A_{10}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>-1</td>
<td>(6, 3, 1, 0)</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>(3, 6, 1, 0)</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>(5, 4, 1, 0)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>(4, 5, 1, 0)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td>(4, 3, 1, 2)</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>(3, 4, 1, 2)</td>
</tr>
</tbody>
</table>
Theorem 20. If $A$ is a zero-nonzero pattern with weighted digraph $D(A)$ isomorphic to digraph $P_{11}$ or $P_{12}$, then

$$ri(A) = ri(A_1) \cup ri(A_3) \cup ri(A_5) - \{(0, 0, 10, 0), (2, 2, 4, 2)\}.$$ 

Proof. Let $A$ be a matrix realization of either $A_{11}$ or $A_{12}$. The characteristic polynomial of $A$ is of the form

$$x^{10} - \alpha x^5 - \beta x^4 - \gamma x^2 - \zeta x,$$

where $\alpha, \beta, \gamma, \zeta \in \mathbb{R}$. Ignoring parameter restrictions (i.e., assuming $\alpha, \beta, \gamma, \zeta$ can attain any values in $\mathbb{R}$ independently), if $\gamma = \zeta = 0$, then (3.6) is the form of (3.2), and thus $ri(A_1) \subseteq ri(A)$. If $\gamma \neq 0$ and $\zeta = 0$, then (3.6) is of the form (3.3), and thus $ri(A_3) \subseteq ri(A)$. If $\gamma \neq 0$ and $\zeta \neq 0$, then (3.6) is of the form (3.5), and thus $ri(A_5) \subseteq ri(A)$. This latter inclusion also holds if $\gamma = 0$ and $\zeta \neq 0$ since the proof of Theorem 8 does not require the coefficient $-C$ of $x^2$ to be nonzero. Hence,

$$ri(A) \subseteq ri(A_1) \cup ri(A_3) \cup ri(A_5).$$

However, $A$ has a characteristic polynomial with restricted coefficients. Specifically, if $\alpha = \beta = \gamma = 0$ in $C_{11}$, then

$$bf = -ad, \quad fcd = -e - f, \quad dbf + afe = 0.$$ 

Thus since $\zeta = df + fce$, the above three conditions give

$$\zeta ad = ad(df + fce) = -(bf + afe)d = afe + ae(-e - f) = -ae^2 \neq 0,$$

a contradiction. A similar contradiction occurs for $C_{12}$. Hence, no realization $A$ of either $A_{11}$ or $A_{12}$ can have a characteristic polynomial with $\alpha = \beta = \gamma = \zeta = 0$, which eliminates refined inertia $(0, 0, 10, 0)$. If $A$ is a realization of $A_{11}$ with $ri(A) = (2, 2, 4, 2)$, then $\alpha = \gamma = \zeta = 0$ and $\beta < 0$, as in the proof of Theorem 17. If $\alpha = 0$, then $bf + adf = 0$, implying $b = -ad$. If $\gamma = 0$, then $bd + aec = 0$ and it follows that $e = d^2$. If $\zeta = 0$, then $df + fce = df + d^2fc = fd(1 + cd) = 0$, implying $c = -1/d$. Thus $\beta = e + f + fcd$, then becomes $d^2 + f + fd(-1/d) = d^2 > 0$, and consequently the refined inertia $(2, 2, 4, 2) \notin ri(A_{11})$. A similar analysis and contradiction show
that \((2, 2, 4, 2) \notin \text{ri}(A_{12})\). Numerical values for the parameters \(a, b, c, d, e, f\) have been found for each of the remaining allowed refined inertias \(\text{ri}(A_1) \cup \text{ri}(A_3) \cup \text{ri}(A_5) - \{(0, 0, 10, 0), (2, 2, 4, 2)\}\), but for brevity are not given here.

\[\text{Theorem 21. If } A \text{ is a zero-nonzero pattern with weighted digraph } D(A) \text{ isomorphic to } P_{13}, P_{14}, P_{15}, \text{ or } P_{16}, \text{ then}\]

\[\text{ri}(A) = \{(7, 3, 0, 0), (3, 7, 0, 0), (6, 4, 0, 0), (4, 6, 0, 0), (5, 3, 0, 2), (3, 5, 0, 2), (4, 4, 0, 2), (5, 5, 0, 0)\}.

\[\text{Proof. Let } A \text{ be a matrix realization of } A_{13}, A_{14}, A_{15}, \text{ or } A_{16}. \text{ The characteristic polynomial of } A \text{ is of the form}\]

\[p(x) = x^{10} - \alpha x^5 - Bx^4 - Cx^2 - Dx + E \quad (3.7)\]

or

\[p(x) = x^{10} - \alpha x^5 - \beta x^4 - \gamma x^2 - Dx + E, \quad (3.8)\]

where \(\alpha, \beta, \gamma, \in \mathbb{R}\) and \(B, C, D, E \neq 0\). Since \(\det(A) \neq 0\), by Theorems 12, 13, and 14 the only possible refined inertias of \(A\) are \((8, 2, 0, 0), (7, 3, 0, 0), (6, 4, 0, 0), (5, 5, 0, 0), (6, 2, 0, 2), (5, 3, 0, 2), (4, 4, 0, 2), (4, 2, 0, 4), (3, 3, 0, 4), (2, 2, 0, 6), (0, 0, 0, 10)\), and their reversals.

If (3.7) has a complex zero \(x = i\sqrt{\phi}\) with \(\sqrt{\phi} > 0\), then \(p(i\sqrt{\phi}) = 0\) implies that \(D = -\alpha \phi^2\) and \(B = -\phi^3 + C/\phi + E/\phi^2\). (Note that \(D \neq 0\) implies that \(\alpha \neq 0\).) Substituting these conditions into \(p(x)\), the polynomial becomes

\[x^{10} - \alpha x^5 - (C/\phi + E/\phi^2 - \phi^3)x^4 - Cx^2 + \alpha \phi^2 x + E,
\]

which factors into the form

\[\left(x^2 + \phi\right)\left(x^8 - \phi x^6 + \phi^2 x^4 - \alpha x^3 - \left(\frac{E}{\phi^2} + \frac{C}{\phi}\right) x^2 + \alpha \phi x + \frac{E}{\phi}\right) q(x).
\]

Since \(E, C, \alpha \neq 0\) and \(\phi > 0\), if \(q(x)\) has a complex zero \(x = it\) with \(t > 0\), then

\[-\alpha(-it^2) + \alpha \phi (it) = 0,\]
which implies that $\alpha = 0$, a contradiction. Thus $q(x)$ can have no zeros with real part equal to 0. It can be shown numerically that if $E > 0$, then the refined inertia corresponding to the zeros of $q(x)$ is $(4, 4, 0, 0)$; and if $E < 0$, then for $\alpha > 0$ or $\alpha < 0$ the refined inertia corresponding to the zeros of $q(x)$ is $(5, 3, 0, 0)$ or $(3, 5, 0, 0)$, respectively. Consequently, if $A$ has two fixed eigenvalues $\pm i\sqrt{\phi}$ for any $\phi > 0$, then since $E \neq 0$, the other eight eigenvalues are such that $\text{ri}(A) = (4, 4, 0, 2), (5, 3, 0, 2), \text{or} (3, 5, 0, 2)$ as $\alpha, C$ and $E$ vary. In particular, the refined inertias $(4, 2, 0, 4), (2, 4, 0, 4), (3, 3, 0, 4)$, and $(2, 2, 0, 6)$ cannot occur.

If $\text{ri}(A) = (0, 0, 0, 10)$, then the characteristic polynomial $\prod_{i=1}^{5} (x^2 + \phi_i)$ implies that $S_2(A) = S_1(\phi_1, ..., \phi_5) > 0$, a contradiction. If $\text{ri}(A) = (8, 2, 0, 0)$, then the characteristic polynomial of $A$ can be factored into the form

$$(x + a_1)(x + a_2)\prod_{i=1}^{8} (x - b_i),$$

where the real parts of $a_1, a_2, b_1, ..., b_8 > 0$. Using the same argument as for the refined inertia $(6, 2, 2, 0)$ in Theorem 18, on replacing $b_1, ..., b_6$ by $b_1, ..., b_8$, a contradiction is found, namely $S_4(A) > 0$ as in $(3.4)$. Thus $(8, 2, 0, 0)$ and its reversal cannot occur. Parameter values that demonstrate all allowed refined inertias of $A_{13}, A_{14}, A_{15}$ and $A_{16}$ are given in Table 3.5.

**Theorem 22.** If $A$ is a zero-nonzero pattern with weighted digraph $D(A)$ isomorphic to $P_{17}$ or $P_{18}$, then

$$\text{ri}(A) = \text{ri}(A_{13}) \cup \{(3, 3, 0, 4)\}.$$  

**Proof.** Let $A$ be a matrix realization of $A_{17}$ or $A_{18}$. The characteristic polynomial of $A$ is of the form

$$x^{10} - \alpha x^5 - Bx^4 - Cx^2 - \zeta x + E,$$  

where $\alpha, \zeta \in \mathbb{R}$ and $B, C, E \neq 0$. Many parts in the proof of Theorem 21 are applicable to eliminate certain refined inertias of $A$. Since $\det(A) \neq 0$, by Theorems 12, 13, and 14 the only possible refined inertias of $A$ are $(8, 2, 0, 0), (7, 3, 0, 0), (6, 4, 0, 0), (5, 5, 0, 0), (6, 2, 0, 2), (5, 3, 0, 2), (4, 4, 0, 2), (4, 2, 0, 4), (3, 3, 0, 4), (2, 2, 0, 6), (0, 0, 0, 10)$, and their reversals. Refined inertias $(8, 2, 0, 0), (2, 6, 0, 2), (4, 2, 0, 4), (2, 2, 0, 6)$, and $(0, 0, 0, 10)$ are eliminated by using the same analysis as in the proof for Theorem 21. However, unlike Theorem 21, pattern $A$ allows refined inertia $(3, 3, 0, 4)$ since $S_0(A) = \zeta$ can equal zero. If $A$ has parameter values giving $\alpha = \zeta = 0$, $B = 1,$
$C = 3$, and $E = -1$, then $\text{ri}(A) = (3, 3, 0, 4)$. Parameter values that demonstrate the other allowed refined inertias for $A_{17}$ and $A_{18}$ are given in Table 3.5.

Table 3.5: Examples of values for the coefficients of the characteristic polynomials $C_{13}$ to $C_{18}$ that can be obtained by choosing nonzero values of $a, b, c, d, e$, and $f$ to give all possible refined inertias of $A_{13}$ to $A_{18}$. For brevity we do not list the parameter values $a, ..., f$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta, B$</th>
<th>$\gamma, C$</th>
<th>$D, \zeta$</th>
<th>$E$</th>
<th>$\text{ri}(A_{13})$ to $\text{ri}(A_{18})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-1</td>
<td>2</td>
<td>-2</td>
<td>-1</td>
<td>$(7, 3, 0, 0)$</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>-2</td>
<td>-1</td>
<td>$(5, 3, 0, 2)$</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>$(5, 5, 0, 0)$</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>-1</td>
<td>$(3, 5, 0, 2)$</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>-1</td>
<td>$(3, 7, 0, 0)$</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>$-\frac{1}{2}$</td>
<td>1</td>
<td>$(6, 4, 0, 0)$</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>$(4, 4, 0, 2)$</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-2</td>
<td>1</td>
<td>$(4, 6, 0, 0)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$B$</th>
<th>$C$</th>
<th>$\zeta$</th>
<th>$E$</th>
<th>In addition $\text{ri}(A_{17}), \text{ri}(A_{18})$ can equal</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>$(3, 3, 0, 4)$</td>
</tr>
</tbody>
</table>

3.4 A Sign Pattern

We now consider $A$ to be a sign pattern, i.e., each nonzero entry of $A_i$ is signed one of $+$ or $-$. If sign pattern $A$ is a signing of one of the zero-nonzero patterns $A_i$ defined in Section 3.3, then its weighted digraph $D(A)$ is isomorphic to digraph $P_i$. It follows from the definitions of zero-nonzero patterns and sign patterns that the refined inertia of a sign pattern is a subset of the refined inertia of its corresponding zero-nonzero pattern.

**Theorem 23.** There exists a sign pattern $A_c$ that is a signing of $A_3$ having refined inertia $\text{ri}(A_c) = \{(4, 4, 2, 0), (2, 2, 2, 4)\}$. 
Proof. Let $A_c$ be the matrix in Figure 3.1 with the parameters $a$, $b$, $c$, $d$, $e'$, $f > 0$, and let $A_c$ denote the sign pattern of $A_c$. Then $A_c$ is a signing of the zero-nonzero pattern $A_3$ defined in Section 3.3, and thus $\text{ri}(A_c) \subseteq \text{ri}(A_3)$ as given in Theorem 18. The characteristic polynomial of $A_c$ is

$$x^{10} + (f - fbe' - e')x^5 + (fce' + dbae')x^4 + fbae'x^2.$$  

Parameter values $a = 1$, $b = 1/2$, $c = 1/2$, $d = 2$, $e' = 1$, and $f = 2$, which set the coefficient of $x^5$ equal to zero, give $\text{ri}(A_c) = (2, 2, 2, 4)$. If $f = 3$ with the other parameters fixed as above, then the coefficient of $x^5$ is nonzero and $\text{ri}(A) = (4, 4, 2, 0)$. Since $fbae' > 0$, it follows that the product of all nonzero eigenvalues of $A_c$ is positive; hence, there cannot be an odd number of eigenvalues with negative real parts. Thus, the refined inertias $(3, 5, 2, 0)$, $(5, 3, 2, 0)$ and $(3, 3, 2, 2)$ that the nonzero pattern $A_3$ also allows (see Theorem 18) are not allowed by the sign pattern $A_c$. \hfill \Box

![Figure 3.1: $D(A_c)$, a weighted digraph of $P_3$](image)

Note that for any realization of $A_c$ that has refined inertia $(2, 2, 2, 4)$, any perturbation in the parameters that makes the coefficient of $x^5$ nonzero results in one pair of pure imaginary eigenvalues moving into the left half plane while the other pair moves into the right half plane.

In conclusion, we remark that each pattern resulting from a weighted strongly connected orientation of the Petersen graph has a very small set of refined inertias, given that the total number of distinct refined inertias for a 10-by-10 pattern is 161, or is 91 for a zero-nonzero pattern if reversals are excluded [8, Theorem 1.1].
Chapter 4
Discussion and Conclusions

We summarize our results and discuss their implications. In Chapter 2 we highlighted several biological systems in which the linear stability of a positive steady state, described by the eigenvalues of the Jacobian matrix $J$, is analyzed. We are able to use the refined inertia of $\text{sgn}(J)$ to determine whether or not a system has the potential to undergo a Hopf bifurcation, resulting in periodic solutions. This is particularly useful when investigating many different models in which a parameter has fixed sign but unknown magnitude.

One potential shortcoming of sign pattern analysis is having to account for restricted realizations. For example, system (2.19) in Section 2.7.1 has a sign pattern that allows $\mathbb{H}_4$; however, system (2.19) has a Jacobian matrix (restricted realization) that is semi-stable. In this example, the sign pattern leads us to the incorrect conclusion that system (2.19) would allow periodic solutions. However, the sign pattern gives us insight into how different functional forms may affect the dynamics of the model. That is, the compartments of the model interact in such a way that a change in the functional response may result in the system exhibiting periodic solutions or other behavior. In fact, for the example above, if the linear functional response is replaced with a Holling type II functional response, then the system undergoes a Hopf bifurcation [19]. In [19] the sign pattern of the Jacobian matrix contains fewer restrictions and as a result the restricted sign pattern allows $\mathbb{H}_4$. In general, a system with sign pattern $\mathcal{A}$ that allows $\mathbb{H}_n$ has the potential for a Hopf bifurcation given the right functional responses, as is the case with predator–prey systems.

In Chapter 3 we determined the refined inertias of all strongly connected orientations of the Petersen graph. Using the elementary symmetric functions, we are able to eliminate some possible refined inertias. This type of analysis is novel and
may provide insight into the search for a sign pattern that requires $\mathbb{H}_8$, although the existence of such a sign pattern is not known. We also explore a sign pattern that requires the set of refined inertias $\{(4, 4, 2, 0), (2, 2, 2, 4)\}$. This is interesting because two pairs of pure imaginary eigenvalues simultaneously transition from the left and right halves of the complex plane onto the imaginary axis.
Bibliography


Appendix A

Petersen Graph
The following are the characteristic polynomials for the matrices $A_i$ in A.2. Greek letters denote real-valued parameters (possibly zero) and Roman letters denote nonzero real-valued parameters.

<table>
<thead>
<tr>
<th>Characteristic Polynomial $C_i$</th>
<th>Equivalent Polynomial Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^{10} - (c + de + ae + f) x^5 - (bae + d) x^4$</td>
<td>$x^{10} - \alpha x^5 - \beta x^4$ (C₁)</td>
</tr>
<tr>
<td>$x^{10} - (c + de + be + f) x^5 - (d + ac) x^4$</td>
<td>$x^{10} - \alpha x^5 - \beta x^4$ (C₂)</td>
</tr>
<tr>
<td>$x^{10} - (f + fbe + e) x^5 - (f + dbae) x^4 - fbaex^2$</td>
<td>$x^{10} - \alpha x^5 - \beta x^4 - C x^2$ (C₃)</td>
</tr>
<tr>
<td>$x^{10} - (b + ce + f) x^5 - (ae + d) x^4 - ax^2$</td>
<td>$x^{10} - \alpha x^5 - \beta x^4 - C x^2$ (C₄)</td>
</tr>
<tr>
<td>$x^{10} - (bf + edf) x^5 - (df + f) x^4 - adfx^2 - cdf x$</td>
<td>$x^{10} - \alpha x^5 - \beta x^4 - C x^2 - Dx$ (C₅)</td>
</tr>
<tr>
<td>$x^{10} - (f + adf + a) x^5 - (fbde + c) x^4 - fadex^2 - fedex$</td>
<td>$x^{10} - \alpha x^5 - \beta x^4 - C x^2 - Dx$ (C₆)</td>
</tr>
<tr>
<td>$x^{10} - f (a + c) x^5 - (f + b) x^4 - bfx^2 - fbdex$</td>
<td>$x^{10} - \alpha x^5 - \beta x^4 - C x^2 - Dx$ (C₇)</td>
</tr>
<tr>
<td>$x^{10} - (ae + f + ba) x^5 - (e + d) x^4 - caex^2 - cex$</td>
<td>$x^{10} - \alpha x^5 - \beta x^4 - C x^2 - Dx$ (C₈)</td>
</tr>
<tr>
<td>$x^{10} - f (1 + d + cde) x^5 - cfx^4 - fadex^2 - fbcxex$</td>
<td>$x^{10} - \alpha x^5 - B x^4 - C x^2 - Dx$ (C₉)</td>
</tr>
<tr>
<td>$x^{10} - (b + d + c) x^5 - fx^4 - cx^2 - cafex$</td>
<td>$x^{10} - \alpha x^5 - B x^4 - C x^2 - Dx$ (C₁₀)</td>
</tr>
<tr>
<td>$x^{10} - (bf + adf) x^5 - (e + f + fcd) x^4 - (dbf + afe) x^2 - (df + fce) x$</td>
<td>$x^{10} - \alpha x^5 - \beta x^4 - \gamma x^2 - \zeta x$ (C₁₁)</td>
</tr>
<tr>
<td>$x^{10} - (bf + a) x^5 - (f + fbe + e + be) x^4 - (f + bce + bde) x$</td>
<td>$x^{10} - \alpha x^5 - \beta x^4 - \gamma x^2 - \zeta x$ (C₁₂)</td>
</tr>
<tr>
<td>$x^{10} - (f + ce + dbac) x^5 - fex^4 - fbaex^2 - cafex + dbacf$</td>
<td>$x^{10} - \alpha x^5 - B x^4 - C x^2 - Dx + E$ (C₁₃)</td>
</tr>
<tr>
<td>$x^{10} - (be + fcd + b) x^5 - fbcex^4 - dbex^2 - edcbfx + dbcf$</td>
<td>$x^{10} - \alpha x^5 - B x^4 - C x^2 - Dx + E$ (C₁₄)</td>
</tr>
<tr>
<td>$x^{10} - (e + df + ac) x^5 - e(a + bd) x^4 - (dae + fac) x^2 - bcaex + cae$</td>
<td>$x^{10} - \alpha x^5 - \beta x^4 - \gamma x^2 - D x + E$ (C₁₅)</td>
</tr>
<tr>
<td>$x^{10} - (e + df + ac) x^5 - (f + dbde + ad) x^4 - (ecad + a) x^2 - adbfx + fcd$</td>
<td>$x^{10} - \alpha x^5 - \beta x^4 - \gamma x^2 - D x + E$ (C₁₆)</td>
</tr>
<tr>
<td>$x^{10} - (e + cde + d + f) x^5 - f + dx^4 - dbae x^2 - ea(b + df) x + facde$</td>
<td>$x^{10} - \alpha x^5 - B x^4 - C x^2 - \zeta x + E$ (C₁₇)</td>
</tr>
<tr>
<td>$x^{10} - (bae + c + edf + d) x^5 - df x^4 - caex^2 - (cde + dafe) x + dbae$</td>
<td>$x^{10} - \alpha x^5 - B x^4 - C x^2 - \zeta x + E$ (C₁₈)</td>
</tr>
</tbody>
</table>
A.2 Strongly Connected Orientations of the Petersen Graph

Up to isomorphism, the following are all of the weighted digraphs $\mathcal{D}(A_i)$ isomorphic to $P_i$, together with their matrix representations $A_i$.

Figure A.1: $\mathcal{D}(A_1)$, a weighted digraph of $P_1$ with directed spanning tree (solid arcs)

Figure A.2: $\mathcal{D}(A_2)$, a weighted digraph of $P_2$ with directed spanning tree (solid arcs)
Figure A.3: $D(A_3)$, a weighted digraph of $P_3$ with directed spanning tree (solid arcs)

\[
A_3 = \begin{bmatrix}
0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & c & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e \\
0 & 0 & f & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}
\]

Figure A.4: $D(A_4)$, a weighted digraph of $P_4$ with directed spanning tree (solid arcs)

\[
A_4 = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c & 0 & 0 & 0 & d & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & f & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}
\]
Figure A.5: $D(A_5)$, a weighted digraph of $P_5$ with directed spanning tree (solid arcs)

$A_5 = 
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f \\
0 & 0 & 0 & 1 & 0 & 0 & e & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}$

Figure A.6: $D(A_6)$, a weighted digraph of $P_6$ with directed spanning tree (solid arcs)

$A_6 = 
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & a & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & b & 0 & 0 \\
c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 \\
\end{bmatrix}$
Figure A.7: $\mathcal{D}(A_7)$, a weighted digraph of $P_7$ with directed spanning tree (solid arcs)

$A_7 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & a & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 
\end{bmatrix}$

Figure A.8: $\mathcal{D}(A_8)$, a weighted digraph of $P_8$ with directed spanning tree (solid arcs)

$A_8 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\
0 & b & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c & 0 & 0 & 0 & d & 0 \\
e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & f & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 
\end{bmatrix}$
Figure A.9: $D(A_9)$, a weighted digraph of $P_9$ with directed spanning tree (solid arcs)

$$A_9 = \begin{bmatrix}
0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0
\end{bmatrix}$$

Figure A.10: $D(A_{10})$, a weighted digraph of $P_{10}$ with directed spanning tree (solid arcs)

$$A_{10} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & b & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}$$
Figure A.11: $\mathcal{D}(A_{11})$, a weighted digraph of $P_{11}$ with directed spanning tree (solid arcs)

$$A_{11} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
\end{bmatrix}$$

Figure A.12: $\mathcal{D}(A_{12})$, a weighted digraph of $P_{12}$ with directed spanning tree (solid arcs)

$$A_{12} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & a & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c & d & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 1 & 0 \\
\end{bmatrix}$$
Figure A.13: $\mathcal{D}(A_{13})$, a weighted digraph of $P_{13}$ with directed spanning tree (solid arcs)

$$A_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 \\ 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & e \\ 0 & 0 & f & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Figure A.14: $\mathcal{D}(A_{14})$, a weighted digraph of $P_{14}$ with directed spanning tree (solid arcs)

$$A_{14} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 \\ 0 & e & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$
Figure A.15: $D(A_{15})$, a weighted digraph of $P_{15}$ with directed spanning tree (solid arcs)

$A_{15} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\
0 & 0 & e & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & f & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 
\end{bmatrix}$

Figure A.16: $D(A_{16})$, a weighted digraph of $P_{16}$ with directed spanning tree (solid arcs)

$A_{16} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\
0 & b & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e \\
0 & 0 & 1 & 0 & 0 & f & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 
\end{bmatrix}$
Figure A.17: $\mathcal{D}(A_{17})$, a weighted digraph of $P_{17}$ with directed spanning tree (solid arcs)

\[
A_{17} = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & b \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

Figure A.18: $\mathcal{D}(A_{18})$, a weighted digraph of $P_{18}$ with directed spanning tree (solid arcs)

\[
A_{18} = \begin{bmatrix}
0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & b \\
0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\
0 & 0 & e & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]