Isometries of a Generalized Numerical Radius

by

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B.A., Federal University of Santa Catarina, 1998
M.A., Federal University of Santa Catarina, 2000

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Abstract

For $0 < |q| < 1$, the $q$-numerical range is defined on the algebra $\mathcal{M}_n$ of all $n \times n$ complex matrices by

$$W_q(A) = \{x^*Ay : x, y \in \mathbb{C}^n, x^*x = y^*y = 1, x^*y = q\}.$$  

The $q$-numerical radius is defined by $r_q(A) = \max\{|\mu| : \mu \in W_q(A)\}$. We characterize isometries of the metric space $(\mathcal{M}_n, r_q)$, i.e., the maps $\phi : \mathcal{M}_n \rightarrow \mathcal{M}_n$ that satisfy $r_q(A - B) = r_q(\phi(A) - \phi(B))$. We also characterise maps on $\mathcal{M}_n$ that preserve the $q$-numerical range.
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Maria Inez Cardoso Gonçalves
Dedication

To my love, Daniel
Chapter 1

Introduction

One of the areas in matrix and operator theory that has attracted the attention of many researchers in the past few decades is the study of Numerical Ranges and Radius [2, 12, 13, 28] and [34, chapter 1]. Another important research topic is the study of Linear Preserver Problems, [44, 55, 66]. Combining these two areas, one is then interested in characterizing the linear preservers of the numerical range or numerical radius [42]. In the space of all \( n \times n \) complex matrices, the problem of preservers of the classical numerical range and radius is completely characterized [39, 47]. Linear preservers of some generalizations of the numerical range and radius have also been characterized [42].

Let \( A \) be an \( n \times n \) complex matrix. The **numerical range** of \( A \) (or the **field of values** of \( A \)) was introduced in 1918, by Toeplitz [74], for finite dimensional Hilbert Spaces. It is the collection of all the complex numbers of the form:

\[
W(A) = \{ x^*Ax : x \in \mathbb{C}^n \text{ and } x^*x = 1 \}.
\]

The **numerical radius** is the quantity defined by:

\[
r(A) = \max \{ |z| : z \in W(A) \}.
\]

The concepts of numerical range and radius have applications in many areas of pure and applied mathematics such as operator theory, Banach algebras, matrix norms, numerical analysis, matrix polynomials, etc. In \( \mathcal{M}_n \), since the numerical range of \( A \) contains the spectrum of \( A \), it can be used to locate eigenvalues and also to deduce algebraic and analytic properties of \( A \). The numerical radius is useful in
studying perturbation, convergence, matrix inequalities, etc. For further information on numerical range and radius applications and properties we refer to [28, 34].

There are many generalizations of the concepts of numerical range and radius that are sometimes motivated by theory and sometimes by applications, [7, 10, 28, 34, 46]. In Chapter 2 we present some of these generalizations.

Since we are interested in one of these generalizations, the $C$-numerical range and the $C$-numerical radius for a class of matrices $C$, in what follows we introduce their definitions.

Let $U_n$ be the group of all $n \times n$ unitary matrices. Let $C$ be an $n \times n$ complex matrix. For any $n \times n$ complex matrix $A$, the $C$-numerical range of $A$ is the set defined by

$$W_C(A) = \{\text{tr}(CU^*AU) : U \in U_n\}.$$ 

The $C$-numerical radius of $A$ is the quantity defined by:

$$r_C(A) = \max\{||z|| : z \in W_C(A)\}.$$ 

If $C$ is a matrix of rank one, norm one and trace $q$, where $q \in \mathbb{C}$, $0 < |q| \leq 1$ the $C$-numerical range can be written as:

$$W_C(A) = W_q(A) = \{x^*Ay : x, y \in \mathbb{C}^n, \|x\| = \|y\| = 1 \quad \text{and} \quad x^*y = q\}.$$ 

Also by taking $C$ as the matrix whose $(1, 1)$ entry is 1 and all other entries are zeros, the $C$-numerical range and radius reduce to the classical numerical range and radius.

As one can easily verify, the $C$-numerical range and radius are invariant under unitary similarity transformations. If $C$ is a non-scalar matrix and $\text{tr}(C) \neq 0$, then the $C$-numerical radius is a norm. In this case, the $C$-numerical radii can be viewed as the building blocks of the unitary similarity invariant norms [60], so it is useful when studying properties that are invariant under unitary similarities. The $C$-numerical range and radius derive their importance, in part, from this and from applications, e.g., Quantum Dynamics [72] or NMR Spectroscopy [32].

Many important objects in mathematics are preservers of certain properties, such as the isometry group of a Banach space, the group of automorphisms of an algebra, preservers of singular matrices, preservers of matrices of rank one, etc. In matrix theory, the characterization of linear maps that leave certain matrix functions, subsets or relations invariant has been an active research area in the past 100 years. These problems are called Linear Preserver Problems. The first result was obtained in 1897.
by Frobenius [21]. He characterized all the linear maps \( \phi \) on the space of real or complex matrices that preserve the determinant, that is, linear maps that satisfy \( \det(\phi(A)) = \det(A) \) for all matrices \( A \). He showed that such maps have the form:

\[
A \mapsto MAN \quad \text{or} \quad A \mapsto MA^tN
\]

for nonsingular matrices \( M, N \) with \( \det(MN) = 1 \).

Since then, many papers [18, 21, 36, 44, 55, 58, 65, 66, 67, 69, 76] have been written in this area. Old and new problems have been solved, some are still open, and dozens of new techniques have been developed. Usually, the tools used to solve linear preserver problems include basic matrix theory, differential geometry, group theory, projective geometry, Lie groups and Lie algebras, etc. For more details about techniques see [44, 55, 66].

Linear preserver problems are also important in many applied problems. For example, as pointed out in [44], in matrix models in systems theory one often needs to transform a complex system to a simpler one by linear transformations that do not affect the nature of the system, that is linear transformations that preserve controllability or observability [22].

In Chapter 3 we present some classical linear preserver problems as well as some linear preserver problems that are relevant to the development of this work. We also present some of the techniques commonly used to solve such problems. In particular, we describe two of these techniques in some detail. It is not our intention in this chapter to give a detailed survey in the topic, since as mentioned before, there exists a large amount of research literature in the area. We also discuss a generalization of linear preserver problems: the additive preserver problems. There are many other generalizations of linear preserver problems, for example one can consider multiplicative preserver problems, quadratic or bilinear preserver problems, etc.

Before we present the main results of this thesis, we pause to fix notation and terminology. By \( M_n \) we denote the complex vector space of all \( n \times n \) matrices. All vectors in \( \mathbb{C}^n \), the complex vector space of vectors with \( n \) components, are assumed to be column vectors so that, for \( x, y \in \mathbb{C}^n \), the expression \( x^*y \) is the usual inner product of \( x \) and \( y \) while \( xy^* \) or \( xy^t \) is a matrix of rank at most one, where \( x^* \) is the conjugate transpose of \( x \). We also observe that every matrix of rank one is of this form.

The standard basis of \( \mathbb{C}^n \) is denoted by \( \{ e_1, e_2, \ldots, e_n \} \), where \( e_j \) is the column vector whose \( j \)th entry is 1 and all other entries are 0. We denote the standard matrix units by \( E_{ij} \), i.e., \( E_{ij} = e_i e_j^* \). The trace of a matrix \( A \) is denoted by \( \text{tr}(A) \). For
a matrix $A = [a_{jk}]$, the matrix $\bar{A}$ denotes the matrix $A = [\overline{a_{jk}}]$ obtained by taking the entry-wise complex-conjugate of $A$.

The unitary group is denoted by $U_n$. The unitary orbit $U(A)$ of a matrix $A$ is the set

$$\mathcal{U}(A) = \{U^*AU : U \in U_n\}.$$  \hfill (1.1)

The set

$$SU(A) = \{\mu U^*AU : \mu \in \mathbb{C}, |\mu| = 1, U \in U_n\}$$  \hfill (1.2)

will be called the saturated unitary orbit of $A$.

The operator norm of a matrix $A$ is denoted by $\|A\|$ and is defined as $\|A\| = \sup\{|Ax| : x \in \mathbb{C}^n, \|x\| = 1\}$ or as the largest singular value of $A$. The Hilbert-Schmidt (Frobenius) norm of $A$ is denoted by $\|A\|_2$ and is defined as $\|A\|_2 = \left(\sum_{i=1}^{n} |s_i(A)|^2\right)^{1/2}$, where $s_1(A), \ldots, s_n(A)$ denote the singular values of $A$. We note that $\|xy^*\|_2 = \|x\| \|y\|$, where the vector norms are the Euclidean ($\ell^2$) norms on $\mathbb{C}^n$.

The space $\mathcal{M}_n$ is a complex-Hilbert space under the (complex) inner product:

$$\langle A, B \rangle_{\mathbb{C}} = \text{tr}(AB^*).$$  \hfill (1.3)

It is also a real Hilbert space under the (real) inner product

$$\langle A, B \rangle_{\mathbb{R}} = \text{Re}(\text{tr}(AB^*)), \hfill (1.4)$$

where $\text{Re}(\cdot)$ denotes the real part of a complex number.

A real linear map on $\mathcal{M}_n$ is a linear map on $\mathcal{M}_n$ as a real space, without the additional structure of multiplication by $i$. And an additive map on $\mathcal{M}_n$ is a map that is only assumed to be additive. It is easy to verify that if $T$ is an additive map that is continuous then $T$ is real linear.

By the dual $T'$ of a real linear operator $T$ on $\mathcal{M}_n$, we mean the "adjoint" relative to the real inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}}$, i.e., the unique real linear operator $T' : \mathcal{M}_n \to \mathcal{M}_n$ that satisfies:

$$\langle T(A), B \rangle_{\mathbb{R}} = \langle A, T'(B) \rangle_{\mathbb{R}} \quad \text{for all} \quad A, B \in \mathcal{M}_n. \hfill (1.5)$$

For example, if $T(A) = \overline{A}$, a real linear mapping on $\mathcal{M}_n$, then $T'(A) = \overline{A}$, since $\langle T(A), B \rangle_{\mathbb{R}} = \text{Re}(\text{tr}(\overline{AB}^*)) = \text{Re}(\text{tr}(AB^*)) = \langle A, (B^*)^* \rangle_{\mathbb{R}} = \langle A, B \rangle_{\mathbb{R}}$.

In Chapter 4 we give a characterization of isometries of the metric space $(\mathcal{M}_n, r_{\mathbb{C}}(\cdot))$, i.e., maps $\phi : \mathcal{M}_n \to \mathcal{M}_n$ that satisfy $r_{\mathbb{C}}(A - B) = r_{\mathbb{C}}(\phi(A) - \phi(B))$,
where $C$ is a matrix of rank one, norm one and trace $q$, with $0 < |q| < 1$. We also characterize maps on $M_n$ that preserve the $C$-numerical range. No linearity is assumed, but such problems are quickly reduced to studying real linear isometries. Observe that when complex linearity is assumed, a large part of the isometry group is lost. For example in the plane, the isometry group of the Euclidean distance is generated by translations, rotations and reflections. Complex linear maps on the plane have the form $f(z) = cz$, where $c \in \mathbb{C}$. These maps will preserve Euclidean distance if and only if $|c| = 1$. Thus, if complex linearity is assumed, then translations and reflections will be eliminated. While real linear isometries include rotations and reflections.

The problem of characterizing the real linear isometries of the classical numerical radius was completely solved by Li and Šemrl [47]. Observe that if $q = 1$ and $C$ is a matrix of rank one, norm one and trace $q$, then the $C$-numerical radius reduces to the classical numerical radius. So we will consider $0 < |q| < 1$, since there are no characterizations of real linear isometries of $r_C(\cdot)$ in this case.

Complex linear preservers of the $C$-numerical radius and $C$-numerical range for $C$ a matrix of rank one, norm one and trace $q$, with $0 < |q| \leq 1$ were characterized in 1994 by Li, Mehta and Rodman [43]. Since we do not assume any linearity, the results obtained in this work are more general than the ones obtained in [43], also in the complex linear case our proof is an alternative proof to the one obtained in [43].

To solve the problem we divided it into two cases: $n > 2$ and $n = 2$. The case $n > 2$ was solved using a combination of two techniques used in preserver problems: the duality technique and the reduction to an existing preserver result. More specifically, we show that the dual of the mapping $\phi$ preserves rank one matrices. Since rank one preservers have been already characterized [64], we then use their characterization to obtain a characterization of $\phi$. Unfortunately our proof for $n > 2$ cannot be applied to the case $n = 2$, so we had to find an alternative solution for this case. For $n = 2$ we start characterizing the real linear mappings on $M_2$ that preserve the $C$-numerical range, and we show that they also preserve rank one matrices. Then, we show that $\phi$ satisfy a restrict complex linearity condition. Using this restrict complex linearity we show that $\phi$ preserves the $C$-numerical radius if and only if there exist a complex number $\mu$, $|\mu| = 1$, such that $\mu \phi$ or $\overline{\mu} \phi$ preserves the $q$-numerical range. To conclude the proof for the case $n = 2$, we use the characterization of rank one preservers and properties of the saturated unitary orbit of the matrix $C$ of rank one, trace $q$ and norm $1$. 
Chapter 2

Numerical Range and Radius

We start this chapter by introducing concepts that have been studied extensively in the last decades: the numerical range and radius. Then we present some of their generalizations, in particular, we introduce the concepts of $C$-numerical range and radius as well as some of the properties and results that are important to the development of Chapter 4.

2.1 Numerical Range, Numerical Radius and their generalizations

Let $A \in M_n$. The numerical range of $A$, or the field of values of $A$, is the collection of all the complex numbers that satisfy:

$$W(A) = \{ x^*Ax : x \in \mathbb{C}^n \text{ and } x^*x = 1 \} = \{ (Ax, x) : x \in \mathbb{C}^n \text{ and } x^*x = 1 \}$$

$$= \{ \text{tr}(Ax^*x) : x \in \mathbb{C}^n \text{ and } x^*x = 1 \} = \{ (A, xx^*)_{\mathbb{C}} : x \in \mathbb{C}^n \text{ and } x^*x = 1 \}.$$

(2.1)

Many researchers have dedicated their work to the study of numerical ranges not only in Hilbert spaces [28, 30, 34], but also in Banach spaces and Banach Algebras [5, 11, 12, 13, 35, 46]. The numerical range can be seen as the image of the Euclidean unit ball in $\mathbb{C}^n$, which is a compact and connected set, under the continuous mapping
$x \mapsto x^*Ax$. Therefore, $W(A)$ is a compact and connected set. $W(A)$ is also a convex set, and this fact is known as the Toeplitz-Hausdorff theorem, since Toeplitz [74] proved that the outer boundary of $W(A)$ is a convex curve and Hausdorff [31] showed that the interior of this curve is filled out with points of $W(A)$.

From equation (2.1) we also have that the numerical range can be seen as the image of $A$ under all vector states, or the image of the unitary orbit of $A$ under one vector state, where the set of vector states with respect to the Euclidean norm is defined by:

$$\mathcal{R}_2 = \{xx^*: x \in \mathbb{C}^n, \|x\| = 1\}.$$  

Related to the numerical range there is an important number, the numerical radius, which is defined by:

$$\rho(A) = \max\{|z| : z \in W(A)\}. \quad (2.2)$$

For infinite dimensional Hilbert spaces we have the following: Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, the numerical range of $T$ is defined by:

$$W(T) = \{(Tx, x) : \langle x, x \rangle = 1\}.$$  

In this context, $W(T)$ is convex, bounded but not necessarily closed. The numerical radius is defined by:

$$\rho(T) = \sup\{|z| : z \in W(T)\}.$$  

For properties on numerical range and radius on infinite dimensional Hilbert spaces we refer to [28, 30].

The concepts of numerical range and numerical radius have lots of applications in both pure and applied mathematics. In $M_n$, since the numerical range of $A$ contains the spectrum of $A$, it can be used to locate eigenvalues and also to deduce algebraic and analytic properties of $A$. The numerical radius is useful in studying perturbation, convergence, matrix inequalities, etc. For further information on numerical range and radius applications and properties we refer to [28, 34].

Next we list some properties and results about the numerical range and radius that are important to the development of this thesis. If not specified, proofs can be found in [28] and [34, Chapter 1].

**Some Properties of the Numerical Range and Radius** - Let $A, B \in M_n$ and $\alpha, \beta \in \mathbb{C}$. Then we have:
(i) \( W(\alpha A + \beta I) = \alpha W(A) + \beta \);

(ii) \( W(UAU^*) = W(A) \), for any \( U \in \mathcal{U}_n \);

(iii) \( r(UAU^*) = r(A) \), for any \( U \in \mathcal{U}_n \);

(iv) \( W(A) = \{ \alpha \} \) if and only if \( A = \alpha I \);

(v) \( r(\cdot) \) is a norm on \( \mathcal{M}_n \);

(vi) \( \rho(A) \leq r(A) \leq \|A\| \), where \( \rho(A) \) denotes the spectral radius and \( \|A\| \) the operator norm of \( A \);

(vii) \( r(A^m) \leq r^m(A) \), \( m = 1, 2, \ldots \).

For \( n \geq 3 \), the geometry of \( W(A) \) is usually complicated, unless \( A \) is a normal matrix, and in this case \( W(A) \) is the convex hull of the spectrum of \( A \). But for \( n = 2 \), Murnaghan [63] (see also Donoghue [19]) proved the following:

**Theorem 2.1.** [63] Let \( A \in \mathcal{M}_2 \) such that \( A \) has eigenvalues \( \lambda_1 \) and \( \lambda_2 \). Then \( W(A) \) is an elliptical disk with foci \( \lambda_1, \lambda_2 \) and minor axis with length \( \{ \text{tr}(A^*A) - |\lambda_1|^2 - |\lambda_2|^2 \}^{1/2} \).

Generalizations of the concepts of numerical range and numerical radius are generally motivated by pure or applied mathematics. But essentially one can say that there exist two types of generalizations of the concept of numerical range. Before they are described, we need to introduce some definitions.

A *state* on a normed algebra \( \mathcal{A} \), with unit \( e \) of norm 1, is a linear functional \( f \) on \( A \) such that \( f(e) = \|f\| = 1 \). With the usual identification of \( \mathcal{M}_n \) and its dual, the set of states with respect to the norm \( \|\cdot\| \) is then identified with the set \( S \in \mathcal{M}_n \),

\[
S = \{ B \in \mathcal{M}_n : \text{tr} B = \|B\|^D = 1 \},
\]

where \( \|B\|^D = \{ |\langle B, A \rangle|, \|A\| = 1 \} \) is the dual of the norm \( \|\cdot\| \).

Now we are ready to describe the two types of generalizations mentioned above:

(a) Having a general normed space instead of a Hilbert space.

(b) Replacing the states by arbitrary linear functionals.
Next, to illustrate the first generalization type we give a brief example since generalizations of this type are only touched upon in this work.

Let $A$ be a unital normed algebra over a field $F$, where $F$ denotes either $\mathbb{R}$ or $\mathbb{C}$. The unit element is denoted by $e$ and satisfies $\|e\| = 1$. Let $A'$ denote the dual space of $A$, that is, the Banach space of all continuous linear functionals on $A$.

Given $a \in A$ the numerical range of $a$ is defined by:

$$V(a) = \{ f(a) : f \in S \},$$

where $S$ is the set of states, that is, $S = \{ f \in A' : f(e) = \|f\| = 1 \}$. The numerical radius of $a$ is defined by:

$$v(a) = \sup \{|\lambda| : \lambda \in V(a)\}.$$

In the next sections we present some of this generalizations that are related to this work, in the context of $\mathcal{M}_n$. For more details and generalizations we refer to [5, 6, 7, 10, 11, 28, 30, 34, 45, 49, 56, 71].

### 2.1.1 $k$-numerical Range and Radius

Let $k$ be a given positive integer. In 1967, Halmos [30] introduced the concept of the $k$-numerical range:

$$W_k(A) = \{ \text{tr}(PAP) : P = P^2 = P^*, P \text{ has rank } k \}$$

(2.3)

Note that for $k = 1$, the $k$-numerical range reduces to the classical numerical range.

An important property of the $k$-numerical range is that $W_k(A)$ is convex for all $A \in \mathcal{M}_n$ and all positive integer $k$, the proof can be found in [30, Chapter 22].

Analogous to the numerical radius, the $k$-numerical radius is the number defined by:

$$r_k(A) = \max \{|z| : z \in W_k(A)\}.$$

### 2.1.2 $e$-numerical Range and Radius

Given a vector $e = (e_1, \ldots, e_n)^t \in \mathbb{C}^n$, in 1975, Westwick [77] introduced the $e$-numerical range:

$$W_e(A) = \{ \sum_{j=1}^n e_j x_j^* A x_j : \{x_1, \ldots, x_n\} \text{ is an orthonormal basis for } \mathbb{C}^n \}.$$  

(2.4)
By taking $c = (1, 0, \cdots, 0)^T$, the c-numerical range reduces to the numerical range.

Unlike the $k$-numerical range, the c-numerical range is not always convex for all vectors $c \in \mathbb{C}^n$. For example if $c = (0, 1, i)^T$ and $A$ is a diagonal matrix with $(0, 1, i)$ as the diagonal entries, then the c-numerical range of $A$ is not convex, [28, page 129].

In the same paper Westwick [77] proved that if $c \in \mathbb{R}^n$, then $W_c(A)$ is convex for all $A \in \mathcal{M}_n$.

Associated with the c-numerical range is the c-numerical radius which is defined by:

$$r_c(A) = \max\{|z| : z \in W_c(A)\}.$$  

For $n \geq 2$, if $c_1 + \cdots + c_n \neq 0$ and not all the $c_i$'s are equal, then $r_c(\cdot)$ is a norm, [24].

2.1.3 $q$-numerical Range and Radius

Let $q \in \mathbb{C}$, $|q| \leq 1$. Given $A \in \mathcal{M}_n$, Andersen and Marcus [1] defined the $q$-numerical range of $A$ as the set:

$$W_q(A) = \{x^*Ay : x^*x = 1, y^*y = 1 \text{ and } x^*y = q\}. \quad (2.5)$$

Clearly, if $q = 1$, then $W_q(A) = W(A)$.

Regarding the convexity of the $q$-numerical range, Tsing [75] proved that $W_q(A)$ is always convex for all $A \in \mathcal{M}_n$ and all $q \in \mathbb{C}$, $|q| \leq 1$.

As before, associated to the $q$-numerical range is the $q$-numerical radius defined by:

$$r_q(A) = \max\{|z| : z \in W_q(A)\}.$$  

Also, by specializing the result in [24], we have that $r_q(\cdot)$ is a norm if and only if $q \neq 0$.

2.2 $C$-numerical Range and $C$-numerical Radius

Let $C \in \mathcal{M}_n$. For any $A \in \mathcal{M}_n$, the $C$-numerical range of $A \in \mathcal{M}_n$ is the set defined by

$$W_C(A) = \{tr(CU^*AU) : U \in U_n\}. \quad (2.6)$$
The $C$-numerical radius of $A$ is the quantity defined by:

$$r_C(A) = \max\{|z| : z \in W_C(A)\}. \tag{2.7}$$

The $C$-numerical range and the $C$-numerical radius were introduced in 1975 by Westwick, [77], and generalized in 1977 by Goldberg and Strauss, [23]. If $C = E_{11}$, then they reduce to the classical numerical range and classical numerical radius, respectively. Note that going from $W(A)$ to $W_C(A)$ only replaces the states by an arbitrary linear functional, since $W_C(A) = \{(B^*, C) : B \in U(A)\}$. The $C$-numerical radius is easily seen to be invariant under unitary conjugation, i.e., $r_C(U^*AU) = r_C(A)$ for every unitary $U$, but unlike the operator norm or the Hilbert-Schmidt (Frobenius) norm is not invariant under one-sided multiplication by a unitary matrix. The $C$-numerical radii may be viewed as building blocks of norms invariant under unitary conjugation, (see [53] or [9, p. 106]). They derive their importance, in part, from this and from applications to, e.g., Quantum Dynamics [72] or NMR Spectroscopy [32].

As one can easily verify, the $C$-numerical range generalizes the three numerical ranges described above, that is:

1. If $C$ is a projection of rank $k$, then

$$W_C(A) = W_k(A), \text{ for all } A \in \mathcal{M}_n,$$

for $C = I_k \oplus 0_{n-k}$.

2. If $C$ is a normal matrix with eigenvalues $\{c_1, ..., c_n\}$, then:

$$W_C(A) = W_c(A), \text{ for all } A \in \mathcal{M}_n.$$

3. If $C$ is a matrix of rank one, trace $q$ and norm 1, then the $C$-numerical range reduces to the $q$-numerical range. In particular if $C = C_q = qE_{11} + \sqrt{1 - q^2}E_{12}$, then we have

$$W_{C_q}(A) = W_q(A), \text{ for all } A \in \mathcal{M}_n.$$

The $C$-numerical range is the image of the unitary group $U_n$ under the continuous map $U \mapsto \text{tr}(CU^*AU)$, therefore it is a compact and connected subset of the complex plane $\mathbb{C}$. In particular the maximum in equation (2.7) exists.

The study of the convexity of $W_C(A)$ is still an open problem; there are some results for some particular cases of matrices $C$, for more details see [40]. In particular,
if $C$ is a rank one matrix we have the following result from [75], which will be used in Chapter 4.

**Theorem 2.2.** [75] Let $C \in \mathcal{M}_n$ be a rank one matrix. Then $W_C(A)$ is convex for all $A \in \mathcal{M}_n$.

**Theorem 2.3.** [17] Let $C \in \mathcal{M}_n$. Then $W_C(A)$ is star-shaped with $\text{tr}(C)\text{tr}(A)/n$ as a star-center.

For a *star-shaped* set $S$ with a star-center $s_0$, we mean a set such that for all $s \in S$, the whole line segment $\{\alpha s + (1 - \alpha)s_0 : 0 \leq \alpha \leq 1\}$ lies in $S$.

Next we present some basic properties of $W_C(A)$ and $r_C(A)$. If not specified, their proofs can be found in [34, 40].

**Properties of the $C$-numerical Range**

(i) $W_C(A) = W_A(C)$;

(ii) The set $W_C(A)$ is invariant under unitarily similarities of $C$ and $A$, that is, for any unitary matrix $U$, $W_C(UAU^*) = W_{UCU^*}(A) = W_C(A)$;

(iii) For any $\mu, \eta \in \mathbb{C}$ and any $A \in \mathcal{M}_n$, $W_C(\mu A + \eta I) = \mu W_C(A) + \eta \text{tr}(C)$;

(iv) For any $A, B \in \mathcal{M}_n$, $W_C(A + B) \subseteq W_C(A) + W_C(B)$.

**Properties and Norm Properties of $r_C(\cdot)$**

As mentioned in the beginning of this section, the $C$-numerical radius is invariant under unitary conjugation. Goldberg and Straus [24] showed conditions on the matrix $C$ such that $r_C(\cdot)$ is a norm. They obtained the following Theorem which will be used on Chapter 4.

**Theorem 2.4.** [24] The $C$-numerical radius $r_C(\cdot)$ is a norm if and only if $C$ is a non-scalar matrix with $\text{tr} C \neq 0$.

Thus, if $r_C(\cdot)$ is a norm, we have that $r_C(\cdot)$ is a unitary similarity invariant norm. Moreover, Mathias [60] (see also [54]) proved the following:
Theorem 2.5. [60] For any similarity invariant norm $N(\cdot)$ on $M_n$, there is a compact subset $\mathcal{K}$ of $M_n$ such that

$$N(A) = \max \{r_C(A) : C \in \mathcal{K}\}$$

for all $A \in M_n$.

So, the $C$-numerical radii can be seen as building blocks of the unitary similarity invariant norms.

The next theorem gives necessary and sufficient conditions for two rank one matrices to be unitarily equivalent, as well as necessary and sufficient conditions for a rank one matrix to be in the saturated unitary orbit of another rank one matrix. The norm used is either the operator norm or the Hilbert-Schmidt norm, since they are equal for rank one matrices.

Theorem 2.6. Let $A$ and $B$ be rank one matrices. Then

(a) $A$ is unitarily equivalent to $B$ if and only if $\text{tr}(A) = \text{tr}(B)$ and $\|A\| = \|B\|$.

(b) $A \in SU(B)$ if and only if $|\text{tr}(A)| = |\text{tr}(B)|$ and $\|A\| = \|B\|$.

Proof. (a) If $A$ is unitarily equivalent to $B$ then it is easy to verify that they have the same trace and norm.

Suppose now that $\text{tr}(A) = \text{tr}(B)$ and $\|A\| = \|B\|$. Since $A$ is a rank one matrix there exists a unitary matrix $U_1$ such that $U_1AU_1^* = xE_{11} + yE_{12}$, where $x, y \in \mathbb{C}$. Analogously, there exists $U_2 \in \mathcal{U}_n$ such that $U_2BU_2^* = zE_{11} + wE_{12}$, where $z, w \in \mathbb{C}$. Since $\text{tr}(A) = \text{tr}(B)$, we have that $x = z$. And from $\|A\| = \|B\|$ we have that $|y| = |w|$. Therefore, there exists $\theta \in [0, 2\pi)$ such that $w = ye^{i\theta}$. Then $A = VBV^*$, where $V = U_1^*QU_2$ and $Q$ is a diagonal matrix with diagonal $(1, e^{-i\theta}, 1, \ldots, 1)$. Since $V$ is a unitary matrix, we have that $A$ and $B$ are unitarily equivalent.

(b) If $A \in SU(B)$, then there exists $U_0 \in \mathcal{U}_n$ and $\theta_0 \in \mathbb{C}$, $|\theta| = 1$ such that $A = \theta_0U_0BU_0^*$. By part (a) we have that $|\text{tr}(A)| = |\theta_0||\text{tr}(U_0BU_0^*)| = |\text{tr}(B)|$ and $\|A\| = |\theta_0| \|U_0BU_0^*\| = \|B\|$.

Since $A$ and $B$ are rank one matrices, we have that there exists $V \in \mathcal{U}_n$ such that $VAV^* = xE_{11} + yE_{12}$, where $x, y \in \mathbb{C}$. Analogously, there exists $U \in \mathcal{U}_n$ such that $UBU^* = zE_{11} + wE_{12}$, where $z, w \in \mathbb{C}$.

From $|\text{tr}(A)| = |\text{tr}(B)|$ we have that $|x| = |z|$, i.e., there exists $\theta \in [0, 2\pi)$ such that $z = xe^{i\theta}$. And from $\|A\| = \|B\|$ we have that $|y| = |w|$, so there exists $\beta \in [0, 2\pi)$ such that $w = ye^{i\beta}$.
Therefore, $UBU^* = xe^{i\theta}E_{11} + ye^{i\beta}E_{12} = e^{i\theta}(xe_{11}^{(y^{2})}E_{11}^{(y^{2})})Q^* = e^{i\theta}QVAV^*Q^*$, where $Q$ is a diagonal matrix with diagonal $(1, e^{i\theta-y}, 1, \ldots, 1)$. Hence, $A = e^{-i\theta}WBW^*$, where $W = V^*Q^*U$. Since $W \in \mathcal{U}_n$, we have that $A \in \text{SU}(B)$. \hfill \Box

Corollary 2.7. Let $C_q = qE_{11} + \sqrt{1-|q|^2}E_{12}$, $q \in \mathbb{C}$, $0 < |q| \leq 1$. Letting $A \in \mathcal{M}_n$, $A$ belongs to $\text{SU}(C_q)$ if and only if $\text{rank}(A) = 1$, $|\text{tr}(A)| = |q|$ and $\text{tr}(A^*A) = 1$.

The next two results will be used to prove the last theorem of this chapter and will be used in Chapter 4 as well.

Lemma 2.8. Let $C \in \mathcal{M}_n$. Then $r_C(A) = \max\{|A, X^*|_R : X \in \text{SU}(C)\}$

Proof. 

$$r_C(A) = \max\{|z| : z \in W_C(A)\} = \max\{|\text{tr}(AX)| : X \in \mathcal{U}(C)\} = \max\{|\text{Re}(\text{tr}(AX))| : \theta \in \mathbb{C}, |\theta| = 1 \text{ and } X \in \mathcal{U}(C)\} = \max\{|\text{Re}(\text{tr}(AY))| : Y \in \text{SU}(C)\} = \max\{|A, Y^*|_R : Y \in \text{SU}(C)\}$$

\hfill \Box

In the next theorem, $\text{conv}(Y)$ denotes the convex hull of a set $Y$ and $\text{ext}(X)$ denotes the set of extreme points of a convex set $X$.

Theorem 2.9. Let $C \in \mathcal{M}_n$. The convex hull of $\text{SU}(C)$ is a compact and convex set and $\text{ext}[\text{conv}(\text{SU}(C))] = \text{SU}(C)$.

Proof. Since $\text{SU}(C)$ is the image of the unitary group and the unitary circle under a continuous map, we have that $\text{SU}(C)$ is a compact set. By theorem 17.2[68] we have that $\text{conv}(\text{SU}(C))$ also is compact. So, $\text{conv}(\text{SU}(C))$ is a compact convex set.

For the second affirmation, first let’s show that $\text{SU}(C_q \subseteq \text{ext}[\text{conv}(\text{SU}(C_q))]$. Let $A \in \text{SU}(C_q)$. By Corollary 2.7, $|A| = 1$. Suppose that there exist $C_1, \ldots, C_k \in \text{conv}(\text{SU}(C_q))$ and $\alpha_1 > 0, 1 \leq i \leq k, \sum_{i=1}^{k} \alpha_i = 1$ such that $A = \alpha_1 C_1 + \cdots + \alpha_k C_k$. 

By taking the norm on both sides of the previous equation we have:

\[ I = \|A\| \leq \alpha_1 \|C_1\| + \cdots + \alpha_k \|C_k\| = \alpha_1 + \cdots + \alpha_k = 1. \]

Since each \( \|C_i\| = 1 \), \( 1 \leq i \leq k \), the triangle inequality becomes an equality, which implies that \( C_1 = \cdots = C_k \). Therefore \( A \) is an extreme point of \( \text{conv}(SU(C_q)) \).

Clearly we have \( \text{ext}[\text{conv}(SU(C_q))] \subseteq SU(C_q) \).

Given two matrices \( C \) and \( C' \), the next theorem gives necessary and sufficient conditions for the \( C \)-numerical radius to be equal to the \( C' \)-numerical radius.

**Theorem 2.10.** Let \( C \) and \( C' \in \mathcal{M}_n \). Then

\[ r_C(A) = r_{C'}(A) \text{ for all } A \in \mathcal{M}_n \]

if and only if \( C \) is unitarily equivalent to \( \alpha C' \), where \( \alpha \in \mathbb{C} \), \( |\alpha| = 1 \).

**Proof.** First suppose that \( r_C(A) = r_{C'}(A) \) for all \( A \in \mathcal{M}_n \).

From Lemma 2.8, we have that \( r_C(A) = \max\{\langle A, Y^* \rangle_\mathbb{R} : Y \in SU(C)\} \) and \( r_{C'}(A) = \max\{\langle A, W^* \rangle_\mathbb{R} : W \in SU(C')\} \).

Now, \( r_C(A) = r_{C'}(A) \) for all \( A \in \mathcal{M}_n \) is equivalent to:

\[ \max\{\langle A, Y^* \rangle_\mathbb{R} : Y \in SU(C)\} = \max\{\langle A, W^* \rangle_\mathbb{R} : W \in SU(C')\}, \forall A \in \mathcal{M}_n \]

i.e.,

\[ \max\{\phi(Y) : Y \in SU(C)\} = \max\{\phi(W) : W \in SU(C')\}, \quad (2.8) \]

for all functionals \( \phi \), where \( \phi(X) = \langle A, X \rangle_\mathbb{R} \).

By Theorem 2.9 we have that \( \text{conv}(SU(C)) \) and \( \text{conv}(SU(C')) \) are compact and convex sets.

Suppose that \( \text{conv}(SU(C)) \not\subseteq \text{conv}(SU(C')) \). Then there exists \( x_0 \in \text{conv}(SU(C)) \) such that \( x_0 \notin \text{conv}(SU(C')) \). Now, there exists \( c \in \mathbb{R} \) such that \( \max\{\phi(Y) : Y \in \text{conv}(SU(C))\} \leq c < \phi(x_0) \leq \{\phi(W) : W \in SU(C')\} \). Which is a contradiction with equation \( (2.8) \). Analogously if \( x_0 \in \text{conv}(SU(C')) \) and \( x_0 \notin \text{conv}(SU(C)) \) we also get a contradiction with equation \( (2.8) \). Therefore, we have:

\[ \text{conv}(SU(C)) = \text{conv}(SU(C')). \quad (2.9) \]
Now using Theorem 2.9 and equation (2.9) we have that

$$SU(C) = \text{ext}(\text{conv}(SU(C))) = \text{ext}(\text{conv}(SU(C'))) = SU(C').$$

Therefore there exists \( \alpha \in \mathbb{C}, \ |\alpha| = 1 \) such that \( C \) is unitarily equivalent to \( \alpha C' \).

Clearly, if \( C \) is unitarily similar to \( \alpha C' \), then \( r_C(A) = r_{C'} \) for all \( A \in \mathcal{M}_n \).  

\( \square \)
Chapter 3

Preserver Problems

In this chapter we give a brief introduction to a very active topic in matrix theory; the study of Linear Preserver Problems. We give a gentle introduction to the topic, by presenting some classical results as well as some results that are relevant to this thesis. We also introduce some of the techniques commonly used. In particular we give more emphasis to the techniques that will be used in this work.

3.1 Preserver Problems - Some examples and techniques

Linear Preserver Problems deal with the characterization, that is, the structural description, of linear operators on matrix or operator spaces (or more generally on rings or algebras) that leave certain matrix functions, subsets or relations invariant. As mentioned in [44], linear preserver problems are interesting because their formulation is usually simple and the results are often very elegant.

More specifically, let $\mathcal{M}$ be a matrix space over a field $F$. The solution of a linear preserver problem typically requires the characterization of linear operators $\phi : \mathcal{M} \rightarrow \mathcal{M}$ with some special properties such as:

(a) $\phi$ preserves functions: $F(\phi(A)) = F(A)$ for all $A \in \mathcal{M}_n$, where $F$ is a given function on $\mathcal{M}$;

(b) $\phi$ preserves sets: $\phi(S) \subset S$ or $\phi(S) = S$ for a certain subspace $S \subset \mathcal{M}$;
(c) \( \phi \) preserves relations: \( \phi(A) \sim \phi(B) \) in \( M_n \) whenever \( A \sim B \) in \( M_n \) for a certain relation \( \sim \) on \( M \).

For illustrative purposes, next we present classical examples of each one of the three categories above:

**Preserver of Function**

One of the first linear preserver problems was solved in 1897 by Frobenius [21]. He showed that all the linear maps \( \phi : M_n \rightarrow M_n \) that satisfy

\[
\text{det}(\phi(A)) = \text{det}(A), \quad \text{for all } A \in M_n
\]

have the form:

\[
A \mapsto MAN \text{ or } A \mapsto MA^tN,
\]

for some \( M, N \in M_n \) with \( \text{det}(MN) = 1 \).

**Preserver of Sets**

Another classical example is the non-invertibility preservers result obtained in 1949 by Dieudonné [18]. He showed that all the bijective linear maps \( \phi : M_n \rightarrow M_n \), that map the set of singular matrices into itself have the form:

\[
A \mapsto MAN \text{ or } A \mapsto MA^tN,
\]

for some invertible matrices \( M, N \in M_n \).

**Preserver of Relations**

In 1976, Watkins [76] characterized commutativity preservers, that is, the linear maps \( \phi : M_n \rightarrow M_n \) that satisfy: \( \phi(A)\phi(B) = \phi(B)\phi(A) \) whenever \( AB = BA \), for all \( A, B \in M_n \). He solved the problem for \( n \geq 4 \) and gave a counterexample for \( n = 2 \).

**Theorem 3.1.** [76] Let \( n \geq 4 \) and \( \phi \) be a non-singular linear map on \( M_n \) that preserves commutativity. Then either:

\[
\phi(A) = cS^{-1}AS + f(A)I
\]

or

\[
\phi(A) = cS^{-1}A^tS + f(A)I
\]

for all \( A \in M_n \), for some scalar \( c \), non-singular matrix \( S \) and linear functional \( f \).
During the past few decades many researchers in matrix and operator theory dedicated they work to the study of preservers problems. A large amount of research literature was developed in this subject. Many papers [18, 21, 36, 65, 67, 69, 76], monographs [44, 55, 66] and Ph.D. Theses [25, 73] were written and published in this area. Various problems were solved, some are still open, and lots of new techniques were developed to solve them.

Some of the techniques used to solve linear preserver problems include: the duality technique [37, 50, 51], algebraic geometric technique [33], reduction to existing preserver results [36, 69], group theory techniques [20, 26], functional identity [14], algebra [29]. For more techniques and details we refer to [29, 44, 51, 66].

Now we give a brief description of two techniques used to solve linear preserver problems that will be used in this thesis. As Li and Pierce [44] pointed out, the duality technique consists of obtaining information about a linear preserver \( \phi \) by studying its dual transformation \( \phi^* \), or by studying simultaneously both \( \phi \) and \( \phi^* \). The dual transformation might be easier to characterize or might be itself an interesting linear preserver. Also, by using this approach, one can use additional techniques related to the dual to solve the problem. The first numerical range preserver result was obtained by Pellegrini [65], using this approach. Li and Tsing [51] also used this technique to characterize the linear preservers of the \( C \)-numerical range and radius, when \( C \) is a Hermitian matrix. The second method is the reduction of the given preserver problem to a well-studied linear preserver problem. In other words, one shows that the given linear preserver also preserves another function, subset or relation, whose linear preserver has been characterized, so the known results can be applied. As observed by different authors, [8, 58, 64, 70], reduction to rank one preservers is one of the most used approaches. For example the characterization of: spectrum preservers [36], commutativity preservers [15, 67, 76] and invertibility preservers [69] involve rank one preservers.

Besides linear preserver problems, researchers have also been interested in more general problems such as Additive and Multiplicative preservers on matrix or operator spaces. In some cases one can characterize preservers without any assumption such as linearity, additivity and multiplicity.

**Additive Preserver Problems** are a generalization of the linear preserver problems, where now the algebra is considered as a ring and the maps are assumed to be additive only. In most of the cases, for a given problem either the set of linear preservers and the set of additive preservers is the same or the set of additive preservers contains the set of linear preservers. For example, in [38], additive preservers of the numerical
range were characterized on various subalgebras of $\mathcal{M}_n$. In [38] it was shown that for $\mathcal{M}_n$ all the additive preservers are also linear, but for $\mathcal{T}_n$ (the set of $n \times n$ triangular matrices) there exists some additive preservers that are not linear.

As for the linear preserver problems, there are some techniques commonly used to solve additive preserver problems. The first idea is to try to adapt the proofs used on the linear analogue problem. Unfortunately, this may not be an easy task and there are various problems where this technique does not work. As in linear preserver problems, one can reduce the given additive preserver problem to a well-studied one and use the known results to solve the given problem. Another idea is to construct a new mapping, which is linear and is related to the original additive one, and then use the results of linear preserver problems. Finally, one can try to prove that the given additive preserver is linear, and then again use results from linear preserver problems.

As mentioned before, there are many different linear preservers problems and therefore it would be impossible to cover all of them in this chapter. Thus, in what follows, we present examples of linear and additive preserver problems that are relevant to this thesis. We start with preservers of rank one, since as mentioned above, some other problems can be solved by reducing them to a rank one preserver problem.

### 3.1.1 Rank one Preserver Problems

We say that a map $\phi$ from a matrix space $M_1$ to a matrix space $M_2$ preserves rank one if $\phi(A)$ is of rank one whenever $A$ is of rank one.

The linear rank one preservers on the space $\mathcal{M}_{m\times n}(F)$, where $F$ is an algebraically closed field of characteristic zero, were characterized in 1959 by Marcus and Moyls [59], using tensor spaces. In 1976, another proof was given by Minc [62] using elementary matrix theory.

The additive rank one preservers on the space $\mathcal{M}_n$ were characterized by Omladič and Šemrl [64]. They established the following result:

**Theorem 3.2.** [64] Suppose that $\phi$ is an additive surjective mapping on $\mathcal{M}_n$, preserving operators of rank one. Then there exists a ring automorphism $h : \mathbb{C} \to \mathbb{C}$ such that $\phi$ is either of the form

$$\phi([A_{ij}]) = P[h(A_{ij})]Q \quad \text{or} \quad \phi([A_{ij}]) = P[h(A_{ij})]^TQ,$$

where $P$ and $Q$ are invertible matrices.
To prove the previous result, the authors first showed that there is a ring automorphism \( h : \mathbb{C} \to \mathbb{C} \) such that for every rank one matrix \( A \) and every \( \lambda \in \mathbb{C} \), \( h \) satisfies

\[
\phi(\lambda A) = h(\lambda)\phi(A).
\]

Then they introduced a new mapping \( \psi : \mathcal{M}_n \to \mathcal{M}_n \) defined by:

\[
\psi([A_{ij}]) = \phi([h^{-1}(A_{ij})]),
\]

which is linear and preserves rank one matrices. Using linear rank one preservers they concluded the proof.

Next, we present the linear preservers of the classical numerical range and radius.

### 3.1.2 Numerical Range and Numerical Radius Preservers

As mentioned before, the study of numerical range and radius is an active area in matrix and operator theory. There has been also considerable interest in studying linear operators that preserve the numerical range or radius. In the following we present some of the results obtained, not only on the complex space of \( n \times n \) matrices, but also in some more general spaces. For more results and details we refer to [3, 16, 39, 48, 65] and [66, Chapter 5].

The first numerical range preserver result was obtained by Pellegrini [65]. He showed that the linear operators \( \phi \) on a unital algebra \( \mathcal{A} \) satisfy

\[
V(\phi(A)) = V(A) \quad \text{for all } A \in \mathcal{A}
\]

if and only if the dual transformation \( \phi' \) satisfies \( \phi'(S) = S \), where \( V(A) \) denotes the numerical range of \( A \) and \( S \) the set of states defined in Chapter 2.

In 1987, Li [39] showed that a linear mapping \( \phi : \mathcal{M} \to \mathcal{M} \), where \( \mathcal{M} \) either denotes \( \mathcal{M}_n \) or \( \mathcal{H}_n \), satisfies

\[
r(\phi(A)) = r(A) \quad \text{for all } A \in \mathcal{M}
\]

if and only if there exist \( U \in \mathcal{U}_n \) and \( \mu \in \mathbb{F} \) (\( \mathbb{F} = \mathbb{C} \) if \( \mathcal{M} = \mathcal{M}_n \) or \( \mathbb{F} = \mathbb{R} \) if \( \mathcal{M} = \mathcal{H}_n \)) with \( |\mu| = 1 \) such that:

\[
\phi(A) = \mu U \tilde{A} U^* \quad \text{for all } A \in \mathcal{M},
\]

where \( \tilde{A} \) either denotes \( A \) or \( A^t \).
Later, Chan [16] considered linear operators $\phi : B(\mathcal{H}) \rightarrow B(\mathcal{H})$, where $B(\mathcal{H})$ denotes the vector space of all bounded linear operators on a complex Hilbert space that preserve the numerical radius. He proved that a linear isomorphism $\phi : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ preserves the numerical radius if and only if it is a multiple of a $C^*$-isomorphism by a scalar of modulus one, where a $C^*$-isomorphism is a linear isomorphism of $B(\mathcal{H})$ such that $\phi(A^*) = \phi(A)^*$ for all $A$ in $B(\mathcal{H})$ and $\phi(A^n) = \phi(A)^n$ for all self-adjoint $A$ in $B(\mathcal{H})$ and all natural numbers $n$.

The additive preservers of the numerical range on $\mathcal{M}_n$ were obtained by Lešnjak, [38]. He proved the following result:

**Theorem 3.3.** [38] An additive mapping $\phi : \mathcal{M}_n \rightarrow \mathcal{M}_n$ preserves numerical range on $\mathcal{M}_n$ if and only if there exists a unitary matrix $U \in \mathcal{U}_n$ such that for all $A \in \mathcal{M}_n$, either

$$\phi(A) = UAU^* \quad \text{or} \quad \phi(A) = UA^*U^*.$$  

So, the additive preservers of the numerical range on $\mathcal{M}_n$ coincide with the linear preservers of the numerical range.

In the same year Li and Semrl [47] characterized numerical radius isometries, that is, maps $\phi : \mathcal{M}_n \rightarrow \mathcal{M}_n$ that satisfy

$$r(\phi(A) - \phi(B)) = r(A - B) \quad \text{for all} \quad A, B \in \mathcal{M}_n,$$

where $\phi$ is not assumed to be linear a priori. The mapping $L : \mathcal{M}_n \rightarrow \mathcal{M}_n$ defined by $L(A) = \phi(A) - \phi(0)$ satisfies $r(L(A)) = r(A)$ for all $A \in \mathcal{M}_n$ and is easily shown to be real linear. Therefore, the numerical radius isometries are in fact real linear.

They obtained the following result:

**Theorem 3.4.** [47] A map $\phi : \mathcal{M}_n \rightarrow \mathcal{M}_n$ satisfies

$$r(\phi(A) - \phi(B)) = r(A - B) \quad \text{for all} \quad A, B \in \mathcal{M}_n$$

if and only if there exist $U \in \mathcal{U}_n$, $R \in \mathcal{M}_n$ and a complex $\mu$, with $|\mu| = 1$, such that $\phi$ has the form:

$$A \mapsto \mu U^* A^\dagger U + R,$$

where $A^\dagger$ either denotes $A$, $A^\dagger$, $A^*$, $A^*$.
The results of Li and Sëmrl were generalized in [3] for $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space.

Many authors have studied linear preservers of different generalizations of numerical ranges and radii. Next we present the complex linear preservers of the $C$-numerical range and $C$-numerical radius. For preservers of other generalized numerical ranges and radius we refer to [42] and [66, Chapter 5].

### 3.1.3 $C$-numerical Range and $C$-numerical Radius (Complex)

#### Linear Preservers

Complex linear preservers of the $C$-numerical range and $C$-numerical radius were characterized in [52, 51, 57] for normal $C$. Unfortunately, the methods used to characterize $C$-numerical range and radius linear preservers when $C$ is a normal matrix are not applicable when $C$ is non-normal.

The first approach to $C$-numerical range and radius complex linear preservers for non-normal $C$ was made by Li, Mehta and Rodman, [43], in 1994. They studied the problem for matrices $C$ of rank one, and obtained two results: One for matrices $C$ of nonzero trace and one for matrices $C$ of zero trace. Their results are the following:

**Theorem 3.5.** [43] Let $C$ be a rank one matrix and $\phi : M_n \rightarrow M_n$ be a complex linear operator.

(i) If $\text{tr } C \neq 0$, then $C$-numerical range preservers have the form:

$$A \mapsto U^*AU \quad \text{or} \quad A \mapsto U^*A^tU,$$

for some $U \in U_n$.

And $C$-numerical radius preservers have the form:

$$A \mapsto \mu U^*AU \quad \text{or} \quad A \mapsto \mu U^*A^tU$$

for some $U \in U_n$ and $\mu \in \mathbb{C}$, $|\mu| = 1$.

(ii) If $\text{tr } C = 0$, then $C$-numerical range and $C$-numerical radius preservers have the form:

$$A \mapsto \mu U^*AU + f(A)I \quad \text{or} \quad A \mapsto \mu U^*A^tU + f(A)I$$

for some $U \in U_n$, $\mu \in \mathbb{C}$, $|\mu| = 1$ and $f$ a linear functional on $M_n$. 
To prove the previous theorem the authors used the fact that $W_{C_q}(A) = W_q(A)$ for $C_q = qE_{11} + \sqrt{1-|q|^2}E_{12}$, with $q \in \mathbb{C}$ and $0 < |q| \leq 1$, together with properties of the unitary orbit $U(C_q)$. Their proof was very computational.

Finally, in [27] using a group scheme, Li and Guralnick characterized all the complex linear operators that preserve the $C$-numerical range and the $C$-numerical radius for any nonzero matrix $C$. First, they characterized all the linear operators of the unitary orbit $U(C)$, then they used the fact that a linear operator preserves the $C$-numerical range if and only if its dual preserves the unitary orbit, [51], to obtain the desired result. The next theorem states their result:

**Theorem 3.6.** [27] Let $C$ be a nonscalar matrix. A complex linear operator $\phi : \mathcal{M}_n \rightarrow \mathcal{M}_n$ satisfies $W_C(\phi(A)) = W_C(A)$ for all $A \in \mathcal{M}_n$ if and only if there exist $U \in U_n$ and $\mu \in \mathbb{C}$ with $\mu C_0 \in U(C_0)$, where $C_0 = C - \text{tr}(C)I/n$, such that $\phi$ is of the form

$$A \mapsto \text{tr}(DA)I + \mu U^*AU,$$

or

$$A \mapsto \text{tr}(DA)I + \mu U^*A^tU,$$

for some $D \in \mathcal{M}_n$ satisfying

(a) $D = 0$ if $\text{tr}(C) = 0$,

(b) $D = (1 - \mu)I/n$ if $\text{tr}(C) \neq 0$.

The $r_C(\cdot)$ preservers were shown to be unit multiples of $W_C(\cdot)$ preservers.

This completely solves the problem of characterizing the complex linear preservers of the $C$-numerical range and $C$-numerical radius on $\mathcal{M}_n$. 
Chapter 4

Mappings that Preserve the Numerical Radii

In this chapter, we characterize isometries of the $q$-numerical radius when $q \in \mathbb{C}$, $0 < |q| < 1$ and we also characterize maps that preserve the $q$-numerical range. The case $q = 1$ (the classical numerical range and radius) is due to Lešnjak [38] and Li and Šemrl [47]. The complex linear isometries of $r_q(\cdot)$ were characterized by Li, Mehta and Rodman [43]. By an isometry of a metric space, we mean a function that preserves distance. In particular, a $q$-numerical radius isometry is a mapping $f : \mathcal{M}_n \to \mathcal{M}_n$ that satisfies

$$r_q(f(A) - f(B)) = r_q(A - B),$$

for all $A, B \in \mathcal{M}_n$. We assume no additivity or linearity of our maps, but such problems are quickly reduced to studying real linear isometrics as seen below.

4.1 Main Theorems

We begin by stating the two main theorems of this chapter.

Theorem 4.1. Let $q \in \mathbb{C}$ with $0 < |q| < 1$ and let $\phi : \mathcal{M}_n \to \mathcal{M}_n$. Then $\phi$ is an isometry of the $q$-numerical radius if and only if there exist a matrix $S_0 \in \mathcal{M}_n$, a
unitary $U \in \mathcal{U}_n$ and a complex number $\mu$, with $|\mu| = 1$, such that $\phi$ has one of the following four forms for all $A \in \mathcal{M}_n$:

$$\phi(A) = S_0 + \mu U^* AU, \quad \phi(A) = S_0 + \mu U^* A' U, \quad \phi(A) = S_0 + \mu U^* A^* \bar{U}$$

or

$$\phi(A) = S_0 + \mu U^* A U.$$

**Theorem 4.2.** Let $0 < |q| < 1$ and let $\phi : \mathcal{M}_n \rightarrow \mathcal{M}_n$. Then $\phi$ preserves the $q$-numerical range, i.e., $W_q(\phi(A)) = W_q(A)$ for every $A \in \mathcal{M}_n$, if and only if there exists a unitary $U \in \mathcal{U}_n$ such that

$$\phi(A) = U^* AU \quad \text{or} \quad \phi(A) = U^* A' A,$$

for all $A \in \mathcal{M}_n$.

We end this section by describing the reduction to real linear maps alluded to above. Let $(\mathcal{X}, ||\cdot||)$ be a Banach space and $\phi : \mathcal{X} \rightarrow \mathcal{X}$ satisfy $||\phi(x) - \phi(y)|| = ||x - y||$. Define $\rho : \mathcal{X} \rightarrow \mathcal{X}$ by $\rho(x) = \phi(x) - \phi(0)$. Then $||\rho(x)|| = ||x||$. The Mazur-Ulam Theorem [61] (see [4, p.166]) asserts that every such map $\rho$ is real linear.

In view of the above, we shall assume henceforth that all isometries studied in this thesis are real linear. Also, since $r_q(A) = r_q(A)$ and $W_q(A) = e^{i\theta} W_{|q|}(A)$, where $\theta = \arg q$, we can assume that $0 < q < 1$.

### 4.2 Proofs of Main Theorems for $n > 2$

We shall prove Theorem 4.1 directly for $n > 2$. Theorem 4.2 (for $n > 2$) then easily follows. We shall assume in what follows that $\phi : \mathcal{M}_n \rightarrow \mathcal{M}_n$ is a real linear mapping and satisfies $r_q(\phi(A)) = r_q(A)$, for $0 < q < 1$ and every $A \in \mathcal{M}_n$. We denote the dual of $\phi$ by $\psi$.

The following result describes the condition on the dual of $\phi$ that is equivalent to the isometric property of $\phi$.

**Theorem 4.3.** Let $0 < q \leq 1$, $p = \sqrt{1 - q^2}$ and $C_q = qE_{11} + pE_{12}$. Assume that $\phi : \mathcal{M}_n \rightarrow \mathcal{M}_n$ is a real linear map with dual $\psi$. Then $\phi$ is an $r_q(\cdot)$ isometry if and only if $\psi$ satisfies $\psi(SU(C_q)) = SU(C_q)$, where $SU(C_q)$ is the saturated unitary orbit of $C_q$. 
Proof. The proof is based on the same ideas used in the proof of Theorem 3.1 in [51] and the definition of inner product (1.4).

First observe that if \( Y \in SU(C_q) \), then there exist \( U_0 \in U_n \) and \( \theta_0 \in \mathbb{C} \) with \( |\theta_0| = 1 \) such that \( Y = \theta_0 U_0 C_q U_0^* \). Since \( Y^* \) is a rank one matrix that satisfies \( \text{tr}(Y^*) = q \) and \( \text{tr}(YY^*) = 1 \) by Corollary 2.7 we have that \( Y^* \in SU(C_q) \).

Suppose that \( \psi(SU(C_q)) = SU(C_q) \). Then using Lemma 2.8 and the observation above we have:

\[
\begin{align*}
  r_q(\phi(A)) &= \max\{\langle \phi(A), X^* \rangle : X \in SU(C_q) \} \\
               &= \max\{\langle A, \psi(X^*) \rangle : X \in SU(C_q) \} \\
               &= \max\{\langle A, Y \rangle : Y \in SU(C_q) \} = r_q(A).
\end{align*}
\]

Therefore \( \phi \) preserves the \( q \)-numerical radius.

Suppose now that \( r_q(\phi(A)) = r_q(A) \) for all \( A \in M_n \). Then we have that:

\[
\begin{align*}
  \max\\{\langle A, X \rangle : X \in SU(C_q) \} &= r_q(A) = r_q(\phi(A)) \\
  &= \max\{\langle \phi(A), X^* \rangle : X \in SU(C_q) \} \\
  &= \max\{\langle A, \psi(X^*) \rangle : X \in SU(C_q) \} \\
  &= \max\{\langle A, Y \rangle : Y \in \psi(SU(C_q)) \} \quad \text{for all} \quad A \in M_n.
\end{align*}
\]

That is, \( \max\\{\langle A, X \rangle : X \in SU(C_q) \} = \max\{\langle A, Y \rangle : Y \in \psi(SU(C_q)) \} \) for all \( A \in M_n \). Or,

\[
\max\{\Phi(X) : X \in SU(C_q) \} = \max\{\Phi(Y) : Y \in \psi(SU(C_q)) \} \quad \text{(4.1)}
\]

for all functionals \( \Phi \), where \( \Phi(X) = \langle A, X \rangle \).

Since \( SU(C_q) \) is compact, by Theorem 17.2 [68] we have that \( \text{conv}(SU(C_q)) \) also is compact. So \( \text{conv}(SU(C_q)) \) is a compact convex set. Analogously \( \text{conv}(\psi(SU(C_q))) \) is a compact convex set.

Suppose that \( \text{conv}(SU(C_q)) \not\subset \psi(\text{conv}(SU(C_q))) \). Then there exists at least one \( x_0 \in \text{conv}(SU(C_q)) \) such that \( x_0 \notin \psi(\text{conv}(SU(C_q))) \). Now, there exists \( c \in \mathbb{R} \) such that \( \max\{\Phi(X) : X \in \psi(\text{conv}(SU(C_q))) \} \leq c < \Phi(x_0) \leq \{\Phi(Y) : Y \in SU(C_q) \} \). Which is a contradiction with equation (4.1). Analogously if \( x_0 \in \psi(\text{conv}(SU(C_q))) \) and \( x_0 \notin \text{conv}(SU(C_q)) \) we also get a contradiction with equation (4.1). Therefore, we have:

\[
\psi(\text{conv}(SU(C_q))) = \text{conv}(SU(C_q)). \quad \text{(4.2)}
\]
Since \( q \neq 0 \), we have that the linear span of \( SU(C_q) \) is \( \mathcal{M}_n \). One can easily write \( E_{11} \) and \( E_{12} \) as linear combinations of elements of \( SU(C_q) \). And since the linear span of \( SU(C_q) \) is invariant under unitary similarity transformations, we also get that \( E_{ij} \in \text{span}\{SU(C_q)\} \), for all \( i, j \).

So, by (4.2) we have that \( \mathcal{M}_n \) is invariant under \( \psi \).

Also, since \( \phi \) is onto, we have that \( \psi \) is one-to-one. So, \( \psi \) is a bijection and therefore maps a convex set bijectively onto itself, hence it maps its extreme points onto itself.

By Theorem 2.9 we have that \( \text{ext}[\text{conv}(SU(C_q))] = SU(C_q) \), thus

\[
\psi(SU(C_q)) = \psi(\text{ext}[\text{conv}(SU(C_q))]) = \text{ext}[\text{conv}(SU(C_q))] = SU(C_q).
\]

\( \square \)

Now, let \( R \in \mathcal{M}_n \) be the sum of two members of \( SU(C_q) \), \( A \) and \( B \). Since \( A \) and \( B \in SU(C_q) \) by Corollary 2.7, we have that \( A \) and \( B \) are rank one matrices. So \( R \) can have rank two, one or zero.

The next two Lemmas compute the real dimension of the span of such matrices \( A \) and \( B \), when \( \text{rank}(R) = 2 \) and \( \text{rank}(R) = 1 \) for \( n > 2 \). The idea is to use the dimension of such spans to distinguish matrices with rank one from matrices with rank two, we show that if \( R \) is a rank one matrix, then \( \psi(R) \) also has rank one. The case \( \text{rank}(R) = 0 \) it is trivial.

**Lemma 4.4.** Let \( n > 2 \). Let \( R \) be a fixed rank two matrix such that \( R = A + B \), where \( A \) and \( B \) are any rank one matrices. Then the real dimension of the span of all such matrices \( A \) and \( B \) is at most 8.

**Proof.** Let \( U = \text{range}(R) \) and \( V = \text{null}(R) \). If \( A = ab^* \) and \( B = cd^* \), then \( \text{range}(R) = \text{range}(A + B) = \text{span}\{a, c\} \) and \( \text{range}(R^*) = \text{span}\{b, d\} \). So \( U = \text{span}\{a, c\} \) and \( V^\perp = \text{span}\{b, d\} \). We also have that \( \text{range}(A) \subset U \), \( \text{range}(B) \subset U \) and \( \text{null}(A) \supset V \), \( \text{null}(B) \supset V \). Hence \( A \) and \( B \) are essentially operators from \( V^\perp \) to \( U \), and the set of all such operators has 4 complex dimensions, i.e., 8 real dimensions.

\( \square \)

**Lemma 4.5.** Let \( n > 2 \). Let \( R \) be a rank one matrix of sufficiently small norm. The span of all matrices \( A, B \in SU(C_q) \) such that \( R = A + B \) has real dimension \( 4n - 2 \).
Proof. If $R = uw^t$ and $R$ has sufficiently small norm, then it is unitarily equivalent to $R_1 = \epsilon E_{11} + \delta E_{12}$, for $\epsilon$ and $\delta$ sufficiently small. Observe that $\epsilon$ can be written as a sum of two complex numbers $z_1$ and $z_2$, such that $|z_1| = |z_2| = q$, and $\delta$ as a sum of two complex numbers $z_3$ and $z_4$, such that $|z_3| = |z_4| < \sqrt{1 - q^2}$. Thus, $R_1 = A + B$, where $A = z_1 E_{11} + z_2 E_{12} + \zeta E_{1k}$, $B = z_2 E_{11} + z_4 E_{12} - \zeta E_{1k}$, $\zeta \in \mathbb{C}$ is such that $\|A\| = \|B\| = 1$ and $k = 3, 4, \ldots, n$. Since $A$ and $B$ are rank one matrices, $\|A\| = \|B\| = 1$ and $|\text{tr}(A)| = |\text{tr}(B)| = q$, we have that $A, B \in SU(C_q)$. For each different value of $k$, we can vary $\zeta$, as long as its modulus is fixed. Proceeding this way, we get a family of different matrices $A$ and $B$

Now, for each value of $k$, subtracting the appropriate matrices $A$ and $B$ we get multiples of $E_{1k}$ and $iE_{1k}$, $k = 3, 4, \ldots, n$. Also, since $|z_3| = |z_4| < \sqrt{1 - q^2}$, there exist different values of $z_3$ and $z_4$ and consequently different matrices $A$ and $B$. Once again, subtracting the appropriate matrices, we get multiples of $E_{11}$, $iE_{11}$, $E_{12}$ and $iE_{12}$. Thus the span of some of the possible $A$'s and $B$'s such that $R_1 = A + B$ is the span $E_{1j}$ and $iE_{1j}$, for $j = 1, \ldots, n$. In terms of the original $R$, we have $A$'s and $B$'s spanning the space $uw^t$ for every possible $w$. This has $n$ complex dimensions or $2n$ real dimensions. But by symmetry we also get all of $uw^t$. The total dimension is now $4n - 2$ real dimensions.

\[ \square \]

Lemma 4.6. Let $n > 2$. Let $R \in SU(C_q)$ be a rank one matrix and $\psi$ a real linear mapping that preserves $SU(C_q)$. Then $\psi$ preserves rank one, that is, $\text{rank}(\psi(R)) = 1$.

Proof. Since $n > 2$, we have that $8 < 4n - 2$. Let $R' = \lambda R$, where $\lambda$ is a constant that makes $R'$ a rank one matrix with small norm. By Lemma 4.5, the span of all matrices $A, B \in SU(C_q)$ such that $R' = A + B$ has dimension $4n - 2$. Since $\psi$ preserves $SU(C_q)$, we have that the span of all matrices $A, B \in SU(C_q)$ such that $\psi(R') = A + B$ also has dimension $4n - 2$. By Lemma 4.4, we have that $\psi(R')$ cannot be of rank 2, so $\text{rank}(\psi(R')) = 1$. Since $\psi(R) = \lambda \psi(R)$, we can remove the norm condition, so we have that $\psi$ preserves rank one.

\[ \square \]

By Theorem 4.3 and Lemma 4.6 we obtain the following result.

Theorem 4.7. Let $n > 2$ and $\phi: M_n \to M_n$ be a real linear map that preserves the $q$-numerical radius. Then the dual $\psi$ of $\phi$ preserves rank one matrices.
As mentioned in chapter 3, the additive rank one preservers on the space $\mathcal{M}_n$ were characterized by Omladič and Šemrl [64], see Theorem 3.2 in Chapter 3.

**Theorem 4.8.** Let $T$ be an additive mapping that preserves rank one matrices. Then $T$ is of the form:

$$T(A) = PA^tQ, \text{ for all } A \in \mathcal{M}_n,$$

where $A^t$ denotes either $A$ or $A^t$ or $A^*$ or $A^*$, $P$ and $Q$ are invertible matrices.

So, using Theorem 4.7 and Theorem 4.8 we have that $\phi$ is of the same form as the mapping described in (4.3). This partially solves the problem for $n > 2$, since we still need to show that the matrices $P$ and $Q$ satisfy $PQ = \mu U$ for some $\mu \in \mathbb{C}$, $|\mu| = 1$ and $U \in \mathcal{U}_n$. This will be done at the end of this chapter. Now, we solve the case $n = 2$.

### 4.3 Proofs of Main Theorems for $n = 2$

We start by characterizing the mappings on $\mathcal{M}_2$ that preserve the $q$-numerical range. The next step is to show that if $\phi$ is a real linear map that preserves the $q$-numerical radius, then there exists $\mu \in \mathbb{C}$, $|\mu| = 1$, such that $\phi(I) = \mu I$ and $\phi(iI) = \pm \mu i I$. Finally, we show that $\phi$ preserves the $q$-numerical radius if and only if $\phi$ is a unit multiple of a $q$-numerical range preserver.

**Theorem 4.9.** Let $\rho : \mathcal{M}_2 \to \mathcal{M}_2$ be a real linear mapping that satisfies $W_{C_q}(\rho(A)) = W_{C_q}(A)$ for all $A \in \mathcal{M}_2$, that is, $\rho$ preserves the $q$-numerical range. Then $\rho$ also preserves rank one matrices.

**Proof.** Any matrix $A \in \mathcal{M}_2$ is unitarily equivalent to

$$e^{it} \begin{pmatrix} \alpha & a \\ b & \alpha \end{pmatrix},$$

with $t \in [0, 2\pi)$, $\alpha \in \mathbb{C}$ and $0 \leq b \leq a$, [34]. So, there exist $t_1, t_2 \in [0, 2\pi)$, $\alpha, \gamma \in \mathbb{C}$, $0 \leq b \leq a$ and $0 \leq d \leq c$ such that $C_q$ is unitarily equivalent to $e^{it_1} \begin{pmatrix} \gamma & c \\ d & \gamma \end{pmatrix}$ and $A$ is unitarily equivalent to $e^{it_2} \begin{pmatrix} \alpha & a \\ b & \alpha \end{pmatrix}$. 


Since the $C$-numerical range is invariant under unitary similarity transformations of both $A$ and $C_q$, we have that $W_{C_q}(A) = W_{e^{i\theta}C_q}(e^{i\theta}A)$, where $\bar{A} = \begin{pmatrix} a & \alpha \\ b & \alpha \end{pmatrix}$ and $\tilde{C}_q = \begin{pmatrix} \gamma & c \\ d & \gamma \end{pmatrix}$. We also have that $W_{\eta_1 C}(\eta_2 A) = \eta_1 \eta_2 W_{C}(A)$ for any $\eta_1, \eta_2 \in \mathbb{C}$ and $A, C \in M_n$. Therefore, $W_{C_q}(A) = e^{i(t_1 + t_2)}W_{\tilde{C}_q}(\bar{A})$.

From $\det(C_q) = 0$, we get that $\gamma^2 = cd$ and from $\text{tr}(C_q) = q$ we get that $\gamma = e^{-\mu_1} \frac{q}{d}$.

Therefore, $cd = e^{-2\mu_1} \frac{q^2}{d^2}$.

By Theorem 1 in [41], we have that

$$W_{C_q}(A) = W\left(2e^{i(t_1 + t_2)}\begin{pmatrix} \gamma \alpha & ac \\ bd & \gamma \alpha \end{pmatrix}\right),$$

which is an ellipse with foci $\lambda_1 = qe^{i\alpha}(\alpha + \sqrt{ab})$ and $\lambda_2 = qe^{i\alpha}(\alpha - \sqrt{ab})$, where $\lambda_1$ and $\lambda_2$ are the eigenvalues of the new matrix $A' = 2e^{i(t_1 + t_2)}\begin{pmatrix} \gamma \alpha & ac \\ bd & \gamma \alpha \end{pmatrix}$, see [30],

and they satisfy $\lambda_1 = q\lambda_1$ and $\lambda_2 = q\lambda_2$, where $\lambda_1$ and $\lambda_2$ are the eigenvalues of $\bar{A}$.

Now, since $\rho$ preserves the $C$-numerical range we have $W_{C_q}(\rho(A)) = W_{C_q}(A) = W(A')$. So, both $\rho(A)$ and $A$ have the same eigenvalues, which implies that $\rho$ preserves the spectrum. And since $\rho$ preserves the spectrum it will also preserve rank one matrices.

\[\square\]

To show that $q$-numerical radius isometries satisfy $\phi(I) = \mu I$ and $\phi(iI) = \pm \mu iI$, for some $\mu \in \mathbb{C}$ with $|\mu| = 1$, we will need to prove first some auxiliary results. The first series of results will allow us to prove that $\phi(iI) = \pm i\phi(I)$.

**Lemma 4.10.** Let $a, b$ and $d \in \mathbb{C}$. If $|ae^{it} + be^{-it} + d| = 1$ for all $t \in \mathbb{R}$, then two of $a, b$ and $d$ are zero.

**Proof.** By expanding $|ae^{it} + be^{-it} + d|^2 = 1$ and using the orthogonality of the trigonometric system we have that

$$a\bar{b} = 0$$

(4.4)
and

\[ ad + db = 0 \]  \hspace{1cm} (4.5)

From (4.4) we have that \( a = 0 \) or \( b = 0 \).

If \( a = 0 \), then by (4.5) we have that \( d = 0 \) or \( b = 0 \). And if \( b = 0 \), then by (4.5) again we have that \( a = 0 \) or \( d = 0 \). So, we have that either \( a = d = 0 \), or \( a = b = 0 \) or \( b = d = 0 \).

\[ \square \]

**Corollary 4.11.** Let \( a, b \) and \( d \in \mathbb{C} \). If \( |a \cos t + b \sin t + d| = 1 \) for all \( t \in \mathbb{R} \), then either

(i) \( a = b = 0 \) or

(ii) \( d = 0 \) and \( b = \pm ia \).

**Proof.** The equation in the statement becomes

\[
\left| \frac{a - bi}{2} e^{it} + \frac{a + bi}{2} e^{-it} + d \right| = 1.
\]

By Lemma 4.10, we have that two of \( a + ib \), \( a - ib \) and \( d \) are zeros.

If \( a + ib = 0 \) and \( a - ib = 0 \), then \( a = b = 0 \).

If \( a + ib = 0 \) and \( d = 0 \), then \( a = -ib \) and \( d = 0 \) and if \( a - ib = 0 \) and \( d = 0 \), then \( a = ib \) and \( d = 0 \). Hence, \( d = 0 \) and \( a = \pm ib \).

\[ \square \]

The next Theorem establishes another preserver property of the dual of \( \phi \).

**Theorem 4.12.** Let \( \phi : M_2 \to M_2 \) be a real linear map that preserves the \( q \)-numerical radius and let \( \psi \) be its dual map. Then \( \psi \) preserves matrices of trace zero.

**Proof.** Let \( u \) and \( v \) be orthonormal vectors in \( \mathbb{C}^n \). Consider \( A = \psi(uu^*) \), \( B = \psi(iuv^*) \) and \( D = \psi( uv^*) \). By Corollary 2.7 we have that

\[ q(uu^*) \cos t + qi(uu^*) \sin t + p( uv^*) \in SU(C_q) \] for all \( t \).
By Theorem 4.3 \( \psi \) preserves \( SU(C_q) \), hence
\[
\psi(q(uu^*) \cos t + qii(uu^*) \sin t + p(uu^*)) \in SU(C_q) \text{ for all } t,
\]
that is,
\[
qA \cos t + qB \sin t + pD \in SU(C_q) \text{ for all } t.
\]

By Corollary 2.7 we have that \( |\text{tr}(qA \cos t + qB \sin t + pD)| = q \) for all \( t \), that is
\[
|\text{tr}(A) \cos t + \text{tr}(B) \sin t + \frac{p}{q} \text{tr}(D)| = 1 \text{ for all } t.
\]

By Corollary 4.11 we have that either \( \text{tr}(D) = 0 \) and \( \text{tr}(B) = \pm i\text{tr}(A) \) or \( \text{tr}(A) = \text{tr}(B) = 0 \) and \( |\text{tr}(D)| = \frac{q}{p} \).

Suppose that \( \text{tr}(D) = 0 \), that is \( \text{tr}(\psi(uu^*)) = 0 \) for orthonormal vectors \( u \) and \( v \).

Then, since any zero trace matrix is a real linear combination of matrices of the form \( zw^* \), where \( z \) and \( w \) are orthonormal vectors, we have that \( \psi \) maps each zero trace matrix into a zero trace matrix.

Suppose now that \( \text{tr}(A) = \text{tr}(B) = 0 \) and \( |\text{tr}(D)| = \frac{q}{p} \). Let's show that this case cannot happen. Let \( u = e_1 \) and \( v = e_2 \), then \( A = \psi(E_{11}), B = \psi(E_{21}) \) and \( D = \psi(E_{12}) \).

In what follows, under the above hypothesis, we are going to use the image under \( \psi \) of different matrices in \( SU(C_q) \) together with the trace condition given by Corollary 2.7 to obtain relations between the trace of the images of \( iE_{12}, E_{21}, iE_{21}, E_{22} \) and \( iE_{22} \). Let
\[
N_1 = qE_{11} + pE_{21},
N_2 = \frac{q}{2}E_{11} + \frac{1+p}{2}E_{12} + \frac{1-p}{2}E_{21} + \frac{q}{2}E_{22},
\]
and
\[
N_3 = \frac{iq}{2}E_{11} + \frac{1+p}{2}E_{12} + \frac{-(1-p)}{2}E_{21} + \frac{iq}{2}E_{22}.
\]

Due to Corollary 2.7, we have that \( N_k \in SU(C_q), k = 1, 2, 3 \) and therefore \( \psi(N_k) \in SU(C_q), k = 1, 2, 3 \).

Hence, \( |\text{tr}(\psi(N_1))| = q \), i.e., \( |\text{tr}(A) + p\text{tr}(H)| = q \), where \( H = \psi(E_{21}) \). Since \( \text{tr}(A) = 0 \), we have that \( |\text{tr}(H)| = \frac{q}{p} \).

Using the same ideas at the beginning of the proof with \( F = \psi(vv^*), G = \psi(ivv^*) \) and \( D = \psi(uu^*) \), together with the fact that \( |\text{tr}(\psi(E_{21}))| = \frac{q}{p} \), one can show that \( \text{tr}(F) = \text{tr}(G) = 0 \).
Using the fact that $|\text{tr}(\psi(N_3))| = q$, we have
\[ \left| \frac{(1 + p)}{2} \text{tr}(D) + \frac{(1 - p)}{2} \text{tr}(H) \right| = q. \]

Since $|\text{tr}(D)| = |\text{tr}(H)|$, there exists $\theta$ such that $\text{tr}(H) = e^{i\theta}\text{tr}(D)$, thus
\[ \left| \frac{(1 + p)}{2} \text{tr}(D) + \frac{(1 - p)}{2} e^{i\theta} \text{tr}(D) \right| = q, \]
that is,
\[ |(1 + p) + (1 - p)(\cos(\theta) + i\sin(\theta))| = 2p. \]

By expanding $[(1 + p) + (1 - p)\cos(\theta)] + i[(1 - p)\sin(\theta)]|^2 = 4p^2$ we have that $\cos(\theta) = -1$, hence $\text{tr}(H) = -\text{tr}(D)$.

Finally, $\psi(N_3) \in SU(C)$, but
\[ |\text{tr}(\psi(N_3))| = \left| \frac{(1 + p)}{2} \text{tr}(D) - \frac{(1 - p)}{2} \text{tr}(H) \right| = \left| \frac{(1 + p)}{2} \text{tr}(D) + \frac{(1 - p)}{2} \text{tr}(D) \right| = \frac{q}{2p} \neq q. \]

Therefore $\text{tr}(A) = \text{tr}(B) = 0$ cannot happen.

\[ \square \]

**Corollary 4.13.** Let $\phi : M_2 \to M_2$ be a real linear map that preserves the $q$-numerical radius. Then $\phi$ maps the space $Cl = \{\alpha I : \alpha \in \mathbb{C}\}$ into itself.

**Proof.** By Theorem 4.12, we have that $\psi$ maps the space of zero trace matrices into itself. Therefore, $\phi$ leaves the orthogonal complement of this space invariant, that is, $\phi$ leaves the space $Cl$ invariant.

\[ \square \]

**Theorem 4.14.** Let $\phi : M_2 \to M_2$ be a real linear map that preserves the $q$-numerical radius. Then we have:

(a) $\phi(iI) = \pm i\phi(I)$
(b) \( \phi(\lambda I) = \lambda \phi(I) \) or \( \phi(\lambda I) = \overline{\lambda \phi(I)} \), where \( \lambda \in \mathbb{C} \).

**Proof.** (a) From Corollary 4.13, we have that \( \phi(I) = \alpha I \) and \( \phi(iI) = \beta I \), for some \( \alpha, \beta \in \mathbb{C} \).

Since \( \phi \) preserves the q-numerical radius and \( r_q(I) = r_q(iI) = q \), we have that \( |\alpha| = 1 \) and \( |\beta| = 1 \). We also have that \( r_q(I + iI) = \sqrt{2} \), that is \( |\alpha + \beta| = \sqrt{2} \).

Therefore \( \beta = \pm i\alpha \) and \( \phi(iI) = \pm i\phi(I) \).

(b) Let \( \lambda \in \mathbb{C}, \lambda = \alpha + i\beta \). Then \( \phi(\lambda I) = \phi((\alpha + i\beta)I) = a\phi(I) + b\phi(iI) = a\phi(I) \pm ib\phi(I) \). Therefore \( \phi(\lambda I) = (a + ib)\phi(I) = \lambda \phi(I) \) or \( \phi(\lambda I) = (a - ib)\phi(I) = \overline{\lambda \phi(I)} \).

\[ \square \]

**Proposition 4.15.** Suppose \( F: \mathcal{M}_2 \to \mathcal{K}(\mathbb{C}) \), where \( \mathcal{K}(\mathbb{C}) \) is the set of nonempty compact convex subsets of \( \mathbb{C} \) such that
\[
F(\alpha A + \beta I) = \alpha F(A) + \beta \quad \text{for any} \quad \alpha, \beta \in \mathbb{C},
\]
and define \( f: \mathcal{M}_2 \to \mathbb{R} \) by
\[
f(A) = \max\{|z| : z \in F(A)\}.
\]

If \( \phi \) is a real operator on \( \mathcal{M}_2 \) satisfying \( \phi(I) = I \), \( \phi(iI) = iI \) and \( f(\phi(A)) = f(A) \) for all \( A \in \mathcal{M}_2 \), then \( F(\phi(A)) = F(A) \) for all \( A \in \mathcal{M}_2 \).

**Proof.** The proof is analogous to the proof of Proposition 4.19 in [46].

\[ \square \]

The next theorem relates q-numerical range real linear preservers and q-numerical radius real-linear preserves. Before we present and prove the Theorem we need to introduce the following notation: For a real linear mapping \( \phi: \mathcal{M}_n \to \mathcal{M}_n \), \( \overline{\phi}: \mathcal{M}_n \to \mathcal{M}_n \) is defined by \( \overline{\phi}(A) = \phi(\overline{A}) \) for all \( A \in \mathcal{M}_n \).

**Theorem 4.16.** A real linear map \( \phi: \mathcal{M}_2 \to \mathcal{M}_2 \) preserves the q-numerical radius if and only if there exists a complex \( \mu \), with \( |\mu| = 1 \), such that \( \mu \phi \) or \( \mu \overline{\phi} \) preserves the q-numerical range.
Proof. If $\phi$ is a unit multiple of a $q$-numerical range preserver, then one can easily show that $\phi$ preserves the $q$-numerical radius.

Suppose now that $\phi$ preserves the $q$-numerical radius. By Corollary 4.13 and Theorem 4.14, there exists $\mu \in \mathbb{C}$, $|\mu| = 1$ such that $\phi(I) = \mu I$ and $\phi(\mu I) = \pm \mu I$. Thus, $\overline{\mu} \phi$ or $\overline{\mu} \phi$ satisfies the hypothesis of Proposition 4.15 with $F(A) = W_q(A) / \text{trace}(A)$, where $\overline{\phi}(A) = \phi(A)$.

Therefore, $W_q(\overline{\phi}(A)) = W_q(A)$ for all $A \in \mathcal{M}_2$, where $\overline{\phi}(A) = \overline{\mu} \phi(A)$ or $\overline{\phi}(A) = \overline{\mu} \phi(A)$. That is, $\phi$ is a unit multiple of a $q$-numerical range preserver.

The next series of lemmas will allow us to obtain the form of $\phi$ stated in Theorem 4.1.

**Lemma 4.17.** Let $a$, $b$ and $c$ be complex numbers such that $a + be^{is} + ce^{-is} = 0$ for every $s$. Then $a = b = c = 0$

*Proof.* This follows from the orthogonality of the trigonometric system. 

**Lemma 4.18.** Let $\mathcal{H}$ be a Hilbert space. If $u, v \in \mathcal{H}$ and $\|u + e^{is}v\| = k$, where $k$ is a positive real number, for all $s$, then $u$ is orthogonal to $v$.

*Proof.* From $k = \langle u + e^{is}v, u + e^{is}v \rangle$ and Lemma 4.17 one can easily obtain $\langle u, v \rangle = 0$. Therefore, $u$ is orthogonal to $v$.

**Corollary 4.19.** Let $u, v \in \mathbb{C}$. If $|u + e^{is}v| = 1$ for all $s$, then either $u = 0$ or $v = 0$.

**Lemma 4.20.** Let $T \in \mathcal{B}(\mathcal{H})$. If $T(x)$ is orthogonal to $T(y)$ whenever $x$ is orthogonal to $y$, then $T$ is a scalar multiple of an isometry.
Proof. The proof is divided into three parts:

(i) First, let \( x \) and \( y \) be orthonormal. If \( x \) and \( y \) are orthonormal, then \( x + e^{is}y \) is orthogonal to \( x - e^{is}y \) for all \( s \). By hypothesis we have \( \langle T(x), T(y) \rangle_{\mathcal{C}} = 0 \) and \( \langle T(x + e^{is}y), T(x - e^{is}y) \rangle_{\mathcal{C}} = 0 \), this implies that \( \|T(x)\| = \|T(y)\| \).

(ii) Suppose now that \( n > 2 \), and \( u \) and \( v \) are any vectors such that \( \|u\| = \|v\| = 1 \). Then there exists a unit vector \( w \) which is orthogonal to both \( u \) and \( v \). From the previous part, we have \( \|T(x)\| = \|T(w)\| = \|T(y)\| \).

(iii) Finally, for \( n = 2 \), let \( u \) and \( v \) be any vectors such that \( \|u\| = \|v\| = 1 \). If \( w = v - \langle v, u \rangle_{\mathcal{C}}u \), then \( w \) is orthogonal to \( u \). Since \( T(w) \) is orthogonal to \( T(u) \), we have \( \langle T^{*}T(u), u \rangle_{\mathcal{C}} = \langle v, u \rangle_{\mathcal{C}} \langle T^{*}T(v), u \rangle_{\mathcal{C}} \). This implies that

\[
\langle T^{*}T(u), u \rangle_{\mathcal{C}} = \langle T^{*}T(v), u \rangle_{\mathcal{C}}
\]

(4.6)

for all unitary vectors \( u \) and \( v \). Using (4.6) and the extremal eigenvalues of \( T^{*}T \), that is, the smallest and the largest eigenvalues of \( T^{*}T \), we have that \( T^{*}T \) is a scalar matrix.

Therefore, \( T \) is a scalar multiple of an isometry.

\[\square\]

Now we are able to prove Theorem 4.1.

Proof. Suppose that there exists a matrix \( S_{0} \in \mathcal{M}_{n} \), a unitary \( U \in \mathcal{U}_{n} \) and a complex number \( \mu \) with \( |\mu| = 1 \) such that \( \phi(A) = S_{0} + \mu A^{*}A^{\dagger}U \) for all \( A \in \mathcal{M}_{n} \), where \( A^{\dagger} \) denotes either \( A \) or \( A^{*} \) or \( A^{*} \). Let’s us show that \( \phi \) is a \( q \)-numerical radius isometry.

First observe that since \( C_{q}^{*} \), \( \overline{C}_{q} \) and \( C_{q}^{r} \) satisfy the hypothesis of Corollary 2.7, we have that \( C_{q}^{*} \), \( \overline{C}_{q} \) and \( C_{q}^{r} \) belong to \( SU(C_{q}) \). In fact, one can easily show that they are unitarily equivalent to \( C_{q} \).

We also have that:

\[
W_{C_{q}}(A^{*}) = W_{A}(C_{q}) = \overline{W_{A}(C_{q}^{*})} = \overline{W_{A}(C_{q})} = \overline{W_{C_{q}}(A)},
\]

\[
W_{C_{q}}(A^{\dagger}) = W_{A}(C_{q}) = W_{A}(C_{q}^{r}) = W_{A}(C_{q}) = W_{C_{q}}(A),
\]

\[
W_{C_{q}}(A) = W_{A}(C_{q}) = W_{A}(C_{q}) = W_{A}(C_{q}) = W_{C_{q}}(A).
\]

Therefore \( r_{C_{q}}(A^{*}) = r_{C_{q}}(A) = r_{C_{q}}(A^{\dagger}) = r_{C_{q}}(A) \) for all \( A \in \mathcal{M}_{n} \). And now it is straightforward to verify that \( \phi \) is a \( q \)-numerical radius isometry.
Let $n > 2$. Since $\phi$ preserves the $C$-numerical radius, by Theorem 4.7 and Theorem 4.8 we have that $\psi$, the dual of $\phi$, satisfies $\psi(A) = PA^tQ$, where $P$ and $Q$ are invertible matrices and $A^t$ denotes $A$ or $A^t$ or $A^*$ or $\bar{A}$. Suppose $\psi(A) = PAQ$, the other three possibilities are analogous. Let $x, y \in \mathbb{C}^n$ such that $x$ is orthogonal to $y$ and $\|x\| = \|y\| = 1$. Let $z = qx + e^{is}(\sqrt{1-q^2})y$. Then $xz^* = qxz^* + e^{-is}(\sqrt{1-q^2})xy^*$, that is $xz^* \in SU(C_n)$. Thus $Pxz^*Q \in SU(C_q)$, which implies that:

(a) $\|Pxz^*Q\| = 1$
(b) $|\text{tr}(Pxz^*Q)| = q$

From condition (a) we have $1 = \|Pxz^*Q\| = \|Px\|\|Q^*z\|$, therefore $|qQ^*x + e^{is}\sqrt{1-q^2}Q^*y| = K$ for all $s$, where $K$ is a positive constant. By Lemma 4.18 we have that $Q^*x$ is perpendicular to $Q^*y$.

Now, since $Q^*x$ is orthogonal to $Q^*y$, by Lemma 4.20 we have that $Q$ is a multiple of an isometry. Since $Q$ is invertible we have that $Q$ is a multiple of a unitary matrix. Analogously, we have that $P$ is a multiple of a unitary matrix.

The trace condition (b) implies that $|q(QPx, x)_C + e^{is}(Q Px, y)_C| = q$, whenever $x$ and $y$ are orthonormal. By Corollary 4.19, $(QPx, x)_C = 0$ or $(QPx, x)_C = 1$. Since the numerical range is a convex set, $(QPx, x)_C = 0$ for some unit vectors $x$ and $(QPx, x)_C = 1$ for others is not possible. Therefore, $(QPx, x)_C = 0$ for every unit vector $x$ or $(QPx, x)_C = 1$ for every unit vector $x$. But $(QPx, x)_C = 0$ for every unit vector $x$, implies that $QP = 0$, which is not true. So, $(QPx, x)_C = 1$ for every unit vector $x$, which again by the convexity of the numerical range implies that $QP = \lambda I$, with $|\lambda| = 1$. Therefore, there exist a unitary matrix $U$ and $\mu \in \mathbb{C}$ such that $\phi(A) = \mu U^* A U$.

Suppose now $n = 2$. By Theorem 4.16 we have that there exists $\xi \in \mathbb{C}$, $|\xi| = 1$, such that $\xi \phi$ or $\xi \bar{\phi}$ preserve the $q$-numerical range. And by Theorem 4.9 we have that $q$-numerical range preservers are in fact rank one preservers. So by the above we have that $\phi$ is of the desired form.
Chapter 5

Conclusion

Generalizations of the concepts of numerical range and radius have been the objects of study in much research in the past years. The characterization of preservers of some set, function or property has also been another important area of research in matrix and operator theory.

In this thesis we characterized isometries of one of these generalizations: the $q$-numerical radius, which is equal to the $C$-numerical range if $C$ is a matrix of rank one, norm 1 and trace $q$, $0 < q < 1$. When complex linearity is assumed, a part of the isometry group is lost. So no linearity of these maps was assumed and we showed that in fact these maps are real linear. Hence, we solved the problem of characterization of real linear maps $\phi$ that preserve the $q$-numerical radius. Real linear maps that preserve the $q$-numerical radius were also characterized, since they are easily deduced from the radius preservers.

Complex linear preservers of the $C$-numerical radius and $C$-numerical range for $C$ a matrix of rank one, norm one and trace $q$, with $0 < |q| \leq 1$ were characterized in 1994 by Li, Mehta and Rodman [43]. Since we did not assume any linearity, the results obtained in this work are more general than the ones obtained in [43]. Also in the complex linear case our proof is an alternative proof to the one obtained in [43].

For $n > 2$ the key step to solve the problem was the use of the dual map, $\psi$, of the $q$-numerical radius preserver. We showed that $\psi$ preserves the saturated unitary orbit and preserves rank one matrices, and rank one preservers have been already characterized. The proof that $\psi$ preserves the saturated unitary orbit does not depend on $n$, so it is also true for $n = 2$. Unfortunately the proof that $\psi$ preserves the set of
rank one matrices depends on $n$, and it is not true for $n = 2$. So we had to find an alternative proof for $n = 2$.

To solve the case $n = 2$, first we showed that a real linear map $\rho$ that preserves the $q$-numerical range also preserves rank one matrices. Then, using the dual map $\psi$ of $\phi$, we showed that $\phi$ maps the set of complex scalar matrices into itself and that $\phi(iI) = \pm i\phi(I)$. Using these two results about $\phi$ we proved that $\phi$ preserves the $q$-numerical radius if and only if there exist a complex number $\mu$, $|\mu| = 1$, such that $\mu\phi$ or $\overline{\mu}\phi$ preserves the $q$-numerical range.

The characterization of a $q$-numerical radius isometries was then concluded using rank one preservers and properties of the saturated unitary orbit of the matrix $C_q$.

Since the concepts of $q$-numerical range and radius can be extended to Hilbert spaces of infinite dimension, it would be interesting to characterize maps that preserve the $q$-numerical range and radius in such spaces. Also, it would be interesting to characterize real linear maps that preserve the $C$-numerical radius when $C$ is not a rank one matrix.
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