Part I. The $\pi^0\gamma\gamma$ Form Factor

Part II. Validity of Soft Photon Amplitudes

Part III. Soft Photon Excess in Hadron Scattering

BY

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A dissertation submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy in the Department of Physics and Astronomy.

We accept this dissertation as conforming to the required standard.

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Abstract

Part I: The $\pi^0\gamma\gamma$ form factor is studied in the context of chiral perturbation theory. The form factor is computed with all particles off-shell and up to one-loop level. The contribution of the complete $O(p^4)$ parity-odd sector as constructed by Fearing and Scherer is computed for the first time. It is found that the vertex depends on only four independent low energy constants from this part of the chiral Lagrangian. Several processes are studied which involve this form factor and the low energy constants are constrained through comparison with experiment.

Part II: A review is made of the construction and use of the soft photon approximation in particle physics. It is found that problems can arise in the practical application of the approximation which prevent one from making a model independent relation between a bremsstrahlung process and its non-radiative counterpart. The difficulty is traceable to the selection of expansion points which one is required to make during the construction of a soft photon amplitude. The body of literature which has employed the traditional Low form of the amplitude is found to be, for practical purposes, unaffected by this model-dependency. However, it is found that certain recently proposed alternative forms of the soft photon approximation are subject to the problems and their usefulness in a model-independent analysis of, for instance, the off-shell behaviour of the $pp$ elastic scattering amplitude is therefore called into question.

Part III: In several high energy hadron scattering experiments the production of low energy photons has been observed to be greatly in excess of the yield expected from an application of the Low soft photon approximation. A study has been made of possible sources for these excess photons. The contribution of higher order terms in the soft photon amplitude is found to be negligible, as are interference effects between the bremsstrahlung amplitude and amplitudes describing the decay of hadronic resonances. Multiple photon emission within a simple multi-hadron production model is found to have a substantial effect on the soft photon spectrum, but cannot by itself explain the very large experimental yields.

Examiners:

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Part I

The $\pi^0\gamma\gamma$ Form Factor
Chapter 1

Introduction

Quantum chromo-dynamics (QCD) is the sector of the standard model of particle physics which is believed to describe the strong interaction. QCD is written in terms of quark degrees of freedom and, due to a property called confinement, these quarks are not directly observable in experiments. We instead observe hadrons such as pions and kaons which are thought to be strongly bound collections of quarks and their antiparticles and of the force carrying gauge bosons of QCD, gluons. Confinement arises because the effective coupling of quarks to gluons is very large for interactions with a small characteristic momentum transfer squared and tends to zero as this momentum transfer is increased. At small momentum, where the coupling is large, a perturbative treatment of QCD will fail to converge.

One may tackle this problem by using non-perturbative techniques like lattice field theory, or using models such as the Nambu–Jona-Lasinio model [2] or QCD sum rules [3]. Still another approach is to distill the symmetries obeyed by QCD into another field theory which has as its dynamic fields the physical hadrons observed in the laboratory. One may then recover the powerful framework of perturbation theory by expanding this field theory simultaneously about the chiral limit, at which the pseudoscalar meson octet containing the pions and kaons becomes massless, and about the low energy limit in a momentum expansion. So long as the momenta involved are small enough this perturbative expansion will be convergent. Something is lost in the transition from QCD to this chiral perturbation theory, since there is more dynamical structure to QCD than is contained in its symmetries. This loss of information is parameterized in the chiral Lagrangian by low energy constants which act as weights for each of the Lagrangian's terms. While at the lower orders in the momentum expansion there are only a few such low energy constants, as one goes to higher orders their numbers increase rapidly—in principle, to all orders in the momentum expansion, there are an infinite number of
them. Their values are not predicted within the context of chiral perturbation theory (χPT) but must be taken from a model or by fitting χPT calculations to experiment.

The chiral Lagrangian which we shall consider here contains only the mesons of the lowest mass pseudoscalar octet as dynamic fields. In Chp. 2 we write down the chiral Lagrangian to $O(p^4)$ in the parity-even sector and $O(p^6)$ in the parity-odd. With only parity conserving external fields present, such as the electromagnetic field, the parity even and parity odd terms describe respectively the interactions of even and odd numbers of mesons at a vertex.

The renormalized low energy constants of $O(p^4)$ parity-even and $O(p^6)$ parity-odd will enter during our calculation of the $\pi^0\gamma\gamma$ and $\eta\gamma\gamma$ vertex functions. In Chp. 3 we use the background field method to construct an effective action containing the one-loop structure of the lowest order part of the chiral Lagrangian. The divergent terms of this one-loop structure are then isolated using the results of the heat kernel method and are shown to renormalize the $O(p^4)$ parity-even sector of the Lagrangian. We then briefly describe the method of extending this work to the $O(p^6)$ parity-odd sector, which would form an interesting future project.

In Chp. 4 we compute the vertex factors $\pi^0\gamma\gamma$ and $\eta\gamma\gamma$ with all the interacting particle off mass shell. Taking the virtual meson off shell introduces the controversial issue of whether terms from the Lagrangian proportional to the classical equation of motion should be included in the calculation. It was recently noted [4, 5] for the $\pi\gamma\gamma$ form factor that though such terms appear in the calculated form factors they vanish when one uses this form factor to compute a measurable process such as Compton scattering. We will consider the contribution of such terms though they are irrelevant for our later calculations of physical processes, all of which have the neutral meson on-shell.

In Chp. 5 we compute several decays and cross sections which involve the $\pi^0\gamma\gamma$ vertex in order to constrain the values of the low energy constants upon which it depends. Most of these processes have either been measured or will be accessible when the high luminosity $e^+e^-$ collider DAΦNE [6] comes on-line.
Chapter 2

The Chiral Lagrangian

In this chapter we will define our notation for the chiral Lagrangian and, where appropriate, relate our notation to that of other authors. We shall consider only the pseudoscalar meson octet as dynamic fields of the Lagrangian, and we use the conventional momentum power counting scheme of Weinberg [7] to define the ordering of terms in the perturbative expansion.

We write the chiral Lagrangian in the form

\[ \mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_4^A + \mathcal{L}_6 + \mathcal{L}_6^A + \ldots \]

where the subscripts denote the order in powers of momentum of the contained terms. The terms with superscript \( A \) are parity odd or anomalous while the remaining terms are parity even. Parity odd or even terms, in the absence of parity-violating external fields, define the interactions of odd or even numbers of mesons.

2.1 Parity even sector

The lowest order term of the parity even sector is

\[ \mathcal{L}_2 \equiv \frac{F^2}{4} \left( D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U \right) \]

where

- \( \langle \ldots \rangle = \text{Tr}(\ldots) \), a trace over flavour degrees of freedom.
- \( U = \exp(i\lambda_a \phi_a / F) \), where a summation on index \( a \) is implied.
- \( \chi \equiv 2Bm \), where \( B \) is a constant and \( m \equiv \text{diag}(m_u, m_d, m_s) \) is a matrix of quark masses.
To this order in \( \chi PT \) we have \( F = F_* = (92.4 \pm 0.3) \text{ MeV} \), the charged pion decay constant. Care must be taken with this definition since the charged pseudoscalar meson decay constants are often defined to contain an extra factor of \( \sqrt{2} \)—see, for example, Ref. [8, pp. 1443]. This would imply for the charged pion a value \( f_* = \sqrt{2} F_* = (130.7 \pm 0.5) \text{ MeV} \).

- \( D_\mu U \equiv \partial_\mu U - i r_\mu U + i U t_\mu \), where \( r_\mu \) and \( t_\mu \) are right and left handed external fields.

- \( \lambda_a \) are the Gell-Mann matrices of flavour SU(3); we use the representation of Ref. [8, pp. 1288].

\[
\phi_a \equiv \left\{ \frac{1}{\sqrt{2}}(\pi^+ - \pi^-), \frac{1}{\sqrt{2}}(\pi^+ + \pi^-), \pi^0, \frac{1}{\sqrt{2}}(K^+ + K^-), \frac{1}{\sqrt{2}}(K^+ - K^-), \frac{1}{\sqrt{2}}(K^0 + \bar{K}^0), \frac{1}{\sqrt{2}}(K^0 - \bar{K}^0), \eta_8 \right\}.
\]

Hence,

\[
\phi \equiv \lambda_a \phi_a = \begin{pmatrix}
\pi^0 + \frac{1}{\sqrt{3}}\eta_8 & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\
\sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta_8 & \sqrt{2}K^0 \\
\sqrt{2}K^- & \sqrt{2}K^0 & -\frac{3}{2}\eta_8
\end{pmatrix}
\]

where \( \pi^+, \bar{K}^0 \ldots \) are the pseudoscalar meson fields. In particular, \( \eta_8 \) is the flavour-SU(3) octet part of the physical \( \eta \) and \( \eta' \) mesons. They are related by

\[
\eta = \eta_8 \cos \theta_P - \eta_1 \sin \theta_P
\]
\[
\eta' = \eta_8 \sin \theta_P + \eta_1 \cos \theta_P
\]

where \( \eta_1 \) is a flavour-SU(3) singlet, and \( \theta_P \) is a mixing angle. Combined fits to experiment give this mixing angle as \( \theta_P \approx -14^\circ \) [9] where SU(3) symmetry breaking effects are included and \( \theta_P \approx -20^\circ \) [10] assuming SU(3) symmetry. We shall not consider the \( \eta_1 \) singlet contribution and compute only the octet component of the meson-\( \gamma \) vertices. This clearly restricts our ability to compare with experiment to any great degree of accuracy in the case of the \( \eta \). We shall, however, be mainly concerned with the \( \pi^0 \gamma \gamma \) vertex.

We also define field strength tensors for the left and right handed external fields.

\[
F^{\mu\nu}_L \equiv \partial^\mu \ell^\nu - \partial^\nu \ell^\mu - i [\ell^\mu, \ell^\nu]
\]
\[
F^{\mu\nu}_R \equiv \partial^\mu r^\nu - \partial^\nu r^\mu - i [r^\mu, r^\nu]
\]  

(2.1)
2.1 Parity even sector

While the external fields could potentially include contributions from, for instance, the $W^\pm$ vector bosons of the weak interaction we shall require for our calculation only the external vector field of the photon. This gives

$$r_\mu = v_\mu + a_\mu = -eA_\mu Q$$

$$l_\mu = v_\mu - a_\mu = -eA_\mu Q$$

where $v_\mu$, $a_\mu$ are vector and axial components of the external fields, $Q \equiv \text{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ is a matrix of light quark charges, and $e = \sqrt{4\pi\alpha}$ is the positron charge. The covariant derivative and the field strength tensors then have rather simple forms.

$$D_\mu U = \partial_\mu U - ieA_\mu [U, Q]$$

$$D_\mu U^\dagger = \partial_\mu U^\dagger - ieA_\mu [U^\dagger, Q]$$

$$F_{\mu\nu}^\mu = F_{\mu\nu}^R = -eQ(\partial_\mu A_\nu - \partial_\nu A_\mu) = -eQ F_{\mu\nu}$$

With this notation defined we are in a position to write down the chiral counting rules which determine the order, in powers of momentum, of the terms in the chiral Lagrangian. Following Weinberg [7] and Gasser and Leutwyler [11] we assign

$$U \quad \rightarrow \quad O(p^0)$$

$$D_\mu U, r_\mu, \quad \rightarrow \quad O(p)$$

$$F_{\mu\nu}^\mu, \quad \rightarrow \quad O(p^2)$$

$$B \quad \rightarrow \quad O(p)$$

$$m_u, m_d, m_s \quad \rightarrow \quad O(p).$$

As they stand these rules would lead to two concurrent expansions: an expansion in powers of momentum about the low energy limit, and an expansion in quark masses about the chiral limit at which $m_u = m_d = m_s = 0$ and flavour SU(3) is an exact symmetry. We follow the conventional course of combining these expansions by imposing that the constant $B$ and the mass matrix $m$ only appear together in the Lagrangian in the combination $\chi = 2Bm$.

A recently proposed alternative framework is that of Generalized $\chi$PT (see [12] for a recent review) where $m$ is allowed to appear independently of $B$ in the Lagrangian. This leads to both even and odd powers of momentum in the Lagrangian, gives a modified prediction for the Gell-Mann–Okubo mass relation, and allows the value of the quark condensate in the chiral limit, a quantity proportional to $\chi$, to become much smaller than the value given by conventional $\chi$PT. The unmodified mass relation for the pseudoscalar mesons, which is replicated by conventional $\chi$PT,

$$m_{ns}^2 = \frac{1}{3}(4m_K^2 - m_\pi^2)$$
appears, however, to be in good agreement with measured masses. Also in apparent
coradiction of Generalised χPT are the recent lattice calculations of Refs. [13, 14]
which indicate that the quark condensate \( \langle 0|\bar{q}q|0 \rangle \) in the chiral limit is close in value to
the prediction of the conventional expansion of the chiral Lagrangian [15].

We shall now rewrite this lowest order part of the chiral Lagrangian, \( \mathcal{L}_2 \), in terms
of the physical pseudoscalar meson masses. This will, to \( \mathcal{O}(p^2) \), give us the relationship
between the constants \( B_{m_u}, B_{m_d}, B_m \), and these physical masses, and will also allow us
to reproduce the Gell-Mann–Okubo mass formula for the pseudoscalar meson octet. We
set the external fields to zero, implying \( D_\mu \rightarrow \partial_\mu \), and expand \( U \) in terms of \( \phi \equiv \lambda_8 \phi_a \).

\[
\mathcal{L}_2 = \frac{F^2}{4} \left( \partial_\mu \frac{1}{F} \phi + \ldots \right) \partial^\mu \left( -\frac{i}{F} \phi + \ldots \right) + 2Bm \left( 1 + \frac{i}{F} \phi - \frac{1}{2F^2} \phi^2 + \ldots \right) + 2Bm \left( 1 - \frac{i}{F} \phi - \frac{1}{2F^2} \phi^2 + \ldots \right) \]

\[
= \frac{1}{4} \left( \partial^\mu \phi \partial_\mu \phi \right) - \frac{B}{2} \left( 2 \phi^2 \right) + \text{(constants)} + \mathcal{O}(\phi^4)
\]

Here, as in several later calculations, we have used the algebraic manipulator MATHEMATICA [16] to multiply and take trace of explicit representations of the matrices \( \phi, m \) and, when it is required, \( Q \). The result is

\[
\mathcal{L}_2 = (\partial^\mu \pi^+ \partial_\mu \pi^- - (2Bm)\pi^+\pi^-) + \frac{1}{2} \left( \partial^\mu \pi^0 \partial_\mu \pi^0 - (2Bm)\pi^0\pi^0 \right) + \left( \partial^\mu K^+ \partial_\mu K^- - B(m_u + m_s)K^+K^- \right) + \left( \partial^\mu K_0 \partial_\mu K_0 - B(m_d + m_s)K_0K_0 \right) + \frac{1}{2} \left( \partial^\mu \eta_8 \partial_\mu \eta_8 \right) - \frac{1}{3} B(\bar{m} + 2m_s)\eta_8^2 + \frac{1}{3} B(m_d - m_u)\pi^0\eta_8 + \ldots \tag{2.2}
\]

where \( \bar{m} \equiv \frac{1}{2}(m_u + m_d) \). These are the kinetic and mass terms for three complex scalar
fields and two real scalar fields. The term \( \pi^0\eta_8 \) implies that the fields we have designated
as \( \pi^0 \) and \( \eta_8 \) are shifted from the physical fields. Ignoring this term for a moment we
may pick off the meson masses in terms of the constants \( B_{m_u}, B_{m_d} \) and \( B_{m_s} \).

\[
\begin{align*}
m_{\pi \pm}^2 & = 2Bm \\
m_{\pi^0}^2 & \approx 2Bm \\
m_{K^\pm}^2 & = B(m_u + m_s) \\
m_{K^0}^2 & = B(m_d + m_s) \\
m_{\eta_8}^2 & \approx \frac{1}{3} B(\bar{m} + 2m_s)
\end{align*}
\tag{2.3}
\]

The \( \pi^0\eta_8 \) mixing may be removed by diagonalizing \( \mathcal{L}_2 \) with a redefinition of the
neutral scalar fields.

\[
\pi^0 = \pi^0 \cos \theta + \eta_8 \sin \theta
\]
\[ \hat{\eta}_8 = -\pi^0 \sin \theta + \eta_8 \cos \theta \]

With the appropriate choice for the mixing angle \( \theta \), the Lagrangian will be free of the term proportional to \( \pi^0 \eta_8 \).

\[ \mathcal{L}_2 = \frac{1}{2} \left( \partial^\mu \pi^0 \partial_\mu \pi^0 - m_{\pi^0}^2 \pi^0 \right) + \frac{1}{2} \left( \partial^\mu \tilde{\eta}_8 \partial_\mu \tilde{\eta}_8 - m_{\tilde{\eta}_8}^2 \tilde{\eta}_8 \right) + \ldots \]

\[ = \frac{1}{2} \left( \partial^\mu \pi^0 \partial_\mu \pi^0 - (m_{\pi^0}^2 \cos^2 \theta + m_{\tilde{\eta}_8}^2 \sin^2 \theta) \pi^0 \right) \]

\[ + \frac{1}{2} \left( \partial^\mu \eta_8 \partial_\mu \eta_8 - (m_{\eta_8}^2 \sin^2 \theta + m_{\tilde{\eta}_8}^2 \cos^2 \theta) \eta_8 \right) \]

\[ - \cos \theta \sin \theta (m_{\pi^0}^2 - m_{\tilde{\eta}_8}^2) \pi^0 \eta_8 + \ldots \]

Matching terms with the expression of Eq. (2.2) we have

\[ \frac{1}{2} \sin 2\theta \left( m_{\eta_8}^2 - m_{\pi^0}^2 \right) = \frac{1}{3} B(m_d - m_u) \]

\[ m_{\eta_8}^2 \cos^2 \theta + m_{\eta_8}^2 \sin^2 \theta = 2B \overline{m} \]

\[ m_{\eta_8}^2 \sin^2 \theta + m_{\eta_8}^2 \cos^2 \theta = \frac{2}{3} B(\overline{m} + 2m_s) \]

(2.4)

which may be solved for \( \theta \) by dividing the first equation above by the difference of the second and third.

\[ \tan 2\theta = \sqrt{3} \left( \frac{m_d - m_u}{m_s - \overline{m}} \right) \]

This mixing angle is clearly very small; less than 2° if we take the isospin breaking scale to be around \( |m_u - m_d| \approx 10 \text{ MeV} \) and flavour SU(3) breaking at \( m_s - \overline{m} \approx 150 \text{ MeV} \). Making a small angle expansion in Eq. (2.4), the first order corrections to the \( \pi^0 \) and \( \eta_8 \) masses are

\[ m_{\pi^0}^2 = 2B \overline{m} - \frac{1}{4} B \left( \frac{(m_d - m_u)^2}{m_s - \overline{m}} \right) \]

\[ m_{\eta_8}^2 = \frac{2}{3} B(\overline{m} + 2m_s) + \frac{1}{4} B \left( \frac{(m_d - m_u)^2}{m_s - \overline{m}} \right) \]

We consider this neutral meson mixing to be negligible in comparison with the error introduced in our calculations by computing to a low order in the momentum expansion. To simplify our calculations we will therefore continue to write the Lagrangian in terms of the mixed fields \( \pi^0 \) and \( \eta_8 \) rather than the diagonalized fields \( \tilde{\pi}^0 \) and \( \tilde{\eta}_8 \).

By taking the appropriate combination of the meson masses it is clear that we can now reproduce the Gell-Mann-Okubo mass relation for the pseudoscalar mesons.

\[ 2(m_{\pi^0}^2 + m_{\eta_8}^2) - 3m_{\pi^0}^2 - m_{\eta_8}^2 \approx \]

\[ 2(B(m_u + m_s) + B(m_d + m_s)) - 3\frac{2}{3} B(\overline{m} + 2m_s) - 2B \overline{m} \]

\[ = 0 + \mathcal{O} \left( B \left( \frac{(m_d - m_u)^2}{m_s - \overline{m}} \right) \right) + \mathcal{O}(p^4) \]
2.1 Parity even sector

If we use the physical kaon and charged pion masses in this formula \( m_{K^\pm} = 493.7 \text{ MeV}, \)
\( m_{K^0} = 497.7 \text{ MeV} \) and \( m_{\pi} = 139.6 \text{ MeV} \) we find the predicted value for the octet \( \eta_8 \)
mass to be \( m_{\eta_8} = 566.7 \text{ MeV} \) as compared to the physical \( \eta \) mass of \( m_{\eta} = 547.5 \text{ MeV}, \)
a difference of less than 4%. Indeed this agreement seems fortuitously good since the
relation does not take into account \( \eta - \eta' \) mixing or, within this context of \( \chi \text{PT}, \)
the truncation of the chiral expansion at lowest order.

We now consider the terms of the parity even sector which are of higher order in
momentum. Each of the expressions \( \mathcal{L}_4, \mathcal{L}_6, \ldots \) may be constructed by writing down all
independent terms of the required order in momentum, according to the chiral counting
rules, which may be formed of the building blocks \( U, D_\mu U, F^{\mu\nu}_{L,R} \) and \( \chi. \) The full
Lagrangian, and therefore each term of a particular order in momentum, must be Lorentz
covariant and invariant under parity and charge conjugation since these are symmetries
of the strong interaction. We quote the most general such expression for \( \mathcal{L}_4 \) from the
papers of Gasser and Leutwyler—Refs. [11, 15] considered the case of two flavours of
light quark (\( u \) and \( d \)), while Ref. [17] considered the case with which we are concerned
involving three light flavours (\( u, d \) and \( s \)).

\[
\mathcal{L}_4 = L_1 \langle D_\mu U^\dagger D_\mu U \rangle^2 + L_2 \langle D_\mu U^\dagger D_\nu U \rangle \langle D_\mu U^\dagger D_\nu U \rangle \\
+ L_3 \langle D_\mu U^\dagger D_\mu U D_\sigma U^\dagger D_\sigma U \rangle + L_4 \langle D_\mu U^\dagger D_\mu U \rangle \langle \chi^\dagger U + \chi U^\dagger \rangle \\
+ L_5 \langle D_\mu U^\dagger D_\mu U (\chi^\dagger U + \chi U^\dagger) \rangle + L_6 \langle \chi^\dagger U + \chi U^\dagger \rangle^2 \\
+ L_7 \langle \chi^\dagger U - \chi U^\dagger \rangle^2 + L_8 \langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger \rangle \\
- iL_9 \langle F^{\mu\nu}_{L,R} D_\mu U D_\nu U^\dagger + F^{\mu\nu}_{L,R} D_\mu U^\dagger D_\nu U \rangle + L_{10} \langle U^\dagger F^{\mu\nu}_{L,R} F_{L,\mu\nu} \rangle \\
+ H_1 \langle F^{\mu\nu}_{L,R} F_{R,\mu\nu} + F^{\mu\nu}_{L,R} F_{L,\mu\nu} \rangle + H_2 \langle \chi^\dagger \chi \rangle \quad (2.5)
\]

The factors \( L_1 - L_{10} \) are the low energy constants of the \( \mathcal{O}(p^4) \) parity even sector.
Their values are not given by \( \chi \text{PT} \) and must be determined within the context of a
particular model of low energy hadronic interactions or by comparison with experiment.
These constants are a parameterization of those aspects of the strong interaction which
are not determined solely by the symmetry constraints placed on the chiral Lagrangian.
In principle the low energy constants may be obtained from quantum chromodynamics
(QCD), if we believe that theory to be the full and correct description of the strong
interaction, using a non-perturbative method of calculation such as lattice field theory.
Unfortunately, lattice calculations are not currently of a sufficient level of sophistication
to give predictions for these constants. All of the terms in Eq. (2.5) except those with
coefficients \( L_8 \) and \( L_7 \) act as counter-terms during renormalization of the divergent \( \mathcal{O}(p^4) \)
loop integrals which arise from the \( \mathcal{L}_2 \) Lagrangian. The terms with coefficients \( H_1 \) and
2.2 Parity odd sector

$H_2$ are required as renormalization counter-terms but have no contribution to physical processes involving incoming or outgoing mesons. We will conduct our discussion of renormalization in the following chapter.

While $\mathcal{L}_2$ has the constant $F$ to be extracted from experiment, and $\mathcal{L}_4$ has the ten $L_4$, it has been found by Fearing and Scherer in Ref. [18] that the $O(p^6)$ parity even sector $\mathcal{L}_8$ contains 111 independent terms, each with its own low energy constant to be fit to experiment or derived from a model. The authors of Ref. [18] in addition found terms which are proportional to the classical equation of motion derived from $\mathcal{L}_2$. They argue that those terms cannot contribute to physical processes since their effects may be removed by making unitary transformations of the field $\mathcal{U}$—such transformations leave the $S$-matrix unchanged.

In our $O(p^6)$ calculation of the $\pi^0\gamma^*\gamma^*$ form factor we will not require terms from the parity even expression $\mathcal{L}_8$ or from any higher order terms. We will therefore now leave the parity even sector and consider the form of the parity odd terms $\mathcal{L}_4^A$ and $\mathcal{L}_6^A$.

2.2 Parity odd sector

2.2.1 The axial anomaly in chiral perturbation theory

The chiral Lagrangian as described so far has all of the symmetries of QCD, but in addition has an unwanted symmetry not found in QCD. The Lagrangian of §2.1 is invariant under the parity operation $\vec{x} \rightarrow -\vec{x}$, $t \rightarrow t$. It is also invariant under $\mathcal{U} \rightarrow \mathcal{U}^\dagger$—this operation preserves the number of mesons in an interaction, modulo two. Thus only even numbers of mesons can interact at a vertex. This is not a property of QCD, which is invariant only under the combined transformation

$$\vec{x} \rightarrow -\vec{x}, t \rightarrow t, \mathcal{U} \rightarrow \mathcal{U}^\dagger$$

and does allow the interaction of odd numbers of mesons at a vertex. We therefore wish to introduce the most general terms to the chiral Lagrangian which break this unwanted symmetry. At lowest order in the momentum expansion these extra terms arise from the axial anomaly.

The anomaly terms will account for the coupling of odd numbers of mesons to the electromagnetic external field. In addition they will give rise to purely hadronic vertices coupling odd numbers of mesons. In Ref. [19] Wess and Zumino derived a set of consistency relations to be satisfied by an action describing the axial anomaly. Witten [20] later developed an explicit representation of such an action. Interestingly, when this action is written in terms of the exponentiated representation, $\mathcal{U}$, of the meson field it
cannot be expressed in the usual fashion as a space-time integral over some Lagrangian density. Witten instead wrote the action as an integral over a five dimensional open surface labelled $Q$. This surface has as its boundary a four dimensional closed surface which is defined to be normal four dimensional space-time.

$$S_{WZW} \equiv \int_Q d^5xe^{ijklm} \omega_{ijklm} \text{ where } i, \ldots, m = 0, 1, 2, 3, 4$$

$$\omega_{ijklm} \equiv -iN_c \frac{1}{240\pi^2} \left\langle (U'^1\partial_i U)(U'^1\partial_j U)(U'^1\partial_k U)(U'^1\partial_m U) \right\rangle \quad (2.6)$$

Here $e^{ijklm}$ is the totally antisymmetric object in five dimensions, with $\epsilon^{01234} = +1$, and so $d^5xe^{ijklm}$ is an infinitesimal element of the five-surface $Q$. One can expand the field $U$ to any desired order in the meson field $\phi$ and manipulate the resulting expression into a four dimensional integral over a Lagrangian density by writing the integrand as a total derivative and applying a higher dimensional analog of Stoke’s theorem,

$$\int_Q d^5x e^{ijklm} \partial_i \tau_{jklm} = \oint d^4x e^{\mu\alpha\beta} \tau_{\mu\alpha\beta} \quad (2.7)$$

with $\tau_{\mu\alpha\beta}$ being an arbitrary tensor. Examples of such Stoke’s theorem extensions are given for other dimensionalities in Ref. [21, §6].

As an example we treat the lowest order in such an expansion of the field $U$, which contains five occurrences of the meson field.

$$S_{WZW} = -iN_c \frac{1}{240\pi^2} \int d^5xe^{ijklm} \left( \frac{i}{F} \right)^5 \left\langle \left( \partial_i \phi \right) \left( \partial_j \phi \right) \left( \partial_k \phi \right) \left( \partial_m \phi \right) \right\rangle + O(\phi^7)$$

$$= \frac{N_c}{240\pi^2 F^5} \int d^5xe^{ijklm} \left\langle \partial_i \left( \phi(\partial_j \phi)(\partial_k \phi)(\partial_m \phi) \right) \right\rangle$$

$$- \left\langle \phi(\partial_i \partial_j \phi)(\partial_k \phi)(\partial_m \phi) \right\rangle$$

$$- \left\langle \phi(\partial_i \phi)(\partial_j \partial_k \phi)(\partial_m \phi) \right\rangle$$

$$- \left\langle \phi(\partial_i \phi)(\partial_j \phi)(\partial_k \partial_m \phi) \right\rangle$$

$$- \left\langle \phi(\partial_i \phi)(\partial_j \phi)(\partial_k \partial_m \phi) \right\rangle + O(\phi^7)$$

Due to the antisymmetry of $e^{ijklm}$ the four terms with symmetric derivatives like $(\partial_i \partial_j \phi)$ vanish. We may apply Eq. (2.7) to find the result

$$S_{WZW} = \int d^4x L^A_{4(\phi^7)} + O(\phi^7)$$

where

$$L^A_{4(\phi^7)} = \frac{N_c}{240\pi^2 F^5} e^{\mu\alpha\beta} \left\langle \phi(\partial_\mu \phi)(\partial_\nu \phi)(\partial_\alpha \phi)(\partial_\beta \phi) \right\rangle$$
2.2 Parity odd sector

As it stands, this action gives only the coupling of the meson fields to one another. Witten used a trial-and-error method to find the gauge invariant addition to $\Gamma_{WZW}$ which would give the photon couplings. His result, though gauge invariant, was flawed in that it failed to respect parity invariance, which is of course a symmetry of the electromagnetic and strong interactions we wish to model. This flaw was corrected by Kaymakcalan, Rajeev and Schechter [22] whose result we quote below.

$$\tilde{S}_{WZW} = S_{WZW} + N_c \int d^4 x \left( -e A_\mu J^\mu + \frac{ie^2}{24\pi^2} \epsilon^{\mu\nu\sigma\beta} (\partial_\mu A_\nu) A_\sigma T_\beta \right)$$  \hspace{1cm} (2.8)

where

\begin{align*}
J^\mu &\equiv \frac{1}{48\pi^2} \epsilon^{\mu\nu\alpha\beta} \left< (Q_\nu U U^\dagger)(\partial_\nu U U^\dagger)(\partial_\rho U U^\dagger) + Q(U^\dagger \partial_\nu U)(U^\dagger \partial_\alpha U)(U^\dagger \partial_\beta U) \right> \\
T_\beta &\equiv \left< Q^2 (\partial_\beta U U^\dagger) + Q^2 (U^\dagger \partial_\beta U) + a Q U Q U^\dagger (\partial_\beta U) + b Q U U^\dagger Q (\partial_\beta U) \right>.
\end{align*}

Witten's original work had $a = 1$, $b = 0$ in the above expression, and while the authors of Ref. [22] found that any such expression having $a + b = 1$ satisfies gauge invariance, they also found it necessary to impose $a - b = 0$ in order to make the action parity invariant. With the definitions $L^\mu \equiv U^\dagger \partial^\mu U$ and $R^\mu \equiv \partial^\mu U U^\dagger$ we have our final form for $J^\mu$ and $T_\beta$.

\begin{align*}
J^\mu &= \frac{1}{48\pi^2} \epsilon^{\mu\nu\alpha\beta} \left< Q(R_\nu R_\alpha R_\beta + L_\nu L_\alpha L_\beta) \right> \\
T_\beta &= \left< Q^2 (R_\beta + L_\beta) + \frac{1}{2} Q U Q U^\dagger R_\beta + \frac{1}{2} Q U U^\dagger Q L_\beta \right>.
\end{align*}

More recently several authors [23, 24, 25] have rederived the action of the axial anomaly coupled to external fields, avoiding the trial-and-error technique used by Witten. They instead begin with the Bardeen form [26] of the anomalous axial current and from it derive the action. When the only gauge couplings are to vector fields, such as the photon field, each of their results reduces to the action given above.

In Ref. [27] Adkins, Nappi and Witten note that solitonic solutions of the non-linear SU(2) $\times$ SU(2) model, the Skyrme model, have precisely the quantum numbers of QCD baryons provided one includes the effects of the Wess–Zumino–Witten term given in Witten's earlier papers [20, 28]. This is contradicted by the later claim of Bijnens et al. [29] that the WZW term must enter the chiral Lagrangian with sign opposite to that given by Witten. No detailed argument is made in Ref. [29] in support of this claim, and so we take the relative sign of the WZW term from Witten's original work. Though this sign does not enter any observables computed using the chiral action, it is important that we establish the convention since it determines how we define the sign of the low energy constants of the $\mathcal{O}(p^6)$ parity odd sector of the action.
In computing the tree level and one-loop contributions to the $\pi^0 \gamma \gamma$ and $\eta_8 \gamma \gamma$ vertices we shall not require the Wess–Zumino–Witten action $S_{WZW}$ itself, but only the meson-photon couplings given in Eq. (2.8).

### 2.2.2 The parity odd Lagrangian of $O(p^6)$

We take the $\mathcal{L}_\delta^A$ tree level contribution to the $\pi^0 \gamma \gamma$ and $\eta_8 \gamma \gamma$ vertices from the $O(p^6)$ parity odd Lagrangian of Fearing and Scherer [18]. These authors use a very compact notation, designed to make the construction of the general form for the chiral Lagrangian a more straightforward task. Our goal in this section is to remove this notational shell and to see which of the terms in the Lagrangian of Ref. [18] contribute to our processes of interest. At an intermediate stage we shall also be in a position to relate the low energy constants of the Fearing and Scherer Lagrangian to those of other authors who have previously considered this part of the $O(p^6)$ parity odd sector.

The additional notation used in Ref. [18] is

\[
G^{\mu\nu} \equiv F_R^{\mu\nu} U + U F_L^{\mu\nu} \\
H^{\mu\nu} \equiv F_R^{\mu\nu} U - U F_L^{\mu\nu} \\
\left[ A \right]_\pm \equiv \frac{1}{2} \left( A U^\dagger \pm U A^\dagger \right) \tag{2.9}
\]

where $A$ is any operator. Since in our case we deal with only an electromagnetic external field, we have

\[
F^\mu_\nu = F^\mu_\nu = -e F^{\mu\nu} Q \quad \text{where} \quad F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu
\]

and hence

\[
G^{\mu\nu} = -e F^{\mu\nu} \{ Q, U \} \\
H^{\mu\nu} = -e F^{\mu\nu} [ Q, U ].
\]

We are interested only in those terms of the $\mathcal{L}_\delta^A$ Lagrangian which might contribute to the meson-photon-photon vertices. From the classification scheme of Ref. [18] we see that we need consider the terms listed in their Table V.

\[
\mathcal{L}_\delta^A = A_1 \epsilon_{\mu\nu\alpha\beta} \langle [D^\mu U ]_+ \left( \left[ D^\nu \chi \right]_+ + [G^{\alpha\beta}]_+ \right) - \left[ \chi \right]_+ + \left[ D^\nu G^{\alpha\beta} \right]_+ \rangle \\
\quad + i A_2 \epsilon_{\mu\nu\alpha\beta} \langle [D^\mu U ]_- \left( \left[ D^\nu \chi \right]_+ + [G^{\alpha\beta}]_+ \right) \rangle \\
\quad + i A_3 \epsilon_{\mu\nu\alpha\beta} \langle [D^\mu U ]_- \left( \left[ D^\alpha G^{\lambda\beta} \right]_+ + [G^{\lambda\beta}]_+ \right) \rangle \\
\quad + i A_4 \epsilon_{\mu\nu\alpha\beta} \langle \left[ \chi \right]_- \left[ D^\nu \right]_+ \left[ G^{\alpha\beta} \right]_+ \rangle \\
\quad + i A_5 \epsilon_{\mu\nu\alpha\beta} \langle \left[ \chi \right]_+ \left[ H^{\mu\nu} \right]_+ \left[ G^{\alpha\beta} \right]_+ \rangle \\
\quad + i A_6 \epsilon_{\mu\nu\alpha\beta} \langle \left[ \chi \right]_- \left[ G^{\mu\nu} \right]_+ \left[ G^{\alpha\beta} \right]_+ \rangle + \ldots
\]
2.2 Parity odd sector

The notation used here is extremely terse. The brackets (...) imply a trace over flavour indices, [...] and {...} are regular commutators and anti-commutators, while the [...]± are the operators defined in Eq. (2.9) above. Only the braces (...) are simple grouping symbols. During our calculation we shall ignore terms containing two or more meson fields or three or more photon fields. We are interested in the $\mathcal{L}_d^A$ couplings of the $\pi^0$ and $\eta_8$, and since terms from the $\mathcal{L}_d^A$ part of the Lagrangian enter only at tree level in our calculation we may take $U$ to be composed only of neutral mesons. This implies that $[U,Q] = 0$. In particular the covariant derivative reduces to an ordinary derivative in this circumstance; $D^\mu U = \partial^\mu U$. In order that we can unambiguously compare our notation to that of Refs. [30, 31] we do not yet use the relation $[U,Q] = 0$ to further simplify the remaining terms. The building blocks of the expressions above are

$$
[x]_\pm = B(mU^\dagger \pm Um)
$$

$$
[G^\alpha\beta]_+ = -\epsilon F^\alpha\beta \{Q,U\} U^\dagger
$$

$$
[H^\alpha\beta]_+ = -\epsilon F^\alpha\beta \{Q,U\} U^\dagger
$$

$$
[D^\mu x]_+ = -2ieBA \left[ [m,Q] \right]_+ = 0 \text{ since } [m,Q] = 0
$$

$$
[D^\mu U]_- = \delta^\mu U U^\dagger
$$

$$
[D^\mu G^{\alpha\beta}]_+ = -e\delta^\mu F^{\alpha\beta} \{Q,U\} U^\dagger - \frac{1}{2} e F^{\alpha\beta} \left( \{Q,\partial^\mu U\} U^\dagger + U \{Q,\partial^\mu U^\dagger\} \right).
$$

We now consider each term’s contribution to the meson-photon-photon vertex in turn.

- $A_1$:

$$
A_1 \epsilon_{\mu\nu\lambda\beta} \left( [D^\mu U]_- \left( \left[ [D^\nu x]_+, [G^{\alpha\beta}]_+ \right] \right) \right)
$$

This term has, with the loss of the covariant derivatives due to the neutral meson condition, only a single photon field. It does not, therefore, contribute to the vertices $\pi^0\gamma\gamma$ or $\eta_8\gamma\gamma$.

- $A_2$:

$$
i A_2 \epsilon_{\mu\nu\lambda\beta} \left( [D^\mu U]_- \left\{ [D^\nu G^{\alpha\lambda}]_+, [G^{\beta}]_+ \right\} \right)
$$

$$
= i e^2 A_2 \epsilon_{\mu\nu\lambda\beta} \partial^\nu F^{\alpha\lambda} F^\beta_\lambda \left\{ \partial^\mu U\{Q,U\} U^\dagger + \{Q,U\} U^\dagger \right\}
$$

$$
= 2ie^2 A_2 \epsilon_{\mu\nu\lambda\beta} \partial^\nu F^{\alpha\lambda} F^\beta_\lambda \left( \partial^\mu U \{U^\dagger Q^2 + QU^\dagger Q\} - \partial^\mu U^\dagger \{QUQ + UQ^2\} \right)
$$

We choose to write

$$
\epsilon_{\mu\nu\lambda\beta} \partial^\nu F^{\alpha\lambda} F^\beta_\lambda = \epsilon_{\mu\nu\lambda\beta} \left( -\partial^\lambda F^{\alpha\nu} F_{\lambda\beta} - \partial^\alpha F^{\nu\lambda} F^\beta_\lambda \right)
$$

$$
= \epsilon_{\mu\nu\lambda\beta} \partial^\lambda F^{\nu\alpha} F_{\lambda\beta} - \epsilon_{\mu\nu\lambda\beta} \partial^\nu F^{\lambda\alpha} F^\beta_\lambda
$$

$$
= \epsilon_{\mu\nu\lambda\beta} \partial^\nu F^{\lambda\alpha} F^\beta_\lambda = \frac{1}{2} \epsilon_{\mu\nu\lambda\beta} \partial^\lambda F^{\nu\alpha} F^\beta_\lambda
$$
where in the first line we used the Bianchi identity for $F_{\mu\nu}$
\[ \partial^\lambda F_{\mu\nu} + \partial^\mu F_{\nu\lambda} + \partial^\nu F_{\lambda\mu} = 0 \]
and in the second line we used the antisymmetry under index interchanges of $F_{\mu\nu}$ and of $\epsilon_{\mu\nu\rho\sigma}$. The $A_2$ term is finally
\[ ie^2 A_2 \epsilon_{\mu\nu\rho\sigma} \partial^\lambda F^\nu_{\mu} F^\rho_{\lambda} \langle \partial^\mu U (U^\dagger Q^2 + QU^\dagger Q) - \partial^\mu U^\dagger (QU Q + U Q^2) \rangle. \]

- $A_3$:
\[ iA_3 \epsilon_{\mu\nu\rho\sigma} \langle [D^\mu U] - (\{D_{\lambda} G^{\lambda\nu}\} + [G_{\alpha\beta}]) \rangle \]
\[ = ie^2 A_3 \epsilon_{\mu\nu\rho\sigma} (\partial^\lambda F^\nu_{\mu} F^\rho_{\alpha\beta} - \partial^\rho F^\nu_{\alpha\beta} F^\lambda_{\mu}) \langle \partial^\mu U U^\dagger \{Q, U\} U^\dagger \{Q, U\} U^\dagger \{Q, U\} U^\dagger \rangle \]
\[ = 2ie^2 A_3 \epsilon_{\mu\nu\rho\sigma} (\partial^\lambda F^\nu_{\mu} F^\rho_{\alpha\beta} - \partial^\rho F^\nu_{\alpha\beta} F^\lambda_{\mu}) \times \langle \partial^\mu U (U^\dagger Q^2 + QU^\dagger Q) - \partial^\mu U^\dagger (QU Q + U Q^2) \rangle \]

- $A_4$:
\[ iA_4 \epsilon_{\mu\nu\rho\sigma} \langle [x] - [G^\mu] + [G^\nu] \rangle \]
\[ = ie^2 A_4 \epsilon_{\mu\nu\rho\sigma} F^\mu_{\nu} F^\rho_{\alpha\beta} \langle B (m U^\dagger - U m) \{Q, U\} U^\dagger \{Q, U\} U^\dagger \rangle \]
\[ = ie^2 A_4 \epsilon_{\mu\nu\rho\sigma} F^\mu_{\nu} F^\rho_{\alpha\beta} \langle B m (3Q^2(U^\dagger - U) + U^\dagger QU U^\dagger - U QU^\dagger Q U) \rangle \]

where we have used $[m, Q] = 0$ in the last step.

- $A_5$:
\[ iA_5 \epsilon_{\mu\nu\rho\sigma} \langle [x] + [H^\mu] + [G^\alpha] \rangle \]
\[ = ie^2 A_5 \epsilon_{\mu\nu\rho\sigma} F^\mu_{\nu} F^\rho_{\alpha\beta} \langle B (m U^\dagger + U m) \{Q, U\} U^\dagger \{Q, U\} U^\dagger \rangle \]

This is proportional to $[Q, U]$ and so gives no contribution when applied to neutral mesons.

- $A_6$:
\[ iA_6 \epsilon_{\mu\nu\rho\sigma} \langle [x] - [G^\mu] + [G^\nu] \rangle \]
\[ = ie^2 A_6 \epsilon_{\mu\nu\rho\sigma} F^\mu_{\nu} F^\rho_{\alpha\beta} \langle B m (U^\dagger - U) \{Q, U\} U^\dagger \{Q, U\} U^\dagger \rangle \]
\[ = 2ie^2 A_6 \epsilon_{\mu\nu\rho\sigma} F^\mu_{\nu} F^\rho_{\alpha\beta} \langle B m (U^\dagger - U) \{Q, U\} U^\dagger \{Q, U\} U^\dagger \rangle \]
There are also terms in the $\mathcal{L}^A_6$ Lagrangian which are proportional to the classical equation of motion. It has been proposed in Ref. [18] that such terms can always be removed by a suitable unitary transformation of the field $U$. The S-matrix is invariant under such unitary transformations and so it seems that these equation of motion terms have no physical content. For completeness, however, we shall compute the contribution of these terms to the $\pi^0 \gamma \gamma$ and $\eta \gamma \gamma$ vertices. By examination of Table VIII in Ref. [18] we find that there is only one equation of motion term which we must consider. Defining

$$\mathcal{O}^{(2)}_{\text{som}} = 2[D_\mu D_\nu U]_+ - 2\langle |x| \rangle + \frac{2}{3} \langle |x| \rangle$$

this term is

$$i E_{29} \epsilon_{\mu\nu\alpha\beta} \left( \mathcal{O}^{(2)}_{\text{som}} [G^{\mu\nu}]_+ [G^{\alpha\beta}]_+ \right)$$

$$= i e^2 E_{29} \epsilon_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta} \left( \left( \partial^2 U U^1 - U \partial^2 U^1 \right)(Q + UQU^1)^2 \right)$$

$$- 2B \left( (mU^1 - Um)(Q + UQU^1)^2 \right)$$

$$+ \frac{4}{3} B \left( m(U^1 - U) \right) \left( Q \left( Q, U^1 \right) U \right).$$

For later reference we collect the terms of $\mathcal{L}^A_6$ which contribute to the meson-photon-photon vertex.

$$\mathcal{L}^A_6 = i e^2 A_2 \epsilon_{\mu\nu\alpha\beta} \partial^\lambda F^{\mu\nu} F^{\alpha\beta} \left( \partial U \left( U^1 Q^2 + QU^1 Q \right) - \partial U^1 \left( QUQ + UQ^2 \right) \right)$$

$$+ 2i e^2 A_3 \epsilon_{\mu\nu\alpha\beta} \left( \partial_\lambda F^{\mu\nu} F^{\alpha\beta} - \partial_\alpha F^{\mu\beta} F^{\lambda\nu} \right)$$

$$\times \left( \partial^\nu U \left( U^1 Q^2 + QU^1 Q \right) - \partial U^1 \left( QUQ + UQ^2 \right) \right)$$

$$+ i e^2 A_4 \epsilon_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta} \left( Bm \left( QU^1 U^1 - U \right) + U^1 QUQU^1 - U QUQ U \right)$$

$$+ 2i e^2 A_5 \epsilon_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta} \left( Bm(U^1 - U) \right) \left( Q \left( Q, U \right) U \right)$$

$$+ i e^2 E_{29} \epsilon_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta} \left( \left( \partial^2 UU^1 - U \partial^2 U^1 \right)(Q + UQU^1)^2 \right)$$

$$- 2B \left( (mU^1 - Um)(Q + UQU^1)^2 \right)$$

$$+ \frac{4}{3} B \left( m(U^1 - U) \right) \left( Q \left( Q, U^1 \right) U \right).$$

We are now in a position to compare our four terms with the parity odd, $\mathcal{O}(p^6)$ Lagrangians of other authors: Bijnens, Bramon and Cornet in Ref. [30], and Donoghue and Wyler in Ref. [31]. Neither of these groups considered terms proportional to the classical equation of motion and so we set $E_{29} = 0$ for the comparisons. The Lagrangian used in Ref. [31] is

$$\mathcal{L}^A_{\gamma\gamma} = \frac{N_c}{4\pi F^3} \left( \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \left( \left( 2Bm(U - U^1) \right) \left( Q^2 \right) c_1 + c_2 \left( 2Bm(Q^2 U - U^1 Q^2) \right) \right)$$

where $N_c$ is the number of colors.
The relationship between their low energy constants \( c_i \) and our \( A_2, A_3, A_4 \) and \( A_6 \) may then be extracted by inspection. Though the condition \( [U, Q] = 0 \) has been used to exclude terms from our Lagrangian it has not been used to manipulate the form of the remaining terms. The following relations between these remaining terms are therefore quite general.

\[
\begin{align*}
(c_1 + c_3) & = \left( -\frac{32i\pi^2 F^2}{N_c} \right) A_6 \\
(c_2 + c_4) & = \left( -\frac{32i\pi^2 F^2}{N_c} \right) A_4 \\
(c_5 + c_6) & = \left( -\frac{32i\pi^2 F^2}{N_c} \right) (A_2 - 2A_3) \\
(c_7 + c_8) & = \left( -\frac{64i\pi^2 F^2}{N_c} \right) A_3 
\end{align*}
\]

(2.10)

This would imply that the coefficients of the Donoghue and Wyler Lagrangian can only appear in a meson-photon-photon process in the linear combinations \( (c_1 + c_3), (c_2 + c_4), (c_5 + c_6) \) and \( (c_7 + c_8) \). The authors of Ref. [31] do indeed find this to be the case when they compute the \( \mathcal{O}(p^6) \) contribution to \( \pi^0 \to \gamma \gamma \) and \( \eta_8 \to \gamma \gamma \). The orthogonal linear combinations of \( c_i \) coefficients contribute only to vertices with three or more mesons. In the Fearing and Scherer formalism employed by us the terms in \( \mathcal{L}_5^{\text{sym}} \) coupling three or more mesons to two photons have the coefficients \( A_7 \to A_{25} \). We find that, for instance, \( (c_1 - c_3) \sim A_{28} \) and \( (c_2 - c_4) \sim A_{35} \). The \( \mathcal{O}(p^6) \) parity odd Lagrangian presented in Ref. [31] for the \( \pi^0 \to \gamma \gamma \) and \( \eta_8 \to \gamma \gamma \) vertices is therefore missing no terms in comparison with our work but does have four excess degrees of freedom. Also, it appears that the coefficients \( c_i \) have been defined rather oddly in that their numeric values would be purely imaginary.

We may repeat this comparison for the case of Ref. [30]. The relevant terms from their Lagrangian are

\[
\mathcal{L}_5^{\text{sym}} = i\epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \left( a_1 \langle mQ^2(U - U^\dagger) \rangle + a'_1 \langle Q^2 \rangle \langle m(U - U^\dagger) \rangle \right)
\]
2.2 Parity odd sector

\[ +a_2\left(m(UQU'Q U - U'QUQU'^{\dagger})\right) \]
\[ +ie^{\mu\nu\alpha\beta}\partial^\lambda F_{\mu\nu}F_{\alpha\beta}\left(3\left(Q^2 U'\partial_\mu U - Q^2 U\partial_\mu U'^{\dagger}\right) + B_1\left(QU'\partial_\mu U - QU\partial_\mu U'^{\dagger}\right)\right). \]

Here we have omitted several terms which, as the authors of Ref. [30] stated in a later publication [29], provide no contribution to the $\pi^0\gamma\gamma$ or $\eta_8\gamma\gamma$ vertices. We also omit terms which are specific to the $\eta_1\gamma\gamma$ vertex since we have not included the singlet eta in our treatment. The Lagrangian above also seems to be missing some terms proportional to $\epsilon^{\mu\nu\alpha\beta}\partial^\lambda F_{\mu\nu}F_{\alpha\beta}$—these are the terms with coefficients $c_5$ and $c_6$ in Donoghue and Wyler’s work and with coefficient $(A_2 - 2A_3)$ in our analysis—and in addition are missing a term of the form $\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}(QUQU'^{\dagger})(m(U - U'^{\dagger}))$. For neutral mesons only composing $U$ we have $[U, Q] = 0$ and so this last term reduces to the form of that in the Bijnens, Bramon and Cornet Lagrangian having coefficient $a'_4$. In making comparisons with this term of coefficient $a'_4$ we therefore assume that $[U, Q] = 0$. Below we relate as many coefficients as possible between our work and that of Ref. [30], given the apparent omissions in the form of their Lagrangian.

\[
\begin{align*}
a_1 &= (-12\pi\alpha B)A_4 \\
a'_1 &= (-16\pi\alpha B)A_6 \\
a_2 &= (-4\pi\alpha B)A_4 \\
b_1 &= (8\pi\alpha)A_3 \\
b_2 &= (8\pi\alpha)A_3
\end{align*}
\]

In the above $B$ is the constant appearing in our definition $\chi \equiv 2Bm$. This is of less utility than the previous comparison with the work of Donoghue and Wyler [31]. A calculation made by the authors of Ref. [30] with particular values for their constants $B_1$ and $B_2$ would have a different result than the same calculation made with our Fearing–Scherer Lagrangian where we had set $A_3 \equiv B_{1,2}/8\pi\alpha$ using the expressions above. Agreement would only be achieved if we were to ignore the terms of the form $\sim A_3\epsilon^{\mu\nu\alpha\beta}\partial^\lambda F_{\mu\nu}F_{\alpha\beta}$ in our Lagrangian, since these terms have been omitted during the construction of the Lagrangian of Ref. [30].

Before proceeding with the calculation of particular processes we shall in the next chapter consider the one-loop renormalization of the chiral Lagrangian.
Chapter 3

Renormalization to One Loop

In chiral perturbation theory the low energy constants at each order in the momentum expansion can pick up renormalization contributions from a limited number of sources. For example the constants $L_1 \ldots L_{10}, H_1, H_2$ of $\mathcal{L}_4$ are required to absorb divergences only from the one-loop structure of $\mathcal{L}_2$. Similarly the thirty-two low energy constants of $\mathcal{L}_6^A$ absorb divergences from one-loop diagrams composed of a vertex from $\mathcal{L}_4^A$ and vertices from $\mathcal{L}_2$. The parameters of this effective theory can therefore be renormalized once and for all, with a finite amount of effort. This is to be contrasted with theories such as quantum electrodynamics, referred to as renormalizable theories, which have a finite number of constants to be renormalized but which must absorb new divergent contributions in these constants at each level of the perturbative expansion.

As has been shown previously [15, 17, 29, 31] it is possible to perform the one-loop part of this renormalization by expanding the appropriate part of the chiral action about a field configuration $\tilde{U}$ which satisfies the classical equation of motion. Taking the path integral over field configurations of this expanded action one may write down a formal expression for the action's one-loop structure. The divergent terms in this one-loop effective action may then be extracted using the heat kernel method, which will be described shortly. Using this technique we shall first compute the one-loop structure of the lowest order part of the chiral action, $S_2 \equiv \int d^4x \mathcal{L}_2$. The divergent parts of this provide the renormalization coefficients for the parameters $L_1 \ldots L_{10}, H_1, H_2$ of the next order in $p^2$ in the parity even sector, $S_4 \equiv \int d^4x \mathcal{L}_4$. In §3.2 we briefly describe how this procedure could be extended in a future project to calculate the renormalization of the $O(p^6)$ parity odd sector.
3.1 Renormalization of $\mathcal{L}_4$

The action we must consider is

$$S_2 = \int d^4x \mathcal{L}_2 = \int d^4x \frac{F^2}{4} \left\langle D_\mu U D^\mu U^\dagger + \chi^\dagger U + U^\dagger \chi \right\rangle.$$ 

We expand the field $U$ about a configuration $\bar{U}$ which we shall set to be a solution of the classical equation of motion for $S_2$. This expansion may be parameterized in many equivalent ways. We choose to use $U \equiv \bar{U} e^{i\Delta}$ with $\Delta \equiv \Delta^a \lambda^a$ Hermitian so that $U$ and $\bar{U}$ may be kept unitary simultaneously. The $\Delta^a$ are therefore a set of real fields.

The action $S_2(U)$ is now expanded to second order in the fields $\Delta^a$ about $\bar{U}$. This will allow us to extract the first and second variations of $S_2(U)$—the first variation will be set to zero, giving us the classical equation of motion to this order in $p^2$, while the second variation after path integration over the field variable $\Delta$ will give us the one-loop effective action for $S_2$.

$$S_2(U) = \int d^4x \mathcal{L}_2 = \int d^4x \frac{F^2}{4} \left\langle D_\mu (\bar{U} e^{i\Delta}) D^\mu (e^{-i\Delta} \bar{U}^\dagger) + \chi^\dagger (\bar{U} e^{i\Delta}) + (e^{-i\Delta} \bar{U}^\dagger) \chi \right\rangle$$

Expanding to second order in $\Delta$ we have

$$D_\mu (\bar{U} e^{i\Delta}) = \partial_\mu (\bar{U} e^{i\Delta}) - i\partial_\mu \bar{U} e^{i\Delta} + i\bar{U} e^{i\Delta} \partial_\mu$$

$$= \left( \partial_\mu \bar{U} + i\bar{U} \partial_\mu \Delta - i\partial_\mu \bar{U} + i\bar{U} e^{i\Delta} \partial_\mu e^{-i\Delta} \right) e^{i\Delta}$$

and the adjoint of this operator is

$$D^\mu (e^{-i\Delta} \bar{U}^\dagger)$$

$$= e^{-i\Delta} \left( \partial^\mu \bar{U}^\dagger - i\partial^\mu \Delta \bar{U}^\dagger + i\bar{U}^\dagger \partial^\mu - i\bar{U}^\dagger \Delta \bar{U}^\dagger + \frac{1}{2} [\partial^\mu, \Delta] \bar{U}^\dagger \right) + O(\Delta^3)$$

while for the mass terms we have

$$\left\langle \chi^\dagger \bar{U} e^{i\Delta} + e^{-i\Delta} \bar{U}^\dagger \chi \right\rangle = \left\langle \chi^\dagger \bar{U} + \bar{U}^\dagger \chi \right\rangle + i\Delta \left\langle \chi^\dagger \bar{U} - \bar{U}^\dagger \chi \right\rangle$$

$$- \frac{1}{2} \Delta^2 \left\langle \chi^\dagger \bar{U} + \bar{U}^\dagger \chi \right\rangle + O(\Delta^3).$$

The action is then

$$S_2(U) = \int d^4x \frac{F^2}{4} \left\langle D_\mu \bar{U} D^\mu \bar{U}^\dagger + \chi^\dagger \bar{U} + \bar{U}^\dagger \chi \right\rangle$$

$$+ \left\langle -2i\bar{U}^\dagger D_\mu \bar{U} D^\mu \Delta + i\Delta (\chi^\dagger \bar{U} - \bar{U}^\dagger \chi) \right\rangle$$

$$+ \left\langle D_\mu \Delta D^\mu \Delta + i\bar{U}^\dagger D_\mu \bar{U} [\partial^\mu, \Delta] - \frac{1}{2} \Delta^2 (\chi^\dagger \bar{U} + \bar{U}^\dagger \chi) \right\rangle$$
where we have defined a derivative operator for \( \Delta \)

\[
D_\mu \Delta \equiv \partial_\mu \Delta - i[\ell_\mu, \Delta]
\]

and have used the relation

\[
D^\mu \tilde{U}^\dagger \tilde{U} = D^\mu (\tilde{U}^\dagger \tilde{U}) - \tilde{U}^\dagger D^\mu \tilde{U} = -\tilde{U}^\dagger D^\mu \tilde{U} \quad \text{since} \quad \tilde{U}^\dagger \tilde{U} = 1.
\]

We match this expression against variations of the action \( S_2(U) \) with respect to the fields \( \Delta^a \) about the so-called background field \( \tilde{U} \).

\[
S_2(U) = S_2(\tilde{U}) + \Delta^a \frac{\delta S}{\delta \Delta^a}(\tilde{U}) + \frac{1}{2} \Delta^a \frac{\delta^2 S}{\delta \Delta^a \delta \Delta^b}(\tilde{U}) \Delta^b + O(\Delta^3)
\]

Here we have defined \( \tilde{D}_\mu \equiv \tilde{U}^\dagger D_\mu \tilde{U} \). Setting the first variation to zero will give us the equation of motion satisfied by \( \tilde{U} \),

\[
\Delta^a \frac{\delta S}{\delta \Delta^a}(\tilde{U}) = \int d^4x \frac{F^2}{4} (-2i) \langle \tilde{L}_\mu \tilde{D}^\mu \Delta - \frac{1}{2} \Delta (\chi^\dagger \tilde{U} - \tilde{U}^\dagger \chi) \rangle
\]

Here we have defined \( \tilde{L}_\mu \equiv \tilde{U}^\dagger D_\mu \tilde{U} \). Setting the first variation to zero will give us the equation of motion satisfied by \( \tilde{U} \),

\[
\Delta^a \frac{\delta S}{\delta \Delta^a}(\tilde{U}) = \int d^4x \frac{F^2}{2} \langle - D^\mu \tilde{L}_\mu \Delta - \frac{1}{2} \Delta (\chi^\dagger \tilde{U} - \tilde{U}^\dagger \chi) \rangle = 0
\]

after integration by parts of the first term. We have used the fact that the covariant derivatives have suitable definitions that they may be manipulated like ordinary derivatives, and also that the volume integral over all space of a total ordinary derivative is zero. In particular we used

\[
\partial^\mu (\tilde{L}_\mu \Delta) = D^\mu \tilde{L}_\mu \Delta + \tilde{L}_\mu D^\mu \Delta
\]

which may be checked explicitly using the definitions of the covariant derivatives of \( U \) and \( \Delta \). Setting the first variation of the action to zero gives us

\[
\langle \chi^a \left( D^\mu \tilde{L}_\mu \Delta + \frac{1}{2} (\chi^\dagger \tilde{U} - \tilde{U}^\dagger \chi) \right) \rangle = 0.
\]

In matrix form this is

\[
D^\mu (\tilde{U}^\dagger D_\mu \tilde{U}) + \frac{1}{2} (\chi^\dagger \tilde{U} - \tilde{U}^\dagger \chi) - \frac{1}{6} \langle \chi^\dagger \tilde{U} - \tilde{U}^\dagger \chi \rangle = 0
\]

where the trace term arises due to the minimization of the action under the constraint \( \det(\tilde{U}) = 1 \), since \( \tilde{U} \) is an SU(3) matrix. We take the form of this extra term directly from Ref. [17].
3.1 Renormalization of $\mathcal{L}_4$

We now define an effective action $Z$ which is equal, up to $\mathcal{O}(\Delta^2)$, to the action $S_2(U)$ after it has been path integrated over all field configurations $\Delta^a$.

$$
\int [d\Delta^a] e^{\delta S_i(U)} = \int [d\Delta^a] \exp \left( iS_2(\bar{U}) + \frac{i}{\Delta^a} \frac{\delta^2 S}{\delta \Delta^a \delta \Delta^b}(\bar{U}) \Delta^b + \mathcal{O}(\Delta^3) \right)
$$

We take the result of this integration over a Gaussian function directly from Ref. [32, pp. 75]. The second variation is written in the form

$$
\Delta^a \frac{\delta^2 S}{\delta \Delta^a \delta \Delta^b} \Delta^b = C \int d^4x \Delta^a (d^\mu d_\mu + m^2 + \sigma)^{ab} \Delta^b
$$

where $C$, $m^2$ are constants, the operator $d_\mu$ is defined as $d_\mu \equiv \delta^{ab} \partial_\mu + \Gamma_\mu^{ab}$, and $\sigma^{ab}$, $\Gamma_\mu^{ab}$ contain no derivatives. The result of the path integral is then

$$
\int [d\Delta^a] e^{\delta S_i(U)} = \mathcal{N} \exp \left( iS_2(\bar{U}) - \frac{1}{2} \int d^4x \left( \langle x | \log (d^\mu d_\mu + m^2 + \sigma) | x \rangle \right) \right).
$$

Here $a, b$ are $SU(N)$ indices and $\mathcal{N}$ is a divergent constant resulting from the path integration. The $|x\rangle$ are some complete set of coordinate-space states. We shall find it convenient to work in $N$ flavours throughout our calculation since the various powers of $N$ which arise provide useful tags with which to group the many terms that appear and therefore help to keep the algebra under control—at the end we shall reduce our result to $N = 3$.

The effective action is defined by

$$
Z \equiv S_2(\bar{U}) + Z_{\text{loop}}
$$

$$
Z_{\text{loop}} \equiv \frac{1}{2} \int d^4x \left( \langle x | \log (d^\mu d_\mu + m^2 + \sigma) | x \rangle \right)
$$

(3.2)

where $Z_{\text{loop}}$ contains the full one-loop structure of our original action $S_2(U)$. This one-loop effective action is of $\mathcal{O}(p^4)$ and contains divergences which must be absorbed within the low energy constants of the $\mathcal{O}(p^4)$ part of the chiral action

$$
S_4 \equiv \int d^4x L_4(\bar{U}; L_1 \ldots L_{10}, H_1, H_2).
$$

To $\mathcal{O}(p^4)$ the full effective action is therefore

$$
S_2(\bar{U}) + (Z_{\text{loop}}^{\text{finite}} + Z_{\text{loop}}^{\text{dir}}) + S_4(\bar{U}; L_1 \ldots L_{10}, H_1, H_2)
$$

which we wish to write as

$$
S_2(\bar{U}) + Z_{\text{loop}}^{\text{finite}} + S_4(\bar{U}; L_1' \ldots L_{10}', H_1', H_2').
$$
3.1 Renormalization of $L_4$

Clearly we have to define renormalized constants $L'_1 \ldots L'_{10}, H'_1, H'_2$ such that the following is satisfied,

$$S_4(\bar{U}; L'_1 \ldots L'_{10}, H'_1, H'_2) = S_4(\bar{U}; L_1 \ldots L_{10}, H_1, H_2) + Z_{\text{loop}}^{\text{div}}.$$  

The divergent structure $Z_{\text{loop}}^{\text{div}}$ may be extracted from $Z_{\text{loop}}$ of Eq. (3.2) using a technique called the heat kernel method. The renormalization of the $O(p^4)$ parity-even sector's low energy constants may then be performed by comparing the functional forms in $Z_{\text{loop}}^{\text{div}}$ with those in $S_4(\bar{U})$. We shall quote the necessary results of the heat kernel approach directly from Appendix B of Ref. [33] and from Ref. [34].

Using the dimensional method to regulate divergences, the one-loop effective action may be written

$$\langle x| \log D|x \rangle = \frac{i}{16\pi^2} \left( \frac{m^2}{4\pi\mu^2} \right)^{-\epsilon} \sum_{n=0}^{\infty} a_n (m^2)^{2-n}(n-2+\epsilon).$$

Here the space-time dimension is defined to be $2(2-\epsilon)$. The divergences which occur in this expression in the limit $\epsilon \to 0$ are contained within the $\Gamma$-function, and $\Gamma(x)$ is singular for $x = 0, -1, -2, \ldots$. It can be seen, therefore, that divergences arise only in the terms $n = 0, 1$ and 2. We quote the heat kernel coefficients $a_0, a_1$ and $a_2$ from Ref. [34],

$$a_0 = 1, \quad a_1 = -\sigma, \quad a_2 = \left\langle \frac{1}{2}\sigma^2 + \frac{1}{12} \Gamma_{\mu\nu} \Gamma^{\mu\nu} \right\rangle$$

where we have defined

$$\Gamma^{ab}_{\mu\nu} = \partial_{\mu} \Gamma^{ab}_{\nu} - \partial_{\nu} \Gamma^{ab}_{\mu} + \Gamma^{ac}_{\mu} \Gamma^{c}_{\nu} - \Gamma^{ac}_{\nu} \Gamma^{c}_{\mu}.$$  

The divergent part of the $O(p^4)$ one-loop effective action is then

$$Z_{\text{loop}}^{\text{div}} = \frac{i}{32\pi^2} \left( \frac{m^2}{4\pi\mu^2} \right)^{-\epsilon} \left( \left\langle a_0 \right\rangle m^4 \Gamma(-2+\epsilon) + \left\langle a_1 \right\rangle m^2 \Gamma(-1+\epsilon) + \left\langle a_2 \right\rangle \Gamma(\epsilon) \right).$$

Given this equation our procedure is straightforward—we must write the second variation of our expanded action in the form of Eq. (3.1), pick off the structures $\Gamma^{ab}_{\mu}$ and $\sigma^{ab}$, and then insert them into the equation above for $Z_{\text{loop}}^{\text{div}}$. The renormalization of $S_4(\bar{U}; L_1 \ldots L_{10}, H_1, H_2)$ may then be done by inspection.
The second variation of the action is
\begin{align*}
\frac{1}{2} \Delta^2 \frac{\delta^2 S}{\delta \Delta^a \delta \Delta^b} &= \int d^4z \frac{F^2}{4} \left( D_\mu \Delta D^\mu \Delta + i \hat{L}_\mu [\ell', \Delta] - \frac{1}{2} \Delta^2 (\chi' \hat{U} + \hat{U}' \chi) \right) \\
&= \int d^4z \frac{F^2}{4} \left( \frac{1}{2} \Delta^2 \left( \chi' \hat{U} + \hat{U}' \chi \right) \right)
\end{align*}
where we have used the relation
\[ \Delta D^\mu \Delta - D^\mu \Delta \Delta = \Delta \partial^\mu \Delta - i \Delta [\ell, \Delta] - \partial^\mu \Delta \Delta + i [\ell', \Delta] \Delta = i \left[ [\ell', \Delta], \Delta \right]. \]

We complete the square on \( D^\mu \Delta \) by defining
\[ \tilde{D}^\mu \Delta \equiv D^\mu \Delta + \frac{1}{2} \hat{L}_{\mu} \Delta \]
where \( \hat{L}_{\mu} \equiv \ell_{\mu} + \frac{1}{2} \hat{L}_{\mu} \).
\begin{align*}
\frac{1}{2} \Delta^2 \frac{\delta^2 S}{\delta \Delta^a \delta \Delta^b} &= \int d^4z \frac{F^2}{4} \left( \tilde{D}_\mu \Delta \tilde{D}^\mu \Delta - \frac{1}{4} \hat{L}_\mu, \Delta][\hat{L}_\mu, \Delta] - \frac{1}{2} \Delta^2 (\chi' \hat{U} + \hat{U}' \chi) \right)
\end{align*}
With a small amount of further manipulation we have the form of the differential operator appearing in the path integral of Eq. (3.1). We must compare terms in the expression
\[ \int d^4z \left( \left( \partial^\mu - i [\tilde{\Gamma}^a, \Delta] \right) (\partial_\mu - i [\tilde{\Gamma}^a, \Delta]) - \frac{i}{4} [\tilde{L}^a, \Delta][\tilde{L}^a, \Delta] - \frac{1}{2} \Delta^2 (\chi' \hat{U} + \hat{U}' \chi) \right) \]
\[ \equiv \int d^4z \Delta^a \left( (\delta^{ac} \partial_\mu + \Gamma_{\mu}^{ac})(\delta^{cb} \partial_\mu + \Gamma_{\mu}^{cb}) + \sigma^{cb} \right) \Delta^b. \]

The overall constant \( C \) is determined by comparison of the ordinary derivatives,
\[ \int d^4z \left( \partial^\mu \Delta \partial_\mu \Delta \right) = \int d^4z \left( \partial^\mu (\Delta \partial_\mu \Delta) - \Delta \partial^\mu \partial_\mu \Delta \right) = - (\lambda^a \lambda^b) \int d^4z \Delta^a \partial^\mu \partial_\mu \Delta^b = (-2) \delta^{ab} \int d^4z \Delta^a \partial^\mu \partial_\mu \Delta^b \]
and so \( C = -2 \). We may now identify by inspection that suitable definitions for \( \Gamma_{\mu}^{ab} \) and \( \sigma^{ab} \) are
\[ \Gamma_{\mu}^{ab} \equiv \frac{i}{2} \left( [\lambda^a, \lambda^b] \tilde{\Gamma}_\mu \right) \] 
\[ \sigma^{ab} \equiv \frac{1}{8} \left( [\tilde{L}^a, \lambda^a][\tilde{L}^a, \lambda^b] + (\lambda^a, \lambda^b) (\chi', \hat{U}') \right). \]

The only term where it is not obvious that these are the correct choices is
\[ \Gamma_{\mu}^{ac} \Gamma_{\mu}^{cb} = - \frac{1}{2} \left( [\lambda^a, \lambda^c] \tilde{\Gamma}_\mu \right) \left( [\lambda^c, \lambda^b] \tilde{\Gamma}_\mu \right). \]
3.1 Renormalization of $\mathcal{L}_4$

We use the SU($N$) completeness relation for $N$ flavours

$$
\sum_{a=1}^{N^2-1} (\lambda^a)_{ij}(\lambda^a)_{kl} = 2 \left( \delta_{ij}\delta_{lk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right)
$$

(3.4)

to write this as

$$
\Gamma_\mu^a \Gamma^c_{\mu}^b = -\frac{1}{4} \left< \lambda^a \lambda^c \tilde{\Gamma}^\mu \lambda^b \lambda^c \tilde{\Gamma}^\mu \right> - \frac{1}{2} \left< \lambda^a \lambda^b \tilde{\Gamma}^\mu \lambda^a \lambda^b \tilde{\Gamma}^\mu \lambda^a \lambda^b \tilde{\Gamma}^\mu \lambda^a \lambda^b \tilde{\Gamma}^\mu \lambda^a \lambda^b \tilde{\Gamma}^\mu \lambda^a \lambda^b \tilde{\Gamma}^\mu \right>
$$

$$
= \frac{1}{N} \left< \lambda^a \tilde{\Gamma}^\mu \lambda^a \tilde{\Gamma}^\mu \right> - \frac{1}{2} \left< \left[ \tilde{\Gamma}^\mu, \lambda^a \right] \left[ \tilde{\Gamma}^\mu, \lambda^b \right] \right>
$$

which is exactly the term we require.

With the identification of $\sigma^{ab}$ and $\Gamma_\mu^a$ we are in a position to compute the necessary heat kernel coefficients $a_n$ and hence the one-loop divergent structure $Z^\text{div}_{\text{loop}}$. At this point we note that we have defined $\Gamma_\mu^a$ and $\sigma^{ab}$ such that the mass-type term $m^2$ in our derivative operator $D_{\mu}^{ab} \equiv (d_\mu d^\mu + m^2 + \sigma)^{ab}$ is zero. The expression we require therefore simplifies to

$$
Z^\text{div}_{\text{loop}} = \frac{1}{32\pi^2} \Gamma(\epsilon) \int d^4z \left< a_2 \right> = \frac{1}{32\pi^2} \frac{\Gamma(1+\epsilon)}{\epsilon} \int d^4z \left< a_2 \right> = \frac{-1}{32\pi^2} \left( -\frac{1}{\epsilon} - \Gamma'(1) + \mathcal{O}(\epsilon) \right) \int d^4z \left< a_2 \right>.
$$

We in fact choose to redefine $Z^\text{div}_{\text{loop}}$ as

$$
Z^\text{div}_{\text{loop}} = -D(\epsilon) \int d^4z \left< a_2 \right> \quad \text{where} \quad D(\epsilon) \equiv \frac{1}{32\pi^2} \left( -\frac{1}{\epsilon} - 1 + \gamma_E - \log 4\pi \right)
$$

and Euler's constant is $\gamma_E = -\Gamma'(1) \approx 0.577$. We have introduced some finite constants into the definition of $Z^\text{div}_{\text{loop}}$ in order that we be consistent with the \MS\-scheme dimensional regularization procedure to be used in our later perturbation theory calculations. The way we choose to split $Z_{\text{loop}}$ into its finite and divergent parts $Z^{\text{div}}_{\text{loop}}$ and $Z^{\text{finite}}_{\text{loop}}$ is not unique, and we are free to move finite constants in this fashion between the two parts.

It now remains to compute $\left< a_2 \right>$ in terms of $\Gamma_\mu^a$ and $\sigma^{ab}$. We recall that

$$
\left< a_2 \right> = \left< \frac{1}{2} \sigma^2 + \frac{1}{12} \Gamma_\mu \Gamma^{\mu} \right>
$$
where
\[ 
\Gamma_{\mu}^{ab} = \partial_\mu \Gamma_{\nu}^{ab} - \partial_\nu \Gamma_{\mu}^{ab} + [\Gamma_{\mu}^{a c}, \Gamma_{\nu}^{b c}] \\
\Gamma_{\mu}^{ab} = \frac{1}{2} \left\langle [\lambda^a, \lambda^b] \bar{\Gamma}_\mu \right\rangle \\
\sigma_{ab} = \frac{1}{6} \left\langle [\bar{L}_\mu, \lambda^a][\bar{L}_\nu, \lambda^b] + \{\lambda^a, \lambda^b\} M \right\rangle \\
M = (\chi^\dagger \bar{U} + \bar{U}^\dagger \chi).
\]

The first term in \( \left\langle a_2 \right\rangle \) is
\[ 
\left\langle \frac{1}{2} \sigma^2 \right\rangle = \frac{1}{2} \sum_{a,b} \sigma_{ab} \sigma_{ba} \\
= \frac{1}{138} \sum_{a,b} \left\langle [\bar{L}_\mu, \lambda^a][\bar{L}_\nu, \lambda^b] + \{\lambda^a, \lambda^b\} M \right\rangle \left\langle [\bar{L}_\nu, \lambda^a][\bar{L}_\mu, \lambda^b] + \{\lambda^a, \lambda^b\} M \right\rangle \\
= \frac{1}{138} \sum_{a,b} \left(2\bar{\tilde{L}}_\mu \lambda^a \bar{L}_\nu \lambda^b - \{\lambda^a, \lambda^b\}(\bar{L}_\mu \bar{L}_\nu - M)\right) \left(2\bar{\tilde{L}}_\nu \lambda^a \bar{L}_\mu \lambda^b - \{\lambda^a, \lambda^b\}(\bar{L}_\nu \bar{L}_\mu - M)\right).
\]

Using the \( SU(N) \) completeness relation of Eq. (3.4) to sum over the indices \( a \) and \( b \) we find, after some algebra,
\[ 
\left\langle \frac{1}{2} \sigma^2 \right\rangle = \frac{1}{8} \langle \bar{L}_\mu \bar{L}_\nu \rangle \langle \bar{L}_\mu \bar{L}_\nu \rangle + \frac{1}{16} \langle \bar{L}_\mu \bar{L}_\nu \rangle^2 + \frac{N}{16} \langle \bar{L}_\mu \bar{L}_\nu \rangle \langle \bar{L}_\mu \bar{L}_\nu \rangle \\
- \frac{1}{8} \langle M \rangle \langle \bar{L}_\mu \bar{L}_\mu \rangle - \frac{N}{8} \langle M \rangle \langle \bar{L}_\mu \bar{L}_\nu \rangle + \frac{N^2 + 2}{16N^2} \langle M \rangle^2 + \frac{N^2 - 4}{16N} \langle M^2 \rangle.
\]

The remaining term in \( \left\langle a_2 \right\rangle \) is of the form
\[ 
\left\langle \frac{1}{138} \Gamma_{\mu\nu} \Gamma_{\mu\nu} \right\rangle.
\]

Recalling that
\[ 
\Gamma_{\mu}^{ab} = \frac{1}{2} \left\langle [\lambda^a, \lambda^b] \bar{\Gamma}_\mu \right\rangle,
\]
we can write the field strength \( \Gamma_{\mu\nu}^{ab} \) as
\[ 
\Gamma_{\mu\nu}^{ab} = \frac{1}{2} \left\langle [\lambda^a, \lambda^b] \partial_\mu \bar{\Gamma}_\nu \right\rangle - \frac{1}{2} \left\langle [\lambda^a, \lambda^b] \partial_\nu \bar{\Gamma}_\mu \right\rangle \\
- \frac{1}{4} \left( \left\langle [\lambda^a, \lambda^c] \bar{\Gamma}_\mu \right\rangle \left\langle [\lambda^c, \lambda^b] \bar{\Gamma}_\nu \right\rangle - \left\langle [\lambda^a, \lambda^c] \bar{\Gamma}_\nu \right\rangle \left\langle [\lambda^c, \lambda^b] \bar{\Gamma}_\mu \right\rangle \right) \\
= \frac{1}{2} \left\langle [\lambda^a, \lambda^b] \bar{\Gamma}_{\mu\nu} \right\rangle
\]

where in the final line we have used Eq. (3.4) to sum on index \( c \), and where we have defined a field strength for \( \bar{\Gamma}_\mu \),
\[ 
\bar{\Gamma}_{\mu\nu} \equiv \partial_\nu \bar{\Gamma}_\mu - \partial_\mu \bar{\Gamma}_\nu - i[\bar{\Gamma}_\mu, \bar{\Gamma}_\nu].
\]
The term in \( \langle a_2 \rangle \) is of the form
\[
\left\langle \frac{1}{12} \Gamma_{\mu\nu} \Gamma^{\mu\nu} \right\rangle = \frac{1}{12} \sum_{a,b} \Gamma_{\mu\nu}^{ab} \Gamma^{\mu\nu ba} \\
= \frac{1}{16}\sum_{a,b} \left\langle [\lambda^a, \lambda^b] \bar{\Phi} \right\rangle \left\langle [\lambda^a, \lambda^b] \Phi \right\rangle \\
= -\frac{N}{8} \left\langle \bar{\Phi}_{\mu\nu} \Phi^{\mu\nu} \right\rangle + \frac{1}{8} \left\langle \bar{\Phi}_{\mu\nu} \Phi^{\mu\nu} \right\rangle
\]
where in the last step we have once more used the SU(N) completeness relation to sum on \( a \) and \( b \).

We must now compute \( \bar{\Phi}_{\mu\nu} \) in terms of the objects \( D_{\mu} \bar{U}^\dagger, F_+^{\mu\nu} \) et cetera which appear in the \( O(p^4) \) parity even action. Recall that \( F_+^{\mu\nu} \) and \( F_-^{\mu\nu} \) are defined in Eq. (2.1) as field strengths for the left-handed and right-handed external fields \( l^\mu \) and \( r^\mu \). We expand the function \( \bar{\Phi} \) to give
\[
\bar{\Phi}_{\mu\nu} = \partial^\mu \bar{\Phi} - \partial^\nu \bar{\Phi} - i[\bar{\Phi}, \Phi] \\
= \frac{1}{2} \left( 2\partial^\mu (\bar{U}^\dagger \partial^\nu \bar{U} - i\bar{U}^\dagger r^\nu \bar{U} - il^\nu) - 2\partial^\nu (\bar{U}^\dagger \partial^\mu \bar{U} - i\bar{U}^\dagger l^\mu \bar{U} - il^\mu) \\
+ [\bar{U}^\dagger D^\mu \bar{U} - 2il^\mu, \bar{U}^\dagger D^\nu \bar{U} - 2il^\nu] \right), \quad (3.5)
\]
employ the result
\[
\partial^\mu (\bar{U}^\dagger \partial^\nu \bar{U}) - \partial^\nu (\bar{U}^\dagger \partial^\mu \bar{U}) \\
= \partial^\mu \bar{U}^\dagger \partial^\nu \bar{U} - \partial^\nu \bar{U}^\dagger \partial^\mu \bar{U} + \bar{U}^\dagger \partial^\nu \partial^\mu \bar{U} - \bar{U}^\dagger \partial^\mu \partial^\nu \bar{U} \\
= -[\bar{U}^\dagger \partial^\nu \bar{U}, \bar{U}^\dagger \partial^\mu \bar{U}] \quad (3.6)
\]
and the relation between the commutators of partial and covariant derivatives
\[
[\bar{U}^\dagger D^\mu \bar{U}, \bar{U}^\dagger D^\nu \bar{U}] = [\bar{U}^\dagger \partial^\nu \bar{U}, \bar{U}^\dagger \partial^\mu \bar{U}] \\
+ [-i\bar{U}^\dagger r^\mu \bar{U} - il^\mu, \bar{U}^\dagger \partial^\nu \bar{U}] + [\bar{U}^\dagger \partial^\mu \bar{U}, -i\bar{U}^\dagger r^\nu \bar{U} + il^\nu] \\
+ [-i\bar{U}^\dagger l^\mu \bar{U} + il^\mu, -i\bar{U}^\dagger r^\mu \bar{U} + il^\nu] \quad (3.7)
\]
to write \( \bar{\Phi}_{\mu\nu} \) as
\[
\bar{\Phi}_{\mu\nu} = \frac{1}{2} \left( -[\bar{U}^\dagger D^\mu \bar{U}, \bar{U}^\dagger D^\nu \bar{U}] - 2i\partial^\mu \ell^\nu + 2i\partial^\nu \ell^\mu - 2[l^\mu, l^\nu] \\
- 2i\bar{U}^\dagger \partial^\mu r^\nu \bar{U} + 2i\bar{U}^\dagger \partial^\nu r^\mu \bar{U} - 2\bar{U}^\dagger [l^\mu, l^\nu] \bar{U} \right).
\]

There are a great many terms which arise upon the substitution of Eqs. (3.5,3.6) into Eq. (3.7), and subsequently cancel amongst themselves. Finally then we have
\[
\bar{\Phi}_{\mu\nu} = -\frac{1}{4} [\bar{U}^\dagger D^\mu \bar{U}, \bar{U}^\dagger D^\nu \bar{U}] + \frac{1}{2} (F_+^{\mu\nu} + \bar{U}^\dagger F_-^{\mu\nu} \bar{U}).
\]
3.1 Renormalization of $\mathcal{L}_4$

Since the trace of a commutator vanishes and since $\ell_\mu$, $r_\mu$ have the SU($N$) flavour structure $\ell_\mu \equiv \ell_\mu^a \lambda^a$, $r_\mu \equiv r_\mu^a \lambda^a$ with $\langle \lambda^a \rangle = 0$, we see that $\langle \bar{\Gamma}^{\mu\nu} \rangle$ vanishes. The remaining term is

$$\langle \frac{1}{12} \Gamma^{\mu\nu} \Gamma_{\mu\nu} \rangle = -\frac{N}{6} \langle \bar{\Gamma}^{\mu\nu} \bar{\Gamma}_{\mu\nu} \rangle$$

$$= -\frac{N}{6} \left( -\frac{1}{16} \langle [\bar{U}^\dagger D_\mu \bar{U}, \bar{U}'^\dagger D_\nu \bar{U}'][\bar{U}'^\dagger D^\mu \bar{U}, \bar{U}^\dagger D^\nu \bar{U}] \rangle \right)$$

$$- \frac{i}{4} \langle (F_L^{\mu\nu} + \bar{U}'^\dagger F_R^{\mu\nu} \bar{U}) [\bar{U}'^\dagger D_\mu \bar{U}, \bar{U}^\dagger D_\nu \bar{U}] \rangle$$

$$+ \frac{1}{4} \langle (F_L^{\mu\nu} + \bar{U}'^\dagger F_R^{\mu\nu} \bar{U}) (F_L^{\mu\nu} + \bar{U}'^\dagger F_R^{\mu\nu} \bar{U}) \rangle$$

$$= -\frac{N}{48} \langle D_\mu \bar{U}^\dagger D^\mu \bar{U}'^\dagger D^\nu \bar{U}' \rangle + \frac{N}{48} \langle D_\mu \bar{U}^\dagger D_\nu \bar{U}' D^\mu \bar{U}^\dagger D^\nu \bar{U}' \rangle$$

$$- \frac{iN}{12} \langle F_R^{\mu\nu} D_\mu \bar{U}'^\dagger D_\nu \bar{U}' + F_L^{\mu\nu} D_\nu \bar{U}' D_\mu \bar{U}' \rangle$$

$$- \frac{N}{24} \langle F_L^{\mu\nu} F_L^{\mu\nu} + F_R^{\mu\nu} F_R^{\mu\nu} \rangle - \frac{N}{12} \langle \bar{U}'^\dagger F_R^{\mu\nu} \bar{U} F_L^{\mu\nu} \rangle.$$
we have the full $O(p^4)$ divergent structure of the chiral action, written in the same operator basis as was used to define the $\mathcal{L}_4$ Lagrangian in Eq. (2.5).

$$Z_{\text{loop}}^{\text{div}} = -D(\epsilon) \int d^4x \left( \frac{3}{32} \langle D_\mu \bar{U}^\dagger D_\nu \bar{U} \rangle^2 + \frac{3}{16} \langle D_\mu \bar{U}^\dagger D_\nu \bar{U} \rangle \langle D^\nu \bar{U}^\dagger D^\mu \bar{U} \rangle \right. \right.$$  

$$+ \frac{1}{8} \langle D_\mu \bar{U}^\dagger D^\mu \bar{U} \rangle \langle \chi^\dagger \bar{U} + \chi \bar{U}^\dagger \rangle \right.$$  

$$+ \frac{3}{2} \langle D_\mu \bar{U}^\dagger D^\nu \bar{U} \rangle \langle \chi^\dagger \bar{U} + \chi \bar{U}^\dagger \rangle + \frac{11}{144} \langle \chi^\dagger \bar{U} + \chi \bar{U}^\dagger \rangle^2 \right.$$  

$$+ \frac{5}{48} \langle \chi^\dagger \bar{U} \chi^\dagger \bar{U} + \chi \bar{U}^\dagger \chi \bar{U}^\dagger \rangle \right.$$  

$$- \frac{1}{4} \left( F_\mu^\nu D_\mu \bar{U} D_\nu \bar{U} \right) + F_L^\mu \nu D_\mu \bar{U} D_\nu \bar{U} \right) - \frac{1}{4} \langle \bar{U}^\dagger F_\nu^\mu \bar{U} F_\nu \mu \rangle \right.$$  

$$- \frac{1}{8} \left( F_R^\mu \nu F_{R \mu \nu} + F_L^\mu \nu F_{L \mu \nu} \right) + \frac{5}{24} \langle \chi^\dagger \chi \rangle \right)$$

The appropriate renormalizations of the low energy constants in $\mathcal{L}_4$ are therefore

$$L_i' = L_i - \gamma_i D(\epsilon)$$  

$$H_i' = H_i - \delta_i D(\epsilon)$$

where

$$\gamma_1 = \frac{3}{32}, \quad \gamma_2 = \frac{3}{16}, \quad \gamma_3 = 0, \quad \gamma_4 = \frac{1}{8}, \quad \gamma_5 = \frac{3}{8}$$  

$$\gamma_6 = \frac{11}{144}, \quad \gamma_7 = 0, \quad \gamma_8 = \frac{5}{48}, \quad \gamma_9 = \frac{1}{4}, \quad \gamma_{10} = \frac{1}{4}$$  

$$\delta_1 = -\frac{1}{8}, \quad \delta_2 = \frac{5}{24}$$

(3.8)

This is in agreement with the results obtained by Gasser and Leutwyler in Ref. [17].

### 3.2 Renormalization of $\mathcal{L}_6^A$

An interesting future extension of this work would be to use this background field method in the renormalization of higher orders in the chiral Lagrangian. In particular we are interested in the renormalized low energy constants of $\mathcal{L}_6^A$, some of which enter in our later calculation of the meson-$\gamma$-$\gamma$ vertex. This $\mathcal{L}_6^A$ renormalization has been treated by several groups previously [29, 31, 35, 36], but their results are inconsistent with one another and, it appears, with the most general $\mathcal{L}_6^A$ Lagrangian written down recently by Fearing and Scherer [18].

Since we know from momentum power-counting arguments that the one-loop structure renormalizing $\mathcal{L}_6^A$ will arise in the perturbative treatment from diagrams containing vertices from $\mathcal{L}_2$ and from the Wess-Zumino-Witten action $S_{WZW}$ it seems natural to
3.2 Renormalization of $\mathcal{L}_6^A$

proceed in an analogous fashion to the work of the last section. We should take the first and second variations of the action $S_2 + S_{WZ}$ with respect to a field parameter such as $\Delta^a$, about a background field $\bar{U}$. This background field will be a solution of a new, higher-order equation of motion derived by setting the action's first variation to zero. The second variation can then be made to yield the divergent one-loop anomalous structure of $O(p^6)$ by application of the path integral and heat kernel techniques used previously. The coefficients describing the renormalization of the 32 low energy constants of Fearing and Scherer's $\mathcal{L}_6^A$ can then be extracted by inspection.

One might be concerned about the use of a background field which is a solution of a higher-order equation of motion. To be consistent with the perturbative approach one should surely make an expansion always about a solution of the lowest order classical equation of motion. While we agree that this is correct, it is not difficult to see from momentum power-counting arguments that the parity-odd $O(p^6)$ divergent structures derived using these two different background fields can differ only at an irrelevant $O(p^6)$. This issue, of the correct classical equation of motion to use, becomes more important at the two-loop level as is discussed by Ecker in a recent review article [37].
Chapter 4

The $\pi^0\gamma\gamma$ Form Factor

In $\chi$PT the $\pi^0\gamma\gamma$ vertex function may be written

$$iA(q^2, k_1^2, k_2^2)\epsilon^{\mu\nu\alpha\beta}k_{1\mu}k_{2\nu}\epsilon_{1\alpha}\epsilon_{2\beta}$$

where contributions to the form factor may be expanded in powers of momentum according to the chiral counting rules.

$$A(q^2, k_1^2, k_2^2) = A^{(p^4)}(q^2, k_1^2, k_2^2) + A^{(p^6)}(q^2, k_1^2, k_2^2) + \ldots$$  \hspace{1cm} (4.1)

The leading, $O(p^4)$ term arises in a tree level calculation from $L_4^\Delta$. At $O(p^6)$ we have one-loop contributions involving vertices from $L_4^\Delta$ and $L_2$, and a tree level contribution from $L_0^\Delta$.

In the following sections we shall compute each of these contributions in turn, giving as our final result the form factor $A(q^2, k_1^2, k_2^2)$ to $O(p^6)$.

4.1 Tree level

We begin by computing the $L_4^\Delta$ tree level contribution to the meson-$\gamma-\gamma$ vertices using the field theoretical methods described in the classic text of Bjorken and Drell [38]. We will then set down a Feynman rule prescription, consistent with that of Bjorken and Drell, for the extraction of vertex factors from the chiral Lagrangian. This Feynman rule prescription will later be applied to the computation of $O(p^6)$ one-loop and tree level diagrams for the $\pi^0\gamma\gamma$ and $\eta\gamma\gamma$ vertices.

Our starting point in the field theory calculation is equation 16.81 of Ref. [38] which relates the $S$-matrix to a time-ordered product of interaction fields. We write it for the specific case of an incoming meson and two final state photons.

$$S_{fi} = \frac{i}{\sqrt{Z_\phi}} \left( \frac{-i}{\sqrt{Z_\gamma}} \right)^2 \int d^4x \int d^4y_1 \int d^4y_2 f_\phi(x) (\bar{\psi}_x + m^2)$$
4.1 Tree level

\[ \times \langle 0 \mid \mathcal{T} \left( A_\mu(y_1) A_\nu(y_2) \varphi(z) \right) \rangle \! \rangle = \mathbb{D}_{y_1} \mathbb{D}_{y_2} A_{\mu(k_1, \lambda_1)}^* A_{\nu(k_2, \lambda_2)}^* \] \tag{4.2}

In the above we use the definitions

\[ f_{\ell}(x) = \frac{1}{\sqrt{(2\pi)^3 2\omega_\ell}} e^{-i\ell \cdot x} \]
\[ A_{\mu(k, \lambda)}^* = \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} e^{-i k \cdot x} \epsilon_{\lambda}^\mu \]

where \( q^\mu \equiv (\omega_q, \vec{q}) \), \( k^\mu \equiv (\omega_k, \vec{k}) \) and \( \epsilon_{\lambda}^\mu \) is the photon polarization four-vector of helicity \( \lambda \). The charge renormalization \( Z_3 \) may be set to one since we treat electromagnetism as an external field and so calculate to lowest order in the charge \( e \). Although we shall also set the meson wavefunction renormalization \( Z_\phi = 1 \) for this tree level calculation, we shall retain the \( Z_\phi \) dependencies where they appear in order to guide us in the construction of Feynman rules for higher orders in the momentum expansion.

The transition amplitude \( M \) which enters in calculations of cross sections or decays is related to the scattering matrix \( S_{\ell i} \) for our process as follows (Bjorken and Drell [38, eq. (17.49)])

\[ S_{\ell i} = -i(2\pi)^4 \delta^4(q - k_1 - k_2) \frac{1}{\sqrt{(2\pi)^3 2\omega_q}} \frac{1}{\sqrt{(2\pi)^3 2\omega_{k_1}}} \frac{1}{\sqrt{(2\pi)^3 2\omega_{k_2}}} M. \]

The fields \( \varphi \) and \( A_\mu \) in the time-ordered product of Eq. (4.2) are solutions of the field equations including interactions. The time-ordered product of these interacting fields may be rewritten in terms of the fields \( \varphi_{\ell n} \) and \( A_{\mu n}^* \) which are solutions of the free field equations [38, eq. (17.22)].

\[ \langle 0 \mid \mathcal{T} \left( A_\mu(y_1) A_\nu(y_2) \varphi(z) \right) \rangle \! \rangle = \frac{\langle 0 \mid \mathcal{T} \left( A_{\mu_{\ell n}}(y_1) A_{\nu_{\ell n}}(y_2) \varphi_{\ell n}(z) \exp \left[ i \int d^4x L_\ell \right] \right) \rangle \! \rangle}{\langle 0 \mid \mathcal{T} \left( \exp \left[ i \int d^4x L_\ell \right] \right) \rangle \! \rangle} \tag{4.3} \]

The exponentials may be expanded to the required order in the fields and Wick's theorem [39] used to rewrite the expression in terms of the following elementary time-ordered products and their derivatives [40].

\[ \langle 0 \mid \mathcal{T} \left( \varphi_{\ell n}(x) \varphi_{\ell n}(y) \right) \rangle \! \rangle \equiv i \Delta_F(x - y) = i \int \frac{d^4\tilde{q}}{(2\pi)^4} \frac{e^{-i\tilde{q} \cdot (x - y)}}{\tilde{q}^2 - m^2 + i\varepsilon} \]
\[ \langle 0 \mid \mathcal{T} \left( A_{\mu_{\ell n}}(x) A_{\nu_{\ell n}}(y) \right) \rangle \! \rangle \equiv i D_{F\mu\nu}(x - y) = -ig^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot (x - y)}}{k^2 + i\varepsilon} \]
\[ \langle 0 \mid \mathcal{T} \left( \partial_\beta \varphi_{\ell n}(x) \partial^\beta \varphi_{\ell n}(y) \right) \rangle \! \rangle \equiv \frac{i}{\partial y^\beta} \Delta_F(x - y) \]
\[ \langle 0 \mid \mathcal{T} \left( \partial_\beta A_{\mu_{\ell n}}(x) \partial^\beta A_{\nu_{\ell n}}(y) \right) \rangle \! \rangle \equiv \frac{i}{\partial y^\beta} D_{F\mu\nu}^\beta(x - y) \]
The $+i\varepsilon$ in the denominators are to be taken in the limit $\varepsilon \to 0^+$ and specify the contour used in the integrals on $q^2$ or $k^0$. When computing the higher order terms in the meson-$\gamma-\gamma$ vertex function in this way we would find that terms arise which, in the language of Feynman diagrams, are one-particle reducible. That is, the Feynman diagram representing the term may be cut in two by breaking a single internal line. It is equivalent to instead compute only the one-particle irreducible diagrams—those which cannot be decomposed in this way—while using the exact propagator $i\Delta_F(x - y) \equiv \int \mathcal{D} p \phi(p) = \int \mathcal{D} p \phi(p(x - y))$ in the meson external legs of the vertex. This factor $Z_\phi$ and the factor $Z_\phi^{-\frac{1}{2}}$ in Eq. (4.2) gives the Feynman rule for wavefunction renormalization of mesons:

- For each meson entering the diagram, include a factor $Z_\phi^{\frac{1}{2}}$.

Having reacquainted ourselves with the notation of Bjorken and Drell we can compute the $\mathcal{L}_4^A$ tree level contribution to the meson-$\gamma-\gamma$ vertex.

$$-i(2\pi)^4\delta^4(q - k_1 - k_2)M = -iZ_\phi^{\frac{1}{2}} \int d^4x \int d^4y_1 \int d^4y_2 e^{-i(q.x - k_1.y_1 - k_2.y_2)} \times (\bar{\psi}_z + m^2)(0)|T(A^\mu(y_1)A^\nu(y_2)\varphi(z))|0_i \bar{\psi}_{y_1} \bar{\psi}_{y_2} \epsilon_\mu(\lambda_1)\epsilon_\nu(\lambda_2)$$

(4.4)

We must now extract the relevant interaction terms of the chiral Lagrangian. The $\pi^0\gamma\gamma$ vertex to $O(p^4)$ is generated from terms in $\mathcal{L}_4^A$ involving two photon fields; namely, from Eq. (2.8) of §2.2,

$$\mathcal{L}_4^A = N_c \frac{ie^2}{24\pi^2} \epsilon^{\mu\nu\alpha\beta}(\partial_\mu A_\nu)A_\alpha T_\beta + \ldots$$

where

$$T_\beta = \left\langle Q^2(R_\beta + L_\beta) + \frac{1}{2} QUQU^\dagger R_\beta + \frac{1}{2} QU^\dagger QUL_\beta \right\rangle$$

and $L_\beta = U^\dagger(\partial_\beta U)$, $R_\beta = (\partial_\beta U)U^\dagger$.

At tree level we are interested only in the couplings of the neutral mesons $\pi^0$ and $\eta_8$ to photons. With the charged meson fields omitted, $U$ commutes with the charge matrix $Q$. This simplifies the form of $T_\beta$,

$$T_\beta = \left\langle Q^2(R_\beta + L_\beta) + \frac{1}{2} Q^2(UU^\dagger)R_\beta + \frac{1}{2} Q^2(U^\dagger U)L_\beta \right\rangle$$

$$= \frac{1}{2} \left\langle Q^2(R_\beta + L_\beta) \right\rangle$$

and allows us to carry out the trace over flavour degrees of freedom by hand—this will let us illustrate the method. Later, more complicated extractions of vertices will be done with the assistance of Mathematica [16].
We have

\[ Q^- = \frac{1}{\phi} \text{diag}(4, 1, 1) \]

\[ U = \exp(i\phi/F) = 1 + \frac{i}{F} \phi - \frac{1}{2F^2} \phi^2 - \frac{i}{6F^3} \phi^3 + \ldots \]

\[ U^\dagger = \exp(-i\phi/F) = 1 - \frac{i}{F} \phi - \frac{1}{2F^2} \phi^2 + \frac{i}{6F^3} \phi^3 + \ldots \]

\[ L_\beta = U^\dagger(\partial_\beta U) = \frac{i}{F} \partial_\beta \phi \ldots \]

\[ R_\beta = (\partial_\beta U)U^\dagger = \frac{i}{F} \partial_\beta \phi + \ldots \]

giving for \( T_\beta \),

\[ T_\beta = \frac{3}{2} \left( Q^2 \frac{2i}{F} \partial_\beta \phi \right) + O(\phi^3) \]

\[ = \frac{i}{F} \left( \partial_\beta \pi^0 + \frac{1}{\sqrt{3}} \partial_\beta \eta_8 \right). \tag{4.5} \]

The interaction Lagrangian is then

\[ i \mathcal{L}_I = \frac{-ie^2 N_c C_P \epsilon^{\mu \nu \sigma \beta} : \partial_\mu A_\nu A_\sigma \partial_\beta \varphi :}{24\pi^2 F} \]

where \( C_P = 1, \frac{1}{\sqrt{3}} \) for \( \varphi = \pi^0, \eta_8 \). In the above we have explicitly inserted the normal ordering operator, represented by placing colons—or as they are called in Wick’s paper, semicolons—on either side of the argument. This has the effect that all creation operators within the colons are understood to be written to the left of all destruction operators, and the device is introduced to prevent the sum of zero-mode energies of the various fields from becoming divergent.

The time-ordered product is, from Eq. (4.3),

\[ \langle 0 | T \left( A^\mu(y_1) A^\nu(y_2) \varphi(x) \right) | 0 \rangle \]

\[ = \frac{\langle 0 | T \left( A^\mu_{\alpha}(y_1) A^\mu_{\alpha}(y_2) \varphi_{\alpha}(x) \left[ 1 + i \int d^4 \bar{z} \mathcal{L}_I + \ldots \right] \right) | 0 \rangle}{\langle 0 | T \left( 1 + \ldots \right) | 0 \rangle} \]

\[ = \frac{-iaC_P}{2\pi F} \epsilon_{\mu \rho \sigma \delta} \int d^4 \bar{z} \langle 0 | T \left( A^\mu_{\alpha}(y_1) A^\mu_{\alpha}(y_2) \varphi_{\alpha}(x) : \partial^\rho A^\sigma_{\alpha}(\bar{z}) A^\delta_{\alpha}(\bar{z}) \partial^\delta \varphi_{\alpha}(\bar{z}) : \right) | 0 \rangle} \]

\[ = \frac{-aC_P}{2\pi F} \epsilon_{\mu \rho \sigma \delta} \int d^4 \bar{z} \langle 0 | T \left( \varphi_{\alpha}(x) \partial^\rho A^\sigma_{\alpha}(\bar{z}) \right) | 0 \rangle \]

\[ \times \left( \langle 0 | T \left( A^\mu_{\alpha}(y_1) \partial^\rho A^\sigma_{\alpha}(\bar{z}) \right) | 0 \rangle \langle 0 | T \left( A^\mu_{\alpha}(y_2) A^\delta_{\alpha}(\bar{z}) \right) | 0 \rangle \right) \]

\[ + \langle 0 | T \left( A^\mu_{\alpha}(y_1) A^\delta_{\alpha}(\bar{z}) \right) | 0 \rangle \langle 0 | T \left( A^\mu_{\alpha}(y_2) \partial^\rho A^\delta_{\alpha}(\bar{z}) \right) | 0 \rangle \)
where we used Wick’s theorem in arriving at the last line. We substitute the expressions of Eq. (4.4), taking derivatives where necessary, to find

\[
\langle 0 | T \left( A^{\mu}(y_1) A^{\nu}(y_2) \varphi(x) \right) | 0 \rangle \\
= \frac{\alpha C_P}{2 \pi F} \epsilon_{\lambda \alpha \delta} \int d^4 x \int d^4 \hat{q} \int d^4 \hat{k}_1 \int d^4 \hat{k}_2 \exp \left( i(\hat{q} + \hat{k}_1 + \hat{k}_2) \cdot \hat{z} \right) \\
\times \left( \frac{e^{-iq \cdot x}}{(\hat{q}^2 - m^2 + i\epsilon)} \right) \left( \frac{e^{-ik_1 \cdot x}}{(k_1^2 + i\epsilon)} \right) \left( \frac{e^{-ik_2 \cdot x}}{(k_2^2 + i\epsilon)} \right) \delta^{\beta} \left( \hat{k}_1^{\beta} g^{\mu \nu} g^{\nu \delta} + \hat{k}_2^{\beta} g^{\mu \alpha} g^{\alpha \delta} \right).
\]

Inserting this into the expression for the transition amplitude \( M \) in Eq. (4.4) and using the results

\[
\left( \delta^{\lambda} + m^2 \right) e^{-iq \cdot x} = -(\hat{q}^2 - m^2) e^{-iq \cdot x} \\
e^{-ik_1 \cdot y} \delta^{\alpha} = -\hat{k}_1^{\alpha} e^{-ik_1 \cdot y}
\]

we have

\[
-i(2\pi)^4 \delta^4(q - k_1 - k_2) M \\
= i Z_{\phi}^{\frac{1}{2}} \frac{\alpha C_P}{2 \pi F} \epsilon_{\lambda \alpha \delta} \int d^4 x \int d^4 \hat{q} \int d^4 y_1 \int d^4 y_2 \int d^4 \hat{k}_1 \int d^4 \hat{k}_2 \\
\times \exp \left[ -i(\hat{q} + \hat{q}) \cdot x - i(\hat{k}_1 - k_1) \cdot y_1 - i(\hat{k}_2 - k_2) \cdot y_2 + i(\hat{q} + \hat{k}_1 + \hat{k}_2) \cdot \hat{z} \right] \\
\times \hat{q}^{\delta}(\hat{k}_1^{\alpha} g^{\mu \nu} g^{\nu \delta} + \hat{k}_2^{\beta} g^{\mu \alpha} g^{\alpha \delta}) \epsilon^{\lambda}_{\gamma \alpha \beta}(\lambda_1) \epsilon^{\alpha}_{\gamma \lambda \beta}(\lambda_2).
\]

We use the results

\[
\int \frac{d^4 x}{(2\pi)^4} e^{-i\alpha \cdot x} = \delta(\alpha) \\
\int d^4 \hat{q} \delta^4(\hat{q} - q) f(\hat{q}) = f(q)
\]

to find

\[
-i(2\pi)^4 \delta^4(q - k_1 - k_2) M \\
= -i(2\pi)^4 \delta^4(q - k_1 - k_2) Z_{\phi}^{\frac{1}{2}} \frac{\alpha C_P}{2 \pi F} \epsilon_{\mu \nu \alpha \delta} \left( k_1^{\mu} \epsilon_1^{\nu} \epsilon_1^{\alpha} \epsilon_1^{\delta} + k_2^{\mu} \epsilon_2^{\nu} \epsilon_2^{\alpha} \epsilon_1^{\delta} \right) \\
\Rightarrow M = -Z_{\phi}^{\frac{1}{2}} C_P \frac{\alpha}{\pi F} \epsilon_{\mu \nu \alpha \delta} k_1^{\mu} \epsilon_1^{\nu} k_2^{\alpha} \epsilon_1^{\delta}.
\]

This result could have been obtained by using the Feynman rules for the construction of \(-i M\) set down in appendix B of Ref. [38], so long as we supplement those rules with an algorithm for the extraction of vertex factors from the interaction part of the chiral Lagrangian. A suitable algorithm, which will return the result of Eq. (4.6) above for the \( \mathcal{L}_4^4 \) tree level term, is as follows:
4.2 One loop diagrams

Figure 4.1: The one-loop diagrams contributing at $\mathcal{O}(p^5)$ to the $\pi^0\gamma\gamma$ vertex

- Write down $i\mathcal{L}_I$ in momentum space, where $\mathcal{L}_I$ is the interaction part of the chiral Lagrangian. This implies that $\partial_\mu = \pm ip_\mu$ for momenta defined as outgoing from the vertex.

- Replace the fields by unity.

- Where we have $n$ identical particles at a vertex, we accumulate $n!$ terms in our vertex factor; one for each permutation of associating the field operators in the interaction Lagrangian with the creation or destruction of the particles.

In the notation defined at the start of this chapter, the meson-$\gamma\cdot\gamma$ vertex factor to $\mathcal{O}(p^4)$ is

$$A^{(s)}(q^2, k_1^2, k_2^2) = C_p \frac{\alpha}{\pi F^3}.$$  \hspace{1cm} (4.7)

where $C_p = 1, \frac{1}{\sqrt{3}}$ for $\pi^0\gamma\gamma, \eta_8\gamma\gamma$. We now proceed to compute the $\mathcal{O}(p^5)$ contribution using the Feynman rule technique.

4.2 One loop diagrams

One determines the contributing Feynman diagrams of a particular order in the momentum expansion by naive power counting. The order in momentum of a diagram is defined to be the sum of the powers of each of the diagram's vertices (2 from an $\mathcal{L}_2$ vertex, 4 from an $\mathcal{L}_4$ or $\mathcal{L}_4^\dagger$ vertex, et cetera), plus 4 from each integral over a loop.
momentum, minus 2 from each propagator. Using this scheme we find that two one-loop diagrams contribute to the $\pi^0\gamma\gamma$ vertex function at $O(p^0)$ in the chiral expansion. They are shown in Fig. 4.1, with the loops being composed of any of the pseudoscalar mesons which are found to have finite coefficients at each of the loop vertices—it will turn out that only the charged kaons and pions contribute to each loop. Throughout our calculations we shall compute results for the $\pi^0\gamma\gamma$ vertex and state the corresponding results for $\eta_8\gamma\gamma$. We have chosen not to include the singlet $\eta_1$ in our analysis, and must therefore view with caution any results based on directly equating the $\eta_8$ with the physical $\eta$ meson.

Diagram (a) has amplitude

$$-iM(a) = \sum \mu^2 \int \frac{d^N \tau}{(2\pi)^N} B^{\mu \nu}_{\pi^0\gamma\gamma M \bar{M}} \left( \frac{i}{\tau^2 - m_\Delta^2} \right) \epsilon_\mu \epsilon^\nu$$

where $B^{\mu \nu}_{\pi^0\gamma\gamma M \bar{M}}$ is the three meson, two photon vertex arising from $\mathcal{L}_4^A$ and the summation is over the various mesons in the loop. We shall use the dimensional method to regulate divergent integrals, with the space-time dimension being taken as $N = 2(2 - \epsilon)$. The factor $\mu^2\epsilon$ has been introduced to retain the correct dimension for the physical amplitude—$\mu$ has units of mass and is called the renormalization scale. In $\chi$PT this scale is usually chosen to be the mass of the $\rho$, 770 MeV, since this is the lightest meson which does not appear as a dynamic field in the chiral Lagrangian.

For the extraction of the 3-meson/2-photon vertex the relevant part of $\mathcal{L}_4^A$ is once again

$$\mathcal{L}_4^A = N_\pi \frac{i \epsilon^3}{24\pi^2} \epsilon^{\mu \nu \alpha \beta} (\partial_\mu A_\nu) A_\alpha T_\beta + \ldots$$

where

$$T_\beta = \left( Q^2 (R_\beta + L_\beta) + \frac{1}{2} Q U Q U^\dagger R_\beta + \frac{1}{2} Q U^\dagger Q U L_\beta \right)$$

We expand the exponential representation of $U$ in terms of $\phi$ retaining only terms involving three meson fields. The result is

$$T_\beta = \frac{i}{2F^3} \left( Q^2 (\phi \partial_\beta \phi - \phi^2 \partial_\beta - \partial_\beta \phi^2) + Q \phi Q \phi \partial_\beta + \phi Q \phi \partial_\beta \phi - \phi^2 Q \partial_\beta \phi \right).$$

We multiply the matrices $Q$, $\phi$ and $\partial_\beta \phi$ and take the trace on flavour indices using Mathematica. The terms which contribute to the loop of diagram (a) with an incoming $\pi^0$ or $\eta_8$ are

$$\mathcal{L}_4^A = \frac{-\alpha}{4\pi F^3} \epsilon^{\mu \nu \alpha \beta} (\partial_\mu A_\nu) A_\alpha \times \left( -\frac{8}{3} \pi^- \pi^+ \partial_\beta \pi^0 + \frac{4}{3} \pi^0 (\pi^+ \partial_\beta \pi^- + \pi^- \partial_\beta \pi^+) \right)$$
4.2 One loop diagrams

\[ -\frac{10}{3} K^- K^+ \partial_\beta \pi^0 + \frac{2}{3} \partial_\beta \left( K^+ \partial_\beta K^- + K^- \partial_\beta K^+ \right) \]
\[ - \frac{4 \sqrt{3}}{3} \pi^+ \partial_\beta \eta_8 \]
\[ - \frac{2}{3} K^- K^+ \partial_\beta \eta_8 + \frac{2}{3} \eta_8 \left( K^+ \partial_\beta K^- + K^- \partial_\beta K^+ \right) + \ldots \].

The vertex factor is found in this case, where we have no net four-momentum being carried off by the loop, to be of the form

\[ \frac{i \alpha}{2 \pi^3} F^\mu \alpha \beta k_1 \epsilon k_2 \]

where the coefficient \( \alpha \) is \(-\frac{10}{3} \) for the \( \pi^0 \rightarrow K^\pm \) loop, \(-\frac{8}{3} \) for the \( \pi^0 \rightarrow \pi^\pm \) loop, \(-\frac{3}{\sqrt{3}} \) for the \( \eta_8 \rightarrow K^\pm \) loop, and \(-\frac{4}{\sqrt{3}} \) for the \( \eta_8 \rightarrow K^\pm \) loop.

The loop integral in \( M(a) \) may now be computed, using the results of App. E,

\[ \mu^2 \int \frac{d^N \tau}{(2\pi)^N} \frac{i}{\tau^2 - m_M^2} = \frac{(m_M^2)^{1-\epsilon}}{16\pi^2(4\pi\mu^2)^{-\epsilon}} \Gamma(1+\epsilon) \]
\[ = \frac{1}{16\pi^2} \left( \frac{m_M^2}{4\pi\mu^2} \right)^{-\epsilon} \Gamma(1+\epsilon) \]
\[ = \frac{1}{16\pi^2} \left( -1 + \epsilon \left( \log \frac{m_M^2}{4\pi\mu^2} - \Gamma(1) - 1 \right) + \ldots \right) \]
\[ = \frac{(m_M^2) \log(m_M^2/\mu^2)}{16\pi^2} + \frac{m_M^2}{16\pi^2} \left( \frac{1}{\epsilon} - 1 + \gamma_E - \log(4\pi) \right) + O(\epsilon) \]

where the Euler constant is \( \gamma_E = -\Gamma'(1) \approx 0.577 \). The contribution to the \( \pi^0 \gamma \gamma \) vertex from diagram (a) is then

\[ -iM(a) = \frac{i \alpha}{\pi^F} \epsilon^\mu \alpha \beta k_1 \epsilon k_2 \left( -\frac{10}{3} \mu_{K^\pm} - \frac{8}{3} \mu_{\pi^\pm} \right) \]

where we have defined

\[ \mu_M^D \equiv \mu_M + \frac{m_M^2}{2} D(\epsilon) \]
\[ \mu_M \equiv \frac{m_M^2}{32\pi^2 F^2} \log(m_M^2/\mu^2) \]
\[ D(\epsilon) \equiv \frac{1}{32\pi^2} \left( \frac{1}{\epsilon} - 1 + \gamma_E - \log(4\pi) \right). \]  

(4.8)

In the notation of Eq. (4.1) we have for the \( \pi^0 \)

\[ A_{\pi^0, \text{loop} (a)}(\pi^0, m) = \frac{\alpha}{\pi^F} \left( -\frac{10}{3} \mu_{K^\pm} ^D - \frac{8}{3} \mu_{\pi^\pm} ^D \right) \]

and the for \( \eta_8 \)

\[ A_{\eta_8, \text{loop} (a)}(\eta_8, m) = \frac{\alpha}{\pi^F} \frac{1}{\sqrt{3}} \left( -2 \mu_{K^\pm} ^D - 4 \mu_{\pi^\pm} ^D \right). \]
4.2 One loop diagrams

Diagram (b) has amplitude

\[ -i M(b) = \sum_{\mu} \mu^2 \frac{d^N r}{(2\pi)^N} B^\mu_{\gamma \gamma M \overline{M}} \frac{i}{r^2 - m_M^2} \frac{i}{(r + k_1)^2 - m_M^2} \epsilon^*_\mu \epsilon^*_\nu + \{k_1 \leftrightarrow k_2\} \]

where \( B^\mu_{\gamma \gamma M \overline{M}} \) is the 3-meson/1-photon vertex from \( L_4 \) and \( B^\nu_{\gamma M \overline{M}} \) is the meson-meson-photon vertex from \( L_2 \). The part of \( L_4 \) containing the 3-meson/1-photon vertex is

\[ L_4^A = \frac{-e N_c}{48\pi^2} \epsilon^{\mu \nu \alpha \beta} A_\mu \left( Q(R_\nu R_\alpha R_\beta + L_\nu L_\alpha L_\beta) \right) + \ldots \]

We expand the fields \( U \) keeping only terms with three mesons.

\[ \frac{L_4^A}{48\pi^2} = \frac{-e N_c}{48\pi^2} \epsilon^{\mu \nu \alpha \beta} A_\mu \left( 2\partial_\nu \pi^0 - \partial_\alpha \partial_\beta \pi^0 + \partial_\alpha \pi^+ \partial_\beta \pi^- \right) \]

Here we have excluded terms like \( \partial_\alpha \pi^0 \partial_\beta \pi^0 \) since neutral mesons are disallowed at the other vertex in the loop. The 3-meson/1-photon vertex is then

\[ B^\mu_{\gamma \gamma M \overline{M}} = \frac{-ie N_c}{4\pi^2 F_\pi^3} \epsilon^{\mu \nu \alpha \beta} \left( -i q_\nu (i r_\alpha + i k_1 \theta) \right) C_P \]

where, once again, \( C_P = 1 \) for the vertices \( \pi^0 \gamma K^+K^- \) and \( \pi^0 \pi^+\pi^- \) and \( C_P = \frac{1}{\sqrt{3}} \) for \( \eta_8 \gamma K^+K^- \) and \( \eta_8 \pi^+\pi^- \).

To get the meson-meson-photon vertex we expand \( L_2 \).

\[ L_2 = \frac{F_\pi^2}{4} \left( D_\mu U D^\mu U^T + \chi U^T + \chi^T U \right) \]

\[ = \frac{F_\pi^2}{4} \left( \partial_\mu U - ie A_\mu [U, Q] \right) \partial^\mu U^T - ie A_\mu [U^T, Q] \right) + \ldots \]

\[ = -ie A_\mu F_\pi^2 \left( [U, Q] \partial^\mu U^T + \partial^\mu U [U^T, Q] \right) + \ldots \]

\[ = -\frac{1}{2} ie A_\mu \left( \partial^\mu (\phi Q - Q \phi) \right) + \ldots \text{ retaining } O(\phi^3) \text{ only} \]

\[ = -ie A_\mu \left( (\pi^- \partial^\mu \pi^+ - \pi^+ \partial^\mu \pi^-) + (K^- \partial^\mu K^+ - K^+ \partial^\mu K^-) \right) + \ldots \]

The vertex factor is then \( B^\nu_{\gamma M \overline{M}} = ie(2\pi^\nu + k_1 \gamma) \) for the \( \pi^\pm/K^\pm \) loop on the photon line labelled \( k_1^\nu \). The resulting amplitude for diagram (b) is

\[ -i M(b) = \sum_M 2\alpha C_P \frac{\epsilon^{-\nu \alpha \beta}}{F_\pi^3} k_{2\nu} k_{1\alpha} \epsilon^*_\nu \epsilon^*_\alpha \left[ \mu^2 \int \frac{d^N r}{(2\pi)^N (r^2 - m_M^2)((r + k_1)^2 - m_M^2)} \right] + \{k_1 \leftrightarrow k_2\} \]
The integral may be performed with the assistance of the relations developed in App. E.

\[
\mu^{2\epsilon}\int \frac{d^N r}{(2\pi)^N} \frac{r^\alpha r^\beta}{(r^2 - m_M^2)((r + k_1)^2 - m_M^2)}
\]

\[
= \int_0^1 dz \mu^{2\epsilon}\int \frac{d^N r}{(2\pi)^N} \frac{r^\alpha r^\beta}{[z(r^2 - m_M^2) + (1 - z)((r + k_1)^2 - m_M^2)]^2}
\]

\[
= \frac{i}{16\pi^2} \int_0^1 dx \left( \frac{m_M^2 - (z - x^2)k_1^2}{4\pi\mu^2} \right)^{-\epsilon} \left[ \Gamma(1 - x)k_2^\alpha k_2^\beta \right.
\]

\[
+ \frac{1}{2} [(z - x^2)k_1^2 - m_M^2] \Gamma(-1 + \epsilon)g^\alpha g^\beta \left. \right] \text{ using Eq. (E.5)}
\]

It is immediately clear that the \(k_1^\alpha k_2^\beta\) term in the above will have no contribution to our process due to the presence of the antisymmetric factor \(\epsilon^{\mu_1\nu_1\rho_1\sigma_1} k_{2\nu} \) in \(M_{(b)}\). We Taylor expand in powers of \(\epsilon\)

\[
\left( \frac{m_M^2 - (z - x^2)k_1^2}{4\pi\mu^2} \right)^{-\epsilon} \Gamma(-1 + \epsilon)
\]

\[
= \left( \frac{m_M^2 - (z - x^2)k_1^2}{4\pi\mu^2} \right)^{-\epsilon} \frac{\Gamma(1 + \epsilon)}{\epsilon(-1 + \epsilon)}
\]

\[
= \log \left( \frac{m_M^2 - (z - x^2)k_1^2}{4\pi\mu^2} \right) - \frac{1}{\epsilon} + \gamma_E - 1 - \log(4\pi) + O(\epsilon)
\]

giving for the integral

\[
= ig^\alpha g^\beta \int_0^1 dx \left\{ [(z - x^2)k_1^2 - m_M^2] \left( \frac{1}{32\pi^2} \log \frac{m_M^2}{\mu^2}
\right.
\]

\[
+ \frac{1}{32\pi^2} \log \left( \frac{k_1^2}{m_M^2}(z - x^2) \right) + D(\epsilon)) \right\}.
\]

The integral over the Feynman parameter \(x\) may now be performed using the results of Eq. (E.3) in App. E.

\[
= -ig^\alpha g^\beta \left( 1 - \frac{k_1^2}{6m_M^2} \right) F^2 \mu_M^{D} + ig^\alpha g^\beta \left( \frac{1}{18} k_1^2 - \frac{2}{3} m_M^2 \left( 1 - \frac{k_1^2}{4m_M^2} \right) \right) H_0(k_1^2/m_M^2)
\]

Including contributions from both \(K^\pm\) and \(\pi^\pm\) loops, and including the \(\{k_1 \rightarrow k_2\}\) interchange term, the amplitudes for diagram (b) is

\[
-iM_{(b)} = iC_p \alpha \epsilon^{\nu_1\alpha_1} k_1^\nu_2 k_2^\alpha_2
\]

\[
\times \left\{ \left( \frac{4}{ \mu_2^D + \mu_K^D} - \frac{1}{3}(k_1^2 + k_2^2) \right) \left( \frac{\mu_\pi^D}{m_\pi^2} + \frac{\mu_K^D}{m_K^2} + \frac{1}{48\pi^2 F^2} \right)
\]

\[
+ \frac{1}{96\pi^2 F^2} \left[ (4m_\pi^2 - k_1^2)H_0(k_1^2/m_\pi^2) + (4m_K^2 - k_1^2)H_0(k_1^2/m_K^2) + (4m_\pi^2 - k_2^2)H_0(k_2^2/m_\pi^2) + (4m_K^2 - k_2^2)H_0(k_2^2/m_K^2) \right] \right\}
\]
4.2 One loop diagrams

We find the contribution of diagram (b) to the $\pi^0\gamma\gamma$ or $\eta\gamma\gamma$ vertex factor to be

$$A^{(p^4,\text{loop (b)})} = C_p \frac{\alpha}{\pi F} \left\{ 4(\mu^D + \mu_K) - \frac{3}{2}(k_1^2 + k_2^2) \left( \frac{\mu^D}{m_{\pi^0}^2} - \frac{\mu_K}{m_{K^0}^2} + \frac{1}{48\pi^2 F^2} \right) 
+ \frac{1}{96\pi^2 F^2} \left[ (4m_{\pi^0}^2 - \mu^D)H_0(k_1^2/m_{\pi^0}^2) + (4m_{K^0}^2 - \mu_K)H_0(k_2^2/m_{K^0}^2) + (4m_{\pi^0}^2 - k_1^2)H_0(k_1^2/m_{\pi^0}^2) + (4m_{K^0}^2 - k_2^2)H_0(k_2^2/m_{K^0}^2) \right] \right\}. $$

In the limit $k_1^2, k_2^2 \to 0$ this reduces to

$$A^{(p^4,\text{loop (b)})} = C_p \frac{\alpha}{\pi F} 4(\mu^D + \mu_K).$$

To see this one must take the representation of the function $H_0(a)$ in the region $0 < a < 4$, given in Eq. (E.2) of App. E, and expand it about small values of $a$. The leading term is

$$H_0(a) = 2 \left( \frac{1}{y} \arctan(y) - 1 \right) \text{ where } y \equiv (\frac{4}{a} - 1)^{-\frac{1}{2}}$$

$$\to -\frac{a}{6} + \ldots \text{ for a small.}$$

Therefore $H_0(k^2/m^2) \to 0$ as $k^2 \to 0$.

At $O(p^4)$, in addition to the one-loop diagrams (a) and (b), there are one-loop contributions which enter as $O(p^2)$ corrections to the $L_4^a$ tree level diagram calculated in §4.1. These corrections arise in the meson wavefunction and in its decay constant. The corrections to the relationship between the physical decay constants $F_\pi$, $F_{\eta}$ and the parameter $F$ are well known. We quote them directly from Ref. [17].

$$F_\pi = F \left( 1 - 2\mu^D - \mu_K + \frac{4B(m_u + m_d)}{F^2}L_5 + \frac{8B(m_u + m_d + m_s)}{F^2}L_4 \right)$$

$$F_{\eta} = F \left( 1 - 3\mu_K + \frac{4B(m_u + m_d + 4m_s)}{3F^2}L_5 + \frac{8B(m_u + m_d + 4m_s)}{F^2}L_4 \right)$$

It will be more convenient for our later calculation if we now express $F_\pi$ and $F_{\eta}$ in terms of the divergent quantities $\mu^D$, $\mu_K^D$ and the unrenormalized low energy constants $L_4$, $L_5$. The result is

$$F_\pi = F \left( 1 - 2\mu^D - \mu_K^D + \frac{4B(m_u + m_d)}{F^2}L_5 + \frac{8B(m_u + m_d + m_s)}{F^2}L_4 \right)$$

$$F_{\eta} = F \left( 1 - 3\mu_K^D + \frac{4B(m_u + m_d + 4m_s)}{3F^2}L_5 + \frac{8B(m_u + m_d + 4m_s)}{F^2}L_4 \right)$$

where careful note should be made of the superscripts $^D$ and $^r$. Use was made of the lowest order mass relations of Eq. (2.3), the definition of $\mu^D$ in Eq. (4.8) and the
4.2 One loop diagrams

renormalization coefficients $\gamma_4$, $\gamma_5$ of Eq. (3.8) in deriving this, at first sight trivial, expression.

The relationship between the bare and the one-loop corrected meson wavefunctions are defined to be $\pi^0 \equiv (Z^*_\phi)^{-\frac{1}{2}} \pi^0$ and $\eta_8 \equiv (Z^*_\phi)^{-\frac{1}{2}} \eta_8$. We compute the constants $Z^*_\phi$ and $Z^*_\phi$ by constructing a one-loop effective Lagrangian as was outlined in Ref. [33, pp. 169] for the flavour $SU(2)$ case. One would normally compute the wavefunction and mass renormalizations by calculating the one-loop corrections to the meson propagators using tree level graphs from $L_2$ and $L_4$ expanded to $O(\phi^3)$ and one-loop graphs from the $O(\phi^3)$ part of $L_2$. During the loop calculation we would contract over all pairs of meson fields in the vertices arising from the $O(\phi^3)$ part of $L_2$.

It is equivalent to instead compute the propagator at tree level using an effective Lagrangian of apparent $O(\phi^3)$ but which has within it the one-loop structure of $L_2$ and the tree level structure of $L_4$. The one-loop part of this effective Lagrangian is formed by summing all possible contractions among pairs of fields in the $O(\phi^4)$ part of $L_2$.

With external fields set to zero the parity-even Lagrangians $L_2$ and $L_4$ are

$$L_2 = \frac{1}{4} \langle \partial^\mu \phi \partial_\mu \phi \rangle - \frac{B}{2} \langle m^2 \phi \rangle$$

$$+ \frac{1}{24 F^2} \langle \phi \partial_\mu \phi \partial^\mu \phi - \phi^2 \partial_\mu \phi \partial^\mu \phi \rangle + O(\phi^6)$$

$$L_4 = \frac{4B}{F^2} \langle \partial^\mu \phi \partial_\mu \phi \rangle \langle m^2 \phi \rangle - \frac{4B}{F^2} \langle \partial^\mu \phi \partial_\mu \phi \rangle \langle m^2 \phi \rangle$$

$$- \frac{16B^2}{F^2} \left( L_6 \langle m^2 \phi \rangle + L_7 \langle m^2 \phi \rangle \right) + O(\phi^6)$$

We define an effective Lagrangian $L_{\text{eff}}$ of $O(\phi^3)$ which has the following canonical form in terms of renormalized fields and masses

$$L_{\text{eff}} \equiv \frac{1}{2} \left( \partial_\mu \pi^0 \partial^\mu \pi^0 - (m_{\pi^0})^2 \pi^0 \right) + \frac{1}{2} \left( \partial_\mu \eta_8 \partial^\mu \eta_8 - (m_{\eta_8})^2 \eta_8 \right) + \ldots$$

We shall treat only the $\pi^0$ and $\eta_8$ fields in the remainder of this discussion. In terms of bare quantities this is

$$L_{\text{eff}} = \frac{1}{2} \left( (Z^*_\phi)^{-1} \partial_\mu \pi^0 \partial^\mu \pi^0 - (Z^*_M)^{-1} \partial_\mu \pi^0 \partial^\mu \pi^0 \right)$$

$$+ \frac{1}{2} \left( (Z^*_\phi)^{-1} \partial_\mu \eta_8 \partial^\mu \eta_8 - (Z^*_M)^{-1} \partial_\mu \eta_8 \partial^\mu \eta_8 \right). \quad (4.9)$$

We are interested here only in the wavefunction renormalization $Z_{\phi}$ and so we shall ignore all functional forms in the Lagrangian except $\partial_\mu \pi^0 \partial^\mu \pi^0$ and $\partial_\mu \eta_8 \partial^\mu \eta_8$.

This effective Lagrangian may also be written as

$$L_{\text{eff}} = L_2 (O(\phi^3)) + C L_2 (O(\phi^4)) + L_4 (O(\phi^3))$$
where $\mathcal{C}$ is an operator which implies all possible pairwise contractions be taken among the meson fields within each term.

\[
\mathcal{L}_{\text{eff}} = \frac{1}{2} \partial_{\mu} \pi^0 \partial^{\mu} \pi^0 + \frac{1}{2} \partial_{\mu} \eta_{\delta} \partial^{\mu} \eta_{\delta} - \frac{1}{12 F^2} \left( K^+ K^- + K^0 \bar{K}^0 + 4 \pi^+ \pi^- \right) \partial_{\mu} \pi^0 \partial^{\mu} \pi^0 - \frac{1}{12 F^2} \left( 3 K^+ K^- + 3 K^0 \bar{K}^0 \right) \partial_{\mu} \eta_{\delta} \partial^{\mu} \eta_{\delta} + \left( \frac{8 B}{F^2} (m_u + m_d + m_s) L_4 + \frac{4 B}{F^2} (m_u + m_d) L_5 \right) \partial_{\mu} \pi^0 \partial^{\mu} \pi^0 + \left( \frac{8 B}{F^2} (m_u + m_d + m_s) L_4 + \frac{4 B}{3 F^2} (m_u + m_d + 4 m_s) L_5 \right) \partial_{\mu} \eta_{\delta} \partial^{\mu} \eta_{\delta} + \ldots
\]

The contractions represent a loop integral over the appropriate meson propagator; for example,

\[
\frac{K^+ K^-}{\langle p^2 \rangle} = \mu^2 \int \frac{d^N r}{(2\pi)^N} \frac{i}{r^2 - m_{K^\pm}^2} = 2F^2 \mu_{K^\pm}^D.
\]

At this point one can in fact forego the calculation of the propagator altogether and simply extract the renormalization parameters by comparison of this effective Lagrangian with the canonical form of the $O(\phi^2)$ Lagrangian written in terms of the renormalized fields and masses. Comparison with the canonical form in Eq. (4.9) allows us to extract the wavefunction renormalizations.

\[
Z_\phi^{\pi^0} = 1 + \frac{4}{3} \mu_\pi^D + \frac{2}{3} \mu_K^D - \frac{8 B (m_u + m_d)}{F^2} L_5 - \frac{16 B (m_u + m_d + m_s)}{F^2} L_4,
\]

\[
Z_\phi^{\eta_{\delta}} = 1 + 2 \mu_K^D - \frac{8 B (m_u + m_d + 4 m_s)}{3 F^2} L_5 - \frac{16 B (m_u + m_d + m_s)}{F^2} L_4.
\]

When computing the $O(\phi^4)$ vertex factor in §4.1 we worked in terms of the bare wavefunction and the parameter $F$. Each meson field brings with it to the amplitude a factor $(Z_\phi)^{\frac{1}{2}} [38]$. Our $\mathcal{L}_4^a$ tree level result for $A_{\pi^0}(\pi)$ was then, strictly speaking, equal to $(Z_\phi)^{\frac{3}{2}} \alpha/\pi F$. In the spirit of our perturbative expansion in momentum, to $O(p^4)$ we can write $Z_\phi^{\pi^0} = 1$ and $F = F_\pi$ giving our previous tree level result.

\[
A_{\pi^0}^{(\pi^0)} = \frac{\alpha}{\pi F_\pi}.
\]

Including the additional $O(p^3)$ corrections to $Z_\phi^{\pi^0}$ and $F_\pi$ gives us a result correct to $O(p^6)$ for the $\mathcal{L}_4^a$ tree level diagram.

\[
A_{\pi^0 + \pi + F}^{(\pi^0 + \pi + F)} = (Z_\phi^{\pi^0})^{\frac{1}{2}} \left( \frac{F_\pi}{F} \right) \frac{\alpha}{\pi F_\pi} = \frac{\alpha}{\pi F_\pi} \left( 1 + \left( \frac{1}{8} \mu_\pi^D + \frac{1}{8} \mu_K^D \right) (Z_\phi) - \left( 2 \mu_\pi^D + \mu_K^D \right) (F) \right) + O(p^6)
\]
The corresponding expression for the \( \eta_8 \) is

\[
A_{\eta_8}^{(q^4+z+F)} = \left( Z_{\eta_8}^\gamma \right)^{\frac{1}{2}} \left( \frac{F_{q_8}}{F} \right) \frac{\alpha}{\pi F_{q_8}}
\]

\[
= \frac{\alpha}{\pi F_{q_8}} \left( 1 + (\mu_{2\gamma})_{(Z_\gamma)} - (3\mu_{D})(F) \right) + O(p^8).
\]

Here we have used the fact that \( Z_{\eta_8} = 1 + x \) where \( x \) is of \( O(p^2) \) in order to write \( Z_{\eta_8}^{\frac{1}{2}} \approx 1 + \frac{1}{2}x + O(p^4) \). The dependencies on \( L_4 \) and \( L_5 \) have cancelled between these two sets of one-loop corrections.

### 4.3 Renormalization

In computing the \( \mathcal{L}_6^4 \) contribution to the \( \pi^0\gamma\gamma \) and \( \eta_8\gamma\gamma \) vertices we begin with the Lagrangian of Fearing and Scherer [18] which we studied in §2.2, and apply the neutral meson condition \([U, Q] = 0\).

\[
\mathcal{L}_6^4 = 16\pi\alpha(A_2 - 2A_3)\epsilon_{\mu\nu\alpha\beta} \partial_\lambda F^{\mu\nu} F^{\lambda\beta} \left\langle Q^2 U^\dagger \partial^\mu U \right\rangle
\]

\[
+ 32\pi\alpha A_3 \epsilon_{\mu\nu\alpha\beta} \partial_\lambda F^{\mu\nu} F^{\lambda\beta} \left\langle Q^2 U^\dagger \partial^\mu U \right\rangle
\]

\[
+ 16\pi\alpha A_4 \epsilon_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta} \left\langle BmQ^2(U^\dagger - U) \right\rangle
\]

\[
+ 16\pi\alpha A_6 \epsilon_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta} \left\langle Bm(U^\dagger - U) \right\rangle \langle Q^2 \rangle
\]

\[
+ 16\pi\alpha F_{2\sigma} \epsilon_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta} \left\langle \left\langle Q^2(U^\dagger - U) \right\rangle + \frac{3}{4} B \left\langle m(U^\dagger - U) \right\rangle \langle Q^2 \rangle \right\rangle
\]

The traces are computed explicitly, giving

\[
\left\langle Q^2 U^\dagger \partial^\mu U \right\rangle = \frac{i}{F} \left\langle Q^2 \partial^\mu \phi \right\rangle
\]

\[
= \frac{i}{3F} (\partial^\mu \pi^0 + \frac{1}{\sqrt{3}} \partial^\mu \eta_8)
\]

\[
\left\langle Q^2(\partial^2UU^\dagger - U\partial^2U^\dagger) \right\rangle = \frac{2i}{F} \left\langle Q^2 \partial^2 \phi \right\rangle
\]

\[
= \frac{2i}{3F} (\partial^2 \pi^0 + \frac{1}{\sqrt{3}} \partial^2 \eta_8)
\]

\[
\left\langle BmQ^2(U^\dagger - U) \right\rangle = \frac{-2iB}{F} \left\langle mQ^2 \phi \right\rangle
\]

\[
= \frac{-2iB}{9F} \left( (4m_u - m_d)\pi^0 + \frac{1}{\sqrt{3}}(4m_u + m_d - 2m_s)\eta_8 \right)
\]

\[
\left\langle Bm(U^\dagger - U) \right\rangle = \frac{-2iB}{F} \left\langle m\phi \right\rangle
\]

\[
= \frac{-2iB}{F} \left( (m_u - m_d)\pi^0 + \frac{1}{\sqrt{3}}(m_u + m_d - 2m_s)\eta_8 \right).
\]
The vertex factors for these interaction terms may then be extracted. For the $\pi^0\gamma\gamma$ we have

$$i \left( \frac{\alpha}{\pi F} \right) \epsilon_{\mu\nu\rho} k_1^\mu \epsilon_1^\nu k_2^\rho \epsilon_2^\nu \left( \frac{32\pi^2}{3} \right) \left\{ A_2(q^2 - k_1^2 - k_2^2) - 2A_3(q^2 - 2k_1^2 - 2k_2^2) \right. \right.$$  

$$- \frac{8}{3} A_4 B(4m_u - m_d) - 16A_5 B(m_u - m_d)$$  

$$- \frac{8}{3} E_{20}(q^2 - \frac{3}{2} B(4m_u - m_d) + \frac{3}{2} B(m_u - m_d)) \}$$

and for the $\eta\gamma\gamma$

$$i \left( \frac{\alpha}{\sqrt{3} F} \right) \epsilon_{\mu\nu\rho} k_1^\mu \epsilon_1^\nu k_2^\rho \epsilon_2^\nu \left( \frac{32\pi^2}{3} \right) \left\{ A_2(q^2 - k_1^2 - k_2^2) - 2A_3(q^2 - 2k_1^2 - 2k_2^2) \right. \right.$$  

$$- \frac{8}{3} A_4 B(4m_u + m_d - 2m_\pi) - 16A_5 B(m_u + m_d - 2m_\pi)$$  

$$- \frac{8}{3} E_{20}(q^2 - \frac{3}{2} B(4m_u + m_d - 2m_\pi) + \frac{3}{2} B(m_u + m_d - 2m_\pi)) \}$$

We may now write down the full expression for the $\mathcal{O}(p^6)$ $\pi^0\gamma\gamma$ vertex function with all particles off mass-shell, including the $\mathcal{L}_4^A$ tree contribution, the various one-loop corrections, and the $\mathcal{L}_d^A$ tree level term which we have just derived.

$$A_{\pi^0}(q^2, k_1^2, k_2^2) = A_{\pi^0}(p^{+} F + Z) + A_{\pi^0}(p^{+, \text{loop (a)}}) + A_{\pi^0}(p^{+, \text{loop (b)}}) + A_{\pi^0}(p^{+, \text{tree}})$$  

$$= \frac{\alpha}{\pi F} \left[ 1 - \frac{8}{3} + 4 + \frac{8}{3} - 2 \right] \mu^D + \left( - \frac{10}{3} + 4 + \frac{1}{3} - 1 \right) \mu^D_R$$  

$$- \frac{1}{3}(k_1^2 + k_2^2) \left( \frac{\mu^D}{m_{\pi^0}^2} + \frac{\mu^D_R}{m_{\pi^0}^2} + \frac{1}{48\pi^2 F^2} \right)$$  

$$+ \frac{1}{96\pi^2 F^2} \left\{ (4m_{\pi^0}^2 - k_1^2)H_0 \left( \frac{k_1^2}{m_{\pi^0}^2} \right) + (4m_{\pi^0}^2 - k_2^2)H_0 \left( \frac{k_2^2}{m_{\pi^0}^2} \right) \right\}$$  

$$+ \frac{32\pi^2}{3} \{ A_2(q^2 - k_1^2 - k_2^2) - 2A_3(q^2 - 2k_1^2 - 2k_2^2) - 8E_{20}q^2 \}$$  

$$\quad - \frac{8}{3} (A_4 - 2E_{20})B(4m_u - m_d) - 16(A_6 + \frac{3}{3} E_{20})B(m_u - m_d) \}$$

The bracketed coefficients of $\mu^D$ and $\mu^D_R$ in the first line of this expression contain contributions from, in order, diagram (a), diagram (b), the wavefunction correction, and the decay constant correction. Thus, as has been noted previously in Refs. [30, 41], there is no net contribution to the $\pi^0\gamma\gamma$ vertex factor from the various one-loop diagrams when we place the photons on mass-shell ($k_1^2 = k_2^2 = 0$).

We now turn to the renormalization of this expression. Divergent contributions arise only in the terms $\mu^D$ and $\mu^D_R$, most of which have cancelled. The remaining ones appear in the second line of the equation above with a factor $(k_1^2 + k_2^2)$ and so must clearly be
absorbed through renormalization of the $\mathcal{L}_a^A$ coefficients $A_2$ and $A_3$. The terms which concern us are

$$-\frac{1}{3}(k_1^2 + k_2^2) \left( \frac{1}{F_2} D(\epsilon) + \frac{1}{F_2} D(\epsilon) \right) + \frac{32\pi^2}{3}(A_2 - 2A_3 - 8E_{2\theta})q^2 + \frac{32\pi^2}{3}(4A_3 - A_2)(k_1^2 + k_2^2)$$

and so we require the renormalized low energy constants to be defined by

$$A_2' - 2A_3' - 8E_{2\theta}' \equiv A_2 - 2A_3 - 8E_{2\theta}$$

$$4A_3' - A_2' \equiv 4A_3 - A_2 - \frac{1}{16\pi^2 F_2^2} D(\epsilon)$$

$$A_4' \equiv A_4$$

$$A_6' \equiv A_6$$

$$E_{2\theta}' \equiv E_{2\theta}$$

which implies that

$$A_2' = A_2 - \frac{1}{16\pi^2 F_2^2} D(\epsilon)$$

$$A_3' = A_3 - \frac{1}{32\pi^2 F_2^2} D(\epsilon)$$

$$A_4' \equiv A_4$$

$$A_6' \equiv A_6$$

$$E_{2\theta}' \equiv E_{2\theta}.$$ 

The divergent term $D(\epsilon)$ is defined in Eq. (4.8). These renormalized coefficients should also arise in a background field calculation, as discussed in Chp. 3.

Our final result for the $\pi^0\gamma\gamma$ vertex to $O(p^8)$ in chiral perturbation theory is then

$$A_{\pi^0}(q^2, k_1^2, k_2^2) = \frac{\alpha}{\pi F_{\pi}} \left[ 1 - k_1^2 \mathcal{F}(m_{\pi^0}^2, k_1^2) - k_2^2 \mathcal{F}(m_{\pi^0}^2, k_2^2) - k_1^2 \mathcal{F}(m_{K^S}^2, k_1^2) - k_2^2 \mathcal{F}(m_{K^S}^2, k_2^2) - \frac{32\pi^2}{3}\left\{ A_2'(q^2 - k_1^2 - k_2^2) - 2A_3'(q^2 - 2k_1^2 - 2k_2^2) - 8E_{2\theta}'(q^2 - m_{\pi^0}^2) - 4A_6'(m_{\pi^0}^2 + \frac{5}{3}\Delta_{u,d}) - 16(A_6' - \frac{1}{6}F_{2\theta}')\Delta_{u,d}\right\}\right]$$

(4.10)

where we have defined

$$\mathcal{F}(m^2, k^2) \equiv \frac{1}{96\pi^2 F_2^2} \left( \log(m^2/m^2) + \frac{1}{3} + (1 - 4m^2/k^2)H_0(k^2/m^2) \right)$$
4.4 Models of off-shell behaviour

and $\Delta_{u,d} \equiv B(m_u - m_d)$ is the isospin breaking term, which is expected to be small in comparison with the meson masses squared. The corresponding expression for the $\eta_8$ is

$$A_{\eta_8}(q^2, k_1^2, k_2^2) = \rac{1}{\sqrt{3}} \frac{\alpha}{\pi F_{\eta_8}} \left[ 1 - k_1^2 \mathcal{F}(m_{\pi^0}^2, k_1^2) - k_2^2 \mathcal{F}(m_{\pi^+}^2, k_1^2) - k_2^2 \mathcal{F}(m_{\pi^+}^2, k_2^2) - k_2^2 \mathcal{F}(m_{K^+}^2, k_2^2) + 32\pi^2 \left\{ 2 A_4^*(q^2 - 2k_1^2 - 2k_2^2) + 4 A_4^*(m_{\eta_8}^2 - 2m_{\pi^+}^2 - \Delta_{u,d}) \right. \right.$$

$$\left. + 8 E_{20}^*(q^2 - m_{\eta_8}^2) + 8 E_{20}^* \Delta_{u,d} + 24 A_6^*(m_{\eta_8}^2 - m_{\pi^+}^2) \right\} \right].$$

(4.11)

These expressions agree with those derived in Refs. [30, 31] once one takes account of the differing definitions of the $\mathcal{L}_0^A$ part of the Lagrangian in each case. A comparison between these definitions was made in §2.2.

The low energy constants of the $\mathcal{L}_0^A$ Lagrangian have brought with them off-shell behaviour of the vertex factors with respect to the neutral meson momentum as well as the photon momenta. This meson off-shell behaviour was not noted in previous treatments of the $\pi^0\gamma\gamma$ vertex to $\mathcal{O}(p^6)$ in Refs. [30, 31]. We shall henceforth set the coefficient $E_{20}^* = 0$ since it is associated with a term proportional to the equation of motion, and according to Refs. [5, 18] should therefore have no contribution to measurable quantities.

4.4 Models of off-shell behaviour

In Chp. 5 we shall compute observables for various processes in terms of the $\pi^0\gamma\gamma$ form factor. We now define several options for the form of this vertex function, $A(m_{\pi^0}^2, k_1^2, k_2^2)$.

1. Tree level:

$$A_{(1)}(m_{\pi^0}^2, k_1^2, k_2^2) = \frac{C}{\pi F_{\pi}}$$

This is the tree level, $\mathcal{O}(p^4)$ chiral perturbation theory calculation of the vertex function, as given in §4.1. We take $F_{\pi} = 0.093$ GeV in this expression.

2. Naive VMD:

$$A_{(2)}(m_{\pi^0}^2, k_1^2, k_2^2) = A_{(1)}(m_{\pi^0}^2, k_1^2, k_2^2) \left( \frac{1}{2} P_\rho(k_1^2) + \frac{1}{2} P_\omega(k_2^2) \right) \left( \frac{1}{2} P_\rho(k_1^2) + \frac{1}{2} P_\omega(k_2^2) \right)$$

where

$$P_{\rho,\omega}(k^2) \equiv \frac{m_{\rho,\omega}^2}{m_{\rho,\omega}^2 - k^2 + i m_{\rho,\omega} \Gamma_{\rho,\omega}(k^2)}$$

$$\Gamma_{\rho,\omega}(k^2) \equiv \left\{ \begin{array}{ll} \Gamma_{\rho,\omega} \sqrt{k^2/m_{\rho,\omega}} & \text{if } k^2 > 0 \\ 0 & \text{if } k^2 \leq 0 \end{array} \right.$$
4.4 Models of off-shell behaviour

Here we have taken the tree level $\chi$PT result of option (1) and added vector meson propagator factors to represent in a simple fashion vector meson dominance in the the off-shell behaviour of the two photons. We have assumed equal amplitude, constructively interfering contributions from the $\rho$ and $\omega$ resonances and we take the masses and widths of the vectors mesons to be $m_\rho = 0.770$ GeV, $m_\omega = 0.782$ GeV, $\Gamma_\rho = 0.151$ GeV, $\Gamma_\omega = 0.0084$ GeV.

(3) One-loop: This is the one-loop $\chi$PT result of Eq. (4.10) where the renormalized low energy constants $A_2^\prime, A_3^\prime, A_4^\prime$ and $A_6^\prime$ are set to zero.

(4) Fitted $O(p^4)$: This calculation option is defined to be the same as option (3) except that the low energy constants $A_2^\prime, A_3^\prime, A_4^\prime$ and $A_6^\prime$ introduced during renormalization are taken to have values consistent with the experimental data on the $^0\gamma^\gamma^\prime$ slope parameter taken from Ref. [42]. The slope parameter is defined as

$$ \frac{1}{A_\pi^+(m_\pi^+,0,0)} \frac{d}{dk_1^2} A_\pi^+(m_\pi^+,k_1^2,0)|_{k_1^2=0} $$

and so from Eq. (4.10) we see it is

$$ = -\frac{1}{6\pi F^2} \left( \log \frac{m_{\pi^+}^2}{\mu^2} + \log \frac{m_{\pi^0}^2}{\mu^2} + 2 \right) \left( -A_5^\prime + 4A_6^\prime \right) \frac{32\pi^2}{3} (m_{\pi^0}^2 - 2A_5^\prime) - 4A_4^\prime (m_{\pi^+}^2 + \frac{2}{3} \Delta_{u,d}) - 16A_6^\prime \Delta_{u,d}. $$

We let the on-shell value of the vertex factor remain the same as the tree level prediction of option (1), while having a photon off-shell dependence which reproduces the slope parameter of value $(1.79 \pm 0.14)$ GeV$^{-2}$ extracted from the CELLO experiment [42]. The explicit values of the renormalized constants are then

$$ A_2^\prime = \frac{\beta^{a_1}}{16\pi^2 F^2} $$
$$ A_3^\prime = \frac{\beta^{a_1}}{32\pi^2 F^2} $$
$$ A_4^\prime = 0 $$
$$ A_6^\prime = 0 $$

(4.12)

where the dimensionless constant fitted to the data is $\beta^{a_1} = (9.6 \pm 1.6) \times 10^{-3}$.

Where data exists we shall also place constraints on the low energy constants $A_2^\prime, A_3^\prime, A_4^\prime$ and $A_6^\prime$ by comparison with other experiments.

The power-of-$p^2$ expansion used in chiral perturbation theory is thought to converge only when all momenta in a process are less than some characteristic hadronic energy.
Figure 4.2: Behaviour of our various calculational options with one photon taken off-shell in the space-like direction. Options (1), (2), (3) and (4) are represented by the solid, the long dashed, the dot-dashed and the short dashed curves, respectively. The double-lined curve is the QCD-inspired model of Ref. [43]. The data points are from the CELLO experiment of Ref. [42].

scale, usually assumed to be the mass of the rho. Thus we would expect the vertex factor $A(m_\pi^2, k_1^2, k_2^2)$ computed using $O(p^6)$ chPT to be a good approximation only for virtual photon invariant masses of less than a few hundred MeV. The simple vector meson propagators used in option (2) can be viewed as extending the low energy result of chiral perturbation theory into the region of the rho resonance, within the context of a particular model.

Fig. 4.2 shows the behaviour of our various form factor options with one photon taken off-shell in the space-like direction. Also shown are the results of a QCD-inspired model calculation by Frank et al. [43], and the data points measured in the CELLO $e^+e^- \rightarrow \pi^0e^+e^-$ experiment [42]. Our tree level calculation of option (1) is constant, while the one-loop result of option (3) has only a slight $k^2$ dependence. The naive vector
meson dominance model of option (2) is consistent with the CELLO data and differs little from the predictions of the QCD-inspired model of Frank et al.

Our option (4) is a one-loop result with most of the $k^2$ dependence coming from the values of the low energy constants of $O(p^6)$. Recall that the $k^2$ dependence of this option was fit to the slope parameter that the CELLO group obtained by extrapolating the data shown in Fig. 4.2 back to $k^2 = 0$. We see that the simple linear dependence of option (4) matches the VMD model of option (2) and the QCD-inspired model of Ref. [43] only in the region $\sqrt{-k^2}$ less than a few hundred MeV.

The QCD-inspired model of Ref. [43] also gives a prediction for the behaviour of the form factor as the pion is taken off-shell. As noted in Refs.[5, 18] for the case of the charged pion form factor, it seems that off-shell behaviour is dependent upon the representation one takes for the meson fields. This prevents a direct comparison between the off-shell behaviour of different models from being meaningful. Any physical S-matrix computed with this off-shell vertex as one of its components would of course be independent of the field representation, so long as all the vertices involved in the full process were computed in a consistent fashion, without any ad-hoc phenomenological modifications being made.
Chapter 5

Processes Containing the $\pi^0\gamma\gamma$ Vertex

We shall now compute various processes which include the $\pi^0\gamma\gamma$ vertex using the models for the vertex function $A(m^2_{\pi^0}, k_1^2, k_2^2)$ described in §4.4. In Fig. 5.1 we have illustrated the regions in the $(k_1^2, k_2^2)$ plane that each of the considered processes can probe. In drawing the diagram we have multiplied the electron mass by a factor of 50 for clarity. In the interactions under examination the pion is always an on-shell particle. The processes we consider are $\pi^0 \rightarrow \gamma\gamma$, $\pi^0 \rightarrow \gamma e^+e^-$, $e^+e^- \rightarrow \gamma\pi^0$, $e^+e^- \rightarrow \mu^+\mu^-\pi^0$, $e^-e^- \rightarrow e^-e^-\pi^0$, and $e^+e^- \rightarrow e^+e^-\pi^0$. In the following sections we shall compute observables for each process in terms of the vertex factor $A(m^2_{\pi^0}, k_1^2, k_2^2)$. Each of the options for this factor detailed in §4.4 which are relevant to the process are then used to calculate numerical results for the observables. We compare these results with experimental data wherever it is available.

Throughout this chapter we shall write the phase space integrals for decays and cross sections in the following form.

Decay rate ($1 \rightarrow 2 \ldots n$):

$$\Gamma = \frac{S}{2m_1(2\pi)^{3(n-1)-4}} R_{n-1}(m_1^2, m_2^2, \ldots, m_n^2) \sum |M_{1-2\ldots n}|^2$$

Cross section ($12 \rightarrow 3 \ldots n$):

$$\sigma = \frac{S}{2\lambda^*(s, m_1^2, m_2^2)(2\pi)^{3(n-2)-4}} R_{n-2}(s; m_3^2, \ldots, m_n^2) \sum |M_{12-3\ldots n}|^2$$

He.e $S$ is composed of a factor $1/m!$ for each group of $m$ identical particles in the final state; this factor compensates for over-counting of these indistinguishable particles during the phase space integration. The phase space operator $R_{n-1}$ for the decay case
Chapter 5 Processes Containing the $\pi^0\gamma\gamma$ Vertex

Figure 5.1: The regions of the vertex function $A(m_e^2, k_1^2, k_2^2)$ accessible to each of the considered processes. All limits shown are correctly to scale except for those involving the electron mass. This mass has been increased by a factor of 50 in order to clarify the diagram.

is defined

$$R_{n-1}(m_1^2; m_2^2, \ldots, m_n^2) = \left( \prod_{i=2}^{n} \frac{\delta^4(p_i)}{2E_i} \right) \delta^4(p_1' - p_2' - \ldots - p_n').$$

The phase space operator for the cross section is defined similarly. These operators may be decomposed into simpler functions using time-like recursion relations, as is demonstrated in appendix §B.1.

When computing certain processes we shall also require the results

$$\epsilon^{\mu\nu\sigma\beta} \epsilon_{\mu'\nu'\sigma'\beta'} g_{\nu\nu'} g_{\beta\beta'} = -2(g^{\mu\nu'} g^{\alpha\beta'} - g^{\mu\alpha'} g^{\nu\beta'})$$
5.1 $\pi^0 \rightarrow \gamma\gamma$

The first process involving the $\pi^0\gamma\gamma$ vertex which we consider is the decay $\pi^0 \rightarrow \gamma\gamma$. The rate for this process, where we define $k_1^\mu$, $k_2^\mu$, $\epsilon_1^\mu$ and $\epsilon_2^\mu$ as the photon momenta and polarization vectors, is

$$\Gamma = \frac{1}{2!} \frac{1}{2m_{\pi^0}(2\pi)^2} \frac{\lambda^4(m_{\pi^0}^2,0,0)}{8m_{\pi^0}^2} \int d\Omega \sum_{\lambda_1,\lambda_2} |A(m_{\pi^0}^2,0,0)\epsilon^\mu_{\lambda_1} \epsilon^\nu_{\lambda_2} k_{1\mu}k_{2\nu}|^2$$

$$= \frac{|A(m_{\pi^0}^2,0,0)|^2}{32\pi m_{\pi^0}} (-\epsilon^\mu_{\lambda_1} \epsilon^\nu_{\lambda_2} (-g^{\mu\nu})(-g^{\alpha\beta})k_{1\nu}k_{2\alpha}k_{2\beta})$$

$$= \frac{|A(m_{\pi^0}^2,0,0)|^2}{32\pi m_{\pi^0}} (-2)(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\beta}g^{\alpha\nu})k_{1\nu}k_{1\alpha}k_{2\beta}k_{2\alpha}$$

$$= \frac{m_{\pi^0}^3}{64\pi} |A(m_{\pi^0}^2,0,0)|^2$$

where we have used the result $\sum_{\lambda} \epsilon_{(\lambda)}^\nu \epsilon_{(\lambda)}^\mu = -g^{\mu\nu}$.

Using the tree level option (1) of §4.4, that is setting $A(m_{\pi^0}^2,0,0) = \alpha/(\pi F_\pi)$ with $F_\pi = 93$ MeV, we find that $\Gamma_{\pi^0 \rightarrow \gamma\gamma} = 7.63$ eV. Experimentally the decay rate is found to be $(7.7 \pm 0.6)$ eV. This is computed using the mean life of $(8.4 \pm 0.6) \times 10^{-17}$ s and branching ratio to two photons of $(98.798 \pm 0.032)$% given in Ref. [8, pp. 1449]. The lowest order $\chi$PT result given by option (1) is clearly well within the experimental errors. Options (2), (3) and (4) are defined to differ from option (1) only in their photon off-mass-shell behaviour, and so gives identical results for this process. In general the extension to $O(p^6)$ gives the on-shell $\pi^0$ vertex factor as

$$A_{\pi^0}(q^2, k_1^2, k_2^2) = \frac{\alpha}{\pi F_\pi} \left[ 1 + \frac{32\pi^2}{3} \left\{ A_4^s m_{\pi^0}^2 + 2A_5^s m_{\pi^0}^2 - 4A_4^s m_{\pi^0}^2 \right\} \right]$$

We may therefore set a constraint on this combination of low energy constants in order that their contribution not spoil the tree level agreement with experiment.

$$[(A_4^s - 2A_5^s - 4A_4^s)m_{\pi^0}^2 - (\frac{32}{3} A_4^s + 16A_5^s)\Delta_{u,d}] = (5 \pm 38) \times 10^{-6}$$
The $\eta$ decay rate to two photons is $\Gamma(\eta \rightarrow \gamma\gamma) = (0.46 \pm 0.04)$ keV [8, pp. 1452] in comparison with our calculated tree level result of $\Gamma(\eta_b \rightarrow \gamma\gamma) = 0.51$ keV. At $O(p^6)$ we find the constraint on the low energy constants to therefore be

$$\left[(A_s^s - 2A_s^a + 4A_s^4 + 24A_s^6)m_{\eta_s}^2 - 8(A_s^s + 6A_s^a)m_{\eta_s}^2 - 4A_s^6\Delta_\eta,d\right] = (-47 \pm 38) \times 10^{-8}$$

where we have set $m_{\eta_s} = m_\eta = 547$ MeV. Here we have been forced to assume that the physical $\eta$ is well described by the octet field $\eta_b$. This is clearly a poor approximation given the known mixing with the singlet $\eta_1$, but it is the best that can be done without extending our chiral Lagrangian to accommodate the $\eta_1$. Such an extension would bring several more terms to $L_\eta^8$ along with new low energy constants, and so it seems unlikely we could gain predictive power in this way.

5.2 $\pi^0 \rightarrow \gamma e^+e^-$

The decay $\pi^0 \rightarrow \gamma e^+e^-$ is sensitive to the $\pi^0\gamma\gamma$ vertex function with one photon off-shell. The decay rate is given by

$$\Gamma = \frac{1}{2m_{\pi^0}(2\pi)^3} \int_{m_{\pi^0}^2}^{m_{\pi^0}^2} dk_1 \lambda_\pi^2 \left(\frac{m_{\pi^0}^2 + k_1^2}{8m_{\pi^0}^2}\right) (4\pi) \frac{\lambda_\pi^2}{8k_1^2} \int d\Omega_{\pi^0} \sum |M_{\pi^0 \rightarrow \gamma e^+ e^-}|^2$$

where the amplitude is

$$M_{\pi^0 \rightarrow \gamma e^+ e^-} = A(m_{\pi^0}^2, k_1^2, 0) e^{\mu\nu\alpha\beta} k_{\nu\alpha} \frac{(4\pi\alpha)^\frac{1}{2}}{k_1^2} [\bar{u}_e - \gamma e_+] k_2 e_+ e^- \cdot p$$

Squaring the amplitude and defining the lepton tensor to be

$$L^{\mu\nu} \equiv \sum_{\text{spin}} [\bar{u}_e - \gamma \mu u_+][\bar{\nu}_e + \gamma \nu u_-]$$

we have for the decay rate

$$\Gamma = \frac{\alpha}{64\pi^2 m_{\pi^0}^2} \int_{m_{\pi^0}^2}^{m_{\pi^0}^2} \frac{dk_1^2}{k_1^4} (m_{\pi^0}^2 - k_1^2)^{\frac{3}{2}} \left(1 - \frac{4m_e^2}{k_1^2}\right)^\frac{1}{2} |A(m_{\pi^0}^2, k_1^2, 0)|^2 \times \epsilon^{\mu\nu\alpha\beta} \epsilon^{\mu'\nu'\alpha'\beta'} k_{\nu\alpha} k_{\nu'\alpha'} k_1 k_{\nu\alpha} k_{\nu'\alpha'} (-g_{\beta\beta'}) \int \frac{d\Omega_{\pi^0}}{4\pi} L_{\mu\nu}$$

In App. E we find the integral of the lepton tensor over angle to be

$$\int \frac{d\Omega_{\pi^0}}{4\pi} L_{\mu\nu} = \frac{4}{3} \left(1 + \frac{2m_e^2}{k_1^2}\right) (k_{1\nu} k_{1\nu'} - k_{1\nu}^2 g_{\nu\nu'})$$

The first term in this integral vanishes upon contraction with the totally antisymmetric objects. After some algebra we have the result,

$$\Gamma = \frac{\alpha m_{\pi^0}^3}{96\pi^2} \int_{m_{\pi^0}^2}^{m_{\pi^0}^2} \frac{dk_1^2}{k_1^4} \left(1 - \frac{k_1^2}{m_{\pi^0}^2}\right) \left(1 - \frac{4m_e^2}{k_1^2}\right)^\frac{1}{2} \left(1 + \frac{2m_e^2}{k_1^2}\right) |A(m_{\pi^0}^2, k_1^2, 0)|^2.$$
This may be integrated numerically on $k_1^2$ in order the yield the total rate. Actually we make a change of integration variable to $x = \log(k_1^2/m_{e\gamma}^2)$ in order to smooth out the $1/k_1^2$ divergence in the integrand. We note here that setting the electron mass to zero, which might naively seem to be a good approximation, leads to an infinite total rate since the virtual photon decaying to the lepton pair is allowed to become arbitrarily soft. This must also be borne in mind when we come to calculate the more involved process $e^+e^- \rightarrow e^+e^-\pi^0$ which includes a contribution from a diagram with a time-like virtual photon decaying to an electron-positron pair.

The experimental decay rate for the process $\pi^0 \rightarrow \gamma e^+e^-$ is $(93.9 \pm 7.2)$ meV—this is derived from the information in the latest publication of the Particle Data Group Ref. [8, pp. 1449]. The rates with each of our options for the vertex factor are given below.

| Rate in meV |
|-----|-----|
| Option (1): tree level | 90.419 |
| Option (2): VMD | 90.597 |
| Option (3): one-loop | 90.423 |
| Option (4): $O(p^6)$ fitted result | 90.613 |

We have displayed the above rates to accuracies of $1\mu$eV not because we believe this to be a measure of the precision of our calculation—rather we intend to show the very tiny differences between the various calculational options for this process. Indeed it is known that these results receive electromagnetic radiative corrections [44, 45] which increase the above rates by 0.9%.

Due to the $1/k_1^2$ behaviour of the integrand it is clear that the total rate receives most of its value from the vertex function $A(m_{e\gamma}^2, k_1^2, 0)$ in the region where the virtual photon is close to being on-shell, and where all of the options coincide in value. The total rate is therefore a very poor probe of off-shell behaviour.

By measuring the differential rate with respect to dilepton invariant mass, $d\Gamma/dM_{ee}$ with $M_{ee} = \sqrt{k_1^2}$, one can extract from experiment the $\pi^0\gamma\gamma^*$ slope parameter,

$$\frac{1}{A(m_{e\gamma}^2, 0, 0)} \frac{d}{dk_1^2} A(m_{e\gamma}^2, k_1^2, 0)_{k_1^2=0}.$$

The tree level $\chi$PT calculation of option (1) has no off-shell dependence in the photons and so gives zero for the slope parameter. The vector meson dominance model of option (2) gives 1.64 GeV$^{-2}$ for this observable—this is approximately just $1/m_{\rho,\omega}^2$. The intrinsic photon off-shell dependence seen in the $F(m_r^2, k^2)$ functions of the one-loop model, option (3), gives a contribution to the slope parameter of 0.28 GeV$^{-2}$. The $O(p^6)$ calculation of option (4) uses low energy constants fit from the experiment of
Ref. [42]—the extracted slope parameter therefore reproduces the value of 1.79 GeV$^{-2}$ found in that experiment. As mentioned above, proper consideration must be given to the electromagnetic radiative corrections to this decay—these corrections have a substantial effect on the value of the slope parameter extracted from experiment [46]. The existing experimental measurements of the $\pi\gamma\gamma^*$ slope parameter with radiative corrections accounted for are shown below. They are clearly in disagreement with one another and the uncertainties are far too large for the measurements to be used to discriminate between the predictions of our options (2) through (4). The central values of the most recent two experiments are, however, close to the vector meson dominance prediction.

<table>
<thead>
<tr>
<th>Slope parameter (GeV$^{-2}$)</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+1.4 \pm 3.0$</td>
<td>F. Farzanpay et al. (1992) [47]</td>
</tr>
<tr>
<td>$+1.4 \pm 1.6$</td>
<td>SINDRUM I Collaboration (1992) [48]</td>
</tr>
<tr>
<td>$-6.0 \pm 4.7$</td>
<td>F. Fonvieille et al. (1989) [49]</td>
</tr>
<tr>
<td>$+5.5 \pm 1.7$</td>
<td>J. Fischer et al. (1978) [50]</td>
</tr>
<tr>
<td>$-13.2 \pm 8.8$</td>
<td>S. Devons et al. (1969) [51]</td>
</tr>
<tr>
<td>$-8.2 \pm 5.5$</td>
<td>N. P. Samios (1961) [52]</td>
</tr>
<tr>
<td>$+0.6 \pm 6.0$</td>
<td>H. Kobrak (1961) [53]</td>
</tr>
</tbody>
</table>

We can also compute the slope parameter for the $\eta_8\gamma\gamma$ vertex using our various calculational options. Using these models the results are identical to the slope parameter of the $\pi^0\gamma\gamma$ vertex. The $\eta\gamma\gamma$ slope parameter has been measured to much higher accuracy than in the $\pi^0\gamma\gamma$ case. Measured values are given below.

<table>
<thead>
<tr>
<th>Slope parameter (GeV$^{-2}$)</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+1.42 \pm 0.21$</td>
<td>CELLO Collaboration (1991) [42]</td>
</tr>
<tr>
<td>$+2.0 \pm 0.4$</td>
<td>TPC/Two-Gamma Collaboration (1990) [54]</td>
</tr>
</tbody>
</table>

We hesitate to compare our predictions for the octet $\eta_8$ with these experimental results on the physical $\eta$. However, it does appear that the experimental value is in the neighbourhood of $1/m_\rho^2$ which is approximately the slope parameter of our vector meson dominance model, option (2).

5.3 \( e^+e^- \rightarrow \pi^0\gamma \)

The $e^+e^- \rightarrow \pi^0\gamma$ interaction, shown with labelled momenta in Fig. 5.2, probes the $\pi^0\gamma\gamma^*$ vertex function $A(m_{s*}, k_1^2, 0)$ in the region $k_1^2 \geq m_{s*}^2$. This interaction is the first of several we shall consider which will be accessible at the $e^+e^-$ collider DAΦNE [6].

The cross section for this process is given by

\[
\sigma(s) = \frac{1}{2\lambda^4(s, m_{s*}^2, m_{s*}^2)} \frac{\lambda^2(s, m_{s*}^2, 0)}{8s} \int d\Omega \sum |M_{e^+e^-\rightarrow \pi^0\gamma}|^2
\]
where
\[ M_{e^{+}e^{-} \rightarrow \pi^{0}\gamma} = \frac{(4\pi\alpha)^{\frac{3}{2}}}{s} A(m_{\pi^{0}}^2, s, 0) \epsilon^{\mu\nu\alpha\beta} k_{1\mu} [\bar{\nu} + \gamma_{\nu} u_{-}] k_{2\alpha} c_{2\beta}. \]

The absolute value of the amplitude, squared and summed over final and averaged over initial state spins is then
\[
\sum |M_{e^{+}e^{-} \rightarrow \pi^{0}\gamma}|^2 = \frac{1}{4} \frac{4\pi\alpha}{s^2} |A(m_{\pi^{0}}^2, s, 0)|^2 \epsilon^{\mu\nu\alpha\beta} \epsilon^{\mu'\nu'\alpha'\beta'} k_{1\mu} k_{1\mu'} L_{\nu\nu'} k_{2\alpha} k_{2\alpha'} (-g_{\beta\beta'})
\]

where
\[ L_{\nu\nu'} = \text{Tr} \left( (p_+ + m_e) \gamma_\nu (p_- - m_e) \gamma_{\nu'} \right) = 4 \left( p_+ p_{-\nu'} + p_{-\nu} p_{+\nu'} - \frac{s}{2} g_{\nu\nu'} \right). \]

\[ \Rightarrow \sum |M_{e^{+}e^{-} \rightarrow \pi^{0}\gamma}|^2 = \frac{\alpha \pi}{s^2} |A(m_{\pi^{0}}^2, s, 0)|^2 \]
\[ \times (k_1^2 k_2^2 L_{\mu\mu} - k_1^2 k_2 L_{\mu2} L_{\nu\nu} - k_2^2 k_{1\mu} k_{2\nu} L_{\mu\nu} + 2 k_1 \cdot k_2 L_{\mu\nu} k_{1\mu} k_{2\nu} - (k_1 \cdot k_2)^2 L_{\mu\mu}) \]

The required Lorentz scalar objects may be computed in terms of the interaction energy \( \sqrt{s} \) and the CM frame scattering angle \( \theta \);
\[ k_1^2 = s, \quad k_2^2 = 0, \quad L_{\mu\mu} = -4(s + 2m_e^2), \quad k_1 \cdot k_2 = \frac{1}{2}(s - m_{\pi^{0}}^2), \]
\[ L_{\mu2} L_{\nu\nu} = 8 p_+ \cdot k_2 p_- \cdot k_2 = \frac{1}{2}(s - m_{\pi^{0}}^2)^2 \left( 1 - \frac{4m_e^2}{s} \right) \cos^2 \theta, \]
\[ L_{\mu\nu} k_{1\mu} k_{2\nu} = 4 \left( k_1 p_+ k_2 p_+ + k_1 p_- k_2 p_- - \frac{s}{2} k_1 \cdot k_2 \right) = 0. \]
\[ \sum |M_{e^+e^-\rightarrow\pi^0\gamma}|^2 = \frac{\alpha\pi}{s^2} |A(m_{\pi^0}^2, s, 0)|^2 (s - m_{\pi^0}^2)^{1/2} \left(1 + \cos^2 \theta\right) + \frac{4m_e^2}{s} (1 - \cos^2 \theta). \]

The scattering angle may now be integrated out, giving for the total cross section

\[ \sigma(s) = \frac{\alpha}{24} |A(m_{\pi^0}^2, s, 0)|^2 \left(1 - \frac{m_{\pi^0}^2}{s}\right)^3 \left(1 - \frac{4m_e^2}{s}\right)^{-\frac{1}{2}} \left(1 + \frac{2m_e^2}{s}\right). \]

Plots of the total cross section for this process versus interaction energy are given in Fig. 5.3 for the vertex factor options (1) to (4) specified in §4.4. The region of low interaction energy in this plot is reproduced with a linear vertical scale in Fig. 5.4. In this low energy region one may expect the \( \chiPT \) expansion in powers of momentum to converge with the inclusion of only a few terms. The tree level result of option (1) and the one-loop result of option (3) can be seen to differ only very slightly over the energy range considered. Much larger than these, and almost identical to one another for the low energy region up to \( \sqrt{s} \approx 300 \text{ MeV} \), are the naive vector meson dominance model of option (3) and the fitted \( \mathcal{O}(p^6) \) result of option (4).

Currently there is data on this process only for \( \sqrt{s} \approx m_\omega \). The experimental analysis [55] depended upon a vector meson dominance model for the separation of signal from an uncorrelated three-photon background. An assumption of an omega resonance gave a cross section of \( \sigma_{e^+e^-\rightarrow\pi^0\gamma}(s = m_\omega^2) = 176 \pm 8 \text{ nb} \); contributions from the \( \omega \) and \( \rho \) equal in amplitude and destructively interfering gave \( \sigma_{e^+e^-\rightarrow\pi^0\gamma}(s = m_\rho^2) = 246 \pm 11 \text{ nb} \); equal and constructively interfering \( \rho \) and \( \omega \) resonances gave \( \sigma_{e^+e^-\rightarrow\pi^0\gamma}(s = m_\omega^2) = 153 \pm 7 \text{ nb} \). This last result was favoured by the experimentalists and is marked in Fig. 5.3. Since we have used a similar model of constructively interfering, equal amplitude resonances for our VMD model we should not be surprised at the very close agreement between the experimental data point and our vector meson dominance result for the cross section.

The agreement of options (2) and (4) suggests that values of the low energy constants \( A_2 \) and \( A_3 \) which were fit to data from CELLO [42] are consistent with the naive vector meson dominance prediction. Consideration of where the predictions differ gives us a clue to the region where each order in momentum squared of the \( \chiPT \) expansion is a good approximation. The tree level result differs from the VMD result of option (2) by less than 20% for \( \sqrt{s} = \sqrt{k_f^2} < 220 \text{ MeV} \), while the \( \mathcal{O}(p^6) \) result of option (4) is within 20% of the VMD result for \( \sqrt{s} = \sqrt{k_f^2} < 450 \text{ MeV} \). The DA\$NE experiment should bring a wealth of new information about this process at these lower energies, allowing the validity of chiral perturbation theory to be tested in the region where the momentum expansion is convergent.
Figure 5.3: The cross section for $e^+e^- \rightarrow \pi^0\gamma$.

The different curves above correspond to the different choices for the $\pi^0\gamma^*\gamma^*$ form factor $A(m_{\pi}^2, k_1^2, k_2^2)$ described in the text. The solid curve is the tree level result of option (1), the long dashed curve is the VMD model of option (2), the dot-dashed curve is the one-loop calculation of option (3), while the short dashed curve is $O(p^8)$ fit to data of option (4). Also shown is the single experimental data point existing for this process, taken from Ref. [55]. The apparently high degree of agreement between the data point and our VMD model is not very significant since the experimental analysis depended upon a similar vector meson dominance model for the subtraction of the $3\gamma$ background.
Figure 5.4: An expansion of the low energy region of Fig. 5.3
5.4 $e^+e^- \rightarrow \pi^0\mu^+\mu^-$

This interaction is similar to the preceding one except that the real final state photon in $e^+e^- \rightarrow \pi^0\gamma$ is here allowed to become off-shell through decay to a muon pair. Due to the additional electromagnetic vertex and the constraint that the massive leptons place on phase space we will find the cross section for $e^+e^- \rightarrow \pi^0\mu^+\mu^-$ to be several orders of magnitude smaller than that for $e^+e^- \rightarrow \pi^0\gamma$, making this a more challenging process to study experimentally.

The introduction of the final state lepton pair also brings a theoretical problem. Both of the virtual photons from the $\pi^0\gamma\gamma^*$ vertex are very far off-shell, as can be seen from the phase space limits on their invariant masses.

$$\sqrt{s} = \sqrt{k_1^2} \geq (m_\pi^2 + 2m_\mu); \quad 2m_\mu \leq \sqrt{k_2^2} \leq \left( \sqrt{k_1^2} - m_\pi^0 \right).$$

This may well place us outside the region of validity for chiral perturbation theory, within which the first few terms in the powers-of-$p^2$ expansion can be expected to provide a good approximation to the full result.

The cross section for $e^+e^- \rightarrow \pi^0\mu^+\mu^-$, with four-momenta labelled as in Fig. 5.5, is

$$\sigma(s) = \frac{1}{2\lambda^4(s, m_\pi^2, m_\mu^2)(2\pi)^5} \int_{4m_\mu^2}^{(\sqrt{s} - m_\pi^2)^2} \frac{dk_2^2}{8s} \frac{\lambda^2(s, m_\pi^2, k_2^2)}{8s} \int d\Omega_{\gamma^-\pi^+\gamma^+} \times \frac{\lambda^4(k_2^2, m_\mu^2, m_\pi^2)}{8k_2^4} \int d\Omega_{\gamma^-\mu^-\mu^+} \sum |M_{e^+e^-\rightarrow \pi^0\mu^+\mu^-}|^2$$
with amplitude as follows.

\[
M_{e^+e^- \rightarrow \pi^0\mu^+\mu^-} = \frac{4\pi\alpha}{s k_1^2} A(m_{\pi^0}^2, s, k_2^2) \epsilon^{\nu\alpha\beta\gamma} k_{1\mu} [\bar{\nu}_e + \gamma_{\nu} u_{e^-}] k_{2\nu} [\bar{\nu}_{\mu^-} + \gamma_{\mu} v_{\mu^+}]
\]

\[
\sum |M_{e^+e^- \rightarrow \pi^0\mu^+\mu^-}|^2 = \frac{1}{4} \frac{(4\pi\alpha)^2}{s^2 k_1^2} |A(m_{\pi^0}^2, s, k_2^2)|^2 \epsilon^{\nu\alpha\beta\gamma} \epsilon^{\nu'\alpha'\beta'\gamma'} k_{1\mu} k_{1\mu'} L^{(e)}_{\nu\nu'} k_{2\alpha} k_{2\alpha'} L^{(\mu)}_{\beta\beta'}
\]

The electron pair and muon pair lepton tensors are defined by

\[
L^{(e)}_{\nu\nu'} = 4 \left( p_{\nu} p_{-\nu'} + p_{-\nu} p_{\nu'} - \frac{s}{2} g_{\nu\nu'} \right)
\]

\[
L^{(\mu)}_{\beta\beta'} = 4 \left( q_{\beta} q_{-\beta'} + q_{-\beta} q_{\beta'} - \frac{k_2^2}{2} g_{\beta\beta'} \right).
\]

We now perform the phase space integration over the \( \gamma^- \rightarrow \mu^+\mu^- \) decay angles; details of this integral are contained in App. E.

\[
\int \frac{d\Omega}{4\pi} L^{(\mu)}_{\beta\beta'} = \frac{4}{3} \left( 1 + \frac{2m^2}{k_2^2} \right) (k_{2\beta} k_{2\beta'} - k_2^2 g_{\beta\beta'})
\]

Remaining now are the integrals on \( k_1^2 \) and on the electron to \( \pi^0 \) scattering angle \( \theta \).

\[
\sigma(s) = \frac{\alpha^2}{96\pi s^3} \left( 1 - \frac{4m_e^2}{s} \right)^{-\frac{1}{2}} \int_{4m_e^2}^{(\sqrt{s} - m_{\pi^0})^2} \frac{dk_1^2}{k_1^2} \lambda^{\frac{1}{2}}(s, m_{\pi^0}^2, k_2^2) |A(m_{\pi^0}^2, s, k_2^2)|^2 \times \left( 1 - \frac{4m_e^2}{k_1^2} \right)^{\frac{1}{2}} \left( 1 + \frac{2m_e^2}{k_1^2} \right) \int_{-1}^{+1} \cos \theta \epsilon^{\nu\alpha\beta\gamma} \epsilon^{\nu'\alpha'\beta'\gamma'} k_{1\mu} k_{1\mu'} L^{(e)}_{\nu\nu'} k_{2\alpha} k_{2\alpha'} (-g_{\beta\beta'})
\]

The \( k_1^2 \) integral will be performed numerically due to its complexity. The angular integral is quite straightforward, however, once we have contracted on the remaining tensor indices in our expression. As before, in the calculation of the \( e^+e^- \rightarrow \pi^0\gamma \) cross section, we have

\[
e^{\nu\alpha\beta\gamma} \epsilon^{\nu'\alpha'\beta'\gamma'} k_{1\mu} k_{1\mu'} L^{(e)}_{\nu\nu'} k_{2\alpha} k_{2\alpha'} (-g_{\beta\beta'}) = (k_1^2 k_2^2 - (k_1 \cdot k_2)^2) L_{\mu}^{\mu'} - k_1^2 k_2^2 L^{\mu\nu} - k_2^2 k_1^2 L^{\mu\nu} + 2k_1 \cdot k_2 L^{\mu\nu} k_{1\mu} k_{2\nu}
\]

The Lorentz scalar objects composing this expression have values almost identical to those given previously for the \( e^+e^- \rightarrow \pi^0\gamma \) calculation, except for the non-vanishing value of \( k_1^2 \) wherever it appears.

\[
k_1^2 = s, \quad k_2^2 \neq 0, \quad L^{\mu}_{\mu} = -4(s + 2m_e^2), \quad k_1 \cdot k_2 = \frac{1}{2}(s + k_2^2 - m_{\pi^0}^2),
\]

\[
L^{\mu\nu} k_{2\mu} k_{2\nu} = \frac{1}{2} \lambda(s, m_{\pi^0}^2, k_2^2) \left( 1 - \left( \frac{s - 4m_e^2}{s} \right) \cos^2 \theta \right), \quad L^{\mu\nu} k_{1\mu} k_{2\nu} = 0
\]
5.5 \( e^-e^- \rightarrow \pi^0e^-e^- \)

Substituting these values into the expression for the cross section and integrating over the angular dependence we have the result,

\[
\sigma(s) = \frac{\alpha^2}{72\pi s^3} \left(1 - \frac{4m_e^2}{s}\right)^{-\frac{1}{2}} \left(1 + \frac{2m_e^2}{s}\right) 
\times \int_{4m_e^2}^{(\sqrt{s} - m_\omega)^2} \frac{dk_3^2}{k_3^2} \lambda^{\frac{3}{2}}(s, m_{\omega}^2, k_3^2) |A(m_{\omega}^2, s, k_3^2)|^2 \left(1 - \frac{4m_e^2}{k_3^2}\right)^{\frac{1}{2}} \left(1 + \frac{2m_e^2}{k_3^2}\right).
\]

In Fig. 5.6 the \( k_3^2 \) integral has been performed numerically to give the total cross section as a function of interaction energy. The plot for option (2) shows the familiar peaking in the region of the \( \omega \) mass as \( s = k_3^2 \) passes through the resonance.

Another observable of interest is the differential cross section with respect to di-muon invariant mass,

\[
\frac{d\sigma}{dM_{\mu\mu}} = 2\sqrt{k_3^2} \frac{d\sigma}{dk_3^2}.
\]

This spectrum is plotted for a variety of interaction energies in Figs. 5.7–5.10, using options (1)–(4) for the vertex function \( A(m_{\omega}^2, k_1^2, k_3^2) \). For the vector meson dominance model of option (2) peaking behaviour due to the vector meson resonances can be seen in both the overall normalization of the spectrum, as \( k_1^2 = s \) is varied, and in the di-muon invariant mass spectrum, which corresponds to a variation of \( k_3^2 \). No experimental data currently exists for this process due to its very small cross section. It will, however, be accessible at DAΦNE in the future.

5.5 \( e^-e^- \rightarrow \pi^0e^-e^- \)

This is the first process we have studied which probes the function \( A(m_{\omega}^2, k_1^2, k_3^2) \) in the space-like region of \( k_1^2 \) and \( k_3^2 \). This process has a contribution from two diagrams at tree level, which means that we no longer have a simple relationship between an experimental observable and an evaluation of \( |A(m_{\omega}^2, k_1^2, k_3^2)|^2 \) at a single point in \( (k_1^2, k_3^2) \).

Fig. 5.11 shows the contributing diagrams with their momentum labels. In particular, the photon momenta are now defined by \( q_1^\mu \equiv p_1^\mu - p_2^\mu, \ q_2^\mu \equiv p_4^\mu - p_3^\mu \) for the first diagram and by \( q_1^\mu \equiv p_1^\mu - p_4^\mu, \ q_2^\mu \equiv p_3^\mu - p_2^\mu \) for the exchange diagram. The amplitude for this process is sufficiently complicated that we choose to perform the entire phase space integration numerically—we shall define our method of calculation shortly. First, however, we shall compute the amplitude squared for the process in terms of a small number of Lorentz scalars.

The amplitude for \( e^-e^- \rightarrow \pi^0e^-e^- \) is

\[
M_{e^-e^- \rightarrow \pi^0e^-e^-} = [\bar{u}_2(-i\gamma_\mu)u_1] i q_1^\lambda \frac{1}{q_1^2} A(m_{\omega}^2, q_1^2, q_2^2) e^{i\mu\nu+p_\alpha q_1^\mu q_2^\nu} \frac{1}{q_2^2} [\bar{u}_4(-i\gamma_\rho)u_3]
\]
Figure 5.6: The cross section for $e^+e^- \rightarrow \pi^0\mu^+\mu^-$. The curves correspond to the different choices for the $\pi^0\gamma^*\gamma^*$ form factor $A(m_{\pi}^2, k_1^2, k_2^2)$ described in the text. The solid curve is the tree level result of option (1), the long dashed curve is the VMD model of option (2), the dot-dashed curve is the one-loop calculation of option (3), while the short dashed curve is $O(p^6)$ fit to data of option (4).
Figure 5.7: The spectrum $d\sigma/dM_{\mu\mu}$ for the process $e^+e^- \rightarrow \pi^0\mu^+\mu^-$.  

The form factor $A(m_{\pi^0}^2, k_1^2, k_2^2)$ is given here by the tree level calculation of option (1), as described in the text. The dilepton invariant mass spectra are shown for several values of the interaction energy $\sqrt{s}$; the values of $\sqrt{s}$ are marked in GeV next to each curve.
Figure 5.8: Identical to Fig. 5.7 but uses the VMD model of option (2) for the $\pi^0\gamma^*\gamma^*$ form factor.
Figure 5.9: Identical to Fig. 5.7 but uses the one-loop result of option (3) for the $\pi^0\gamma^*\gamma^*$ form factor.
Figure 5.10: Identical to Fig. 5.7 but uses the $\mathcal{O}(p^0)$ fit to data of option (4) for the $\pi^0\gamma^*\gamma^*$ form factor.

Figure 5.11: Four-momentum labelling for the process $e^-e^-\rightarrow\pi^0e^-e^-$. 
Neglecting the electron mass, the amplitude squared and summed on spins is given by
\[
\sum |M_{e^- e^- \rightarrow \pi^0 e^+ e^-}|^2 = \frac{1}{4} (4\pi\alpha)^2 \left[ \frac{|A(m_{e^+}, q_1, q_2)|^2}{q_1 q_2^2} \right]_\ell \frac{1}{\ell_1 \ell_2} + \frac{|A(m_{e+}, q_1, q_2)|^2}{q_1 q_2^2} \left( |A(m_{e^+}, q_1, q_2)|^2 \right)_\ell \frac{1}{\ell_1 \ell_2} 
\]

This expression is complicated and the computer algebra program Mathematica [16] was used to assist in its evaluation. We present the result in terms of the Lorentz scalars \( q_1, q_2, \ell_1, \ell_2, p_1, p_2 \) and \( p_3, p_4 \).

\[
\sum |M_{e^- e^- \rightarrow \pi^0 e^+ e^-}|^2 
= -2 \left( 2q_1^2 q_2^2 + 2(4p_1 \cdot p_2 p_3 + p_4) - \ell_1 \ell_2 \right)^2 \left( q_1^2 q_2^2 \right) \left( \ell_1 \ell_2 \right)^2 
- \left( q_1^2 q_2^2 \ell_1 \ell_2 \right)^2 \left( \ell_1 \ell_2 \right)^2 
- \left( q_1^2 q_2^2 \ell_1 \ell_2 \right)^2 \left( \ell_1 \ell_2 \right)^2 
+ 8p_1 \cdot p_2 p_3 + p_4 \left( (p_1 \cdot p_2)^2 + (p_3 \cdot p_4)^2 \right) 
\]

A check can be made by using crossing symmetry to relate this expression to the amplitude squared for the process \( e^+ e^- \rightarrow \pi^0 e^+ e^- \). That process has previously been calculated in Ref. [56]. The crossing calculation will be given in the next section and we will find that we are in agreement with the result from the literature.
For scattering processes with three particles in the final state the differential cross section may be written [57, pp. 137, eq. V.9.2]

\[
\frac{d^4\sigma}{ds_1 ds_2 dt_1 dt_2} = \frac{\pi}{32\lambda(s, m^2, m^2)(2\pi)^5\sqrt{-\Delta_4}} \sum |M_{12 \to 345}|^2
\]

where the physical region of phase space is defined by \( \Delta_4 \leq 0 \). Here \( \Delta_4 \) is the symmetric Gram determinant whose arguments are four linearly independent combinations of the momenta \( p_1^\mu \ldots p_5^\mu \). The momenta are defined with \( p_1^\mu \) and \( p_2^\mu \) in the initial state, and \( p_3^\mu \), \( p_4^\mu \) and \( p_5^\mu \) in the final state. The invariants are then

\[
s \equiv (p_1 + p_2)^2 \quad s_1 \equiv (p_3 + p_4)^2 \quad s_2 \equiv (p_4 + p_5)^2
\]

\[
t_1 \equiv (p_1 - p_3)^2 \quad t_2 \equiv (p_2 - p_4)^2.
\]

We choose to define \( \Delta_4 \equiv \Delta_b(p_4 + p_5, p_2, p_1 + p_3, p_4) \), giving

\[
\Delta_4 = \frac{1}{16} |2s_2 \begin{vmatrix} s_2 - t_1 + m^2_2 & s + s_2 - m^2_3 & 2s - m^2_3 + m^2_4 \\ 2s - m^2_3 + m^2_4 & s - m^2_1 + m^2_2 & m^2_2 + m^2_4 - t_2 \\ s_2 - m^2_1 + m^2_4 & m^2_3 + m^2_4 - t_2 & s - s_1 + m^2_4 \end{vmatrix} |
\]

When applied to the interaction \( e^- e^- \to \pi^0 e^- e^- \) a more useful quantity to consider is

\[
\frac{d^4\sigma}{dE_3 dE_4 d\cos\theta_{13} d\cos\theta_{24}},
\]

the differential cross section with respect to the CM frame energies and scattering angles of the final state leptons. The relations between the invariant quantities \( s, s_1, s_2, t_1, t_2 \) and these energies and angles are straightforward to derive, using the same arguments as were used in §12.1.

\[
t_1 = \frac{1}{2s} \left( -s^2 + (3m^2_e + s_2)s + \cos\theta_{13} \lambda^\frac{1}{2}(s, m^2_e, m^2_e) \lambda^\frac{1}{2}(s, m^2_e, s_2) \right)
\]

\[
t_2 = \frac{1}{2s} \left( -s^2 + (3m^2_e + s_1)s + \cos\theta_{24} \lambda^\frac{1}{2}(s, m^2_e, m^2_e) \lambda^\frac{1}{2}(s, m^2_e, s_1) \right)
\]

\[
E_3 = (s + m^2_e - s_2)/2\sqrt{s}
\]

\[
E_4 = (s + m^2_e - s_1)/2\sqrt{s}
\]

The Jacobian for this variable transformation is

\[
dE_3 dE_4 d\cos\theta_{13} d\cos\theta_{24} = \frac{s}{\lambda(s, m^2_e, m^2_e) \lambda^\frac{1}{2}(s, m^2_e, m^2_e) \lambda^\frac{1}{2}(s, m^2_e, s_2)} ds_1 ds_2 dt_1 dt_2
\]

giving for the differential cross section

\[
\frac{d^4\sigma}{dE_3 dE_4 d\cos\theta_{13} d\cos\theta_{24}} = \frac{\lambda^\frac{1}{2}(s, m^2_e, s_1) \lambda^\frac{1}{2}(s, m^2_e, s_2) \pi^{\frac{1}{2}}}{32s(2\pi)^5\sqrt{-\Delta_4}} \sum |M_{e^- e^- \to \pi^- e^- e^-}|^2. \tag{5.2}
\]
The factor of $\frac{1}{2}$ appearing before the amplitude squared in this expression is the usual statistical factor included to compensate for double counting of the indistinguishable final state electrons. The variables $q_1^2$, $q_2^2$, $t_1^2$, $t_2^2$, $p_1 \cdot p_2$ and $p_3 \cdot p_4$ which appear in the expression for the process $e^- e^- \rightarrow \pi^0 e^- e^-$ must now be written in terms of the invariants $s_1$, $s_2$, $t_1$ and $t_2$.

\[
\begin{align*}
q_1^2 &= t_1 & q_2^2 &= t_2 & p_1 \cdot p_2 &= \frac{1}{2}(s - 2m_e^2) \\
t_1^2 &= (p_1 - p_4)^2 &= s_1 - s - t_2 + 3m_e^2 \\
t_2^2 &= (p_2 - p_3)^2 &= s_2 - s - t_1 + 3m_e^2 \\
p_3 \cdot p_4 &= \frac{1}{2}(s - s_1 - s_2 + m_{\pi^0})
\end{align*}
\]

These last three relations may be obtained in the following manner:

\[
\begin{align*}
(p_3 + p_4)^2 &= (p_1 + p_2 - p_3)^2 = (p_1 + p_2)^2 + m_{\pi^0}^2 - 2(p_1 + p_2) \cdot p_5 \\
&= s + m_{\pi^0}^2 - 2(p_3 + p_4 + p_5) \cdot p_5 \\
&= s - m_{\pi^0}^2 - (s_1 - m_e^2 - m_{\pi^0}^2) - (s_2 - m_e^2 - m_{\pi^0}^2) \\

\Rightarrow p_3 \cdot p_4 &= \frac{1}{2}(s - s_1 - s_2 + m_{\pi^0})
\end{align*}
\]

\[
\begin{align*}
s_1 &= (p_1 + p_2 - p_3)^2 = s - 2p_4 \cdot (p_1 + p_2) + m_e^2 \\
&= s - 2p_1 \cdot p_4 + (t_2 - 2m_e^2) + m_e^2 \\

\Rightarrow p_1 \cdot p_4 &= \frac{1}{2}(s - s_1 + t_2 - m_e^2)
\end{align*}
\]

\[
\begin{align*}
s_2 &= (p_1 + p_2 - p_3)^2 = s - 2p_3 \cdot (p_1 + p_2) + m_e^2 \\
&= s - 2p_2 \cdot p_3 + (t_1 - 2m_e^2) + m_e^2 \\

\Rightarrow p_2 \cdot p_3 &= \frac{1}{2}(s - s_2 + t_1 - m_e^2)
\end{align*}
\]

On the left side of Figs. 5.12–5.15 we have plotted the differential cross section as a function of the final state electron scattering angles $\theta_{12}$ and $\theta_{24}$ and integrated over the electron energies $E_3$ and $E_4$. The center of mass energy for each case is $\sqrt{s} = 400$ MeV. The range for the energy integrals is from a minimum value, representing a detector resolution and taken to be 20 MeV, up to the kinematically allowed maximum. The graph to the right of each figure is a section taken through the corresponding contour plot at $\cos \theta_{24} = 0$. One can see that there is little difference between the various calculation options for this process, although the tree level and one-loop results of options (1) and (3) are somewhat larger than the VMD model and the fitted $O(p^0)$ results of options (2) and (4).
Figure 5.12: The differential quantity $d^2\sigma/d\cos\theta_{13}d\cos\theta_{24}$ for the process $e^-e^-\rightarrow\pi^0e^+e^-$ as a function of the final state electrons' scattering angles. The center of mass energy is 400 MeV.

In order to obtain this plot from the expression in Eq. (5.2) the electrons' energies were integrated from 20 MeV up to their kinematic maximum. The $\pi^0\gamma^-\gamma^+$ vertex factor given by the tree level result option (1) of §4.4 was used in computing this cross section. The graph to the right shows a section through the contour plot taken at $\cos\theta_{24} = 0$.

The singular behaviour of the cross section with respect to the variables $q_1^2$, $q_2^2$, $\ell_1^2$ and $\ell_2^2$ can be seen in the contour plots. The correspondence between singular behaviour in the invariants and in the angles is as follows:

- $q_1^2$ small $\Rightarrow$ $\cos\theta_{13} \rightarrow +1$
- $q_2^2$ small $\Rightarrow$ $\cos\theta_{24} \rightarrow +1$
- $\ell_1^2$ small $\Rightarrow$ $\cos\theta_{24} \rightarrow -1$
- $\ell_2^2$ small $\Rightarrow$ $\cos\theta_{13} \rightarrow -1$.

5.6 \( e^+e^- \rightarrow \pi^0e^+e^- \)

The process $e^+e^- \rightarrow \pi^0e^+e^-$ is related to $e^-e^- \rightarrow \pi^0e^-e^-$ by crossing symmetry. Explicitly, if we replace

- $p_1^\mu \rightarrow p_1^\mu$, $p_2^\mu \rightarrow -p_2^\mu$, $p_3^\mu \rightarrow -p_3^\mu$, $p_4^\mu \rightarrow p_3^\mu$
Figure 5.13: Identical to Fig. 5.12 except that the VMD model of option (2) was used for the vertex factor.

Figure 5.14: Identical to Fig. 5.12 except that the one-loop result of option (3) was used for the vertex factor.
Figure 5.15: Identical to Fig. 5.12 except that the fitted $\mathcal{O}(p^8)$ result of option (4) was used for the vertex factor.

Figure 5.16: Four-momentum labelling for the process $e^+e^- \rightarrow \pi^0e^+e^-$. 
in Eq. (5.1) for $\sum |M_{e^- e^- e^+ e^-}|^2$, we have the expression $\sum |M_{e^{+e^-} \rightarrow \pi^0 e^+ e^-}|^2$. The momentum labels on the two diagrams for $e^+ e^- \rightarrow \pi^0 e^+ e^-$ are shown in Fig. 5.16. In terms of the new, crossed four-momenta we have

$$q_1'' = p_1'' + p_2'' \quad q_2'' = -(p_3'' + p_4'') \quad \ell_1'' = p_1'' - p_3'' \quad \ell_2'' = p_2'' - p_4''.$$  

With these definitions the scalars $q_1''$, $q_2''$, $\ell_1''$ and $\ell_2''$ remain unchanged under our crossing transformation. The scalar products $p_1 p_2$ and $p_3 p_4$ transform as

$$p_1 p_2 \rightarrow -p_1 p_4 \quad \text{and} \quad p_3 p_4 \rightarrow -p_2 p_3.$$  

Finally then we have

$$\sum |M_{e^{+e^-} \rightarrow \pi^0 e^+ e^-}|^2 \equiv \frac{1}{4} (4\pi\alpha)^2 \left[ \frac{|A(m^2_{e^+}, q_1'', q_2'', \ell_1'', \ell_2'')|^2}{q_1'' q_2''} D_{q''} + \frac{|A(m^2_{e^-}, \ell_1'', \ell_2'')|^2}{\ell_1'' \ell_2''} D_{\ell''} \right. \left. - \frac{(A(m^2_{e^+}, q_1'', q_2'', \ell_1'', \ell_2'')A^*(m^2_{e^-}, \ell_1'', \ell_2'') + A^*(m^2_{e^+}, q_1'', q_2'', \ell_1'', \ell_2'')A(m^2_{e^-}, \ell_1'', \ell_2''))}{q_1'' q_2'' \ell_1'' \ell_2''} D_{q\ell''} \right]$$

where now $q_1'' = (p_1 + p_2)^2$, $q_2'' = (p_3 + p_4)^2$, $\ell_1'' = (p_1 - p_3)^2$, $\ell_2'' = (p_2 - p_4)^2$ and

$$D_{q''} = -2(2q_1'' q_2'' + 2(p_1 p_2 p_4 p_2 - \ell_1'' \ell_2'') - q_1'' q_2'' (\ell_1'' + \ell_2'')^2 - 4q_1'' q_2'' (p_1 p_4 + p_2 p_3)^2)$$

$$D_{\ell''} = -2(2\ell_1'' \ell_2'' + 2(p_1 p_4 p_2 p_3 - q_1'' q_2'') - \ell_1'' \ell_2'' (q_1'' + q_2'')^2 - 4\ell_1'' \ell_2'' (p_1 p_4 + p_2 p_3)^2)$$

$$D_{q\ell''} = -2((q_1'' q_2'' - \ell_1'' \ell_2'')^2 - 2(p_1 p_4 + p_2 p_3)^2 (\ell_1'' \ell_2'' + q_1'' q_2'') + 8p_1 p_4 p_2 p_3 ((p_1 p_4)^2 + (p_2 p_3)^2)).$$

This expression is identical to that given in Ref. [56].

The expression for the differential cross section is almost the same as that given for the preceding process,

$$\frac{d^4\sigma}{dE_3 dE_4 d\cos \theta_3 d\cos \theta_4} = \frac{\lambda^4(s, m^2_e, s_1)\lambda^4(s, m^2_e, s_2)\pi}{32 s (2\pi)^5 \sqrt{-\Delta_4}} \sum |M_{e^{+e^-} \rightarrow \pi^0 e^+ e^-}|^2.$$

with the only difference being in the definition of the invariants appearing in the amplitude squared $\sum |M_{e^{+e^-} \rightarrow \pi^0 e^+ e^-}|^2$ in terms of the invariants $s_1$, $s_2$, $t_1$ and $t_2$.

$$q_1'' = s \quad \ell_1'' = t_1 \quad \ell_2'' = t_2$$

$$q_2'' = s - s_1 - s_2 + 2m^2_e + m^2_{\pi^0}$$

$$p_1 p_4 = \frac{1}{2}(s + t_2 - s_1 - m^2_{\pi^0})$$

$$p_2 p_3 = \frac{1}{2}(s + t_1 - s_2 - m^2_{\pi^0}).$$

In Figs. 5.17-5.20 we plot this differential cross section integrated over the energies of the two final state leptons as a function of their scattering angles. Once again,
the integration range for the energies is 20 MeV up to the kinematic maximum and the interaction energy is \( \sqrt{s} = 400 \) MeV. Just as in the case of the \( e^-e^- \rightarrow \pi^0e^-e^- \) interaction, the rate is singular in the quantities \( q_1^2, q_2^2, \ell_1^2 \) and \( \ell_2^2 \). The relation of these quantities to the two scattering angles is different for this \( e^+e^- \rightarrow \pi^0e^+e^- \) process, however. Firstly, since we now have \( q_1^2 = s \) and since \( s \geq (2m_e + m_\pi)^2 \) we see that the singularity in \( q_1^2 \) lies outside of the physical region for the process. With regard to the other three invariants we see that the differential cross section becomes large in the forward scattering region where \( \cos \theta_{13} \rightarrow +1 \), corresponding to \( \ell_1^2 \rightarrow \) small, and in the region \( \cos \theta_{24} \rightarrow +1 \), corresponding to \( \ell_2^2 \rightarrow \) small, and in the region \( \theta_{13} \approx (180^\circ - \theta_{24}) \). The relation of this last region to a singularity in the invariants is a little harder to see. When \( \theta_{13} \approx (180^\circ - \theta_{24}) \) the two leptons are permitted to emerge from the interaction travelling in the same direction. This occurs when the energies of the two leptons are both close in value to \( (s - m_\pi^2 + 4m_e^2)/(4\sqrt{s}) \). In this situation the leptons emerge with small relative momentum, and in the opposite direction to the neutral pion. The invariant mass of the lepton pair is given by the quantity \( q_1^2 \). The \( 1/q_1^2 \) dependence of \( \sum |M_{e^+e^-\rightarrow e^+e^-}|^2 \) therefore causes the 'ridge' which can be seen running diagonally across Figs. 5.17-5.20.

This process has been studied by the CELLO Collaboration [42] and it currently provides the best measurement of the \( \pi^0\gamma\gamma^* \) slope parameter which we discussed in the context of the decay \( \pi^0 \rightarrow \gamma e^+e^- \) in §5.2. The CELLO measurement was made in the range of space-like photon off-shell momentum \(-2.4 \text{ GeV}^2 < k^2 < -0.6 \text{GeV}^2 \). The off-shell form factor was then extrapolated back to \( k^2 = 0 \), where the slope parameter is defined, by assuming a simple pole structure for the vertex factor,

\[
A(m_{e^+e^-}^2, k^2, 0) \sim \left(1 - \frac{k^2}{\Lambda_{e^+e^-}^2}\right)^{-1}.
\]

Since it is thought that the chiral perturbation theory momentum expansion converges only when we consider values of \( \sqrt{|k^2|} \) which are much less than the \( \rho \) mass of 770 MeV, and given that the closest to being on-shell that the CELLO experiment actually measures is \( k^2 \approx -0.6 \text{ GeV}^2 \approx -m_\rho^2 \), the extrapolation to \( k^2 = 0 \) using such a simple pole behaviour to interpolate is rather worrisome. It is, however, this extrapolated data from CELLO upon which our calculational option (4) is based.
Figure 5.17: The differential quantity $d^2\sigma/d\cos\theta_{13} d\cos\theta_{24}$ for the process $e^+e^- \rightarrow \pi^0e^+e^-$ as a function of the electron and positron scattering angles. The center of mass energy is 400 MeV.

In order to obtain this plot from the expression in Eq. (5.3) the electron and positron energies were integrated from 20 MeV up to their kinematic maximum. The $\pi^0\gamma^*\gamma^*$ vertex factor given by the tree level result option (1) of §4.4 was used in computing this cross section. The graph on the right shows a section through the contour plot along the line $\cos\theta_{24} = 0$. 
Figure 5.18: Identical to Fig. 5.17 except that the vertex factor is defined by the VMD model of option (2).

Figure 5.19: Identical to Fig. 5.17 except that the vertex factor is defined by the one-loop calculation, option (3).
Figure 5.20: Identical to Fig. 5.17 except that the vertex factor is defined by the fitted $O(p^6)$ result of option (4).
Chapter 6

Discussion

In Part I of the thesis we have computed the $\pi^0\gamma\gamma$ and $\eta\gamma\gamma$ vertex factors using for the first time the general expression for the parity-odd $O(p^6)$ sector of the chiral Lagrangian written down by Fearing and Scherer [18]. This calculation was done with all particles off-mass-shell. It should be noted, however, that the behavior of the vertex as the neutral meson is taken off-shell is not a physical observable and appears not to be uniquely defined. According to the work of Ref. [5] on pion Compton scattering computed within chiral perturbation theory, it seems likely that such hadronic off-shell dependencies can be removed by making field transformations which leave the physical S-matrix unchanged. An interesting extension of our work would be to investigate the effect of such field transformations on the neutral meson off-shell behavior of the $\pi^0\gamma\gamma$ and $\eta\gamma\gamma$ vertices calculated here.

A number of decay and scattering processes which involve these vertices have been computed, all of which involve the meson on-shell as an external particle. An attempt has been made to constrain the four low energy constants of the $O(p^6)$ parity-odd sector which enter in the vertex factors. The tightest constraint was found to come from a measurement of $e^+e^- \rightarrow \pi^0e^+e^-$ by the CELLO Collaboration [42]. This is effectively a measurement of the $\pi^0\gamma\gamma$ vertex with one photon far off-shell in the space-like direction: $0.6 \text{ GeV}^2 < -k^2 < 2.5 \text{ GeV}^2$ where $k^\mu$ is the four-momentum of the off-shell photon. This is outside of the expected region of convergence for chiral perturbation theory's momentum expansion. If, however, one accepts the extrapolation of the CELLO analysis to $k^2 = 0$ one finds a constraint on the combination of low energy constants $A_2^* - 4A_5^*$:

$$A_2^* - 4A_5^* = \frac{1}{16\pi^2 F^*_\pi}(-19.6 \pm 1.6) \times 10^{-3}$$

where we take $F_\pi = 0.093 \text{ GeV}$.

Constraints may be placed on other linear combinations of the low energy constants
$A_5^*$, $A_3^*$, $A_4^*$ and $A_6^*$ by comparison with the measured decays $\pi^0 \rightarrow \gamma\gamma$ and $\eta \rightarrow \gamma\gamma$. To the accuracy of current experimental data the resulting combinations of low energy constants are consistent with zero. From the $\pi^0 \rightarrow \gamma\gamma$ rate we find

$$[(A_5^* - 2A_3^* - 4A_4^*)m_{\pi^0}^2 - (\frac{20}{9}A_4^* + 16A_6^*)\Delta_{u.d}] = (5 \pm 38) \times 10^{-5} \tag{6.2}$$

and, assuming the physical $\eta$ to be well represented by the octet $\eta_8$, from $\eta \rightarrow \gamma\gamma$ we have

$$[(A_5^* - 2A_3^* + 4A_4^* + 24A_6^*)m_{\eta_8}^2 - 8(A_4^* + 6A_6^*)m_{\eta_8}^2 - 4A_4^*\Delta_{u.d}] = (-47 \pm 38) \times 10^{-5}. \tag{6.3}$$

If we arbitrarily set $A_4^* \equiv A_6^* \equiv 0$ we can use this data together with the CELLO data above to fix the constants $A_5^*$ and $A_3^*$ independently. Using the CELLO constraint of Eq. (6.1) and the $\pi^0 \rightarrow \gamma\gamma$ constraint of Eq. (6.2) we have

$$A_2^* = \frac{1}{16\pi^2 F^2} (27 \pm 55) \times 10^{-3}$$
$$A_3^* = \frac{1}{16\pi^2 F^2} (12 \pm 14) \times 10^{-3}$$
$$A_4^* \equiv 0, \quad A_6^* \equiv 0$$

while the use of Eq. (6.1) and Eq. (6.3) gives a much improved result

$$A_2^* = \frac{1}{16\pi^2 F^2} (15.4 \pm 3.8) \times 10^{-3}$$
$$A_3^* = \frac{1}{16\pi^2 F^2} (8.8 \pm 1.2) \times 10^{-3}$$
$$A_4^* \equiv 0, \quad A_6^* \equiv 0.$$

Information from the soon to be commissioned $e^+ e^-$ collider DAPHNE should provide tighter constraints upon these low energy constants, particularly through study of the process $e^+ e^- \rightarrow \pi^0 \gamma$ in the region of low interaction energy $s < m^2_\rho$, where chiral perturbation theory is expected to be valid. This process was studied in §5.3 using our $\mathcal{O}(p^6)$ $\pi^0\gamma\gamma$ vertex factor with one photon off-shell and time-like. Even in principle, however, it will not be possible to fix more than three of the four low energy constants $A_5^*$, $A_3^*$, $A_4^*$ and $A_6^*$ independently, by study of processes involving the $\pi^0\gamma\gamma$ and $\eta_8\gamma\gamma$ vertex factors with off-shell photons.
Part II

Validity of Soft Photon Amplitudes
Chapter 7

Introduction and Background

The electro-magnetic interaction is an excellent probe of the internal structure of hadrons and of hadronic interactions. The small coupling of photons to charged particles simplifies theoretical analysis in two ways. First, we can confidently calculate the photon's interactions using a powerful and well established tool—perturbative quantum field theory. The contribution of each higher term in the perturbative expansion of quantum electro-dynamics is smaller than the last by a factor $\sim \alpha / \pi \approx \frac{1}{4} \%$. Thus the leading term in this expansion is an excellent approximation to the full result. Secondly, a photon can interact deep within a hadronic scattering process and carry information about that process directly to experimental detector systems without engaging in complicated rescatterings with the hadrons. For this reason the photon is a much cleaner probe of the off-mass-shell\(^1\) behaviour of, for example, low energy pion-proton scattering through the bremsstrahlung processes $\pi^\pm p \rightarrow \pi^\pm p\gamma$ than is the neutral pion in the interactions $\pi^\pm p \rightarrow \pi^\pm p\pi^0$. In the latter case the final state hadrons interact strongly with the emitted pion, clouding our understanding of the fundamental scattering process.

The soft photon approximation is a technique providing a model-independent expression for the leading terms of a radiative amplitude as a function of the amplitude for the corresponding non-radiative process. A prominent application of this technique is the analysis of nucleon-nucleon bremsstrahlung. It transpires that the spectrum of low energy photons from a bremsstrahlung interaction such as $pp \rightarrow pp\gamma$ can be well described in terms of information about the on-mass-shell $pp$ elastic scattering process, despite the fact that the $pp \rightarrow pp$ sub-process within the bremsstrahlung scattering takes

\(^1\)A particle which is off-mass-shell, or virtual, has the square of its four-momentum not equal to its rest mass squared. Such particles occur in the interior of interactions, as legs which connect one sub-process to another. In contrast, an on-mass-shell or real particle has $p^\mu p_\mu = m^2$ and can occur as the external leg of a scattering process. A very detailed discussion of off-shell processes and their analysis is given in Ref. [58].
place with one charged particle off-mass-shell. Here we would wish to remove the trivial part of the photon spectrum resulting from the on-shell $NN$ interaction by using the soft photon approximation. The remainder must presumably arise due to the off-shell behaviour of the $NN$ interaction. The various theoretically motivated $NN$ potential models such as the Paris [59] and Bonn [60], which fit on-shell nucleon-nucleon scattering data very well, could then be tested in this off-shell region.

Unfortunately the construction of a soft photon amplitude is not a unique procedure. An infinite number of such amplitudes can be written down, whose predictions for the photon spectrum can be shown to differ from one another in the higher order terms of an expansion in powers of photon momentum. Since the scattering process's off-shell behaviour manifests itself within these same higher order terms, the extraction of this behaviour is obscured.

In this thesis we analyse the ambiguity in the soft photon approximation, and impose a constraint upon the derived amplitudes. This constraint states that a valid soft photon amplitude should require evaluations of the non-radiative amplitude only at physically accessible points in phase space. No model-dependent extrapolations of the non-radiative amplitude outside of its measurable region should be required. This turns out to be a strict constraint, and eliminates most choices of soft photon amplitude from being applied to commonly studied bremsstrahlung processes.

The answer as to whether or not a particular choice of soft photon amplitude can be constructed in terms of only experimentally measurable information about the non-radiative amplitude depends upon the masses of the scattering particles and upon the kinematic region studied in the bremsstrahlung experiment under analysis. It will be found that the soft photon amplitude used most extensively in the literature, usually termed the Low amplitude, is suitable for use in all kinematic regions of the $pp$ bremsstrahlung interaction, and in the vast majority of the $\pi N$ bremsstrahlung interaction's kinematic region. We also find that a recent extension of the soft photon approximation by Liou and collaborators [61, 62] which employs different expansion points for each charged particle cannot be used in the analysis of experimental data on $pp$ bremsstrahlung without making a model-dependent extrapolation of the $pp$ elastic amplitude far outside of its physical region. We suggest that this observation invalidates the conclusions drawn in Ref. [62] about the off-shell behaviour of the $pp$ elastic scattering amplitude based on such an analysis of the bremsstrahlung reaction.

The format of Part II of the thesis is as follows. In Chapter 8 we review the derivation of soft photon amplitudes, first for interactions involving only spin-0 particles and then extending to processes with spin-$\frac{1}{2}$ particles. A formalism is given in Chapter 9.
which parameterizes the ambiguity in construction of soft photon amplitudes in the most general fashion possible, consistent with the constraints of radiative phase space. This formalism is used to reproduce several amplitudes from the literature. In Chapter 10 we introduce the constraint that valid soft photon amplitudes should be expressed solely in terms of measurable information about the non-radiative scattering process. Chapter 11 considers a particular process, proton-proton bremsstrahlung, which has been studied recently by another group. An additional restriction on the validity of the amplitudes is found in this case of identical particle scattering—it concerns correctly symmetrizing or anti-symmetrizing the radiative amplitude, for identical boson or fermion scattering, while still employing only measurable information about the non-radiative process. Having placed these restrictions on the construction of valid soft photon amplitudes, in Chapter 12 we present certain choices of radiative amplitude and prove that they satisfy our restrictions when applied to a large category of two body bremsstrahlung processes. In particular, they may be applied to identical particle scatterings such as nucleon-nucleon bremsstrahlung, and to non-identical scattering processes like \( \pi N \) bremsstrahlung. Finally, in Chapter 13 we drawn some conclusions from our work.
Chapter 8

Review of Soft Photon Results

In this chapter we will review the literature of the soft photon approximation, which was initiated by the work of Low in 1958 [64]. We begin by treating soft photon radiation from the interaction of only spin-0 particles. It is known that there is an ambiguity in the choice of variables with which one parameterizes the non-radiative matrix element as it appears in the soft photon approximation to the corresponding radiative matrix element. This ambiguity has no physical basis but rather is an artifact of the approximations made in the derivation of soft photon amplitudes. The result is rederived here that, despite this ambiguity, the first two terms in a power-of-$k^+$ expansion of the radiative matrix element are uniquely given by the soft photon procedure. This result has previously been shown for a single choice of variables parameterizing the full soft photon amplitude [65, 66]. It is extended here to the case where a different choice of variables is made for photon emission from each charged particle. It is noted that, when such multiple sets of expansion variables are used, difficulties can arise in maintaining the correct gauge invariance of the soft photon amplitude while retaining its useful form, solely in terms of the non-radiative amplitude.

The soft photon approximation is then extended in §8.2 to include radiation from particles of higher spin.

8.1 Spin-0 derivations

We consider a process involving $n$ spin-zero particles of momenta $p_i^\mu$ ($i = 1, \ldots, n$) and having charges $Q_i$ in units of the positron charge $e = \sqrt{4\pi\alpha}$. In order to treat the particles on an equal footing we take them all to be in the final state. To transform our results to a physical decay or cross section, we would move one or two particles into the initial state; $p^\mu \rightarrow -p^\mu$ and $Q \rightarrow -Q$. With this choice of momenta we find charge
conservation, four-momentum conservation and the on-shell conditions to be

\[ \sum_{i=1}^{n} Q_i = 0 \quad (8.1) \]

\[ \sum_{i=1}^{n} p_i^\mu = 0 \quad (8.2) \]

\[ p_i^2 = m_i^2 \text{ (a constant)} \quad (i = 1, \ldots, n). \quad (8.3) \]

The results we derive will generalize straightforwardly to the situation involving spin-\(\frac{1}{2}\) particles in the scattering process. By discussing only the spin-0 case the points we wish to make will not be obscured by algebra.

In the following pages we shall derive the general spin-0 soft photon amplitudes for two cases: a single non-radiative amplitude function used as the expansion point for all photon emission vertices; and many such functions—one for each charged leg in the non-radiative process. For both cases, we shall also show that differing choices of these functions give rise to changes in the soft photon amplitude only at orders-(\(k^\mu\)) and higher.

### 8.1.1 Single expansion point

We write the matrix element for a scattering process without bremsstrahlung photons as a function of the particle four-momenta \(p_i^\mu\), where \(i = 1, \ldots, n\)

\[ A \equiv A(p_1 \ldots p_n). \]

We refer to this as the non-radiative amplitude, shown in Fig. 8.1(a); upon this we shall build soft photon approximations to the corresponding radiative process. The radiative matrix element can be written as a sum of operators for photon emission from each charged particle, with each operator acting on the non-radiative amplitude evaluated off mass shell in that particle's four-momentum.

\[ M^\mu \epsilon_\nu = \sum_{m=1}^{n} J_m^\mu \epsilon_\nu A(p_1 \ldots p_m + k \ldots p_n); \quad J_m^\mu = eQ_m \frac{p_m^\nu}{p_m \cdot k} \quad \text{for spin-0 particles} \quad (8.4) \]

Here \(k^\mu\) and \(\epsilon^\mu\) are the four-momentum and polarization four-vector of the radiated photon. This corresponds to the external leg emission diagrams shown in Fig. 8.1(b); these are occasionally called 'inner bremsstrahlung diagrams' in the literature.

The spin-0 electro-magnetic operator for the emission of a real \((k^2 = 0)\) photon used here can be shown to be exact (see Ref. [64]) to all orders in perturbation theory. This result comes from the Ward–Takahashi identity [67, 68],

\[ (p' - p)^\mu \Gamma_\mu(p, p') = eQ [G^{-1}(p') - G^{-1}(p)] \quad (8.5) \]
8.1 Spin-0 derivations

(a) Non-radiative process with four-momenta labelled.

(b) Photon emission from external charged particles.

(c) Radiation from the interaction's internal charge structure.

Figure 8.1: Feynman diagrams representative of terms in the soft photon approximation.

where $\Gamma_\mu(p,p')$ is the exact, renormalized vertex factor for a particle of charge $eQ$ coupling to a photon of momentum $p^\mu - p'^\mu \equiv k^\mu$, and $G(p)$ is the exact, renormalized propagator for a particle of momentum $p^\mu$. The photon emission operator is defined to be the product of the photon vertex factor and the charged particle's propagator

$$ J^\mu \equiv \Gamma^\mu(p,p')G(p'). \quad (8.6) $$

For the case of a spin-0 charged particle we have at our disposal only two independent four-vectors, say, $p^\mu$ and $p'^\mu$. Three independent scalars can be formed by their contraction, $p^2$, $p'^2$ and $p.p' = \frac{1}{2}(k^2 - p^2 - p'^2)$. The most general form for a vector-valued function such as $J^\mu$ can therefore be written as

$$ J^\mu \equiv (p' + p)^\mu f(p^2, p'^2, k^2) + (p' - p)^\mu g(p^2, p'^2, k^2). \quad (8.7) $$

Since the operator always appears in a physical amplitude in the form $J^\mu \varepsilon_\mu$ we can choose to work in the class of gauges specified by imposition of the Lorentz condition...
$k_{\mu}e^{\mu} = 0$, such that the function $g()$ does not contribute. From Eqs. (8.6) and (8.7) we form two expressions for $k \cdot J$;

$$k \cdot J = k \cdot \Gamma(p, p')G(p')$$

$$= eQ \left[ G^{-1}(p') - G^{-1}(p) \right] G(p')$$

$$= eQ \left[ 1 - G^{-1}(p)G(p') \right]$$

and

$$k \cdot J = (2p \cdot k + k^2)f(p^2, (p + k)^2, k^2) + k^2g(p^2, (p + k)^2, k^2).$$

(8.8)

We are interested in the case where $p'$ and $k'$ are the momenta of final state particles; that is, they are on-shell, $p^2 = m^2$ and $k^2 = 0$. The propagator $G(p)$ has as its denominator $p^2 - m^2 + im\Gamma$. The term $G^{-1}(p)G(p + k)$ in Eq. (8.8) is therefore proportional to

$$\frac{p^2 - m^2 + im\Gamma}{p^2 - m^2 + 2p \cdot k + im\Gamma}$$

and, as $p^2 - m^2 \to 0$, this term becomes negligible to the extent that $p \cdot k \gg m\Gamma/2$. This inequality limits the validity of our arguments to the region of photon energy

$$k \gg \frac{m\Gamma/2}{E - |\vec{p}|};$$

here we have taken the charged particle and photon to be collinear, which provides the most severe bound on $k$. For low energy scattering processes we have $E - |\vec{p}| \approx m + O(|\vec{p}|)$ and the photon energy restriction is $k \gg \Gamma/2$. For high energy scattering processes $E - |\vec{p}| \approx m^2/(2|\vec{p}|) + O(m^4/|\vec{p}|^2)$ and the limit acquires a dependence on the charged particle momentum; $k \gg |\vec{p}|\Gamma/m$. Fortunately, this restriction is of little practical importance since any charged particle which is long-lived enough to be used as a target or be employed in a beam has a small enough decay width that the corresponding limit on photon energy is well below the experimentally detectable region. A charged pion, for example, has width $25 \times 10^{-9}$eV implying that the term $G^{-1}(p)G(p + k)$ in Eq. (8.8) is completely negligible for measurable photon momenta.

Assuming the term $G^{-1}(p)G(p + k)$ to vanish in Eq. (8.8) we can equate Eqs. (8.8) and (8.9) to find

$$eQ = 2p \cdot kf(m^2, m^2 + 2p \cdot k, 0).$$

Substituting this value for $f(m^2, m^2 + 2p \cdot k, 0)$ back into Eq. (8.7) we have our result.

$$J_{\mu}|_{k^2 = (p^2 - m^2) = 0} = 2p_{\mu}f(m^2, m^2 + 2p \cdot k, 0)$$

$$= eQ \frac{p_{\mu}}{p \cdot k}$$

(8.10)
8.1 Spin-0 derivations

For charged particles with spin the expression corresponding to Eq. (8.7) can have a considerably more complicated form. The arguments given above for the spinless case would not then follow. When we come to consider the soft photon approximation for emission from spin-\( \frac{1}{2} \) particles in §8.2 we will use the lowest order term in a perturbative expansion for the operator \( J^\mu \).

Returning now to the expression for the external leg radiation amplitude of Eq. (8.4), we expand \( A(p_1 \ldots p_m + k \ldots p_n) \) about the point \( A(p_1 \ldots p_m \ldots p_n) \) in a formal Taylor expansion, which is then truncated at next to leading order in \( k^\mu \).

\[
\approx \left( 1 + k^\nu \frac{\partial}{\partial p_m^\nu} \right) A(p_1 \ldots p_m \ldots p_n) \tag{8.11}
\]

The point about which the expression has been expanded is usually referred to as an "on-shell point", since the explicit dependence of the non-radiative amplitude \( A \) on the photon momentum \( k^\mu \) has been removed. This terminology is somewhat misleading, however, since there remains an implicit dependence on the photon four-momentum—the charged particle momenta \( p_m^\nu \) are still defined by the constraints of radiative rather than non-radiative phase space. This means that the point \( A(p_1 \ldots p_m \ldots p_n) \) need not lie within the physical region of non-radiative phase and, if it does not, it cannot be obtained from an experiment analysis of the non-radiative process. This is the main idea we wish to convey in this Part II of the thesis. It implies that certain choices of expansion point—or, in the current notation, certain choices of the non-radiative amplitude \( A() \)—give rise to soft photon amplitudes for the bremsstrahlung process which require information about the unphysical region of the corresponding non-radiative process. This fact destroys much of the utility of such soft photon approximations.

The truncated expression is a good approximation to Eq. (8.4) when

\[
k^\mu \frac{\partial}{\partial p_m^\nu} \left( \log A(p_1 \ldots p_n) \right) \ll 1. \tag{8.12}
\]

That is, the non-radiative amplitude should be slowly varying with respect to variations in the momenta of the scattering particles, and the photon momentum \( k^\mu \) should be small. From Eqs. (8.4) and (8.11) we define the approximate external leg contribution to the radiative process to be

\[
\mathcal{M}_{\text{ext} \cdot \epsilon^\nu} = \sum_{m=1}^{n} eQ_m \left( \frac{p_n \cdot \epsilon^\nu}{p_n \cdot k} + \frac{p_m \cdot \epsilon^\nu}{p_m \cdot k} k^\nu \frac{\partial}{\partial p_m^\nu} \right) A(p_1 \ldots p_n).
\]

This amplitude will be integrated over radiative phase space and so the momenta \( p_m^\nu \ldots p_n^\nu \) are now constrained by radiative kinematics—they satisfy \( \sum_{m=1}^{n} p_m^\mu + k^\mu = 0 \) instead of the non-radiative relation Eq. (8.2). Notice that this truncated expansion is not gauge
8.1 Spin-0 derivations

invariant. We therefore add a term chosen to restore gauge invariance. We interpret this term as arising from internal radiation graphs, represented by Fig. 8.1(c). This choice of the internal contribution is not unique—we add the minimal term required to restore gauge invariance, but any additional terms which are independently gauge invariant and are regular in \( k^\nu \) as \( k^\nu \to 0 \) could also be included. It is important to note that such internal contributions cannot include terms of order \((1/k^\mu)\), as do the external leg radiation diagrams; the largest terms are of order \((1)\). In perturbation theory this internal contribution would be represented by diagrams involving radiation from a charged particle which is free to be off-shell both before and after photon emission. To show that these diagrams cannot give rise to terms of order \((1/k^\mu)\) it suffices to consider the operator representing charged particle propagation and photon emission which would arise in the amplitude for such internal radiation graphs.

\[
\mathcal{H} = G(p')\Gamma^\mu(p,p')G(p')
\]

Here \( G(p) \) and \( \Gamma^\mu(p,p') \) are the charged particle propagator and photon emission vertex factor. We shall now demonstrate the absence of order \((1/k^\mu)\) terms for the case of internal radiation from a spinless particle.

The most general form allowed for the vector-valued operator \( \mathcal{H}^\mu \) is, in the spinless case,

\[
\mathcal{H}^\mu = (p' + p)^\mu h(p^2, p'^2, k^2) + (p' - p)^\mu h'(p^2, p'^2, k^2).
\]

This operator arises in the amplitude contracted with the photon polarization vector \( \epsilon^\ast \). Imposing the Lorentz condition, we see that the function \( h'() \) does not contribute. The situation in which we are interested has the photon taken on-shell \((k^2 = 0)\) with the charged particle momenta \( p^\mu \) and \( p'^\mu \) allowed to be off-shell. Our operator is then

\[
\mathcal{H}^\mu = 2p^\mu h(p^2, p'^2 + 2p\cdot k, 0). \tag{8.13}
\]

We may also use the Ward identity of Eq. (8.5) to write

\[
\mathcal{H} \cdot k = G(p)\Gamma(p,p+k)kG(p+k) = \epsilon q G(p) [G^{-1}(p+k) - G^{-1}(p)] G(p+k) = \epsilon q [G(p) - G(p+k)]. \tag{8.14}
\]

The function \( G(p + k) \) can be written in terms of \( G(p) \)

\[
G(p + k) \equiv [1 - \alpha(p^2, p\cdot k)] G(p)
\]
where \( \alpha(p^2, p \cdot k) \) is a scalar function which must have the property \( \alpha(p^2, p \cdot k) \to 0 \) as \( k^\mu \to 0 \). Substituting this into Eq. (3.14) we find that \( \mathcal{H} \cdot k = eQ \alpha(p^2, p \cdot k) G(p) \). This and the expression for \( \mathcal{H} \cdot k \) derived from Eq. (8.13) may now be equated.

\[
2p \cdot k (p^2, (p + k)^2, 0) = eQ \alpha(p^2, p \cdot k) G(p)
\]

Replacing this expression for \( h() \) in Eq. (8.13) we have

\[
\mathcal{H}^\mu = eQ \frac{p^\mu}{p \cdot k} \alpha(p^2, p \cdot k) G(p).
\]

A Maclaurin expansion of \( \alpha() \) in its second argument yields

\[
\alpha(p^2, p \cdot k) = \alpha(p^2, 0) + p \cdot k \alpha^{(1)}(p^2, 0) + \frac{1}{2} [p \cdot k]^2 \alpha^{(2)}(p^2, 0) + \ldots
\]

where \( \alpha^{(n)}(p^2, 0) \equiv \frac{\partial^n}{\partial z^n} \alpha(p^2, z) |_{z=0} \). From its definition we see that \( \alpha(p^2, 0) = 0 \) implying that the internal photon emission operator has the form

\[
\mathcal{H}^\mu = eQ p^\mu \left[ \alpha^{(1)}(p^2, \ell) + \frac{1}{2} p \cdot k \alpha^{(2)}(p^2, 0) + \ldots \right] G(p)
\]

We see that it has no order-\((1/k^\mu)\) part and is independent of \( k^\mu \) as \( k^\mu \) is taken to zero, and so we have our result.

We define the soft photon amplitude to be \( \mathcal{M}_{\text{soft}} \cdot e^\nu \equiv \mathcal{M}_{\text{ext}} \cdot e^\nu + \mathcal{M}_{\text{int}} \cdot e^\nu \). Gauge invariance implies that the radiative amplitude \( \mathcal{M}_{\text{soft}} \cdot e^\nu \) should be invariant under the transformation \( e^\nu \to e^\nu + a k^\mu \) (\( a \) arbitrary). Thus we must have

\[
\mathcal{M}_{\text{soft}}^\mu k^\mu \equiv 0
\]

\[
\implies \mathcal{M}_{\text{int}}^\mu k^\mu = - \mathcal{M}_{\text{ext}}^\mu k^\mu = - \sum_{m=1}^{n} \left( eQ \cdot m A(p_1 \ldots p_n) + eQ \cdot m k^\nu \frac{\partial}{\partial p_m} A(p_1 \ldots p_n) \right).
\]

Due to the charge conservation shown in Eq. (8.1), the first term is zero. The gauge invariant soft photon matrix element is then

\[
\mathcal{M}_{\text{soft}} \cdot e^\nu = \sum_{m=1}^{n} eQ \cdot m \left[ \frac{p_m \cdot e^\nu}{p_m \cdot k} + \left( \frac{p_m \cdot e^\nu}{p_m \cdot k} - e^\nu \right) \frac{\partial}{\partial p_m^\nu} \right] A(p_1 \ldots p_n). \quad (8.15)
\]

Terms which are separately gauge invariant could, in general, be added to this expression. The approximate radiative matrix element has been written as an operator acting on the non-radiative function \( A(p_1 \ldots p_n) \), with the \( p_m^\mu \) defined by the radiative kinematics. The prescription defined by F.E. Low (Ref. [64]) for the construction of a soft photon approximation to a radiative amplitude may then be stated as follows:
• The contribution from an external charged particle is written as an operator, representing the emission of a real photon, acting on the non-radiative amplitude evaluated with that particle's four-momentum taken to be off-mass-shell due to the radiation. Contributions from all charged legs are summed.

• The non-radiative amplitudes are expanded about a point with the explicit $k^\mu$ dependencies set to zero. The expansion is truncated at next to leading order in the photon four-momentum.

• Finally a term is added which is chosen to restore gauge invariance to the radiative amplitude. The addition of this term is justified by stating that it could have arisen due to radiation from internal charged lines in the non-radiative process.

The choice of expansion point is not unique; functions representing the non-radiative amplitude can differ from one another under the constraint of radiative kinematics even though they describe the same scattering process when non-radiative kinematics are applied. The question of how the derived soft photon amplitude differs when these differing non-radiative amplitudes are used was considered originally by Bell and Van Royen [65] and by Ferrari and Rosa-Clot [66]. We shall repeat their arguments here, though with rather more detail than was given in their papers.

Let us choose two different functions representing the non-radiative matrix element,

$$A_{(1)}(p_1 \ldots p_n) = A_1(p_1 \ldots p_n); \quad A_{(2)}(p_1 \ldots p_n) = A_2(p_1 \ldots p_n)$$

with the restriction that the values of the functions coincide when the $p_i^\mu$ are constrained by non-radiative kinematics:

$$A_{(2)}(p_1 \ldots p_n) = A_{(1)}(p_1 \ldots p_n) \quad \text{when} \quad p_1^\mu + \ldots + p_n^\mu = 0.$$

The functions may differ when radiative kinematics constrain the $p_i^\mu$:

$$A_{(2)}(p_1 \ldots p_n) \neq A_{(1)}(p_1 \ldots p_n) \quad \text{when} \quad p_1^\mu + \ldots + p_n^\mu + k^\mu = 0 \quad \text{in general.}$$

We can write

$$A_{(2)}(p_1 \ldots p_n) \equiv A_{(1)}(p_1 \ldots p_n) + k^\nu W_\nu(p_1 \ldots p_n)$$

with $W_\nu$ regular in $k^\nu$ as $k^\nu \rightarrow 0$. How do the two soft photon expressions derived from $A_{(1)}$ and $A_{(2)}$ differ?

$$\left( M_{\text{soft,2}}^\mu - M_{\text{soft,1}}^\mu \right)$$
8.1 Spin-0 derivations

\[
\begin{align*}
&= \sum_{m=1}^{n} eQ_m \left[ \frac{P_m^\mu}{p_m \cdot k} + \left( \frac{P_m^\mu k^\nu}{p_m \cdot k} - g^{\mu\nu} \right) \frac{\partial}{\partial p_m^\nu} \right] k^\sigma W_\sigma(p_1 \ldots p_n) \\
&= \sum_{m=1}^{n} eQ_m \left[ \frac{P_m^\mu}{p_m \cdot k} + \left( \frac{P_m^\mu k^\nu}{p_m \cdot k} - g^{\mu\nu} \right) \frac{\partial}{\partial p_m^\nu} \right] \left( -\sum_{j=1}^{n} P_j^\sigma \right) W_\sigma(p_1 \ldots p_n)
\end{align*}
\]

We use the result

\[
\frac{\partial}{\partial p_m^\nu} \left[ \left( -\sum_{j=1}^{n} P_j^\sigma \right) W_\sigma(p_1 \ldots p_n) \right]
\]

\[
= - \left( -\sum_{j=1}^{n} \frac{\partial P_j^\sigma}{\partial p_m^\nu} \right) W_\sigma(p_1 \ldots p_n) + k^\sigma \left( \frac{\partial W_\sigma}{\partial p_m^\nu}(p_1 \ldots p_n) \right)
\]

\[
= -g_\sigma^\nu W_\sigma(p_1 \ldots p_n) + k^\sigma \left( \frac{\partial W_\sigma}{\partial p_m^\nu}(p_1 \ldots p_n) \right)
\]

to obtain

\[
\left( M_\text{soft}_{(1)} - M_\text{soft}_{(1)} \right) = \sum_{m=1}^{n} eQ_m \left[ \frac{P_m^\mu}{p_m \cdot k} \left( -\sum_{j=1}^{n} P_j^\sigma - k^\nu g_\nu^\sigma \right) W_\sigma(p_1 \ldots p_n) + g_\mu^\nu g_\nu^\sigma W_\sigma(p_1 \ldots p_n) \right.
\]

\[
+ \left. \left( \frac{P_m^\mu k^\nu}{p_m \cdot k} - g^{\mu\nu} \right) k^\sigma \frac{\partial W_\sigma}{\partial p_m^\nu}(p_1 \ldots p_n) \right].
\]

The term \((-\sum_{j=1}^{n} P_j^\sigma) - k^\sigma\) = 0 due to the radiative four-momentum conservation relation, while \(\sum_{m=1}^{n} eQ_m g^{\mu\nu} g_\nu^\sigma W_\sigma(p_1 \ldots p_n) = (\sum_{m=1}^{n} eQ_m) W_\mu(p_1 \ldots p_n) = 0\) due to charge conservation.

Finally, then, we have

\[
\left( M_\text{soft}_{(1)} - M_\text{soft}_{(1)} \right) = \sum_{m=1}^{n} eQ_m \left( \frac{P_m^\mu k^\nu}{p_m \cdot k} - g^{\mu\nu} \right) k^\sigma \frac{\partial W_\sigma}{\partial p_m^\nu}(p_1 \ldots p_n)
\]

which involves terms of order-\((k^\mu)\) only. We have shown that the soft photon approximation gives a unique result for the first two terms of an order-in-\(k^\mu\) expansion of the radiative matrix element\(^1\). Since the soft photon expression Eq. (8.15) contains only

\(^1\)These leading two terms are expected to provide a good estimate of the full radiative spectrum only when there are no narrow resonances in the structure of the non-radiative scattering process. This was first demonstrated in Ref. [69] and is related to the restriction on the validity of the soft photon approximation given in Eq. (8.12). In the presence of such a resonant state, the region of photon energy \(k\) where the Low approximation is expected to be good is limited to \(k < \Gamma\), the resonance width. Notice that this upper bound on photon energy arising due to a resonant state in the scattering process is separate from the lower bound on photon energy discussed earlier, which was due to the finite decay width of the external charged particles involved in the process.
order-\((1/k^\nu)\), order-\((k^\mu/k^\nu)\) and order-\((1)\) operators, it might seem that \(M^\mu_{\text{soft}}\) will be fully specified, regardless of the choice of non-radiative matrix element \(A(p_1 \ldots p_n)\). This is not the case, however, since \(A(p_1 \ldots p_n)\) is evaluated using radiative kinematic constraints and so implicitly contains non-unique terms of order-\((k^\mu)\) and higher, which may change with different choices of the non-radiative function.

This apparent lack of clarity in the order-in-\(k^\nu\) expansion prompted M.K. Liou and W.T. Nutt in Refs. [70, 71, 72] to investigate the possibility of expanding the radiative kinematics about small photon energy \(k^\nu\), in addition to the expansion of the matrix element. This approach involves a further approximation in one’s calculation and seems to offer little compensation other than a simplification in the phase space calculation. One can obtain the lowest order term in this modified expansion by ignoring the photon four-momentum in the momentum conserving delta-function of the phase space integration. This approach has been used recently in Refs. [73, 74, 75] for the calculation of soft radiation from inelastic channels in intermediate energy nucleon-nucleon scattering.

### 8.1.2 Many expansion points

We now consider the case where a different choice of the non-radiative function is made for emission from each of the external charged legs. There are \(n\) functions \(A_i(p_1 \ldots p_n)\), for \(i = 1 \ldots n\), whose values coincide when we impose non-radiative kinematics:

\[
A_i(p_1 \ldots p_n) = A_j(p_1 \ldots p_n) \quad \text{when} \quad p_i^\nu + \ldots + p_n^\nu = 0; \quad 1 \leq i \neq j \leq n.
\]

Under the constraints of radiative kinematics we can write

\[
A_i(p_1 \ldots p_n) \equiv A(p_1 \ldots p_n) + k^\nu W_i^\nu(p_1 \ldots p_n) \quad \text{(8.16)}
\]

where we define \(A(p_1 \ldots p_n)\) to be some particular choice of non-radiative amplitude; it is a reference function with respect to which we expand the other amplitudes.

Following our previous procedure for the construction of a soft photon approximation we find the truncated contribution of the external emission diagrams to be

\[
M^\mu_{\text{ext}} \equiv \sum_{m=1}^{n} eQ_m \left( \frac{p_m^\mu}{p_m \cdot k} + \frac{p_m^\mu k^\nu}{p_m \cdot k \partial p_m^\nu} \right) A_m(p_1 \ldots p_n)
\]

and so we choose the minimal term to restore gauge invariance.

\[
M^\mu_{\text{int}} k^\mu = -M^\mu_{\text{ext}} k^\mu = -\sum_{m=1}^{n} eQ_m \left( 1 + k^\nu \frac{\partial}{\partial p_m^\nu} \right) (A(p_1 \ldots p_n) + k^\nu W_m^\nu(p_1 \ldots p_n))
\]
The second term in Eq. (8.17) above did not arise in the previous case, where we had only one choice of non-radiative function for all external leg emissions. We must decide how to deal with this term, which is defined only indirectly in Eq. (8.16) in terms of the non-radiative amplitude functions.

One possibility is to impose the condition

$$\sum_{m=1}^{n} eQ_m W^m(p_1 \ldots p_n) = 0 \quad (8.18)$$

which, through restricting the difference functions $W^m(p_1 \ldots p_n)$, places a restriction on our choice of the functions $A_m(p_1 \ldots p_n)$ and upon the charges of the scattering particles. Recall that the functions $W^m()$ are defined entirely by the choice of amplitude functions $A_m()$ or equivalently, as we shall see in Chp. 9, by the choice of radiative variables about which Taylor expansions are to be made during the construction of a soft photon amplitude. Another possibility in dealing with this term is to find a suitable explicit form for the functions $W^m(p_1 \ldots p_n)$ in terms of the non-radiative amplitude $A_m(p_1 \ldots p_n)$. Then, since the extra term is independent of $k^\nu$ as $k^\nu \to 0$, we can treat it as a valid part of $\mathcal{M}_{\text{int}}^\mu$ and add it, to form

$$\mathcal{M}_{\text{soft}}^\mu = \sum_{m=1}^{n} eQ_m \left\{ \left[ \frac{p^\mu_m}{p_m \cdot k} + \frac{p^\nu_m k^\nu}{p_m \cdot k} - g^{\mu\nu} \right] \frac{\partial}{\partial p_m^{\nu}} A_m(p_1 \ldots p_n) - W^m(p_1 \ldots p_n) \right\}.$$

This is the approach taken recently by Liou, Gibson and collaborators (Refs. [61, 62]) in the construction of their soft photon amplitudes. In effect they perform manipulations of the sum on $W^m$, in order to re-express the sum in terms of the amplitudes $A_m$; the notation used is rather different from that used here, however. As we shall see in Chapter 11 the resulting radiative amplitudes are beset by other problems in practise.

The result that different choices of expansion point give rise to only order-$k^\nu$ changes in the soft photon matrix element can be extended to the case where many expansion points are used to define the soft photon amplitude. The argument proceeds in an almost identical fashion to the one give in the previous section, where the use of a single choice of expansion point was treated. We now have two sets of non-radiative...
functions,

\[ A_{m_{(1)}}(p_1 \ldots p_n) \quad \text{and} \quad A_{m_{(2)}}(p_1 \ldots p_n) \quad m = 1, \ldots, n \]

and, as before, we consider the difference

\[
\left( M_{\text{soft}_{(1)}}^\mu - M_{\text{soft}_{(1)}}^\mu \right) = \sum_{m=1}^{n} e Q_m \left\{ \left( \frac{p_m^\mu}{p_m \cdot k} + \left( \frac{p_m^\mu k}{p_m \cdot k} - g_{\mu \nu} \right) \frac{\partial}{\partial p_m^\nu} \right) k^\nu \left( W_{m_{(1)}}^\sigma - W_{m_{(1)}}^\sigma \right) - \left( W_{m_{(1)}}^\mu - W_{m_{(1)}}^\mu \right) \right\}
\]

\[
= \sum_{m=1}^{n} e Q_m \left\{ \left[ \frac{p_m^\mu}{p_m \cdot k} + \left( \frac{p_m^\mu k}{p_m \cdot k} - g_{\mu \nu} \right) \frac{\partial}{\partial p_m^\nu} \right] \left( - \sum_{j=1}^{n} p_j^\rho \right) \left( W_{m_{(1)}}^\rho - W_{m_{(1)}}^\rho \right) \right\}
\]

The term on the first line above is zero due to momentum conservation

\[
\left( - \sum_{j=1}^{n} p_j^\rho - k^\rho \right) = 0,
\]

while the terms on the second line cancel one another. This once more leaves contributions of order \((k^\mu)\) only.

\[
\left( M_{\text{soft}_{(1)}}^\mu - M_{\text{soft}_{(1)}}^\mu \right) = \sum_{m=1}^{n} e Q_m \left( \frac{p_m^\mu k}{p_m \cdot k} - g_{\mu \nu} \right) k^\nu \frac{\partial}{\partial p_m^\nu} \left( W_{m_{(1)}}^\sigma - W_{m_{(1)}}^\sigma \right)
\]

8.2 The extension to higher spin

We shall now extend our formalism to include bremsstrahlung from a process involving \(n\) particles, \(2r\) of which are spin-\(\frac{1}{2}\), with the remaining \(n - 2r\) being spin-0. We again take all particles to be in the final state when writing the general formalism, and move either one or two particles into the initial state when describing a physical decay or scattering process.
8.2 The extension to higher spin

Figure 8.2: Labelling of four-momenta for processes involving both spin-$\frac{1}{2}$ and spin-0 particles. Lines labelled $1, 2, \ldots, r$ are incoming spin-$\frac{1}{2}$ particles or outgoing spin-$\frac{1}{2}$ anti-particles, while $r+1, r+2, \ldots, 2r$ represent incoming spin-$\frac{1}{2}$ anti-particles or outgoing spin-$\frac{1}{2}$ particles. The remainder, labelled $(2r+1), (2r+2), \ldots, n$, are spin-0 particles. All lines are labelled as though in the final state. We have moved particles 1 and 2 into the initial state as an example of how a scattering process would be represented in our formalism.

8.2.1 Non-radiative amplitude

The amplitude for the non-radiative process may be written in the form

$$A(p_1 \ldots p_n) = \langle \alpha_{-1}^{\alpha_1} \ldots \alpha_{-1}^{\alpha_r} | \Gamma_{\alpha_1 \alpha_2 \ldots \alpha_r} (p_1 \ldots p_n) | \nu_{\alpha_r}(1) \nu_{\alpha_{r+1}}(2) \ldots \nu_{\alpha_{n}}(r) \rangle .$$

We have labelled the spin-$\frac{1}{2}$ and spin-0 particles as is shown in Fig. 8.2. The indices $\alpha_i$ on Dirac space will be written explicitly throughout much of this derivation and have a summation implicit over them. The function $\Gamma_{\alpha_1 \alpha_2 \ldots \alpha_r} (p_1 \ldots p_n)$ is, at its most general, a sum of terms each of which consists of a product of $r$ Dirac space operators; these operators for the different spinor chains may be cross-linked by contraction over Lorentz indices.

Illustrative examples are (i) $\pi^+ p$ elastic scattering and (ii) $np$ elastic scattering.

(i) Here we have $n = 4$ and $r = 1$. We take particles 1 and 3 from the final to the initial state ($p_1 \rightarrow -p_1, p_3 \rightarrow -p_3$, and $\nu_{\alpha_1}(1) \rightarrow u_{\alpha_1}(1)$), such that the scattering particles are labelled $\pi^+_1 p(1) \rightarrow \pi^+_2 p(2)$. The amplitude is

$$A(p_1 \ldots p_4) = \langle u_{\alpha_2}(2) \Gamma_{\alpha_2 \alpha_3} (p_1 \ldots p_4) u_{\alpha_1}(1) \rangle .$$

where the most general form for $\Gamma_{\alpha_2 \alpha_3}$ consistent with parity invariance of the strong interaction and on-shell conditions for the external particles is (Ref. [76,
8.2 The extension to higher spin

\[ \Gamma_{\alpha_1,\alpha_2} = \Gamma_1(p_1 \cdots p_n)[1]_{\alpha_1,\alpha_2} + \gamma^\mu [\gamma_\mu]_{\alpha_1,\alpha_2,\alpha_3} \Gamma_2(p_1 \cdots p_n). \]

The invariant scalar functions \( \Gamma_1 \) and \( \Gamma_2 \) must be extracted from experiment or computed within a particular model of the \( \pi N \) interaction.

(ii) For np elastic scattering, labelled \( n(1)p(2) \rightarrow n(3)p(4) \), we have \( n = 4 \) and \( r = 2 \). The amplitude is then

\[ A(p_1 \cdots p_4) = \bar{u}_\alpha(3)\bar{u}_\lambda(4)\Gamma_{\alpha_1,\alpha_2,\alpha_3}(p_1 \cdots p_4)u_\alpha(1)u_\lambda(2). \]

The most general form for \( \Gamma_{\alpha_1,\alpha_2,\alpha_3} \) consistent with the physical constraints of the strong interaction is (see, for example, Refs. [77, 78])

\[ \Gamma_{\alpha_1,\alpha_2,\alpha_3}(p_1 \cdots p_4) \equiv \sum_{\lambda=1}^5 F_\lambda(p_1 \cdots p_n) [t^\lambda]_{\alpha_1,\alpha_3} [t^\lambda]_{\alpha_2,\alpha_4}, \]

where \( t_\lambda \equiv \{1, \gamma^\nu, \gamma^\nu, \gamma^\nu, \gamma^\nu, \gamma^\nu\} \) for \( \lambda = \{1, \ldots, 5\} \). Contraction between the Lorentz indices of the \( [t^\lambda]_{\alpha_1,\alpha_3} \) and the \( [t^\lambda]_{\alpha_2,\alpha_4} \) is implied. The amplitude and the invariant functions \( F_\lambda \), which must again be measured or calculated within a model, are usually written in terms of the Mandelstam invariants \( s \) and \( t \) rather than the four-momenta \( p_1 \cdots p_4 \). This gives for the non-radiative np amplitude

\[ A(s, t) = \sum_{\lambda=1}^5 F_\lambda(s, t) [\bar{u}(3)t^\lambda u(1)] [\bar{u}(4)t_\lambda u(2)]. \]

8.2.2 Soft photon amplitude

The soft photon amplitude is constructed just as in §8.1 where bremsstrahlung from only spin-0 particles was treated. We write the radiation from external charged particles as a sum of real photon emission operators acting on the off-shell non-radiative amplitude. For spin-\( \frac{1}{2} \) particles we use the lowest order contribution as given by perturbation theory.

\[ \mathcal{M}^\sigma e_\mu^* = \left[ \bar{u}_{\alpha_{r+1}}(r+1) \bar{u}_{\alpha_{r+2}}(r+2) \cdots \bar{u}_{\alpha_r}(2r) \right] \]

\[ \times \left\{ \sum_{\ell=r+1}^{2r} [J^{\mu}(\ell)]_{\alpha_\ell} e_\mu^* \Gamma_{\alpha_1 \cdots \alpha_{\ell-1} \delta_{\alpha_{\ell+1} \cdots \alpha_r}}(p_1 \cdots p_\ell + k \cdots p_n) \right. \]

\[ + \sum_{\ell=1}^r \Gamma_{\alpha_1 \cdots \alpha_{\ell-1} \delta_{\alpha_{\ell+1} \cdots \alpha_r}}(p_1 \cdots p_\ell + k \cdots p_n) [J^{\mu}(\ell)]_{\delta_{\alpha_\ell} e_\mu^*} \]

\[ + \left[ \sum_{\ell=2r+1}^{n} J^{\mu}(\ell) e_\mu^* \right] \Gamma_{\alpha_1 \cdots \alpha_r}(p_1 \cdots p_\ell + k \cdots p_n) \]

\[ \times [v_{\alpha_1}(1)v_{\alpha_2}(2) \cdots v_{\alpha_r}(r)]. \]
The operators are defined

\[ J'^{\mu}(\ell) \equiv eQ_{\ell} \frac{\not p_{\ell} + \not k - m_{\ell}}{2p_{\ell} \cdot k} \left( \gamma^{\mu} - \frac{i\kappa_{\ell}}{2m_{\ell}} \sigma^{\mu\nu} k_{\nu} \right) \text{ for } \ell = 1, 2, \ldots r \]

\[ J'^{\mu}(\ell) \equiv eQ_{\ell} \left( \gamma^{\mu} - \frac{i\kappa_{\ell}}{2m_{\ell}} \sigma^{\mu\nu} k_{\nu} \right) \frac{\not p_{\ell} + \not k + m_{\ell}}{2p_{\ell} \cdot k} \text{ for } \ell = r + 1, r + 2, \ldots 2r \]

\[ J'^{\mu}(\ell) \equiv eQ_{\ell} \frac{p_{\mu}^{\ell}}{p_{\ell} \cdot k} \text{ for } \ell = 2r + 1, \ldots n \]

where \( Q_{\ell}, \kappa_{\ell} \) and \( m_{\ell} \) are the charge (in units of positron charge), anomalous magnetic moment and mass of the \( \ell^{th} \) particle. We can manipulate the spin-\( \frac{1}{2} \) photon emission terms into a more compact form. Consider the term

\[ \bar{u}(\ell) J'^{\mu}(\ell) = eQ_{\ell} \bar{u}(\ell) \left( \gamma^{\mu} \not p_{\ell} + \gamma^{\mu} \not k + m_{\ell} \gamma^{\mu} - \frac{i\kappa_{\ell}}{2m_{\ell}} \sigma^{\mu\nu} k_{\nu} (\not p_{\ell} + \not k + m_{\ell}) \right) / 2p_{\ell} \cdot k \]

\[ = eQ_{\ell} \bar{u}(\ell) \left( 2p_{\mu}^{\ell} + (- \not p_{\ell} + m_{\ell}) \gamma^{\mu} + \gamma^{\mu} \not k - \frac{i\kappa_{\ell}}{2m_{\ell}} \sigma^{\mu\nu} k_{\nu} (\not p_{\ell} + \not k + m_{\ell}) \right) / 2p_{\ell} \cdot k \]

\[ \equiv eQ_{\ell} \bar{u}(\ell) \left( \frac{p_{\mu}^{\ell} + \mathcal{R}^{\mu}(p_{\ell})}{p_{\ell} \cdot k} \right) \]

since \( \bar{u}(\ell) (- \not p_{\ell} + m_{\ell}) = 0 \), and where we have used the notation

\[ \mathcal{R}^{\mu}(p_{\ell}) \equiv \frac{1}{2} \left( \gamma^{\mu} \not k - \frac{i\kappa_{\ell}}{2m_{\ell}} \sigma^{\mu\nu} k_{\nu} (\not p_{\ell} + \not k + m_{\ell}) \right). \]

Furthermore we may rewrite \( \mathcal{R}(p_{\ell}) \epsilon^{\ast} \) by employing these relations

1. \( k \cdot \epsilon = k^{2} = 0 \)
2. \( \sigma^{\mu\nu} \epsilon_{\mu} k_{\nu} \not k = i(\gamma k^{2} - k \cdot \epsilon \not k) = 0 \)
3. \( \not k = \frac{i}{2} \left[ \gamma, \not k \right] \)
4. \( \bar{u}(\ell)(\not p_{\ell} - m_{\ell}) \frac{\gamma_{\mu} \sigma^{\mu\nu}}{2m_{\ell}} = 0 \), so we are free to introduce a term of this form.

Thus we can write

\[ \mathcal{R}(p_{\ell}) \epsilon^{\ast} = \frac{1}{2} \left( \gamma^{\mu} \not k - \frac{i\kappa_{\ell}}{2m_{\ell}} \sigma^{\mu\nu} \epsilon^{\ast} k^{\nu} \right) \]

\[ = \frac{1}{4} \left[ \gamma, \not k \right] + \frac{i\kappa_{\ell}}{8m_{\ell}} \langle \gamma, \not k \rangle (\not p_{\ell} + m_{\ell}) - \frac{i\kappa_{\ell}}{8m_{\ell}} (\not p_{\ell} - m_{\ell}) \left[ \gamma, \not k \right] \]

\[ = \frac{1}{4} \left[ \gamma, \not k \right] + \frac{i\kappa_{\ell}}{8m_{\ell}} \left[ \left[ \gamma, \not k \right], \not p_{\ell} \right]. \]  \((8.19)\)

A Taylor expansion is made about one or more points

\[ M^{\mu\nu}\epsilon^{\ast} = [\bar{u}_{\alpha}(r + 1) \bar{u}_{\alpha}(r + 2) \ldots \bar{u}_{\alpha}(2r)] \]
8.2 The extension to higher spin

\[ \times \left\{ \sum_{t=r+1}^{2r} eQ_t \frac{p_t^a e^* + R(p_t) e^*}{p_t k} \right\}_{a_1 \alpha} \]

\[ \times \left( 1 + k^\nu \frac{\partial}{\partial p_t^\nu} + \cdots \right) \Gamma_{a_1 \ldots a_{t-1} a_{t+1} \ldots a_r} (p_1 \ldots p_t \ldots p_n) \]

\[ + \sum_{t=1}^{r} \left( 1 + k^\nu \frac{\partial}{\partial p_t^\nu} + \cdots \right) \Gamma_{a_1 \ldots a_{t-1} \alpha a_{t+1} \ldots a_r} (p_1 \ldots p_t \ldots p_n) \]

\[ \times \left[ eQ_t \frac{p_t^a e^* + R(p_t) e^*}{p_t k} \right] \delta_{\alpha t} \]

\[ + \left\{ \sum_{t=r+1}^{n} eQ_t \left( \frac{p_t^a e^*}{p_t k} \right) \right\}_{x} \Gamma_{a_1 \ldots a_r} (p_1 \ldots p_n) \}

and the expansion is truncated at next to leading order in the photon momentum.

\[ \mathcal{M}_{\text{ext}}^\mu \epsilon^\nu = \left[ \tilde{u}_{a_{r+1}} (r+1) \tilde{u}_{a_{r+2}} (r+2) \ldots \tilde{u}_{a_{2r}} (2r) \right] \]

\[ \times \left\{ \sum_{t=r+1}^{2r} eQ_t \frac{p_t^a e^* + R(p_t) e^*}{p_t k} \right\}_{a_1 \alpha} \Gamma_{a_1 \ldots a_{t-1} a_{t+1} \ldots a_r} (p_1 \ldots p_n) \]

\[ + \sum_{t=1}^{r} \Gamma_{a_1 \ldots a_{t-1} \alpha a_{t+1} \ldots a_r} (p_1 \ldots p_n) eQ_t \left[ \frac{p_t^a e^*}{p_t k} + \frac{p_t^a e^*}{p_t k} k^\mu \frac{\partial}{\partial p_t^\mu} + \frac{\tilde{\Sigma}(p_t) e^*}{p_t k} \right] \delta_{\alpha t} \]

\[ + \left\{ \sum_{t=r+1}^{n} eQ_t \left( \frac{p_t^a e^*}{p_t k} \right) \right\}_{x} \Gamma_{a_1 \ldots a_r} (p_1 \ldots p_n) \}

\times [v_{a_1}(1)v_{a_2}(2) \ldots v_{a_r}(r)] .

Here we have chosen to use a single Taylor expansion point, for simplicity.

Gauge invariance of the amplitude is re-established by the addition of a term which is presumed to arise due to bremsstrahlung from charged lines which are internal to the scattering process.

\[ \mathcal{M}_{\text{int}}^\mu \epsilon^\nu = \left[ \tilde{u}_{a_{r+1}} (r+1) \tilde{u}_{a_{r+2}} (r+2) \ldots \tilde{u}_{a_{2r}} (2r) \right] \]

\[ \times \left\{ \sum_{t=1}^{n} eQ_t \left( -eQ_t e^* \right) \right\}_{a_1 \alpha} \Gamma_{a_1 \ldots a_r} (p_1 \ldots p_n) \]

\times [v_{a_1}(1)v_{a_2}(2) \ldots v_{a_r}(r)] .

The resulting amplitude is

\[ \mathcal{M}_{\text{soft}}^\mu \epsilon^\nu = \left[ \tilde{u}_{a_{r+1}} (r+1) \tilde{u}_{a_{r+2}} (r+2) \ldots \tilde{u}_{a_{2r}} (2r) \right] \]
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\[ \times \left\{ \sum_{\ell=1}^{2r} eQ_\ell \left[ \frac{p_\ell \cdot e^\bullet}{p_\ell \cdot k} + D^\mu(p_\ell) \frac{\partial}{\partial p_\ell^\mu} + \frac{\mathcal{R}(p_\ell) \cdot e^\bullet}{p_\ell \cdot k} \right] \Gamma_{a_1 \ldots a_2a_{2r+1} \ldots a_{2s}}(p_1 \ldots p_n) \right. \\
+ \sum_{\ell=1}^{2r} \Gamma_{a_1 \ldots a_{2r+1}a_{2r+2} \ldots a_{2s}}(p_1 \ldots p_n)eQ_\ell \left[ \frac{p_\ell \cdot e^\bullet}{p_\ell \cdot k} + D^\mu(p_\ell) \frac{\partial}{\partial p_\ell^\mu} + \frac{\mathcal{R}(p_\ell) \cdot e^\bullet}{p_\ell \cdot k} \right] \delta_{a_1} \\\n+ \left[ \sum_{\ell=2r+1}^{2s} eQ_\ell \left( \frac{p_\ell \cdot e^\bullet}{p_\ell \cdot k} + D^\mu(p_\ell) \frac{\partial}{\partial p_\ell^\mu} \right) \Gamma_{a_1 \ldots a_{2r}}(p_1 \ldots p_n) \right] \right\} \\
\times \{ \nu_{a_1}(1)\nu_{a_2}(2) \ldots \nu_{a_{2s}}(r) \} . \]

We have used the compact notation,

\[ D^\mu(p_\ell) = \frac{p_\ell \cdot e^\bullet}{p_\ell \cdot k} - e^\mu \]

originally introduced in Ref. [79].

Just as in the non-radiative case we may apply this general formalism to some commonly studied examples: (i) \[ \pi^\pm_2(p_3) \rightarrow \pi^0(p_2)\gamma(p_4) \] and (ii) \[ \eta_1(p_3) \rightarrow \eta_2(p_2)\gamma(p_4) \], taking the relevant particles from the final to the initial state in each case. The subscripts here show the labelling of the external particles.

(i)

\[ \mathcal{M}^{\mu}_{\text{soft}} e^\bullet = \bar{u}(2) \left\{ \sum_{\ell=1}^{4} eQ_\ell \left( \eta_\ell \frac{p_\ell \cdot e^\bullet}{p_\ell \cdot k} + D^\mu(p_\ell) \frac{\partial}{\partial p_\ell^\mu} \right) \right. \]
\[ \times (\Gamma_1(p_1 \ldots p_n) + (p_3 + p_4) \Gamma_2(p_1 \ldots p_n)) u(1) \]
\[ + \bar{u}(2) \left\{ \eta_2 eQ_2 \frac{\mathcal{R}(p_2) \cdot e^\bullet}{p_2 \cdot k} (\Gamma_1(p_1 \ldots p_n) + (p_3 + p_4) \Gamma_2(p_1 \ldots p_n)) \right\} u(1) \]
\[ + \bar{u}(2) \left\{ (\Gamma_1(p_1 \ldots p_n) + (p_3 + p_4) \Gamma_2(p_1 \ldots p_n)) \eta_2 eQ_2 \right. \]
\[ \left. \frac{\mathcal{R}(p_2) \cdot e^\bullet}{p_2 \cdot k} \right\} u(1) \]

where \( \eta_\ell = +1 \) for final state particles (\( \ell = 2, 4 \) in this example), and \( \eta_\ell = -1 \) for initial state particles (\( \ell = 1, 3 \) here).

(ii)

\[ \mathcal{M}^{\mu}_{\text{soft}} e^\bullet = \sum_{\lambda=1}^{5} \left[ \bar{u}(3) t^\lambda u(1) \right] \left[ \bar{u}(4) t^{\lambda'} u(2) \right] \]
\[ \times \left\{ \sum_{\ell=1}^{4} eQ_\ell \left( \eta_\ell \frac{p_\ell \cdot e^\bullet}{p_\ell \cdot k} + D^\mu(p_\ell) \frac{\partial}{\partial p_\ell^\mu} \right) \right. \]
\[ \times F_\lambda(s', t') \]
\[ + \sum_{\lambda=1}^{5} F_\lambda(s', t') \left\{ \eta_2 eQ_2 \frac{\mathcal{R}(p_2) \cdot e^\bullet}{p_2 \cdot k} \right. \left. \left[ \bar{u}(3) t^\lambda u(1) \right] \left[ \bar{u}(4) t^{\lambda'} u(2) \right] \right. \]
\[ + \frac{\eta_1 eQ_1}{p_1 \cdot k} \right. \left. \left[ \bar{u}(3) t^\lambda \mathcal{R}(p_1) \cdot e^\bullet u(1) \right] \left[ \bar{u}(4) t^{\lambda'} u(2) \right] \right. \]
8.2 The extension to higher spin

\[\begin{align*}
+ \frac{\eta_4 e Q_4}{p_4 \cdot \mathbf{k}} \left[ \bar{u}(3) t^4 u(1) \right] \left[ \bar{u}(4) R(p_4) e^\ast t^3 u(2) \right] \\
+ \frac{\eta_2 e Q_2}{p_2 \cdot \mathbf{k}} \left[ \bar{u}(3) t^4 u(1) \right] \left[ \bar{u}(4) e^\ast t^3 R(p_2) u(2) \right].
\end{align*}\]

Here \(s'\) and \(t'\) are Lorentz variables which are defined in terms of the momenta \(p_1 \ldots p_n\) describing a point in radiative phase space. These variables are constrained to reduce to the Mandelstam \(s\) and \(t\) of non-radiative phase space in the case that photon energy is taken to zero. Expressions similar to this will be used in Chapter 11 in our reproduction of the proton-proton bremsstrahlung calculation of Ref. [62].

We now briefly mention the Burnett-Kroll theorem of Ref. [80]. This result takes the soft photon amplitude, which is written in terms of the corresponding non-radiative amplitude, and uses it to express the differential cross section for the radiative process as a simple operator acting on the non-radiative differential cross section. In Ref. [80] the relation was made between the cross sections summed and averaged over final and initial state spin states. In Ref. [79] H. W. Fearing extended the result to relate polarized radiative and non-radiative differential cross sections. Such relations can simplify the construction of soft photon cross sections in comparison to directly squaring the soft photon amplitude. For example, in Ref. [81] Fearing computed \(NN\) bremsstrahlung using this approach and noted that whereas a simple squaring of the matrix element gave rise to many thousands of terms each containing traces of up to 12 gamma matrices, use of the Burnett-Kroll theorem gave a very simple form for the leading terms in the radiative differential cross section in terms of the non-radiative cross section. Since in our work we choose to numerically evaluate any soft photon amplitudes using a computer, we will not require the Burnett-Kroll theorem.
Chapter 9

The Soft Photon Ambiguity

9.1 A general formalism

In the previous chapter we noted that we can choose non-radiative matrix elements which are identical under the imposition of non-radiative kinematics but which differ under radiative kinematics. This gives rise to differences at order \( (k^\mu)^2 \) in the resulting soft photon approximation to the radiative matrix element.

Until now we have written the non-radiative amplitudes as differing functions of the particle four-momenta \( p_1^\mu \ldots p_n^\mu \). While adequate for our previous arguments, this is not the form in which soft photon theorems are usually derived. Rather we choose to have a single function \( A(\{w_\mu\}) \) representing the non-radiative matrix element which now depends on a set of Lorentz scalar variables \( w_1, w_2 \ldots \). These variables are taken to be functions of the four-momenta, with different sets of variables coinciding as the photon momentum vanishes, \( k^\mu \to 0 \). In the case of two body scattering such a variable set might be the Mandelstam variables \( s \) and \( t \)—then the non-radiative amplitude defined with non-radiative kinematics (that is, with \( k^\mu = 0 \)) is \( A(s, t) \). Examples of non-radiative amplitudes evaluated using variable sets which differ for radiative kinematics are \( A(s_{12}, t_{18}) \) and \( A(s_{34}, t_{24}) \), where we have defined

\[
\begin{align*}
  s_{12} &\equiv (p_1 + p_2)^2 \\
  s_{34} &\equiv (p_3 + p_4)^2 \\
  t_{18} &\equiv (p_1 + p_3)^2 \\
  t_{24} &\equiv (p_2 + p_4)^2.
\end{align*}
\]

These variables differ for \( k^\mu \neq 0 \). Clearly though, as \( k_k \to 0 \) we see that \( s_{12} \to s_{34} \) and \( t_{18} \to t_{24} \).

We shall now develop a formalism describing this ambiguity in the most general
fashion which respects on-shell conditions for the external particles and the appropriate four-momentum conservation laws. Once more we consider the scattering of only spin-0 charged particles in order to avoid introducing unnecessary algebraic complexity; the arguments extend in a straightforward fashion to the scattering of higher spin particles. Another parameterization of soft photon amplitudes was given in Ref. [32]; that work is somewhat less general than the formalism developed here, dealing as it does only with two body radiative scattering processes and with a limited class of Taylor expansion points.

The scattering matrix for our process involving only spin-0 particles is a function of the Lorentz scalars $p_i p_j (1 \leq i \leq j \leq (n - 1))$. Lorentz scalars involving the four-vector $p^a \mu$ have been excluded using the four-momentum conservation relation of Eq. (8.2). We impose on these $\frac{1}{2} n(n - 1)$ scalars, the $n$ on-shell conditions of Eq. (8.3). The process then appears to depend on $\frac{3}{2} n(n - 3)$ scalars. There are, however, a further $\frac{1}{2} (n - 4)(n - 5)$ conditions on our remaining scalars. These are an expression of the linear dependencies of five or more otherwise unconstrained vectors in a four dimensional space. Explicitly the relations may be written as conditions on certain Gram determinants; Ref. [57, pp. 209 and 289].

$$G \left( \begin{array}{c} p_1 \ p_2 \ p_3 \ p_4 \ p_i \\ p_1 \ p_2 \ p_3 \ p_4 \ p_j \end{array} \right) = 0 \quad (5 \leq i \leq j \leq (n - 1))$$

Using these constraints we are left with $(3n - 10)$ independent Lorentz scalars.

We write

$$A = A(w_1 \ldots w_{2n-10}; p_1^2 \ldots p_n^2)$$

where we have taken the set of Lorentz scalar variables $\{w_k\}$ to be certain linear combinations of the $p_i p_j$ and of the particles' rest masses,

$$w_k \equiv \sum_{i,j=1}^{n} a_{k,ij} p_i p_j + \sum_{i=1}^{n} a_{k,i} m_i^2$$

These variables $w_k$ are functions only of the four-momenta $p_i^\mu$. We can define several variable sets $\{w_k\}$ which coincide in value when the momenta $p_i^\mu$ are defined by non-radiative kinematics, but which differ when the $p_i^\mu$ are constrained by radiative phase space. In particular, we will find that certain choices of variable have the property that their value when defined by radiative kinematics lies outside of their range for non-radiative kinematics. This property, that the radiative region of the variable set can be larger than the physically allowed non-radiative region, is at the heart of the problem with certain soft photon amplitudes which we shall consider in Chp. 10.
The diagonal elements \( a_{k,ni} \) in the definition of the \( w_k \) are coefficients of masses. They play no role in the derivation of the radiative amplitude and may be chosen at one's convenience. The coefficients \( a_{kj} \) are taken symmetric in indices \( i \) and \( j \). The remaining elements may be freely chosen, with only one restriction—the resulting set of scalar variables \( \{ w_k \} \) must be linearly independent of one another once the four-momentum, on-shell and Gram determinant conditions have been imposed—they must be sufficient to span the non-radiative interaction's phase space.

The source of the soft-photon ambiguity is that we must construct our \( 3n - 10 \) scalar variables \( w_k \) as a linear combination of the \( 3(n + 1) - 10 \) independent scalars we have at our disposal in the radiative phase space. Different choices of the \( w_k \) may be equivalent under non-radiative kinematics, but correspond to different regions of the radiative phase space.

We shall now re-derive the soft photon approximation using our modified notation. The prescription for deriving the radiative amplitude and the results obtained are identical to those of §8.1; they are merely expressed in terms of Lorentz scalar variables instead of four-momenta. The scattering matrix for our \( n \)-particle interaction with a photon being emitted from one of the charged particles can be written

\[
\mathcal{M}^\mu = \sum_{m=1}^{n} \mathcal{J}_m^\mu A(w_1^{(m)}' \cdots w_{3n-10}^{(m)}'; p_1^2 \cdots (p_m + k)^2 \cdots p_n^2)
\]

where the spin-0 photon emission operator is \( \mathcal{J}_m^\mu = \frac{\epsilon_{\mu\nu\rho\sigma} p_{m\nu} k_{\rho}}{p_{m\nu} k_{\rho}} \), and \( \{ w_k^{(m)} \} \) is the set of variables \( \{ w_k^{(m)} \} \) with the \( m \)-th particle taken off-shell due to the emitted photon; \( p_m^\nu \to p_m^\nu + k^\nu \). Notice that the variables have now been labelled with particle label \( (m) \)—this is because we are free to choose a different set of variables when computing photon emission from each of the particles in turn. This is equivalent to the choice of many expansion points considered in §8.1.2.

We do not know the value of the function \( A \) at these off-shell points. For small enough photon momentum \( k^\nu \) we can, however, make a Taylor expansion of \( \mathcal{M}^\mu \) about the points \( A(w_1^{(m)} \cdots w_{3n-10}^{(m)}; p_1^2 \cdots p_m^2 \cdots p_n^2) \).

\[
\mathcal{M}^\mu = \sum_{m=1}^{n} \mathcal{J}_m^\mu \left\{ \sum_{r=0}^{\infty} \frac{1}{r!} \left[ k^\sigma \frac{\partial}{\partial k^\sigma} \right]^{(r)} A(k^\nu) \big|_{k^\nu=0} \right\}
\]

We use the results

\[
\frac{\partial}{\partial k^\sigma} A(k^\nu) = \left( \sum_{m=1}^{3n-10} \frac{\partial w_k^{(m)\nu}}{\partial k^\sigma} \frac{\partial}{\partial w_k^{(m)}}, + \sum_{k=1}^{n} \frac{\partial p_k^2}{\partial k^\sigma} \frac{\partial}{\partial p_k^2} \right) \times A(w_1^{(m)} \cdots w_{3n-10}^{(m)}; p_1^2 \cdots (p_m + k)^2 \cdots p_n^2) \big|_{k^\nu=0}
\]
9.1 A general formalism

\[ \frac{\partial w_i^{(m)}}{\partial k^\sigma} = \sum_{j=1}^{n} a_{k,m_j}^{(m)} 2p_{j\sigma} \]

\[ \frac{\partial p_k^2}{\partial k^\sigma} = \delta_{k,m} 2p_{k\sigma} \]

to give the external emission amplitude as

\[ M_\mu = \sum_{m=1}^{n} \mathcal{J}_m \sum_{r=0}^{\infty} \frac{1}{r!} \left[ \sum_{k=1}^{3n-10} \sum_{j=1}^{n} a_{k,m_j}^{(m)} 2p_{j\sigma} \frac{\partial}{\partial w_k^{(m)}} + 2p_m \cdot k \frac{\partial}{\partial p_m^2} \right] \right. \times A(w_i^{(m)} \ldots w_{3n-10}^{(m)}; p_1^2 \ldots p_n^2). \quad (9.1) \]

The expansion is truncated as before at the first derivative terms.

\[ M_{\mu \text{ext}} = \sum_{m=1}^{n} \mathcal{J}_m \left[ 1 + \sum_{k=1}^{3n-10} \sum_{j=1}^{n} a_{k,m_j}^{(m)} 2p_{j\sigma} \frac{\partial}{\partial w_k^{(m)}} + 2p_m \cdot k \frac{\partial}{\partial p_m^2} \right] \right. \times A(w_i^{(m)} \ldots w_{3n-10}^{(m)}; p_1^2 \ldots p_n^2) \]

We add a contribution \( M_{\mu \text{int}} \) taken to be from internal radiation graphs, and chosen to make the full radiative amplitude gauge invariant.

\[ M_{\mu \text{int}} k_\mu = -M_{\mu \text{ext}} k_\mu \]

\[ = -\sum_{m=1}^{n} eQ_m \left[ 1 + \sum_{k=1}^{3n-10} \sum_{j=1}^{n} a_{k,m_j}^{(m)} 2p_{j\sigma} \frac{\partial}{\partial w_k^{(m)}} + 2p_m \cdot k \frac{\partial}{\partial p_m^2} \right] \right. \times A(w_i^{(m)} \ldots w_{3n-10}^{(m)}; p_1^2 \ldots p_n^2) \]

The first term in square brackets above can be written

\[ \sum_{m=1}^{n} eQ_m A(\{w_k^{(m)}\}) = \sum_{m=1}^{n} eQ_m \left[ A(\{w_k\}) + k_\sigma W^\sigma(\{w_k^{(m)}\}) \right] \quad (9.2) \]

where \( \{w_k\} \) is some particular but arbitrary set of variables; just as in §8.1.2 the function \( A(\{w_k^{(m)}\}) \) is used as a reference function about which the other non-radiative amplitudes are expanded. The functions \( W^\sigma() \) are defined entirely by the choice of variable sets \( \{w_k^{(m)}\} \) and by the reference set \( \{w_k\} \). The introduction of these \( W^\sigma() \) is a method of making explicit the order-\( (k^\rho) \) part of the difference between two evaluations of the non-radiative amplitude which employ different radiative variable sets. The terms involving \( A(\{w_k\}) \) vanish due to conservation of charge. The \( \frac{\partial}{\partial p_m^2} \) term in Eq. (9.2) above exactly
cancels its companion in $\mathcal{M}^\mu_{\text{ext}}$. Derivatives with respect to invariant masses never arise in soft-photon amplitudes. The charge-conserving soft-photon amplitude is now,

$$
\mathcal{M}_{\text{soft}}^\mu = \sum_{m=1}^{n} e Q_m \left\{ \left[ \frac{p_{m}^\mu}{p_{m} \cdot k} + \sum_{k=1}^{3n-10} \sum_{j=1}^{n} 2 a_{k,m,j}^{(m)} \left( \frac{p_{j} \cdot k}{p_{m} \cdot k} p_{m}^\mu - p_{j}^\mu \right) \frac{\partial}{\partial u^{(m)}_{k}} \right] \times \mathcal{A}(w_1^{(m)} \ldots w_{2n-10}^{(m)} p_1^2 \ldots p_n^2) \\
+ W^\mu(w_1^{(m)} \ldots w_{2n-10}^{(m)} p_1^2 \ldots p_n^2) \right\}.
$$

(9.3)

With different choices of the coefficients $a_{k,i,j}^{(m)}$ forming each variable set $\{w_n^{(m)}\}$, we calculate different values for the soft photon matrix element $\mathcal{M}^\mu_{\text{soft}}$. As shown in Refs. [65, 66], and repeated in §8.1 these changes only occur at order-$\mathcal{O}(k^n)$ in the radiative matrix element and so should be small in regions where the soft photon amplitude is expected to be a good approximation to the full radiative amplitude. It is still troublesome though, that we appear to have no physical motivation for choosing one collection of Taylor expansion points over another.

Given that we have such an ambiguity in the construction of soft-photon theorems, how should we proceed in constraining the systems of variables? One possibility is to impose a physical constraint upon the soft photon amplitude, such as the constraint that it must be calculable entirely in terms of experimentally measurable (i.e. model independent) information about the non-radiative process. This line of argument will be pursued in Chps. 10, 11 and 12.

Alternatively, we might choose to set to zero as many of the derivative terms as possible in the soft-photon amplitude by suitable choice of the coefficients $a_{k,i,j}^{(m)}$. This has the merit of simplifying the form of the soft-photon expansion somewhat. It may be seen as being particularly desirable in the region of narrow resonances in the non-radiative scattering process since in these regions the amplitude $\mathcal{A}()$ is rapidly varying.

In order to remove the derivative terms, we must set the coefficients $a_{k,i,j}^{(m)} = 0$ for $j \neq m$. Thus, for every variable in each variable set, we impose $n - 1$ conditions upon our previously independent coefficients. Another way to state this is that the variables $u \ e^i$ in the expression for emission from each external particle must have no explicit dependence on that particle’s four-momentum. Thus no derivatives with respect to these variables arise. It is interesting to notice that this rather draconian set of conditions still leaves some freedom in the construction of each variable $w_n^{(m)}$, even for the minimal non-trivial case of $n = 4$ (two body scattering or a three body decay). The soft photon
amplitudes of Refs. [61, 62] are of this form, and lack derivative terms. We shall study these amplitudes in some detail in Chapter 11.

At this stage there is a point of interest to consider. The condition required to remove the first derivative terms also serves to remove all derivatives with respect to the variables \( w \) in the full Taylor expansion Eq. (9.1), before truncation of the series. All that remain are multiple derivatives with respect to invariant masses—these are cancelled when we impose gauge invariance by adding the internal radiation contribution, just as occurred with the single derivatives with respect to invariant masses. For such a choice of variables it might appear that the truncation of the Taylor series was unnecessary because all the higher order derivative terms vanish anyway. Since we have not truncated the series, does this imply that we have exactly computed the full radiative amplitude? Unfortunately, it does not. The soft photon expression is still only uniquely predictive at order-(1/\( k^n \)) and order-(1) as is evidenced by the fact that the Taylor expansion with the derivative terms removed is not gauge invariant. An internal radiation contribution was still required to restore gauge invariance. Just as before a separately gauge invariant contribution could be added which would alter the radiative amplitude at orders-(\( k^n \)) and higher. No matter how cleverly we choose our sets of variables the soft photon procedure is incapable of providing any information about orders-(\( k^n \)) or higher of the full radiative amplitude. Such information can only come from a detailed model of the contributing internal radiation graphs.

9.2 Reduction to cases in the literature

In this section we shall reduce the general formalism developed above to some particular cases describing two body scattering which have appeared in the literature. In particular we shall emphasize the fact that using more than one expansion point in the construction of a soft photon theorem can remove derivative terms from the soft photon amplitude, but can also cause the condition of Eq. (8.18) to be violated—this violation can lead to difficulty in expressing the radiative amplitude explicitly in terms of the non-radiative amplitude.

- F. E. Low, 1958 Ref. [64] This original soft photon theorem involves a single set of expansion variables for photon emission from all charged particles. For 2 \( \rightarrow \) 2 scattering and with a single expansion point Eq. (9.3) becomes

\[
\mathcal{M}_{\text{soft}}^\mu = \sum_{m=1}^{n} \lambda Q_m \left[ \frac{p_m^\mu}{p_m \cdot k} + \sum_{k=1}^{2} \sum_{j=1}^{4} 2a_{k,m,j} \left[ \frac{p_j \cdot k}{p_m \cdot k} p_m^\nu - p_j^\nu \right] \frac{\partial}{\partial w_k} \right]
\]
9.2 Reduction to cases in the literature

\[ xA(w_1, w_2; p_1^2, p_2^2, p_3^2, p_4^2). \]  

(9.4)

We make the particular choice of variables

\[ w_1 = \bar{s} = \frac{1}{2}(s_{12} + s_{24}) = \frac{1}{2}(m_1^2 + m_2^2 + 2p_1p_2 + m_3^2 + m_4^2 + 2p_2p_4) \]

\[ w_2 = \bar{t} = \frac{1}{2}(t_{12} + t_{24}) = \frac{1}{2}(m_1^2 + m_2^2 + 2p_1p_2 + m_3^2 + m_4^2 + 2p_2p_4) \]

giving for our coefficients \( a_{k;ij} \),

\[ a_{k;12} = \frac{1}{2} \quad (k = 1, 2; i = 1 \ldots 4; \text{mass terms}) \]

\[ a_{1;24} = a_{2;12} = a_{2;34} = \frac{1}{2} \quad (\text{and symmetric in 2nd and 3rd indices}) \]

(\text{others are zero}).

We now redefine our four-momenta and charges such that particles 1 and 2 are taken from the final into the initial state,

\[ p_1 \rightarrow -p_1; \quad p_2 \rightarrow -p_2; \quad Q_1 \rightarrow -Q_1; \quad Q_2 \rightarrow -Q_2; \]

Upon substituting these values for the \( a_{k;ij} \) into Eq. (9.4) we have the usual Low soft photon theorem result.

\[
M_{\text{soft}}^\mu = \left\{ eQ_4 \frac{p_4^\mu}{p_4 \cdot k} - eQ_3 \frac{p_3^\mu}{p_3 \cdot k} + eQ_2 \frac{p_2^\mu}{p_2 \cdot k} - eQ_1 \frac{p_1^\mu}{p_1 \cdot k} \right. \\
+ eQ_4 \left[ \frac{p_2 \cdot k}{p_4 \cdot k} p_4^\mu - p_2^\mu \right] \frac{\partial}{\partial \bar{s}} - eQ_3 \left[ \frac{p_1 \cdot k}{p_4 \cdot k} p_4^\mu - p_2^\mu \right] \frac{\partial}{\partial \bar{t}} \\
+ eQ_2 \left[ \frac{p_1 \cdot k}{p_2 \cdot k} p_2^\mu - p_1^\mu \right] \frac{\partial}{\partial \bar{s}} - eQ_3 \left[ \frac{p_1 \cdot k}{p_2 \cdot k} p_2^\mu - p_2^\mu \right] \frac{\partial}{\partial \bar{t}} \\
+ eQ_3 \left[ \frac{p_3 \cdot k}{p_2 \cdot k} p_2^\mu - p_3^\mu \right] \frac{\partial}{\partial \bar{s}} - eQ_1 \left[ \frac{p_3 \cdot k}{p_2 \cdot k} p_2^\mu - p_3^\mu \right] \frac{\partial}{\partial \bar{t}} \\
+ eQ_3 \left[ \frac{p_3 \cdot k}{p_1 \cdot k} p_1^\mu - p_3^\mu \right] \frac{\partial}{\partial \bar{s}} - eQ_1 \left[ \frac{p_3 \cdot k}{p_1 \cdot k} p_1^\mu - p_3^\mu \right] \frac{\partial}{\partial \bar{t}} \\
\left. \times A(\bar{s}, \bar{t}, m_1^2, m_2^2, m_3^2, m_4^2) \right) \]

Points to note in this example are the single choice of expansion point, \( A(\bar{s}, \bar{t}) \),

and the resulting derivative terms with respect to both of the variables \( \bar{s} \) and \( \bar{t} \).
In addition the terms $W^\mu$, which represent differences between the non-radiative amplitude evaluated with differing sets of radiative variable, do not arise where we have only a single choice of radiative variables as the origin of the soft photon theorem's Taylor expansion.

- **H. Feshbach and D. R. Yennie, 1962** Refs. [86] Soft photon amplitudes of this kind are distinguished by their use of two Taylor expansion points. Terms describing radiation from final state charged particles are expanded about a point involving initial state energies, in our case $(s_{12}, \tilde{t})$, while terms for initial state radiation are expanded about a point involving final state energies, in our case $(s_{34}, \tilde{t})$. This definition of variables allows us to pick off the coefficients $a_{n,mj}^{(m)}$,

$$w_2^{(m)} = \tilde{t} = \frac{1}{2}(t_{12} + t_{34}) \text{ for all } m$$

$$a_{2;i} = \frac{1}{2} (i = 1 \ldots 4)$$

$$a_{2,13}^{(m)} = a_{2,34}^{(m)} = \frac{1}{2} \text{ and symmetric in 2nd/3rd index}$$

$$w_1^{(1)} = w_1^{(2)} \equiv s_{34}$$

$$\Rightarrow \left\{ a_{1,33}^{(1)} = a_{1,44}^{(1)} = a_{1,34}^{(1)} = 1 \text{ and symmetric in 2nd/3rd index} \right\}$$

Identically for the $a_{1,ij}^{(2)}$

$$w_1^{(3)} = w_1^{(4)} \equiv s_{12}$$

$$\Rightarrow \left\{ a_{1,11}^{(3)} = a_{1,22}^{(3)} = a_{1,12}^{(3)} = 1 \text{ and symmetric in 2nd/3rd index} \right\}$$

Identically for the $a_{1,ij}^{(4)}$

All other $a_{k,ij}^{(m)}$ are zero

With these values of $a_{n,mj}^{(m)}$ in Eq. (9.3) and bringing particles 1 and 2 into the initial state as we did in the previous example, we have

$$\mathcal{M}_\text{soft}^\mu = \left\{ eQ_3 \left[ \frac{p_3^\mu}{p_3 \cdot k} - \left[ \frac{p_1 \cdot k}{p_3 \cdot k} p_3^\mu - p_1^\mu \right] \frac{\partial}{\partial t} \right] \right\} A (s_{12}, \tilde{t}) +$$

$$eQ_4 \left[ \frac{p_4^\mu}{p_4 \cdot k} - \left[ \frac{p_2 \cdot k}{p_4 \cdot k} p_4^\mu - p_2^\mu \right] \frac{\partial}{\partial t} \right] \right\} A (s_{12}, \tilde{t}) +$$

$$+ \left\{ eQ_1 \left[ - \frac{p_1^\mu}{p_1 \cdot k} - \left[ \frac{p_3 \cdot k}{p_1 \cdot k} p_3^\mu - p_1^\mu \right] \frac{\partial}{\partial t} \right] \right\} A (s_{34}, \tilde{t}) +$$

$$+ eQ_2 \left[ - \frac{p_2^\mu}{p_2 \cdot k} - \left[ \frac{p_4 \cdot k}{p_2 \cdot k} p_4^\mu - p_2^\mu \right] \frac{\partial}{\partial t} \right] \right\} A (s_{34}, \tilde{t}) - B^\mu$$

where we have defined

$$B^\mu \equiv eQ_3 W_3^\mu + eQ_4 W_4^\mu - eQ_1 W_1^\mu - eQ_2 W_2^\mu.$$
From our imposition of gauge invariance we see that $B^\mu$ must satisfy

$$B^\mu k^\mu = (eQ_3 + eQ_4)A(s_{12}, \bar{t}) - (eQ_1 + eQ_3)A(s_{34}, \bar{t}).$$

In this example, the derivative terms with respect to $s$-type variables have been eliminated. This is due to the condition $a^{(m)}_{ij} = 0$ being satisfied by the variables forming the two expansion points. An unfortunate side effect of using more than one expansion point, as was noted in §8.1.2, is that gauge invariance introduces the function $B^\nu()$. It is not immediately apparent how to express this function explicitly in terms of the non-radiative amplitude $A()$. We note in passing that if the scattering particles are of opposite charge, the term $B^\nu$ will vanish.

Having eliminated the derivatives with respect to $s$-type variables, one might wonder if it is possible to eliminate all derivative terms, as we commented upon at the end of §9.1. This issue has been studied previously by Liou and collaborators [82, 61, 62], and we shall reconstruct one of their amplitudes next.

* M. K. Liou, Dahang Lin and B. F. Gibson, 1993 Refs. [61] We now consider a particular example where a different Taylor expansion point is used for the terms involving radiation from each charged particle. Care is taken to choose these expansion points such that they have no explicit dependence upon the four-momentum of the radiating charge. This implies that no derivative terms will be present in the resulting amplitude. As was noted in §9.1 this condition may be expressed in terms of our coefficients $a^{(m)}_{j;i}$; to remove the derivative terms we require,

$$a^{(m)}_{j;i} \equiv 0$$

for each $m$ (each expansion point)

for each $k$ (each variable defining that point)

for each $j \neq m$ (each 4-momentum except that of radiating particle).

(9.5)

The particular choice of variables used by the authors of Ref. [61] in the construction of their Two-$u$-Two-$t$-special ($TuTuTs$) amplitude is, for emission from particle 1:$(u_{35}, t_{24})$, from particle 2:$(u_{14}, t_{13})$, from particle 3:$(u_{41}, t_{34})$, and from particle 4:$(u_{29}, t_{13})$. This choice of expansion points satisfies the condition of Eq. (9.5) above. The resulting amplitude may be written

$$\mathcal{M}^\mu_{\text{soft}} = \frac{eQ_3 p_3^\mu}{p_3 \cdot k} A'(u_{14}, t_{24}) + \frac{eQ_4 p_4^\mu}{p_4 \cdot k} A'(u_{13}, t_{13})$$

$$- \frac{eQ_1 p_1^\mu}{p_1 \cdot k} A'(u_{35}, t_{24}) - \frac{eQ_2 p_2^\mu}{p_2 \cdot k} A'(u_{41}, t_{34}) - B^\mu$$
where $B^\mu$ must satisfy the constraint

$$B^\mu k_\mu = eQ_3A'(u_{14}, t_{24}) + eQ_4A'(u_{28}, t_{18}) - eQ_1A'(u_{28}, t_{24}) - eQ_2A'(u_{14}, t_{13})$$

(9.6)

and where $A'(\cdot)$ is the non-radiative amplitude evaluated in terms of the Mandelstam variables $u$ and $t$ instead of the usual $s$ and $t$.

As in the last example we have the problem of writing the four-vector function $B^\mu$ in terms of the non-radiative amplitude $A'(\cdot)$ when the only condition we have is upon the scalar function $B^\gamma$. The specific choice made for $B^\mu$ in Ref. [61] is

$$B^\mu = \frac{(p_1 - p_4)^\mu}{(p_1 - p_4) \cdot k} (eQ_3A'(u_{14}, t_{24}) + eQ_4A'(u_{28}, t_{13})$$

$$- eQ_1A'(u_{28}, t_{24}) - eQ_2A'(u_{14}, t_{13}))$$

which satisfies Eq. (9.6)$^8$. The $1/k^\mu$ dependence of this term appears to make it an illegal choice, since $B^\mu$ can arise only from internal radiation graphs and it is well-known (see §8.1.1) that radiation from internal lines can give a contribution to the radiative amplitude only at orders-(1) and higher. This important point will be analysed in some detail in §11.2.

In this chapter we have developed a general formalism for the construction of soft photon amplitudes. Next we shall impose the reasonable requirement that a soft photon amplitude describing a bremsstrahlung process should, for arbitrary radiative kinematics, be expressible in terms of only the measurable, on-shell region of the corresponding non-radiative amplitude. We will discover that certain recently proposed soft photon amplitudes fail to meet this requirement. This failure implies that these soft photon expressions cannot be used to extract the off-shell behaviour of the non-radiative amplitude in a model-independent way.

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$^2$The function $A'(u, t)$ is defined such that it has the same numerical value as $A(s, t)$ when evaluated at the same point in non-radiative phase space, despite being expressed in terms of different variables. Specifically, $A'(u, t) \equiv A((\text{sum of } m_i^2 - u - t), t)$ since we have the non-radiative phase space constraint $s + t + u = \sum_i m_i^2$.

$^3$The notation of Ref. [61] and their method of development of this amplitude are rather different from those given here. The final form of the amplitude is identical, however.
Chapter 10

The Phase Space Problem

The soft photon approximation is useful in that it provides a relatively simple link between the low energy part of a measured photon spectrum and the measured cross section for the corresponding non-radiative process. If one chose to use some particular model for the non-radiative process as opposed to experimental data then off-shell information about the scattering is available within the context of that model. The soft photon procedure would then not be useful; a direct calculation of the radiative process could be made from the model, without need for a low energy approximation.

It is therefore quite reasonable to state that a useful soft photon amplitude must not require evaluations of the non-radiative cross section at unphysical, unmeasurable points. Unfortunately, as we shall show in this chapter, this condition is not satisfied by certain soft photon theorems in the literature. Whether the condition is upheld or not depends both upon the choice of radiative phase space variables one uses to parameterize the non-radiative amplitude, and upon the the masses of the particles involved in the scattering.

In this chapter we shall first attempt to give a clear picture of how this problem arises. Then in Chapter 11 we consider certain soft photon amplitudes from the literature which are shown to fail in experimental situations of interest. We will spend some time reproducing this calculation of Ref. [62] which contrasted the application of several different soft photon amplitudes to the proton-proton bremsstrahlung process. One of these amplitudes, called Two-s-Two-t-special (TsTτrτs) in that paper, is shown to fail when applied to pp bremsstrahlung in certain kinematic regions due to the phase space problem described here. The other amplitude, labelled Two-u-Two-t-special (TuTτrτs), requires no such unphysical evaluations of the non-radiative matrix element. Both of these amplitudes are, however, found to also suffer from a quite separate problem; it seems impossible to write the radiative amplitude, when anti-symmetrized with respect...
to identical particles, purely in terms of the anti-symmetric non-radiative amplitude.

In Chapter 12 we prove a class of soft photon amplitudes to be free of the aforementioned phase space problems when used to compute bremsstrahlung from the majority of two body scattering processes. For the special case of identical particle scattering a further choice of amplitude is proven to require only measurable input. This amplitude employs a single choice of expansion point at \((\hat{s}, \hat{t})\) and is commonly referred to in the literature as the Low amplitude. As has been previously demonstrated by Fearing [81] this amplitude may be anti-symmetrized without difficulty, in contrast to the \(T_s T_r s\) and \(T_T r_s s\) amplitudes.

The crucial step in the construction of a soft photon amplitude is the expansion of any off-mass-shell non-radiative amplitudes about points where the kinematic variables have had their explicit dependencies on photon momentum \(k^\mu\) removed. We will show that even after such an expansion the value of the non-radiative amplitude may still be required at points outside of the region where it is measurable by experiment.

A typical off-shell non-radiative amplitude appearing in a soft photon derivation might be

\[ A(\bar{s} + k \cdot (p_3 + p_4), \bar{t} - k \cdot (p_1 - p_3); m_1, m_2, m_3 + 2k \cdot p_3, m_4) \]

where we are considering radiation from particle 3, and we have chosen, as a specific example, to use the radiative phase space variables \(\bar{s}\) and \(\bar{t}\) to parameterize the amplitude. The soft photon prescription states that we Taylor expand about the point where explicit dependencies on \(k^\mu\) have been set to zero. For our example this point would be

\[ A(\bar{s}, \bar{t}; m_1, m_2, m_3, m_4) \]

This point can be termed on-shell because it is evaluated with \(p_3^2 = m_3^2\). However, the function \(A(s, t; m_1, m_2, m_3, m_4)\) is only physically measurable within the region of the \((s, t)\) plane defined by non-radiative kinematics. We have no guarantee that the locus of points \((\bar{s}, \bar{t})\) lies within this same region. Indeed, for most choices of radiative variable pairs and for most sets of masses \(m_1, m_2, m_3, m_4\) defining phase space, we find that the soft photon amplitude can require evaluations of the non-radiative amplitude at points which are not physically measurable.

Let us now illustrate this with a simple example; a definition of four-momenta and Lorentz invariants used is given in Fig. 10.1. Suppose we have measurements of an amplitude \(A(s, t)\) at all beam energies and scattering angles; that is, at all physical \(s\) and \(t\). This amplitude represents the elastic scattering of two spin-0 particles of masses \(m_1\) and \(m_2\). For a particular choice of masses representing \(\pi^- p\) scattering this physical
Figure 10.1: This chapter's definitions for momenta and Lorentz invariants of the radiative process.

Figure 10.2: The region shown in Fig. 10.2 (shaded areas). We now consider the corresponding radiative process for some particular beam energy; this defines a certain value for $s_{12} = (p_1 + p_2)^2$. We consider a pion beam of kinetic energy 298 MeV in order to replicate the kinematics of the experiment documented in Ref. [87]. A soft photon approximation might require the calculation of the expansion points $A(s_{12}, t_{13})$, $A(s_{12}, t_{24})$, $A(s_{34}, t_{13})$, or $A(s_{34}, t_{24})$. The loci of points required in such calculations are shown in Figs. 10.2(a,b,c,d), respectively. The region of $(s_{12}, t_{13})$ is contained within the elastic process's phase space, as is the region of $A(s_{12}, t_{24})$; in Chapter 12 we shall show this to be true in general for elastic scattering processes using these choices of variables. In contrast, the regions of $(s_{34}, t_{13})$ and $(s_{34}, t_{24})$ go well outside of the measurable region of $A(s, t)$; a soft photon amplitude defined with these variable choices would be unusable in practice.

We can see intuitively how this problem arises. The quantities $s_{12}$ and $s_{34}$ are related by

$$s_{34} = s_{12} - k \cdot (p_1 + p_2 + p_3 + p_4).$$

As photon energy increases $s_{34}$ becomes progressively smaller than $s_{12}$. Even though a range $(s_{12}, t_{13})$ as defined by radiative kinematics might lie within the non-radiative region for $s = s_{12}$, if we take $s = s_{34}$ we find the allowed range of the non-radiative variable $t$ to be much smaller. The points $(s_{34}, t_{24})$ are not contained within this non-radiative physical region.

This problem is by no means isolated to the one example shown above. We can also make the same comparison between the physical elastic region of phase space and the regions mapped out by certain radiative variable pairs in the case of proton-proton scattering. The results of this comparison are shown in Fig. 10.4. In this case also a
Figure 10.2: Example showing the phase space problem for a soft photon amplitude. The shaded region in each diagram shows the physical region of phase space for the elastic process of $\pi^- p$ scattering at pion beam kinetic energy 298 MeV (i.e. $m_1 = m_8 = m_{\pi^-}$, $m_2 = m_4 = m_{\text{proton}}$). Diagrams (a), (b), (c) and (d) show the regions covered by the radiative phase space points $(s_{12}, t_{13})$, $(s_{12}, t_{23})$, $(s_{23}, t_{12})$ and $(s_{23}, t_{34})$ respectively. (Diagrams (c) and (d) are shown on the next page). The regions mapped out by $(s_{12}, t_{13})$ and $(s_{12}, t_{23})$ are lines, which are contained within the elastic region. The areas mapped out by $(s_{23}, t_{12})$ and $(s_{23}, t_{34})$ extend far outside of the physical elastic region. The range of $s_{23}$ marked $15 < k < 150$ MeV in (c) and (d) is the approximate region of radiative phase space studied in the $\pi^- p\gamma$ experiment of Ref. [87].
Figure 10.3: Diagrams (c) and (d) of Fig. 10.2.
Figure 10.4: The phase space problem in proton-proton bremsstrahlung. Diagram (a) shows the physical region of phase space for the pp elastic process (shaded), along with the line mapped out by the radiative phase space variable pair \((s_{12}, t_{13})\). The hatched region in diagram (b) shows the same for the variable pair \((s_{34}, t_{13})\). The proton beam energy for the bremsstrahlung process is \(T = 200\) MeV. Due to the charges in this process being of identical mass the region of \((s_{12}, t_{24})\) is identical to that of \((s_{12}, t_{13})\), while \((s_{34}, t_{24})\) is the same as that of \((s_{34}, t_{13})\).
Figure 10.5: Region of phase space covered in the $pp\gamma$ experiment of Ref. [1].
In figure (a) the regions of phase space mapped out as a function of photon angle during the 200 MeV $pp$ bremsstrahlung experiment of Ref. [1] are shown. The points $(s_{34}, t_{13})$ and $(s_{34}, t_{24})$ are seen to be outside of the measurable region of elastic phase space for most photon angles. Figure (b) is an expansion of a region in figure (a) and shows the photon angle measured in the target frame for the regions $(s_{34}, t_{13})$ and $(s_{34}, t_{24})$. To guide the eye points have been marked at $30^\circ$ intervals in photon lab angle along these trajectories.
soft photon amplitude employing the expansion points $A(s_{12}, t_{12})$ or $A(s_{12}, t_{24})$ would require only measurable information about the non-radiative, elastic amplitude. A radiative amplitude using the points $A(s_{54}, t_{13})$ or $A(s_{54}, t_{24})$, however, would require unphysical information and would be incalculable unless one resorted to model-dependent extrapolations of the elastic amplitude.

This difficulty might be avoided by considering only certain experimental kinematics for the bremsstrahlung process. It is unsatisfactory, however, that only limited radiative kinematics should be calculable, particularly in light of the existence of the variable choices which we prove in Chapter 12 to have no such kinematic restrictions. Additionally, when we consider the radiative regions covered by certain experiment in the past, we find that soft photon amplitudes employing the points $(s_{54}, t_{13})$ or $(s_{54}, t_{24})$ would still be incalculable. The $\pi^- p \gamma$ experiment of Ref. [87] covered the majority of radiative phase space, with photon energy in the range 15–150 MeV being measured. This region is shown in Figs. 10.2(c,d). The 200 MeV $pp$ bremsstrahlung experiment of Ref. [1] measured the photon spectrum as a function of angle with outgoing proton angles fixed at 16.4° on either side of the beam axis in the lab frame and with all particles coplanar. As can be seen in Fig. 10.5 the resulting trajectories through radiative phase space of the points $A(s_{54}, t_{13})$ and $A(s_{54}, t_{24})$ also fall outside of the physical region of phase space for the elastic $pp$ process. This would make any soft photon amplitude which was based on these radiative expansion points unusable, unless model-dependent extrapolations of the $\pi^- p$ and $pp$ elastic amplitudes were to be made.
Chapter 11

Proton-Proton Bremsstrahlung

In this chapter we shall study the recent application in Ref. [62] of certain soft photon amplitudes to proton-proton bremsstrahlung for the kinematic regions studied by the Manitoba, Harvard and TRIUMF experimental groups (Refs. [88, 89, 90]). Our purpose in studying these amplitudes is that they exhibit the behaviour we discussed in the preceding chapter. Namely, one requires evaluations of the non-radiative amplitude outside of its measurable region in order to compute the soft photon approximation to the radiative process.

The soft photon amplitudes used in Ref. [62] and derived in a previous paper [61] are called by the authors the Two-s-Two-t-special (abbreviated to $T_sT_t$) and the Two-u-Two-t-special ($T_uT_t$) amplitudes. These names refer to the fact that the amplitudes each employ four expansion points during their derivation, at points defined either by $(s, t)$-type or $(u, t)$-type radiative variable pairs.

As a precursor to our examination of these amplitudes' failings we shall briefly go over their derivation, first for the spin-0 case and then for the spin-$\frac{1}{2}$ example of $pp$ bremsstrahlung.

The derivation of these soft amplitudes was motivated in Refs. [61, 62] by arguments which claim to distinguish them as being preferred over other amplitudes for the analysis of certain processes. Briefly, the motivations are:

1. For a non-radiative process whose amplitude is rapidly varying in the region of a resonance it might seem desirable to eliminate any derivative terms from the corresponding soft photon amplitude. This is done for the $T_sT_t$ and $T_uT_t$ amplitudes by careful choice of Taylor expansion points during the development of the soft photon amplitude.

2. The decision about whether to use $(s, t)$-type or $(u, t)$-type variables for the soft expansion is made by assuming the $(s, t)$-type to be more applicable to processes
dominated by s-channel resonances; (u, t)-type are to be used when t-channel or u-channel exchanges are thought to dominate the non-radiative scattering.

We shall ignore these motivations here, and merely attempt to derive the amplitudes in as straightforward a fashion as possible. The amplitudes are special cases of the general soft photon formalism developed in Chapter 9. This derivation is done first for the case of spin-0 scattering. We then consider a constraint upon the internal emission part of the amplitudes which limits the validity of the TsTtTs and TtTsTt results to elastic scattering only. Next, the impact of the phase space problem described in Chapter 10 is assessed. We note the impossibility of correctly symmetrizing these spin-0 bremsstrahlung amplitudes for identical particle scattering processes while expressing the amplitudes purely in terms of only measurable information about the non-radiative process. Finally the work is extended to the case of spin-\( \frac{1}{2} \) identical particle scattering and the pp bremsstrahlung calculation of Ref. [62] is reproduced in order to confirm numerically the various problems found with these soft photon amplitudes.

11.1 Spin-0 derivation

We consider radiation from a scattering of spin-0 particles labelled 1, 2 in the initial state to 3, 4 in the final state. The non-radiative amplitude \( A(s, t) \) may also be written \( A'(u, t) \) where \( A(s, t) = A'(u, t) \) for \( s + t + u = \sum_{i=1}^{4} m_i^2 \). The external leg contribution to the radiative process may be written,

\[
\mathcal{M} = \frac{e Q^\mu_3 p^\mu_2}{p_3 \cdot k} A (s_{12}, t_{24}; m_1^2, m_2^2, m_3^2 + 2k \cdot p_2, m_4^2) \\
+ \frac{e Q^\mu_4 p^\mu_4}{p_4 \cdot k} A (s_{12}, t_{13}; m_1^2, m_2^2, m_3^2 + 2k \cdot p_4) \\
- \frac{e Q^\mu_1 p^\mu_1}{p_1 \cdot k} A (s_{34}, t_{24}; m_1^2 + 2k \cdot p_1, m_2^2, m_3^2, m_4^2) \\
- \frac{e Q^\mu_3 p^\mu_3}{p_3 \cdot k} A (s_{34}, t_{13}; m_1^2, m_2^2 + 2k \cdot p_2, m_3^2, m_4^2) .
\]

Notice that the radiative variables have been chosen carefully such that the only explicit \( k^\mu \) dependence is in the invariant masses. When the usual expansion and truncation is made around the points with the explicit dependence on \( k^\mu \) set to zero, we find that only derivative terms with respect to the invariant masses arise. Once the gauge-invariance restoring contribution representing internal radiation is added, the resulting amplitude is

\[
\mathcal{M}^\mu_{\text{soft}} = \frac{e Q^\mu_3 p^\mu_2}{p_3 \cdot k} A (s_{12}, t_{24}; m_1^2, m_2^2, m_3^2, m_4^2) 
\]
where, due to the imposition of gauge invariance on the soft photon amplitude, we see that \( B^\mu \) must satisfy the constraint

\[
B^\mu k_\mu = eQ_8 A(s_{12}, t_{24}; m_1^2, m_2^2, m_3^2, m_4^2) + eQ_4 A(s_{12}, t_{13}; m_1^2, m_2^2, m_3^2, m_4^2)
- eQ_1 A(s_{34}, t_{24}; m_1^2, m_2^2, m_3^2, m_4^2) - eQ_2 A(s_{34}, t_{13}; m_1^2, m_2^2, m_3^2, m_4^2) \quad (11.1)
\]

This formula for \( \mathcal{M}_{\text{soft}}^\mu \) could have been obtained directly from Eq. (9.3) in which case we can make the following identification between \( B^\mu \) and the \( W^\mu \) function of Chps. 8 and 9,

\[
B^\mu = eQ_8 W^\mu (s_{12}, t_{24}; m_1^2, m_2^2, m_3^2, m_4^2) + eQ_4 W^\mu (s_{12}, t_{13}; m_1^2, m_2^2, m_3^2, m_4^2)
- eQ_1 W^\mu (s_{34}, t_{24}; m_1^2, m_2^2, m_3^2, m_4^2) - eQ_2 W^\mu (s_{34}, t_{13}; m_1^2, m_2^2, m_3^2, m_4^2).
\]

Many functions \( B^\mu \) could be found which satisfy this constraint; we make the appropriate choice giving rise to the \( T_s T_{Ts} \) amplitude as defined in Ref. [61];

\[
B^\mu \equiv \left( \frac{p_3 + p_4}{p_3 + p_4} \right)^\mu \left( eQ_3 A(s_{12}, t_{24}) + eQ_4 A(s_{12}, t_{13}) - eQ_1 A(s_{34}, t_{24}) - eQ_2 A(s_{34}, t_{13}) \right)
\quad (11.2)
\]

where the invariant mass arguments of \( A() \) have been suppressed, for brevity. We note that four-momentum conservation and the relations \( k^2 = k \cdot e_k = 0 \) give us the relation

\[
\frac{(p_3 + p_4) \cdot e_k^\mu}{(p_3 + p_4) \cdot k} = \left( \frac{p_1 + p_2}{p_1 + p_2} \right) \cdot e_k^\mu
\]

which allows us to write the \( T_s T_{Ts} \) amplitude in the form quoted by Refs. [61, 62],

\[
\mathcal{M}_{T_s T_{Ts}}^\mu = eQ_3 \left[ \frac{p_3^\mu}{p_3 \cdot k} - \frac{(p_3 + p_4)^\mu}{(p_3 + p_4) \cdot k} \right] A(s_{12}, t_{24})
+ eQ_4 \left[ \frac{p_4^\mu}{p_4 \cdot k} - \frac{(p_3 + p_4)^\mu}{(p_3 + p_4) \cdot k} \right] A(s_{12}, t_{13})
- eQ_1 \left[ \frac{p_1^\mu}{p_1 \cdot k} - \frac{(p_1 + p_2)^\mu}{(p_1 + p_2) \cdot k} \right] A(s_{34}, t_{24})
- eQ_2 \left[ \frac{p_2^\mu}{p_2 \cdot k} - \frac{(p_1 + p_2)^\mu}{(p_1 + p_2) \cdot k} \right] A(s_{34}, t_{13}).
\]
We have no guarantee that the form given above in Eq. (11.2) for the contribution $B^\mu$ could actually arise from the internal radiation graphs of a perturbative analysis. Recall that graphs which involve a photon coupled to an internal, off-shell charged line cannot give rise to an order-$(1/k)$ dependence in the bremsstrahlung amplitude. The function $B^\mu$ must be independent of $k^\mu$ as $k^\mu \to 0$. We shall find in the following section that the imposition of this constraint on the function $B^\mu$ restricts the charges of scattered particles for which the associated $T_s T_s$ and $T_T T_T$ amplitudes are well-defined.

The $T_T T_s$ amplitude may be derived in similar fashion to that given for $T_s T_T$. This time we express the external radiation terms as depending on $A'(u, t)$ instead of $A(s, t)$—recall that $A'(u, t)$ is just a short-hand notation for $A(\sum_{i=1}^4 m_i^2 - t - u, t)$, where $A(s, t)$ is the non-radiative amplitude.

\[ \mathcal{M}_u = \frac{e Q_3 P_3^\mu}{p_3 \cdot k} A'(u_{14}, t_{24}; m_1^2, m_2^2, m_3^2 + 2 k \cdot p_3, m_4^2) \]
\[ + \frac{e Q_4 P_4^\mu}{p_4 \cdot k} A'(u_{23}, t_{13}; m_1^2, m_2^2, m_3^2 + 2 k \cdot p_4) \]
\[ - \frac{e Q_1 P_1^\mu}{p_1 \cdot k} A'(u_{28}, t_{28}; m_1^2 + 2 k \cdot p_1, m_1^2, m_4^2) \]
\[ - \frac{e Q_2 P_2^\mu}{p_2 \cdot k} A'(u_{14}, t_{18}; m_1^2, m_2^2 + 2 k \cdot p_2, m_3^2, m_4^2) \]

The amplitude is expanded around $k^\mu = 0$ and truncated, and a gauge-invariance restoring term from the internal radiation diagrams is added. The result is

\[ \mathcal{M}_{soft} = \frac{e Q_3 P_3^\mu}{p_3 \cdot k} A'(u_{14}, t_{24}; m_1^2, m_2^2, m_3^2, m_4^2) \]
\[ + \frac{e Q_4 P_4^\mu}{p_4 \cdot k} A'(u_{23}, t_{13}; m_1^2, m_2^2, m_3^2, m_4^2) \]
\[ - \frac{e Q_1 P_1^\mu}{p_1 \cdot k} A'(u_{28}, t_{28}; m_1^2, m_2^2, m_3^2, m_4^2) \]
\[ - \frac{e Q_2 P_2^\mu}{p_2 \cdot k} A'(u_{14}, t_{18}; m_1^2, m_2^2, m_3^2, m_4^2) \]
\[ - B^\mu. \]

This time we choose

\[ B^\mu \equiv \frac{(p_1 - p_4)^\mu}{(p_1 - p_4) \cdot k} (e Q_3 A'(u_{14}, t_{24}) + e Q_4 A'(u_{23}, t_{13}) - e Q_1 A'(u_{28}, t_{28}) - e Q_2 A'(u_{14}, t_{13})) \]

giving for the final $T_T T_T$ amplitude,

\[ \mathcal{M}_{T_T T_T}^u = \frac{e Q_3}{P_3 \cdot k} \left[ \frac{P_3^\mu}{(p_3 - p_3) \cdot k} - \frac{(p_2 - p_3)^\mu}{(p_2 - p_3) \cdot k} \right] A'(u_{14}, t_{24}) \]
11.2 Limitation to elastic scattering

\[ +eQ_4 \left[ \frac{p_4^\mu}{p_4 \cdot k} - \frac{(p_1 - p_4)^\mu}{(p_1 - p_4) \cdot k} \right] A' (u_{23}, t_{13}) \]
\[ -eQ_1 \left[ \frac{p_1^\mu}{p_1 \cdot k} - \frac{(p_1 - p_4)^\mu}{(p_1 - p_4) \cdot k} \right] A' (u_{23}, t_{24}) \]
\[ -eQ_2 \left[ \frac{p_2^\mu}{p_2 \cdot k} - \frac{(p_2 - p_4)^\mu}{(p_2 - p_1) \cdot k} \right] A' (u_{14}, t_{13}) . \]

With the form of the amplitudes chosen, we shall now show that the choices of \( B^\mu \) made above are consistent with having come from internal radiation graphs only if the constraints \( Q_1 = Q_3 \) and \( Q_2 = Q_4 \) are satisfied.

11.2 Limitation to elastic scattering

In §8.1.1 we presented the well-known result that a radiative amplitude describing photon emission from the internal charge structure of a scattering process must have the property that it is independent of photon four-momentum \( k^\mu \) as one takes \( k^\mu \rightarrow 0 \). It is reasonable to make the same requirement on the internal emission part of a soft photon approximation to this amplitude. This implies that the internal part of the soft photon amplitude cannot contain terms with a factor \( k^\mu \) in the denominator; such terms include the apparently \( O(k^0) \) term \( a \cdot k / b \cdot k \), where \( a^\mu \) and \( b^\mu \) are composed of the charged particle four-momenta. Although the form \( a \cdot k / b \cdot k \) is independent of photon energy \( k \) as \( k \rightarrow 0 \), it retains its dependence on photon angle and so fails our condition. We will now show that the terms \( B^\mu \) forming the internal radiation parts of the \( TsTt_s \) and \( TuTt_s \) amplitudes do indeed contain such illegal forms unless the charge constraints \( Q_1 = Q_3 \) and \( Q_2 = Q_4 \) are satisfied.

We shall constrain the term

\[ B^\mu \equiv \frac{(p_a + p_b)^\mu}{(p_a + p_b) \cdot k} \left( eQ_3 A (s_{12}, t_{24}) + eQ_4 A (s_{12}, t_{12}) - eQ_1 A (s_{34}, t_{24}) - eQ_2 A (s_{34}, t_{13}) \right) \]

arising in the derivation of the \( TsTt_s \) amplitude by expanding three of the evaluations of \( A() \) about the fourth. It does not matter which of these we choose as the expansion point—we select \( (s_{12}, t_{24}) \). We make a Taylor expansion in both arguments of \( A() \), even when such an expansion is unnecessary, in order to make term-by-term arguments about the cancellation of the denominator term \( (p_a + p_b) \cdot k \).

\[ B^\mu = \frac{(p_a + p_b)^\mu}{(p_a + p_b) \cdot k} \times \]
\[ \left[ eQ_3 \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} (s_{12} - s_{12})^m (t_{24} - t_{24})^{n-m} \frac{\partial^n}{\partial s_{12} \partial t_{24}^{n-m}} \right\} A(s, t) \right]_{s=s_{12}, t=t_{24}} \]
11.2 Limitation to elastic scattering

\[ + eQ_4 \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} (s_{12} - s_{13})^m (t_{13} - t_{24})^{n-m} \frac{\partial^n}{\partial s^m \partial t^{n-m}} \right\} A(s,t) |_{s=s_{13}, t=t_{24}} \]

\[ - eQ_1 \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} (s_{24} - s_{13})^m (t_{24} - t_{24})^{n-m} \frac{\partial^n}{\partial s^m \partial t^{n-m}} \right\} A(s,t) |_{s=s_{13}, t=t_{24}} \]

\[ - eQ_2 \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} (s_{24} - s_{13})^m (t_{13} - t_{24})^{n-m} \frac{\partial^n}{\partial s^m \partial t^{n-m}} \right\} A(s,t) |_{s=s_{13}, t=t_{24}} \]

and we use the results

\[ s_{24} - s_{13} = -2(p_3 + p_4) \cdot k \]
\[ t_{13} - t_{24} = -2(p_2 - p_4) \cdot k \]

to reduce this to the form

\[ B^\mu = e^{(p_3 + p_4)^\mu} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} \times \]
\[ \{ Q_3(0)^n + Q_4(0)^m [-2(p_2 - p_4) \cdot k]^{n-m} - Q_1[-2(p_3 + p_4) \cdot k]^{m}(0)^{n-m} \]
\[ - Q_2[-2(p_3 + p_4) \cdot k]^{m}[-2(p_3 - p_4) \cdot k]^{n-m} \}
\[ \times A(s,t) |_{s=s_{13}, t=t_{24}} \]

Our goal is to find the conditions whereby the unphysical denominator term \((p_3 + p_4) \cdot k\) is cancelled by a term in the numerator. The conditions we must impose for this to be true are:

- \((m = 0)\): The term in curly braces is

\[ \left\{ (Q_3 - Q_4)(0)^{-m} + (Q_4 - Q_3)[-2(p_2 - p_4) \cdot k]^{n-m} \right\}. \]

Since no cancelling term \([(p_3 + p_4) \cdot k]\) appears in this numerator we must remove the whole term by imposing the conditions \(Q_3 = Q_1\) and \(Q_4 = Q_2\).

- \((m > 0)\): The term in the braces is now

\[ \{-(-2)^m[(p_3 + p_4) \cdot k]^m(Q_1(0)^{-m} + Q_3[2(p_2 - p_4) \cdot k]^{n-m})\}. \]

Since a term \([(p_3 + p_4) \cdot k]^m\) arises, the denominator \((p_3 + p_4) \cdot k\) in which we are interested is cancelled automatically.

The argument for \(TTT^TTS\) proceeds in an identical fashion, with the same resulting condition. Thus we see that the \(T^TS^TTS\) and \(TTT^TTS\) amplitudes are well-defined only
for the case $Q_1 = Q_3$ and $Q_2 = Q_4$, which effectively limits their use to elastic scattering processes.

There is the possibility of a more general choice for the form of $B^\mu$. In general we may write

$$B^\mu \equiv \frac{X^\mu}{X\cdot k} (B\cdot k)$$

where $X^\mu$ is some combination of the momenta $p_1^\mu \rightarrow p_4^\mu$ and $(B\cdot k)$ is constrained by imposition of gauge invariance to satisfy a relation similar to that of Eq. 11.1. We have shown that the conventional choices of $X^\mu \equiv (p_3 + p_4)^\mu$ for the $T_s T_s T_s$ amplitude, and $X^\mu \equiv (p_1 - p_4)^\mu$ for the $T_u T_s T_s$ amplitude give rise to the required cancellation of the unphysical $1/k \cdots$ terms so long as the charge conditions $Q_1 = Q_3$ and $Q_2 = Q_4$ hold. It should be clear that the choice $X^\mu \equiv (p_2 - p_4)^\mu$ could have been made for either the $T_s T_s T_s$ or the $T_u T_s T_s$ amplitudes while still ensuring the vanishing of the $1/k \cdots$ terms; the charge condition would change in this case to $Q_1 = -Q_2$ and $Q_3 = -Q_4$. Thus the constraint on the internal part of the soft amplitude would disallow us from using such a modified $X^\mu$ for identical particle scattering processes such as $p p \rightarrow p p \gamma$, but would not disallow us from using it for processes such as $\pi^- p \rightarrow \pi^- p \gamma$.

The constraint that internal radiation contributions should only come from physical diagrams has not been mentioned by the authors of Refs.[61, 62]. Neither has it been previously discussed that choices can be made for the function $X^\mu$ other than the particular cases which give rise to the $T_s T_s T_s$ and $T_u T_s T_s$ amplitudes.

### 11.3 Impact of the phase space problem

In this section we consider the impact of the phase space problem upon the amplitudes $T_s T_s T_s$ and $T_u T_s T_s$; that is, we consider whether the points $(s_{12}, t_{24})$, $(s_{28}, t_{13})$ etc. required by $T_s T_s T_s$ and the points $(u_{14}, t_{24})$, $(u_{28}, t_{13})$ etc. required by $T_u T_s T_s$ fall within the physical non-radiative phase space region. As concrete examples we consider the kinematics of the Manitoba, Harvard and TRIUMF $pp$ bremsstrahlung experiments (Refs. [88, 89, 90]). In these experiments a proton beam is directed into a hydrogen target and the outgoing protons and photon are measured in a limited kinematic region, with all particles coplanar\(^1\). The quoted results are differential cross sections $d^4 \sigma / d\Omega_3 d\Omega_4 d\theta_4$ for fixed outgoing proton angles $\theta_3$ and $\theta_4$ and for the full range of photon angle $\theta_4$.

\(^1\)Due to finite detector sizes the particles measured are only approximately coplanar. In the Harvard experiment, Ref. [89], protons were measured up to $\pm 1.7^\circ$ out of the horizontal plane; photon momenta were deduced using energy-momentum conservation. In the TRIUMF experiment, Ref. [90], the non-coplanarity was "a few degrees" for protons and was between $\pm 2.5^\circ$ and $\pm 5.5^\circ$ for photons, depending on the geometry of the lead glass Čerenkov counters. In the Manitoba experiment, Ref. [88], the non-coplanarity of particles was large, but not well quantified in their paper.
Figure 11.1: Trajectories of radiative variable pairs through non-radiative phase space. The trajectories of \((s,t)\)-type variables with varying photon angle are shown in diagram (a) for the case with proton angles \(\theta_c = 26^\circ\), \(\theta_d = 26^\circ\) and beam kinetic energy of 42 MeV. Diagram (b) shows the same for the trajectories of \((u,t)\)-type variables, projected onto the \((s,t)\) plane. The physical region for the non-radiative process is marked.
Figure 11.2: Identical to Fig. 11.1 but with $\theta_3 = 35^\circ$, $\theta_4 = 35^\circ$ and beam kinetic energy of 157 MeV.
11.3 Impact of the phase space problem

Figure 11.3: Identical to Fig. 11.1 but with $\theta_8 = 12.4^\circ$, $\theta_4 = 12^\circ$ and beam kinetic energy of 280 MeV.
11.4 Symmetrization problem

Shown in Figs. 11.1–11.3 are the trajectories of the radiative variable pairs required for evaluation of the $T_sT-ts$ and $T_uT-ts$ amplitudes for the particular experimental configuration studied in Ref. [62]. Part (a) of each figure shows the trajectories of $(s,t)$-type variables as a function of photon angle, while part (b) shows the same for $(u,t)$-type variables. Also shown on each plot are the limits of the physical region of non-radiative phase space for this identical particle scattering interaction. Each full figure is drawn for a certain pair of fixed outgoing proton angles.

It is found that the $(u,t)$-type variables required by the $T_uT-ts$ soft photon amplitude all lie within the physical non-radiative region. The $T_uT-ts$ amplitude therefore appears to be calculable in terms of measured $pp$ elastic scattering data.

In contrast, the $(s_{4\ell}, t_{4\ell})$ and $(s_{2\ell}, t_{2\ell})$ variable pairs required by the $T_sT-ts$ amplitude lie sometimes inside and sometimes outside of the physical $pp$ elastic region. This makes it impossible to compute the $T_sT-ts$ amplitude correctly within an arbitrary region of radiative phase space, as we shall see in §11.6 where we attempt to reproduce the calculations of Ref. [62] numerically.

11.4 Symmetrization problem

In §11.5 we will show that a problem exists with correctly anti-symmetrizing the spin-$\frac{1}{2}$ $T_sT-ts$ and $T_uT-ts$ amplitudes of Ref. [62]. To illustrate the source of this problem we consider in this section the quite analogous and algebraically much less daunting task of symmetrizing the spin-0 $T_sT-ts$ and $T_uT-ts$ amplitudes.

The spin-0 amplitudes given in §11.1 would have to be explicitly symmetrized if applied to the case of identical particle scattering. Upon attempting to symmetrize the $T_sT-ts$ and $T_uT-ts$ amplitudes we find that the connection to measurable non-radiative scattering data is lost. This problem is quite independent of the phase space problem discussed previously. The Low amplitude, employing a single choice of Taylor expansion point at $(\bar{s}, \bar{t})$, is also treated for comparison. It is found that the symmetrized $(\bar{s}, \bar{t})$ soft photon amplitude may be written in terms of the measurable, symmetrized non-radiative scattering amplitude; this is due to a special property of the $(\bar{s}, \bar{t})$ variables.

We denote the unsymmetrized non-radiative scattering amplitude by $A(s,t)$. The symmetrized amplitude is then

$$A^S(s,t) \equiv A(s,t) + A(s,u)$$

$$= A(s,t) + A(s,4m^2 - s - t).$$

(11.4)

Low-$(\bar{s}, \bar{t})$ amplitude:
The unsymmetrized Low-(s, \bar{s}) amplitude may be written
\[ \mathcal{M}_{(s, \bar{s})} e^* = eQ \left\{ \left[ \frac{p_3 \cdot e^*}{p_3 \cdot k} + D^\mu(p_3) \frac{\partial}{\partial p_3^\mu} \right] A(s, \bar{s}) \right. 
+ \left[ \frac{p_4 \cdot e^*}{p_4 \cdot k} + D^\mu(p_4) \frac{\partial}{\partial p_4^\mu} \right] A(s, \bar{s}) \right. 
- \left[ \frac{p_1 \cdot e^*}{p_1 \cdot k} - D^\mu(p_1) \frac{\partial}{\partial p_1^\mu} \right] A(s, \bar{s}) 
- \left[ \frac{p_2 \cdot e^*}{p_2 \cdot k} - D^\mu(p_2) \frac{\partial}{\partial p_2^\mu} \right] A(s, \bar{s}) \bigg\} \]

where, following Ref. [81], we used the notation \( D^\mu(p) \), defined in Eq. (8.20). This is easily equated to the usual form for the amplitude by noting that
\[ \frac{\partial}{\partial p_1^\mu} = \frac{\partial s}{\partial p_1^\mu} \frac{\partial}{\partial s} + \frac{\partial t}{\partial p_1^\mu} \frac{\partial}{\partial t} = p_1^\mu \frac{\partial}{\partial s} - p_3^\mu \frac{\partial}{\partial t} \]
with similar expressions for the other derivatives.

The symmetrized amplitude is then
\[ \mathcal{M}^S_{(s, \bar{s})} e^* = eQ \left\{ \left[ \frac{p_3 \cdot e^*}{p_3 \cdot k} + D^\mu(p_3) \frac{\partial}{\partial p_3^\mu} \right] A(s, \bar{s}) \right. 
+ \left[ \frac{p_4 \cdot e^*}{p_4 \cdot k} + D^\mu(p_4) \frac{\partial}{\partial p_4^\mu} \right] A(s, \bar{s}) \right. 
- \left[ \frac{p_1 \cdot e^*}{p_1 \cdot k} - D^\mu(p_1) \frac{\partial}{\partial p_1^\mu} \right] A(s, \bar{s}) 
- \left[ \frac{p_2 \cdot e^*}{p_2 \cdot k} - D^\mu(p_2) \frac{\partial}{\partial p_2^\mu} \right] A(s, \bar{s}) \bigg\} \]

A common factor in this expression is
\[ A(s, \bar{s}) + A(s, \bar{\bar{s}}). \]

In order to have the soft photon amplitude solely a function of the measurable, symmetric part of the non-radiative amplitude we must be able to write this factor in terms of the symmetric function \( A^S() \). From Eq. (11.4) it is clear that if the relation \( s + \bar{s} + \bar{\bar{s}} = 4m^2 \) holds among the radiative variables we can write
\[ A(s, \bar{s}) + A(s, \bar{\bar{s}}) = A(s, \bar{s}, 4m^2 - s - \bar{\bar{s}}) \]
\[ = A^S(s, \bar{s}). \]

The relation among the 'barred' radiative variables may be proved by consideration of four momentum conservation:
\[ (k^\mu)^2 = (p_1^\mu + p_2^\mu - p_3^\mu - p_4^\mu)^2 \]
11.4 Symmetrization problem

\[ 0 = 4m^2 + 2p_1 \cdot p_2 + 2p_3 \cdot p_4 - 2p_1 \cdot p_3 - 2p_1 \cdot p_4 - 2p_2 \cdot p_3 - 2p_2 \cdot p_4 \]
\[ s_{12} + s_{54} + t_{13} + t_{24} + u_{14} + u_{23} = 8m^2 \]
\[ \bar{s} + \bar{t} + \bar{u} = 4m^2. \]  

(11.5)

The symmetrized radiative amplitude now takes on the form of the unsymmetrized amplitude, but with \( A(\bar{s}, \bar{t}) \) replaced by the measurable, symmetrized non-radiative amplitude \( A^S(\bar{s}, \bar{t}) \).

\[
M^S_{(\bar{s}, \bar{t})} = eQ \left\{ \left[ \frac{p_3 \epsilon^*}{p_3 \cdot k} + D^\nu(p_3) \frac{\partial}{\partial p_3^\nu} \right] A^S(\bar{s}, \bar{t}) \right. \\
+ \left[ \frac{p_4 \epsilon^*}{p_4 \cdot k} + D^\nu(p_4) \frac{\partial}{\partial p_4^\nu} \right] A^S(\bar{s}, \bar{t}) \\
- \left[ \frac{p_1 \epsilon^*}{p_1 \cdot k} - D^\nu(p_1) \frac{\partial}{\partial p_1^\nu} \right] A^S(\bar{s}, \bar{t}) \\
- \left[ \frac{p_2 \epsilon^*}{p_2 \cdot k} - D^\nu(p_2) \frac{\partial}{\partial p_2^\nu} \right] A^S(\bar{s}, \bar{t}) \right\}
\]

This procedure of correctly symmetrizing the radiative amplitude by simply replacing \( A() \rightarrow A^S() \) works only for the Low-(\( \bar{s}, \bar{t} \)) case, due to the special relationship \( \bar{s} + \bar{t} + \bar{u} = 4m^2 \). We will now show explicitly that such a replacement in the \( TsTts \) or \( TuTts \) amplitudes is incorrect.

\textbf{TsTts amplitude:}

For the radiative process, the unsymmetrized \( TsTts \) amplitude is

\[
M^\mu_{TsTts} = eQ \left( \left[ \frac{p_3^\mu}{p_3 \cdot k} - \left( \frac{p_3 + p_4}{p_3 + p_4} \right)^\mu \right] A(s_{12}, t_{24}) \right. \\
+ \left[ \frac{p_4^\mu}{p_4 \cdot k} - \left( \frac{p_3 + p_4}{p_3 + p_4} \right)^\mu \right] A(s_{12}, t_{13}) \\
- \left[ \frac{p_1^\mu}{p_1 \cdot k} - \left( \frac{p_1 + p_2}{p_1 + p_2} \right)^\mu \right] A(s_{54}, t_{24}) \\
- \left[ \frac{p_2^\mu}{p_2 \cdot k} - \left( \frac{p_1 + p_2}{p_1 + p_2} \right)^\mu \right] A(s_{54}, t_{13}) \right).
\]

We define the symmetrized amplitude \( M^S_{(\text{1,2} ightarrow \text{3,4})} \)

\[
M^S_{TsTts} = eQ \left( \left[ \frac{p_3^\mu}{p_3 \cdot k} - \left( \frac{p_3 + p_4}{p_3 + p_4} \right)^\mu \right] (A(s_{12}, t_{24}) + A(s_{12}, u_{14})) \right. \\
+ \left[ \frac{p_4^\mu}{p_4 \cdot k} - \left( \frac{p_3 + p_4}{p_3 + p_4} \right)^\mu \right] (A(s_{12}, t_{13}) + A(s_{12}, u_{23})) \\
- \left[ \frac{p_1^\mu}{p_1 \cdot k} - \left( \frac{p_1 + p_2}{p_1 + p_2} \right)^\mu \right] (A(s_{54}, t_{24}) + A(s_{54}, u_{23})) \\
- \left[ \frac{p_2^\mu}{p_2 \cdot k} - \left( \frac{p_1 + p_2}{p_1 + p_2} \right)^\mu \right] (A(s_{54}, t_{13}) + A(s_{54}, u_{14})) \right). \]  

(11.6)
We define symmetrized functions in analogy to the non-radiative case:

\[ A^S_1(s_{34}, t_{24}, u_{23}) \equiv A(s_{34}, t_{24}) + A(s_{24}, u_{23}) \]
\[ A^S_2(s_{34}, t_{13}, u_{14}) \equiv A(s_{34}, t_{13}) + A(s_{24}, u_{14}) \]
\[ A^S_3(s_{12}, t_{24}, u_{14}) \equiv A(s_{12}, t_{24}) + A(s_{13}, u_{14}) \]
\[ A^S_4(s_{12}, t_{13}, u_{23}) \equiv A(s_{12}, t_{13}) + A(s_{13}, u_{23}) \]

Notice that these functions are not the same as the symmetric non-radiative function; an internal constraint similar to that for the non-radiative phase space variables, \( s + t + u = 4m^2 \), does not hold for \( s_{34}, t_{24} \) and \( u_{23} \), or for the other sets of radiative variables which appear as arguments in the \( A^S_i \). Direct replacement of, for example, \( A^S_1(s_{34}, t_{24}, u_{23}) \) by \( A^S(s_{34}, t_{24}) \) will give an error of the form

\[ A^S_1(s_{34}, t_{24}, u_{23}) - A^S(s_{34}, t_{24}) = \left[ A(s_{34}, t_{24}) + A(s_{24}, u_{23}) \right] - \left[ A(s_{34}, t_{24}) + A(s_{24}, 4m^2 - s_{34} - t_{24}) \right] \]
\[ = A(s_{34}, u_{23}) - A(s_{34}, 4m^2 - s_{34} - t_{24}) \]
\[ = [u_{23} - (4m^2 - s_{34} - t_{24})] \frac{\partial}{\partial u} A(s_{34}, u) \bigg|_{u=u_{23}} + \ldots \]

The amplitude \( M^S_{T^u T^w} \) would, after such a replacement, be no longer invariant under the interchange of particle labels 3 and 4, which it should be. Using the radiative phase space constraint \( s_{12} + s_{34} + t_{13} + t_{24} + u_{14} + u_{23} = 8m^2 \) we can show that the term \( [u_{23} - (4m^2 - s_{34} - t_{24})] \) is -2p1·k. One would naively expect this order-(k) error which is being introduced to give rise to an order-(k/k) or order-(1) error in the amplitude \( M^S_{T^u T^w} \). Due to a cancellation in this leading order between the four such \( A^S_i() \) terms appearing in \( M^S_{T^u T^w} \), the error introduced by these replacements is instead only of order-(k).

**TuT TS amplitude:**

For the TuT TS amplitude another difficulty arises in trying to symmetrize. The unsymmetrized amplitude is

\[ M^S_{TuT TS} = eQ \left( \frac{p_0^\mu}{p_3 \cdot k} - \frac{(p_2 - p_3)^\mu}{(p_2 - p_3) \cdot k} \right) A^' (u_{14}, t_{24}) \]
\[ + \left[ \frac{p_4^\mu}{p_4 \cdot k} - \frac{(p_1 - p_4)^\mu}{(p_1 - p_4) \cdot k} \right] A^' (u_{23}, t_{13}) \]
\[ - \left[ \frac{p_2^\mu}{p_1 \cdot k} - \frac{(p_1 - p_2)^\mu}{(p_1 - p_2) \cdot k} \right] A^' (u_{23}, t_{13}) \]
\[ - \left[ \frac{p_3^\mu}{p_2 \cdot k} - \frac{(p_2 - p_3)^\mu}{(p_2 - p_3) \cdot k} \right] A^' (u_{14}, t_{24}) \].
and we define the symmetrized amplitude as $\mathcal{M}^{S\mu}_{TuTs} = M^{\mu}_{(1,2 \leftrightarrow 3,4)} + M^{\mu}_{(1,3 \leftrightarrow 2,4)}$

\[
\begin{align*}
\mathcal{M}^{S\mu}_{TuTs} &= \\
&= eQ \left( \left[ \frac{p_3^\mu}{p_3 \cdot k} - \frac{(p_2 - p_3)^\mu}{k} \right] A'(u_{14}, t_{24}) + \left[ \frac{p_4^\mu}{p_4 \cdot k} - \frac{(p_1 - p_4)^\mu}{k} \right] A'(t_{24}, u_{14}) \right) \\
&\quad + \left[ \frac{p_3^\mu}{p_4 \cdot k} - \frac{(p_1 - p_4)^\mu}{k} \right] A'(u_{23}, t_{13}) + \left[ \frac{p_4^\mu}{p_4 \cdot k} - \frac{(p_2 - p_4)^\mu}{k} \right] A'(t_{13}, u_{23}) \\
&\quad - \left[ \frac{p_3^\mu}{p_1 \cdot k} - \frac{(p_1 - p_4)^\mu}{k} \right] A'(u_{23}, t_{24}) + \left[ \frac{p_1^\mu}{p_1 \cdot k} - \frac{(p_2 - p_4)^\mu}{k} \right] A'(t_{24}, u_{23}) \\
&\quad - \left[ \frac{p_3^\mu}{p_2 \cdot k} - \frac{(p_2 - p_3)^\mu}{k} \right] A'(u_{14}, t_{13}) + \left[ \frac{p_2^\mu}{p_2 \cdot k} - \frac{(p_1 - p_3)^\mu}{k} \right] A'(t_{13}, u_{14}) \\
&= (11.7)
\end{align*}
\]

Consistent with our previous definition of $A'(\cdot)$ we have that, for example, $A'(t_{24}, u_{14}) = A((\sum, m_1^2 - t_{24} - u_{14}), u_{14})$ where $A(s, t)$ is the measured amplitude for the non-radiative process. The procedure followed in Ref. [62] is to state that this symmetrized amplitude is the original unsymmetrized $TuTs$ amplitude with the functions $A'(\cdot)$ replaced by their counterparts from the symmetrized elastic process. This prescription would give the result

\[
\begin{align*}
eQ \left( \left[ \frac{p_3^\mu}{p_3 \cdot k} - \frac{(p_2 - p_3)^\mu}{k} \right] A^{S}(u_{14}, t_{24}) \right) \\
&\quad + \left[ \frac{p_4^\mu}{p_4 \cdot k} - \frac{(p_1 - p_4)^\mu}{k} \right] A^{S}(u_{23}, t_{13}) \\
&\quad - \left[ \frac{p_3^\mu}{p_1 \cdot k} - \frac{(p_1 - p_4)^\mu}{k} \right] A^{S}(u_{23}, t_{24}) \\
&\quad - \left[ \frac{p_3^\mu}{p_2 \cdot k} - \frac{(p_2 - p_3)^\mu}{k} \right] A^{S}(u_{14}, t_{13}) \\
&= (11.8)
\end{align*}
\]

where, for example,

\[
A^{S}(u_{23}, t_{24}) = A'(u_{23}, t_{24}) + A'(t_{24}, u_{23}) = A((4m^2 - u_{23} - t_{24}, u_{23}) + A((4m^2 - u_{23} - t_{24}, u_{23}) = A^{S}(4m^2 - u_{23} - t_{24}, t_{24})
\]

A comparison between the symmetrized expression of Eq. (11.7) and the form of Eq. (11.8) shows them to be unequal. Eq. (11.8) is not invariant under the interchange of particle labels 3 and 4. We have found no way to express the correct expression, Eq. (11.7), solely in terms of the measurable symmetrized non-radiative amplitude, $A^S(s, t)$.

The arguments above have shown that the $TsTs$ and $TuTs$ amplitudes for a spin-0 scattering process cannot be made correctly symmetric under interchange of identical
11.5 Extension to spin-\(\frac{1}{2}\) scattering

As we showed at the end of §8.2.2, the soft photon amplitudes \(\text{Low-(s, t)}\), \(\text{TsTts}\) and \(\text{TuTts}\) for the radiative scattering of spin-1 on spin-\(\frac{3}{2}\) particles may be written in the common form,

\[
M_\mu = \sum_{\alpha=1}^{5} \left[ eQ_1 \bar{u}_3 X_\mu^{\alpha} u_1 \bar{u}_4 t_\alpha u_2 + eQ_2 \bar{u}_3 t_\alpha u_1 \bar{u}_4 Y_\mu^{\alpha} u_2 \right]
\]  

(11.9)

where \(Q_1\) and \(Q_2\) are the charges of the scattering particles, and

\[
t_\alpha \equiv \{1, \frac{1}{2}, \gamma^\mu, \gamma^\tau, \gamma^\tau, \gamma^\mu, \gamma^\nu, \gamma^\nu\}
\]

with summation over the Lorentz indices of the \(t_\alpha\) being implied. The particular notation used here is that of Liou and collaborators, Ref. [62], and is closely related to the invariant amplitude formalism of Nyman, Ref. [78]. Through a re-labelling of the four-momenta it may also be related to our general formalism of Chapter 9, in particular to the final example of §9.2. The different soft photon amplitudes give rise to different sets of functions \(X_\alpha^\mu\) and \(Y_\alpha^\mu\). A derivation of the \(\text{Low-(s, t)}\) amplitude will be given in a moment. The construction of the \(\text{TsTts}\) and \(\text{TuTts}\) amplitudes should be clear, given this example derivation and given the spin-0 derivations of §11.1. Thus we will simply write down the expressions for the \(\text{TsTts}\) and \(\text{TuTts}\) amplitudes.

The unsymmetrized amplitude for elastic scattering is

\[
A(12 \rightarrow 34) \equiv \sum_{\alpha=1}^{5} F_\alpha(s, t)[\bar{u}_3 t_\alpha u_1][\bar{u}_4 t^\alpha u_2]
\]

(11.10)

and this is taken to define the the Lorentz invariant functions \(F_\alpha(s, t)\) which will also appear in the unsymmetrized soft photon amplitudes. These functions \(F_\alpha(s, t)\) may be related to measured phase shift fits for the non-radiative process at hand. For the case of \(pp\) elastic scattering this relation has been outlined previously in the literature, and is rather lengthy. Its derivation is given in full in App. A.
11.5 Extension to spin-\(\frac{1}{2}\) scattering

The Low-(\(\vec{s}, \vec{t}\)) expression is now derived. The contribution of the four diagrams where the photon is radiated from an incoming or outgoing spin-\(\frac{1}{2}\) particle is then, to lowest order in perturbation theory,

\[
M_\mu = \sum_{\alpha=1}^8 \left[ F_\alpha(\vec{s} + (p_3 + p_4) \cdot k, \vec{t} + (p_4 - p_3) \cdot k) \bar{u}_a J_\mu^{(3)} t^\alpha u_1 \bar{u}_4 t_\alpha u_2 
+ F_\alpha(\vec{s} - (p_1 + p_2) \cdot k, \vec{t} + (p_4 - p_3) \cdot k) \bar{u}_a t^\alpha J_\mu^{(1)} u_1 \bar{u}_4 t_\alpha u_2 
+ F_\alpha(\vec{s} + (p_3 + p_4) \cdot k, \vec{t} + (p_3 - p_1) \cdot k) \bar{u}_a t^\alpha u_1 \bar{u}_4 J_\mu^{(4)} t_\alpha u_2 
+ F_\alpha(\vec{s} - (p_1 + p_2) \cdot k, \vec{t} + (p_3 - p_1) \cdot k) \bar{u}_a t^\alpha u_1 \bar{u}_4 J_\mu^{(2)} t_\alpha u_2 \right] (11.11)
\]

where

\[
J_\mu^{(3,4)} = e Q_{3,4} \left( \gamma_\mu - \frac{i \kappa_{3,4}}{2m_{3,4}} \sigma_{\mu \nu} k^\nu \right) \left( \frac{p_{3,4} + k + m_{3,4}}{2p_{3,4} \cdot k} \right)
\]

\[
J_\mu^{(1,2)} = -e Q_{1,2} \left( \frac{p_{1,2} + k + m_{1,2}}{2p_{1,2} \cdot k} \right) \left( \gamma_\mu - \frac{i \kappa_{1,2}}{2m_{1,2}} \sigma_{\mu \nu} k^\nu \right)
\]

and \(\kappa\) is the anomalous magnetic moment of the scattering particle. As we showed in \(\S 8.2.2\) we may write these operators more compactly. For example, the operator for particle 4 may be written

\[
\bar{u}_a J_\mu^{(4)} = e Q_4 \bar{u}_4 \left( \frac{p_{4 \mu} + \mathcal{R}_\mu(p_4)}{p_4 \cdot k} \right)
\]

where

\[
\mathcal{R}(p) \cdot e^* = \frac{1}{4} [\vec{f}, \vec{k}] + \frac{i \kappa}{8m} \{[[\vec{f}, \vec{k}], \vec{p}]\}.
\]

We then follow the standard procedure of Taylor expanding Eq. (11.11) about the explicit \(k^\mu\) dependencies in the functions \(F_\alpha()\) and truncating at next to leading order in \(k^\mu\); this gives rise to derivative terms. A term, taken to arise from internal photon emission diagrams, is added to restore gauge invariance. The result of this procedure is shown in Eq. (11.12) where the \(X^a_\alpha\) and \(Y^a_\alpha\) are the functions appearing in Eq. (11.9).

- Low-(\(\vec{s}, \vec{t}\)) amplitude:

\[
X^a_\alpha e^* = \left[ F_\alpha(\vec{s}, \vec{t}) \frac{p_3 \cdot e^* + \mathcal{R}(p_3) \cdot e^*}{p_3 \cdot k} + \mathcal{D}^\mu(p_3) \frac{\partial F_\alpha(\vec{s}, \vec{t})}{\partial p_3^\mu} \right] t^\alpha 
- t^\alpha \left[ F_\alpha(\vec{s}, \vec{t}) \frac{p_1 \cdot e^* + \mathcal{R}(p_1) \cdot e^*}{p_1 \cdot k} - \mathcal{D}^\mu(p_1) \frac{\partial F_\alpha(\vec{s}, \vec{t})}{\partial p_1^\mu} \right]
\]

\[
Y^a_\alpha e^* = \left[ F_\alpha(\vec{s}, \vec{t}) \frac{p_4 \cdot e^* + \mathcal{R}(p_4) \cdot e^*}{p_4 \cdot k} + \mathcal{D}^\mu(p_4) \frac{\partial F_\alpha(\vec{s}, \vec{t})}{\partial p_4^\mu} \right] t^\alpha 
- t^\alpha \left[ F_\alpha(\vec{s}, \vec{t}) \frac{p_2 \cdot e^* + \mathcal{R}(p_2) \cdot e^*}{p_2 \cdot k} - \mathcal{D}^\mu(p_2) \frac{\partial F_\alpha(\vec{s}, \vec{t})}{\partial p_2^\mu} \right] (11.12)
\]
11.5 Extension to spin-$\frac{1}{2}$ scattering

For the other two amplitudes, $T_{sT\bar{t}s}$ and $TuT\bar{t}s$, considered in Ref. [62] the resulting values of $X^\alpha$ and $Y^\alpha$ are:

- $T_{sT\bar{t}s}$ amplitude:

$$X^\alpha \equiv F_\alpha(s_{12}, t_{24}) \left[ \frac{p_{5\mu} + R_\mu(p_3)}{p_3 \cdot k} - \frac{(p_3 + p_4)_\mu}{(p_3 + p_4) \cdot k} \right] t^\alpha$$

$$- F_\alpha(s_{24}, t_{24}) t^\alpha \left[ \frac{p_{1\mu} + R_\mu(p_1)}{p_1 \cdot k} - \frac{(p_1 + p_2)_\mu}{(p_1 + p_2) \cdot k} \right]$$

$$Y^\alpha \equiv F_\alpha(s_{12}, t_{13}) \left[ \frac{p_{4\mu} + R_\mu(p_4)}{p_4 \cdot k} - \frac{(p_3 + p_4)_\mu}{(p_3 + p_4) \cdot k} \right] t^\alpha$$

$$- F_\alpha(s_{24}, t_{13}) t^\alpha \left[ \frac{p_{2\mu} + R_\mu(p_2)}{p_2 \cdot k} - \frac{(p_1 + p_2)_\mu}{(p_1 + p_2) \cdot k} \right]. \quad (11.13)$$

- $TuT\bar{t}s$ amplitude:

$$X^\alpha \equiv F'_\alpha(u_{14}, t_{24}) \left[ \frac{p_{3\mu} + R_\mu(p_3)}{p_3 \cdot k} - \frac{(p_3 - p_0)_\mu}{(p_3 - p_0) \cdot k} \right] t^\alpha$$

$$- F'_\alpha(u_{23}, t_{24}) t^\alpha \left[ \frac{p_{1\mu} + R_\mu(p_1)}{p_1 \cdot k} - \frac{(p_1 - p_4)_\mu}{(p_1 - p_4) \cdot k} \right]$$

$$Y^\alpha \equiv F'_\alpha(u_{23}, t_{13}) \left[ \frac{p_{4\mu} + R_\mu(p_4)}{p_4 \cdot k} - \frac{(p_3 - p_4)_\mu}{(p_3 - p_4) \cdot k} \right] t^\alpha$$

$$- F'_\alpha(u_{14}, t_{13}) t^\alpha \left[ \frac{p_{2\mu} + R_\mu(p_2)}{p_2 \cdot k} - \frac{(p_1 - p_3)_\mu}{(p_1 - p_3) \cdot k} \right]. \quad (11.14)$$

The phase space problem noted for spin-0 identical particle scattering in §11.3 carries over directly to the spin-$\frac{1}{2}$ case. Even if the non-radiative elastic phase shifts are measured throughout their physical region, functions such as $F_\alpha(s_{24}, t_{24})$ may still be unknown since the point $(s_{24}, t_{24})$ can fall outside of this region.

We will now specialize our treatment to the case of proton-proton bremsstrahlung. The amplitudes given above must be anti-symmetrized if we are to treat identical spin-$\frac{1}{2}$ particle scattering. The anti-symmetrization of the spin-$\frac{1}{2}$ amplitudes is no different in principle than the symmetrization of spin-0 amplitudes given in the preceding section. The results are identical: the Low-$(\bar{z}, \bar{t})$ amplitude can be successfully anti-symmetrized while still being expressed solely in terms of the measured elastic phase shifts; the $T_{sT\bar{t}s}$ and $TuT\bar{t}s$ amplitudes cannot be so expressed. We now show these attempts at anti-symmetrization explicitly and demonstrate how the problems arise. In order to introduce the technique to be used we shall first consider the definition of a
11.5 Extension to spin-$\frac{1}{2}$ scattering

Correctly anti-symmetrized $pp$ elastic amplitude. This involves writing down the unsymmetrized amplitude in its most general form, subtracting that amplitude with the final state identical particles explicitly interchanged, and then performing some simple manipulations in order to regain the same form with which we began, but with different coefficients of the invariant components.

Given the unsymmetrized form in Eq. (11.10) above, the anti-symmetrized elastic amplitude is then taken to be

$$A(12 \rightarrow 34) - A(12 \rightarrow 43)$$

where

$$A(12 \rightarrow 43) \equiv \sum_{\alpha=1}^{5} F_{\alpha}(s, u) [\bar{u}_{4}t_{\alpha}u_{1}] [\bar{u}_{5}t^{\alpha}u_{2}].$$

(11.15)

Using the Fierz relation (Ref. [91])

$$(t_{\alpha})_{\sigma\nu}(t^{\alpha})_{\tau\nu} = \sum_{\beta=1}^{5} C_{\alpha\beta}(t_{\beta})_{\phi\nu}(t^{\beta})_{\tau\sigma}$$

(11.16)

with

$$C_{\alpha\beta} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 6 & -2 & 0 & 0 & 6 \\ 4 & 0 & -2 & 2 & -4 \\ 4 & 0 & 2 & -2 & -4 \\ 1 & 1 & -1 & -1 & 1 \end{pmatrix}$$

(11.17)

we can put this back into the same form as the unsymmetrized amplitude,

$$A(12 \rightarrow 43) \equiv \sum_{\alpha=1}^{5} \sum_{\beta=1}^{5} C_{\alpha\beta} F_{\beta}(s, u) [\bar{u}_{4}t_{\alpha}u_{2}] [\bar{u}_{5}t^{\alpha}u_{1}].$$

The correctly anti-symmetrized amplitude is then identical to the unsymmetrized amplitude but with $F_{\alpha}(s, t)$ replaced by

$$F_{\alpha}^{A}(s, t) \equiv F_{\alpha}(s, t) - \sum_{\beta=1}^{5} C_{\alpha\beta} F_{\beta}(s, u)$$

(11.18)

where $s + t + u = 4m^2$. These $F_{\alpha}^{A}(s, t)$ are the functions which are measurable in $pp$ elastic experiments. The unsymmetrized functions $F_{\alpha}(s, t)$ cannot be measured in $pp$ elastic scattering.

We shall now attempt to perform a similar anti-symmetrization on the three soft photon amplitudes under consideration. As has been shown previously by Fearing
11.5 Extension to spin-$\frac{1}{2}$ scattering

(Ref. [81]) one can anti-symmetrize the Low ($\bar{s}, \bar{t}$) amplitude using an analogous procedure to that shown above for $pp$ elastic scattering. One takes the amplitude of Eq. (11.12), exchanges $p^\sigma_s \rightarrow p^\sigma_t$, and applies the Fierz manipulation. The result is then in the same form as Eq. (11.12), but with $F_a(\bar{s}, \bar{t})$ replaced by $\sum_\beta^5 C_{\beta\alpha} F_\beta(\bar{s}, \bar{u})$. The final anti-symmetrized form is then also identical to Eq. (11.12) but with $F_a(\bar{s}, \bar{t})$ replaced by $F^A_{\alpha(\text{Low})}(\bar{s}, \bar{t})$ where

$$F^A_{\alpha(\text{Low})}(\bar{s}, \bar{t}) \equiv F_a(\bar{s}, \bar{t}) - \sum_{\beta=1}^5 C_{\beta\alpha} F_\beta(\bar{s}, \bar{u})$$

and $\bar{s}, \bar{t}, \bar{u}$ satisfy the radiative phase space constraint $\bar{s} + \bar{t} + \bar{u} = 4m^2$. By their identical definitions we see that $F^A_{\alpha(\text{Low})} \equiv F^A_a$. Operationally, therefore, one only has to take the anti-symmetrized $F^A_a$ from a phase shift analysis of $pp$ elastic scattering data and insert these functions in the unsymmetrized Low $(\bar{s}, \bar{t})$ amplitude of Eq. (11.12) in order to ensure the correct anti-symmetrization of this radiative amplitude.

When we attempt the same procedure for the $T_T T_T$ amplitude a problem arises. The calculation goes just as with the $(\bar{s}, \bar{t})$ case except that the definition of the four anti-symmetrized $F_a$ functions is not the same. To anti-symmetrize the $T_T T_T$ amplitude we must replace the $F_a$ of the unsymmetrized Eq. (11.13) as follows:

$$F_a(s_{34}, t_{24}) \rightarrow F^{A(1)}_a(s_{34}, t_{24}, u_{23}) = F_a(s_{34}, t_{24}) - \sum_{\beta=1}^5 C_{\beta\alpha} F_\beta(s_{34}, u_{23})$$

$$F_a(s_{34}, t_{13}) \rightarrow F^{A(2)}_a(s_{34}, t_{13}, u_{14}) = F_a(s_{34}, t_{13}) - \sum_{\beta=1}^5 C_{\beta\alpha} F_\beta(s_{34}, u_{14})$$

$$F_a(s_{12}, t_{24}) \rightarrow F^{A(3)}_a(s_{12}, t_{24}, u_{14}) = F_a(s_{12}, t_{24}) - \sum_{\beta=1}^5 C_{\beta\alpha} F_\beta(s_{12}, u_{14})$$

$$F_a(s_{12}, t_{13}) \rightarrow F^{A(4)}_a(s_{12}, t_{13}, u_{23}) = F_a(s_{12}, t_{13}) - \sum_{\beta=1}^5 C_{\beta\alpha} F_\beta(s_{12}, u_{23})$$

The various Lorentz variables in these definitions do not satisfy the same internal constraints as do the $s, t, u$ of $F^A_a(s, t)$—for example, in $F^{A(3)}_a(s_{12}, t_{24}, u_{14})$ we have that

$$s_{12} + t_{24} + u_{14} \neq 4m^2$$

in general. These $F^{A(1,2,3,4)}_a$ cannot be simply replaced by the $F_a(s, t)$ of $pp$ elastic scattering—the functions are not identically defined. To make such a replacement will result in the radiative amplitude having improper symmetry properties.

While attempting to anti-symmetrize the $T_T T_T$ amplitude we find another difficulty. After replacing $p^\sigma_s \rightarrow p^\sigma_t$ in the unsymmetrized version of $T_T T_T$ and applying
the Fierz manipulation we find

\[
M^{3-4}_\mu = eQ_x \sum_{\alpha=1}^{5} \sum_{\beta=1}^{5} C_{\alpha \beta} \left( F_{\alpha}(t_{13}, u_{28}) \bar{u}_4 \left[ \frac{p_{4\mu} + R_\mu(p_4)}{p_4 \cdot k} - \frac{(p_3 - p_4)_\mu}{(p_3 - p_4) \cdot k} \right] t_\alpha u_2 \bar{u}_3 t_\alpha u_1 
- F_{\alpha}(t_{24}, u_{22}) \bar{u}_4 t_\alpha u_2 \bar{u}_3 t_\alpha \frac{p_{1\mu} + R_\mu(p_1)}{p_1 \cdot k} - \frac{(p_1 - p_3)_\mu}{(p_1 - p_3) \cdot k} \left[ \frac{p_{2\mu} + R_\mu(p_2)}{p_2 \cdot k} - \frac{(p_2 - p_3)_\mu}{(p_2 - p_3) \cdot k} \right] \bar{u}_3 t_\alpha u_1 
+ F_{\alpha}(t_{13}, u_{14}) \bar{u}_4 t_\alpha u_2 \bar{u}_3 t_\alpha \frac{p_{1\mu} + R_\mu(p_1)}{p_1 \cdot k} - \frac{(p_1 - p_3)_\mu}{(p_1 - p_3) \cdot k} \left[ \frac{p_{2\mu} + R_\mu(p_2)}{p_2 \cdot k} - \frac{(p_2 - p_3)_\mu}{(p_2 - p_3) \cdot k} \right] \bar{u}_3 t_\alpha u_1 \right)
\]

which cannot be put into the original form of the amplitude given in Eq. (11.14). Thus it is impossible to define anti-symmetrized $F_{\alpha}(s, t)$ functions for the $T_T T_T$ amplitude. The amplitudes $T_S T_T$ and $T_T T_T$ could be correctly anti-symmetrized by computing $M_\mu - M_\mu(p_3 \leftrightarrow p_4)$ directly in terms of the unsymmetrized functions $F_{\alpha}(s, t)$. As stated previously, the $F_{\alpha}(s, t)$ cannot be derived from $pp$ elastic scattering data alone. Thus we would lose the direct connection between a process and its radiative counterpart which gives the soft photon theorem its utility.

From Ref. [62] we see that Liou et al. have used the anti-symmetric $F_{\alpha}(s, t)$ $pp$ elastic functions directly in their unsymmetrized $T_S T_T$ and $T_T T_T$ amplitudes. Their results cannot have the correct symmetry properties.

## 11.6 Application to $pp$ bremsstrahlung

A computer program has been written to calculate the three soft photon approximations to $pp$ bremsstrahlung which were derived in the last section. It takes as input a set of $pp$ elastic nuclear bar phase shifts\(^2\) measured at many energies, computes the functions $F_{\alpha}(s, t)$ using the formulae of App. A, and gives as output the differential cross section $d^3\sigma/d\Omega_d d\Omega_4 d\theta_4$ for fixed outgoing proton angles and as a function of photon angle $\theta_4$.

The expression for the phase space factor in the quantity $d^3\sigma/d\Omega_d d\Omega_4 d\theta_4$ is that of Ref. [92]. As a check of the program the Low-$\langle \bar{s}, \bar{t} \rangle$ result was compared with the earlier and completely independent calculation of Fearing (Ref. [93]); see Figs. 11.4-11.6 for representative examples. When the same set of phase shifts are used as were employed by Fearing the calculations agreed at the ~2% level. The same calculation of the Low-$\langle \bar{s}, \bar{t} \rangle$ soft photon results was made with more modern phase shifts: namely, those from the recent analysis of the Nijmegen group, Refs. [94, 95, 96], and those obtained from the SAID database [97] under the reference SM95. These are also shown in Figs. 11.4-11.6.

\(^2\)See p. A for a definition of the term 'nuclear bar phase shift'.
along with the calculations employing phase shifts extracted from the the Nijmegen, Bonn and Paris potential models.

We found in the preceding sections that the TsTs amplitude should be incalculable in terms of the physical phase shifts because the TsTs amplitude requires evaluations of the non-radiative amplitude to be made far outside of the physically allowed non-radiative phase space region. Indeed we are unable to present numeric results for this amplitude. Yet, Liou and collaborators in Ref. [62] present such a result. We shall now give a speculative but plausible explanation of how the TsTs computation of Ref. [62] could have been made without the authors being aware of the phase space problem.

We noted in §11.3 that the variable pairs \((s_{84},t_{13})\) and \((s_{84},t_{24})\) leave the physical non-radiative region for the kinematics of the TRIUMF pp bremsstrahlung experiment. Consider how this would affect an attempt to calculate, say, the function \(F^{A}(s_{84},t_{13})\). Phase shifts, and hence the are numerically evaluated in terms of the centre of mass frame proton momentum \(k_{CM}\) and the cosine of the scattering angle \(\theta_{CM}\). A point \((s_{84},t_{13})\) which is outside of the physical non-radiative region would translate into a \((k_{CM},\cos\theta_{CM})\) pair with \(\cos\theta_{CM}\) outside of its physical range \([-1,+1]\). During the evaluation of the function \(F^{A}\) this value of \(\cos\theta_{CM}\) is passed as an argument to various functions, such as the Legendre functions of the first and second kind. If library functions have been used which, as is usual, accept arguments in the complex domain then some numeric result would always be returned. This would correspond to an extrapolation of the phase shifts to a point far from the physical region where their values were originally fitted and extracted. Such an extrapolation is certainly not to be trusted, particularly when one considers how far outside the physical region one can be: the point \((s_{84},t_{13})\) can give rise to values such as \(\cos\theta_{CM} \sim -20\). A plot of \(\cos\theta_{CM}\) versus \(k_{CM}\) is shown in Fig. 11.7 with the physical non-radiative region marked as well as the trajectory of evaluations required during a typical TsTs pp bremsstrahlung spectrum calculation.

If we suppose that this unjustified and probably accidental extrapolation of the phase shifts by the authors of Ref. [62] actually occurred it is hardly a surprise that their TsTs calculation of pp bremsstrahlung is found to differ by up to an order of magnitude from other soft photon calculations, from potential model calculations, and from experimental data. This difference appears in spite of the well known result which states that the leading two orders of the photon spectrum are uniquely predicted. The authors of Ref. [62] suggest that this large discrepancy is evidence that the TuTs amplitude should be preferred over the TsTs amplitude for calculations of identical particle scattering. We feel that the large discrepancy between the TsTs calculation of Ref. [62] and other calculations is due to the extrapolation of phase shifts in an arbitrary
11.6 Application to $pp$ bremsstrahlung

$\theta_3 = \theta_4 = 22^\circ$

$E_{\text{lab}} = 42$ MeV

Figure 11.4: Comparison of different $pp$ phase shift sets, used to compute bremsstrahlung.

Shown is the Low-{\theta,} soft photon approximation to $pp$ bremsstrahlung, with beam kinetic energy 42 MeV and outgoing proton angles at $22^\circ$ on either side of the beam direction. The calculation was made with three sets of multi-energy $pp$ phase shifts and three sets of potential model phase shifts. The long dashed line shows the results of the older dataset of Workman and Fearing [98], while the solid line is the SM95 dataset obtained from the SAID database [97]. The short dashed line uses the recent Nijmegen partial wave analysis of Refs. [94, 95]; the data was obtained from the “NN-Online” database located at Nijmegen[96]. The long dash-dot, the short dash-dot and the dotted lines are the results of the phase shifts extracted from the Nijmegen, Bonn and Paris potential models, respectively. These were also obtained from the SAID database.
Figure 11.5: The same as Fig. 11.4 but with beam kinetic energy 200 MeV and outgoing proton angles at 16.4° on either side of the beam direction.
Figure 11.6: The same as Fig. 11.4 but with beam kinetic energy 280 MeV and outgoing proton angles at 10° on either side of the beam direction.
Figure 11.7: Trajectories of radiative variable pairs projected into the \((k_{CM}, \cos(\theta_{CM}))\) plane.

The physical non-radiative region in terms of centre-of-mass frame proton momentum and scattering angle, and the trajectories of various radiative variable pairs. The process is \(pp\) bremsstrahlung at beam kinetic energy of 280 MeV, with outgoing protons at 12.4° and 12° on either side of the beam direction.
fashion far outside of their fitted region.

We then reproduced the results of Ref. [62] which use the TuTs amplitude. In that paper it appears to be the case that the pp bremsstrahlung amplitude has been obtained by substituting the anti-symmetric elastic scattering functions $F^*$ directly into the unsymmetrized form for the TuTs soft photon amplitude of Eqs. (11.9) and (11.14). We have seen in §11.5 that this gives rise to errors in the resulting amplitude's anti-symmetry properties. We have confirmed such an error by direct calculation. For kinematic situations where the outgoing protons are at equal angles on either side of the beam direction it is clear from the amplitude's anti-symmetry that the photon spectrum should be symmetric under the reflection $\theta_k \rightarrow 360° - \theta_k$. Since the authors of Ref. [62] give their results only in the region $\theta_k = 0° \rightarrow 180°$ this cannot be checked directly from their paper. We obtain results shown in Fig. 11.3 which agree very well with theirs' for that range of $\theta_k$ but which are not mirror symmetric about $\theta_k = 180°$, which they should be. This is consistent with an error in the anti-symmetrization of the TuTs amplitude. We do not know of a way to correct this problem while maintaining the necessary link to pp elastic scattering data.

In summary we find that the TsTs and TuTs amplitudes of Refs. [61, 62] cannot be correctly computed for the case of identical particle scattering. This is due in the TsTs case to the inappropriate extrapolation of pp elastic phase shifts far outside of the phase space region where they were originally fitted. When we attempted to replicate the work of Ref. [62] it was found that the anti-symmetrized radiative amplitudes could not be written in terms of only the anti-symmetric part of the non-radiative amplitude. Upon following the prescription of Ref. [62] we found that the resulting radiative spectra lose their expected symmetry properties; this has been shown by direct numeric computation for the TuTs case. No alternative prescription which might correct the TsTs and TuTs soft photon amplitudes has been discovered.

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*It appears that the published version of Ref. [62] has an error in the ordering and labelling of the figures, as can be seen by comparison with the paper's preprint (Ref. [63]) and in order to make sense of references to the figures from the text. To correct this one would replace the bodies of figures 1, 2 and 3 with those of figures 2, 3 and 1. The figure captions are correctly ordered, however.
11.6 Application to pp bremsstrahlung

The solid line shows the TuTts soft photon approximation to pp bremsstrahlung for the case \( T = 157 \text{ MeV}, \theta_3 = \theta_4 = 10^\circ \). The Low-\((s,t)\) SPA is shown by the dashed line. Both are computed using the SAID SP94 phase shift dataset. The spectra should exhibit mirror symmetry in \( \theta_\gamma \) about the point \( \theta_\gamma = 180^\circ \), since the protons are emitted at equal angles on either side of the beam direction. The Low-\((s,t)\) result is symmetric, but the TuTts result is clearly not. This is in agreement with our analytical work of §11.5.
Chapter 12

A Solution, for Common Processes

In this chapter we present certain choices of radiative variable pair to be used as expansion points in the development of a soft photon amplitude and prove that their ranges, defined by radiative phase space constraints, lie entirely within physical non-radiative phase space. These choices therefore do not fall victim to the phase space problem described in Chp. 10. The variable pairs are

- \((s_{12}, t_a)\) where \(t_a = (1 - a)t_{12} + at_{24}; 0 \leq a \leq 1\). This expansion point lies in the physical region of non-radiative phase space so long as the scattering masses satisfy \((m_3 + m_4) \geq (m_1 + m_2)\). This constraint includes the most commonly studied situation, elastic scattering.

- \((\bar{s}, \bar{t})\) for the special case of identical particle scattering.

We shall now prove these results in some detail.

12.1 \((s_{12}, t_a)\)

This proof goes in two stages:

1) we show that \(t_{12}\) and \(t_{24}\) are correctly limited within the physical region of the non-radiative variable \(t\), where we have set the non-radiative variable \(s = s_{12}\);

2) we show that a linear combination of variables which are confined to a certain range is also confined to that same range, with certain restrictions upon the form of the linear combination.

Part(1):
The Mandelstam variable \( t \) for non-radiative phase space is defined

\[
\begin{align*}
  t &= (p_2 - p_1)^2 \\
  &= m^2_2 + m^2_3 - 2p_1 \cdot p_3 \\
  &= m^2_2 + m^2_3 - 2E_1E_3 + 2|\vec{p}_1||\vec{p}_3| \cos \theta_{13}
\end{align*}
\]

in the centre-of-mass frame. The variable \( t \) takes on its minimum and maximum values \( t^- \) and \( t^+ \) at \( \cos \theta_{13} = -1, 1 \). Expressing the c.m. frame energies and momenta in terms of the Mandelstam variable \( s \),

\[
\begin{align*}
  |\vec{p}_1| &= \frac{\lambda^\frac{1}{3}(s, m^2_1, m^2_2)}{2\sqrt{s}} \\
  |\vec{p}_3| &= \frac{\lambda^\frac{1}{3}(s, m^2_1, m^2_2)}{2\sqrt{s}} \\
  E_1 &= \frac{(s + m^2_2 - m^2_1)}{2\sqrt{s}} \\
  E_3 &= \frac{(s + m^2_3 - m^2_2)}{2\sqrt{s}}
\end{align*}
\]

where

\[
\lambda(x, y, z) \equiv [x - (\sqrt{y} + \sqrt{z})^2] [x - (\sqrt{y} - \sqrt{z})^2] = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz
\]

we have for the limits on \( t \),

\[
t^\pm = \frac{1}{2s} \left[ -s^2 + (m^2_1 + m^2_2 + m^2_3 + m^2_4)s - (m^2_1 - m^2_2)(m^2_3 - m^2_4) \right. \\
&\left. \pm \lambda^\frac{1}{3}(s, m^2_1, m^2_2) \lambda^\frac{1}{3}(s, m^2_3, m^2_4) \right].
\]

To get the limits on \( t_{13} \) in the corresponding radiative phase space we consider particle 4 to be composed of the true particle of rest mass \( m_4 \) and of the photon. The rest mass of this fictitious kinematic composite is \( m_{4k} \) where \( m^2_{4k} = m^2_4 + 2k \cdot p_4 \) (since \( p^2_{4k} \equiv p^2_4 + k^2 \)).

The limits \( t_{13}^\pm \) are obtained by replacing \( m^2_1 \rightarrow m^2_{4k} \) and \( s \rightarrow s_{12} \) in the expression for \( t^\pm \) above;

\[
t_{13}^\pm(x) = \frac{1}{2s_{12}} \left[ -s_{12}^2 + (m^2_1 + m^2_2 + m^2_3 + m^2_4)s_{12} + xs_{12} \\
- (m^2_1 - m^2_2)(m^2_3 - m^2_4) + x(m^2_1 - m^2_2) \right. \\
&\left. \pm \lambda^\frac{1}{3}(s_{12}, m^2_1, m^2_2) \lambda^\frac{1}{3}(s_{12}, m^2_3, m^2_4 + x) \right]
\]

(12.1)
where we define \( z = 2k \cdot p_4 \). For \( t_{13} \) to be contained within the physical range of \( t \) (for \( s = s_{12} \)) we must have

\[
\begin{align*}
t_{13}^+(z) & \leq t^+ \quad \text{for} \quad 0 \leq z \leq z_{\text{max}}. \\
t_{13}^-(z) & \geq t^- 
\end{align*}
\]

The limit \( z_{\text{max}} \) comes when particle 3 and the fictitious \( 4k \) composite are produced at rest; \( m_{4k}^2 \) and hence \( z \) are maximized. The limit is explicitly \( m_{4k}^{\text{max}} = \sqrt{s_{12} - m_3} \) giving \( z_{\text{max}} = s_{12} - 2\sqrt{s_{12}m_3 + m_3^2} - m_4^2 \).

We first manipulate the form of the conditions on \( t_{13}^\pm(z) \) in order to clarify them:

\[
f^\pm(z) = \frac{2s_{12}}{\lambda^4(s_{12},m_3^2,m_4^2)\lambda^4(s_{12},m_3^3,m_4^3)} \frac{z}{(s_{12} + m_3^2 - m_4^2)} \pm \left\{ \frac{\lambda^4(s_{12},m_3^3,m_4^3 + z)}{\lambda^4(s_{12},m_3^3,m_4^3)} - 1 \right\}
\]

we note that

\[
\frac{\lambda(a,b,c+z)}{\lambda(a,b,c)} = 1 + \frac{z^2}{\lambda(a,b,c)} - \frac{2z(a+b-c)}{\lambda(a,b,c)}
\]

and define the quantities

\[
X = \frac{z}{\lambda^4(s_{12},m_3^3,m_4^3)}
\]

\[
c_{12} = \frac{s_{12} + m_3^2 - m_4^2}{\lambda^4(s_{12},m_3^3,m_4^3)}
\]

\[
c_{34} = \frac{s_{12} + m_3^2 - m_4^2}{\lambda^4(s_{12},m_3^3,m_4^3)}
\]

giving us the simplified result

\[
f^\pm(X) = Xc_{12} \pm \left\{ [1 + X^2 - 2Xc_{34}]^{\frac{1}{2}} - 1 \right\}.
\]

The condition that must be satisfied is now

\[
\begin{align*}
f^+(X) & \leq 0 \quad \text{for} \quad 0 \leq X \leq X_{\text{max}} = \frac{(\sqrt{s_{12} - m_3})^2 - m_4^2}{\lambda^4(s_{12},m_3^3,m_4^3)}. \\
f^-(X) & \geq 0
\end{align*}
\]

In a moment we shall prove that

\[c_{12} > 1 \quad \text{and} \quad c_{34} > 1 \quad \text{for} \quad m_1, m_2, m_3, m_4 \text{ non-zero}\]

\[1 + X^2 - 2Xc_{34} > 0 \quad \text{for} \quad 0 \leq X < X_{\text{max}}.\]
With these two facts in hand our argument proceeds as follows: we compute the derivatives of $f^\pm(X)$,

\[
\frac{df^\pm}{dX}(X) = c_{12} \mp \frac{c_{34} - X}{[1 + X^2 - 2Xc_{34}]^{\frac{1}{2}}}
\]

\[
\frac{d^2f^\pm}{dX^2}(X) = \mp \frac{c_{34} - 1}{[1 + X^2 - 2Xc_{34}]^{\frac{3}{2}}}.
\]

At $X = 0$, $f^\pm(0) = 0$ and $df^\pm/dX(0) = c_{12} \mp c_{34}$. Also, since $c_{34} > 1$ and $[1 + X^2 - 2Xc_{34}] > 0$, we have that $d^2f^+/dX^2(X) \leq 0$ and $d^2f^-/dX^2(X) \geq 0$ in the full range of $X$.

- For $f^-(X)$: at $X = 0$, $f^-(0) = 0$ and $df^-/dX(0) = c_{12} + c_{34} > 2$. Since $d^2f^-/dX^2(X) > 0$ we have that $f^-(X)$ is positive in the full range of $X$.

- For $f^+(X)$: If we assume that $c_{34} \geq c_{12}$ then we have the results, $f^+(0) = 0$ and $df^+/dX(0) = c_{12} - c_{34} \leq 0$. Since $d^2f^+/dX^2(X) < 0$ we then have that $f^+(X)$ is negative in the full range of $X$.

Thus so long as the condition $c_{34} \geq c_{12}$ is satisfied we have our desired result; that the physical range of the radiative variable $t_{13}$ is enclosed within the range $[t^-, t^+]$ for the corresponding non-radiative variable $t$.

We now go back to prove the assertions we made above:

- $c_{12} > 1$, $c_{34} > 1$ for masses non-zero: We simply note the relation of these quantities to centre-of-mass frame particle velocities in the non-radiative process. We write,

\[
c_{12} = \frac{s_{12} + m_1^2 - m_2^2}{\lambda^2(s_{12}, m_1^2, m_2^2)}
\]

\[
= \frac{s_{12} + m_1^2 - m_2^2}{2\sqrt{s_{12}}} \cdot \frac{2\sqrt{s_{12}}}{\lambda^2(s_{12}, m_1^2, m_2^2)}
\]

\[
= \frac{E_{1CM}^{CM}}{p_{1CM}^{CM}}
\]

\[
= \frac{1}{v_{1CM}^{CM}}
\]

where $E_{1CM}^{CM}$, $p_{1CM}^{CM}$ and $v_{1CM}^{CM}$ are the centre-of-mass frame energy, magnitude of 3-momentum and the velocity of particle 1. Similarly we find that $c_{34} = 1/v_{2CM}$. For masses non-zero, these velocities are less than 1 (in units of $c$); hence $c_{12} > 1$ and $c_{34} > 1$. 
- \( 1 + X^2 - 2X c_{94} > 0 \) for \( 0 \leq X < X_{\text{max}} \): This parabola has the value 1 for \( X = 0 \); thus it remains to show that the curve does not have its first zero crossing before the point \( X = X_{\text{max}} \). This first zero of the curve occurs for \( X = c_{94} - \sqrt{c_{94}^2 - 1} \).

It is simple to show that this is precisely \( X_{\text{max}} \), by writing each in terms of \( s_{12}, m_3 \), and \( m_4 \). Hence the function decreases from 1 to 0 as \( X \) goes from 0 to \( X_{\text{max}} \), and the result is proved.

An identical argument may be performed for \( t_{24}(x) \) by exchanging 1 \( \leftrightarrow \) 2 and 3 \( \leftrightarrow \) 4 throughout. Finally then we conclude that \( t_{13}(x) \) and \( t_{24}(x) \) are contained within the allowed range of non-radiative \( t \) (for \( s = s_{12} \)) if the conditions \( v_1^{CM} \geq v_3^{CM} \) and \( v_3^{CM} \geq v_4^{CM} \) are upheld. By considering these conditions on the centre-of-mass frame velocities at the threshold for the interaction (that is, when \( s_{12} \) equals the larger of \( (m_1 + m_2)^2 \) and \( (m_3 + m_4)^2 \)), we find that they both reduce to a simple condition on the masses of the scattering particles

\[
(m_0 + m_4) \geq (m_1 + m_2).
\]

Part (2):

Let the variables \( y_i \) \( (i = 1, \ldots, n) \) lie within the range \([y^-, y^+]\). We define a new variable \( Y = \sum_{i=1}^{n} a_i y_i \), with the constraint \( \sum_{i=1}^{n} a_i = 1 \). A sufficient condition to ensure that \( Y \) also lies in the same range is that \( 0 \leq a_i \leq 1 \), \( i = 1, \ldots, n \).

The proof goes as follows: We define \( Y = \sum_{i=1}^{n} a_i y_i \), where \( \sum_{i=1}^{n} a_i = 1 \), and impose on the variables \( y_i \) the conditions \( y^- \leq y_i \leq y^+ \) for \( i = 1, \ldots, n \). If we state that the coefficients \( a_i \) must be positive we have

\[
a_i y^- \leq a_i y_i \leq a_i y^+ \quad \text{for } i = 1, \ldots, n.
\]

Summing on \( i \) we find

\[
\left( \sum_{i=1}^{n} a_i \right) y^- \leq \left( \sum_{i=1}^{n} a_i y_i \right) \leq \left( \sum_{i=1}^{n} a_i \right) y^+ \quad \Rightarrow \quad y^- \leq Y \leq y^+.
\]

The constraints \( \sum_{i=1}^{n} a_i = 1 \) and \( a_i \geq 0 \) also imply that \( a_i \leq 1 \). Thus we have our result.

We use this theorem to show that \( t_{a} = (1 - a)t_{13} + at_{24} \) lies in the range of non-radiative \( t \) so long as \( t_{13} \) and \( t_{24} \) lie in that same range. Thus we have extended the results of Part (1) above from two discrete cases to a larger class of radiative variable pairs, \((s_{12}, t_{a})\). The condition that the coefficients sum to 1 just ensures that \( t_{a} \) reduces to \( t \) as \( k_{\mu} \to 0 \).

These choices of radiative variables lie within the non-radiative physical region so long as the sum of the final state particles' masses is greater than the sum of the initial state masses. This condition excludes interactions where we have heavy particles.
scattering to light, such as $p\bar{p} \rightarrow \pi^+\pi^-$. For such cases, we know of no suitable choice of variable pair to be used as an expansion point in the construction of soft photon amplitudes. Interactions where light particles scatter to heavy, or where we have elastic scattering, give rise to points $(s_{12}, t_a)$ which are always within the physical non-radiative region. This is useful since elastic scattering processes, particularly pion-nucleon and nucleon-nucleon bremsstrahlung interactions, dominate the literature of the soft photon approximation.

12.2 \((\vec{s}, \vec{t})\)

For identical particle scattering the radiative \((\vec{s}, \vec{t})\) region is identical to the physical non-radiative \((s, t)\) region. This is due to the fact that the Mandelstam variables $s, t, u$ of a non-radiative identical particle scattering process satisfy the same phase space constraints as the $\vec{s}, \vec{t}, \vec{u}$ of the corresponding radiative process.

For the non-radiative process we have the familiar constraints for equal mass, two-body elastic scattering

\[
\begin{align*}
    s + t + u &= 4m^2 \\
    s &\geq 4m^2 \\
    0 &\geq t, u &\geq \frac{1}{s} \lambda(s, m^2, m^2).
\end{align*}
\]

For the radiative process, as we previously noted in Eq. (11.5) above, we can use four-momentum conservation to write

\[
(k^\mu)^2 = (p_{i1}^\mu + p_{i2}^\mu - p_{i3}^\mu - p_{i4}^\mu)^2
\implies \vec{s} + \vec{t} + \vec{u} = 4m^2.
\]  

(12.2)

We also have the constraints,

\[
\begin{align*}
    \vec{s} &\geq 4m^2 \\
    0 &\geq \vec{t}, \vec{u} &\geq \frac{1}{\vec{s}} \lambda(\vec{s}, m^2, m^2).
\end{align*}
\]

(12.3)

The threshold condition on $\vec{s}$ is clear, however the $\vec{t}, \vec{u}$ constraints require some thought. It may be seen from Eq. (12.1) that for identical particle scattering the variables $t_{18}$, and symmetrically $t_{24}$, have zero as their upper bounds. Thus the average $\vec{t} = \frac{1}{2}(t_{18} + t_{24})$ is bounded above by zero. Putting $\vec{t} \leq 0$ into Eq. (12.2) we find the lower bound for $\vec{u}$; $\vec{u} \geq -\lambda(\vec{s}, m^2, m^2)/\vec{s}$. Finally, noting that the constraints on $\vec{t}$ and on $\vec{u}$ must be the same for identical particle scattering through the symmetry of the kinematics under the exchange of final state particles, we have the result of Eq. (12.3).
Thus for identical particle scattering, all points \((s, t)\) defined by radiative phase space constraints lie within the allowed \((s, t)\) region of the corresponding non-radiative process. From Chapter 11, we recall also that the choice \((s, t)\) allows one to anti-symmetrize the radiative amplitude correctly while still expressing this amplitude solely in terms of measurable non-radiative data.

The success of the \((s, t)\) expansion point is limited to identical particle scattering and does not carry over to the case of unequal particle masses. Certain calculations in the literature (Ref. [83] and Ref. [84]) have employed the \((s, t)\) choice of expansion point for the analysis of the processes \(\pi^\pm p \rightarrow \pi^\pm p\gamma\). Strictly, such soft photon amplitudes cannot be applied to the full kinematic region for the radiative process without extrapolating the non-radiative amplitude outside of its measurable region. Practically though the region of the radiative variable pair \((s, \bar{t})\) falls only very slightly outside of the physical region of the non-radiative variable pair \((s, t)\) in the case of low energy \(\pi^\pm p \rightarrow \pi^\pm p\gamma\) scattering. The phase space problem is clearly not so much of a practical concern here as it was in the case of \(pp\) bremsstrahlung calculated using the \(TsTts\) amplitude, which we analysed in §11.6.

In conclusion, in this section and the preceding one, we have presented choices of radiative variable pair which generate soft photon amplitudes requiring only physically measurable non-radiative data, for most two-body scattering processes. We do not claim that these are the only valid choices of expansion point—there may well be others which can be proven to lie within the physical non-radiative phase space region, for certain masses of the scattering charged particles. The two choices presented here are simply those for which we have been able to build an analytic argument, rather than relying on numeric trial-and-error to demonstrate their validity. In addition, the only types of interaction considered in this chapter are two body scattering processes. With a greater number of final state particles one would again have to ensure that the set of radiative variables chosen as the soft photon expansion point map out a region of phase space contained within that for the physical, on-shell non-radiative process. Practically, such checks on interactions having more than two final state particles would probably best be done numerically for the regions of radiative phase space in which one is interested.

---

1 In Ref. [83] the variables \(\nu \equiv p_1 \cdot p_2 + p_3 \cdot p_4\) and \(\Delta \equiv p_1 \cdot p_3 + p_2 \cdot p_4\) were used (our notation for the four-momenta). These are identical to the variables \(s\) and \(-\bar{t}\) except for the subtraction of some irrelevant mass terms.

2 The authors of Ref. [84] recognise the fact that the choice of expansion point in the construction of the soft photon expression is not unique. In addition to using the \((s, \bar{t})\) point, they constructed an expression using the expansion point \((s_{12}, t_{12})\) (in our notation) where particles 1 and 3 are the incoming and outgoing pions. As we demonstrated in §12.1 this latter expression would be free of the phase space problem. We note that they also chose to use only the leading term of the soft photon expansion in their results, calling this the “external-emission dominance” approximation.
Chapter 13

Conclusions

In what follows we will summarize the work performed in this thesis on the construction of useful soft photon amplitudes.

We have seen that many different soft photon amplitudes may be written down, as is shown in our formalism of Chapter 9, but that only a very few choices of expansion point give rise to useful amplitudes. Several problems were encountered in the construction of soft photon amplitudes and in their application to particular processes.

1. Amplitudes constructed using multiple expansion points can contain, as part of their internal radiation contribution, terms of the form $a.k/b.k$ where $a^\mu$ and $b^\mu$ are linear combinations of the charged particle four-momenta. Such terms could not have arisen within a full perturbative calculation of the radiative process and so the soft photon amplitudes containing them should be deemed illegal.

2. Many choices of expansion point have domains which extend outside of the physical region of non-radiative phase space. This limits the usefulness of the resulting soft photon amplitudes. Upon application to a bremsstrahlung process we would be forced to extrapolate the non-radiative amplitude sometimes very far outside of its measurable region. This is unsatisfactory since the most significant use of the soft photon approximation is to extract the behaviour of the non-radiative amplitude in off-shell and unphysical regions.

3. For identical particle scattering we have found that certain soft photon amplitudes advocated recently in the literature cannot be symmetrized or anti-symmetrized, as appropriate, while maintaining contact with measurable non-radiative scattering data.

In Chapter 12 we discussed expansion points which would give rise to soft photon amplitudes free of the above-noted problems. One may use a single expansion point at
(s_{12}, t_{2a}) (where \( t_{2a} = (1 - a)t_{13} + a t_{24} \) for \( 0 \leq a \leq 1 \)) to develop a satisfactory amplitude for two body scattering processes so long as the sum of initial state particle masses is less than or equal to the sum of final state masses. For identical particle scattering it has been proven that a single choice of expansion point at \((\vec{s}, \vec{t})\) is valid. Fortunately, the resulting 'Low' amplitude has been used in much of the nucleon-nucleon bremsstrahlung literature [78, 79, 93]. This previous work has not, therefore, been invalidated by our study.

Before closing we wish to make one further point. The argument has been made by the authors of Refs. [61, 62] that one can systematically improve the fit of the soft photon approximation to data by making certain choices of expansion points—namely, the ones giving rise to their Two-s-Two-t-special and Two-u-Two-t-special amplitudes. These soft photon amplitudes have been shown in this thesis to have problems with their relation to the measurable non-radiative amplitude and with their correct symmetrization or anti-symmetrization. Even disregarding these difficulties it is our opinion that such improved agreement is coincidental. It seems hardly surprising that some choice of soft photon amplitude would happen to fit a particular set of data rather well—to be of value one would have to establish a physically motivated procedure for constructing the optimal amplitude for a given bremsstrahlung process. Moreover, if one finds that the difference between the predictions of one soft amplitude and another is large, we feel that this should be taken as an indication that one is working in a region of phase space where the photon momentum is relatively large and hence any soft photon procedure gives an unreliable approximation to the data. In such regions, to improve one's understanding of the bremsstrahlung spectrum, one must introduce a model for the internal charge structure of the non-radiative scattering process. While the order-(\(k^\mu\)) and higher contributions of such structure dependent terms in the spectrum might be mimicked by a particular choice of soft photon amplitude, what is lacking is predictive power—in one case a clear understanding of the physical structure of the non-radiative scattering process would provide us with an improved agreement with experiment; in the other we would have a coincidental agreement, having no physical significance.

There have been previous attempts to construct photon amplitudes claimed to contain unambiguous information about the \(O(k^\mu)\) part of the radiative process, expressed purely in terms of the on-shell non-radiative amplitude. These were the so-called hard photon theorems of Haddock and Leung [99] who wrote the radiative amplitude as a finite difference expression in terms of the non-radiative. Picciotto [100] and later Grammer [101] showed that such theorems were still ambiguous in their construction at the \(O(k^\mu)\) level, and so added nothing to the usual soft photon treatment.
Part III

Soft Photon Excess in Hadron Scattering
Chapter 14

Introduction

In several high energy hadron scattering experiments [102, 103, 104, 105] soft photon rates have been found which are in excess of theoretical expectation. In a hadron scattering process having interaction energy $\sqrt{s}$ of several tens of GeV the vast majority of the measured photon spectrum comes from the two photon decay of neutral pions and from other radiative hadron decays. A smaller contribution also appears due to neutral hadrons like $K^0_S$ or $\pi$ faking a photon signal inside the detector. These hadronic sources of real and faked photons may be modelled within a Monte Carlo simulation and subtracted from the data. The remaining photon spectrum for small photon energies, $E_\gamma < 1$ GeV in the lab frame, and small transverse momenta, $p_T < 100$ MeV/c, is expected to arise due to bremsstrahlung from the incoming and outgoing charged hadrons. The soft photon theorem of Low [64], studied in Part II of this thesis, states that for small enough photon energy the spectrum must be given by this external particle radiation. The measured spectrum has been found to be a factor of around five (Refs. [102, 103]) to eight (Ref. [104]) times larger than the Low prediction. An excess of soft $e^+ e^-$ pairs has also been noted in several experiments [106, 107] and an anomalously large $e/\pi$ ratio—defined as the ratio of charged leptons to pions in the interaction's final state—has been seen at low lepton and pion energies in $\sqrt{s} = 63$ GeV $pp$ experiments [108, 109]. The source of these anomalies in soft lepton rates could well be related to that for the real soft photon rates, since soft (low energy and small invariant mass) virtual photons give rise to dilepton pairs.

In this work we will first review the body of experimental data, and make some general arguments about possible sources for these excess soft photons. The soft photon theorem of Low is very restrictive—for small enough photon energy this approximation must account for the full bremsstrahlung spectrum, in terms of the large distance behaviour of the scattering hadrons. We will then make general arguments about the
distance scales involved in the emission of the observed excess of soft photons.

We then review models from the literature which select particular charge carrying hadronic degrees of freedom as being the relevant ones—quarks and anti-quarks [110, 111], pions [112, 113], or colour-flux tubes [114]. The excess of soft photons and dilepton pairs could be the signature of interesting physics in the hadronization phase of scattering processes. Before conclusions may be drawn about these models, however, one must be confident that contributions from more conventional sources of soft photons have been well understood.

In chapter 16 we look at soft photon sources which have not previously been considered: the contribution of multiple photon emission, a possible interference effect between hadronic resonance decay amplitudes and bremsstrahlung amplitudes sharing the same final states, and the contribution of the higher order terms in the soft photon expansion. Finally in chapter 17 we summarize our results.
Chapter 15

Review

15.1 Experiment

The first measurement of direct photons in a high-energy hadron scattering process [115] involved \( \pi^+ p \) scattering, with incident pion lab momentum of \( p_{LAB}^{+} = 10.5 \) GeV/c (or \( \sqrt{s} = 4.54 \) GeV). The spectrum was measured in the momentum range \( 9 < (k_T, k'_L) < 20 \) MeV/c, where \( k_T \) is the photon transverse momentum and \( k'_L \) is its longitudinal momentum in the centre-of-mass frame of the interaction. This spectrum was found to be well accounted for by the expected \( \pi^0 \rightarrow \gamma\gamma \) hadronic decays and by hadronic bremsstrahlung computed using the soft photon theorem of LOW. As input to the calculation of the \( \pi^0 \rightarrow \gamma\gamma \) spectrum, \( \pi^0 \) production in the final state was assumed to be the same as the measured \( \pi^- \) production rate.

More recent \( \pi^+ p \) and \( K^+ p \) experiments at higher beam energies have all found an excess of soft photons over the LOW bremsstrahlung prediction, after subtraction of the hadronic radiative decay spectrum. Ref. [102] reported an excess factor of around \( 5 \pm 2 \) (my estimate from Fig. 2 of Ref. [102]) of soft photons over hadronic bremsstrahlung in \( K^+ p \) scattering at \( p_{LAB}^{+} = 70 \) GeV/c (\( \sqrt{s} = 11.5 \) GeV). The excess photon spectrum was observed in the region of photon kinematics with transverse momentum \( k_T < 60 \) MeV/c and with \( z \) in the range \(-0.001 < z < 0.008\). Given that the Feynman \( z \) variable is defined as \( z \equiv k_L/k_{MAX}^{+} \approx 2k'_L/\sqrt{s} \) (see, for example, Ref. [8, pp. 1292]) this corresponds to the range of CM frame longitudinal momentum \(-5 < k'_L < 45 \) MeV/c. The dependence of the measured photon spectrum on energy \( k \) after the subtraction of hadronic radiative decays is approximately \( 1/k \), the same as for the LOW hadronic bremsstrahlung prediction, but with a larger normalization. This characteristic spectrum shape is to be found in the other experiments where a soft photon excess has been observed.

The EHS-NA22 Collaboration [103] measured the direct soft photon spectrum in \( K^+ p \) and \( \pi^+ p \) interactions with \( p_{LAB}^{+} = 250 \) GeV/c (\( \sqrt{s} = 21.7 \) GeV for both meson...
beams). An excess of around $5 \pm 2$ (my estimate from their Fig. 7) was seen in both experiments for the spectrum as compared to the bremsstrahlung prediction, with hadronic radiative decays and neutral hadron fake events having been subtracted using Monte Carlo simulation as usual. The kinematic region of the excess was $k_T < 40$ MeV/c, $-0.001 < z < 0.010$ (corresponding to $-10 \leq k_L \leq 100$ MeV/c).

The SOPHIE-WA83 Collaboration [104] reported an excess factor of $7.9 \pm 1.4$ in $p_T = 280$ GeV/c ($\sqrt{s} = 22.9$ GeV) $\pi^- p$ interactions. The quoted kinematic region for the excess is $k_T < 10$ MeV/c and $0.2 < k_{LAB} < 1.0$ GeV where $k_{LAB}$ is the photon energy measured in the target proton's rest frame. This corresponds roughly to a region $10 \leq k_L \leq 45$ MeV/c.

In these experiments the background of hadronic radiative decays was generated using a Monte Carlo simulation. One might be concerned that weaknesses in this simulation could give rise to a false excess of soft photons by failing to predict the hadronic production rates accurately enough. Though the radiative decays dominate most of the photon spectrum, for the very low energy region the decay contribution falls to zero while the bremsstrahlung contribution rises like $1/k$. Bremsstrahlung is therefore expected on quite general grounds to completely dominate the very soft photon region. Any inaccuracies in the Monte Carlo program in this region would then be of little consequence. In addition it is clear in, for example, the SOPHIE-WA83 experiment that the simulation of hadronic decays and neutral hadron fake events fits the photon spectrum very well for higher photon energies (see Fig. 3 of Ref. [104]), where hadronic bremsstrahlung is expected to be negligible.

Another concern might be that the prediction of the bremsstrahlung spectrum is derived from an application of the leading order part of the Low theorem to a Monte Carlo simulation of the final state charged hadron distribution. The measured soft photon spectrum is so much in excess of the resulting hadronic bremsstrahlung prediction, though, that it seems most unlikely the soft photon excess could result from an inaccurate Monte Carlo model. The Low theorem soft photon production rate varies in an approximately linear fashion with the final state charged hadron multiplicity—any model of the final state which could under-predict the soft photon spectrum by the large factor measured would also under-predict the average number of final state charged hadrons by the same amount. It seems highly unlikely that such a gross error could lie undiscovered in the Monte Carlo simulations used in these experimental analyses. As an additional check of this method, in the SOPHIE-WA83 experiment the bremsstrahlung rate was computed using measured final state charged tracks as input [116]—this rate and the rate derived from the FRITIOF [117] Monte Carlo were found to agree to within a few
percent.

The results of proton-nucleus scattering [118, 119] might seem, at first sight, to contradict the π±p and K±p experiments discussed above. The HELIOS Collaboration [119] conducted p-Be scattering at \( p_{LAB} = 450 \) GeV/c and set a 90% upper confidence limit on the production of soft photons in excess of the hadronic bremsstrahlung prediction. For \( k_T < 3 \) MeV/c the limit on the excess is around 30%-50%, while for \( 3 < k_T < 10 \) MeV/c the limit is around 100%-150%. This limit is much smaller than the excess spectrum found in the previously mentioned experiments. As we shall see in our discussion in the following section, however, these low-\( k_T \) results are not necessarily in conflict with the other data. An upper confidence limit is set with this data, rather than an actual measurement of the spectrum being given, because of the great problem of correctly subtracting the photon spectrum due to π⁰ decays for nuclear target interactions. According to Ref. [119] this problem worsens with increasing \( k_T \) and it therefore seems unlikely that the results of these proton-nucleus interactions could be extended into the region \( 10 \leq k_T \leq 50 \) MeV/c measured by the π±p and K±p experiments. These p-Be results do not confirm the excess of soft photons previously claimed in a smaller data sample [120]. The previous results found an excess factor which varied with the square of the hadronic multiplicity.

For completeness we mention the excess soft photon signal observed in the European Muon Collaborative experiment [105] which studied \( \mu^\pm p \) interactions at \( p_{LAB} = 200 \) GeV/c. This signal was found to be the same, within experimental errors, for \( \mu^+ \) and \( \mu^- \) projectiles and seemed correlated to the direction of the struck parton in the proton. Our discussion will hereafter be limited to hadron-hadron interactions.

A possibly related phenomenon to the soft photon excess is the excess of low invariant mass and low energy dilepton (e⁺e⁻) pairs seen in π±p interactions [106, 107] and the large \( e/\pi \) ratio at low energy measured in pp interactions [108, 109]. The dominant source for such soft leptons is expected to be the conversion to dilepton pairs of virtual photons arising from hadronic bremsstrahlung; see, for example, Ref. [121]. We shall not discuss these excess soft lepton rates further except to note the interesting observation at the ISR [109] that the low transverse momentum part of the \( e/\pi \) excess appears to vary like the square of the charged particle rapidity distribution—that is, \( e/\pi \propto (dN_{ch}/dy)^2 \). This is of interest because the dependence upon this charged hadron rapidity distribution in the case of excess soft photon, rather than dilepton, production may be used as a discriminant between the various models of soft photon production to be reviewed in §15.3.
In definition of rapidity is

\[ y = \frac{1}{2} \ln \left( \frac{E + p_t}{E - p_t} \right) \]

where \( E \) and \( p_t \) are the particle's energy and its momentum along the beam direction. Rapidity \( y \) and transverse momentum \( p_T \) are rather more useful variables to use in describing the momenta of final state particles in high energy hadron interactions than are the magnitude of three-momentum \( |\vec{p}| \) and polar angle \( \theta \). Final state particles in hadron interactions tend to have almost flat distributions in rapidity and an exponentially falling distribution with increasing \( p_T \). These distributions are empirically found to be almost independent of interaction energy. In addition, rapidity and transverse momentum have particularly simple Lorentz transformation properties under a boost along the beam axis,

\[
y \rightarrow y + \tanh^{-1}(\beta) \quad \text{and} \quad p_T \rightarrow p_T
\]

which proves to be useful when one models the hard scattering of partons within the hadrons. Here \( \beta \equiv v/c \) is the velocity characterizing the boost. In contrast the density of final state hadrons in a high energy hadron scattering process is very unevenly distributed in polar angle \( \theta \) and in the magnitude of three-momentum \( |\vec{p}| \).

### 15.2 Qualitative discussion

In this section we shall make some qualitative observations about the source of the measured soft photon excess. We may estimate the distances scales appropriate to this problem by assuming that photons emitted in the hadron scattering can couple to, and so could have arisen from, only charged particle scattering processes occurring with characteristic distances scales greater than the photon wavelength. Thus, for example, the photon spectrum around energies of \( \sim 10 \) MeV, and hence wavelengths of \( \sim 120 \) fm, will be defined through the soft photon theorem entirely in terms of the long distance, freely propagating behaviour of the initial and final state hadrons. The photons constituting this part of the spectrum cannot depend directly on the short-distance dynamics of the scattering process. For small enough photon energy the bremsstrahlung spectrum must be given by the soft photon theorem—the measurement of an excess of soft photons in the 10–100 MeV region of \( \text{CM} \) frame energies does not change this fact. What it does tell us is that the boundary in photon energy below which the soft photon theorem holds sway does not lie at several hundred MeV, but instead must be located at less than a few tens of MeV. We also note that the approximate \( \text{CM} \) frame photon energy regions seen by the \( \pi^\pm p/K^+p \) experiments, which find an excess of soft photons, and by the p-Be
15.2 Qualitative discussion

experiment, whose results are consistent with hadronic bremsstrahlung, are distinct and so the experiments are not necessarily in conflict with one another.

Del Duca [122] recently considered the region of applicability for the soft photon theorem in high energy hadron interactions and argued that this region becomes vanishingly small in hadron scattering processes as the interaction energy increases. The region of validity in terms of the CM frame photon energy is suggested to be $k^* \leq m^2/\sqrt{s}$, where $m$ is a characteristic hadron mass; perhaps the pion or nucleon mass. For $\sqrt{s} \approx 10$ GeV and $m \approx m_\pi$ this would limit the region of validity to $k^* \leq$ a few MeV. If this argument is correct then it could account for the accommodation of the p-Be data by the Low prediction, since it has $k_T^* < 10$ MeV/c, while still allowing for the excesses seen in the higher $k^* \pi^+ p/K^+ p$ data. Conversely, the $\pi^+ p/K^+ p$ excesses in the photon spectrum could be viewed as an argument in favour of Del Duca's result—above $k^* \approx m_\pi^2/\sqrt{s}$ contributions to the measured photon spectrum would be allowed from sources other than hadronic bremsstrahlung.

In a recent paper Li chard [123] investigated the experimental soft photon excesses by considering them to be an admixture of Low theorem bremsstrahlung, the spectrum of the modified soft annihilation model (MSAM) of Ref. [124], and the spectrum of the cold quark-gluon plasma, or "glob" model [110]. He found that, except for photons of very low transversity, all of the data can be accommodated within this context, but that certain data do not require one or more parts of the combined model. He concluded that the experiments are at least not inconsistent with one another, but his findings do not indicate that this combination of models is any more than a sophisticated fitting procedure. The ultra-soft photon region, having $k_T^* \leq 10$ MeV/c, is under-predicted in general by this admixture model.

Using our qualitative argument that photons couple only to processes having distance scales greater than the photon wavelength, we can place limits on the source of the excess. Photons of energy less than 100 MeV in the interaction CM frame should couple to processes having distance scales greater than a few tens of fm. This is larger than the scale of processes which occur in conventional models of hadronization. A study of soft photon production within the context of the Lund string fragmentation model [125] finds that only a factor of 2–3 increase in the soft photon spectrum above the Low result can be accommodated within current models of hadronization. The authors state that regions of extent 10–15 fm within which the trajectories of the charged hadrons are correlated in the hadronic final state would be required in order to account for the measured soft photon excesses. In contrast, the strings which represent such hadronic correlations in the Lund model of hadronization typically travel only 1–2 fm before fragmenting.
From these qualitative arguments we can form a picture of the kind of models which could give rise to the photon excess. In the final state of the hadron interaction one must model multiple scattering processes over distance scales of tens of fm, with the charged hadronic degrees of freedom involved being close to on-mass-shell between scatterings. The on-shell requirement is necessary to give the observed bremsstrahlung-like $1/k$ distribution to the soft photon spectrum. In the following section we will review several models from the literature which seek to explain the soft photon excess—they differ in the charged hadronic degrees of freedom which they consider to be relevant, and in the processes which are assumed to be responsible for the multiple scattering of these objects over the required large distance scales.

15.3 Theory

Models in the literature which attempt to explain the excess of very soft photons over the hadronic bremsstrahlung expectation differ from one another primarily in the charged particles which they consider to be relevant in the large distance part of the strong interaction.

Lichard and Van Hove [110, 123, 126] have proposed the cold quark-gluon plasma or "glob" model. They consider the hadronic intermediate particle system, this being the system of particles present after the initial short distance strong scattering but prior to the resolution of the final state into observable hadrons, to be composed of clusters of quarks and gluons. Inside each cluster the partons are assumed to have very small relative momentum; less than around 50 MeV/c. The parton level processes $qg \rightarrow q\gamma$, $qg \rightarrow q\gamma$ and $q\bar{q} \rightarrow g\gamma$ are computed at lowest order in perturbation theory, giving rise to a spectrum of low energy photons. The authors are well aware that a lowest-order QCD calculation is unsatisfactory due to the largeness of the strong coupling constant $\alpha_s$ at such low momentum-transfer-squared. Despite this, and perhaps aided by the free parameters in the model, such as number of clusters per collision, number of partons per cluster and the cluster invariant mass, a reasonable fit to the 70 GeV/c $K^+p$ data of Ref. [102] has been obtained. From a recent analysis of all data by Lichard [123] it seems that this glob model is, however, unable to account for the soft photon data in the very lowest region of $k_\perp$ measured.

Botz, Habel and Nachtmann [111] recently considered a model treating the excess photon spectrum as being a kind of "synchrotron light" caused by accelerations of quarks and anti-quarks in the chromomagnetic fields of the QCD vacuum. They compute, in other words, the bremsstrahlung due to the deflection of charged partons in a potential, with the form of the potential inspired by the QCD gluon condensate. It is found
that, with suitable choice of parameters, their model can accommodate the p-Be data of Ref. [119]. Since this experiment found no statistically significant excess of soft photons over the expected hadronic bremsstrahlung and decay spectra one might conclude that this "synchrotron light" model gives only a tiny yield of soft photon radiation. A more challenging test of this model would be to compute its prediction for the π±p and K+p experiments, where very large soft photon excesses have been observed.

The previous two models have considered radiation at the quark level in the hadronic final state. In contrast, Barshay [112, 127] and Shuryak [113, 128] have each proposed models with charged pions being the radiating objects. Barshay considered a pion condensate, and found it to radiate photons in the transverse momentum range 4 \leq k_T \leq 20 \text{ MeV/c}. The radiation varies like the square of the number of final state charged particles rather than varying linearly, as does the Low soft photon spectrum. In Shuryak's model a dispersion curve was estimated for quasi-pions, whose behaviour has been modified by final state strong interactions. Due to this modified dispersion curve the quasi-pion paths fold over at the edge of a region of high hadron density. Bremsstrahlung may then be computed in a semi-classical fashion from these paths. The model used is very simple—a sphere within which the pions interact and outside of which they propagate freely—and has an unknown parameter, the characteristic time between rescatterings. The velocity of the pions was also varied but was argued to be between \frac{1}{2}c and c. A large excess of photons was found for slowly moving quasi-pions undergoing only a few boundary reflections. The result reverts to the hadronic bremsstrahlung prediction as photon energy is taken to zero, as required by the Low theorem.

In Ref. [114] Czyż and Florkowski computed bremsstrahlung within the boost-invariant colour-flux tube model. The predicted photon spectrum is distinguished from that of other models by its strong angular dependence, reaching a maximum factor of around 10 times the Low theorem rate for radiation emitted transverse to the beam axis in the interaction's CM frame.

In the following chapters we will consider the yield of soft photons from two sources. The first is multiple photon emission, wherein each hadron scattering event may give rise to more than one photon. The second considers the upper limit on the photon spectrum due to bremsstrahlung amplitudes interfering constructively with diagrams containing a radiatively decaying hadronic resonance.
Chapter 16

Possible Soft Photon Sources

16.1 Multiple photon emission

A potential source of soft photons not considered in the experimental analyses is multiple photon emission. It is possible for two or more photons to be emitted during a single hadron scattering event. We would expect these additional emissions to have a small probability since each extra bremsstrahlung photon in an event brings with it a factor of order $\alpha/\pi \sim 1/137\pi$ to the cross section. It transpires that the cross sections for any number of emitted photons can, within certain approximations, be summed to give an exponential form for the total rate.

A complication that arises when calculating the measured differential cross section with respect to photon energy is that one must integrate over the energies of all but one of the photons. Due to photons' zero rest mass, low energy or infrared (IR) divergences appear during these integrations. The divergences are cancelled by the order-$\alpha$ radiative corrections to the charged particles' propagators leaving a small finite remainder. As has been shown by Yennie, Frautschi and Suura (YFS) \cite{129} and later in a different fashion by Grammer and Yennie \cite{130}, these cancelling divergences can also be summed over any number of soft photon emissions.

The final result will be a product of two exponentiated terms, one from the unobserved, cancelling soft photon divergences and one from the observed multiple photon emissions. The dividing line between these two regimes is at the energy resolution of one's experimental detector. The theoretical calculation and the experimental set-up are thus intimately linked. In the analysis of a particular experiment the determination of energy resolution as a function of direction is a crucial precursor to the calculation of multiple photon emission corrections. The resulting complicated calculation can only be carried out using a Monte Carlo integration program—examples are the YFS2/3 (Refs. \cite{131, 132}) and the BHLUMI (Ref. \cite{133}) programs in use at the LEP and SLC.
Multiple photon emission

$e^+e^-$ colliders. These routines respectively compute multiple photon corrections to the $Z^0$ resonance line shape and radiative corrections to the low energy $B$ scattering rate. In our analysis we shall consider only the simplest case, where energy resolution is independent of direction in the interaction's centre-of-mass frame and we have a straightforward division between the unobserved and the observed regions of the photon spectrum.

We consider multiple photon emission from an arbitrary number of initial and final state charged particles. The amplitude for emission of a photon is taken to be that of the lowest order soft photon approximation. Thus if the amplitude for the non-radiative process is

$$M_{12-3...n} \delta^4 \left( \sum_{i=1}^{n} \eta_i p_i^\mu \right)$$

then the amplitude for the corresponding radiative process with $m$ bremsstrahlung photons is

$$\prod_{j=1}^{m} \left( \sum_{i=1}^{n} \eta_i (eQ_i p_i^\mu / p_i \cdot k_j) \right) M_{12-3...n} \delta^4 \left( \sum_{i=1}^{n} \eta_i p_i^\mu + \sum_{j=1}^{m} k_j^\mu \right).$$

Here the $p_i^\mu$ and $Q_i$ are the particle four-momenta and their charges in units of positron charge $e = \sqrt{4\pi\alpha}$, while $\eta_i = \pm 1$ for initial state particles. One would expect the emission of very soft photons to leave the trajectories of the charged hadrons almost unchanged from the case of no bremsstrahlung. This approximation will be made by ignoring the term $\sum_{j=1}^{N} k_j^\mu$ in the momentum conserving $\delta$-function. Each photon then becomes kinematically decoupled from the others and from the particles involved in the original non-radiative scattering process. The differential cross section for the original process is

$$d\sigma_{\text{orig}} = \frac{1}{2\lambda^3(s, m_i^2, m_j^2)(2\pi)^{3n-10}} \left( \prod_{i=3}^{n} \frac{d^2 \vec{p}_i}{2E_i} \right) \delta^4 \left( \sum_{i=1}^{n} \eta_i p_i^\mu \right) \sum_{\text{SPINS}} |M_{12-3...n}|^2,$$

while the cross section for the $m$-photon emission process, under the approximation of kinematic decoupling mentioned above, is

$$d\sigma_m = d\sigma_{\text{orig}} \frac{1}{N!} \left( \prod_{i=1}^{N} \frac{d^2 \vec{k}_i}{2E_k(2\pi)^3} \right) (-4\pi\alpha)^N \left[ \prod_{j=1}^{N} \left( \sum_{i=1}^{n} \eta_i eQ_j p_j^\mu / p_j \cdot k_i \right) \right]^2.$$ 

The factor $1/N!$ in the expression above compensates for over-counting of the photon spectrum due to our explicit labelling of the physically indistinguishable photons. Strictly speaking it should be present only once the photon angular integrals have been carried out, since this is the stage at which the over-counting of identical particles occurs.
Since the experimental photon spectrum is measured without any knowledge of whether the radiation comes from a one, two or many photon emission event, we must convert the calculated cross section into an effective single photon spectrum $d\sigma/dk$. Due to the indistinguishability of the photons it turns out that this spectrum is identical to $N$ times the spectrum for one of our $N$ emitted photons, where the remaining photons' phase space has been integrated over. Here we run into the problem mentioned earlier—integration over a full photon spectrum gives a result which has the behaviour

$$\int_{k_{\text{min}}}^{k_{\text{max}}} \frac{dk}{k} = \ln \left( \frac{k_{\text{max}}}{k_{\text{min}}} \right).$$

This logarithm diverges as $k_{\text{min}}$ is taken to zero. We therefore choose to integrate only within the experimentally observable region denoted by $[K_a, K_b]$. As mentioned above, the limit $K_a$ is taken to be independent of the centre-of-mass frame photon angle—in the analysis of a specific experiment the calculation of this experimental resolution as a function of direction would be crucial.

As demonstrated in Refs. [129, 130], the IR divergences arising from unobserved soft photons with CM frame momenta less than $K_a$ and those from radiative corrections to the scattering process may be shown to cancel to all orders in perturbation theory. It transpires that, within the limits of our approximation that the soft photons' energy be ignored in the phase space four-momentum constraint, the small finite remainder after cancellation of IR divergences is the same for multiple photon and single photon emission. Thus in the quantity of interest to us, which is the ratio of the multiple and single photon emission rates, the finite correction due to a proper treatment of IR divergences vanishes.

Upon integration over the phase space of the $(N-1)$ observed photons we have

$$\frac{d}{dk} d\sigma_N = d\sigma_o \frac{N}{N!} \left( \frac{\alpha A}{\pi} \right)^N \frac{1}{k} \ln(K_b/K_a)^{N-1}$$

where

$$A(p_1, \ldots, p_n) = (-1)^{n-1} \frac{d\Omega_k}{4\pi} \left| \sum_{i=1}^n \gamma_i e Q_i \frac{p_i}{p} \right|^2.$$  \hspace{1cm} (16.1)

Here we have defined $\tilde{k}^\mu = (1, \tilde{k})$, with $\tilde{k}$ being a unit three-vector in the direction of the solid angle element $d\Omega_k$. The expression for $A$ may be integrated exactly over photon emission angle—the calculation is given in App. C.

Summing over any number of emitted photons we have

$$\frac{d}{dk} d\sigma = \sum_{N=1}^{\infty} \frac{d}{dk} d\sigma_N$$
16.1 Multiple photon emission

The process $\pi^- p \rightarrow \pi^- p \gamma$ is studied at incident pion momentum of 280 GeV/c. Photon detectors are assumed to be perfect in the centre-of-mass photon energy range $[K_a, K_b]$. For this elastic channel multiple photon emission is a very small effect.

\begin{equation}
\frac{d\sigma_n}{k} \left( \frac{\alpha A}{\pi} \right) \sum_{N=1}^{\infty} \frac{1}{(N-1)!} \left( \frac{\alpha A}{\pi} \ln(K_b/K_a) \right)^{N-1}
= \frac{d\sigma_n}{k} \left( \frac{\alpha A}{\pi} \right) \exp \left[ \frac{\alpha A}{\pi} \ln(K_b/K_a) \right]
\Rightarrow \frac{d}{dk} d\sigma = \frac{d\sigma_n}{k} \left( \frac{\alpha A}{\pi} \right) \left( \frac{K_b}{K_a} \right)^{\langle q^A \rangle}. (16.2)
\end{equation}

Notice that we still use the symbol $d\sigma_n$ instead of $\sigma_n$ for the non-radiative cross section - this indicates that the charged particle phase space has yet to be integrated. Since $A$ is a function of the charged particle four-momenta, the charged particle phase space must be integrated numerically, which we do by Monte Carlo methods.

16.1.1 Elastic channel—$\pi^- \nu \rightarrow \pi^- p(\gamma)$

In Fig. 16.1 we present the results of a Monte Carlo integration for an example process, $\pi^- p \rightarrow \pi^- p \gamma$ at incident pion momentum of 280 MeV/c. Plotted is the percentage increase in the differential cross section with respect to photon energy for multiple photon emission over single photon emission, computed using Eq. 16.2. The result is shown as a function of the ratio of maximum to minimum measurable CM frame photon energy, $K_b/K_a$. The effects are seen to be less than around 2% for credible detector resolutions. A fitted form for the $\pi^- p$ elastic scattering differential cross section was required for this calculation. The $t$-dependence was taken to be $d\sigma_{\pi^- p}^{\ell}\!/d|t| \sim \exp(-b|t|)$ with slope...
parameter $b = 10.8$ GeV$^{-2}$ taken from Table 7 of Ref. [134].

This very small multiple photon emission rate is somewhat surprising when one considers the form of the electro-magnetic factor $\mathcal{A}$ for $\pi^- p$ elastic scattering as given in Fig. C.1 of App. C. This factor tends to take on values of order $\sim 30$ at this interaction energy. This would imply that the term $(K_s/K_a)^{(\alpha A/\pi)}$, which determines the ratio of multiple to single soft photon emission, should be around $\sim 1.5 \rightarrow 2.0$. Instead we have found it to be no greater than 1.02. This explanation for this lies in the very forward peaked nature of the $\pi^- p$ elastic scattering process. In this kinematic region, where the cosine of the scattering angle tends to be extremely close to 1, it can be seen that the factor $\mathcal{A}$ rapidly approaches zero, leading to the much-suppressed ratio of multiple to single photon emission.

16.1.2 Multi-pion channels—A simple model

At such high interaction energies as we consider here ($\sqrt{s} \sim 10-20$ GeV) the elastic channel forms only $\sim 14\%$ of the total $\pi^- p$ cross section. The ratio of multiple to single photon emission in which we are interested could be very much larger for the more typical inelastic channels, due to the large number of low-mass charged hadrons in the final state—dominantly pions. Using data from several high energy $\pi^- p$ interactions (Refs. [135, 136, 137, 138] for incident pion lab momenta of 50 GeV/c, 100 GeV/c, 147 GeV/c and 205 GeV/c respectively) Fong et al. [137] suggested that the average final state charged particle multiplicity is well fit by the function $<n_{ch}> = a + b \ln(s/s_0)$ where $s_0 = 1$ GeV$^2$, $a = -1.2 \pm 0.3$ and $b = 1.54 \pm 0.07$. For $p_T = 280$ GeV/c this would imply an average final state charged particle multiplicity of $<n_{ch}> \approx 8.5$. Assuming that these charged hadrons are predominantly pions, and that pion production is close to being isospin invariant, this would imply an average final state hadronic multiplicity about 50% larger; $<n> \sim 13$. In order to exclude multiple photon emission as a possible source for the measured soft photon excess one must consider such radiation within a model of hadron scattering including high multiplicity final states.

As can be seen from Eq. (16.2) the rate of multiple photon emission with respect to the single photon rate is determined by the factor $(K_s/K_a)^{(\alpha A/\pi)}$. In this section we examine the behaviour of the function $\alpha \mathcal{A}/\pi$ in the kinematic region of typical multi-pion hadron final states. The function $\mathcal{A}$ depends only upon the four-momenta and charges of the hadrons—we generate these within the context of a very crude model which mimics the general characteristics of multi-pion production.

The model consists of two hadrons in the initial state scattering with interaction energy $\sqrt{s}$ to an $n$-hadron final state. All hadrons are taken to have the same mass,
Figure 16.2: Rapidity distributions in a simple model of multi-pion production. Shown is the distribution in rapidity of 20000 hadrons, generated within the multi-pion production model described in the text. The rapidity distributions were generated for the cases where the hadrons form part of an n-pion final state—the number of final state pions n was varied between 2 and 25. The interaction energy was taken to be $\sqrt{s} = 20$ GeV. Each distribution is normalised to have the same area. The vertical scale is arbitrary.

We are, therefore, no longer dealing with the example of $\pi^-p$ bremsstrahlung. Rather we model the experimentally unrealistic case of $\pi\pi$ scattering at high energies. The particles' four-momenta are chosen such that the kinematic features of true multi-pion production are mimicked—that is, the rapidity distribution of the pions should be as flat as possible within the constraints of four-momentum conservation, the transverse momentum distribution should be of the form $\exp[-p_T/(\sim 350 \text{ MeV}/c)]$, and the pions should be evenly distributed in their azimuthal angles around the beam-line. Recall that rapidity is defined as $y = \frac{1}{2} \ln[(E + p_z)/(E - p_z)]$ where $E$ and $p_z$ are the particle's energy and beam axis momentum. Four-momentum and charge conservation are exactly respected in our model, while isospin invariance is approximately observed with around the same number of $\pi^+, \pi^0$ and $\pi^-$ being produced. The option is given for the initial state particles to be neutral ($\pi^0\pi^0$) or charged ($\pi^+\pi^-$)—the behaviour of $\alpha A/\pi$ differs
considerably between these two cases. The explicit algorithm used to generate four-momenta and charges for this $2 \rightarrow n$ pion interaction is as follows:

- For each of the $n$ final state particles we select at random their rapidity $y$, their azimuthal angle $\phi$ and their transverse momentum $p_\tau$.

$$\begin{align*}
\phi &= 2\pi r_1 \\
p_\tau &= -p_\tau^- \ln \left[ (1 - \exp(-p_\tau^\text{MAX}/p_\tau^-)) r_2 + \exp(-p_\tau^\text{MAX}/p_\tau^-) \right] \\
y &= (2r_3 - 1) \ln \left( \sqrt{s}/m_\tau \right) \quad \text{where} \quad m_\tau \equiv [m^2 + p_\tau^2]^{1/2}
\end{align*}$$

Here $r_1$, $r_2$ and $r_3$ are random variables evenly distributed over the interval $[0,1)$. The point where the $p_\tau$ distribution is taken to reach $1/e$ of its maximum value is at $p_\tau^- = 0.35$ GeV/c, while the maximum allowed value of $p_\tau$ is taken to be $p_\tau^\text{MAX} = 4 \times p_\tau^-$. The $p_\tau$ distribution is insensitive to this maximum value.
This gives azimuthal angle and rapidity evenly distributed over their physical regions, and transverse momentum exponentially distributed in the few hundred MeV/c region. The particle 3-momenta may then be computed in terms of these quantities.

\[ p_x = p_T \sin \phi \quad p_y = p_T \cos \phi \quad p_z = m_T \sinh y \]

- The resulting collection of 3-momenta do not satisfy momentum conservation with respect to the CM frame of the initial state, nor is the total energy of the particles guaranteed to equal the interaction energy \( \sqrt{s} \). We therefore perform a set of quite arbitrary manipulations of the momenta. The only justification for these manipulations is that they restore four-momentum conservation, and that the rapidity and transverse momentum distributions following the manipulations still mimic those of true multi-pion final states.

The origin of the 3-momenta is first translated such that they sum to zero; we thus have a suitable set of CM frame 3-momenta. This translation does not represent a boost of the momenta to the CM frame. It is merely an arbitrary manipulation which we will find gives a set of four-momenta with reasonable kinematic distributions. The \( p_z \) components are then scaled by a constant factor such that the sum of the particle energies equals \( \sqrt{s} \). Only the \( p_x \) components are scaled in this way in order that the \( p_x \) distribution be left relatively unchanged, as is experimentally observed. This has the effect of expanding or contracting the cone of final state hadron momenta about the beam axis.

These manipulations result in a set of CM frame four-momenta which obey energy-momentum conservation with respect to the initial state. The rapidity and \( p_T \) distributions of these modified vectors may be computed in order to confirm that they have the desired characteristics. Such distributions are shown in Figs. 16.2-16.3.

- Charge assignments are then made to the \( n \) final state pions such that the total charge is zero and that the populations of each charge state are roughly the same. For a given \( n \) the charge composition of a final state is always the same. This is unrealistic, but studying a single channel for each final state multiplicity in this way will make it possible to understand the behaviour of the function \( \alpha A / \pi \) using simple arguments about the pion kinematics.

For each \( n \)-pion final state the function \( \alpha A / \pi \) was then computed, using the expression of Eq. (C.1) derived in App. C. This random generation of final state kinematics
16.1 Multiple photon emission

Figure 16.4: Distribution of values for the function $\alpha A/\pi$ in our simple model, with initial state particles assumed neutral. Shown is the distribution of sampled values of the factor determining multiple soft photon emission rates. The kinematics of 20000 n-pion final states are sampled within the context of the model described in the text. The interaction energy was taken to be $\sqrt{s} = 20$ GeV. The distributions are normalised to have equal area, except for the single-valued $n = 2$ case.

was repeated many times, with the resulting values of $\alpha A/\pi$ being binned. Plots of these binned distributions for various final state pion multiplicities, $n$, are given in Fig. 16.4 for the case where the initial state particles are taken to be neutral, and in Fig. 16.5 for a charged initial state. These binned distributions indicate the importance of multiple photon production for n-pion final states, within the context of our model.

The $n = 2$ plot in Fig. 16.4 represents $\pi^0\pi^0 \rightarrow \pi^+\pi^-$ scattering within our model. Here the only kinematic degree of freedom for the system with interaction energy $\sqrt{s}$ fixed is the cm frame scattering angle. Since the initial state is uncharged the function $A$ is independent of this angle and depends only on the final state charges and on $\sqrt{s}$, and so is a constant. As the number of final state particles $n$, and hence the number of kinematic degrees of freedom, are increased the distribution of values for $\alpha A/\pi$ broadens. Typical values are in the region $\sim 0.05 \rightarrow 0.1$. If we take the detector resolution factor $K_b/K_s$
to be $\sim 1000$ the factor $(K_{b}/K_{a})^{(\alpha A/\pi)}$ appearing in the multiple photon production formula of Eq. (16.2) would have value $\sim 1.5 - 2.0$—recall that this factor reverts to one for single photon emission. In our simple model we have an indication therefore that the multiple photon emission factor can be large.

Between the single-valued distribution of the $n = 2$ case and the broad peaked structure seen for large $n$ ($n > 6$ or so), we find some interesting behaviour. Multiple peaks are seen in the $\alpha A/\pi$ distributions of the $n = 3$ and 4 cases. In the $n = 3$ curve, representing the channel $\pi^{0}\pi^{0} \rightarrow \pi^{+}\pi^{-}\pi^{0}$, we see two peaks—one at $\alpha A/\pi \sim 0.01$ and the other around $\sim 0.04$. Due to the empirical constraint that the transverse momentum of the produced hadrons be very small the configuration of the final state particles is essentially one-dimensional, along the beam-line or longitudinal direction. By analysing the four-momentum of the final state particles for events contained in each of the peaks we find the following. The $\sim 0.01$ peak corresponds to kinematics where the charged pions travel together in one direction along the beam-line, with the $\pi^{0}$ balancing their momentum in the opposite direction. The peak at larger $\alpha A/\pi$ corresponds to the situation where the charged pions have rapidity of opposite sign. The width of the peaks is generated due to the kinematic freedom in assigning longitudinal momentum to the neutral pion and, to a lesser extent, by variations in the transverse momentum components. The relative areas under the two peaks are seen to be in the approximate ratio 2 : 1. This can be understood by noting that there are three distinct situations for assignment of CM frame longitudinal momenta to the final state pions: $\pi^{+}$ and $\pi^{0}$ in the same direction, with $\pi^{-}$ opposed; $\pi^{-}$ and $\pi^{0}$ together, with $\pi^{+}$ opposed; or $\pi^{+}$ and $\pi^{-}$ in the same direction, with $\pi^{0}$ opposed. The first two have the charged particles with large relative longitudinal momentum and so contribute to the $\sim 0.04$ peak, while the third has the charges with small momentum separation and contributes to the $\sim 0.01$ peak, giving the approximate 2 : 1 ratio in areas.

The $n = 4$ channel studied in our model is $\pi^{0}\pi^{0} \rightarrow \pi^{+}\pi^{-}\pi^{0}\pi^{0}$. The same two peaks are seen, again corresponding to the final state charges travelling together or oppositely along the beam-line direction. The peaks are modified from the $n = 3$ case in that their relative areas have changed, and they are slightly broader. The broadening is due to the extra degree of freedom introduced by assignment of longitudinal momentum to the additional $\pi^{0}$. The ratio of areas may again be qualitatively understood by counting the number of independent ways that the sign of CM frame rapidity or longitudinal momentum may be assigned to the final state pions. There are five ways in this channel, with their probabilities now no longer necessarily being equal. Two contribute to the $\sim 0.01$ peak and three to the $\sim 0.04$ peak—the ratio of peak areas is indeed close to
16.1 Multiple photon emission

Figure 16.5: The function $\alpha A/\pi$, with initial state particles taken to be oppositely charged. This plot shows the electro-magnetic factor for an identical situation as is shown above in Fig. 16.4, but with the initial state assumed to be composed of $\pi^+\pi^-$ instead of $\pi^0\pi^0$. The vertical scale is arbitrary.

As $n$ increases the number of configurations of charges increases rapidly and, presumably, so do the number of associated peaks in $\alpha A/\pi$. The width of each peak will increase with $n$ for two reasons: due to greater kinematic freedom in assigning longitudinal momentum as the number of unseen neutrals increases, and due to the reduction in the average longitudinal momentum of each pion as the same interaction energy is shared among more particles—this makes the transverse momentum components and their associated broadening effects relatively more significant. The large number of broad peaks merge to form a single broad distribution, as seen for $n > 6$.

The curves shown in Fig. 16.5 represent the scattering of charged initial state particles and differ somewhat from the case of a neutral initial state. For small numbers of final state particles the distributions are double or triple peaked as before but these peaks occur at much larger values of $\alpha A/\pi$. For $n = 2$, which represents $\pi^+\pi^- \rightarrow \pi^+\pi^-$
scattering, there is a peak around $0.14 < \alpha A/\pi < 0.18$ and another in the $\alpha A/\pi < 0.02$ region. The high-$\alpha A/\pi$ peak corresponds to backward scattering of the charged particles, and reflects the classical idea that charges undergoing greater accelerations emit bremsstrahlung radiation more strongly. The other peak corresponds to a small deflection of the $\pi^+$ and $\pi^-$ with respect to their incoming directions. Experimentally, backward scattering of non-identical charged hadrons is found to be very much smaller than forward scattering—our model does not take any account of such correlations between initial and final state charges and so incorrectly gives rise to forward and backward scattering with equal probability. For larger $n$ the peaks representing particular kinematic configurations of charges increase in number and blend to form single broad distributions. The tails of these distributions extend to larger values of $\alpha A/\pi$ than in the neutral initial state situation.

We have seen that substantial multiple photon emission rates are indicated within our model of multi-pion production. We recognise that this crude model differs in several respects from true multi-hadron production. These differences will now be noted, and we shall make qualitative arguments about how they should affect the photon emission function $\alpha A/\pi$.

The most important restriction of our model is that it considers all particles to be of the pion mass. Heavier hadrons occurring in real scattering tend to make a smaller contribution to $\alpha A/\pi$ and so give rise to less bremsstrahlung radiation. This qualitative behaviour may be observed within the context of our model by varying the mass of the hadrons away from the pion mass.

Another flaw of our model is that all of the final state hadrons are treated on an equal footing. In fact it is the case that two final state hadrons, called leading hadrons, tend to be highly correlated with the trajectories of the initial state hadrons. These leading particles carry off around half of the interaction energy after scattering. Taking proper account of this behaviour would tend to reduce the value of $\alpha A/\pi$ in two ways: there would be less energy to be shared among the remaining final state hadrons leading to a reduction in the bremsstrahlung rate; also the probability of backward scattering of the leading hadrons with respect to the initial state charges would be greatly suppressed—the high-$\alpha A/\pi$ contributions due to such back scattering kinematics which were observed in our model would vanish. This is precisely the situation seen in the $\pi^- p \rightarrow \pi^- p\gamma$ channel studied in §16.1.1. In that example backward scattering was very highly suppressed and the resulting multiple photon emission rate was found to be small.

The previous two defects in our model would, if corrected, lead to a reduction in the expected multiple photon emission rate. We now mention an effect which should
increase this rate. We have assumed the final state pions to be produced independently, with only the imposition of four-momentum conservation giving correlations among their trajectories. In fact the bosonic nature of these identical spin-0 particles would tend to encourage them into clusters. Such clusterings of pion trajectories in high multiplicity final states are referred to as Bose-Einstein correlations. The amplitudes for soft photon emission from charged particles travelling in the same direction tend to add constructively, leading to an increase in the soft photon production rate over the case where charged particle trajectories are uncorrelated.

In conclusion, we have found evidence for a very large multiple photon emission rate within the context of a crude model of multi-pion final state kinematics. Though we end our discussion at this point, it is clear that the next stage in this analysis would be to investigate multiple photon emission effects within a well established Monte Carlo simulation of high energy, high multiplicity hadron scattering processes, such as FRITIOF [117] or PYTHIA [139].

16.2 Decay of Hadronic Resonances

In this section we consider the possible contribution to the low energy photon spectrum of interference terms between certain hadronic resonance diagrams and the bremsstrahlung amplitude, where the two types of diagram have identical final states. As a specific example we shall consider the channel

\[ \pi^- p \rightarrow \pi^- p \gamma \]

where the bremsstrahlung contribution interferes with the amplitudes for processes involving radiative decays of the intermediate hadronic resonances \( \rho^- \), \( N^*(1440) \) and \( \Delta^+(1232) \),

\[ \pi^- p \rightarrow \rho^- p \rightarrow \pi^- p \gamma \]
\[ \pi^- p \rightarrow \pi^- N^*(1440) \rightarrow \pi^- p \gamma \]
\[ \pi^- p \rightarrow \pi^- \Delta^+(1232) \rightarrow \pi^- p \gamma. \]

We shall proceed to estimate the upper limit of such interference contributions to the measured photon spectrum.

Before detailed calculations of these processes would be warranted, we should first check the estimated order of magnitude of the resonance and interference terms. In making this estimation we will use experimental input wherever possible in determining the magnitude and angular dependence of the resonance amplitudes. Our procedure will be to write the \( \pi^- p \rightarrow \Delta^+ \pi^- p \gamma \) cross section, for example, in terms of the sub-processes
18.2 Decay of Hadronic Resonances

$\pi^- p \rightarrow \pi^- \Delta^+(1232)$ and $\Delta^+(1232) \rightarrow p\gamma$. The rate $\Delta^+ \rightarrow p\gamma$ is known, while an upper limit can be placed on the magnitude of the $\pi^- p \rightarrow \pi^- \Delta^+$ matrix element by considering $\pi^- p \rightarrow \pi\pi N$ data at high energies. Certain approximations must be made during this splitting of the process at a resonance line. These approximations are considered in some detail in App. D. For the present case where we assume the sub-process amplitudes to be independent of resonance helicity the approximations reduce to the single assumption that the sub-process amplitudes be slowly varying with respect to resonance invariant mass within the kinematic region where the resonance gives a large contribution.

We choose a very simple parameterization of the sub-process $T$-matrices,

$$T(\pi^- p \rightarrow \pi^- \Delta^+) \equiv A_\Delta \exp \left( -\frac{\beta_\Delta}{2} |t_\Delta| \right) \quad (16.3)$$

$$T(\Delta^+ \rightarrow p\gamma) \equiv B_{\Delta \rightarrow \gamma} \quad (16.4)$$

where $A_\Delta$, $B_{\Delta \rightarrow \gamma}$, and $\beta_\Delta$ are real constants and $t_\Delta \equiv (p_{\pi} - p_{\Delta})^2$. The initial and final state pion and proton have four-momentum labels $p_{\pi}^\mu$, $p_{\pi}^\mu$, $p_{\Delta}^\mu$, and $p_{\Delta}^\mu$, while the photon momentum is $k^\mu$. From the arguments of App. D the amplitude for the full process should then be

$$T(\pi^- p \rightarrow \pi^- \gamma) \equiv \frac{A_\Delta \exp \left( -\frac{\beta_\Delta}{2} |t_\Delta| \right)}{(p_{\Delta}^2 - m_\Delta^2) + \text{Im} \Gamma_\Delta} B_{\Delta \rightarrow \gamma} \quad (16.5)$$

where $p_{\Delta}^\mu \equiv p_{\pi}^\mu + k^\mu$. The resonance mass and width are taken from Ref. [8]. Similar expressions may be written for the $N^*(1440)$ nucleon resonance and the $\rho(770)$ meson resonance. For $\pi^- p$ elastic scattering the angular dependence is well fit using $\sigma_{el} \sim \exp[-\beta_{el}|t|]$ with $\beta_{el} \approx 10.8 \text{ GeV}^{-2}$ (Ref. [134]). We choose the slope parameters for the two-body processes involving the resonances to be the same, $\beta_\Delta = \beta_{N^*} = \beta_{\rho} \equiv 10.8 \text{ GeV}^{-2}$. This is a guess—we have no experimental evidence that this is the case. The constant $B_{\Delta \rightarrow \gamma}$ is then obtained by computing the $\Delta^+ \rightarrow p\gamma$ decay rate in terms of it and fitting to the known $\Delta^+$ width and the branching ratio $BR(\Delta^+ \rightarrow p\gamma) = 0.55-0.61\%$ (P.D.G. estimate, Ref. [8, pp. 1711]).

$$\Gamma_{\Delta^+ \rightarrow \gamma} = \frac{1}{2m_\Delta(2\pi)^2} B_2(m_\Delta^2, m_\Delta^2, 0) |T(\Delta^+ \rightarrow p\gamma)|^2$$

$$= \frac{1}{16\pi m_\Delta} B_{\Delta \rightarrow \gamma}^2$$

$$\Rightarrow B_\Delta^2 = \frac{16\pi m_\Delta \Gamma_\Delta}{\left(1 - \frac{m_\Delta^2}{m_\Delta^2}\right)} \left[BR(\Delta^+ \rightarrow p\gamma)\right].$$

Values of the $B$ parameters, obtained in this way using data from Ref. [8], are shown below.
16.2 Decay of Hadronic Resonances

<table>
<thead>
<tr>
<th>Decay</th>
<th>Width (GeV)</th>
<th>Mass (GeV)</th>
<th>Br. fraction</th>
<th>$B_{\text{res.-}}$ (GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta^+ \to p\pi^0$</td>
<td>0.131</td>
<td>1.232</td>
<td>0.994</td>
<td>4.65</td>
</tr>
<tr>
<td>$\Delta^+ \to p\gamma$</td>
<td>0.131</td>
<td>1.232</td>
<td>0.006</td>
<td>0.340</td>
</tr>
<tr>
<td>$N^{*-} \to p\pi^0$</td>
<td>0.350</td>
<td>1.440</td>
<td>0.65</td>
<td>5.46</td>
</tr>
<tr>
<td>$N^{*-} \to p\gamma$</td>
<td>0.350</td>
<td>1.440</td>
<td>$6.0 \times 10^{-4}$</td>
<td>0.163</td>
</tr>
<tr>
<td>$\rho^- \to \pi^-\pi^0$</td>
<td>0.149</td>
<td>0.767</td>
<td>$\sim 1.00$</td>
<td>2.48</td>
</tr>
<tr>
<td>$\rho^- \to \pi^-\gamma$</td>
<td>0.149</td>
<td>0.767</td>
<td>$4.5 \times 10^{-4}$</td>
<td>0.052</td>
</tr>
</tbody>
</table>

Upon the normalization parameter $A_\Delta$ we place only an upper limit. The cross section for the process $\pi^-p \to \pi^-\Delta^+$ is not directly accessible to experiment. The processes $\pi^-p \to \pi^-\pi^+n$ or $\pi^-p \to \pi^-\pi^0p$ are observed and have a contribution from diagrams with an intermediate $\Delta^+$ resonance. Knowledge of these two cross sections would allow us to decompose the processes into the sub-processes $\pi^-p \to \pi^-\Delta^+$ and $\Delta^+ \to \pi N$. The $\Delta^+ \to \pi N$ sub-process could then be fit to the known partial width as we did above for $\Delta^+ \to \gamma$. An upper bound on the parameter $A_\Delta$ would be obtained by assuming the $\Delta^+$ resonance to be the sole contributor to these final states.

A further complication is that at the beam energies of interest data is only available for topological cross sections; that is, for $\pi^-p \to (n$ charged prongs). This still allows us to place an upper bound on the contribution of the $\Delta^+$ resonance, albeit a weaker one, by assuming the resonance diagram to be the sole contributor to the 2-charged-prong, inelastic $\pi^-p$ cross section. We may then calculate an upper bound on the contribution of the $\Delta^+$ resonance to the process of interest, $\pi^-p \to \pi^-\rho$. Similar upper limits can be placed on contributions of the $N^*(1440)$ resonance and the $\rho(770)$ meson resonance by assuming in each case that the resonance in question provides the full 2-prong inelastic cross section.

The $\pi^-p$ 2-prong inelastic cross section is extrapolated from the lower energy data of Refs. [135, 136, 137, 138] using the fit function

$$\sigma = a + b \log^2(p_{\text{lab}})$$

with parameters $a = 3.8\text{mb}$ and $b = -0.074\text{mb}$. The extrapolated cross section at the energy of the SOPHIE/WA83 experiment [104] is found to be $\sigma(p_{\text{lab}} = 280 \text{ GeV}/c) = 1.46\text{mb}$ as can be seen in Fig. 16.6.

The phase space integrations required in order to fit the 'A' parameters of Eq. 16.5 are calculated numerically using two methods, Gauss-Legendre integration and Monte Carlo sampling. The cross section for $\pi^-p \to \pi^-\rho\pi^0$ is computed, assuming each of the resonances $\Delta$, $\rho$ and $N^*$ in turn to be the sole contributor to the process. This cross section is equated to the upper limit which was computed above from two-prong inelastic scattering data, and the upper limits on the parameters $A_\Delta$, $A_\rho$ and $A_{N^*}$ are extracted.
Figure 16.6: Fit to two-prong, inelastic $\pi^- p$ cross section.
The phase space integral is, explicitly,

\[ \sigma = \frac{1}{2 \lambda^4(s, m_{\pi^-}^2, m_{\pi^0}^2)(2\pi)^6} R_3(s, m_{\pi^-}^2, m_{\pi^0}^2) |T(\pi^- p \rightarrow \pi^- p\pi^0)|^2 \]

where the matrix element \( T() \) is of the form of Eq. 16.5. The phase space operator \( R_3() \) is defined

\[
R_3(s, m_{\pi^-}^2, m_{\pi^0}^2) = \int \frac{d^3 \vec{p}_{\pi^-}}{2E_{\pi^-}} \int \frac{d^3 \vec{p}_{\pi^0}}{2E_{\pi^0}} \delta^4(q^\mu - p_{\pi^-} - p_{\pi^0}^\mu)
\]

\[
\int_{(\sqrt{s}-m_{\pi^-})^2} dp_A^2 R_3(s, m_{\pi^-}^2, p_A^2) R_2(p_A^2, m_{\pi^-}^2, m_{\pi^0}^2)
\]

where \( q^\mu \equiv (\sqrt{s}, \vec{0}) \) in the interaction's overall CM frame. The two body phase space operator \( R_2() \) is defined as

\[
R_2(x, y, z) = \frac{\lambda^2(x, y, z)}{8x} \int d\Omega_x
\]

where the angular integral is defined in the rest frame of the decaying particle or kinematic pseudo-particle whose invariant mass is \( \sqrt{x} \). Due to the strong forward peaking of hadron scattering at these high energies, reflected in the \( \exp(-\beta|t|) \) factor in the matrix element, the integrals converge poorly unless performed with care—the independent methods of integration provide a consistency check of our calculations. The technique of importance sampling was used in the Monte Carlo calculation in order to concentrate the random samples of the integrand in the regions of phase space where the it is largest. The two integration methods agreed to better than the number of significant figures quoted in the table below, where the resulting values of the 'A' parameters are given.

<table>
<thead>
<tr>
<th>Two body sub-process</th>
<th>Upper limit on parameter A</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi^- p \rightarrow \pi^- \Delta^+ )</td>
<td>( 1.20 \times 10^4 )</td>
</tr>
<tr>
<td>( \pi^- p \rightarrow \pi^- N^{*+} )</td>
<td>( 2.18 \times 10^4 )</td>
</tr>
<tr>
<td>( \pi^- p \rightarrow p^- p )</td>
<td>( 1.42 \times 10^4 )</td>
</tr>
</tbody>
</table>

An estimate of the upper bound on the contribution to the photon spectrum of each radiative resonance decay process could then be made. The required integral is, for the \( \Delta \) resonance,

\[
\frac{d\sigma}{dk} = \frac{1}{2 \lambda^4(s, m_{\pi^-}^2, m_{\pi^0}^2)(2\pi)^6} \int \frac{d^3 \vec{p}_{\pi^-}}{2E_{\pi^-}} \int \frac{d^3 \vec{p}_{\pi^0}}{2E_{\pi^0}} \delta^4(q^\mu - p_{\pi^-} - p_{\pi^0}^\mu - k^\mu) |T(\pi^- p \rightarrow \pi^- p\gamma)|^2
\]

where the matrix element is that of Eq. 16.5, \( q^\mu \equiv (\sqrt{s}, \vec{0}) \) in the interaction's CM frame, and \( d\Omega_k \) is the element of solid angle into which the photon is directed. Since we are
16.2 Decay of Hadronic Resonances

Figure 16.7: Upper bound on photon spectrum due to the Δ(1232) resonance in the process π⁻p → π⁻pγ.

The uppermost curve shows the contribution to this final state due to hadronic bremsstrahlung, while the lowest shows the estimated upper limit on the spectrum due to the radiative decay of an intermediate Δ⁺ resonance. The middle curve is just twice the square-root of the product of these two spectra, and is meant to indicate the expected order-of-magnitude of the maximum interference contribution between these two processes.

Figure 16.8: As in Fig. 16.7 but for an intermediate N°(1440) resonance.
interested in the kinematic region where the photon energy is very much smaller than the hadron energies, we made the simplifying assumption that the photon four-momentum could be ignored in the momentum conserving $\delta$-function which appears in the phase space integral. This allows the hadron and photon integrals to be separated and leads to considerable simplification.

$$\frac{d\sigma}{dk} \approx \frac{1}{2\lambda^2(s,m^2_{\pi},m^2_{p})(2\pi)^5} \frac{k}{2}$$

$$\times \int d\Omega_k \frac{\lambda^2(s,m^2_{\pi},m^2_{p})}{8s} 2\pi \int_{-1}^{1} d\cos \theta_{\pi-p} |T(\pi-p \rightarrow \pi-p\gamma)|^2$$

$$\Rightarrow \frac{d\sigma}{dk} \approx \frac{k}{512\pi^4s} \int d\Omega_k \int_{-1}^{1} d\cos \theta_{\pi-p} |T(\pi-p \rightarrow \pi-p\gamma)|^2$$

The $\pi^-p$ scattering angle $\theta_{\pi-p}$ is identical to that of elastic scattering kinematics because of our assumption that the photon four-momentum is small enough to ignore in the four-momentum conserving $\delta$-function. The resulting photon spectra are shown in Figs. 16.7-16.9, along with the bremsstrahlung spectrum for this same $\pi^-p \rightarrow \pi^-p\gamma$ channel, calculated using the lowest order terms of Low's theorem. These upper bounds on the contribution of the resonance diagrams are many orders of magnitude smaller than hadronic bremsstrahlung. The interference contribution between two processes of amplitudes $A_1$ and $A_2$ goes like $\sim 2\text{Re}(A_1^*A_2)$. Unfortunately we cannot compute the integral over phase space of this interference term directly within the context of our crude model.
for the resonance sub-process amplitudes, since they take no explicit account of the photon polarization states. Instead we plot the quantity $2 \times \sqrt{(d\sigma/d\Omega)_{\text{BREM}} \times (d\sigma/d\Omega)_{\text{RBS}}}$ with the intention of doing nothing more than indicating the approximate maximum order of magnitude of the true interference term.

Despite the obvious inadequacies of this simplistic approach, it should be clear that a more detailed calculation of the bremsstrahlung/resonance interference contribution is not warranted. It could not be larger than our result by the factor $\sim 10^2 - 10^3$ necessary for it to become comparable to the bremsstrahlung spectrum. Similarly, though we have only considered the contribution of radiative hadronic resonances to a single channel, it seems unlikely that consideration of such intermediate resonances in higher multiplicity final states would provide the required increase over our estimate.

We conclude that the photon spectrum due to interference between bremsstrahlung amplitudes and radiative resonance decay amplitudes sharing the same final states must be discounted as a possible explanation for the observed soft photon excess.

16.3 Higher order terms in the soft photon expansion

Although the soft photon theorem of Low provides us with the leading two orders in a power-of-$k^\mu$ expansion of a bremsstrahlung amplitude, the experimental analyses of direct photon data use a prediction of the soft photon spectrum based on only the leading order term. In this section we shall confirm that this is a reasonable approximation in the case of the $p^- p \rightarrow p^- p \gamma$ channel by direct calculation of the full Low theorem expression.

The most general form for the $p^- p$ elastic amplitude is

$$A(s,t) = u_1 \left\{ \Gamma_1(s,t) + (p_1 + p_2) \Gamma_2(s,t) \right\} u_2$$

where $\Gamma_1$ and $\Gamma_2$ are scalar functions and where momenta in the interaction $p^- p \rightarrow p^- p \gamma$ have been labelled $12 \rightarrow 34$, respectively.

The Low theorem amplitude for this channel is, using a single Taylor expansion point at $(s_{12}, \tilde{t})$ as advocated in §12.1,

$$\mathcal{M}^\mu = eQ_3 \left( \frac{p_3^\mu}{p_3 \cdot k} - \left( \frac{p_1 \cdot k p_2^\mu}{p_3 \cdot k p_3^\mu} - p_3^\mu \right) \frac{\partial}{\partial \tilde{t}} \right) A(s_{12}, \tilde{t})$$

$$+ eQ_4 \left( \frac{p_4^\mu}{p_4 \cdot k} - \left( \frac{p_2 \cdot k p_3^\mu}{p_4 \cdot k p_4^\mu} - p_2^\mu \right) \frac{\partial}{\partial \tilde{t}} \right) A(s_{12}, \tilde{t})$$

$$- eQ_1 \left( \frac{p_1^\mu}{p_1 \cdot k} - 2 \left( \frac{p_2 \cdot k p_1^\mu - p_1^\mu}{p_1 \cdot k p_2^\mu - p_2^\mu} \right) \frac{\partial}{\partial s_{12}} + \left( \frac{p_3 \cdot k p_1^\mu - p_1^\mu}{p_1 \cdot k p_3^\mu - p_3^\mu} \right) \frac{\partial}{\partial \tilde{t}} \right) A(s_{12}, \tilde{t})$$

$$- eQ_2 \left( \frac{p_3^\mu}{p_3 \cdot k} - 2 \left( \frac{p_2 \cdot k p_3^\mu - p_3^\mu}{p_3 \cdot k p_2^\mu - p_2^\mu} \right) \frac{\partial}{\partial s_{12}} + \left( \frac{p_4 \cdot k p_3^\mu - p_3^\mu}{p_3 \cdot k p_4^\mu - p_4^\mu} \right) \frac{\partial}{\partial \tilde{t}} \right) A(s_{12}, \tilde{t})$$
16.3 Higher order terms in the soft photon expansion

\[ + \frac{eQ_1}{p_4 k} [\bar{u}_4 \mathcal{R}(p_4)\mu (\Gamma(s_{12}, \bar{t}) + (p_1 + p_3)\Gamma_3(s_{12}, \bar{t})) u_2] \]

\[ - \frac{eQ_2}{p_2 k} [\bar{u}_1 (\Gamma(s_{12}, \bar{t}) + (p_1 + p_3)\Gamma_3(s_{12}, \bar{t})) \mathcal{R}(p_2)\mu u_2] \]

The operator \( \mathcal{R}(p)\mu \) is defined in Eq. (8.19), and the value of the proton anomalous magnetic moment which this operator requires is taken as \( \kappa_p = 1.79 \). When squared and summed on spins this amplitude will give the full contribution of the \( \pi^- p \rightarrow \pi^- p\gamma \) channel to the bremsstrahlung spectrum at order-\(1/k^3\) and order-\(k^0\), as well as part of the order-\(k^2\) contribution.

The \( \pi^- p \) elastic amplitude, required as input to the expression above, will be chosen to fit experimental data on the elastic cross section. The \( s \) dependence is constrained by the fit to the total cross section given in Ref. [8, pp. 1335],

\[ \sigma_{el}(s) = \left( \alpha_1 + \alpha_2 (p_{lab}^-)^n + \alpha_3 \log^2(p_{lab}^-) \right) \text{mb} \]

where

\[ \alpha_1 = 1.76 \pm 0.42 \]
\[ \alpha_2 = 11.2 \pm 0.3 \]
\[ \alpha_3 = 0.043 \pm 0.011 \]
\[ n = -0.64 \pm 0.07. \]

The dependence on Mandelstam \( t \) of the differential cross section \( d\sigma/dt \) is taken to be \( \sim \exp(-\beta|t|) \), where \( \beta = (10.8 \pm 0.3)\text{GeV}^{-2} \), in accord with experiment [134]. Our empirical form for the differential cross section is then

\[ \frac{d\sigma}{dt} = \frac{\beta \sigma_{el}(s) \exp(-\beta|t|)}{1 - \exp(-\beta\lambda(s, m_{t^-}^2, m_{t^2}^2)/s)} \text{mb/GeV}^2 \]

This quantity may also be computed in terms of the invariant functions \( \Gamma_1(s, t) \) and \( \Gamma_2(s, t) \) using Eq. (16.7).

\[ \frac{d\sigma}{dt} = \frac{s}{2\pi \lambda^2(s, m_{t^-}^2, m_{t^2}^2)} \left[ (4m_p^2 - t)|\Gamma_1(s, t)|^2 \right. \]

\[ - 2m_p(2s + t - 2m_p^2 - 2m_{t^-}^2)(\Gamma_1\Gamma_2^* + \Gamma_1^*\Gamma_2) \]

\[ + \left. \left[(2s + t - 2m_p^2 - 2m_{t^-}^2)^2 + t(4m_{t^-}^2 - t)\right]|\Gamma_2(s, t)|^2 \right] \]

For the kinematic region of interest we have \( s \) large and \( |t| \ll 1 \text{GeV}^2 \), due to the exponential fall-off of the differential cross section with \( |t| \). In this large \( s \) limit the differential cross section is dominated by the \( |\Gamma_2(s, t)|^2 s^2 \) term. Here we have ignored
16.3 Higher order terms in the soft photon expansion

the contribution of the smaller backward scattering peak in the \( \pi^- p \) data. We therefore choose to set \( \Gamma_1(s, t) \equiv 0 \) and fit \( \Gamma_2(s, t) \) by equating the two expressions for \( d\sigma/dt \) above. The spectrum for the bremsstrahlung process may then be calculated using the same small-\( k' \) approximation in the phase space integral as was employed in §16.2, Eq. (16.6).

It is found that the order-(\( k^0 \)) and order-(\( k'^0 \)) parts of the bremsstrahlung spectrum are factors of \( 10^2-10^4 \) smaller than the leading order-(\( 1/k'' \)) term. The order-(\( k^0 \)) interference term is uniformly negative for the values of \( \sqrt{s} \) considered (\( 5 < \sqrt{s} < 22 \) GeV), while the other two terms are, by necessity, positive. The order-(\( 1/k'' \)) expression for the bremsstrahlung spectrum used in the direct photon experimental analyses therefore seems to be well justified, at least for this channel. Since other high energy hadronic scattering processes have qualitatively similar behaviour to that assumed here for the \( \pi^- p \) elastic channel, it is likely that this conclusion can be extended to include hadronic processes in general.
Chapter 17

Conclusions

We have considered sources for the excess of soft photons over the Low theorem hadronic bremsstrahlung expectation which has been observed in several high energy hadron scattering experiments. After a review of the experimental data and of several theoretical models which have been proposed to explain this excess, we considered possible sources of soft photons which we felt had the potential to explain part of the excess spectrum.

The more promising result was that of multiple soft photon emission from high multiplicity hadron final states. Within the context of the simple model considered in Chp. 16 we found that the function determining photon emission, \( \alpha A/\pi \), tended to be of order \( \sim 0.1 \) for the kinematics and charge assignments presumed to be typical of multi-pion final states. Such values would lead to increases in the soft photon spectrum of \( \sim 50-100\% \) above single photon bremsstrahlung. As we stated previously, this idea would have to be tested within a more realistic model of high energy hadron scattering before being seriously considered as responsible for part of the measured soft photon excess. We also have no indication that this effect could, by itself, account for the factor of 5-8 excess seen in experiment. We feel, however, it is important to account for conventional effects such as this before conclusions can be draw about the more speculative models reviewed in §15.3.

Another effect considered was that of interference terms between bremsstrahlung amplitudes and amplitudes describing the radiative decay of intermediate hadronic resonances. The crude upper limit set on these effects was found to be many orders of magnitude smaller than the bremsstrahlung spectrum in the region of very small photon energy where the excess spectra have been observed. Also briefly considered during our investigations was the contribution of higher order terms in the soft photon expression of Low. These terms are ignored in the experimental analyses because they are expected to be tiny when considering small photon energies and slowly varying hadronic matrix ele-
ments. We found these terms' contribution in the bremsstrahlung channel $\pi^- p \rightarrow \pi^- p\gamma$ to indeed be several orders of magnitude smaller than the leading order $1/k$ term.
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Appendix A

pp Elastic Phase Shift Formalism

As input to the soft photon expressions developed in Part II we require the corresponding non-radiative amplitudes. For our analysis of the process $pp \rightarrow pp\gamma$ in §11.6 we require the experimentally measured proton-proton elastic scattering amplitude. Specifically we require the invariant functions $F^A_\alpha(s,t)$. The $pp$ elastic data is available in terms of a phase shift fit, and we must relate these phase shifts to the $F^A_\alpha(s,t)$. While these relations have certainly been computed by several other researchers (Refs. [62, 77, 93, 140]), their explicit form is rather involved and is not to be found in the literature.

In this appendix we shall derive these relations, quoting intermediate results from the literature wherever they are available. In §A.1 we write down the relations between the phase shifts, which have been fit from experimental $pp$ elastic scattering data, and the helicity matrix elements $M_{ss}$ and $M_{m',m}$. These expressions are quoted directly from the literature and are included here for completeness. We shall use the nuclear bar phase shift formalism given in Ref. [141] with the high-$j$ (large angular momentum) parts of the $pp$ elastic amplitude being taken from a simple one-pion-exchange model (Ref. [142]). The resulting expressions for the amplitude could then be made more accurate by the inclusion of such well-known ingredients as vacuum polarization corrections to the Coulomb scattering amplitude. Such refinements are unnecessary for our purposes, however, and have not been included. In §A.2 we derive the relationship between the helicity matrix elements $M_{ss}$ and $M_{m',m}$, and the invariant functions $F^A_\alpha(s,t)$.

A.1 Helicity matrix elements $M_{ss} / M_{m',m}$ in terms of phase shifts

In this section we shall quote, without derivation, the definitions and results of Ref. [141]. Proton-proton experimental data is parameterized in terms of nuclear bar phase shifts,
A.1 Helicity matrix elements $M_{zz}$ / $M_{m'm}$ in terms of phase shifts

defined by

\[ \delta^N_{\ell} \equiv \delta_\ell - \phi_\ell \]
\[ \overline{\delta}^N_{\ell,j} \equiv \overline{\delta}_{\ell,j} - \phi_\ell \]
\[ \varepsilon^N_{\ell} \equiv \varepsilon_\ell \]

where $\delta_\ell$, $\overline{\delta}_{\ell,j}$, $\varepsilon_\ell$ are termed bar phase shifts. The $\phi_\ell$ are Coulomb phases defined by

\[ \phi_\ell \equiv \sum_{z=1}^{l} \arctan \left( \frac{n}{z} \right); \quad n \equiv \frac{4\pi\alpha_s}{h_v} \]

where $\alpha_s \sim 1/137$ is the fine structure constant and $v$ is the velocity of the incident proton as measured in the laboratory frame.

The matrix elements of a Coulomb subtracted scattering matrix $\alpha$ are defined in terms of the bar phase shifts and the Coulomb phases $\phi_\ell$.

\[ \alpha_\ell \equiv e^{2i\delta_\ell} - e^{2i\phi_\ell} \]
\[ \alpha_{j,j} \equiv e^{2i\overline{\delta}_{j,j}} - e^{2i\phi_j} \]
\[ \alpha_{j\pm 1,j} \equiv \cos 2\varepsilon_j e^{2i\overline{\delta}_{j\pm 1,j}} - e^{2i\phi_{j\pm 1}} \]
\[ \alpha^j \equiv i \sin 2\varepsilon_j e^{i(\overline{\delta}_{j+1,j} + \overline{\delta}_{j-1,j})} \]

We have followed the notation of Stapp (Ref. [141]) here; care must be taken to distinguish the objects defined in the first and fourth lines above. The notation differs only by a raised or lowered index, $\alpha_\ell$ and $\alpha^j$.

The helicity matrix elements $M_{zz}$ for spin singlet to spin singlet scattering and $M_{m'm}$ for triplet to triplet scattering may then be related to the $\alpha_\ell$, $\alpha_{ij}$ and $\alpha^j$, and to the Coulomb scattering amplitude $f_C(\theta)$ given by

\[ f_C(\theta) \equiv \frac{-n}{k(1 - \cos \theta)} \exp \left[ - \ln \log \left( \frac{1}{2}(1 - \cos \theta) \right) \right] \]

where $k$ and $\theta$ are the centre-of-mass frame momentum and scattering angle of the protons. This relation of helicity matrix elements to phase shifts is quoted here in Eq. (A.1) directly from Table III of Ref. [141].

\[ M_{zz}(\theta, \phi) = f_C(\theta) + f_C(\pi - \theta) + \frac{2}{sk} \sum_{\ell \text{ even}} P_\ell(\theta) \left( \frac{2\ell + 1}{2} \right) \alpha_\ell \]
\[ M_{1,1}(\theta, \phi) = f_C(\theta) - f_C(\pi - \theta) + \frac{2}{sk} \sum_{\ell \text{ odd}} P_\ell(\theta) \]
A.1 Helicity matrix elements $M_{\ell s} / M_{m' m}$ in terms of phase shifts

\[ \times \left\{ \left( \frac{\ell + 2}{4} \right) \alpha_{\ell, \ell + 1} + \left( \frac{2\ell + 1}{4} \right) \alpha_{\ell, \ell} + \left( \frac{\ell - 1}{4} \right) \alpha_{\ell, \ell - 1} - \frac{1}{4} [(\ell + 1)(\ell + 2)]^{\frac{1}{2}} \alpha^{\ell + 1} - \frac{1}{4} [(\ell - 1)(\ell)]^{\frac{1}{2}} \alpha^{\ell - 1} \right\} \]

\[ M_{-1,-1}(\theta, \phi) = M_{1,1}(\theta, -\phi) \]

\[ M_{0,0}(\theta, \phi) = f_C(\theta) - f_C(\pi - \theta) + \frac{2}{i k} \sum_{\ell \; \text{odd}} P_\ell(\theta) \]

\[ \times \left\{ \left( \frac{\ell + 1}{2} \right) \alpha_{\ell, \ell + 1} + \left( \frac{\ell}{2} \right) \alpha_{\ell, \ell - 1} + \frac{1}{2} [(\ell + 1)(\ell + 2)]^{\frac{1}{2}} \alpha^{\ell + 1} + \frac{1}{2} [(\ell - 1)(\ell)]^{\frac{1}{2}} \alpha^{\ell - 1} \right\} \]

\[ M_{0,1}(\theta, \phi) = \frac{2}{i k} e^{i \phi} \sum_{\ell \; \text{odd}} P_\ell^1(\theta) \]

\[ \times \left\{ -\frac{\sqrt{2}}{4} \left( \frac{\ell + 2}{\ell + 1} \right) \alpha_{\ell, \ell + 1} + \frac{\sqrt{2}}{4} \left( \frac{2\ell + 1}{\ell(\ell + 1)} \right) \alpha_{\ell, \ell} + \frac{\sqrt{2}}{4} \left( \frac{\ell - 1}{\ell} \right) \alpha_{\ell, \ell - 1} + \frac{\sqrt{2}}{4} \left( \frac{\ell + 2}{\ell + 1} \right)^{\frac{1}{2}} \alpha^{\ell + 1} - \frac{\sqrt{2}}{4} \left( \frac{\ell - 1}{\ell} \right)^{\frac{1}{2}} \alpha^{\ell - 1} \right\} \]

\[ M_{0,-1}(\theta, \phi) = -M_{0,1}(\theta, -\phi) \]

\[ M_{1,0}(\theta, \phi) = \frac{2}{i k} e^{-i \phi} \sum_{\ell \; \text{odd}} P_\ell^1(\theta) \]

\[ \times \left\{ \frac{\sqrt{2}}{4} \alpha_{\ell, \ell + 1} - \frac{\sqrt{2}}{4} \alpha_{\ell, \ell - 1} + \frac{\sqrt{2}}{4} \left( \frac{\ell + 2}{\ell + 1} \right)^{\frac{1}{2}} \alpha^{\ell + 1} - \frac{\sqrt{2}}{4} \left( \frac{\ell - 1}{\ell} \right)^{\frac{1}{2}} \alpha^{\ell - 1} \right\} \]

\[ M_{-1,0}(\theta, \phi) = -M_{1,0}(\theta, -\phi) \]

\[ M_{1,-1}(\theta, \phi) = \frac{2}{i k} e^{-i \phi} \sum_{\ell \; \text{odd}} P_\ell^1(\theta) \]

\[ \times \left\{ \left( \frac{1}{4(\ell + 1)} \alpha_{\ell, \ell + 1} - \frac{2\ell + 1}{4\ell(\ell + 1)} \alpha_{\ell, \ell} + \frac{1}{4\ell} \alpha_{\ell, \ell - 1} - \frac{1}{4} [(\ell + 1)(\ell + 2)]^{-\frac{1}{2}} \alpha^{\ell + 1} - \frac{1}{4} [(\ell - 1)(\ell)]^{-\frac{1}{2}} \alpha^{\ell - 1} \right\} \]

\[ M_{-1,-1}(\theta, \phi) = M_{1,-1}(\theta, -\phi) \quad (A.1) \]

An important point is that the phase convention implicit in the definition of the associated Legendre polynomials $P_\ell^m(\cos \theta)$ must be consistent with the Condon-Shortley conventions of the Clebsch-Gordan coefficients used in the derivation of the helicity matrix elements. A suitable definition of the associated Legendre functions in terms of the Legendre polynomials is

\[ P_\ell^m(z) \equiv (1 - z^2)^{\frac{m}{2}} \frac{d^m}{dz^m} P_\ell(z). \]
A.1 Helicity matrix elements $M_m / M_{m',m}$ in terms of phase shifts

The functions are generated in our computer code by using the starting value

$$P_m^m(z) = (2m + 1)!!(1 - z^2)^{l/2}$$

and employing the recursion relation

$$(l - m)P_{l+1}^m(z) = (2l + 1)zP_l^m(z) - (l + m - 1)zP_l^m(z).$$

Phase shift analyses can be used to extract only the low-$j$ phase shifts, since a limited number of parameters can be fit using a given set of data. Ref. [142] suggests that the high angular momentum, and hence long distance, part of the interaction should be well described by a one pion exchange model of nucleon-nucleon scattering. Since the pion is the lightest meson its exchange should constitute the longest distance part of the $NN$ strong interaction. We take the higher-$j$ $\alpha$ parameters generated in this model directly from Ref. [142] with the result,

$$\alpha_l = \frac{ikg^2}{2E} ((x_0 - 1)Q_l(x_0) - \delta_{l,0})$$

$$\alpha_{j,j} = -\frac{ikg^2}{2E(2j + 1)} (jQ_{j+1}(x_0) + (j + 1)Q_{j-1}(x_0) - (2j + 1)Q_j(x_0))$$

$$\alpha_{j\pm 1,j} = \pm\frac{ikg^2}{2E(2j + 1)} (Q_j(x_0) - Q_{j\pm 1}(x_0))$$

$$\alpha' = -\frac{ikg^2}{2E(2j + 1)} [j(j + 1)]^{1/2} (Q_{j+1}(x_0) + Q_{j-1}(x_0) - 2Q_j(x_0))$$

where

- $E, k$ are the centre-of-mass frame energy and momentum of the protons
- $g$ is the $\pi NN$ coupling constant, taken to be $g^2 = 14.3 \pm 0.3$ (Ref. [143])
- $x_0 \equiv 1 + m^2_{\omega}/2k^2$.

The $Q_l(x)$ are Legendre polynomials of the second kind, and are defined by the relations (Ref. [144])

$$Q_0(x) = \frac{1}{2} \log \left( \frac{x + 1}{x - 1} \right)$$

$$Q_1(x) = xQ_0(x) - 1$$

$$Q_n(x) = [(2n - 1)xQ_{n-1}(x) - (n - 1)Q_{n-2}(x)]/n.$$ 

In our calculations the $\alpha$'s with $j \leq 50$ were computed in this fashion.

The differential cross section and various polarization observables may now be calculated in terms of the helicity matrix elements $M_m$ and $M_{m',m}$. As a check of the
correctness of our computer code, the calculations of the differential cross section and certain polarization observables of Ref. [141] were reproduced. This involved switching off the one-pion-exchange contributions and using the phase shift sets quoted in that reference.

The invariant functions $F^A_\alpha(s, t)$ may now be computed in terms of the helicity matrix elements $M_{ss}$ and $M_{m', m}$, and hence in terms of the nuclear bar phase shifts.

A.2 Invariant functions $F^A_\alpha$ in terms of $M_{ss}/M_{m', m}$

The relations between the matrix elements $M_{ss}$ and $M_{m', m}$, which are taken between states of total initial and final state helicity, and the functions $F^A_\alpha$ are rather involved. We shall therefore derive these relations using several intermediate steps. First we shall relate the $M_{ss}$ and $M_{m', m}$ to matrix elements $(\lambda_3 \lambda_4 | \Phi | \lambda_1 \lambda_2)$ taken between states having particular helicities $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4$ for each proton. This relation is simple to obtain, and depends only on Clebsch-Gordan coefficients and the reduced rotation matrices $d^J_{\lambda\lambda}(\theta)$.

To begin we note that only five of the sixteen $(\lambda_3 \lambda_4 | \Phi | \lambda_1 \lambda_2)$ are independent. This may be shown by making the partial wave decomposition (Ref. [145])

$$\langle \lambda_3 \lambda_4 | \Phi | \lambda_1 \lambda_2 \rangle \equiv \frac{1}{p} \sum J (2J + 1) \langle \lambda_3 \lambda_4 | T^J | \lambda_1 \lambda_2 \rangle d^J_{\lambda\lambda}(\theta)$$

where $\lambda \equiv \lambda_1 - \lambda_2$, $\lambda' \equiv \lambda_3 - \lambda_4$, $d^J_{\lambda\lambda}(\theta)$ are the reduced rotation matrices and the operator $T^J$ is the partial wave sub-matrix of $\Phi$ having total angular momentum $J$. We use the Condon-Shortley phase convention for the reduced rotation matrices and their specific form is taken from Ref. [8, pp. 1287]. With this phase convention they satisfy the relations

$$d^J_{\lambda\lambda}(\theta) = (-)^{\lambda - \lambda'} d^J_{\lambda\lambda}(\theta)$$

$$= d^J_{-\lambda' - \lambda}(\theta)$$

$$= (-)^{\lambda - \lambda'} d^J_{-\lambda' - \lambda}(\theta).$$
The matrix elements of $T^J$ have the following properties due to parity invariance, time reversal invariance and the conservation of total spin,

$$\langle \lambda_2 \lambda_4 | T^J | \lambda_1 \lambda_2 \rangle = \langle -\lambda_2 | T^J | -\lambda_1 \rangle = \langle \lambda_1 \lambda_3 | T^J | \lambda_3 \lambda_4 \rangle$$ Parity

$$= \langle \lambda_1 \lambda_3 | T^J | \lambda_3 \lambda_4 \rangle$$ Time reversal

$$= \langle \lambda_4 \lambda_3 | T^J | \lambda_2 \lambda_1 \rangle$$ Spin.

Using these relations we find

$$\langle ++| \Phi|++\rangle = \langle --| \Phi|--\rangle$$
$$\langle ++| \Phi|--\rangle = \langle --| \Phi|+-\rangle$$
$$\langle ++| \Phi|+-\rangle = \langle --| \Phi|--\rangle = \langle ++| \Phi|--\rangle$$
$$= -\langle ++| \Phi|--\rangle = -\langle ++| \Phi|--\rangle = -\langle ++| \Phi|--\rangle.$$

Following Ref. [77] we choose the five independent matrix elements to be

$$\Phi_1 \equiv \langle ++| \Phi|++\rangle$$
$$\Phi_2 \equiv \langle ++| \Phi|--\rangle$$
$$\Phi_3 \equiv \langle ++| \Phi|--\rangle$$
$$\Phi_4 \equiv \langle ++| \Phi|--\rangle$$
$$\Phi_5 \equiv \langle ++| \Phi|--\rangle.$$

Looking back at Eq. (A.1) we appear to have six independent functions $M_{++}, M_{1,1}, M_{0,0}, M_{1,-1}, M_{1,0}$ and $M_{0,1}$. However, as we shall shortly demonstrate, there is an internal constraint among the triplet matrix elements

$$\sin \theta (M_{1,1} - M_{0,0} - M_{1,-1}) = \sqrt{2}\cos \theta (M_{1,0} + M_{0,1})$$ (A.3)

which reduces the number to five.

We now construct states of total initial and final state spin in terms of the individual proton helicity states. Using these we can relate the $\Phi_1 \ldots \Phi_5$ to the $M_{++}$ and $M_{m',m}$. In the initial state, choosing particle 1 to be travelling in the direction of the spin quantization axis in this centre-of-mass frame, we have the relations

$$\text{singlet} \quad \left| S_{12} \right\rangle = \frac{1}{\sqrt{2}} (|++\rangle - |--\rangle)$$
$$\left| T_{12}^{(++)} \right\rangle = |+-\rangle.$$
A.2 Invariant functions \( F_0^A \) in terms of \( M_{m',m} \)

A triplet \( \{ T^{(+)\downarrow}_{12} \} = \frac{1}{\sqrt{2}} (|++\rangle + |--\rangle) \)

\( \{ T^{(-)\downarrow}_{12} \} = |--\rangle \).

To obtain the final state wavefunctions we rotate the initial state ones by \( \theta \), which is equivalent to rotating the frame by \(-\theta\), and employ the relation

\[
\sum_{M'=-J}^{+J} |J_{12}^{(M')}\rangle \langle J_{12}^{(M')}| \rightarrow e^{i\theta}\sum_{M'=-J}^{+J} |J_{12}^{(M')}\rangle \langle J_{12}^{(M')}| (\theta). 
\]

This gives

\[
|S_{12}\rangle = \frac{1}{\sqrt{2}} (|++\rangle + |--\rangle) \\
|T^{(+)\downarrow}_{12}\rangle = \frac{1}{2} (1 - \cos \theta) |++\rangle + \frac{1}{2} (1 + \cos \theta) |--\rangle - \frac{1}{2} \sin \theta (|++\rangle + |--\rangle) \\
|T^{(0)\downarrow}_{12}\rangle = -\frac{1}{\sqrt{2}} \sin \theta |++\rangle + \frac{1}{2} \sin \theta |--\rangle + \frac{1}{\sqrt{2}} \cos \theta (|++\rangle + |--\rangle) \\
|T^{(-)\downarrow}_{12}\rangle = \frac{1}{2} (1 + \cos \theta) |--\rangle + \frac{1}{2} (1 - \cos \theta) |--\rangle + \frac{1}{2} \sin \theta (|++\rangle + |--\rangle).
\]

We may now relate the \( M'\)'s to the \( \Phi\)'s; for example,

\[
M_{\downarrow} = \langle S_{12} | \Phi | S_{12} \rangle \\
= \frac{1}{2} (\langle ++| \Phi | ++ \rangle \langle ++| \Phi | + - \rangle - \langle ++| \Phi | - + \rangle - \langle ++| \Phi | - - \rangle - \langle - +| \Phi | - - \rangle) \\
= \Phi_1 - \Phi_2.
\]

Similarly,

\[
M_{0,0} = \cos \theta (\Phi_1 + \Phi_2) - 2 \sin \theta \Phi_5 \\
M_{1,1} = M_{-1,-1} = \frac{1}{2} (1 + \cos \theta) \Phi_3 - \frac{1}{2} (1 - \cos \theta) \Phi_4 - \sin \theta \Phi_5 \\
M_{1,-1} = M_{-1,1} = \frac{1}{2} (1 - \cos \theta) \Phi_3 + \frac{1}{2} (1 + \cos \theta) \Phi_4 + \sin \theta \Phi_5 \\
M_{1,0} = -M_{-1,0} = \frac{1}{\sqrt{2}} (-\sin \theta (\Phi_1 + \Phi_2) - 2 \cos \theta \Phi_5) \\
M_{0,1} = -M_{0,-1} = \frac{1}{\sqrt{2}} (-\sin \theta (\Phi_4 - \Phi_3) + 2 \cos \theta \Phi_5) \quad (A.4)
\]

where we have, without loss of generality, chosen scattering to occur in the plane with azimuthal angle set to zero. The relations of Eq. (A.4) can be used to prove the constraint given in Eq. (A.3).

The relation between the matrix elements \( \Phi_1 \ldots \Phi_8 \) and the functions \( F_1^A \ldots F_8^A \) is, from their respective definitions in Eq. (A.2) and Eqs (11.15)-(11.18),

\[
\langle \lambda_2 \lambda_4 | \Phi | \lambda_1 \lambda_2 \rangle = \frac{1}{16\pi E} (F_1^A S + F_2^A T + F_3^A A + F_4^A V + F_5^A P) \quad (A.5)
\]
A.2 Invariant functions $F^A$ in terms of $M_{zz}/M_{m'm}$

where

$$S \equiv \bar{u}_{\lambda_4} u_{\lambda_3} \bar{u}_{\lambda_2} u_{\lambda_1}$$

$$T \equiv \bar{u}_{\lambda_4} \gamma^2 \sigma^{\mu\nu} u_{\lambda_3} \bar{u}_{\lambda_2} \gamma^2 \sigma_{\mu\nu} u_{\lambda_1}$$

$$A \equiv \bar{u}_{\lambda_4} i \gamma^5 \gamma^\mu u_{\lambda_3} \bar{u}_{\lambda_2} i \gamma^5 \gamma^\mu u_{\lambda_1}$$

$$V \equiv \bar{u}_{\lambda_4} \gamma^\nu u_{\lambda_3} \bar{u}_{\lambda_2} \gamma^\nu u_{\lambda_1}$$

$$P \equiv \bar{u}_{\lambda_4} \gamma^5 u_{\lambda_3} \bar{u}_{\lambda_2} \gamma^5 u_{\lambda_1}$$

These spinor chains can in principle be directly computed in terms of scattering angles and energies for each set of proton helicities $\lambda_1 \ldots \lambda_4$. The objects $T, A, V$ are rather complicated, motivating us to reformulate the problem using Fierz manipulations in order to avoid the calculation of $T$ and $V$.

We choose to define auxiliary functions $G_i$ by the expression

$$F^A S + F^A T + F^A A + F^A V + F^A P = G_1 (S - \tilde{S}) + G_2 (T + \tilde{T}) + G_3 (A - \tilde{A}) + G_4 (V + \tilde{V}) + G_5 (P - \tilde{P}) \tag{A.6}$$

where $\tilde{S} \rightarrow \tilde{P}$ are the counterparts of $S \rightarrow P$ under the interchange of particles 3 and 4. We may use the Fierz relation of Eq. (11.16) to eliminate the $\tilde{S}, \tilde{T}, \tilde{A}, \tilde{V}, \tilde{P}$ from Eq. (A.6) and hence give the relation between the $F^A_i$ and the $G_i$:

$$F^A_1 = \frac{1}{4} \begin{bmatrix}
3 & 6 & -4 & 4 & -1 \\
-1 & 2 & 0 & 0 & -1 \\
-1 & 0 & 6 & 2 & 1 \\
-1 & 0 & -2 & 2 & 1 \\
-1 & 6 & 4 & -4 & 3
\end{bmatrix} \begin{bmatrix}
G_1 \\
G_2 \\
G_3 \\
G_4 \\
G_5
\end{bmatrix} . \tag{A.7}$$

We now take linear combinations of certain rows of the Fierz matrix of Eq. (11.17) and show that

$$T + \tilde{T} = S + \tilde{S} + P + \tilde{P} \quad \text{Rows (2) - (1) - (5)}$$

$$V + \tilde{V} = S + \tilde{S} - P - \tilde{P} \quad \text{Rows (4) - (1) + (5)} \tag{A.8}$$

Thus $T, \tilde{T}, V, \tilde{V}$ may be eliminated in favour of $S, \tilde{S}, A, \tilde{A}, P, \tilde{P}$.

Eq. (A.5) may now be rewritten, with the help of Eq. (A.6) and Eq. (A.8), as a relationship between the $\Phi_i$ and the $G_i$ with only the spinor chains $S, \tilde{S}, A, \tilde{A}, P$ and $\tilde{P}$ remaining to be evaluated.

$$16\pi E (\lambda_3 \lambda_4 | \Phi | \lambda_1 \lambda_2) = (G_1 + G_2 + G_4) S + (-G_1 + G_2 + G_4) \tilde{S} + G_3 A - G_3 \tilde{A} + (G_2 - G_4 + G_5) P + (G_2 - G_4 - G_5) \tilde{P} \tag{A.9}$$
A.2 Invariant functions $F^A_2$ in terms of $M_{11}/M_{m',m}$. 

The spinor chains $S$, $\hat{S}$, $A$, $\bar{A}$, $P$, $\bar{P}$ are now computed directly in terms of a particular representation for the spinors and the Dirac matrices. The definitions of the Dirac matrices $\gamma^\mu$ and the Pauli matrices $\sigma_z$, $\sigma_y$, $\sigma_x$ are those of Bjorken and Drell (Ref. [146, pp. 282]). The spinors of definite helicity are defined to be

\[
\begin{align*}
\bar{u}_\lambda &= \frac{1}{\sqrt{E + m}} \left( \begin{array}{c} E + m \\ 2p_\lambda \end{array} \right) \chi_{\lambda}, \\
\bar{u}_\lambda^* &= \frac{1}{\sqrt{E + m}} \left( \begin{array}{c} E + m \\ 2p_\lambda \end{array} \right) e^{-i\sigma_y \frac{1}{2} \theta} \chi_{-\lambda}, \\
\bar{u}_\lambda &= \frac{1}{\sqrt{E + m}} \left( \begin{array}{c} E + m \\ 2p_\lambda \end{array} \right) e^{-i\sigma_y \frac{1}{2} \theta} \chi_{-\lambda}. 
\end{align*}
\]

The adjoint of the final state spinors are required;

\[
\begin{align*}
\bar{u}_\lambda &= \bar{u}_\lambda \gamma^0 = \frac{1}{\sqrt{E + m}} \left( \begin{array}{c} E + m \\ -2p_\lambda \end{array} \right) \chi_{\lambda}^* e^{i\sigma_y \frac{1}{2} \theta}, \\
\bar{u}_\lambda &= \bar{u}_\lambda \gamma^0 = \frac{1}{\sqrt{E + m}} \left( \begin{array}{c} E + m \\ -2p_\lambda \end{array} \right) \chi_{-\lambda}^* e^{i\sigma_y \frac{1}{2} \theta}.
\end{align*}
\]

Thus we have for the spinor chains

\[
\begin{align*}
\bar{u}_\lambda \cdot u_\lambda &= [(E + m) - 4\lambda \lambda'(E - m)] \left( \chi_{\lambda}^* \cdot e^{i\sigma_y \frac{1}{2} \theta} \chi_{\pm \lambda} \right), \\
\bar{u}_\lambda \cdot i\gamma^0 u_\lambda &= 2p(\lambda - \lambda') \left( \chi_{\pm \lambda}^* \cdot e^{i\sigma_y \frac{1}{2} \theta} \chi_{\pm \lambda} \right), \\
\bar{u}_\lambda \cdot i\gamma^0 u_\lambda &= -2ip(\lambda + \lambda') \left( \chi_{\pm \lambda}^* \cdot e^{i\sigma_y \frac{1}{2} \theta} \chi_{\pm \lambda} \right), \\
\bar{u}_\lambda \cdot i\gamma^5 u_\lambda &= -i [(E + m) + 4\lambda \lambda'(E - m)] \left( \chi_{\pm \lambda}^* \cdot e^{i\sigma_y \frac{1}{2} \theta} \sigma \chi_{\pm \lambda} \right).
\end{align*}
\]

In these expressions we have that

\[
\begin{align*}
\pm \lambda' \text{ refers to } & \lambda' = \left\{ \begin{array}{c} \lambda_3 \\ \lambda_4 \end{array} \right. \\
\pm \lambda \text{ refers to } & \lambda = \left\{ \begin{array}{c} \lambda_1 \\ \lambda_2 \end{array} \right.
\end{align*}
\]

By direct matrix multiplication we find that

\[
\begin{align*}
\chi_{\lambda}^* \cdot e^{i\sigma_y \frac{1}{2} \theta} \chi_{\lambda} &= \chi_{\lambda}^* \cdot (\cos \frac{1}{2} \theta + i \sigma_y \sin \frac{1}{2} \theta) \chi_{\lambda} \\
&= |\lambda' + \lambda| \cos \frac{1}{2} \theta + (\lambda' - \lambda) \sin \frac{1}{2} \theta.
\end{align*}
\]
A.2 Invariant functions $F_a^\lambda$ in terms of $M_{m'/m}$

\begin{align*}
\chi_\lambda^1 e^{i\sigma_z \frac{1}{2} \theta} \sigma_x \chi_\lambda &= \chi_\lambda^1 (\sigma_z \cos \frac{1}{2} \theta + \sigma_x \sin \frac{1}{2} \theta) \chi_\lambda \\
&= |\lambda' - \lambda| \cos \frac{1}{2} \theta + (\lambda' + \lambda) \sin \frac{1}{2} \theta \\
\chi_\lambda^1 e^{i\sigma_z \frac{1}{2} \theta} \sigma_y \chi_\lambda &= \chi_\lambda^1 (\sigma_y \cos \frac{1}{2} \theta + i \sin \frac{1}{2} \theta) \chi_\lambda \\
&= -i (\lambda' - \lambda) \cos \frac{1}{2} \theta + i |\lambda' - \lambda| \sin \frac{1}{2} \theta \\
\chi_\lambda^1 e^{i\sigma_z \frac{1}{2} \theta} \sigma_z \chi_\lambda &= \chi_\lambda^1 (\sigma_z \cos \frac{1}{2} \theta - \sigma_x \sin \frac{1}{2} \theta) \chi_\lambda \\
&= (\lambda' + \lambda) \cos \frac{1}{2} \theta - |\lambda' - \lambda| \sin \frac{1}{2} \theta.
\end{align*}

Here we have used the relation

\[ e^{i\sigma_z \frac{1}{2} \theta} = \cos \frac{1}{2} \theta + i \sigma_y \sin \frac{1}{2} \theta \]

which is shown by series expansion and resummation of the exponential.

The explicit expressions for the spinor chains $\mathcal{S}, \tilde{\mathcal{S}}, A, \tilde{A}, \mathcal{P}, \tilde{\mathcal{P}}$ can now be computed.

\begin{align*}
\mathcal{S} &= \bar{u}_{\lambda_1} u_{\lambda_2} \bar{u}_{\lambda_3} u_{\lambda_4} \\
&= [(E + m) - 4 \lambda_4 \lambda_2 (E - m)] |\lambda_4 + \lambda_2| \cos \frac{1}{2} \theta - (\lambda_4 - \lambda_2) \sin \frac{1}{2} \theta \\
&\times [(E + m) - 4 \lambda_3 \lambda_1 (E - m)] |\lambda_3 + \lambda_1| \cos \frac{1}{2} \theta + (\lambda_3 - \lambda_1) \sin \frac{1}{2} \theta \\
\tilde{\mathcal{S}} &= \bar{u}_{\lambda_1} u_{\lambda_2} \bar{u}_{\lambda_3} u_{\lambda_4} \\
&= [(E + m) - 4 \lambda_4 \lambda_1 (E - m)] |\lambda_4 + \lambda_1| \cos \frac{1}{2} \theta - (\lambda_4 + \lambda_1) \sin \frac{1}{2} \theta \\
&\times [(E + m) - 4 \lambda_3 \lambda_2 (E - m)] |\lambda_3 + \lambda_2| \cos \frac{1}{2} \theta + (\lambda_3 + \lambda_2) \sin \frac{1}{2} \theta \\
\mathcal{P} &= \bar{u}_{\lambda_1} \gamma^5 u_{\lambda_2} \bar{u}_{\lambda_3} \gamma_5 u_{\lambda_4} \\
&= 4p^2 (\lambda_4 - \lambda_2) (\lambda_3 - \lambda_1) |\lambda_4 + \lambda_2| \cos \frac{1}{2} \theta - (\lambda_4 - \lambda_2) \sin \frac{1}{2} \theta \\
&\times |\lambda_3 + \lambda_1| \cos \frac{1}{2} \theta + (\lambda_3 - \lambda_1) \sin \frac{1}{2} \theta \\
\tilde{\mathcal{P}} &= \bar{u}_{\lambda_1} \gamma^5 u_{\lambda_2} \bar{u}_{\lambda_3} \gamma_5 u_{\lambda_4} \\
&= 4p^2 (\lambda_4 - \lambda_1) (\lambda_3 - \lambda_2) |\lambda_4 + \lambda_1| \cos \frac{1}{2} \theta - (\lambda_4 + \lambda_1) \sin \frac{1}{2} \theta \\
&\times |\lambda_3 - \lambda_2| \cos \frac{1}{2} \theta + (\lambda_3 + \lambda_2) \sin \frac{1}{2} \theta \\
A &= \bar{u}_{\lambda_1} i\gamma_\mu u_{\lambda_2} \bar{u}_{\lambda_3} i\gamma_\mu u_{\lambda_4} \\
&= -4p^2 (\lambda_4 + \lambda_2) (\lambda_3 + \lambda_1) |\lambda_4 + \lambda_2| \cos \frac{1}{2} \theta - (\lambda_4 - \lambda_2) \sin \frac{1}{2} \theta \\
&\times |\lambda_3 + \lambda_1| \cos \frac{1}{2} \theta + (\lambda_3 - \lambda_1) \sin \frac{1}{2} \theta \\
&+ [(E + m) + 4 \lambda_4 \lambda_2 (E - m)] [(E + m) + 4 \lambda_3 \lambda_1 (E - m)] \\
&\times \{ |\lambda_4 - \lambda_2| \cos \frac{1}{2} \theta - (\lambda_4 + \lambda_2) \sin \frac{1}{2} \theta \} |\lambda_3 - \lambda_1| \cos \frac{1}{2} \theta + (\lambda_3 + \lambda_1) \sin \frac{1}{2} \theta \\
&+ [(\lambda_4 + \lambda_2) \cos \frac{1}{2} \theta + |\lambda_4 + \lambda_2| \sin \frac{1}{2} \theta] [(\lambda_3 - \lambda_1) \cos \frac{1}{2} \theta - (\lambda_3 + \lambda_1) \sin \frac{1}{2} \theta] \\
&- [(\lambda_4 + \lambda_2) \cos \frac{1}{2} \theta + |\lambda_4 - \lambda_2| \sin \frac{1}{2} \theta] [(\lambda_3 + \lambda_1) \cos \frac{1}{2} \theta - (\lambda_3 - \lambda_1) \sin \frac{1}{2} \theta]
\end{align*}

\begin{align*}
\tilde{A} &= \bar{u}_{\lambda_1} i\gamma_\mu u_{\lambda_2} \bar{u}_{\lambda_3} i\gamma_\mu u_{\lambda_4}
\end{align*}
A.2 Invariant functions $F_A^\alpha$ in terms of $M_{ss}/M_m$.

$$
F_A^\alpha = \frac{-4p^2}{2} (\lambda_4 + \lambda_1) (\lambda_3 + \lambda_2) [|\lambda_4 - \lambda_1| \cos \frac{1}{2} \theta - (\lambda_4 + \lambda_1) \sin \frac{1}{2} \theta] \\
\times [(|\lambda_3 - \lambda_2| \cos \frac{1}{2} \theta + (\lambda_3 + \lambda_2) \sin \frac{1}{2} \theta] \\
+ [(E + m) + 4\lambda_4\lambda_1(E - m)][(E + m) + 4\lambda_3\lambda_2(E - m)] \\
\times [(|\lambda_4 + \lambda_1| \cos \frac{1}{2} \theta - (\lambda_4 + \lambda_1) \sin \frac{1}{2} \theta] \\
\times [(|\lambda_3 + \lambda_2| \cos \frac{1}{2} \theta + (\lambda_3 + \lambda_2) \sin \frac{1}{2} \theta] \\
+ [(\lambda_4 + \lambda_1) \cos \frac{1}{2} \theta - |\lambda_4 - \lambda_1| \sin \frac{1}{2} \theta] \\
\times [|\lambda_3 + \lambda_2| \cos \frac{1}{2} \theta + (\lambda_3 + \lambda_2) \sin \frac{1}{2} \theta] \\
- [(\lambda_4 - \lambda_1) \cos \frac{1}{2} \theta + |\lambda_4 + \lambda_1| \sin \frac{1}{2} \theta] \\
\times [|(\lambda_3 + \lambda_2) \cos \frac{1}{2} \theta - (\lambda_3 + \lambda_2) \sin \frac{1}{2} \theta]}
$$

We may now substitute these expressions for $S, \dot{S}, A, \dot{A}, P$ and $\Phi$ into Eq. (A.9) using particular values for the four proton helicities. This gives us the relations between the five independent $\Phi_i$, and the functions $G_i$. The result is surprisingly simple;

\[4\pi E \Phi_1 = m^2 (G_1 + (G_2 + G_4) \cos \theta) - (3E^2 + p^2) G_3\]
\[4\pi E \Phi_2 = -E^2 G_3 + ((E^2 + p^2) G_2 + m^2 G_4) \cos \theta + 3m^2 G_3 - p^2 G_5\]
\[4\pi E \Phi_3 = [2m^2 G_2 + 2E^2 G_4 + p^2 (-G_1 + 2G_3 + G_5)] \cos^2 \frac{1}{2} \theta\]
\[4\pi E \Phi_4 = [2m^2 G_2 + 2E^2 G_4 + p^2 (G_1 - 2G_3 - G_5)] \sin^2 \frac{1}{2} \theta\]
\[4\pi E \Phi_5 = -mE (G_2 + G_4) \sin \theta.\]  

We now have the relations:

- $F_A^\alpha$ in terms of $G_i$; Eq. (A.7)
- $G_i$ in terms of $\Phi_i$; through inversion of Eq. (A.10)
- $\Phi_i$ in terms of $M_{s\bar{s}}, M_{1,1}, M_{0,1}$ and $M_{0,0}$; $M_{0,0}$ is eliminated using Eq. (A.3), and the remaining five relations of Eq. (A.4) are inverted.

These three sets of matrix relations may then be multiplied together giving the desired result; the $F_A^\alpha$ in terms of $M_{s\bar{s}}, M_{1,1}, M_{0,1}$ and $M_{1,-1}$. These complicated relations are shown in Eq. (A.11). This final form has been checked against the relations extracted from Fearing’s well tested pp bremsstrahlung FORTRAN code (used in Refs. [93, 98, 147]), and were found to agree.
\[ F^A_2 = -\frac{\pi}{2E} M_{ss} + \frac{\pi}{2mE(E + m)} \left[ 2E^2 - (2E + m)(E - m)\cos \theta \right] M_{1,1} - \frac{\pi E}{\sqrt{2p^2 \sin \theta}} M_{1,0} \]
\[ + \frac{\pi}{\sqrt{2p^2 \sin \theta} mE} \left[ 2E^3 \cos \theta (\cos \theta - 1) + E^2 m(2 - 3 \cos \theta) - m^3 (1 - \cos^2 \theta) \right] M_{0,1} \]
\[ + \left\{ \frac{2\pi E}{p^2 (1 + \cos \theta)} + \frac{\pi}{2mE(E + m)} \left[ 2E^2 - (2E + m)(E - m)\cos \theta \right] \right\} M_{1,-1} \]

\[ F^A_3 = -\frac{\pi}{2E} M_{ss} + \frac{\pi}{2E(E + m)} \left[ 2E - (E - m)\cos \theta \right] M_{1,1} + \frac{\pi E}{\sqrt{2p^2 \sin \theta}} M_{1,0} \]
\[ + \frac{\pi}{\sqrt{2p^2 \sin \theta} E} \left[ E^2 (1 - 2 \cos \theta + \cos^2 \theta) + 2mE \cos \theta (1 - \cos \theta) - m^2 (1 - \cos^2 \theta) \right] M_{0,1} \]
\[ + \left\{ \frac{2\pi E}{p^2 (1 + \cos \theta)} + \frac{\pi}{2E(E + m)} \left[ -2E + (E - m)\cos \theta \right] \right\} M_{1,-1} \]

\[ F^A_4 = \frac{\pi}{2E} M_{ss} + \frac{\pi}{2E(E + m)} \left[ 2E + (E - m)\cos \theta \right] M_{1,1} - \frac{\pi E}{\sqrt{2p^2 \sin \theta}} M_{1,0} \]
\[ + \frac{\pi}{\sqrt{2p^2 \sin \theta} E} \left[ E^2 (2 - 2 \cos \theta - \cos^2 \theta) + 2mE \cos \theta (1 + \cos \theta) + m^2 (1 - \cos^2 \theta) \right] M_{0,1} \]
\[ + \left\{ \frac{2\pi E}{p^2 (1 + \cos \theta)} - \frac{\pi}{2E(E + m)} \left[ 2E + (E - m)\cos \theta \right] \right\} M_{1,-1} \]

\[ F^A_5 = \frac{\pi}{2E} M_{ss} + \frac{\pi}{2mE(E + m)} \left[ 2E(3E - 2m) + (2E + m)(E - m)\cos \theta \right] M_{1,1} + \frac{\pi E}{\sqrt{2p^2 \sin \theta}} M_{1,0} \]
\[ + \frac{\pi}{\sqrt{2p^2 \sin \theta} mE} \left[ -2E^3 \cos \theta (3 + \cos \theta) + E^2 m(4 + 10 \cos \theta + 3 \cos^2 \theta) - 4E m^2 \cos \theta \right] \]
\[ + m^3 (1 - \cos^2 \theta) \] \[ M_{0,1} \]
\[ + \left\{ \frac{2\pi E(5 + 3 \cos \theta)}{p^2 (1 - \cos^2 \theta)} + \frac{\pi}{2mE(E + m)} \left[ 2E(3E - 2m) + (2E + m)(E - m)\cos \theta \right] \right\} M_{1,-1} \]

(A.11)
Appendix B

Phase Space by Gauss–Legendre Quadratures

The package PHASE++ is a set of C++ classes which allow Gauss–Legendre integration over an arbitrary phase space to be performed with relative ease. It was developed in order to reduce the time taken to write a decay or cross-section calculation from several hours, if one must write a new FORTRAN or C program for each problem, to a few minutes. In comparison with the more commonly used Monte Carlo integration techniques, this program possesses the same strengths and weaknesses as any other Gauss–Legendre phase space integrator. Its strengths lie in the fact that the underlying physics of the problem at hand may be reflected in the construction of the integration grid, leading to rapid convergence for certain types of problems where there are clear resonance structures in the intermediate state of the process; also, the user may exercise a great degree of control over the density of grid point allocation. Its weaknesses are that differential rates are not trivial to obtain and that matrix elements which are strongly peaked with respect to angular variables can cause slow convergence of the integral. This last problem could be handled by making careful changes of integration variable, distributing the grid points more evenly over the integrand. This is equivalent to the technique of importance sampling used in Monte Carlo integration.

In the first section of this appendix we describe the small amount of theoretical work which was required in order to put the n-body phase space problem into a form suitable for solution by a recursive algorithm. The next section gives some specifics of implementation, while the final section goes over a simple example problem in some detail.
B.1 Theory

When computing a cross-section or decay process with \( n \) particles in the final state we encounter the operator (Ref. [57, pp. 54])

\[
R_n(s) = \prod_{i=1}^{n} \frac{d^2 p_i}{2E_i} \delta^4 (q - \sum_{i=1}^{n} \mathbf{p}_i); \quad q = (\sqrt{s}, 0)
\]

where \( \sqrt{s} \) is either the centre of mass energy of the colliding initial state particles or the mass of the decaying particle. Using the identity

\[
\frac{1}{2E_i} = \int d\mathbf{p}_i \delta (p_i^2 - E_i^2) = \int d\mathbf{p}_i \delta (p_i^2 - p_i^2 - m_i^2) = \int d\mathbf{p}_i \delta (p_i^2 - m_i^2)
\]

we can rewrite this as an integral over four-momenta.

\[
R_n(s) = \prod_{i=1}^{n} d^4 \mathbf{p}_i \delta (p_i^2 - m_i^2) \delta^4 (q - \sum_{i=1}^{n} \mathbf{p}_i)
\]

By employing certain decompositions of unity

\[
1 = \int dm_i^2 \delta (q_i^2 - m_i^2); \quad 1 = \int d^4 q_i \delta (q_i - p_i - p_k) \quad (B.1)
\]

the phase space operator can be written in the form we require. For example, a four body final state may be written as

\[
R_4(s) = \int_{(m_1 + m_2)^2}^{(\sqrt{s} - m_3 - m_4)^2} dm_1^2 \int_{(m_1 + m_2)^2}^{(\sqrt{s} - m_1)^2} dm_2^2 \int_{(m_1 + m_2)^2}^{(\sqrt{s} - m_2)^2} dm_3^2 \int_{(m_1 + m_2)^2}^{(\sqrt{s} - m_4)^2} dm_4^2 \times R_2(m_1^2, m_2^2)R_2(m_3^2, m_4^2)R_2(m_3^2, m_4^2)
\]

or in another form

\[
R_4(s) = \int_{(m_1 + m_2)^2}^{(\sqrt{s} - m_3 - m_4)^2} dm_1^2 \int_{(m_1 + m_2)^2}^{(\sqrt{s} - m_1)^2} dm_2^2 \int_{(m_1 + m_2)^2}^{(\sqrt{s} - m_2)^2} dm_3^2 \int_{(m_1 + m_2)^2}^{(\sqrt{s} - m_4)^2} dm_4^2 \times R_2(m_1^2, m_2^2)R_2(m_3^2, m_4^2)R_2(m_3^2, m_4^2)
\]

which differ due to differing choices \( i (i, j, k) \) in the use of Eq. (B.1). The basic unit which makes up each of these expressions is the \( R_2(\ldots) \) function, which can be thought of as describing the decay of a particle of mass \( M \) to two particles of masses \( m_1 \) and \( m_2 \), integrated over all decay angles.

\[
R_2(M^2, m_1^2, m_2^2) = \int d^4 \mathbf{p}_1 \int d^4 \mathbf{p}_2 \delta (p_1^2 - m_1^2)\delta (p_2^2 - m_2^2)\delta (q - p_1 - p_2)
\]

\[
= \frac{\lambda^2 (M^2, m_1^2, m_2^2)}{8M^2} \int_{-1}^{1} d(cos \theta) \int_{0}^{2\pi} d\phi
\]
B.2 Implementation

In the above we have omitted some straightforward algebra—namely, integration of \( p_2^e \) over the \( \delta^e() \)-function, then integration of \( p_1^e \) and \( |p_1| \) over the remain two \( \delta() \)-functions. The angles \( \theta \) and \( \phi \) are defined in the rest frame of the decaying particle.

This algebra has served to reduce our \( n \)-body problem to a series of two body decays, which proceed through a set of \( (n-1) \) fictitious intermediate states. For \( n > 3 \) it is clear that we can construct several different decay chains which are equal upon integration. When we come to implement this procedure on the computer it will turn out that certain decay topologies, though identical analytically, will have better convergence properties than others for particular problems. The decay topology best reflecting any underlying resonance structure in the process's matrix element will tend to converge more rapidly.

B.2 Implementation

The computer language \( C++ \) was chosen for the implementation of this phase space algorithm for several reasons. Firstly, it lends itself to the creation of a user interface which is specific to the application. The user need then be concerned only with how to communicate with the application through a set of interface routines. Specific details of implementation are closed to the user. If a better method of implementing the application is found, its code can be completely rewritten with only the user-interface routines being kept the same. Programs written using the original package may be used unmodified with the new implementation. Secondly, the phase space integration algorithm used here lends itself very naturally to implementation using recursion. Recursive algorithms are difficult to express in some languages, such as the traditional scientific programming language \( FORTRAN \), since they have no straightforward facility through which a routine may call itself. We shall now briefly go over the data structures and methods used in the class library \( PHASE++ \).

The principle data structure is a binary tree which represents the tree structure of our decay problem in a natural fashion. Each element in the tree is an instance of the class \( PS\text{Link} \) and holds information about a single intermediate particle, including pointers to its decay products and its associated integrals, or about a final state particle at the bottom of the decay chain. A collection of these \( PS\text{Links} \) may be fitted together within an instance of another class called \( Tree \). The full structure holds all the data required for a phase space integration. Each link on the tree has an identifying code number; negative for intermediate states and positive for final state particles.

The user communicates with this data structure through an instance of the class \( Rate \), whose routines form the front end for the package. Generally one must pass information about the particle interaction to the instance of \( Rate \) (such as interac-
tion energies, particle masses and the decay topology), and then request that certain quantities be calculated.

The fourth and final class used here is called Integral and holds the information about a single Gauss–Legendre integration grid. While the other three classes are specific to this phase space problem, the class Integral is quite general and has proved useful in other calculations.

Recursive techniques have been heavily used in this package. One steps up and down the links of the tree structure, performing some operation at each stage. Examples would be the totalling of final state particle masses or the computation of the set of nested integrals which make up the phase space integration.

### B.3 Usage

We shall consider the calculation of the muon leptonic decay rate \( \Gamma(\mu^- \rightarrow e^- \nu_{\mu} \bar{\nu}_e) \) along with the concurrent calculation of the outgoing electron's energy spectrum, as measured in the muon rest frame. The program required is shown in Fig. B.1, along with a line by line description of the user's communication with the application.

The first line of the program includes the file containing the definitions of the C++ classes, as well as some auxiliary information such as a file containing definitions of physical constants and particle masses. Next we have a routine to calculate the four-vector dot product. The routine muondecay() accepts two pointers; one to the initial state particle four-vectors and one to the final state four-vectors. It is the user's responsibility to use these four-vectors to compute the spin-averaged amplitude squared for the process at hand.

The routine main() contains the communication with the phase space integrator.

- **Rate rt(decay, 3);** An instance of Rate is created, and a three body decay with the default topology is selected (see Fig. B.2). The first parameter could also be xsection—implying that a cross section is to be computed—while an optional third parameter can be used to select non-default topologies.

- **rt.initialmass(mass.meson);** The initial state particle mass is specified. If this were a cross section calculation then both initial state particle masses would be passed, as well as a third parameter giving the interaction energy \( \sqrt{s} \) in GeV.

- **rt.finalmass(0, 0, mass.electron);** The final state masses are now passed.

- **rt.matrixelement(muondecay);** The name of the user's routine which calculates the amplitude squared is given.
#include "phase.h"

// A 4-vector dot product routine

double VDOT(double *v1, double *v2)
{
    return (v1[0]*v2[0]-v1[1]*v2[1]-v1[2]*v2[2]-v1[3]*v2[3]);
}

// User's matrix-element-squared routine

double muondecay(double* ppi, double* ppf)
{
    double *km, *ne, *nm, *ke;
    km=ppi;
    ke=ppf+8;
    ne=ppf+4;
    nm=ppf;
    return (64*G.Fermi*G.Fermi*VDOT(km,ne)*VDOT(nm,ke));
}

// User's main routine

main()
{
    Rate rt(decay, 3);
    rt.initialmass(massMuon);
    rt.finalmass(0, 0, massElectron);
    rt.matrixelement(muondecay);
    rt.total();
    int magic=rt.invariant(1, 2);
    rt.integrate();
    rt.inv2energy(magic);
    rt.output();
    return 0;
}

Figure B.1: A program to calculate the muon decay rate and the resulting electron energy spectrum, written using the C++ library PHASE++. 

Figure B.2: A listing of the decay topologies supported by the package PHASE++. The initial state of each diagram (the circled cross) can arise from either a single particle's decay or from the interaction of two particles. The (*,1) topology is the default in each case. Notice that topology (3,2) is just a renumbering of (3,1) and has been inserted for convenience.
• `rt.total();` We request the total decay rate.

• `int magic=rt.invariant(1,2);` The decay spectrum with respect to the invariant mass of particles 1 and 2 (the two neutrinos) is requested. This will later be converted to the energy spectrum of the electron. The integer `magic` is a "magic number" which will be passed to the conversion routine `Rate::inv2energy(...)` in order to identify this invariant mass spectrum.

• `rt.integrate();` This is the main routine which does the integration and constructs the requested rates and spectra.

• `rt.inv2energy(magic);` The invariant mass spectrum is now converted to an energy spectrum—it transpires that they are very simply related to one another.

• `rt.output();` Finally the requested rates are output.

A section of the output of this example is shown in Fig. B.3.

For reference, a list of the various decay topologies is given in Fig. B.2. Also shown there are the reference number to be passed as the third parameter of `Rate::Rate(...)` in order to request a particular topology, and the code numbers assigned to each intermediate state and final state particle.
PHASE++ v1.00

Date: Tue Sep 5 23:45:14 1995

Total decay rate (GeV and sec⁻¹)
3.00868e-19 457098

Differential decay rate with respect to C.M. energy
0.000511075 5.44713e-23
0.0005111397 1.25133e-22
0.0005111978 1.96446e-22
0.000512816 2.68222e-22
0.000613912 3.40538e-22
0.000515266 4.13511e-22
0.000516878 4.87268e-22
0.000518748 5.61942e-22
0.000520876 6.37668e-22
0.000523261 7.14584e-22
0.000526004 7.92826e-22
0.000528805 8.72535e-22
0.000531964 9.53852e-22
0.000535381 1.03692e-21
0.000539055 1.12188e-21
0.000542987 1.20887e-21
0.000547176 1.29806e-21
0.000551624 1.38957e-21

Figure B.3: A section of the output of the muon decay rate program shown in Fig. B.1.
Appendix C

Angular Average of Photon Emission Operators

In this appendix we shall compute in closed form a function which commonly arises in soft photon calculations (see for example, Refs. [73, 129] and our Eq. 16.1),

\[ A \equiv \frac{|\vec{k}|^2}{e^2} \int \frac{d\Omega_k}{4\pi} \sum_\lambda |\mathcal{J}_\lambda^\mu(\lambda)|^2 \]

where \( \vec{k} \) is the photon 3-momentum and the operator describing the emission of a real photon is

\[ \mathcal{J}_\lambda^\mu \equiv \sum_{i=1}^{n} \eta_i(cQ_i) \frac{p_i^\mu}{p_i \cdot \vec{k}}. \]

Here we have defined \( \eta_i = \pm 1 \) for \( i = \frac{\text{final}}{\text{initial}} \) state charged particles. This function \( A \) can be decoupled in the calculation of a scattering process involving soft photon emission by assuming that the photon energy is small enough to be ignored in the four-momentum conserving \( \delta \)-function.

Employing the relation for the photon polarization four-vector \( \sum_\lambda \epsilon_\mu(\lambda)\epsilon_\nu^*(\lambda) = -g_\mu\nu \) and defining \( k^\mu \equiv k^\mu / |\vec{k}| \) we have

\[ A = -\int \frac{d\Omega_k}{4\pi} \left( \sum_{i=1}^{n} \frac{Q^2 m_i^2}{(p_i \cdot \vec{k})^2} + 2 \sum_{i<j=1}^{n} \eta_i \eta_j Q_i Q_j \frac{p_i \cdot p_j}{p_i \cdot k p_j \cdot k} \right). \]

We must evaluate two types of integral here; the first is

\[ \int \frac{d\Omega_k}{4\pi} \frac{Q^2 m^2}{(p \cdot k)^2} = \frac{Q^2 m^2}{E^2} \int \frac{d\Omega_k}{4\pi} \frac{1}{(1 - \beta \cos \theta)^2} \]

\[ = \frac{Q^2 m^2}{E^2} \frac{1}{(1 - \beta^2)} \]

\[ = Q^2 \]
where the velocity of the charged particle is \( \frac{\vec{v}}{E} \) and its rest mass is \( m \). The second integral is a little less trivial,

\[
\int \frac{d\Omega_k}{4\pi} \eta Q Q' \frac{p'p}{p' \cdot k} = \eta Q Q' (1 - \vec{\beta} \cdot \vec{\beta}') \int \frac{d\Omega_k}{4\pi} \frac{1}{(1 - \vec{\beta} \cdot \vec{k})(1 - \vec{\beta}' \cdot \vec{k})}.
\]

We insert the Feynman parameterization

\[
\frac{1}{ab} = \int_0^1 \frac{dz}{(az + b(1 - z))^2}
\]

and proceed with the angular integral, and the integral on \( z \).

\[
I(\vec{\beta}, \vec{\beta}') = \int \frac{d\Omega_k}{4\pi} \frac{1}{(1 - \vec{\beta} \cdot \vec{k})(1 - \vec{\beta}' \cdot \vec{k})}
\]

\[
= \int_0^1 dz \int \frac{d\Omega_k}{4\pi} \frac{1}{[1 - (\vec{\beta} \cdot \vec{k})z + (1 - \vec{\beta}' \cdot \vec{k})(1 - z)]^2}
\]

\[
= \int_0^1 \frac{dz}{1 - (\vec{\beta} \cdot \vec{k} + \vec{\beta}'(1 - z))^2}
\]

\[
= \frac{1}{1 - |\vec{\beta} - \vec{\beta}'|^2} \int_0^1 \frac{dz}{\left(z - \frac{(\vec{\beta}' - \vec{\beta} \cdot \vec{\beta}')}{|\vec{\beta} - \vec{\beta}'|^2} \right)^2 - (1 - \vec{\beta}'^2)(1 - \vec{\beta}^2)}
\]

After some algebra and using the standard integral

\[
\int \frac{dx}{x^2 - 1} = \frac{1}{2} \log \left| \frac{z - 1}{z + 1} \right|
\]

we find the result

\[
I(\vec{\beta}, \vec{\beta}') = \frac{1}{2\sqrt{D_{\beta\beta'}}} \log \left| \frac{\left(\beta^2 - \vec{\beta} \cdot \vec{\beta}' + \sqrt{D_{\beta\beta'}}\right) - \vec{\beta} \cdot \vec{\beta}' + \sqrt{D_{\beta\beta'}}}{\left(\beta^2 - \vec{\beta} \cdot \vec{\beta}' - \sqrt{D_{\beta\beta'}}\right) - \vec{\beta} \cdot \vec{\beta}' - \sqrt{D_{\beta\beta'}}} \right|
\]

where

\[
D_{\beta\beta'} \equiv (1 - \vec{\beta} \cdot \vec{\beta}')^2 - (1 - \beta^2)(1 - \beta'^2)
\]

Upon application of these integrals in our expression we find

\[
A = \frac{|\vec{k}|^2}{e^2} \int \frac{d\Omega_k}{4\pi} |f_x e^o(\lambda)|^2 = - \sum_{i=1}^n Q_i^2 - \sum_{i<j}^n 2\eta_i \eta_j Q_i Q_j (1 - \vec{\beta}_i \cdot \vec{\beta}_j) I(\vec{\beta}_i, \vec{\beta}_j).
\]
In Fig. C.1 we graph this electro-magnetic weighting function versus the centre-of-mass frame scattering angle for the interaction $\pi^- p \rightarrow \pi^- p\gamma$ using a number of incident pion momenta. Since we are working under the assumption that the photon momentum is small enough that it does not perturb the hadron kinematics, this scattering angle is exactly that of the corresponding $\pi^- p$ elastic interaction. Figs. C.2–C.3 show similar plots but for the processes $pp \rightarrow pp\gamma$ and $np \rightarrow np\gamma$. 

Figure C.1: The electro-magnetic factor $A$ as a function of $\pi^- p$ centre-of-mass scattering angle for the process $\pi^- p \rightarrow \pi^- p\gamma$, for several incident pion lab momenta.
Figure C.2: The electro-magnetic factor $A$ as a function of $pp$ centre-of-mass scattering angle for the process $pp \rightarrow pp\gamma$, for several incident proton lab momenta.
Figure C.3: The electro-magnetic factor $A$ as a function of $np$ centre-of-mass scattering angle for the process $np \rightarrow np\gamma$, for several incident proton lab momenta.
Appendix D

Decoupling a Process at a Resonance Line

We shall consider the general problem of expressing the cross section for a process containing a resonance in terms of the cross section and decay rate for the sub-processes describing resonance production and decay. The claim is usually made that so long as the resonance is narrow and therefore long-lived the total cross section is just the resonance production cross section multiplied by the resonance's branching ratio to the required final state. That is, in terms of the particle labels of Fig. D.1

\[ \sigma_{12\ldots n} \approx \sigma_{12\ldots R(m+1)\ldots n} \times B.R_{R\rightarrow 3\ldots m}. \]  

(D.1)

There are, however, several approximations which must be made in order to achieve this result. The first of these concerns the case where the resonance is not spin-0—any helicity information carried by the resonance is clearly not accounted for in the simple expression given in Eq. (D.1). The full expression would also depend on the behaviour of the resonance production and decay sub-processes as the resonant state is taken off-shell—this off-shell dependence must also be approximated away.

We write the amplitudes for the full process and for each of the sub-processes with the external particle wavefunctions extracted.

\[
T_A \equiv \phi_0^1 \phi_4^1 \ldots \phi_n^1 [T_A'] \phi_1 \phi_2 \\
T_B^{(a)} \equiv \phi_R^{(a)} \phi_{(m+1)}^1 \ldots \phi_n^1 [T_B'] \phi_1 \phi_2 \\
T_C^{(s')} \equiv \phi_3^1 \ldots \phi_m^1 [T_C'] \phi_R^{(s')}
\]

These wavefunctions may possess helicity indices. We suppress this index notation except for the wavefunctions of the resonances. The function \([T_A']\) for the full process may be expressed in terms of the functions \([T_B']\) and \([T_C']\) for the sub-processes and in terms of
Appendix D  Decoupling a Process at a Resonance Line

Figure D.1: Four-momentum and amplitude labels for a general scattering process containing a resonance line.

the resonant state's propagator.

\[ [T_A'] = [T_C'] P_R(q^2) [T_B'] \]

This propagator is written in terms of a line-shape function \( G(q^2) \)

\[ P_R(q^2) \equiv \sum_{\alpha \alpha'} \phi_R^{(\alpha)} \varphi_R^{(\alpha')} G(q^2) \delta_{\alpha \alpha'} \]

For this propagator to be exact both positive and negative energy states would have to be included in the summation. By only including positive energy particle states we are in effect making a non-relativistic approximation. A Lorentz invariant expression for the line-shape function is, however, used. We choose a Breit-Wigner form for this function, \( G(q^2) = 1/(q^2 - m_R^2 + i m_R \Gamma_R) \). Other forms could be employed without significantly changing our arguments. The amplitude for the full process may then be written

\[
T_A = \sum_{\alpha \alpha'} \frac{\phi^1 \cdots \phi^m [T_C'] \phi_R^{(\alpha)} \phi_R^{(\alpha')} \phi_{(m+1)}^1 \cdots \phi_{n}^1 [T_B'] \phi_1 \phi_2 \delta_{\alpha \alpha'}}{q^2 - m_R^2 + i m_R \Gamma_R} \\
= \sum_{\alpha \alpha'} \frac{T_C^{(\alpha')} T_B^{(\alpha)}}{q^2 - m_R^2 + i m_R \Gamma_R} \delta_{\alpha \alpha'}
\]

We define spin density matrices for the sub-processes which are labelled by the resonance helicity, where \( S_R, S_1 \) and \( S_2 \) are the spins of the resonance and of the initial
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state particles.

\[
\begin{align*}
[\varphi_{12-R(m+1)...n}]_{\alpha'\beta'} & \equiv \frac{1}{(2S_1 + 1)(2S_2 + 1)} T^{(a)} T^{(b)}_B \\
[\varphi_{R-s...m}]_{\alpha'\beta'} & \equiv \frac{1}{(2S_R + 1)} T^{(a')} T^{(b)}_C
\end{align*}
\]

The amplitude squared for the full process, summed and averaged on final and initial state spins, is then

\[
\sum_{\text{SPINS}} |T_A|^2 = \sum_{\alpha\beta'} (2S_R + 1) \left[ \varphi_{R-s...m} \right]_{\alpha\beta} \left[ \varphi_{12-R(m+1)...n} \right]_{\alpha'\beta'} \delta_{\alpha\alpha'} \delta_{\beta\beta'}
\]  

We shall now relate these expressions to the physical cross sections and decay rates for the processes, and impose the approximations required to achieve the form of Eq. (D.1). The cross section for the full process and the cross section and decay rate for the sub-processes may be written

\[
\sigma_{12-s...n}(q^2) = \frac{1}{2\lambda^2(s, m_1^2, m_2^2)(2\pi)^{2n-10}} \sum_{\text{SPINS}} |T_A|^2
\]

\[
\Gamma_{R-s...m}(q^2) = \frac{1}{2m_R(2\pi)^{2m-10}} \sum_{\alpha\beta'} \delta_{\alpha\alpha'} \left[ \varphi_{R-s...m} \right]_{\alpha'\beta'}
\]

Here the \(R()\) are the phase space operators defined in § B.1. We now make an apparently quite arbitrary replacement in Eq. (D.2):

\[
(2S_R + 1) \delta_{\alpha\alpha'} \delta_{\beta\beta'} \rightarrow \delta_{\alpha\beta} \delta_{\alpha'\beta'}
\]

This allows us to write

\[
\sigma_{12-s...n} \approx \frac{m_R}{\pi} \int (\sqrt{s-m_{(m+1)}^2}...m_n)^2 \frac{\sigma_{12-R(m+1)...n}(q^2)\Gamma_{R-s...m}(q^2)}{(q^2 - m_R^2)^2 + m_R^2 T_R^2} \]  

It is straightforward to show that this replacement of the delta-functions leaves Eq. (D.2) unchanged so long as either the resonance's production or decay process is independent of...
resonance helicity. For a spin-0 resonance there is no helicity consideration and Eq. (D.4) always holds.

If we now further assume that the cross section and decay rate for the sub-processes are well approximated by their values at \( q^2 = m_R^2 \)

\[
\sigma_{12 \to R(m+1)\ldots n}(q^2) \approx \sigma_{12 \to R(m+1)\ldots n}(m_R^2)
\]

\[
\Gamma_R \to m_{m+1} \ldots m_n(q^2) \approx \Gamma_R \to m_{m+1} \ldots m_n(m_R^2)
\]

(D.5)

then we may move the sub-process rates outside of the integral. Since the contribution of these off-shell rates is weighted under the integral by the line-shape function, the on-mass-shell approximation will be better for narrower resonance widths.

\[
\int_{q^2_{\text{min}}}^{q^2_{\text{max}}} \frac{dq^2}{\pi (q^2 - m_R^2)^2 + m_R^2 \Gamma_R^2} = \frac{1}{\pi \Gamma_R} \left[ \arctan \left( \frac{q^2_{\text{max}} - m_R^2}{m_R \Gamma_R} \right) + \arctan \left( \frac{m_R^2 - q^2_{\text{min}}}{m_R \Gamma_R} \right) \right]
\]

The \( q^2 \) integral over the resonance line-shape may be performed exactly.

If the line-shape is well contained within the integration range then we may let \( q^2_{\text{max}} \) and \( q^2_{\text{min}} \) go to \(+\infty\) and \(-\infty\), respectively. In this case the integral above becomes \( 1/\Gamma_R \).

This results in the factorized form for the full cross section which was given in Eq. (D.1).

Of the three approximations used above, only the first two, given in Eq. (D.3) and Eq. (D.5), are really necessary—they serve to remove the dependence of the full process upon the resonant state's helicity and its off-shell behaviour. The final approximation, that the resonance line-shape be well contained within the kinematic integration region, merely gave us the intuitively expected form for the factorized cross section. The arguments above would incidentally be unchanged for the case of decomposition of a decay process, rather than a cross section, at a resonance line.
Appendix E

Some Useful Integrals

In this appendix we shall state, and in most cases derive, the form of certain integrals used in Part I of the thesis. All of these integrals arise during the evaluation of amplitudes associated with loop diagrams in perturbation theory.

(i) The introduction of Feynman parameters through the expression

\[ \frac{1}{a_1 a_2 \ldots a_n} = \int_0^1 dz_1 \int_0^{1-z_1} dz_2 \ldots \int_0^{1-\sum_{i=1}^{n-1} z_i} dz_n \frac{\Gamma(n)\delta(1-z_1-z_2-\ldots-z_n)}{[\sum_{i=1}^n z_i a_i]^n} \]  

serves to combine several factors in the denominator of an integral.

(ii) \[ H_n(a) \equiv \int_0^1 dx z^n \log [1 - a(z - z^2)] \]

We shall first integrate \( H_0(a) \) and then express the \( n \geq 1 \) integrals in terms of it.

\[
H_0(z) \equiv \int_0^1 dx \frac{z}{z^2 - z + \frac{3}{4}} \\
= -2 + 2\sqrt{z - 1}\cot^{-1}\sqrt{z - 1}.
\]

This form is clearly valid only for \( z > 1 \) (i.e. for \( 0 < a < 4 \)). For \( z < 1 \) we write

\[
H_0(z) = -2 + 2i\sqrt{1 - z}\tan^{-1}\frac{1}{i\sqrt{1 - z}}
\]

and note that \( \tan^{-1} z = \frac{i}{2}\log((i + z)/(i - z)) \).
This is valid for the range \( z < 0 \) (i.e. \( a < 0 \)). In the remaining region of \( z \), or equivalently \( a \), the argument of the logarithm becomes negative and so we use the relation \( \log(-a) = \log(a) + i\pi \). For any value of \( a \) the integral is then

\[
H_0(a) = \begin{cases} 
-2 + 2\sqrt{\frac{1}{a} - 1} \cot^{-1}\sqrt{\frac{1}{a} - 1} & \text{for } 0 < a < 4 \\
-2 - \sqrt{1 - \frac{4}{a}} \log\frac{\sqrt{1 - \frac{4}{a}} - 1}{\sqrt{1 - \frac{4}{a}} + 1} & \text{for } a < 0 \\
-2 - \sqrt{1 - \frac{4}{a}} \left( \log\frac{1 - \sqrt{1 - \frac{4}{a}}}{1 + \sqrt{1 - \frac{4}{a}}} + i\pi \right) & \text{for } a > 4
\end{cases}
\] (E.2)

For \( n > 1 \) we proceed in similar fashion,

\[
H_n(z) = \int_0^1 dx \, x^n \log \left[ 1 - \frac{4}{x} (x - z^2) \right]
= \frac{1}{n+1} \int_0^1 dx \, x^{n+1} \frac{4(1-2x)}{1 - \frac{4}{x} (x - z^2)}
\quad \text{after int. by parts}
= -\frac{2}{n+1} \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \, \frac{(y + \frac{1}{2})^{n+1}}{y^2 + \frac{1}{4} (z - 1)}
\quad \text{where } y \equiv x - \frac{1}{2}.
\]

Now, for a particular value of \( n \), the ratio of polynomials in \( y \) may be divided out with the result being either trivially integrable or expressible in terms of \( H_0 \). Only the first few integrals are generally required.

\[
H_1(a) = \frac{1}{2} H_0(a)
\]
\[
H_2(a) = -\frac{1}{18} + \frac{1}{3} \left( 1 - \frac{1}{a} \right) H_0(a)
\]
\[
H_3(a) = -\frac{1}{12} + \frac{1}{4} \left( 1 - \frac{2}{a} \right) H_0(a)
\] (E.3)

(iii)

\[
\mu^{2\epsilon} \int \frac{d^N K}{(2\pi)^N (K^2 - C)^M} \quad \text{where } N \equiv 4 - 2\epsilon, \text{ and for } R \geq 0, \, M > 0
\]

Here \( K^\mu \) is a Minkowski four-vector. We choose to evaluate the integral in Euclidean space by integrating over the variable \( \hat{K} \equiv (iK_0, \vec{K}) \) and then analytically continue the result back to Minkowski space at the end. We consider the integral

\[
\mu^{2\epsilon} \int \frac{d^N \hat{K}}{(2\pi)^N (\hat{K}^2 + C)^M}
\] (E.4)

and write the differential element \( d^N \hat{K} \) in terms of \( N \)-dimensional spherical polar coordinates

\[
d^N \hat{K} = \hat{K}^{\nu-1} d\hat{K} \sin^{\nu-2} \theta_1 d\theta_1 \ldots \sin \theta_{\nu-2} d\theta_{\nu-2} d\phi
\]
We now repeatedly use the result
\[ \int_0^\pi \sin^m \theta \, d\theta = \sqrt{\pi} \frac{\Gamma\left(\frac{m}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{m}{2} + 1\right)} \]
which may be derived using the beta function
\[ B(r, s) = \int_0^1 dz \, z^{r-1} (1 - z)^{s-1} = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r + s)} \]
implying
\[ \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{m}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{m}{2} + 1\right)} = \int_0^1 dz \, z^{-1/2} (1 - z)^{m-1/2} = \int_0^1 d(x^2 \theta) \frac{1}{\cos \theta} \sin^{n-1} \theta \quad \text{with} \ x \equiv \cos \theta \]
\[ = \int_0^\pi \sin^m \theta \, d\theta. \]
The element \( d^m \hat{K} \) is now
\[ d^m \hat{K} = \hat{K}^{m-1} \, d\hat{K} \sqrt{\pi^{m-1} \frac{\Gamma\left(\frac{m}{2} - \frac{1}{2}\right)\Gamma\left(\frac{m}{2} - 1\right)}{\Gamma\left(\frac{m}{2} - \frac{1}{2}\right)\Gamma\left(\frac{m}{2} - \frac{1}{2}\right)}} = \frac{2\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)} \hat{K}^{m-1} \, d\hat{K}. \]
The integral of Eq. (E.4) becomes
\[
\mu^2 \int \frac{d^m \hat{K}}{(2\pi)^N (\hat{K}^2 + C)^M} \left(\hat{K}^2\right)^R
\]
\[ = \frac{2\pi^{\frac{m}{2}}}{(2\pi)^N \mu^2 \Gamma\left(\frac{m}{2}\right)} \int_0^\infty d\hat{K} \frac{\hat{K}^{m+M-1}}{(\hat{K}^2 + C)^M}
\]
\[ = \frac{C^{m+M}}{16\pi^3 (4\pi \mu^2)^{-1} \Gamma\left(\frac{m}{2}\right)} \int_0^\infty d(y) \, (y^2)^{m+M-1} \left(1 + (y^2)\right)^{-M} \quad \text{where} \ y \equiv \frac{\hat{K}}{\sqrt{C}}
\]
\[ = \frac{C^{m+M}}{16\pi^3} \left(\frac{C}{4\pi \mu^2}\right)^{-1} \frac{\Gamma\left(\frac{m}{2} + R\right)\Gamma(M - R - \frac{m}{2})}{\Gamma\left(\frac{m}{2}\right)\Gamma(M)} \]
where in arriving at the last line we have used another integral form for the beta function,
\[ B(r, s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r + s)} = \int_0^\infty x^{r-1} (1 + x)^{-m-n} \, dx. \]
Analytic continuation of this result into the Minkowski metric brings with it a factor \( i(-)^{m-n} \). This can be seen directly by consideration of the left-hand-side of the following expression and from the relation of \( K^m \) to \( \hat{K} \).
\[
\mu^2 \int \frac{d^m K}{(2\pi)^N (K^2 - C)^M} \left(K^2\right)^R
\]
\[ = \frac{i(-)^{m-n} C^{m-N+R}}{16\pi^3} \left(\frac{C}{4\pi \mu^2}\right)^{-1} \frac{\Gamma(R + 2 - \epsilon)\Gamma(M - R - 2 + \epsilon)}{\Gamma(2 - \epsilon)\Gamma(M)} \]
For a less pedestrian approach to the integrals (iii) and (iv), see Ref. [32, App. B]

(iv)

\[ \mu^2 \int \frac{d^n \ell}{(2\pi)^n} \frac{\ell \mu \ell \ldots \ell}{(\ell^2 + 2p \cdot \ell + M^2)^4} \]

These can be reduced to the form of the integral (iii) above. The simplest of them is

\[ \mu^2 \int \frac{d^n \ell}{(2\pi)^n} \frac{1}{(\ell^2 + 2p \cdot \ell + M^2)^4} = \mu^2 \int \frac{d^4 K}{(2\pi)^4} \frac{1}{(K^2 - C)^4} \]

with \( K^\mu \equiv \ell^\mu + p^\mu, \ C \equiv p^2 - M^2 \). Upon application of the integral (iii), this reduces to

\[ = \frac{i(M^2 - p^2)^{2-A}}{16\pi^2 \Gamma(A)} \left( -\frac{(M^2 - p^2)}{4\pi \mu^2} \right)^{-1} \Gamma(A - 2 + \epsilon). \]

The integrals with one or more \( l^\mu \) four-vectors in the numerator proceed similarly. Further results required for their integration are:

\[ \int d^n K \ K^\mu K^\nu F(K^2) = \int d^n K \ \frac{K^2}{N} g^{\mu \nu} F(K^2) \quad \text{under symmetric integration} \]

and

\[ \int d^n K \ K^\mu K^\nu K^\alpha K^\beta F(K^2) = \int d^n K \ \frac{K^4}{N(N + 2)} (g^{\mu \nu} g^{\alpha \beta} + g^{\mu \alpha} g^{\nu \beta} + g^{\mu \beta} g^{\nu \alpha}) F(K^2) \]

where \( F(K^2) \) is an arbitrary function; also, terms which are odd in powers of \( K^\mu \) in the numerator vanish under the integral. The factors \( 1/N \) and \( 1/N(N + 2) \) in the above expressions are obtained by contracting both sides with \( g_{\mu \nu} \), for the former relation, or with \( g_{\mu \nu} g_{\alpha \beta} \), for the latter relation—recall that in \( N \)-dimensional space the relation \( g^{\mu \nu} = N \) holds.

Some of the resulting integrals are given below.

\[ \mu^2 \int \frac{d^n \ell}{(2\pi)^n} \frac{\ell}{(\ell^2 + 2p \cdot \ell + M^2)^4} = \frac{-i(M^2 - p^2)^{2-A}}{16\pi^2 \Gamma(A)} \left( -\frac{(M^2 - p^2)}{4\pi \mu^2} \right)^{-1} \Gamma(A - 2 + \epsilon)p^\mu \]

\[ \mu^2 \int \frac{d^n \ell}{(2\pi)^n} \frac{\ell \ell^{\nu}}{(\ell^2 + 2p \cdot \ell + M^2)^4} = \frac{i(M^2 - p^2)^{2-A}}{16\pi^2 \Gamma(A)} \left( -\frac{(M^2 - p^2)}{4\pi \mu^2} \right)^{-1} \Gamma(A - 2 + \epsilon)p^\mu p^\nu + \frac{1}{2}(M^2 - p^2)\Gamma(A - 3 + \epsilon)g^{\mu \nu} \]

(E.5)
\[
\left( \partial_{a} \partial_{d} q + \partial_{a} \partial_{d} \psi + \partial_{a} \partial_{d} \phi \right) (\nabla + \nabla - \nabla) I_{c}(c^{d} - c^{dW}) \right] + \\
\partial_{a} \partial_{d} \partial_{d} (\nabla + \nabla - \nabla) \left[ \left( \frac{dV}{e^{d} - c^{dW}} \right) \left( \frac{dV}{e^{d} - c^{dW}} \right) \right] + \\
\frac{V (c^{d} + c^{dW} - c^{dW})}{\partial_{a} \partial_{d} \partial_{d}} \int_{\mathcal{D}} d^{4} \tau
\]

\[
\left[ \partial_{a} \partial_{d} \partial_{d} q + \partial_{a} \partial_{d} \partial_{d} \psi + \partial_{a} \partial_{d} \partial_{d} \phi \right] (\nabla + \nabla - \nabla) I_{c}(c^{d} - c^{dW}) \right] + \\
\partial_{a} \partial_{d} \partial_{d} (\nabla + \nabla - \nabla) \left[ \left( \frac{dV}{e^{d} - c^{dW}} \right) \left( \frac{dV}{e^{d} - c^{dW}} \right) \right] + \\
\frac{V (c^{d} + c^{dW} - c^{dW})}{\partial_{a} \partial_{d} \partial_{d}} \int_{\mathcal{D}} d^{4} \tau
\]
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The R. M. Pearce Memorial Fellowship 1994-1995
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University of Victoria Fellowship 1991-1992
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University of Victoria Fellowship 1990-1991
University of Victoria Petch Award 1990-1991
University of Victoria Teaching Award 1990
University of Glasgow Mackay Smith Award 1990
University of Glasgow Prize in Experimental Physics 1989

Publications:

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Title of Dissertation:

Part I. The $\pi^0\gamma\gamma$ Form Factor
Part II. Validity of Soft Photon Amplitudes
Part III. Soft Photon Excess in Hadron Scattering

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