Third-Order Differential Subordination and Superordination Results for Meromorphically Multivalent Functions Associated with the Liu-Srivastava Operator

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Third-Order Differential Subordination and Superordination Results for Meromorphically Multivalent Functions Associated with the Liu-Srivastava Operator

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There are many articles in the literature dealing with the first-order and the second-order differential subordination and superordination problems for analytic functions in the unit disk, but only a few articles are dealing with the above problems in the third-order case (see, e.g., Antonino and Miller (2011) and Ponnusamy et al. (1992)). The concept of the third-order differential subordination in the unit disk was introduced by Antonino and Miller in (2011). Let Ω be a set in the complex plane C. Also let p be analytic in the unit disk U = {𝑧: 𝑧 ∈ C and |𝑧| < 1} and suppose that 𝜓: C × U → C. In this paper, we investigate the problem of determining properties of functions p(𝑧) that satisfy the following third-order differential superordination:

Ω ⊂ { 𝜓 (p(𝑧), 𝑧, p′(𝑧), 𝑧2p″(𝑧), 𝑧3p‴(𝑧); 𝑧): 𝑧 ∈ U}. As applications, we derive some third-order differential subordination and superordination results for meromorphically multivalent functions, which are defined by a family of convolution operators involving the Liu-Srivastava operator. The results are obtained by considering suitable classes of admissible functions.

1. Introduction, Definitions, and Preliminaries

Let $\mathcal{H}(U)$ be the class of functions which are analytic in the open unit disk:

$$U = \{ z : z \in \mathbb{C}, \, |z| < 1 \}.$$  \hfill (1)

For $n \in \mathbb{N} := \{1, 2, 3, \ldots \}$ and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \left\{ f : f \in \mathcal{H}(U), \quad f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \right\}$$  \hfill (2)

and suppose that $\mathcal{H} = \mathcal{H}[1, 1]$.

Let $f$ and $F$ be members of the analytic function class $\mathcal{H}(U)$. The function $f$ is said to be subordinate to $F$, or $F$ is superordinate to $f$, if there exists a Schwarz function $w(z)$, analytic in $U$ with

$$w(0) = 0, \quad |w(z)| < 1 \quad (z \in U),$$  \hfill (3)

such that

$$f(z) = F(w(z)).$$  \hfill (4)

In such a case, we write

$$f < F \quad \text{or} \quad f(z) < F(z).$$  \hfill (5)

Furthermore, if the function $F$ is univalent in $U$, then we have the following equivalence (see, for details, [1]):

$$f(z) < F(z) \quad (z \in U) \iff f(0) = F(0), \quad f(U) \subset F(U).$$  \hfill (6)

Let $\Sigma_p$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (p \in \mathbb{N})$$  \hfill (7)
which are analytic and multivalent in the punctured unit disk:
\[ U^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \} = U \setminus \{0\}. \] (8)

For the function \( f \) given by (7) and the function \( g \) given by
\[ g(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} b_k z^k \quad (p \in \mathbb{N}; z \in U^*), \] (9)

the Hadamard product (or convolution) \( f \ast g \) of the functions \( f \) and \( g \) is defined by
\[ (f \ast g)(z) := \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k b_k z^k =: (g \ast f)(z). \] (10)

For parameters \( \alpha_i \in \mathbb{C} (i = 1, 2, \ldots, q) \) and \( \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0 \) \((\mathbb{Z}_0 = 0, 1, 2, \ldots)\), the generalized hypergeometric function \( gF_s(\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_j; z) \) is defined by (see, for example, [2, 3])
\[ gF_s(\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_j; z) = \frac{\sum_{k=0}^{\infty} (\alpha_1)_k \cdots (\alpha_i)_k z^k}{(\beta_1)_k \cdots (\beta_j)_k k!} \] (11)

\[ (q \leq s + 1; \; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U}), \]

where \((v)_k\) denotes the Pochhammer symbol defined, in terms of Gamma function, by
\[ (v)_k = \frac{\Gamma(v + k)}{\Gamma(v)} \]
\[ = \begin{cases} 1 & (k = 0; \; v \in \mathbb{C} \setminus \{0\}), \\ v(v+1) \cdots (v+k-1) & (k \in \mathbb{N}; \; v \in \mathbb{C}). \end{cases} \] (12)

Recently, Tang et al. [4] introduced a function \( h^\lambda_{\mu} (\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_j; z) \) defined by
\[ h^\lambda_{\mu} (\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_j; z) = (1 - \lambda + \mu) z^{-\mu} \sum_{k=0}^{\infty} (\alpha_1)_k \cdots (\alpha_i)_k z^k \]
\[ + \lambda \mu z^{\lambda - 1} \sum_{k=0}^{\infty} (\alpha_1)_k \cdots (\alpha_i)_k (\beta_1)_k \cdots (\beta_j)_k k! z^k \] (13)

\[ (p \in \mathbb{N}; \; \lambda, \mu \geq 0; \; z \in \mathbb{U}). \]

In particular, when \( \lambda = \mu = 0 \), we obtain
\[ h^0_{\mu} (\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_j; z) = h_{\mu} (\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_j; z), \] (14)

which was introduced and studied by Liu and Srivastava [5].

Corresponding to the function \( h^\lambda_{\mu} (\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_j; z) \) given by (13), we consider a convolution operator
\[ H^\lambda_{\mu} (\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_j) : \Sigma_p \longrightarrow \Sigma_p \] (15)
defined by the following Hadamard product (or convolution):
\[ H^\lambda_{\mu} (\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_j; z) \ast f(z) = h^\lambda_{\mu} (\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_j; z) \ast f(z). \] (16)

For the sake of convenience, we write
\[ H^\lambda_{\mu} (\beta_1) = H^\lambda_{\mu} (\beta_1; z) = h^\lambda_{\mu} (\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_j; z). \] (17)

It is easily verified from definition (16) that
\[ z (H^\lambda_{\mu} (\beta_1 + 1) f(z))' = \beta_1 H^\lambda_{\mu} (\beta_1) f(z) \]
\[ - (\beta_1 + p) H^\lambda_{\mu} (\beta_1 + 1) f(z), \] (18)

\[ z (H^\lambda_{\mu} (\alpha_1; z) f(z))' = \alpha_1 H^\lambda_{\mu} (\alpha_1 + 1) f(z) \]
\[ - (\alpha_1 + p) H^\lambda_{\mu} (\alpha_1) f(z). \] (19)

We note that, for \( \lambda = \mu = 0 \), the operator \( H^0_{\mu} (\alpha_1) \) reduces to the Liu-Srivastava operator \( H^0_{\mu} (\alpha_1) \) (see [5, 6]; see also [7]), while the Liu-Srivastava operator is the meromorphic analogous of the Dziok-Srivastava operator (see [8–10]; see also [11, 12]), which includes (as its special cases) the meromorphic analogous of the Carlson-Shaffer convolution operator \( L_p(a, c) = H^0_{\mu} (0; 1, a; c) \) (see [13, 14]), the meromorphic analogous of the Ruscheweyh derivative operator \( D^{\mu+1} = L_p(n + p, 1) \) (see [15]), and the operator
\[ I_{\delta,p} = \frac{\delta}{z^{\frac{\delta+1}{p}}} \int_0^z t^{\frac{\delta+1}{p}} f(t) \, dt = L_p(\delta, \delta + p, 1) \quad (\delta > 0) \] (20)

studied by Uralegaddi and Somanatha [16].

Let \( \Omega \) be any set in \( \mathbb{C} \). Also let \( p \) be analytic in \( \mathbb{U} \) and suppose that \( \psi : \mathbb{C}^4 \times \mathbb{U} \rightarrow \mathbb{C} \). Recently, Antonino and Miller [17] have extended the theory of second-order differential subordinations in \( \mathbb{U} \) introduced by Miller and Mocanu [1] to the third-order case. They determined properties of functions \( p(z) \) that satisfy the following third-order differential subordination:
\[ \{ \psi(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z) : z \in \mathbb{U} \} \subset \Omega. \] (21)

We will now recall some definitions and a theorem due to Antonino and Miller [17], which are required in our next investigations.

**Definition 1** (see [17], p. 440, Definition 1). Let \( \psi : \mathbb{C}^4 \times \mathbb{U} \rightarrow \mathbb{C} \) and \( h(z) \) be univalent in \( \mathbb{U} \). If \( p(z) \) is analytic in \( \mathbb{U} \) and satisfies the following third-order differential subordination:
\[ \psi(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z) < h(z), \] (22)
Abstract and Applied Analysis

then \( p(z) \) is called a solution of the differential subordination. A univalent function \( q(z) \) is called a dominant of the solutions of the differential subordination or, more simply, a dominant if \( p(z) < q(z) \) for all \( p(z) \) satisfying (22). A dominant \( \tilde{q}(z) \) that satisfies \( \tilde{q}(z) < q(z) \) for all dominants \( q(z) \) of (22) is said to be the best dominant.

**Definition 2** (see [17], p. 441, Definition 2). Let \( \mathcal{G} \) denote the set of functions \( q \) that are analytic and univalent on the set \( \mathbb{U} \setminus E(q) \), where

\[
E(q) = \left\{ \xi : \xi \in \partial \mathbb{U} \text{ and } \lim_{z \to \xi} q(z) = \infty \right\},
\]

is such that

\[
\min_{\xi} |q'(\xi)| = \rho > 0
\]

for \( \xi \in \partial \mathbb{U} \setminus E(q) \). Further, let the subclass of \( \mathcal{G} \) for which \( q(0) = a \) be denoted by \( \mathcal{G}(a) \) and

\[
\mathcal{G}(1) = \mathcal{G}_1.
\]

**Definition 3** (see [17], p. 449, Definition 3). Let \( \Omega \) be a set in \( \mathbb{C}, q \in \mathcal{G}, \) and \( n \in \mathbb{N} \setminus \{1\} \). The class of admissible functions \( \Psi_n[\Omega, q] \) consists of those functions \( \psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C} \) that satisfy the following admissibility condition:

\[
\psi(r, s, t, u; z) \notin \Omega
\]

whenever

\[
r = q(\xi), \
s = k\xi q'(\xi), \\
Re\left( \frac{t}{s+1} \right) \geq k \Re\left( \frac{\xi q''(\xi)}{q'(\xi)} + 1 \right), \\
\Re\left( \frac{u}{s} \right) \geq k^2 \Re\left( \frac{\xi^2 q'''(\xi)}{q'(\xi)} \right),
\]

where \( z \in \mathbb{U}, \xi \in \partial \mathbb{U} \setminus E(q), \) and \( k \geq n \).

**Theorem 4** (see [17], p. 449, Theorem 1). Let \( p \in \mathcal{H}[a, n] \) with \( n \in \mathbb{N} \setminus \{1\} \). Also let \( q \in \mathcal{G}(a) \) and satisfy the following conditions:

\[
Re\left( \frac{\xi q''(\xi)}{q'(\xi)} \right) \geq 0, \\ \left| \frac{z p'(z)}{q'(\xi)} \right| \leq k,
\]

where \( z \in \mathbb{U}, \xi \in \partial \mathbb{U} \setminus E(q), \) and \( k \geq n \). If \( \Omega \) is a set in \( \mathbb{C}, \psi \in \Psi_n[\Omega, q] \) and

\[
\psi(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z) \in \Omega,
\]

then

\[
p(z) < q(z).
\]

In this paper, following the theory of second-order differential superordinations in the unit disk introduced by Miller and Mocanu [18], we consider the dual problem of determining properties of functions \( p(z) \) that satisfy the following third-order differential superordination:

\[
\Omega \subset \left\{ \psi \left( p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z \right) : z \in \mathbb{U} \right\}.
\]

In other words, we determine the conditions on \( \Omega, \Delta, \) and \( \psi \) for which the following implication holds true:

\[
\Omega \subset \left\{ \psi \left( p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z \right) : z \in \mathbb{U} \right\} \\
\implies \Delta \subset p(\mathbb{U}),
\]

where \( \Delta \) is any set in \( \mathbb{C} \).

If either \( \Omega \) or \( \Delta \) is a simply connected domain, then (32) can be rephrased in terms of superordination. If \( p(z) \) is univalent in \( \mathbb{U} \), and if \( \Delta \) is a simply connected domain with \( \Delta \neq \mathbb{C}, \) then there is a conformal mapping \( q \) of \( \mathbb{U} \) onto \( \Delta \) such that \( q(0) = p(0) \). In this case, (32) can be rewritten as:

\[
\Omega \subset \left\{ \psi \left( p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z \right) : z \in \mathbb{U} \right\} \\
\implies q(z) < p(z).
\]

If \( \Omega \) is also a simply connected domain with \( \Omega \neq \mathbb{C}, \) then there is a conformal mapping \( h \) of \( \mathbb{U} \) onto \( \Omega \) such that \( h(0) = \psi(p(0), 0, 0, 0, 0) \). In addition, if the function

\[
\psi \left( p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z \right)
\]

is univalent in \( \mathbb{U}, \) then (33) can be rewritten as

\[
h(z) < \psi \left( p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z \right)
\]

\[
\implies q(z) < p(z).
\]

There are three key ingredients in the implication relationship (33): the differential operator \( \psi \), the set \( \Omega \), and the "dominating" function \( q \). If two of these entities were given, one would hope to find conditions on the third entity so that (33) would be satisfied. In this paper, we start with a given set \( \Omega \) and a given function \( q \), and then determine a set of "admissible" operators \( \psi \) so that (33) holds true.

We first introduce the following definition.

**Definition 5.** Let \( \psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C} \) and the function \( h(z) \) be analytic in \( \mathbb{U} \). If the functions \( p(z) \) and

\[
\psi \left( p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z \right)
\]

are univalent in \( \mathbb{U} \) and satisfy the following third-order differential superordination:

\[
h(z) < \psi \left( p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z \right),
\]

then \( p(z) \) is called a solution of the differential superordination. An analytic function \( q(z) \) is called a subordinant of
the solutions of the differential superordination or more simply a subordinant if \( q(z) < p(z) \) for \( p(z) \) satisfying (37). A univalent subordinant \( \tilde{q}(z) \) that satisfies the condition

\[
q(z) < \tilde{q}(z)
\]

for all subordinants \( q(z) \) of (37) is said to be the best subordinant. We note that the best subordinant is unique up to a rotation of \( \mathcal{U} \).

For \( \Omega \) a set in \( \mathbb{C} \), with \( \psi \) and \( p \) as given in Definition 5, we suppose that (37) is replaced by

\[
\Omega \subset \{ \psi(p(z), zp(z), z^2p''(z), z^3p'''(z); z) : z \in \mathbb{U} \}.
\]

(39)

Although this more general situation is a "differential containment," we also refer to it as a differential superordination, and the definitions of solution, subordinant, and best subordinant as given above can be extended to this more general case.

We will use the following lemma [17], p. 445, Lemma D] from the theory of third-order differential subordinations in \( \mathbb{U} \) to determine subordinants of the third-order differential superordinations.

**Lemma 6** (see [17]). Let \( p \in \mathcal{O}(a) \), and let \( q(z) = a + a_nz^n + \cdots \) be analytic in \( \mathbb{U} \) with \( q(z) \neq a \) and \( n \in \mathbb{N} \setminus \{1\} \). If \( q(z) \) is not subordinate to \( p \), then there exists points \( z_0 = r_0e^{i\theta_0} \in \mathbb{U} \) and \( \xi_0 \in \partial \mathbb{U} \setminus \mathcal{E}(p) \), and an \( m \geq n \) for which \( q(\xi_0) \in \{1\} \).

(i) \( q(z_0) = p(\xi_0) \).

(ii) \( \Re(\xi_0 p''''(\xi_0)/p'(\xi_0)) \geq 0 \) and \( |zq(z)/p'(\xi_0)| \leq m \).

(iii) \( z_0q(\xi_0) = m\xi_0p'(\xi_0) \).

(iv) \( \Re(1 + z_0q'''(\xi_0)/q'(\xi_0)) \geq m\Re(1 + \xi_0p'''(\xi_0)/p'(\xi_0)) \).

(v) \( \Re(z_0^2q''''(\xi_0)/q'(\xi_0)) \geq m^2\Re(\xi_0^2p''''(\xi_0)/p'(\xi_0)) \).

2. Admissible Functions and a Fundamental Result

We next define the class of admissible functions referred to in the preceding section.

**Definition 7.** Let \( \Omega \) be a set in \( \mathbb{C} \), \( q \in \mathcal{H}[a, n] \) and \( q'(z) \neq 0 \). The class of admissible functions \( \Psi_q[\Omega, q] \) consists of those functions \( \psi : C^4 \times \mathbb{U} \to \mathbb{C} \) that satisfy the following admissibility condition:

\[
\psi(r, s, t, u; \xi) \in \Omega
\]

whenever

\[
r = q(z), \quad s = \frac{zq'(z)}{m},
\]

\[
\Re\left(\frac{r}{s} + 1\right) \leq \frac{1}{m} \Re\left(\frac{zq''(z)}{q'(z)} + 1\right),
\]

\[
\Re\left(\frac{u}{s} \right) \leq \frac{1}{m^2} \Re\left(\frac{z^2q'''(z)}{q'(z)}\right),
\]

where \( z \in \mathbb{U} \), \( \xi \in \partial \mathbb{U} \), and \( m \geq n \geq 2 \).

If \( \psi : C^2 \times \mathbb{U} \to \mathbb{C} \) and \( q \in \mathcal{H}[a, n] \), then the admissibility condition (41) reduces to the following form:

\[
\psi\left( q(z), \frac{zq'(z)}{m}; \xi \right) \in \Omega \quad \text{for all } z \in \mathbb{U}; \xi \in \partial \mathbb{U}; m \geq n \geq 2.
\]

(43)

If \( \psi : C^3 \times \mathbb{U} \to \mathbb{C} \) and \( q \in \mathcal{H}[a, n] \) with \( q'(z) \neq 0 \), then the admissibility condition (41) reduces to the following form:

\[
\psi(r, s, t; \xi) \in \Omega
\]

whenever \( r = q(z) \), \( s = zq'(z)/m \), and

\[
\Re\left(\frac{zq''(z)}{q'(z)} + 1\right) \leq \frac{1}{m} \Re\left(\frac{z^2q'''(z)}{q'(z)} + 1\right)
\]

\[
(z \in \mathbb{U}; \xi \in \partial \mathbb{U}; m \geq n \geq 2).
\]

The next theorem is a foundation result in the theory of the third-order differential superordinations in \( \mathbb{U} \).

**Theorem 8.** Let \( q \in \mathcal{H}[a, n] \) and \( \psi \in \Psi_q[\Omega, q] \). If

\[
\psi\left( p(z), zp'(z), z^2p''(z), z^3p'''(z); z \right)
\]

is univalent in \( \mathbb{U} \) and \( p \in \mathcal{O}(a) \) satisfy the following conditions:

\[
\Re\left(\frac{zq''(z)}{q'(z)} \right) \geq 0, \quad \left|zq'(z)/q'(z)\right| \leq m
\]

\[
(z \in \mathbb{U}; \xi \in \partial \mathbb{U}; m \geq n \geq 2),
\]

then

\[
\Omega \subset \{ \psi\left( p(z), zp'(z), z^2p''(z), z^3p'''(z); z \right) : z \in \mathbb{U} \}
\]

implies that

\[
q(z) < p(z).
\]

(49)

**Proof.** Suppose that

\[
q(z) \geq p(z).
\]

(50)

Then, by the above lemma, there exists points \( z_0 = r_0e^{i\theta_0} \in \mathbb{U} \) and \( \xi_0 \in \partial \mathbb{U} \setminus \mathcal{E}(p) \), and an \( m \geq n \geq 2 \) that satisfy conditions (i)-(v) of the above lemma. Using these conditions with \( r = p(\xi_0), s = \xi_0p'(\xi_0), t = \xi_0^2p''(\xi_0), u = \xi_0^3p'''(\xi_0), \xi = \xi_0 \) in Definition 7, we obtain

\[
\psi\left( p(\xi_0), \xi_0p'(\xi_0), \xi_0^2p''(\xi_0), \xi_0^3p'''(\xi_0); \xi_0 \right) \in \Omega
\]

(51)

which contradicts (48), so we have

\[
q(z) < p(z).
\]

(52)
Abstract and Applied Analysis

In the special case when \( \Omega \neq C \) is a simply connected domain and \( h \) is a conformal mapping of \( U \) onto \( \Omega \), we denote this class \( \Psi_n^\mu[h(U), q] \) by \( \Psi_n^\mu[h, q] \). The following result is an immediate consequence of Theorem 8.

**Theorem 9.** Let \( q \in \mathcal{H}[a, n] \). Also let the function \( h \) be analytic in \( U \) and suppose that \( \psi \in \Psi_n^\mu[h, q] \). If \( p \in \mathcal{O}(a) \) satisfies condition (47) and

\[
\psi \left( p(z), zp'(z), z^2p''(z), z^3p'''(z); z \right) = h(z)
\]

is univalent in \( U \), then

\[
h(z) < \psi \left( p(z), zp'(z), z^2p''(z), z^3p'''(z); z \right)
\]

implies that

\[
q(z) < p(z).
\]

Theorems 8 and 9 can only be used to obtain subordinants of the third-order differential superordination of the forms (48) or (54).

**Theorem 10.** Let the function \( h \) be analytic in \( U \) and let \( \psi : C^4 \times \mathbb{U} \rightarrow C \). Suppose that the differential equation

\[
\psi \left( q(z), zq'(z), z^2q''(z), z^3q'''(z); z \right) = h(z)
\]

has a solution \( q \in \mathcal{O}(a) \). If \( \psi \in \Psi_n^\mu[h, q] \), \( p \in \mathcal{O}(a) \), and

\[
\psi \left( p(z), zp'(z), z^2p''(z), z^3p'''(z); z \right) = h(z)
\]

is univalent in \( U \), then (54) implies that

\[
q(z) < p(z)
\]

and \( q(z) \) is the best subordinant.

**Proof.** Since \( \psi \in \Psi_n^\mu[h, q] \), by applying Theorem 9, we deduce that \( q \) is a subordinant of (54). Since \( q \) satisfies (56), it is also a solution of the differential superordination (54). Therefore, all subordinants of (54) will be subordinate to \( q \). It follows that \( q(z) \) will be the best subordinant of (54).

In the next two sections, by making use of the third-order differential subordination results of Antonino and Miller [17] in the unit disk \( U \) and the third-order differential superordination results in \( U \) obtained in Section 2 (see, for details, Theorems 8, 9, and 10), we determine certain appropriate classes of admissible functions and investigate some third-order differential subordination and differential superordination properties of meromorphically multivalent functions associated with the operator \( H_{p,q,s}^{\lambda,\mu}(\beta_1) \) defined by (16). It should be remarked in passing that, in recent years, several authors obtained many interesting results involving various linear and nonlinear convolution operators associated with (second-order) differential subordination and superordination, and the interested reader may refer to several earlier works including (for example) [19] to [20–23].

### 3. Third-Order Differential Subordination of the Operator \( H_{p,q,s}^{\lambda,\mu}(\beta_1) \)

We first define the following class of admissible functions, which are required in proving the differential subordination theorem involving the operator \( H_{p,q,s}^{\lambda,\mu}(\beta_1) \) defined by (16).

**Definition 11.** Let \( \Omega \) be a set in \( C \) and \( q \in \mathcal{O}_1 \cap \mathcal{H} \). The class of admissible functions \( \Phi_{p,q}^{\lambda,\mu}[\Omega, q] \) consists of those functions \( \phi : C^4 \times U \rightarrow C \) that satisfy the following admissibility condition:

\[
\phi(a, b, c, d; z) \notin \Omega
\]

whenever

\[
a = q(\xi), \quad b = \frac{kq'(\xi) + \beta_1 q(\xi)}{\beta_1},
\]

\[
\Re \left( \frac{(\beta_1 + 1)(c - a)}{b - a} - (2\beta_1 + 1) \right) \geq k \Re \left( \frac{\xi q'(\xi)}{q'(\xi)} + 1 \right),
\]

\[
\Re \left( \frac{(\beta_1 + 1)(\beta_1 + 2)(d - 3c + 3b - a)}{b - a} \right) \geq k \Re \left( \frac{\xi^2 q'''(\xi)}{q'(\xi)} \right),
\]

where \( z \in U, \beta_1 \in C \setminus \{0, -1, -2, \ldots\}, \xi \in \partial U \setminus E(q), \) and \( k \in \mathbb{N} \setminus \{1\} \).

**Theorem 12.** Let \( \phi \in \Phi_{p,q}^{\lambda,\mu}[\Omega, q] \). If the functions \( f \in \Sigma_p \) and \( q \in \mathcal{O}_1 \) satisfy the following conditions:

\[
\Re \left( \frac{\xi q'(\xi)}{q'(\xi)} \right) \geq 0, \quad \left| z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z) \right| \leq k,
\]

\[
\{\phi \left( z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1) f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z) \right),
\]

\[
z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 1) f(z),
\]

\[
z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 2) f(z); z \in U \} \subset \Omega,
\]

then

\[
z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1) f(z) < q(z).
\]

**Proof.** Define the analytic function \( p(z) \) in \( U \) by

\[
p(z) = z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1) f(z).
\]

Then, differentiating (64) with respect to \( z \) and using (18), we have

\[
z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z) = \frac{z^p(\beta_1 + 1) f(z)}{\beta_1}.
\]
Further computations show that
\[ z^p H_{p,q,\alpha}^{\lambda,\mu} (\beta_1 - 1) f(z) = \frac{z^2 p''(z) + 2(\beta_1 + 1) z p'(z) + \beta_1 (\beta_1 + 1) p(z)}{\beta_1 (\beta_1 + 1)}, \]  
\[ z^p H_{p,q,\alpha}^{\lambda,\mu} (\beta_1 - 2) f(z) = \left( z^3 p'''(z) + 3(\beta_1 + 2) z^2 p''(z) + 3(\beta_1 + 1)(\beta_1 + 2) z p'(z) + \beta_1 (\beta_1 + 1)(\beta_1 + 2) p(z) \right) \times (\beta_1 (\beta_1 + 1)(\beta_1 + 2))^{-1}. \]

We now define the transformation from \( C^4 \) to \( C \) by
\[ a(r,s,t,u) = r, \quad b(r,s,t,u) = \frac{s + \beta_1 r}{\beta_1}, \]
\[ c(r,s,t,u) = \frac{t + 2(\beta_1 + 1)s + \beta_1 (\beta_1 + 1)r}{\beta_1 (\beta_1 + 1)}, \]
\[ d(r,s,t,u) = \frac{u + 3(\beta_1 + 2)t + 3(\beta_1 + 1)(\beta_1 + 2)s + \beta_1 (\beta_1 + 1)(\beta_1 + 2)r}{\beta_1 (\beta_1 + 1)(\beta_1 + 2)} \times (\beta_1 (\beta_1 + 1)(\beta_1 + 2))^{-1}. \]

Hence, clearly, (62) becomes
\[ \psi \left( p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z \right) \in \Omega. \]  
We note that
\[ \frac{t}{s} + 1 = \frac{(\beta_1 + 1)(c-a)}{b-a} - (2\beta_1 + 1), \]
\[ \frac{u}{s} = \frac{(\beta_1 + 1)(\beta_1 + 2)(d - 3c + 3b - a)}{b-a}. \]

Thus, the admissibility condition for \( \phi \in \Phi_{H}[\Omega, q] \) in Definition 11 is equivalent to the admissibility condition for \( \psi \in \Psi_{p}[\Omega, q] \) as given in Definition 3 with \( n = 2 \). Therefore, by using (61) and Theorem 4, we have
\[ \mathbf{p}(z) < \mathbf{q}(z) \]
or, equivalently,
\[ z^p H_{p,q,\alpha}^{\lambda,\mu} (\beta_1 + 1) f(z) < \mathbf{q}(z). \]

Our next result is an extension of Theorem 12 to the case where the behavior of \( \mathbf{q}(z) \) on \( \partial \mathbb{U} \) is not known.

Corollary 13. Let \( \Omega \subset C \) and let the function \( \mathbf{q} \) be univalent in \( \mathbb{U} \) with \( \mathbf{q}(0) = 1 \). Suppose also that \( \phi \in \Phi_{H}[\Omega, q_p] \) for some \( p \in (0,1) \), where \( q_p(z) = \mathbf{q}(pz) \). If the functions \( f \in \Sigma_p \) and \( q_p \) satisfy the following conditions:
\[ \Re \left( \frac{\xi q_p''(\xi)}{q_p'(\xi)} \right) \geq 0, \]
\[ \left| \frac{z^p H_{p,q,\alpha}^{\lambda,\mu} (\beta_1) f(z)}{q_p'(\xi)} \right| \leq k \quad \left( z \in \mathbb{U}; \xi \in \partial \mathbb{U} \setminus E(q_p) \right), \]
\[ \phi \left( z^p H_{p,q,\alpha}^{\lambda,\mu} (\beta_1 + 1) f(z), z^p H_{p,q,\alpha}^{\lambda,\mu} (\beta_1) f(z) \right), \]
\[ z^p H_{p,q,\alpha}^{\lambda,\mu} (\beta_1 - 1) f(z), \]
\[ z^p H_{p,q,\alpha}^{\lambda,\mu} (\beta_1 - 2) f(z); z \in \Omega, \]
then
\[ z^p H_{p,q,\alpha}^{\lambda,\mu} (\beta_1 + 1) f(z) < \mathbf{q}(z). \]

Proof. We note from Theorem 12 that
\[ z^p H_{p,q,\alpha}^{\lambda,\mu} (\beta_1 + 1) f(z) < q_p(z). \]

The result asserted by Corollary 13 is now deduced from the following subordination property:
\[ q_p(z) < \mathbf{q}(z). \]
If $\Omega \neq C$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of $U$ onto $\Omega$. In this case, the class $\Phi_H[\Omega, q]$ is written as $\Phi_H[h, q]$. The following two results are immediate consequences of Theorem 12 and Corollary 13.

**Theorem 14.** Let $\phi \in \Phi_H[h, q]$. If the functions $f \in \Sigma_p$ and $q \in \mathcal{O}_1$ satisfy the following conditions:

$$\Re \left( \frac{\phi'}{\phi} \right) \geq 0,$$

$$\frac{\phi(\beta_1 + 1) | f(z) |}{| \beta_1 |} \leq k,$$  \hspace{1cm} (80)

then

$$\phi(\beta_1 + 1) f(z) < q(z).$$  \hspace{1cm} (82)

**Corollary 15.** Let $\Omega \subset C$ and let the function $q$ be univalent in $U$ with $q(0) = 1$. Suppose also that $\phi \in \Phi_H[h, q]$, for some $\rho \in (0, 1)$, where $q_p(z) = q(\rho^z)$. If the functions $f \in \Sigma_p$ and $q_p$ satisfy the following conditions:

$$\Re \left( \frac{\phi'}{\phi} \right) \geq 0,$$

$$\left| \frac{\phi(\beta_1 + 1) f(z)}{q_p(z)} \right| \leq k (z \in U; \xi \in \partial U \cap E(q_p),$$  \hspace{1cm} (83)

then

$$\phi(\beta_1 + 1) f(z) < q(z).$$  \hspace{1cm} (84)

Our next theorem yields the best dominant of the differential subordination (70).

**Theorem 16.** Let the function $h$ be univalent in $U$. Also let $\phi : \mathbb{C}^4 \times U \to \mathbb{C}$ and $\psi$ be given by (70). Suppose that the differential equation

$$\psi(q(z), q', q''(z), q'''(z); z) = h(z)$$  \hspace{1cm} (85)

has a solution $q(z)$ with $q(0) = 1$, which satisfies condition (61). If the function $f \in \Sigma_p$ satisfies condition (81) and the function

$$\phi(\beta_1 + 1) f(z), \phi(\beta_1 - 1) f(z),$$  \hspace{1cm} (86)

is analytic in $U$, then

$$z^{\beta_1} H_{p,\beta_1}^{\lambda,\mu} (\beta_1 + 1) f(z) < q(z)$$  \hspace{1cm} (87)

and $q(z)$ is the best dominant.

**Proof.** By applying Theorem 12, we deduce that $q$ is a dominant of (81). Since $q$ satisfies (85), it is also a solution of (81). Therefore, $q$ will be dominated by all dominants. Hence $q$ is the best dominant.

In view of Definition 11, in the particular case when $q(z) = 1 + Mz$ ($M > 0$), the class $\Phi_H[\Omega, q]$ of admissible functions, denoted simply by $\Phi_H[\Omega, M]$, is described below.

**Definition 17.** Let $\Omega$ be a set in $C, \beta_1 \in C \setminus \{0, -1, -2, \ldots\}$, and $M > 0$. The class $\Phi_H[\Omega, M]$ of admissible functions consists of those functions $\phi : \mathbb{C}^4 \times U \to C$ such that

$$\phi(1 + M, 1 + \frac{k + \beta_1 - M}{\beta_1 - 1}) \geq 0,$$

whenever $z \in \mathbb{C}, \Re (Le^{-i\theta}) \geq (k - 1)kM$ and $\Re (Ne^{-i\theta}) \geq 0$ for all $\theta \in \mathbb{R}$ and $k \in \mathbb{N} \setminus \{1\}$.

**Corollary 18.** Let $\phi \in \Phi_H[\Omega, M]$. If the function $f \in \Sigma_p$ satisfies the following conditions:

$$|z^{\beta_1} H_{p,\beta_1}^{\lambda,\mu} (\beta_1 + 1) f(z)| \leq kM (k \in \mathbb{N} \setminus \{1\}; M > 0),$$

$$\phi(z^{\beta_1} H_{p,\beta_1}^{\lambda,\mu} (\beta_1 + 1) f(z), z^{\beta_1} H_{p,\beta_1}^{\lambda,\mu} (\beta_1) f(z),$$  \hspace{1cm} (89)

then

$$|z^{\beta_1} H_{p,\beta_1}^{\lambda,\mu} (\beta_1 + 1) f(z) - 1| < M (M > 0).$$  \hspace{1cm} (90)

In the special case when

$$\Omega = q(U) = \{\omega : |\omega - 1| < M (M > 0)\},$$  \hspace{1cm} (91)

the class $\Phi_H[\Omega, M]$ is denoted, for brevity, by $\Phi_H[M]$. Corollary 18 can now be rewritten in the following form.
Corollary 19. Let $\phi \in \Phi_{\tilde{H}[M]}$. If the function $f \in \Sigma_p$ satisfies the following conditions:
\[
\left| z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z) \right| \leq kM \quad (k \in \mathbb{N} \setminus \{1\}; M > 0),
\]
\[
\left| z^p H^{\lambda,\mu}_{p,q,s}(\beta_1 + 1) f(z) \right| \leq kM,
\]
\[
\left| z^p H^{\lambda,\mu}_{p,q,s}(\beta_1 - 1) f(z) \right| < M,
\]
then
\[
\left| z^p H^{\lambda,\mu}_{p,q,s}(\beta_1 + 1) f(z) \right| < M \quad (M > 0). \tag{93}
\]

Corollary 20. Let $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ with $\mathcal{R}(\beta_1) \geq -1/2$ and $M > 0$. If the function $f \in \Sigma_p$ satisfies the following conditions:
\[
\left| z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z) \right| \leq kM \quad (k \in \mathbb{N} \setminus \{1\}),
\]
\[
\left| z^p H^{\lambda,\mu}_{p,q,s}(\beta_1 - 1) f(z) \right| < M,
\]
then
\[
\left| z^p H^{\lambda,\mu}_{p,q,s}(\beta_1 + 1) f(z) \right| < M. \tag{94}
\]

Proof. Corollary 20 follows from Corollary 19 by setting
\[
\phi(a,b,c,d;z) = b = 1 + \frac{k + \beta_1}{\beta_1} Me^{\theta}.
\tag{96}
\]

Corollary 21. Let $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, $k \in \mathbb{N} \setminus \{1\}$, and $M > 0$. If the function $f \in \Sigma_p$ satisfies the following conditions:
\[
\left| z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z) \right| \leq kM,
\]
\[
\left| z^p H^{\lambda,\mu}_{p,q,s}(\beta_1 - 2) f(z) - z^p H^{\lambda,\mu}_{p,q,s}(\beta_1 - 1) f(z) \right| < \frac{M}{|\beta_1|}, \tag{97}
\]
then
\[
\left| z^p H^{\lambda,\mu}_{p,q,s}(\beta_1 + 1) f(z) \right| < M. \tag{98}
\]

Proof. Let
\[
\phi(a,b,c,d;z) = d - c, \quad \Omega = h(\mathbb{U}), \tag{99}
\]
where
\[
h(z) = \frac{Mz}{|\beta_1|} \quad (M > 0). \tag{100}
\]

In order to use Corollary 18, we need to show that $\phi \in \Phi_{\tilde{H}[\Omega,M]}$; that is, the admissibility condition (88) is satisfied. This follows easily, since
\[
\left| \phi \left( 1 + k \frac{L + (k + \beta_1)(\beta_1 + 1) Me^{\theta}}{\beta_1 (\beta_1 + 1) (\beta_1 + 2)} \right) \right| < M
\]
whenever $\theta \in \mathbb{R}$ and $k \in \mathbb{N} \setminus \{1\}$. The required result now follows from Corollary 18. \hfill \Box

4. Third-Order Differential Superordination of the Operator $H^{\lambda,\mu}_{p,q,s}(\beta_1)$

In this section, we obtain the third-order differential superordination results for meromorphically multivalent functions associated with the operator $H^{\lambda,\mu}_{p,q,s}(\beta_1)$ defined by (16). Because of this, the class of admissible functions is given in the following definition.

Definition 22. Let $\Omega$ be a set in $\mathbb{C}$ and $q \in \mathcal{H}$ with $q'(z) \neq 0$. The class of admissible functions $\Phi_{\tilde{H}[\Omega,q]}$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition:
\[
\phi(a,b,c,d;\xi) \in \Omega \tag{102}
\]
Whenever
\[ a = q(z), \quad b = \frac{zq'(z) + mp_1 q(z)}{mp_1}, \]
\[ \Re \left( \frac{(\beta_1 + 1)(c - a)}{b - a} - (2\beta_1 + 1) \right) \leq \frac{1}{m} \Re \left\{ \frac{z^\mu q''(z)}{q'(z)} + 1 \right\}, \]
\[ \Re \left( \frac{(\beta_1 + 1)(\beta_1 + 2)(d - 3c + 3b - a)}{b - a} \right) \leq \frac{1}{m} \Re \left\{ \frac{z^\mu q'''(z)}{q'(z)} \right\}, \]
where \( z \in U, \beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots \}, \xi \in \partial U, \) and \( m \in \mathbb{N} \setminus \{1\} .

**Theorem 23.** Let \( \phi \in \Phi'_{1}\{\Omega, q]\). If the functions \( f \in \Sigma_p \) and \( z^\mu H^\mu_{p,q,s}(\beta_1 + 1) f(z) \in \mathcal{C}_1 \) satisfy the following conditions:
\[ \Re \left( \frac{z^\mu q''(z)}{q'(z)} \right) \geq 0, \quad \left| \frac{z^\mu H^\mu_{p,q,s}(\beta_1) f(z)}{q'(z)} \right| \leq m, \]
\[ \phi \left( z^p H^\mu_{p,q,s}(\beta_1 + 1) f(z), z^p H^\mu_{p,q,s}(\beta_1) f(z), z^p H^\mu_{p,q,s}(\beta_1 - 1) f(z), z^p H^\mu_{p,q,s}(\beta_1 - 2) f(z); z \right) \]
is univalent
\[ \Omega \subset \{ \phi \left( z^p H^\mu_{p,q,s}(\beta_1 + 1) f(z), z^p H^\mu_{p,q,s}(\beta_1) f(z), z^p H^\mu_{p,q,s}(\beta_1 - 1) f(z), z^p H^\mu_{p,q,s}(\beta_1 - 2) f(z); z \right) : z \in U \}\]
implies that
\[ q(z) < z^p H^\mu_{p,q,s}(\beta_1 + 1) f(z). \]

**Proof.** Let the function \( p(z) \) be defined by (64) and \( \psi \) by (70). Since \( \phi \in \Phi'_{1}\{\Omega, q]\), (71) and (106) yield
\[ \Omega \subset \{ \psi \left( p(z), z^p q'(z), z^2 q''(z), z^3 q'''(z); z \right) : z \in U \}\].

We see from (68) and (69) that the admissible condition for \( \phi \in \Phi'_{1}\{\Omega, q]\) in Definition 22 is equivalent to the admissible condition for \( \psi \) as given in Definition 7 with \( n = 2 \). Hence \( \psi \in \Psi'_1\{\Omega, q]\), and by using (104) and Theorem 8, we have
\[ q(z) < p(z) \]
or, equivalently,
\[ q(z) < z^p H^\mu_{p,q,s}(\beta_1 + 1) f(z), \]
which evidently completes the proof of Theorem 23.

If \( \Omega \neq \mathbb{C} \) is a simply connected domain and \( \Omega = h(U) \) for some conformal mapping \( h(z) \) of \( U \) onto \( \Omega \), then the class \( \Phi'_{1}[h(U), q] \) is written simply as \( \Phi'_{1}[h, q] \). With proceedings similar as in the preceding section, the following result is an immediate consequence of Theorem 23.

**Theorem 24.** Let \( \phi \in \Phi'_{1}[h, q] \). Also let the function \( h \) be analytic in \( U \). If the functions \( f \in \Sigma_p \) and \( z^p H^\mu_{p,q,s}(\beta_1 + 1) f(z) \in \mathcal{C}_1 \) satisfy condition (104) and
\[ \phi \left( z^p H^\mu_{p,q,s}(\beta_1 + 1) f(z), z^p H^\mu_{p,q,s}(\beta_1) f(z), z^p H^\mu_{p,q,s}(\beta_1 - 1) f(z), z^p H^\mu_{p,q,s}(\beta_1 - 2) f(z); z \right) \]
is univalent in \( U \), then
\[ h(z) < \phi \left( z^p H^\mu_{p,q,s}(\beta_1 + 1) f(z), z^p H^\mu_{p,q,s}(\beta_1) f(z), z^p H^\mu_{p,q,s}(\beta_1 - 1) f(z), z^p H^\mu_{p,q,s}(\beta_1 - 2) f(z); z \right). \]

Theorems 23 and 24 can only be used to obtain subordinations involving the third-order differential superordination of the forms (106) or (112). The following theorem proves the existence of the best subordinant of (112) for a suitable chosen \( \phi \).

**Theorem 25.** Let the function \( h \) be analytic in \( U \), and let \( \phi : \mathbb{C}^3 \times \mathbb{C} \rightarrow \mathbb{C} \) and \( \psi \) be given by (70). Suppose that the differential equation
\[ \psi \left( q(z), z^p q'(z), z^2 q''(z), z^3 q'''(z); z \right) = h(z) \]
has a solution \( q(z) \in \mathcal{C}_1 \). If the functions \( f \in \Sigma_p \) and \( z^p H^\mu_{p,q,s}(\beta_1 + 1) f(z) \in \mathcal{C}_1 \) satisfy condition (104) and
\[ \phi \left( z^p H^\mu_{p,q,s}(\beta_1 + 1) f(z), z^p H^\mu_{p,q,s}(\beta_1) f(z), z^p H^\mu_{p,q,s}(\beta_1 - 1) f(z), z^p H^\mu_{p,q,s}(\beta_1 - 2) f(z); z \right) \]
is univalent in \( U \), then
\[ h(z) < \phi \left( z^p H^\mu_{p,q,s}(\beta_1 + 1) f(z), z^p H^\mu_{p,q,s}(\beta_1) f(z), z^p H^\mu_{p,q,s}(\beta_1 - 1) f(z), z^p H^\mu_{p,q,s}(\beta_1 - 2) f(z); z \right), \]
implies that
\[ q(z) < z^p H^\mu_{p,q,s}(\beta_1 + 1) f(z), \]
and \( q \) is the best subordinant.
Proof. The proof of Theorem 25 is similar to that of Theorem 16 and it is being omitted here. □

By combining Theorems 14 and 24, we obtain the following sandwich-type result.

Corollary 26. Let the functions $h_1$ and $q_1$ be analytic functions in $U$. Also let the function $h_2$ be univalent in $U$, $q_2 \in \mathcal{E}_1$ with $q_2(0) = q_1(0) = 1$ and $\phi \in \Phi_H[h_2, q_2] \cap \Phi_H[h_1, q_1]$. If the function $f \in \Sigma_{p, q, s}$, $z^p H_{p, q, s}^{1, \mu} (\beta_1 + 1) f(z), z^p H_{p, q, s}^{1, \mu} (\beta_1) f(z), z^p H_{p, q, s}^{1, \mu} (\beta_1 - 1) f(z), z^p H_{p, q, s}^{1, \mu} (\beta_1 - 2) f(z)$ is univalent in $U$, and the conditions (61) and (104) are satisfied, then

$$h_1(z) < \phi \left(z^p H_{p, q, s}^{1, \mu} (\beta_1 + 1) f(z), z^p H_{p, q, s}^{1, \mu} (\beta_1) f(z), z^p H_{p, q, s}^{1, \mu} (\beta_1 - 1) f(z), z^p H_{p, q, s}^{1, \mu} (\beta_1 - 2) f(z)\right) h_2(z)$$

implies that

$$q_1(z) < z^p H_{p, q, s}^{1, \mu} (\beta_1 + 1) f(z) < q_2(z).$$

Remark 27. By setting $\lambda = \mu = 0$ in all results of this paper, we can obtain the corresponding results for the well-known Liu-Srivastava operator $H_{p,q,s}(\beta_1)$.

5. Concluding Remarks and Observations

In our present investigation, we have derived several third-order differential subordination and superordination results for meromorphically multivalent functions in the punctured unit disk involving the operator $H_{p,q,s}^{\lambda, \mu}(\beta_1)$ defined by (16) with respect to the parameter $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, which is associated with the Liu-Srivastava operator $H_{p,q,s}(\beta_1)$. Our results have been obtained by considering suitable classes of admissible functions. Furthermore, if we use relation (19), we can obtain the corresponding third-order differential subordination and superordination results for the operator $H_{p,q,s}^{\lambda, \mu}(\alpha_1)$ with respect to the parameter $\alpha_1 \in \mathbb{C}$ and here we choose to omit the details involved.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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