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Waves Generated by a Load Moving on an Ice Sheet over Water

by

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B.Sc., Bandung Institute of Technology. 1984
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A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of

DOCTOR OF PHILOSOPHY

in the Department of Mathematics and Statistics

We accept this dissertation as conforming to the required standard

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All rights reserved. Dissertation may not be reproduced in whole or in part, by mimeograph or other means, without the permission of the author.
A load moving on a floating ice sheet produces a deflection of the ice sheet. In this Dissertation, three problems associated with mathematical models of the ice water system are examined.

A mathematical model involving a steadily moving rectangular load on an ice sheet where the supporting fluid is of infinite depth is analyzed. The solution is written as a Fourier integral and is estimated using an asymptotic method. The results show that the amplitude of the ice deflection is similar to the case where the supporting fluid is of finite depth. The only significant difference is that, in contrast to the case where the supporting fluid is of finite depth where a quiescent zone appears behind the load when its speed exceeds the speed of gravity waves on shallow water, waves appear behind the load for all supercritical load speeds.

A mathematical model of an ice plate that takes into account the thickness of the ice is derived by assuming that the vertical shearing forces vary linearly through the ice plate. The equations obtained are similar to those used to describe a mathematical model using a thin plate approximation subjected to in-plane forces. A comparison of the dispersion relation is carried out between the mathematical model of an ice plate that takes into account the plate thickness, the mathematical model of an ice plate using the thin plate approximation, and the mathematical model of an ice plate using the thin plate approximation subjected to in-plane forces. The results show that taking the ice thickness into consideration decreases the minimum phase speed. However, this effect is small.

The major contribution of this Dissertation is the determination of the large
time response of the deflection of an ice sheet caused by the steady motion of an impulsively-started point load. The results obtained are new. The solution of the ice deflection is written as a Fourier integral and asymptotic methods are used to estimate the large time behaviour of the rate of change of the ice deflection with respect to time. The large time behaviour of the ice deflection itself is inferred from this estimate. This is done for the full range of load speeds and the results are verified numerically using the Fast Fourier Transform. The results in this Dissertation show that the minimum of the phase speed is the only critical speed, in the sense that no finite steady-state is attainable. At this speed the ice deflection grows logarithmically with time. This is in contrast with the case of a line load where there are two critical speeds: the minimum of the phase speed at which the ice deflection grows as the square-root of time, and the speed of gravity waves in shallow water at which the ice deflection grows as the cube-root of time. For a point load, it is found that the transient part of the ice deflection decays as the cube-root of time when the load speed is the speed of gravity waves in shallow water. The asymptotic estimates also show that the decay or the growth rate of the transient component of the ice deflection does not depend on either the relative orientation of the observation point and the load or on the distance between the load and the observation point.
Examiners:

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Chapter 1

Introduction

1.1 Introduction

For some areas on earth, ice is something that never exists in nature. But in other areas, ice becomes part of daily life that needs to be understood. Working in a cold environment surrounded by ice can pose a certain level of difficulty that is usually associated with a harsh and dynamic environment. Gold [6] states that

As early as the mid-1800s, ice was imposing significant constraints on activities such as the construction and operation of hydro-electric systems and shipping on inland waterways and in coastal regions.

Floating ice sheets on lake water in the Arctic and in the Antarctic regions are routinely used for transportation. For example, ice-covered lakes and rivers serve as seasonal runways and roadways and provide winter access to remote areas. It is now well known that a load travelling above a critical speed on a floating ice sheet generates waves in the ice, whereas the response is quasi-static at slower load speeds (Squire et al. [26]). This wave has an amplitude that depends on the load speed and the ice-water parameters, such as the ice rigidity and the depth of the underlying water. By understanding the response of these floating ice sheets to a moving load, the operational safety of transportation can be assessed. In more temperate climates, the emphasis is on the effort required to break ice. Consequently, it is important to determine loads that can be supported by ice sheets.
1.1 Introduction

This study falls into the more general area of the response of continuously supported beams and plates to moving loads. For example, Timoshenko [33] studied the response of a railroad track to dynamic loads and Kerr [9] analyzed an infinite beam subjected to a moving load and a constant axial force. A large number of these studies are discussed in the survey by Kerr [10] and also in the more recent book by Squire et al. [26].

In order to apply analytic techniques to study the response of a floating ice sheet to a load moving on its surface, the ice sheet is usually treated as an elastic (see, for example, Kerr [11], Davys et al. [5], Schulkes et al. [24], Milinazzo et al. [19]) or a viscoelastic (see Hosking et al. [8]) homogeneous plate of infinite horizontal extent. To model the fluid-ice interaction, one approach is to take the fluid supporting the ice sheet to be a Winkler base where it is assumed that the reaction pressure of the fluid base on the ice sheet is proportional to the local deflection of the ice sheet. This approach is accurate enough to describe the characteristics of a supporting fluid for a static problem (Kerr [10], Livesley [17]).

For dynamic problems, however, a more realistic model of the fluid-ice interaction is needed to take into account the inertia of the fluid base. In this case, the pressure that the fluid exerts on the ice is obtained by solving the equations of motion of the fluid. For example, the fluid base can be considered to consist of an ideal, homogeneous, incompressible fluid and the corresponding mathematical model for the fluid-ice system can be derived. Greenhill [7] used this latter approach. That article, one of the first theoretical studies of the subject, includes an analysis of the response of a floating ice plate to a water wave propagating in one direction. Greenhill assumed the ice plate to be of uniform thickness and the water base to be of uniform depth and gave the derivation of the dispersion relation\(^1\) for the waves in the system.

\(^1\)A dispersion relation states the nature of a dispersive process where, in this case, waves of different lengths, propagating at different speeds, disperse or separate. A dispersive property
Kheisin [12] assumed the fluid base to be ideal, homogeneous, and incompressible, and studied the response of an ice plate to both a moving point load and a moving line load. This pioneering article examined the steady ice deflection for various load speeds and noted that the deflection depends on the load speed. In the article, it was shown that for a line load there are two critical speeds, defined to be those speeds where the ice deflection is infinite. Critical speeds were found only for a line load.

Nevel [21] studied the steady deflection and stress of a floating ice sheet due to a moving load uniformly distributed over a circular area. The ice plate was assumed to be supported by water of uniform depth. The ice deflection under the center of the load was obtained by superimposing the deflections due to concentrated loads. Nevel [21] noted that there does exist a critical speed and that Kheisin ([12], [13]) had incorrectly concluded that for a point load the ice deflection is bounded at all load speeds.

In trying to understand the origin of critical speeds, Kheisin [14] made another major contribution to the field by analysing the associated initial value problem. In that paper, Kheisin analyzed the time-dependence of the ice deflection for a line load and determined that there is a speed at which the ice deflection grows with time. Kerr [11] identified this critical speed as the minimum phase speed\(^2\) of the system of waves. Furthermore, Kerr [11] found that this critical speed varies with the magnitude of in-plane forces which, for example, could be due to thermal strains in the ice. The paper showed that a force field in compression reduces the critical speed, while a force field under tension increases the critical speed. However, for a given set of ice parameters, these force fields must be large in order to have any significant effect (Schulkes et al. [24]).

---

of a wave means that the wave speed depends on the wavelength, and possibly also on the direction of propagation (see Lighthill [16], Kundu [15], also section 2.2 of this Dissertation).

\(^2\)The phase speed of a wave is the rate at which the phase of the wave propagates (see Kundu [15]). The minimum phase speed is calculated from the dispersion relation.
Davys et al. [5] investigated the propagation of flexural-gravity waves in a floating ice sheet over water of finite depth. The Fourier transform was used to examine the hybrid wave patterns generated by a steadily moving point source, where the waves are predominantly flexural ahead and predominantly gravitational behind the source. It was noted that at the critical speed no steady wave pattern can exist, and that a 'shadow zone' appears behind the source when its speed exceeds the shallow water wave speed. Schulkes et al. [24], extending the work of Davys et al. [5], confirmed Kerr's [11] result that a compressive stress in the plane of the ice plate causes a slight decrease in the phase speed. In addition, Schulkes et al. showed that the existence of a uniform flow in the underlying fluid changes the orientation of the wave pattern, and that internal waves are generated if it is stratified.

In further work, Schulkes and Sneyd [25] analyzed the time-dependent response of a floating ice sheet over water of finite depth to a moving line load, as previously done by Kheisin [14]. The Fourier transform and the method of steepest descent were used to obtain the asymptotic expansion of the ice displacement for large time. It was shown that the ice deflection grows with time at the load speed $C_{\text{min}}$, the minimum phase speed of the waves in the system; and also at the load speed $C_{\text{gH}}$, the speed of gravity waves on shallow water. The load speed at which the deflection of a floating ice sheet becomes infinite is referred to in the literature as the critical speed or critical velocity. This critical speed coincides with $C_{\text{min}}$ in the deep water limit provided that the ice sheet acceleration can legitimately be omitted (see Squire et al. [26]). Recently, Milinazzo et al. [19] considered the steady response of an ice sheet of infinite extent to a moving rectangular load. Milinazzo et al. assumed that a steady solution exists and obtained both numerical and asymptotic estimates of the solution. In particular, the paper showed that the deflection of the ice sheet is infinite at the load speed $C_{\text{min}}$, but
finite at the load speed \( C_g H \).

Several experimental studies of waves on ice have also been conducted. A large number of these experimental studies are discussed in detail by Squire \textit{et al.} [26]. It is interesting to note that the theoretical predictions are in remarkably good agreement with experimental results. In Wilson [36], the coupling between moving loads and flexural waves in floating ice sheets was studied experimentally and the results were compared with the theoretical predictions obtained using the analysis of Greenhill [7]. The experiments were carried out for both one and two vehicles as moving loads. It was noted for a single vehicle that the maximum deflection occurs at a critical load speed. For two vehicles, depending on the distance between the vehicles, the maximum deflection differs from that of a single vehicle and the wave pattern appears as the superposition of two separate sets of wave patterns caused by two single vehicles. Beltaos [2] also observed the existence of a critical speed and a dependence of the wave amplitude on the speed of the load.

Takizawa [30, 31] carried out a number of remarkable experiments on ice-covered Lake Saroma on the Japanese island of Hokkaido using a snowmobile. The articles noted the existence of a critical speed below which the response is quasi-static, and above which two ice waves trains appear - one of shorter wavelength ahead of the moving snowmobile, and the other of larger wavelength trailing behind. With increased snowmobile speed the leading wave shortens, while the trailing wave lengthens and eventually vanishes. In subsequent work, Takizawa [32] further analyzed the experimental results and confirmed the earlier findings.

Further experiments were carried out by Squire \textit{et al.} [27] on both lake and sea ice. Again, it was observed that two different waves appear when the load speed exceeds the critical speed. However, for the same load type and the same load speed, the wave amplitude for fresh water lake ice was larger than for sea ice.
Squire et al. [27] pointed out that this is most likely due to the higher damping of sea ice compared to fresh water ice. They also ascertained experimentally the wave patterns theoretically predicted (cf. Davys et al. [5]).

Previous theoretical and experimental studies have concentrated not only on analyzing or measuring the ice deflection near the load region, in order to assess the potential to break ice, but also on analyzing or measuring the ice deflection in the far field - for example, to assess the theoretical and experimental agreement (Squire et al. [26]). Thus Kheisin [14], Nevel [21], and Takizawa [31] concentrated more on the ice deflection near the load region. Davys et al. [5] and Schulkes et al. [24] used asymptotic analysis to predict the wave patterns at some distance from the load, whereas Schulkes and Sneyd [25] considered both the immediate vicinity and the spatial aspects in the case of a line load. Milinazzo et al. [19] calculated the steady ice deflection in the entire flow field and obtained asymptotic estimates for the far field ice deflection.

Several papers, for example Kheisin [14] and Nevel [21], assumed that the supporting fluid is water of unlimited depth; whereas others, Kerr [11], Davys et al. [5], Schulkes et al. [24], and Milinazzo et al. [19] assumed that the fluid is of finite depth. Further analysis is carried out in this Dissertation to examine the ice deflection when the fluid base is assumed to be of infinite depth. The work of Milinazzo et al. [19] is re-visited, but for a fluid of unlimited depth with a view to simplifying the analysis. The results show that the characteristics of the ice deflection are very similar for the two cases (cf. Davys et al. [5], Takizawa [31], Milinazzo et al. [19]). However, it turns out that the analysis is essentially the same as that for the case where the fluid base is of finite depth. The infinite depth assumption does not seem to affect the characteristics of the waves, except that as expected the long waves disappear when the load speed is greater than $C_g H$ for the case where the fluid base is of finite depth. For the case where the fluid base is of infinite depth the waves appear behind the load at all speeds greater
than the load speed $C_{\text{min}}$. As discussed by Squire et al. [26], this distinction corresponds to the respective intersections of an ordinate representing the load speed with the two dispersion curves.

In deriving the mathematical model for the water-ice problem, it is usually assumed that the ice thickness is small so that the stress at the surface and at the bottom of the ice sheet are taken to be the same. Often, a further simplification is made by omitting the term that contains the ice thickness. This is equivalent to assuming that the wavelengths of the waves of interest are much larger than the ice thickness (see for example, Davys et al. [5] or Schulkes and Sneyd [24]). Strathdee et al. [28] analyzed the response of a floating ice sheet to a moving load, where the ice is assumed to be an isotropic viscoelastic plate with finite thickness. They argued that the distribution of stress and strain around a concentrated load receive significant contribution from waves with wavelength comparable to plate thickness and hence the exact description of the thickness effects need to be included in the mathematical model. However, further analysis by Squire et al. [26] where they compared the solution from Strathdee et al. [28] for a stationary load with the solution obtained from the thin plate theory due to Wyman [37] (see Squire et al. [26] section 5.8 for details) showed that the results are in fact identical. In this Dissertation, a mathematical model that takes into account the ice thickness is derived where the ice thickness is no longer considered small. The result shows that the effect of the ice thickness is negligible, but there are some similarities between this model and the model analyzed by Kerr [11] where in-plane forces are taken into account.

As mentioned above, the previous theoretical studies have either assumed a steady state exists or considered the time-dependent response to an impulsively started one-dimensional line load. For the two-dimensional case however, to date no attempt has been made to determine the time-dependent behaviour of the ice deflection as a function of load speed (cf. Squire et al. [26]).
The most important aspect of this Dissertation is the investigation of the time-dependent behaviour of the ice deflection due to an impulsively-started steadily moving point load on an ice sheet. The load is assumed to move over a homogeneous ice sheet of infinite horizontal extent and the fluid base is assumed to be water of finite depth. The Fourier transform is used to solve the equations of motion, and the singularities of the corresponding integrand are analyzed as a function of the load speed. The results show that the time-dependent part of the ice deflection grows with time when the load speed is $C_{\text{min}}$. However, at the load speed $C_{gH}$ the transient part of the ice deflection decays with time. Assuming that a steady solution is possible, for a rectangular load, Milinazzo et al. [19] found that a steady solution exists at the load speed $C_{gH}$, but that paper did not address the attainability of the solution. The findings in this Dissertation suggest that at the load speed $C_{gH}$ the steady state solutions computed in Milinazzo et al. [19] are large time solutions of the corresponding initial value problem and are consequently physically attainable.

The following is the organization of this Dissertation. Chapter 2 analyzes the steady state waves caused by a moving rectangular load on an ice sheet supported by water of infinite depth. Chapter 3 deals with the mathematical modeling of the ice-water system when the thickness of the ice plate is taken into account. Chapter 4 describes the mathematical model of the ice-water system and analyzes in detail the time-dependent solution of the ice deflection due to a point load. Finally, in Chapter 5, a summary is presented.
Chapter 2

Steady Solution due to the Motion of a Rectangular Load on an Ice Sheet over Water of Infinite Depth

2.1 Introduction

In this chapter, the steady state ice deflection caused by a rectangular load moving on an ice sheet supported by a fluid base of infinite depth is considered. The mathematical model is the same as that analyzed by Milinazzo et al. [19], except that in that paper the depth of the fluid base is assumed to be finite. They found that their steady solution is finite at all load speeds except $C_{\text{min}}$. In their analysis, the solution is expressed as a two-dimensional Fourier integral and the poles of the integrand are used to obtain both a numerical description and asymptotic estimates of the ice deflection. The motivation in making the infinite depth assumption in this chapter is to see if the analysis of the ice deflection becomes simpler than when the fluid base is of finite depth. The analysis is also carried out to confirm that the deflection is finite for all load speeds other than the load speed $C_{\text{min}}$, and to compare the results with those obtained for the case where the fluid base is of finite depth. In what follows, the term *the case of finite depth* is used to refer to the problem where the supporting water is of finite depth, and the term *the case of infinite depth* is used to refer to the problem where the supporting water is of infinite depth.

The mathematical model that describes the ice displacement is as follows. The $x,z$ plane is taken to coincide with the bottom of the ice sheet. The elastic, homogeneous, thin ice sheet of infinite extent in the $x,z$ plane is considered to
have constant thickness $h$ and constant density $\rho_{\text{ice}}$. The supporting water is assumed to have constant density $\rho$, the undisturbed water surface is taken to be at $y = 0$, and the bottom is taken to be at $y = -\infty$. The theory of the bending of a thin plate (Timoshenko and Woinowsky-Krieger [35], Timoshenko and Gere [34]) is used to model the floating ice plate. If $\eta(x, z, t)$ represents the ice deflection and $f(x, z)$ represents the downward applied stress due to the uniformly distributed rectangular load on the ice sheet, then the equation of the ice deflection (Szilard [29] section 4.2) in a frame of reference moving with the load at speed $V$ in the $x$ direction is given by

$$
D \left( \frac{\partial^2 \eta}{\partial X^2} + \frac{\partial^2 \eta}{\partial z^2} \right) - \rho h \left( \frac{\partial \eta}{\partial t} + V \frac{\partial \eta}{\partial X} \right) = -f(X, z)
$$

for $-\infty < X, z < \infty$ (2.1)

where $X = x - Vt$. $\eta$ and $f$ have been redefined as functions of $X$ and $z$, and $f(X, z)$ is given by

$$
f(X, z) = \begin{cases} 
P_0 & -a \leq X \leq a, -b \leq z \leq b \\
0 & \text{otherwise.}
\end{cases}$$

(2.2)

Note that in eq. (2.2), $P_0$ is a constant. The constant $D > 0$ is given in terms of Young's modulus $E$ and Poisson's ratio for ice $\nu$ ($0 < \nu < 1$), by the relation

$$
D = \frac{E h^3}{12(1-\nu^2)}.
$$

(2.3)

The upward water pressure $p$ is determined, by assuming that the water base is incompressible and its flow is irrotational, when the flow can be described by the velocity potential $\phi(t, X, y, z)$ satisfying

$$
\frac{\partial^2 \phi}{\partial X^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad -\infty < y < 0, -\infty < X, z < \infty.
$$

(2.4)

The condition of no normal flow at the bottom is enforced by setting the vertical velocity $\frac{\partial \phi}{\partial y}$ to zero at $y = -\infty$. The non-cavitation kinematic condition at the ice-water boundary, which ensures that the ice and water interfaces always coincide.
is enforced by setting the vertical velocity of the water to be equal to that of the
ice-water interface, i.e. \( \frac{\partial \eta}{\partial t} = (\frac{\partial}{\partial t} + \hat{\nu} \frac{\partial}{\partial X}) \eta \). Since the analysis is limited to waves
of small amplitude, the Bernoulli equation at \( y = 0 \) is given by

\[
\rho \eta + \frac{1}{\rho} \frac{\partial \eta}{\partial t} + \left( \hat{\nu} \frac{\partial}{\partial X} \right) \phi(t, X, 0, z) = 0. \tag{2.5}
\]

The gravitational acceleration \( g \) in eq. (2.5) is taken to be in the negative \( y \)
direction.

In order to determine the steady solution of eqs. (2.1)-(2.5) using the Fourier
transform, it is necessary to ensure that the solution satisfies the correct radiation
condition at infinity. This is achieved by using a technique introduced by
Lighthill [16] where the pressure \( f \) is replaced by \( f^\delta = f e^{\delta t}, \delta > 0 \). Thus a pressure
that was zero in the distant past has grown to \( f \) at \( t = 0 \), corresponding to a
radiation condition that eliminates incoming waves. The steady solution is then
obtained by letting \( \delta \to 0^+ \).

The time dependence of the dependent variables can be taken to be \( e^{\delta t} \), con­sequently

\[
\eta(t, X, z) = e^{\delta t} \eta^\delta(X, z). \tag{2.6}
\]

\[
\phi(t, X, y, z) = e^{\delta t} \phi^\delta(X, y, z). \tag{2.7}
\]

\[
p(t, X, z) = e^{\delta t} p^\delta(X, z). \tag{2.8}
\]

Substituting eqs. (2.6)-(2.8) into eqs. (2.1)-(2.5) and transforming the resulting
equations, including the boundary conditions, using Fourier transforms in the \( X \)
and \( z \) directions yield (cf. Milinazzo et al. [19]):

\[
(D \kappa^4 + \rho \omega^2 h(\dot{\delta} - i \nu \kappa_1^2) ) \mathcal{H}^\delta - \mathcal{P}^\delta = -\Pi. \tag{2.9}
\]

\[
\Phi^\delta_{yy} - \kappa^2 \Phi^\delta = 0 \quad \text{for} \quad -\infty < y < 0. \tag{2.10}
\]

\[
\Phi^\delta_y(\kappa_1, \kappa_2, -\infty) = 0, \quad \Phi^\delta_y(\kappa_1, \kappa_2, 0) = (\dot{\delta} - i \nu \kappa_1) \mathcal{H}^\delta. \tag{2.11}
\]
\[ g \mathcal{H}^\delta + \frac{1}{\rho} \mathcal{P}^\delta + (\dot{\delta} - iV\kappa_1)\Phi^\delta(\kappa_1, \kappa_2, 0) = 0. \tag{2.12} \]

where \( \kappa \) is given by \( \kappa^2 = \kappa_1^2 + \kappa_2^2 \) and \( \mathcal{H}^\delta(\kappa_1, \kappa_2), \Phi^\delta(\kappa_1, \kappa_2, y), \mathcal{P}^\delta(\kappa_1, \kappa_2) \), and \( \Pi(\kappa_1, \kappa_2) \) have been introduced to denote the Fourier transforms of \( \eta^\delta, \dot{\delta}^\delta, \rho^\delta \) and \( f \), respectively. The Fourier transform of \( f(X, z) \) in eq. (2.2) is given by

\[
\Pi = \frac{1}{2\pi} \int_{-a}^{a} \int_{-b}^{b} P_0 e^{i(k_1 x + k_2 y)} dz \, dX = \frac{2}{\pi} \frac{P_0 \sin(a\kappa_1)}{\kappa_1} \frac{\sin(b\kappa_2)}{\kappa_2}. \tag{2.13} \]

The solution of eqs. (2.10)-(2.11) is

\[
\Phi^\delta = \frac{(\dot{\delta} - iV\kappa_1)}{|\kappa|} e^{\kappa y} \mathcal{H}^\delta. \tag{2.14} \]

Substituting eq. (2.14) into eq. (2.12) gives

\[
\mathcal{P}^\delta = -\rho \left[ g + \frac{(\dot{\delta} - iV\kappa_1)^2}{|\kappa|} \right] \mathcal{H}^\delta. \tag{2.15} \]

which can be used together with eq. (2.9) to obtain

\[
\mathcal{H}^\delta = \frac{\Pi}{\rho g} \left( \frac{|k|}{F^2(k_1 + i\delta)^2(\beta|k| + 1) - |k|(1 + \alpha k^{-1})} \right), \tag{2.16} \]

where the following terms have been introduced to simplify the notation.

\[
\mu = \frac{\rho_{cv}}{\rho}, \quad \beta = \frac{h}{\alpha}, \quad \delta = \frac{\dot{\delta} a}{V}, \quad \delta = \frac{V^2}{g\alpha}, \quad \alpha = \frac{D}{\rho g a^2}.
\]

\[
k_1 = a\kappa_1, \quad k_2 = a\kappa_2, \quad k = a\kappa. \tag{2.17} \]

Using eq. (2.13) in eq. (2.16), the final form of the steady ice deflection can be seen to be

\[
\eta(X, z) = \frac{P_0}{\pi^2 \rho g} \lim_{\delta \to 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|k| e^{-i\delta (\frac{X}{a} - \frac{k_1}{a} - \frac{k_2}{k_2})}}{F^2(k_1 + i\dot{\delta})^2(\beta|k| + 1) - |k|(1 + \alpha k^{-1})} \sin(\frac{b k_2}{k_2}) \frac{dk_1 dk_2}{k_1}. \tag{2.18} \]
Note that the normalization has been done using $a$, the length of the load in the $X$ direction, as the length scale rather than the water depth as is done for the case of finite depth (see Milinazzo et al. [19]).

From eq. (2.18), the poles of the integrand are given by the zeroes of the denominator $F^2(k_1 + i\delta)^2(\delta |k| + 1) - |k|(1 + \alpha k^4) = 0$, which is a polynomial in $k$. This is in contrast to the case of finite depth (cf. Milinazzo et al. [19]), where the denominator is an equation involving transcendental functions.

### 2.2 The Dispersion Relation

The difference between the case of finite depth and the case of infinite depth can be understood by comparing the dispersion relation for the two problems. A dispersion relation is the relationship between the frequency of a plane wave and its wavelength and is derived by looking for travelling wave solutions of eqs. (2.1)-(2.5) of the form $e^{i(\omega t - \kappa_1 X - \kappa_2 z)}$ with $f(X, z) = 0$. Such solutions exist provided the relation

$$\left(\omega - \kappa_1 V\right)^2 = \omega_n^2 = |\kappa| \frac{\left(\frac{D\kappa_1^4}{\rho} + g\right)}{(1 + \mu h |\kappa|)}$$

is satisfied. The corresponding relation between the phase speed $C$ of the free waves and the wavenumber $\kappa$ is given by

$$C^2 = \left(\frac{\omega_n}{\kappa}\right)^2 = \frac{1}{\kappa} \frac{\left(\frac{D\kappa_1^4}{\rho} + g\right)}{(1 + \mu h |\kappa|)}.$$  \hspace{1cm} (2.20)

The corresponding relationship for water of finite depth, $H$, is given in Chapter 4 (see eq. (4.10). In that equation, taking $H \rightarrow \infty$ gives $C^2 = (Dk^4/\rho + g)/k$, which is the same as eq. (2.20) when wavelengths of interest are much larger than the ice thickness. Davys et al. [5] note that for a steady wave pattern, the component of the load velocity normal to any wave crest must equal the crest phase speed, i.e.

$$C = V \cos \lambda$$  \hspace{1cm} (2.21)
where $\beta$ is the angle between the wavenumber vector and the direction of travel of the load, and $\cos \beta = \frac{\kappa}{k}$. Hence, only waves corresponding to the poles in the integrand of eq. (2.18) will appear in the steady pattern of the ice deflection.

The dispersion relation curve for eq. (2.20) is shown in Figure 2.1. It is plotted using some typical parameters given in Takizawa [31]. Note how the curve approaches infinity as the wavenumber approaches zero. By contrast, in the case of finite depth (see Figure 2.2) the corresponding dispersion curve intersects the vertical axis $k = 0$ at $C = \sqrt{gH}$. The group speed\(^1\), which is defined as $C_{gr} = \frac{d\omega}{dk}$, is also shown in these figures. From the dispersion curves, it is clear that the major difference is for long waves, i.e. waves with a small wavenumber. This difference has been noted previously in the fluid mechanics literature, as well as in present context (e.g. by Davys et al. [5]). In the next section, the solution for the case of infinite depth is analyzed.

\(^1\)A group speed is the speed at which the envelope of a wave group travels. The wave components of the wave group propagate with the speed $C$ (Kundu [15]).
Figure 2.1: The dispersion curve for the case of infinite depth. The plot labeled $C$ denotes the phase speed (eq. (2.20)) and the plot labeled $C_{gr}$ denotes the group speed. The wavenumber is represented by the horizontal axis and the speed is represented by the vertical axis. The parameters are $E = 5 \times 10^8$, $\nu = \frac{1}{3}$, $h = 0.175$ m, $g = 9.8$ m/sec$^2$, $\rho = 1026$ m/kg$^3$, and $\rho_{ice} = 850$ m/kg$^3$. 
2.2 The Dispersion Relation

Dispersion relation for the case of finite depth

\[ \text{Speed} \]

\[ \sqrt{gH} \]

\[ C_{gr} \]

\[ C_{\text{min}} \]

\[ C \]

\text{Wavenumber}

Figure 2.2: The dispersion curve for the case of finite depth. The plot of the dispersion relation for the case of finite depth (eq. (4.10) is shown as the plot of the phase speed \( C \). The plot of the group speed is shown as the plot labeled \( C_{gr} \). The parameters are \( E = 5 \times 10^8 \), \( \nu = \frac{1}{3} \), \( h = 0.175 \, \text{m} \), \( H = 6.8 \, \text{m} \), \( g = 9.8 \, \text{m/sec}^2 \), \( \rho = 1026 \, \text{m/kg}^3 \), and \( \rho_{\text{ice}} = 850 \, \text{m/kg}^3 \).
2.3 The Solution Method

To calculate \( \eta \), the double integral given in eq. (2.18) is written as an iterated integral in \( k_1 \) and \( k_2 \). The inner integral in \( k_1 \) is evaluated for a fixed \( k_2 \) using contour integration. The outer integral in \( k_2 \) is then estimated using the method of stationary phase (Nayfeh [20]). This is in contrast with Milinazzo et al. [19] where the \( k_2 \) integral is also evaluated numerically using an adaptive Gaussian quadrature scheme (see Milinazzo et al. [19] section 4).

The \( k_1 \) integral of eq. (2.18) can be written as a function of \( k_2 \) in the form

\[
\lim_{\delta \to 0} \frac{1}{2i} \int_{-\infty}^{\infty} \frac{e^{-i(\frac{\delta}{\alpha} - 1)k_1} - e^{-i(\frac{\delta}{\alpha} - 1)k_1}}{\mathcal{F}^\delta(k_1, k_2)} dk_1
\]

(2.22)

where \( \mathcal{F}^\delta \) is

\[
\mathcal{F}^\delta(k_1, k_2) = [-F^2(k_1 + i\delta)^2(1 + \beta|k|) - k(1 + \alpha k^2)|k_1|/|k|]
\]

(2.23)

Recall that \( \delta \) in \( \mathcal{F}^\delta \) is used to determine the solution that satisfies the radiation condition at infinity. The poles of eq. (2.22) are the zeroes of the equation \( \mathcal{F}^\delta(k_1, k_2) = 0 \). The zeroes of eq. (2.23) correspond to the poles of the integrand of eq. (2.22) in the \( k_1 \)-plane. For \( \delta = 0 \), the real poles are identified with the waves generated in the system. When these poles are on the real axis, deforming the contour above or below them determines whether the corresponding waves are outgoing waves or incoming waves. By introducing \( \delta > 0 \), these poles move off the real axis, thereby making it possible to identify which poles are to be included when evaluating the integral using contour integration. In what follows, \( \mathcal{F} \) will be used to denote \( \mathcal{F}^\delta \) when \( \delta = 0 \).

From eq. (2.18) it can be seen that it is necessary that \( k > 0 \) for real \( k_1, k_2 \). An appropriate choice of branch cut for the function \( k = \sqrt{k_1^2 + k_2^2} \) is given in the next section.
2.3.1 The Definition of \( k \).

The function \( k = \sqrt{k_1^2 + k_2^2} \) has branch points at \( k_1 = \pm ik_2 \). \( k_2 \) is assumed to be real and \( k_2 > 0 \). Figure 2.3 illustrates the branch cut, denoted by dashed lines along the vertical axis, and the two branch points.

\[
\begin{align*}
\text{\( k_1 \)-plane} \\
\quad \theta_1 \\
\quad \theta_2 \\
\quad i k_2 \\
\quad -i k_2
\end{align*}
\]

Figure 2.3: The branch cut in the \( k_1 \)-plane. The branch cut is shown as dashed lines on the vertical axis starting from the branch points \( \pm ik_2 \).

The angles \( \theta_1 \) and \( \theta_2 \), measured in the counter clockwise direction and shown in Figure 2.3, satisfy

\[
-\frac{3\pi}{2} \leq \theta_1 < \frac{\pi}{2}, \quad -\frac{\pi}{2} \leq \theta_2 < \frac{3\pi}{2}
\]

and the function \( k \) is defined by the expression

\[
k = |k_1 - i k_2|^{1/2} |k_1 + i k_2|^{1/2} \ e^{\frac{i}{2} (\theta_1 - \theta_2)}. \tag{2.25}
\]

This gives a definition of \( k \) in the complex plane for which \( k > 0 \) for all real \( k_1 \) and \( k_2 \). Note that on the horizontal axis in the \( k_1 \)-plane, \( \theta_1 = -\theta_2 \), and hence

\[
k = |k_1 - i k_2|^{1/2} |k_1 + i k_2|^{1/2} \ e^{\frac{i}{2} (\theta_1 - \theta_1)}
\]

\[
= |k_1 - i k_2|^{1/2} |k_1 + i k_2|^{1/2} > 0 \quad \text{for } k_1 \text{ real.}
\]

Thus, from the definition given in eq. (2.25), \( k \) is always real and non-negative on the real \( k_1 \) axis.
2.3.2 Poles in the Complex Plane

As previously noted, to obtain the poles of the integrand of eq. (2.22), it is necessary to determine the zeroes of eq. (2.23). Since the integral given by eq. (2.22) is a function of \( k_2 \), the location of these zeroes, and hence the location of the poles, is also a function of \( k_2 \). To determine these poles, \( k_1 \) is rewritten in terms of \( k \) i.e. \( k_1 = \pm \sqrt{k^2 - k_2^2} \), and the equation,

\[
\mathcal{F} = F^2(k^2 - k_2^2)(1 + 3k) - k(1 + \alpha k^4) = 0
\]

is considered as a function of \( k \) with fixed \( F^2 \) and \( k_2^2 \). The dependence of the

![Figure 2.4: Zeroes of \( \mathcal{F} \). For a fixed \( F \) and \( k_2 > 0 \), the real zeroes are illustrated as the intersection of the two curves in the upper right half of the plane. \( F_m^2 \) indicates the minimum of the left hand side of eq. (2.27).](image)

zeroes in the \( k \)-plane on load speed (i.e. given by \( F^2 \)) and \( k_2^2 \) can be seen by rewriting eq. (2.26) in the form

\[
\frac{1}{k} \frac{(1 + \alpha k^4)}{k} = F^2(1 - \frac{k_2^2}{k^2})
\]

where \( k > k_2 > 0 \). Figure 2.4 illustrates the zeroes of eq. (2.26) for a fixed \( F \) and \( k_2 \). The poles of the integrand of eq. (2.22) correspond only to the zeroes of
eq. (2.26) that occur in the first or the fourth quadrant of the complex k-plane. This assertion can be demonstrated by considering the following four different cases.

Figure 2.3: The relation between the zeroes in the k-plane and the poles in the k_1-plane. The illustration shows the zeroes of eq. (2.23) for \( k_1 = \pm \sqrt{k^2 - k_2^2} \). \( \delta \neq 0 \), \( k_2 \neq 0 \), and the corresponding poles of the integrand of eq. (2.22).

- Assume the zero \( k_1 \) is in the first quadrant of the \( k_1 \)-plane: In this quadrant \(-\pi/2 \leq \theta_1 \leq \pi/2\), \( 0 \leq \theta_2 \leq \pi/2 \), and \( \theta_2 \geq -\theta_1 \). Consequently, \( 0 \leq (\theta_1 + \theta_2) \leq \pi \) and so \( 0 \leq \arg(k) \leq \pi/2 \). \( k \) is in the first quadrant.

- Assume the zero \( k_1 \) is in the second quadrant of the \( k_1 \)-plane: In this quadrant \(-3\pi/2 \leq \theta_1 \leq -\pi/2\), \( \pi/2 \leq \theta_2 \leq \pi \), and \( \theta_2 \leq -\theta_1 \). In this case, \(-\pi \leq (\theta_1 + \theta_2) \leq 0 \) and \(-\pi/2 \leq \arg(k) \leq 0 \). Hence, \( k \) is in the fourth quadrant.

- Assume the zero \( k_1 \) is in the third quadrant of the \( k_1 \)-plane: In this quadrant \( \pi/2 \leq \theta_2 \leq 3\pi/2 \), \(-\pi < \theta_1 \leq -\pi/2 \), and \( \theta_2 \geq -\theta_1 \). Therefore, \( 0 \leq (\theta_1 + \theta_2) \leq \pi \) and \( 0 \leq \arg(k) \leq \pi/2 \). \( k \) is in the first quadrant.

- Assume the zero \( k_1 \) is in the fourth quadrant of the \( k_1 \)-plane: In this quadrant \(-\pi/2 \leq \theta_2 < \pi/2\), \(-\pi/2 \leq \theta_1 \leq 0 \) and \( \theta_2 \leq -\theta_1 \). Consequently,
\[ -\pi < (\theta + \theta_2) \leq 0 \quad \text{and} \quad -\frac{\pi}{2} \leq \arg(k) \leq 0. \] 

\[ k \text{ is in the fourth quadrant.} \]

Since only the zeroes in the first and the fourth quadrant of the \(k\)-plane are important, other zeroes in the left half of the \(k\)-plane can be ignored. Figure 2.5 illustrates the zeroes of eq. (2.23) found in the right half of the \(k\)-plane and the corresponding poles in the \(k_1\)-plane. These zeroes in the \(k\)-plane are computed for a fixed \(F\), \(k_2 \neq 0\), and \(\delta \neq 0\) from eq. (2.23) by substituting \(k_1 = \pm \sqrt{k^2 - k_2^2}\). The zeroes in the \(k\)-plane denoted by \(r_2\) and \(r_3\) correspond to the zeroes of eq. (2.23) when \(k_1 = + \sqrt{k^2 - k_2^2}\); whereas the zeroes denoted by \(r_1\) and \(r_4\) correspond to the zeroes of eq. (2.23) when \(k_1 = - \sqrt{k^2 - k_2^2}\). When \(k_2 = 0\), there is also a root at the origin of the \(k\)-plane. This root corresponds to \(k_1 = 0\) in the \(k_1\)-plane.

In summary, it has been shown that the poles of the integrand of eq. (2.22) can only be the result of zeroes of eq. (2.26) that are in the right half of the \(k\)-plane.

Let the minimum of the left-hand side of (2.27) be defined as \(F_2^m\). If \(F^2 < F_2^m\), the zeroes in the \(k\)-plane are complex for all values of \(k_2\). Using MAPLE, these zeroes can be found from eq. (2.26) for a fixed \(F\) and a given range of \(k_2\). The result in Figure 2.6 shows that the zeroes appear near the origin when \(k_2\) is small and move away from the origin as \(k_2 \to \infty\). Since the zeroes in the \(k\)-plane are complex, the poles are also complex. The behaviour of these poles as a function of \(k_2\) is illustrated in Figure 2.7.

When \(F^2 = F_2^m\), from Figure 2.4 it can be seen that eq. (2.27) has a positive real root when \(k_2 = 0\). In this case, the root is complex when \(k_2 > 0\). The behaviour of the zeroes and the poles as a function of \(k_2\) can be seen in Figures 2.6 and 2.8.

However, when \(F^2 > F_2^m\), eq. (2.27) possesses two real, positive zeroes for a range of \(k_2\) (see Figure 2.9). As a result, the corresponding poles of the integrand of eq. (2.22) are also real for that range of \(k_2\). Figure 2.4 shows the two real zeroes in the positive half of \(k\)-plane for a fixed \(k_2\). Figure 2.6 shows the location of the
zeroes as a function of $k_2$. The zeroes start on the real axis when $k_2$ is small, one near the origin and the other away from the origin. As $k_2$ increases, these zeroes approach each other along the real axis, coalesce and become complex. The behaviour of the corresponding poles in the $k_1$-plane as a function of $k_2$ is illustrated in Figure 2.9.
Figure 2.6: Position of the zeroes in the first and the fourth quadrant of the $k$-plane. The plot shows the location of the zeroes as a function of $k_2$. The corresponding load speed is also shown on the righthand side of the plot.
Figure 2.7: Position of zeroes in the $k$-plane and poles in the $k_1$-plane as a function of $k_2$. Plot (a) shows the zeroes (in the first and fourth quadrant of $k$-plane) of eq. (2.26) as a function of $k_2$. Plot (b) shows the corresponding poles as a function of $k_2$ in the $k_1$-plane.
Figure 2.8: **Position of zeroes in the $k$-plane and poles in the $k_1$-plane as a function of $k_2$.** Plot (a) shows the zeroes (in the first and the fourth quadrant of $k$-plane) of eq. (2.26) as a function of $k_2$. Plot (b) shows the corresponding poles as a function of $k_2$ in the $k_1$-plane.
Figure 2.9: **Position of zeroes in the $k$-plane and poles in the $k_1$-plane as a function of $k_2$.** Plot (a) shows the zeroes (in the first and fourth quadrant of $k$-plane) of eq. (2.26) as a function of $k_2$. Plot (b) shows the corresponding poles as a function of $k_2$ in the $k_1$-plane.
2.3 The Solution Method

2.3.3 The Evaluation of the Integral

In section 2.3.2, the poles of the integrand of eq. (2.22) were analyzed. By analyzing $\mathcal{F}^\delta = 0$ for small $\delta > 0$, it is easy to see that the real poles closest to the origin move into the lower half plane ($k_1^L$ and $-\bar{k}_1^L$) and those farthest from the origin move into the upper half plane ($k_1^U$ and $-\bar{k}_1^U$). For all $k_2$ contour integration can be used to evaluate the integral in eq. (2.22) by closing the path of integration either in the lower or upper half plane.

Since it is necessary that the integrand goes to zero at infinity, whether the integration path is closed in the upper or lower half plane depends on the sign of the exponents in the exponential terms of eq. (2.22). In front of the load, $(\frac{\xi}{a} \pm 1) < 0$, so the path of integration in eq. (2.22) must be taken in the upper half plane. The contour is taken in such a way that all the poles in the upper half plane are included. Using the closed curve $\mathcal{D}$, which consists of the paths $\gamma_0$, $\gamma_1$, $\gamma_2$, $\gamma_3$, $\gamma_4$, and $\gamma$, (see Figure 2.11), the integral in eq. (2.22) can be written by using the residue theorem as follows:

![Figure 2.10: Poles in the $k_1$-plane. The plot shows the position of the poles in the $k_1$-plane. These are the poles of the integrand of the integral given in eq. (2.22).](image)
2.3 The Solution Method

\[
\frac{1}{2i} \int_D \frac{e^{-i \frac{X}{4} - 1}k_1 - e^{-i(\frac{X}{4} - 1)k_1}}{\mathcal{F}(k_1,k_2)} \, dk_1
\]

\[
= \frac{1}{2i} (I_0 + I_1 + I_2 + I_3 + I_4 + I_e)
\]

\[
= \pi \left( \sum \text{residues of } \frac{e^{-i \frac{X}{4} - 1}k_1 - e^{-i(\frac{X}{4} - 1)k_1}}{\mathcal{F}(k_1,k_2)} \text{ in } \mathcal{D} \right). \quad (2.28)
\]

Here, the integrals denoted by \( I_0, I_1, I_2, I_3, I_4, \) and \( I_e \) are correspondingly taken along the paths \( \gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \) and \( \gamma_e \), respectively.

A further evaluation shows that the integrals \( I_1 \) and \( I_2 \) tend to zero as \( R \to \infty \), and \( I_e \) also tend to zero as \( \epsilon \to 0^+ \). Non-zero contributions therefore only come from the integration along the branch cut and from the poles. These poles are simple poles since \( \mathcal{F}_{k_1} \neq 0 \) (subscript denotes a partial derivative) at these poles. Consequently, the integral in eq. (2.28) can be written as

![Integration path in the upper half plane. The direction of the integration is shown by the arrows along the closed curve \( \mathcal{D} \).](image)
2.3 The Solution Method

\[ \frac{1}{2i} \int_{-\infty}^{\infty} \frac{e^{-it(\frac{X}{a} - 1)k_1} - e^{-it(\frac{X}{a} - 1)k_1}}{F^\delta(k_1, k_2)} \, dk_1 = \]

\[ = -\pi \left( \frac{e^{-it(\frac{X}{a} - 1)k_1^\leftrightarrow} - e^{-it(\frac{X}{a} - 1)k_1^\leftrightarrow}}{F^\delta(k_1^\leftrightarrow, k_2)} + \frac{e^{it(\frac{X}{a} - 1)k_1^\leftrightarrow} - e^{it(\frac{X}{a} - 1)k_1^\leftrightarrow}}{F^\delta(-k_1^\leftrightarrow, k_2)} \right) \]

(2.29)

where \( k_1^\leftrightarrow \) denotes the pole in the upper right plane and

\[ B(\lambda, k_2) = F^2\lambda^2 \beta \left| k_2^2 - \lambda^2 \right|^{\frac{1}{2}} + \left| k_2^2 - \lambda^2 \right|^{\frac{1}{2}} \left( 1 + \alpha \right) \left| k_2^2 - \lambda^2 \right| \]

(2.30)

Behind the load, \( (\frac{X}{a} \pm 1) > 0 \), and hence, the path of integration of eq. (2.22) is taken in the lower half plane along the paths \( \gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \) and \( \gamma_s \) (see Figure 2.12). Proceeding as above, it can be shown that

Figure 2.12: Integration path in the lower half plane. The direction of the integration is shown by the arrows along the closed curve \( D \).

\[ \frac{1}{2i} \int_{-\infty}^{\infty} \frac{e^{-it(\frac{X}{a} - 1)k_1} - e^{-it(\frac{X}{a} - 1)k_1}}{F^\delta(k_1, k_2)} \, dk_1 = \]

\[ = -\pi \left( \frac{e^{-it(\frac{X}{a} - 1)k_1^\leftrightarrow} - e^{-it(\frac{X}{a} - 1)k_1^\leftrightarrow}}{F^\delta(k_1^\leftrightarrow, k_2)} + \frac{e^{it(\frac{X}{a} - 1)k_1^\leftrightarrow} - e^{it(\frac{X}{a} - 1)k_1^\leftrightarrow}}{F^\delta(-k_1^\leftrightarrow, k_2)} \right) \]
2.3 The Solution Method

\[ + \int_{k_1}^\infty \frac{F^2 \lambda^2 |k_2^2 - \lambda^2|^\frac{1}{2}}{(F^2 \lambda^2)^2 + B^2(\lambda, k_2)} \left( e^{i(\frac{\lambda}{2} - 1)\lambda} - e^{i(\frac{\lambda}{2} - 1)\lambda} \right) d\lambda \]

(2.31)

where \( k_1^2 \) denotes the pole in the lower right plane.

It is clear that the \( k_1 \)-integrals given by eqs. (2.29) and (2.31) are functions of \( k_2 \). To estimate the \( k_2 \)-integrals, it is necessary to analyze eq. (2.22) as a function of \( k_2 \). This is done in the next section.

2.3.4 The Steady State Solution

The purpose of this section is to analyze the integrand of the \( k_2 \)-integral, i.e. the outer integral of eq. (2.18). In this section, the singularities that occur in the integrand of eq. (2.22) are analyzed further by making use of the properties of the poles discussed in section 2.3.2. This is done to show that the steady ice deflection is bounded at all load speeds not equal to \( C_{\min} \).

In the case \( F^2 < F_m^2 \), the roots of eq. (2.26) are complex and consequently, the corresponding poles in the \( k_1 \)-plane are also complex for all values of \( k_2 \) (see Figure 2.7). The residue at each pole is finite and it is easy to see that the \( k_2 \)-integral and consequently the ice deflection is finite. As \( F^2 \) approaches \( F_m^2 \), the complex pole in the lower half plane and the complex pole in the upper half plane coalesce and form a double real pole (see Figure 2.8). This double pole results in a singularity in the \( k_2 \)-integral of eq. (2.18) at \( k_2 = 0 \). For \( F^2 > F_m^2 \), there is also a singularity in the \( k_2 \)-integral but for a value of \( k_2 > 0 \) (see Figure 2.9). Below, these singularities are analyzed.

For \( F^2 \geq F_m^2 \), let \( k_{2r} \) be the value of \( k_2 \) where the real poles coalesce and become a complex conjugate pair. As a function of \( k_2 \), the poles \( k_1 \) of the integrand of eq. (2.22) are given implicitly by

\[ C(k_1, k_2) \equiv F^2k_1^2 \left( 1 + 3 \sqrt{k_1^2 + k_2^2} \right) - \sqrt{k_1^2 + k_2^2} \left( 1 + \alpha(k_1^2 + k_2^2) \right) = 0. \]

(2.32)

Let \( k_{1c} \) denote the value of \( k_1 \) at \( k_2 = k_{2c} \) and expand eq. (2.32) about \((k_{1c}, k_{2c})\)
to second order to obtain

\[ 0 = \mathcal{C}(k_1, k_2) = \]

\[ \mathcal{C}(k_{1c}, k_{2c}) + \mathcal{C}_{k_1}(k_1 - k_{1c}) + \]

\[ \frac{1}{2} \mathcal{C}_{k_1 k_1}(k_1 - k_{1c})^2 + \mathcal{C}_{k_2}(k_2 - k_{2c}) + \]

\[ + \mathcal{C}_{k_1 k_2}(k_1 - k_{1c})(k_2 - k_{2c}) + \frac{1}{2} \mathcal{C}_{k_2 k_2}(k_2 - k_{2c})^2 + \cdots \]  \hspace{1cm} (2.33)

where the subscripts of \( \mathcal{C} \) indicate partial derivatives. In the right hand side of eq. (2.33), \( \mathcal{C} \) and its derivatives are evaluated at the point \((k_{1c}, k_{2c})\). At the point where the poles coalesce, \( \mathcal{C}_{k_1} = 0 \). Hence, eq. (2.33) can be written up to second order terms as

\[ \frac{1}{2} \mathcal{C}_{k_1 k_1}(k_1 - k_{1c})^2 + \mathcal{C}_{k_2}(k_2 - k_{2c}) + \]

\[ \mathcal{C}_{k_1 k_2}(k_1 - k_{1c})(k_2 - k_{2c}) + \frac{1}{2} \mathcal{C}_{k_2 k_2}(k_2 - k_{2c})^2 = 0. \]  \hspace{1cm} (2.34)

Keeping only the lowest order term, \( k_1 \) can be written in terms of \( k_2 \) as

\[ k_1 = k_{1c} \pm \sqrt{\frac{2 \mathcal{C}_{k_2}}{\mathcal{C}_{k_1 k_1}} (k_{2c} - k_2)}. \]  \hspace{1cm} (2.35)

Eq. (2.35) is an approximation for the functional forms of the two branches of \( \mathcal{C}(k_1, k_2) \) near the point \((k_{1c}, k_{2c})\).

Let \( k_{1P}(k_2) \) and \( k_{1N}(k_2) \) indicate the two branches of eq. (2.35). Near the point \((k_{1c}, k_{2c})\), eq. (2.32) can be factored in terms of the \( k_{1P} \) and \( k_{1N} \) poles to obtain

\[ \mathcal{C}(k_1, k_2) = (k_1^2 - k_{1P}^2) (k_1^2 - k_{1N}^2) \hat{\mathcal{C}}(k_1, k_2) \]  \hspace{1cm} (2.36)

where \( \hat{\mathcal{C}} \) denote the remaining terms of \( \mathcal{C} \) after the factorization. Since the poles are simple, the residue of the integrand of eq. (2.18) can be evaluated at \( k_{1P} \) (similarly for \( k_{1N} \)) for values of \( k_2 \) close to \( k_{2c} \) i.e. the residue is given by

\[ \frac{\hat{\mathcal{C}}(k_{1c}, k_{2c})}{2 k_{1P} (k_{1P}^2 - k_{1N}^2)}. \]  \hspace{1cm} (2.37)
Here, $\tilde{C}$ is used to denote the part of the integrand of eq. (2.18) that is regular at the given poles $k_{1P}$ and $k_{1X}$. Keeping only the lowest order term, it can be seen using eq. (2.35) that at each of the poles $k_{1P}$ and $k_{1X}$, the residue given by eq. (2.37) is proportional to

$$\frac{\tilde{C}(k_{1c}, k_{2c})}{8k_{1c} \sqrt{\frac{\tilde{C}_{k_{1c} k_{2c}}}{2 \tilde{C}_{k_{1c} k_{2c} k_{1c} k_{2c}}} (k_{2c} - k_2)}}.$$  (2.38)

By examining $\tilde{C}_{k_{1c} k_{2c}}$ at the point $(k_{1c}, k_{2c})$, it is easy to verify that the argument of the square root of eq. (2.38) is positive for all values of $|k_2| < |k_{2c}|$. Consequently, for $k_{2c} \neq 0$ the singularity in the $k_2$-integral is integrable, and the ice deflection is finite. On the other hand, if $k_{2c} = 0$, which occurs at the critical speed $V = C_{\text{min}}$, then $C_{k_2} = 0$. Taking this into account in eq. (2.34), solving that equation for $k_1$, and keeping only the lowest order term, $k_1$ becomes

$$k_1 = k_{1c} \pm \sqrt{\frac{C_{k_2} k_{2c}}{C_{k_1 k_2}}}.$$  (2.39)

where $\frac{C_{k_2 k_{2c}}}{C_{k_1 k_2}} > 0$ at the point $(k_{1c}, k_{2c})$. Consequently, the residue is now proportional to

$$\frac{\tilde{C}(k_{1c}, k_{2c})}{8k_{1c} \sqrt{\frac{\tilde{C}_{k_{1c} k_{2c}}}{2 \tilde{C}_{k_{1c} k_{2c} k_{1c} k_{2c}}} k_{2c}^2}}.$$  (2.40)

Therefore, the singularity of the integrand in the $k_2$-integral (see eq. (2.18)) is of the form $\frac{1}{k_2}$ and the integral is not finite.

In summary, it is clear that as long as the load speed is not equal to $C_{\text{min}}$, the $k_2$-integral is finite. However, when the load speed is equal to $C_{\text{min}}$, the $k_2$-integral is not finite.

### 2.3.5 The Far-field Solution

In section 2.3.2, the behaviour of the poles of the integrand in eq. (2.18) was analyzed. In sections 2.3.3 and 2.3.4, the behaviour of the integral in eq. (2.22)
was discussed in terms of its dependence on the load speed. Having done that, it is now possible to evaluate the ice deflection given by the integral in eq. (2.18). The inner integral of eq. (2.18) is evaluated using the residue method as discussed in section 2.3.3, and then the outer integral of eq. (2.18) is estimated asymptotically in the limit of large $|X|$ using the method of stationary phase and only the case $z = 0$ is considered. The integral in eq. (2.18) is rewritten as

$$
\eta(X, z) = \frac{P_0}{\pi^2 \mu g} \lim_{\delta \to 0} \int_{-\infty}^{\infty} (T^\delta_-(k_2) - T^\delta_+(k_2)) e^{-\frac{i}{\delta}k_2 \sin\left(\frac{b}{a}k_2\right)} dk_2 \tag{2.41}
$$

where

$$T^\delta_-(k_2) = \frac{1}{2i} \int_{-\infty}^{\infty} e^{-i(k_2 - k_1)} d(k_1) \tag{2.42}$$
$$T^\delta_+(k_2) = \frac{1}{2i} \int_{-\infty}^{\infty} e^{-i(k_2 - k_1)} d(k_1) \tag{2.43}$$

and the poles of the integrands in eqs. (2.42) and (2.43) are given implicitly, as a function of $k_2$, by eq. (2.26). Taking into consideration the symmetry of the $k_2$ integration, eq. (2.41) can be rewritten as

$$
\eta(X, z) = \frac{2P_0}{\pi^2 \mu g} \lim_{\delta \to 0} \int_{0}^{\infty} (T^\delta_+(k_2) - T^\delta_-(k_2)) \cos\left(\frac{z}{k_2}\right) \frac{\sin\left(\frac{b}{a}k_2\right)}{k_2} dk_2 \tag{2.44}
$$

In the case $(\frac{X}{a} \pm 1) < 0$, i.e. in front of the load, the contour of integration is closed in the upper half plane so $T^\delta_+(k_2) - T^\delta_-(k_2)$ contributes as in eq. (2.29). Hence the first term of the asymptotic estimate of the ice deflection in front of the load is now given by

$$
\eta(X, 0) \approx \frac{P_0b\sqrt{2}}{\pi^2 \mu g} \mathcal{F}_{k_1}(k^U_1, 0) \left[ \frac{2 \text{Re}(e^{-i(\frac{X}{a} - 1)k^U_1 - \frac{\pi}{4})}}{\sqrt{(-\frac{X}{a} + 1) \mathcal{F}_{k_2k_2}(k^U_1, 0)}} - \frac{2 \text{Re}(e^{-i(\frac{X}{a} - 1)k^U_1 - \frac{\pi}{4})}}{\sqrt{(-\frac{X}{a} - 1) \mathcal{F}_{k_2k_2}(k^U_1, 0)}} \right] - \frac{2P_0b}{\pi^2 \mu g} \left[ \frac{F^2}{(\frac{X}{a} - 1)^2} - \frac{F^2}{(\frac{X}{a} + 1)^2} \right] \tag{2.45}
$$

where $\mathcal{F}_{k_2k_2}(k^U_1, 0)$ indicates the second partial derivative of $\mathcal{F}$ (see eq. (2.23)) with respect to $k_2$, evaluated at $k^U_1$. 
Behind the load, \((\frac{X}{a} \pm 1) > 0\), so the contour of the \(k_1\)-integral is closed in the lower half plane. In this case, \(T^s_1(k_2) - T^a_1(k_2)\) gives the integral contribution as in eq. (2.31). The leading term of the asymptotic estimate of the ice deflection behind the load is given by

\[
\eta(X, 0) \approx \frac{P_0 b \sqrt{2}}{\sqrt{\pi \rho g}} \mathcal{F}_{k_1}(k_1^2, 0) \left[ \frac{2 \text{Re}(e^{-it(\frac{X}{a} - 1)k_1^2 - \frac{X}{2})}}{\sqrt{(\frac{X}{a} - 1) \mathcal{F}_{k_2k_2}(k_1^2, 0)}} - \frac{2 \text{Re}(e^{-it(\frac{X}{a} - 1)k_1^2 + \frac{X}{2})}}{\sqrt{(\frac{X}{a} + 1) \mathcal{F}_{k_2k_2}(k_1^2, 0)}} \right] 
\]

\[
+ \frac{2 P_0 b}{\pi^2 \rho g} \left[ \frac{F^2}{(\frac{X}{a} - 1)^2} - \frac{F^2}{(\frac{X}{a} + 1)^2} \right]. \tag{2.46}
\]

It can be seen from the \(k_1\)-integral (see eqs. (2.29) and (2.31)) that the contribution comes from the poles and the integral along the branch cut. The first term approximations to the ice deflection given in eqs. (2.45) and (2.46) show that the poles give contributions that decay away from the load as \(1/\sqrt{X}\), and the integral along the branch cut gives contributions (for both ahead and behind the load) that are comparable to \(1/\sqrt{2}\). Consequently, for \(X\) sufficiently large, the contribution from the integral along the branch cut can be neglected.

As indicated in section 2.3.2, the behaviour of the poles depends on the load speed. In the case \(F^2 < F_m^2\), the poles are complex so eqs. (2.45) and (2.46) show that the ice deflection is static and is not wave like. However, in the case \(F^2 > F_m^2\), the poles are real and eqs. (2.45) and (2.46) show that the ice deflection forms waves ahead and behind the load.

The results obtained for the case of infinite depth for \(F^2 > F_m^2\) show that the leading term of the ice deflection decays as \(X^{-\frac{1}{2}}\), which is the same as in Milinazzo et al. [19]. However, in Milinazzo et al. [19] the behaviour of the ice deflection at \(F^2\) near 1, which corresponds to the load speed \(C_{gH}\), had to be analyzed separately. Milinazzo et al. [19] showed that for a rectangular load a steady deflection exists at this load speed. For the infinite depth case, the ice deflection due to a rectangular load behaves the same way for all load speeds.
$F^2 > F_m^2$. This can be seen from Figure 2.13 where in the case of infinite depth the waves behind the load still exist at all load speeds greater than the load speed $C_{\text{min}}$. In contrast, the results from the finite depth case show that a quiescent zone appears behind the load when the load speed exceeds $C g H$.

In Figure 2.14 the maximum deflection that occurs is shown for the case of finite depth, the experimental results from Takizawa [31], and the case of infinite depth. The results for both the case of finite depth and the case of infinite depth agree well with the experimental data. This means that the depth has little effect on the amplitude of the ice deflection.

Part of the motivation in considering the case of infinite depth was to exploit the fact that the poles of the integrand in eq. (2.18) can be obtained by analysing the roots of a polynomial. This is in contrast with the case of finite depth where the poles satisfy a transcendental equation. However, the analysis is essentially the same and the branch cut that arises leads to a further complication.
Figure 2.13: **Plot of wavelength vs. load speed.** Both the leading wave and the trailing wave are shown on the plot. Takizawa data (Takizawa [31]) and the data from the case of finite depth (Milinazzo *et al.* [19]) are also shown on the plot for comparison. The load speed $C_{\text{min}} = 6.2$ m/sec. and the load speed $C_{gH} = 8.2$ m/sec. The parameters are $E = 5 \times 10^8$, $\nu = \frac{1}{3}$, $h = 0.175$ m, $H = 6.8$ m, $g = 9.8$ m/sec$^2$, and $\rho = 1026$ kg/m$^3$. 
Figure 2.14: **Plot of depression depth vs. load speed.** The depression depth on the vertical axis is the maximum depth measured underneath the load. Takizawa data (Takizawa [31]) and the data from the case of finite depth (Mili­nazzo et al. [19]) are also shown on the plot for comparison. The load speed $C_{\text{min}} = 6.2$ m/sec. and the load speed $C_{gH} = 8.2$ m/sec. The parameters are $E = 5 \times 10^8$, $v = \frac{1}{3}$, $h = 0.175$ m, $H = 6.8$ m, $g = 9.8$ m/sec$^2$ and $\rho = 1026$ kg/m$^3$. 
Chapter 3

Variation on the Mathematical Model

3.1 Ice Plate with Finite Thickness

In the theory of the bending of a thin plate, it is assumed that the plate deflection is small in comparison with the thickness of the plate (Timoshenko and Woinowsky-Krieger [35]). This assumption is equivalent to the condition that the strain in the mid-plane of the plate is neglected, i.e. omitting the effect on bending of the shearing forces and the normal stresses in the vertical direction. This is the usual assumption used for the thin plate equation given in (2.1). Reissner [23] developed a model for the bending of thin elastic plates that takes into account the transverse shear forces in the bending of the plate. That study considered the problem of torsion of a rectangular plate and the problem of the plain bending and the pure twisting of an infinite plate containing a circular hole. Strathdee et al. [28] considered the response of a floating ice sheet to a moving load, where the ice is treated as an isotropic plate with finite thickness and a simple viscoelastic term is also included. The study considers, among others, local and far field asymptotics for a stationary load and far field asymptotics for a moving load. In describing the response of the ice plate to a concentrated load, Strathdee et al. [28] showed that waves with wavelength comparable to the plate thickness are also significant when the load size is small enough, at least in the neighbourhood of the load itself.

In this chapter, a mathematical model for the response of a floating ice plate to a moving load is derived where the plate thickness $h$ is assumed to be comparable
with the plate deflection. The theory of the bending of a thin plate is revisited, and a differential equation similar to eq. (2.1) is derived by taking into account the thickness and the strain in the mid-plane of the plate. The derivation in this chapter follows closely that given in chapter 5-section 39 of Timoshenko and Woinowsky-Krieger [35]. In that chapter, a boundary condition for the normal stress in the vertical direction is specified at the top and at the bottom of the plate. In this Dissertation, instead of specifying a boundary condition for the normal stress in the vertical direction, the evaluation of this normal stress is delayed until the final step of the derivation. In this way, the balance of forces in the vertical direction, which includes the pressure from the load at the top of the ice plate and the pressure from the supporting liquid at the bottom of the ice plate, can still be satisfied.

Figure 3.1: The forces acting on a bending plate. The positive direction of the y-axis is downward. $Q_x$ and $Q_z$ are the vertical shearing forces. $f$ is the pressure per unit area exerted by the load on the ice plate. $M_x$ and $M_z$ are the bending moments. $\sigma_x$ and $\sigma_z$ are the normal stress components.
In the analysis, it is assumed that the $x, z$ plane coincides with the mid-plane of the plate and that the positive direction of the $y$ axis is downward. The aim is to derive the equation for the plate deflection when the plate thickness is taken into account. Using the assumption given in Reissner's [23] theory, the normal stress components $\sigma_x$ and $\sigma_z$, and the shearing stress component $\tau_{xz}$ (see Figure 3.2) are taken to have a linear law distribution through the plate thickness $h$, i.e.\(^1\)

\[
\begin{align*}
\sigma_x &= \frac{12}{h^3}M_x(x,z)y \\
\sigma_z &= \frac{12}{h^3}M_z(x,z)y \\
\tau_{xz} &= -\frac{12}{h^3}M_{xz}(x,z)y.
\end{align*}
\]

\(^1\)In this chapter, subscripts are no longer used to denote partial derivatives.
3.1 Ice Plate with Finite Thickness

Figure 3.3: The moments acting on a bending plate. The positive direction of the $y$ axis is taken to be downward.

Here (see Figures 3.1 and 3.3), $M_x(x, z)$ and $M_z(x, z)$ denote the bending moment per unit length acting on the edges parallel to the $z$ axis and the $x$ axis, respectively, and $M_{xz}(x, z)$ denotes the twisting moment per unit length on the edges parallel to the $z$ axis.

The time-dependent equations for a planar stress distribution are given by (see Timoshenko and Woinowsky-Krieger [35]. Love [18] for the corresponding equilibrium equations)

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + \frac{\partial \tau_{zy}}{\partial y} = \rho \frac{\partial^2 u}{\partial t^2} \tag{3.2}
\]

\[
\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} = \rho \frac{\partial^2 \hat{v}}{\partial t^2} \tag{3.3}
\]

\[
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} = \rho \frac{\partial^2 \hat{w}}{\partial t^2} \tag{3.4}
\]

where $u$, $\hat{v}$, $\hat{w}$ denote the displacement of the plate in the $x$, $z$, $y$ direction, respectively. The phase speeds corresponding to eqs. (3.2)-(3.4) are $C = \sqrt{\mu / \rho}$. $C = \sqrt{\frac{\lambda + 2\mu}{3\rho}}$, where $\lambda$ and $\mu$ are Lamé constants (see Squire et al. [26] sec-
Figure 3.4: The force components acting on a bending plate. The positive direction of the $y$ axis is taken to be downward. $Q_x$ and $Q_z$ are the vertical shearing forces.

The moment due to the load pressure $f(x, z)$ and the moments due to changes in the forces $Q_x$ and $Q_z$ are neglected, because they are terms of higher order. Thus, substituting $\sigma_x$, $\sigma_z$, and $\tau_{xz}$ from eq. (3.1) into eqs. (3.2) and (3.3) without the inertia terms and using eq. (3.5) give
For a thin plate, \( h \) is small and consequently the shearing stress and the normal stress in the \( y \) direction are negligible, i.e. \( \tau_{xy} = \tau_{yz} = \sigma_y = 0 \). If the effect on bending of these vertical shearing stresses is considered, i.e. the plate thickness is no longer negligible, \( \sigma_y \) needs to be included in the analysis. In Timoshenko and Woinowsky-Krieger [3.3], \( \sigma_y \) is evaluated from eq. (3.4) without the inertia term and eq. (3.6) with \( \sigma_y = f \) at the top of the plate and \( \sigma_y = 0 \) at the bottom of the plate. In Squire et al. [26], equations for a plate with finite thickness are also given as six differential equations with six variables where boundary conditions are specified on the plate surfaces. Hence, the conditions for \( \sigma_y \) are specified. In contrast, in this chapter, \( \sigma_y \) is evaluated at the final step of the derivation (see eq. (3.20)).

Making use of the vertical displacement of the plate \( \ddot{u}, \dot{v}, \ddot{w} \), and making use of Hooke's law for the plate displacement (see Timoshenko and Woinowsky-Krieger [35] pp. 166-167),

\[
\frac{\partial \ddot{u}}{\partial x} = \frac{1}{E} [\sigma_x - \nu(\sigma_z + \sigma_y)]
\]
\[
\frac{\partial \ddot{v}}{\partial z} = \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)]
\]
\[
\frac{\partial \ddot{u}}{\partial z} + \frac{\partial \ddot{v}}{\partial x} = \frac{2(1 + \nu)}{E} \tau_{xz}.
\] (3.7)

the following relations are obtained

\[
\sigma_x = \frac{E}{1 - \nu^2} \left( \frac{\partial \ddot{u}}{\partial x} + \nu \frac{\partial \ddot{v}}{\partial z} \right) + \frac{\nu}{1 - \nu} \sigma_y
\]
\[
\sigma_z = \frac{E}{1 - \nu^2} \left( \frac{\partial \ddot{v}}{\partial z} + \nu \frac{\partial \ddot{u}}{\partial x} \right) + \frac{\nu}{1 - \nu} \sigma_y
\]
\[
\tau_{xz} = \frac{E}{2(1 + \nu)} \left( \frac{\partial \ddot{u}}{\partial z} + \frac{\partial \ddot{v}}{\partial x} \right).
\] (3.8)
The plate deflection is usually measured in terms of the average value of the transverse displacement across the plate thickness. Let \( \eta \), the average value of the plate displacement, be defined by noting that the work of the force on the average displacement is equal to the work of the stress components on the plate displacement i.e.

\[
\eta = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} \hat{w} dy / \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} dy. \tag{3.9}
\]

which can also be written using \( \tau_{xy} \) in eq. (3.6) as

\[
\eta = \frac{6}{h^3} Q_x(x, z) \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( \frac{h^2}{4} - y^2 \right) \hat{w} dy / \frac{6}{h^3} Q_x(x, z) \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( \frac{h^2}{4} - y^2 \right) dy
\]

\[
= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( \frac{h^2}{4} - y^2 \right) \hat{w} dy / \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( \frac{h^2}{4} - y^2 \right) dy. \tag{3.10}
\]

Hence, eq. (3.9) can also be written in terms of the stress component \( \tau_{zy} \) as

\[
\eta = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{zy} \hat{w} dy / \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{zy} dy. \tag{3.11}
\]

In a similar way, let \( \phi_x \) and \( \phi_z \) be the average values of the rotation of the plate sections \( x = \text{constant} \) and \( z = \text{constant} \), respectively. These average values are determined by assuming that the work of the average rotation is equal to the work of the corresponding stress components on the plate displacement, i.e.

\[
\phi_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} y \hat{u} dy / \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 dy
\]

\[
= \frac{12}{h^3} M_x(x, z) \int_{-\frac{h}{2}}^{\frac{h}{2}} y \hat{u} dy / \frac{12}{h^3} M_x(x, z) \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 dy
\]

\[
= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \hat{u} dy / \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x dy
\]

\[
= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \hat{u} dy / M_x(x, z). \tag{3.12}
\]
This can also be written as
\[
\sigma_x = -\frac{12}{h^3} M_{xz}(x, z) \int_{-\frac{h}{2}}^{\frac{h}{2}} y \sigma_y dy - \frac{12}{h^3} M_{xz}(x, z) \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 dy
\]
\[
= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} \sigma_y dy - \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} y \sigma_y dy
\]
\[
= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} \sigma_y dy / M_{xz}(x, z).
\]  
(3.13)

The average value \( \sigma_z \) can be obtained in the same way to yield
\[
\sigma_z = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_z \sigma_y dy / M_z(x, z) = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} \sigma_y dy / M_{xz}(x, z).
\]  
(3.14)

With the definitions given in eqs. (3.9)-(3.14), the relations between \( \eta, \sigma_x, \sigma_z \) and \( u, \ddot{u}, \dot{v} \) are now clear:
\[
\sigma_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{12}{h^3} y \sigma_y dy.
\]
\[
\sigma_z = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{12}{h^3} y \dot{v} dy.
\]  
(15)
\[
\eta = \int_{-\frac{h}{2}}^{\frac{h}{2}} 6 \frac{h^2}{4} \dot{u} dy.
\]

Substituting eq. (3.1) into eq. (3.8), multiplying it by \( \frac{12}{h^3} y^2 \), integrating across the plate thickness from \(-\frac{h}{2}\) to \(\frac{h}{2}\), and using eq. (3.15) yield
\[
M_x(x, z) = D \left( \frac{\partial \sigma_x}{\partial x} + \nu \frac{\partial \sigma_z}{\partial z} \right) + \frac{\nu}{1 - \nu} \int_{-\frac{h}{2}}^{\frac{h}{2}} y \sigma_y dy
\]
\[
M_z(x, z) = D \left( \frac{\partial \sigma_z}{\partial z} + \nu \frac{\partial \sigma_x}{\partial x} \right) + \frac{\nu}{1 - \nu} \int_{-\frac{h}{2}}^{\frac{h}{2}} y \sigma_y dy
\]  
(3.16)

\[
M_{xz}(x, z) = -\frac{D(1 - \nu)}{2} \left( \frac{\partial \sigma_x}{\partial z} + \frac{\partial \sigma_z}{\partial x} \right).
\]

where the constant \( D \) is given by eq. (2.3).

Using the general stress-strain relations (see Timoshenko and Woinowsky-Krieger [35] pp. 166-167)
\[
\frac{\partial \ddot{u}}{\partial y} + \frac{\partial \ddot{u}}{\partial x} = \frac{2(1 + \nu)}{E} \tau_{xy}
\]
\[
\frac{\partial \ddot{v}}{\partial y} + \frac{\partial \ddot{u}}{\partial z} = \frac{2(1 + \nu)}{E} \tau_{xy}
\]  
(3.17)
the moments in eq. (3.16) can be written in terms of the vertical displacement \( \eta \) and the vertical shearing forces. To do this, eq. (3.6) is substituted into eq. (3.17). The resulting equations are multiplied by \( \frac{E}{h^3} (\frac{h^2}{4} - y^2) \). integrated across the plate thickness. and \( \sigma_x \) and \( \sigma_z \) given in eq. (3.15) are used to obtain

\[
\begin{align*}
\sigma_x &= -\frac{\partial \eta}{\partial x} + \frac{12(1+\nu)}{5Eh} Q_x(x, z) \\
\sigma_z &= -\frac{\partial \eta}{\partial z} + \frac{12(1+\nu)}{5Eh} Q_z(x, z).
\end{align*}
\]

Substituting eq. (3.18) into eq. (3.16) yields

\[
\begin{align*}
M_x(x, z) &= -D \left( \frac{\partial^2 \eta}{\partial x^2} + \nu \frac{\partial^2 \eta}{\partial z^2} \right) + \frac{h^2}{5(1-\nu)} \left( \frac{\partial Q_x}{\partial x} + \nu \frac{\partial Q_z}{\partial z} \right) \\
&\quad + \frac{\nu}{1-\nu} \int_{-\frac{h}{2}}^{\frac{h}{2}} y \sigma_y dy \\
M_z(x, z) &= -D \left( \frac{\partial^2 \eta}{\partial z^2} + \nu \frac{\partial^2 \eta}{\partial x^2} \right) + \frac{h^2}{5(1-\nu)} \left( \frac{\partial Q_z}{\partial z} + \nu \frac{\partial Q_x}{\partial x} \right) \\
&\quad + \frac{\nu}{1-\nu} \int_{-\frac{h}{2}}^{\frac{h}{2}} y \sigma_y dy \\
M_{xz}(x, z) &= (1-\nu)D \frac{\partial^2 \eta}{\partial x \partial z} - \frac{h^2}{10} \left( \frac{\partial Q_x}{\partial x} + \frac{\partial Q_z}{\partial z} \right).
\end{align*}
\]

The \( \sigma_y \) term can be evaluated by using integration by parts and eq. (3.4) without the inertia term and eq. (3.6) to obtain

\[
\begin{align*}
\int_{-\frac{h}{2}}^{\frac{h}{2}} y \sigma_y dy &= \left. \frac{1}{2} y^2 \sigma_y \right|_{-\frac{h}{2}}^{\frac{h}{2}} - \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2} y^2 \frac{\partial \sigma_y}{\partial y} dy \\
&= -\frac{1}{2} \frac{h^2}{4} \left( \frac{\partial Q_x}{\partial x} + \frac{\partial Q_z}{\partial z} \right) + \frac{3}{h^3} \left( \frac{\partial Q_x}{\partial x} + \frac{\partial Q_z}{\partial z} \right) \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( \frac{h^2}{4} y^2 - y^4 \right) dy \\
&= -\frac{h^2}{10} \left( \frac{\partial Q_x}{\partial x} + \frac{\partial Q_z}{\partial z} \right).
\end{align*}
\]

By differentiating the first line of eq. (3.5) with respect to \( x \) and the second line of eq. (3.5) with respect to \( z \), adding the resulting equations, and making use
of eqs. (3.19)-(3.20), and the Cauchy-Newton equation of motion for the vertical displacement of the neutral surface along the $y$ axis (see Figure 3.1)

$$
\frac{\partial Q_x(x, z)}{\partial x} + \frac{\partial Q_z(x, z)}{\partial z} + f(x, z) - p = \rho_{ee} \frac{\partial^2 \eta}{\partial t^2}.
$$

(3.21)

the differential equation for the plate deflection (neglecting the rotatory inertia terms - cf. Squire et al. [26]) can be written as

$$
D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)^2 \eta(x, z, t) + p(x, z, t) + \rho_{ee} h \frac{\partial^2 \eta(x, z, t)}{\partial t^2} - \gamma\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)p(x, z, t)
$$

$$
= f(x, z, t) - \gamma\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)f(x, z, t)
$$

for $-\infty < x, z < \infty$

(3.22)

where $\gamma = \frac{h^2}{10(1-\nu^2)}$. In eq. (3.22), the term $\gamma\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)p(x, z, t) - f(x, z, t)$ can be considered as a correction to the curvature of the plate deflection when the plate thickness is taken into account i.e. the stresses in the vertical direction also contribute to the deflection of the plate. If $\tau_{xy} = \tau_{yz} = \sigma_y = 0$ then eq. (3.22) becomes

$$
D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)^2 \eta(x, z, t) + p(x, z, t) + \rho_{ee} h \frac{\partial^2 \eta(x, z, t)}{\partial t^2} = f(x, z, t)
$$

for $-\infty < x, z < \infty$

(3.23)

which is the differential equation for the plate deflection when the plate thickness is assumed to be thin (Timoshenko and Woinowsky-Krieger [35]).

In Kerr [11], a mathematical model of an ice plate using a thin plate approximation subjected to in-plane forces was analyzed. Kerr argued that constrained thermal strains cause in-plane forces in ice plates. These forces affect the critical velocity of the moving loads. However, Schulkes et al. [24] found that the compressive stress on the ice has to be very large to have significant effect on the wave propagation.

In deriving the mathematical model where in-plane forces are taken into consideration, besides the equilibrium equations given by eqs. (3.2)-(3.4) without the
inertia terms and eq. (3.5), the equilibrium equations for the in-plane forces given by (see Timoshenko and Woinowsky-Krieger [35] chapter 12-section 90)

\[
\frac{\partial N_x}{\partial x} + \frac{\partial N_{zz}}{\partial z} = 0 \\
\frac{\partial N_{zz}}{\partial x} + \frac{\partial N_z}{\partial z} = 0
\]  

(3.24)

are also used. Here, \(N_x\) and \(N_z\) denote the normal forces per unit length of sections of a plate perpendicular to the \(x\) and the \(z\) axes, and \(N_{zz}\) denotes the shearing force in the direction of the \(z\) axis per unit length of a plate section perpendicular to the \(x\) axis. These forces are acting in the mid-plane of the ice plate (see Figure 3.5).

\[\text{Figure 3.5: The in-plane forces acting on the mid-plane of a plate.}\] The forces act on a small section of the plate. \(N_x\) is the normal force on the section parallel to the \(z\) axis, whereas \(N_z\) is the normal force on the section parallel to the \(x\) axis. The shearing forces in the \(x\) and the \(z\) directions are the same in magnitude, i.e. \(N_{xz} = N_{zx} \).
In this case, the differential equation for the time-dependent response of the plate (Kerr [11], Schulkes et al. [24]) is
\[
D \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right)^2 \eta(x, z, t) + N \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \eta(x, z, t) + p(x, z, t)
\]
\[-\rho_{w} h \frac{\partial^2 \eta(x, z, t)}{\partial t^2} = f(x, z, t) \quad \text{for} \quad -\infty < x, z < \infty. \quad (3.23)
\]
Note that in both Kerr [11] and Schulkes et al. [24] the plate is assumed to be thin.

In the next section, the dispersion relations obtained from the mathematical models given by eqs. (3.22) and (3.25) are compared to determine the effect of the different parameters that appear in the differential equation.

### 3.2 The Dispersion Relation

In this section, the dispersion relation of the mathematical models obtained in section 3.1 are compared for both a supporting base of finite depth and a supporting base of infinite depth. For a supporting base of infinite depth, the dispersion relation for the system given by eq. (3.22) is
\[
C^2 = \frac{g}{k} \frac{1 + \gamma k^2 + \alpha k^{4}}{1 + \gamma k^2 + \beta |k|}, \quad (3.26)
\]
and for the system given by eq. (3.25) is
\[
C^2 = \frac{g}{k} \frac{1 - \hat{N} k^2 + \alpha k^{4}}{1 + \beta |k|}, \quad (3.27)
\]
where \( \hat{N} = \frac{N}{\rho_y} \), and \( \alpha \) and \( \beta \) are given by eq. (2.17) with \( a = 1 \).

In the case of finite depth \( H \), the corresponding dispersion relations are given by
\[
C^2 = \frac{g}{k} \frac{1 + \gamma k^2 + \alpha k^{4}}{(1 + \gamma k^2) \coth(k H) + \beta k}, \quad (3.28)
\]
and
\[
C^2 = \frac{g}{k} \frac{1 - \hat{N} k^2 + \alpha k^{4}}{\coth(k H) + \beta k}. \quad (3.29)
\]
where \( \alpha \) and \( \beta \) are the same as \( \alpha \) and \( \beta \) in eq. (3.26) and eq. (3.27). For this case of finite depth, the dispersion relation where a thin plate assumption is used, can be obtained from eq. (3.29) by setting \( \dot{N} = 0 \).

For a system where the fluid base is of infinite depth, whether the ice thickness is taken into account or not, has little effect on the minimum phase speed. Figures 3.6 and 3.7 show that the plots of the dispersion relation given by eqs. (3.26) and (3.27) with \( \dot{N} = 0 \) are close to each other (within the scale of the plot). For the case of infinite depth, the plate thickness is unlikely to have any significant effect on the behaviour of the waves.

In Figure 3.8 the dispersion relation of the system where the ice thickness is taken into account (eq. (3.28)) shows that its minimum is lower than the minimum of the system where the ice thickness is neglected (eq. (4.10)). In Figures 3.8 and 3.9, for a system where there is an in-plane force, the minimum phase speed is either lower or higher than the system without the in-plane force, depending on whether it is a compression or a tension force (cf. Kerr [11]). This means that it is possible that for certain values of \( \dot{N} \), both dispersion relations given by eqs. (3.28) and (3.29) are approximately the same for a range of wavenumbers.

In summary, taking the ice thickness into account in the modelling of the problem can alter the critical phase speed and the behaviour of the short waves if the fluid base is of finite depth. The approach of taking the ice with finite thickness done by Strathdee et al. [28] confirmed the results obtained from the thin plate approximation both for stationary and for moving loads when \( \rho gh (1 - \nu) / \mu_1 \ll 1 \). It is understood that the far field asymptotic theory is not accurate in the neighbourhood of the load. If the wave behaviour in the neighbourhood of the load is important, i.e. those waves with wavelength comparable with or smaller than the thickness of the ice plate, the ice thickness needs to be considered in the mathematical modelling. If the fluid base is of infinite depth, the ice thickness has little effect on the behaviour of the waves.
For both the case of finite depth and the case of infinite depth, around the origin, which corresponds to the occurrence of long waves, the ice thickness does not have any effects on these waves.
Figure 3.6: The dispersion relation for the case of infinite depth - case 1. Plot number 1 corresponds to eq. (3.27) with $\dot{V} = 0$, plot number 2 corresponds to eq. (3.26), and plot number 3 corresponds to eq. (3.27). The in-plane force $\dot{V}$ used in plot number 3 is a compression force ($\dot{V}$ is set to +1). The plots for the dispersion relation given by eqs. (3.26) and (2.19) are the same within the scale of the plot. The parameters are $E = 5 \times 10^8$, $\nu = \frac{1}{3}$, $h = 0.175$ m, $\alpha = 24.968$, $\beta = 0.145$, $\gamma = 0.008$, and $g = 9.8$ m/sec$^2$. 

Dispersion Relation

1. Thin plate approximation
2. Plate thickness is included
3. Thin plate approximation and in-plane force
3.2 The Dispersion Relation

Figure 3.7: The dispersion relation for the case of infinite depth - case 2. Plot number 1 corresponds to eq. (3.27) with $\dot{\chi} = 0$, plot number 2 corresponds to eq. (3.26), and plot number 3 corresponds to eq. (3.27). The in-plane force $\dot{N}$ used in plot number 3 is a tension force ($\dot{N}$ is set to $-1$). The plots for the dispersion relation given by eqs. (3.26) and (2.19) are the same within the scale of the plot. The parameters are $E = 5 \times 10^8$, $\nu = \frac{1}{3}$, $h = 0.175$ m, $\alpha = 21.968$, $\beta = 0.145$, $\gamma = 0.008$, and $g = 9.8$ m/sec$^2$. 

![Dispersion Relation Diagram]

- 1) Thin plate approximation
- 2) Plate thickness is included
- 3) Thin plate approximation and in-plane force
Figure 3.8: The dispersion relation for the case of finite depth - case 1. Plot number 1 corresponds to eq. (3.29) with $\dot{N} = 0$, plot number 2 corresponds to eq. (3.28), and plot number 3 corresponds to eq. (3.29). The in-plane force $N$ used in plot number 3 is a compression force ($\dot{N}$ is set to +1). The parameters are $E = 5 \times 10^8$, $\nu = \frac{1}{3}$, $h = 0.175$ m, $H = 6.8$ m, $g = 9.8$ m/sec$^2$, $\alpha = 24.968$, $\beta = 0.145$, and $\gamma = 0.008$. 

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3.2 The Dispersion Relation
Figure 3.9: The dispersion relation for the case of finite depth - case 2. Plot number 1 corresponds to eq. (3.29) with $\hat{\mathcal{N}} = 0$. plot number 2 corresponds to eq. (3.28), and plot number 3 corresponds to eq. (3.29). The in-plane force $\mathcal{N}$ used in plot number 3 is a tension force ($\hat{\mathcal{N}}$ is set to $-1$). The parameters are $E = 5 \times 10^8$, $\nu = \frac{1}{3}$, $h = 0.175$ m, $H = 6.8$ m, $g = 9.8$ m/sec$^2$, $\alpha = 21.968$, $\beta = 0.145$, and $\gamma = 0.008$. 

\[ \text{Dispersion Relation} \]
Chapter 4
The Large Time Behaviour of the Ice Deflection for a Point Load

4.1 Introduction

This chapter deals with the time-dependent evolution of waves generated by an impulsively-started steadily moving point load, with particular reference again to the dependence of the deflection on the load speed. The first calculation of this nature is due to Kheisin [14] who recognized that time-dependence might help explain the singular deflection at critical speed noted in the earlier steady-state theory. Inter alia, Kheisin found that for a line load the ice deflection grows as $t^{\frac{1}{2}}$ at the load speed identified as $C_{\text{min}}$. Schulkes and Sneyd [25] carried out a more thorough investigation for the case of a line load, and showed inter alia that the ice deflection in the vicinity of the load grows not only at the load speed $C_{\text{min}}$ but also at the load speed $C_{gH}$. In this chapter, it is shown that the two-dimensional time-dependent behaviour of the ice deflection for a point load is distinctively different, using both asymptotic and numerical analysis.

The notation of chapter 2 is used for all physical variables and parameters, but $f(x,z,t)$ now denotes the pressure of a moving point load on the ice sheet and the equation of motion adopted here for the ice deflection is with reference to a fixed coordinate system:

$$D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)^2 \eta + \rho_{\text{ice}} h \frac{\partial^2 \eta}{\partial t^2} = p - f(x,z,t) \quad \text{for} \quad -\infty < x,z < \infty \quad (4.1)$$

(In contrast, eq. (2.1) is with reference to a coordinate system moving with the load). The equation for the velocity potential of the fluid base is similarly now of
4.1 Introduction

the form

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad -H < y < 0 , -\infty < x, z < \infty \]  

(4.2)

and subject to the boundary condition at the bottom \( \frac{\partial \phi}{\partial y} = 0 \) and the kinematic (non-cavitation) condition \( \frac{\partial \phi}{\partial y} = \frac{\partial n}{\partial t} \) at \( y = 0 \). The Bernoulli equation at \( y = 0 \) is also now given by

\[ y \eta + \frac{1}{\rho} p + \frac{\partial \phi}{\partial t} = 0. \]  

(4.3)

Solving for the water pressure \( p \) in eq. (4.3) and substituting the result into eq. (4.1) yields the fundamental dynamic equation for the ice deflection \( \eta(x, z, t) \):

\[ D(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2})^2 \eta + \rho_{ce} h \frac{\partial^2 \eta}{\partial t^2} = -\rho \frac{\partial \phi}{\partial t} \bigg|_{y=0} - \rho \eta \frac{f(x, z, t)}{2} \quad \text{for } -\infty < x, z < \infty. \]  

(4.4)

For a point load travelling with speed \( V \) in the positive \( x \) direction, the pressure due to the load can be written as \( f(x, z, t) = P_o \delta(x - V t) \delta(z) \mathcal{U}(t) \) where \( P_o \) is the load pressure per unit area, \( \delta \) denotes the Dirac delta function, and \( \mathcal{U}(t) \) is the Heaviside unit step function.

A closed form solution of eqs. (4.2)-(4.4) can be obtained by using the Fourier transform. Taking the Fourier transform of eqs. (4.2)-(4.4) and the corresponding boundary conditions results in the equations

\[ Dk^4 \hat{\eta} + \rho g \hat{\eta} + \rho_{ce} h \hat{\eta}_{tt} + \rho \hat{\phi} = -\frac{P_o}{2\pi} e^{-ik_1 V t} \]  

(4.5)

and

\[ \hat{\phi}_{yy} - k^2 \hat{\phi} = 0 \]  

(4.6)

with \( \hat{\phi}_y = 0 \) at \( y = -H \) and \( \hat{\phi}_y = \hat{\eta}_t \) at \( y = 0 \). Here, \( \hat{\eta} \) and \( \hat{\phi} \) indicate the Fourier transforms of \( \eta \) and \( \phi \), respectively. Subscripts denote partial differentiation, and \( k^2 = k_1^2 + k_2^2 \).
Solving eq. (4.6) for $\phi$ and substituting the result into eq. (4.5) gives

$$Dk^4 \dot{\eta} + \rho g \dot{\eta} + \left( \rho_{ice} h + \frac{\rho}{k} \coth (kH) \right) \ddot{\eta} = \frac{P_o}{2\pi} e^{-ik_1 \nu t}. \tag{4.7}$$

Eq. (4.7) can be simplified by noting that the wavelengths of interest are those much larger than the ice thickness $h$. Hence, for disturbances of wavelength greater than the ice thickness the ice-acceleration term proportional to $\rho_{ice} h$ can be dropped from eq. (4.7) to yield the equation

$$Dk^4 \dot{\eta} + \rho g \dot{\eta} + \frac{\rho}{k} \coth (kH) \ddot{\eta} = \frac{P_o}{2\pi} e^{-ik_1 \nu t}. \tag{4.8}$$

Given that the ice is undisturbed at $t = 0$, eq. (4.8) and the initial conditions $(\dot{\eta})_{t=0} = 0$ and $(\ddot{\eta})_{t=0} = 0$ constitute an initial value problem, which has solution

$$\dot{\eta}(k, t) = \frac{P_o}{4\pi \rho} \frac{e^{-ik_1 \nu t}}{C} \left( \frac{e^{-\nu_1 \nu t} - 1}{\nu_1} + \frac{e^{\nu_2 \nu t} - 1}{\nu_2} \right) \tanh (kH). \tag{4.9}$$

Here, $\nu_1 = kC - k_1 \nu$, $\nu_2 = kC + k_1 \nu$, are the phase functions and the phase speed $C$ is given by

$$C^2 = \left( \frac{Dk^4}{\rho} + g \right) \frac{\tanh (kH)}{k}. \tag{4.10}$$

This form of dispersion relation, corresponding to the neglect of the ice-acceleration term and so applicable to all but extremely short waves, is the same as the relation given by Greenhill [7].

The inverse Fourier transform

$$\eta(x, z, t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dot{\eta}(k, t) e^{i(k_1 x - k_2 z)} dk_1 dk_2 \tag{4.11}$$

is now used to obtain the ice deflection. In a coordinate system moving with the load i.e. $X = x - \nu t$, the result is

$$\eta(X, z, t) = \frac{P_o}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(k_1 X - k_2 z)} \frac{\tanh (kH)}{2\rho C \nu_1} e^{ik_1 X} \frac{\tanh (kH)}{2\rho C \nu_2} e^{ik_2 z} \frac{1}{k_1} \frac{1}{k_2} \tag{4.12}$$
4.2 The Behaviour of the Ice Deflection for Large Time

where \( \eta \) has been redefined as a function of \( X \).

Eq. (4.12) may be compared with the one-dimensional eqs. (2-7)-(2-9) in Schulkes and Sneyd [25] where the ice deflection caused by a line load was analyzed. There, the expression for the ice deflection involves single integrals that respectively represent one time-independent and two time-dependent contributions. For a rectangular load, Milinazzo et al. [19] discussed the steady solution of the ice deflection involving the double integral given by eq. (4.12) without the time-dependent terms.

The following sections are concerned with the large time behaviour of the ice deflection given by eq. (4.12) as a function of load speed.

4.2 The Behaviour of the Ice Deflection for Large Time

To investigate the time-dependent terms, the coordinate system used in eq. (4.12) is transformed into a system of polar coordinates. This is done to simplify the analysis of the integral. Defining \( X = r \cos \xi, z = r \sin \xi, k_1 = k \cos \chi, k_2 = k \sin \chi \), eq. (4.12) becomes

\[
\eta(r, \xi, t) = \frac{P_o}{8\pi^2 \rho} \left( \int_0^\infty \int_0^\infty e^{ikr \cos \xi - \xi} \left( 1 - e^{-ik(C - V \cos \chi)t} \right) \frac{\tanh (kH)}{C(C-V \cos \chi)} dk d\chi \right)
\]

\[
+ \int_0^\infty \int_0^\infty e^{ikr \cos \xi - \xi} \left( 1 - e^{ik(C + V \cos \chi)t} \right) \frac{\tanh (kH)}{C(C+V \cos \chi)} dk d\chi \right)
\]

where \( \eta \) has been redefined as a function of \( r \) and \( \xi \). The range of the \( \chi \) integral can be reduced to \(-\frac{\pi}{2} < \chi < \frac{\pi}{2}\), since the cosine function is even and \( \cos (\chi + \pi) = -\cos \chi \). Combining the resulting terms, eq. (4.13) becomes
\[ \eta(r, \xi, t) = \]

\[- \frac{P_0}{8 \pi^2 \rho} \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{ikr \cos(\chi - \xi)} \left( 1 - e^{-ik(C - V \cos \chi)t} \right) \frac{\tanh(kH)}{C(C - V \cos \chi)} dkd\chi \right) \]

\[+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{ikr \cos(\chi - \xi)} \left( 1 - e^{ik(C - V \cos \chi)t} \right) \frac{\tanh(kH)}{C(C - V \cos \chi)} dkd\chi \]

\[+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-ikr \cos(\chi - \xi)} \left( 1 - e^{-ik(C - V \cos \chi)t} \right) \frac{\tanh(kH)}{C(C + V \cos \chi)} dkd\chi \]

\[+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-ikr \cos(\chi - \xi)} \left( 1 - e^{ik(C - V \cos \chi)t} \right) \frac{\tanh(kH)}{C(C + V \cos \chi)} dkd\chi \]

\[= -\frac{P_0}{4\pi^2} \Re \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\infty} e^{ikr \cos(\chi - \xi)} \left( 1 - e^{-ik(C - V \cos \chi)t} \right) \frac{\tanh(kH)}{C(C - V \cos \chi)} dk d\chi \right) \]

\[+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\infty} e^{-ikr \cos(\chi - \xi)} \left( 1 - e^{-ik(C - V \cos \chi)t} \right) \frac{\tanh(kH)}{C(C + V \cos \chi)} dkd\chi \]

(4.14)

where \( \Re(z) \) has been used to denote the real part of \( z \).

In estimating the integrals in eq. (4.14) for large time \( t \), not all parts contribute to the leading terms of the asymptotic expansion. To see this, recall that the leading contribution to an asymptotic expansion of an integral such as eq. (4.14) comes from the stationary points, where the derivatives of the phase function are zero (Nayfeh [20] section 3.4, Lighthill [16] section 3.7). Since the first derivatives with respect to \( k \) of the phase function \( k(C + V \cos \chi) \) is always positive, no points of stationary phase occur in the term \( e^{ik(C - V \cos \chi)t} \) in the integrand of eq. (4.14), so this term does not contribute to the leading order of the expansion. Taking this into account and rewriting the \( \cos \chi \) term in eq. (4.14) by setting \( \frac{1}{2} = \theta \)
4.2 The Behaviour of the Ice Deflection for Large Time

\[ \eta(r, \xi, t) \approx -\frac{P_o}{2\pi^2 \rho} R e \left( \int_0^\infty \int_{-\pi/4}^{\pi/4} \frac{\tanh (kH) \ e^{ikr \cos (2\theta - \xi)} k}{C_{\ell'}} (1 - e^{-ivt})d\theta dk \right) \]  

(4.15)

where \( v = k(C' - V + 2V \sin^2 \theta) \).

One way to estimate the integral in eq. (4.15) is by analyzing separately the time-dependent and the time independent parts (cf. Schulkes and Sneyd [25]). In the two-dimensional case however, the singularities that occur when \( v = 0 \) are difficult to analyze separately. This difficulty can be circumvented by differentiating the integrand with respect to \( t \) and estimating \( \eta_t \) instead of \( \eta \). The asymptotic estimate of \( \eta_t \) may then be used to infer the large time behaviour of \( \eta \) (see Olver [22] pp.8-10). This is permissible since the integrand of eq. (4.15) is a continuous function of \( k \) and \( \theta \). Thus, taking the derivative with respect to \( t \), from eq. (4.15) the following form is considered

\[ \eta_t(r, \xi, t) \approx -\frac{P_o}{2\pi^2 \rho} R e \left( \int_0^\infty \int_{-\pi/4}^{\pi/4} \frac{\tanh (kH) \ e^{ikr \cos (2\theta - \xi)} k}{C_{\ell'}} e^{-ivt}d\theta dk \right) \]  

(4.16)

As discussed in Chako [3] and Cooke [4], for large values of \( t \) the major contribution to the double integral in eq. (4.16) arises from arbitrarily small neighbourhoods around the points of stationary phase. These stationary points are given by the roots of the equations \( v_k(k, \theta) = C_{gr} - V + 2V \sin^2 \theta = 0 \) and \( v_\theta(k, \theta) = 2kV \sin 2\theta = 0 \). The subscripts indicate partial differentiation, and the notation \( C_{gr} = \frac{d(kC_{\ell'})}{dk} \) is used to denote the group speed that was introduced in section 2.2.

In finding the roots of the equations \( v_k = v_\theta = 0 \), it is important to note how the stationary points depend on the load speed \( V \). To see this, consider the integration domain of eq. (4.16), i.e. \( 0 < k < \infty, -\pi/4 < \theta < \pi/4 \), where the roots of the equations \( v_k = v_\theta = 0 \) must be located. It is easy to see that \( v_\theta = 0 \)
when \( k = 0 \) or \( \theta = 0 \). In what follows, the cases \( \theta = 0 \) and \( k = 0 \) are considered separately.

- **Case 1:** When \( \theta = 0 \), the stationary points are determined by the values of \( k \) for which \( \nu_k = 0 \). Figure 4.1 depicts the roots of \( \nu_k \) for \( \theta = 0 \) as the intersection of \( C_{gr} \) and \( V \). There may be zero, one or two roots that satisfy the equation \( \nu_k = 0 \) depending on the value of \( V \). The stationary points are denoted by \((k_{s1}, 0)\) and \((k_{s2}, 0)\). Denoting the minimum value of \( C_{gr} \) by \((C_{gr})_{\text{min}}\). Figure 4.1 shows that

- for \( V < (C_{gr})_{\text{min}} \), there are no stationary points.
- for \( V = (C_{gr})_{\text{min}} \), there is one stationary point.
- for \((C_{gr})_{\text{min}} < V \leq \sqrt{gH} \), there are two stationary points, and
- for \( V > \sqrt{gH} \), there is only one stationary point.

- **Case 2:** When \( k = 0 \), \( \nu_k \) is given by \( \nu_k = \sqrt{gH} - V + 2V \sin^2 \theta \). For \( V > \sqrt{gH} \), there are two additional stationary points. To see this, \( \nu_k = 0 \) is written in the form \( \frac{V - \sqrt{gH}}{2V} = \sin^2 \theta \). Consequently \( 0 < \frac{V - \sqrt{gH}}{2V} \leq \frac{1}{2} \) when \( V > \sqrt{gH} \). It follows that there are two values of \( \theta \), i.e. \( \theta = \pm \theta_s \neq 0 \) such that \( \nu_k = 0 \).

In summary, the stationary points found in the two cases are listed below.

<table>
<thead>
<tr>
<th>Load speed</th>
<th>Stationary points</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V &lt; (C_{gr})_{\text{min}} )</td>
<td>None</td>
</tr>
<tr>
<td>( V = (C_{gr})_{\text{min}} )</td>
<td>((k_{s1}, 0))</td>
</tr>
<tr>
<td>((C_{gr})_{\text{min}} &lt; V &lt; \sqrt{gH} )</td>
<td>((k_{s1}, 0)), ((k_{s2}, 0))</td>
</tr>
<tr>
<td>( V = \sqrt{gH} )</td>
<td>((0, 0)), ((k_{s2}, 0))</td>
</tr>
<tr>
<td>( V &gt; \sqrt{gH} )</td>
<td>((0, \pm \theta_s)), ((k_{s2}, 0))</td>
</tr>
</tbody>
</table>

Chako [3] gave the analysis of the asymptotic expansion of double and multiple integrals. That article gave a more general analysis than that given by Cooke [4] for the asymptotic expansion of double integrals. For the integral given by
Figure 4.1: The roots of $\nu_k = 0$ for $\theta = 0$. The roots of the equation $\nu_k = C_{gr} - V + 2V \sin^2 \theta = 0$ are plotted as the intersection of $C_{gr}$ and $V$.

eq (4.16) however, an analysis similar to Cooke [4] is often adequate. Cooke shows that the contribution to the integral depends on the location of the stationary points in the domain of integration. In general, when the second and higher order derivatives of $\nu$ do not vanish at the stationary point, the contribution of the largest order comes from stationary points in the interior of the domain of integration. For an integral such as that eq. (4.16), the contribution from these stationary points leads to an expansion in powers of $\frac{1}{t}$. Furthermore, it is shown that the contribution from stationary points that are located on the boundary of
the region of integration yield an expansion of the form $\frac{1}{\sqrt{t}}(a_1/t + a_2/t^2 + \cdots)$. where $a_i, i = 1, 2, \ldots$ may be constants or may contain $e^{-it^2}$ terms. In the case of a point load, when the load speed is below $\sqrt{gH}$ the stationary points are interior points\(^1\). Hence the leading term of the asymptotic expansion of eq. (4.16) for large $t$ will be of $O(t^{-1})$. When the load speed is equal to $\sqrt{gH}$, there are two stationary points. One is a boundary stationary point and the other is an interior stationary point. In this case, the second order derivatives of $\psi$ in the expansion of eq. (4.16) vanish at the boundary stationary point and the condition that the second order derivative does not vanish, required by Cooke \[4\], is not satisfied. Hence those results do not apply. An analysis similar to that given by Cooke \[4\] for the load speed equal to $\sqrt{gH}$ shows that the interior and the boundary stationary points yield an expansion in the form of $\frac{1}{t}(a_1 + a_2/t^{\frac{3}{2}} + \cdots)$. When the load speed is above $\sqrt{gH}$ there are interior and boundary stationary points, and the second order derivatives of $\psi$ in the expansion of eq. (4.16) also vanish at the boundary stationary point. Further analysis for this case shows that the leading term of the asymptotic expansion of eq. (4.16) for large $t$ is of $O(t^{-1})$.

It is convenient to write the integral given in eq. (4.16) in the form

$$
\eta(t, \xi, t) = -\frac{P}{2\pi^2\nu} \Re\left( \int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} iM(k, \theta)e^{-itd\theta dk} \right) \quad \text{(4.17)}
$$

where $M(k, \theta) = (\tanh(kH)e^{ikr\cos(2\theta - \xi)}k)/C$. In applying the method of stationary phase, the contribution to $\eta$ from a stationary point $(k_s, 0)$ can be estimated by expanding the integrand about this point to obtain

$$
-\frac{P}{2\pi^2} \Re\left( \int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} i[kM(k_s, 0) + M(k, 0)(k - k_s)] e^{-it[\nu - \frac{r^2}{2}(k - k_s)^2 - \frac{\nu r^4}{2}\theta^2 - \frac{\nu r^6}{3}(k - k_s)^3 + \cdots]}d\theta dk \right). \quad \text{(4.18)}
$$

\(^1\)A stationary point that lies entirely inside the domain of integration is referred to as an interior stationary point, whereas a stationary point on the boundary of the domain of integration is referred to as a boundary stationary point.
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Then provided that the second derivatives of \( \varphi \) with respect to \( k \) and \( \theta \) are non-zero, the leading term of the asymptotic expansion is given by

\[
-\frac{P_0}{2\pi^2\rho} \text{Re} \left( \int_0^\infty \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} i \mathcal{M}(k_s,0) e^{-it(\varphi - \frac{\varphi''}{2}(k-k_s)^2 - \frac{\varphi'''}{3} k^3)} d\theta dk \right). \quad (4.19)
\]

The next lowest order term given by

\[
\text{Error} \approx \left| \frac{P_0}{2\pi^2\rho} \text{Re} \left( \int_0^\infty \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} i \left[ \left( M_k(k_s,0)(k-k_s) + M_\theta(k_s,0) \theta \right) + (M(k_s,0) + \mathcal{M}_k(k_s,0)(k-k_s) + \mathcal{M}_\theta(k_s,0) \theta ) \left( -it \frac{\varphi'''}{3} (k-k_s)^3 \right) \right] e^{-it(\varphi - \frac{\varphi''}{2}(k-k_s)^2 - \frac{\varphi'''}{3} k^3)} d\theta dk \right) \right| \quad (4.20)
\]

is used to estimate the error.

In the next section, the leading term of the asymptotic expansion of eq. (4.16) is evaluated for different load speeds. In each case, the error term is also estimated.

4.2.1 Case 1: Load speed \( V = (C_{fr})_{\min} \)

For this case, there is a single stationary point \((k_s,0)\) and the Taylor series expansion of the phase function \( \varphi \) about this point is

\[
\varphi(k,\theta) = \varphi + \frac{\varphi'''}{6}(k-k_s)^3 + \frac{\varphi''}{2} \theta^2 (k-k_s) + \frac{\varphi'}{2} \theta^2 + \cdots \quad (4.21)
\]

where \( \varphi \) and its derivatives are evaluated at \((k_s,0)\). At the stationary point, \( \varphi'''_{kk} = (C_{fr})_{kk} > 0 \), \( \varphi'_{\theta\theta} = 4k \), \( \varphi''_{\theta\theta} > 0 \), and \( \varphi''_{\theta\theta k} = 4V > 0 \). The leading term of the approximation to the integral in eq. (4.16) can be obtained as follows. Since the major contribution to the integral in eq. (4.16) comes from the immediate neighbourhood of \((k_s,0)\) then

\[
\eta_h(r,\xi,t) \approx -\frac{P_0}{2\pi^2\rho} \text{Re} \left( \int_{k_s-\epsilon}^{k_s+\epsilon} \int_{-\epsilon}^\epsilon \frac{\tan \left( k_s H \right) e^{ik_s r \cos \xi}}{C(k_s)} \right.
\]

\[
\left. e^{-it\left( \varphi - \frac{\varphi''}{2}(k-k_s)^2 - \frac{\varphi'''}{3} k^3 - 2k_s V \theta^2 - 2V(k-k_s) \theta^2 \right)} d\theta dk \right) \quad (4.22)
\]
where \( \varepsilon > 0 \). Since the integrand is even in \( \theta \), eq. (4.22) simplifies to

\[
\eta(r, \xi, t) \approx -\frac{P_0}{\pi^2 \rho} \text{Re} \left( i \frac{\tanh(k_s H)}{C(k_s)} e^{i k_s r \cos \xi} e^{-ivt} \int_{k_s - \varepsilon}^{k_s + \varepsilon} \int_{0}^{\infty} k e^{-ut \frac{C_s r \cos \xi}{\pi} (k - k_s)^2 - 2V k \theta^2} d\theta dk \right). \tag{4.23}
\]

In evaluating the integral in eq. (4.23), a transformation of variables usually assists. The basic idea is to find a transformation in such a way that the double integral can be evaluated as a product of two single integrals. Using the variable substitution \( k - k_s = S, k \theta^2 = U^3 \) and the corresponding Jacobian \( \frac{1}{2} \sqrt{\frac{1}{S-k_s}} \), the integral in eq. (4.23) can be written as

\[
\eta(r, \xi, t) \approx -\frac{3P_0}{2\pi^2 \rho} \text{Re} \left( i \frac{\tanh(k_s H)}{C(k_s)} e^{i k_s r \cos \xi} e^{-ivt} \int_{-r}^{r} \int_{0}^{\infty} \frac{1}{\sqrt{S-k_s}} e^{-ut \frac{C_s r \cos \xi}{\pi} S^3} e^{-ut^2 V^3} \sqrt{U} dU dS \right)
\]

\[
= -\frac{3P_0}{2\pi^2 \rho} \text{Re} \left( i \frac{\tanh(k_s H)}{C(k_s)} e^{i k_s r \cos \xi} e^{-ivt} \sqrt{k_s} \int_{-r}^{r} e^{-ut \frac{C_s r \cos \xi}{\pi} S^3} \int_{0}^{\infty} \sqrt{U} e^{-ut^2 V^3} dU dS \right). \tag{4.24}
\]

Since only the immediate neighbourhod of \( S = 0 \) and \( U = 0 \) contribute to the integrals in eq. (4.24), \( \varepsilon \) can be replaced by \( \infty \). A further substitution of \( \frac{S}{k_s} = R \) and using the fact that the function \( S^3 \) in eq. (4.24) is an odd function give

\[
\eta(r, \xi, t) \approx -\frac{3P_0}{\pi^2 \rho} \text{Re} \left( i \frac{\tanh(k_s H)}{C(k_s)} e^{i k_s r \cos \xi} e^{-ivt} \sqrt{k_s^3} \int_{0}^{\infty} e^{-ut \frac{C_s r \cos \xi}{\pi} k_s^3} R^3 dR \right) \int_{0}^{\infty} \sqrt{U} e^{-ut^2 V^3} dU - R. \tag{4.25}
\]
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A simple argument involving contour integration demonstrates that each of the integrals in eq. (4.25) can be replaced by an equivalent integral where the integration path has been rotated into the complex plane. The basic idea is to rotate the path in such a way that the original integral is transformed into a Laplace integral, i.e. the integrand contains a real decaying exponential at $\infty$ (Nayfeh [20] section 3.4). This is done by introducing the substitutions

$$\sqrt[6]{t} \frac{(C_{gr})_{kk}}{6} k_s R = \sqrt{\tau} e^{-i\frac{\pi}{6}}.$$ 

$$\sqrt{i2\sqrt{\tau}} \sigma = \sqrt{-}\sigma e^{-i\frac{\pi}{6}}$$

to yield

$$\eta_l(r, \xi, t) \approx -\frac{3P_o}{\pi^2 \rho} \Im \left( i \frac{\tanh (k_s H) e^{i k_r r \cos \xi}}{C(k_s)} e^{-it\nu} \sqrt{k_s} \right) \left( \frac{1}{3} \frac{e^{-i\frac{\pi}{6}}}{\sqrt{t(C_{gr})_{kk}/6}} \int_0^\infty \tau^{-\frac{1}{2}} e^{-\tau} d\tau \right) \left( \int_0^\infty e^{-i\frac{\pi}{6}} e^{-\tau} \sigma^{-\frac{1}{2}} e^{-\sigma} d\sigma \right) \right)$$

which can be evaluated to give

$$\eta_l(r, \xi, t) \approx -\frac{P_o}{6\pi^2 \rho t^{\frac{1}{2}}} \Gamma \left( \frac{1}{3} \right) \Gamma \left( \frac{1}{2} \right) \frac{\sqrt{3}}{\sqrt{2\tau \sqrt{(C_{gr})_{kk}/6}}} \left( \frac{1}{3} \frac{e^{-i\frac{\pi}{6}}}{\sqrt{t(C_{gr})_{kk}/6}} \int_0^\infty \tau^{-\frac{1}{2}} e^{-\tau} d\tau \int_0^\infty e^{-i\frac{\pi}{6}} e^{-\tau} \sigma^{-\frac{1}{2}} e^{-\sigma} d\sigma \right) \right)$$

Here, $\Gamma(z)$ is the Gamma function defined by $\Gamma(z) = \int_0^\infty \tau^{z-1} e^{-\tau} d\tau$. $\Re(z) > 0$ (see Abramowitz and Stegun [1] pp. 255).
From eq. (4.20), the lowest order term for the error of the approximation in eq. (4.27) is

\[
\text{Error} \approx \left| \frac{P_o}{2\pi^2 \rho} \mathcal{R}_0 \left( \int \int \left[ \mathcal{M}(k_\ast, 0) \left( -it\frac{\lambda}{24} (k - k_\ast)^4 \right) \right] e^{-it\left( -\frac{\lambda}{6} (k - k_\ast)^2 \right)} \right|.
\]

Again, for large \( t \), \( \epsilon \) can be replaced by \( \infty \) to give

\[
\text{Error} \approx \left| \frac{P_o}{2\pi^2 \rho} \mathcal{R}_0 \left( i e^{-it\epsilon} \int \int (\mathcal{M}(k_\ast, 0)(k - k_\ast)^4 + \mathcal{M}_k(k - k_\ast)) e^{-it\left( -\frac{\lambda}{6} (k - k_\ast)^2 \right)} \right) \right|.
\]

Applying the steps used in eqs. (4.24)-(4.27), it can be shown that the error term is \( O(t^{-\frac{7}{2}} e^{-it\epsilon}) \). Hence, eq. (4.27) becomes

\[
\eta(t, \xi, t) = -\frac{P_o}{6\pi^2 \rho \frac{\sqrt{3}}{4}} e^{\frac{k_\ast \epsilon}{C(k_\ast)}} \mathcal{R}_t \left( i \frac{\tan \left( k_\ast H \right)}{C(k_\ast)} e^{k_\ast \epsilon \cos \xi} \sqrt{k_\ast} \frac{e^{-t\epsilon}}{\sqrt{2t \sqrt{(C_{gr})_{kk}}/6}} \right) + O(t^{-\frac{7}{2}} e^{-it\epsilon}).
\]
4.2.2 Case 2: Load speed \((C_{gr})_{\text{min}} < V < C_{\text{min}}\) and \(C_{\text{min}} < V < \sqrt{gH}\)

In this case, the stationary points are interior stationary points. These points are denoted by \((k_{s1}, 0)\) and \((k_{s2}, 0)\) (see Figure 4.1 for the case \(C_{\text{min}} < V < \sqrt{gH}\)). After expanding the integrand about the stationary points, the leading term of the approximation to the integral is given by

\[\eta_t(r, \xi, t) \approx \frac{P_0}{4\pi^2 \rho} \Re \left( 2 \int_{k_{s1} - \epsilon}^{k_{s1} + \epsilon} \int_{i}^{+} \frac{\tanh k_{s1} H e^{ik_{s1}r \cos \xi}}{C(k_{s1})} e^{-i \left( V - \frac{V_{kk}(k-k_{s1})^2}{2} - \frac{V_{\theta\theta} g^2}{2} \right)} d\theta dk \right) + 2 \int_{k_{s2} - \epsilon}^{k_{s2} + \epsilon} \int_{i}^{+} \frac{\tanh k_{s2} H e^{ik_{s2}r \cos \xi}}{C(k_{s2})} e^{-i \left( V - \frac{V_{kk}(k-k_{s2})^2}{2} - \frac{V_{\theta\theta} g^2}{2} \right)} d\theta dk \]  

(4.31)

where the derivatives \(V_{kk}\) and \(V_{\theta\theta}\) are given by \(V_{kk} = (C_{gr})_k, V_{\theta\theta} = 4kV \cos 2\theta, \) and \(V_{\theta\theta} = 2V \sin 2\theta\). Also, the symbols \(\hat{V}\) and \(\tilde{V}\) have been used to indicate that \(\hat{V}\) is evaluated at the stationary point \((k_{s1}, 0)\) and \((k_{s2}, 0)\), respectively. Clearly, both \(\hat{V}_{kk}\) and \(\hat{V}_{\theta\theta}\) are positive, both \(\tilde{V}_{kk}\) and \(\tilde{V}_{\theta\theta}\) are negative, and \(\hat{V}_{\theta\theta} = \tilde{V}_{\theta\theta} = 0\). Hence the integral can be written as

\[\eta_t(r, \xi, t) \approx \frac{P_0}{2\pi^2 \rho} \Re \left( \int_{k_{s1} - \epsilon}^{k_{s1} + \epsilon} \int_{i}^{+} \frac{\tanh k_{s1} H e^{ik_{s1}r \cos \xi}}{C(k_{s1})} e^{-i \left( V - \frac{V_{kk}(k-k_{s1})^2}{2} - \frac{V_{\theta\theta} g^2}{2} \right)} d\theta dk \right) \]

\[\eta_t(r, \xi, t) \approx \frac{P_0}{2\pi^2 \rho} \Re \left( \int_{k_{s2} - \epsilon}^{k_{s2} + \epsilon} \int_{i}^{+} \frac{\tanh k_{s2} H e^{ik_{s2}r \cos \xi}}{C(k_{s2})} e^{-i \left( V - \frac{V_{kk}(k-k_{s2})^2}{2} - \frac{V_{\theta\theta} g^2}{2} \right)} d\theta dk \right) \]

and can be evaluated using the same arguments as in section 4.2.1. Rotating the integration path in each of the integrals using the substitutions

\[\sqrt{\left(\frac{V_{kk}(k_{s1}, 0)}{2}\right)} (k - k_{s1}) = Se^{-\text{sgn}(V_{kk}(k, 0)) \pi i / 4} \]
where \( j = 1, 2 \) yields

\[
\eta(r, \xi, t) = \frac{P_a}{2\pi^2 \rho} \Re \left( \int_{-\infty}^{\infty} \frac{\tanh (k_{x_1} H)}{C(k_{x_1})} e^{ik_{x_1} r \cos \xi} k_{x_1} e^{-it \psi} \int_{-\infty}^{\infty} \frac{e^{-s^2}}{\sqrt{t} \sqrt{|\psi_{\theta \theta}|}} e^{-t^2} d\psi dS \right)
\]

\[
- \frac{P_a}{2\pi^2 \rho} \Re \left( \int_{-\infty}^{\infty} \frac{\tanh (k_{x_2} H)}{C(k_{x_2})} e^{ik_{x_2} r \cos \xi} k_{x_2} e^{-it \psi} \int_{-\infty}^{\infty} \frac{e^{-s^2}}{\sqrt{t} \sqrt{|\psi_{\theta \theta}|}} e^{-t^2} d\psi dS \right) + O(t^{-2} e^{-t \psi})
\]

\[
- \frac{P_a}{\pi \rho} \Re \left( \int_{-\infty}^{\infty} \frac{\tanh (k_{x_2} H)}{C(k_{x_2})} e^{ik_{x_2} r \cos \xi} k_{x_2} e^{-it \psi} \int_{-\infty}^{\infty} \frac{e^{-s^2}}{\sqrt{|\psi_{\theta \theta}|}} \frac{1}{\sqrt{|\psi_{kk}|}} e^{-t^2} d\psi dS \right) + O(t^{-2} e^{-t \psi})
\]

(4.32)

4.2.3 Case 3: Load Speed \( \psi = C_{\text{min}} \)

For a line load, in this case Schulkes and Sneyd [25] determined that the ice displacement grows in time as \( t^{1/2} \) as first noted by Kheisin [14]. In addition, Davys et al. [5] argue that at the load speed \( C_{\text{min}} \) the phase speed and the group speed coincide so the energy accumulates underneath the load and a growth in ice displacement is to be expected. As a result, a steady solution cannot exist for a line load travelling at this speed. For a point load, the point \( (k_{x_2}, 0) \) is the point \( (k_{\text{min}}, 0) \) where \( C(k) \) attains its minimum value and \( \psi'(k_{\text{min}}, 0) = 0 \). Hence the expansion of the phase function \( \psi(k, \theta) \) is

\[
\psi(k, \theta) = \frac{\psi_{kk}}{2} (k - k_{\text{min}})^2 + \frac{\psi_{\theta \theta}}{2} \theta^2 + \ldots
\]
4.2 The Behaviour of the Ice Deflection for Large Time

Again, the derivatives are given by $\xi_{kk} = (C_{gr})_k$, $\xi_{\theta \theta} = 4kV \cos 2\theta$, and $\xi_{k\theta} = \xi_{\theta k} = 2V \sin 2\theta$. At the stationary point $(k_{s1}, 0)$, $\xi_{kk}$ is negative and $\xi_{\theta \theta}$ is positive, whereas at the stationary point $(k_{min}, 0)$, both $\xi_{kk}$ and $\xi_{\theta \theta}$ are positive.

The integral in eq. (4.31) can be written in the form

$$\eta_l(r, \xi, t) \approx - \frac{P_0}{2\pi^2 \rho} \Re \left( \int_{k_{s1} - \infty}^{k_{s1}} \frac{\tanh k_{s1} H \epsilon^{ik_{s1} r \cos \xi} k_{s1} \epsilon^{-\frac{t\omega}{2}} - \frac{\xi_{kk} (k-k_{s1})^2 - \frac{\xi_{\theta \theta} \theta^2}{2}}{C(k_{s1})} d\theta dk \right) + \frac{2}{k_{min} - \infty} \int_{k_{min}}^{k_{min} \infty} \frac{\tanh k_{min} H \epsilon^{ik_{min} r \cos \xi} k_{min} \epsilon^{-\frac{t\omega}{2}} - \frac{\xi_{kk} (k-k_{min})^2 - \frac{\xi_{\theta \theta} \theta^2}{2}}{C(k_{min})} d\theta dk \right)$$

(4.33)

Proceeding as in the previous sections, the leading contribution from the stationary points and the error terms can be seen to be

$$\eta_l(r, \xi, t) = - \frac{P_0}{\pi \rho} \Re \left( \frac{\tanh k_{s1} H \epsilon^{ik_{s1} r \cos \xi} k_{s1} \epsilon^{-\frac{t\omega}{2}} - \frac{1}{C(k_{s1})} \frac{1}{t \sqrt{C_{kk} \sqrt{C_{\theta \theta}}}} \right) + O(t^{-2} e^{-it\omega})$$

- $$\frac{P_0}{\pi \rho} \Re \left( \frac{\tanh k_{min} H \epsilon^{ik_{min} r \cos \xi} k_{min} \epsilon^{-\frac{t\omega}{2}} - \frac{1}{C(k_{min})} \frac{1}{t \sqrt{C_{kk} \sqrt{C_{\theta \theta}}}} \right) + O(t^{-2}) \quad (4.34)$$

Here, two error terms are given. At the stationary point $(k_{s1}, 0)$, the phase function $\omega \neq 0$ so the corresponding terms exhibit oscillations. But, at $(k_{min}, 0)$, the phase function $\omega = 0$ and consequently the corresponding terms exhibit no oscillations.
4.2.4 Case 4: Load Speed $V = \sqrt{gH}$

As $V \to \sqrt{gH}$, the stationary point $(k_1, 0)$ near the origin\(^2\) approaches $(0, 0)$, so this stationary point becomes a boundary stationary point. Moreover, at the origin the derivatives of $\nu$ up to second order vanish and the third derivatives are

$$
\nu_{kkk}^2 = (C_{gr})_{kk} < 0
$$
$$
\nu_{\theta\theta}^2 = -8kV \sin 2\theta = 0
$$
$$
\nu_{\theta\theta k}^2 = \nu_{kk\theta}^2 = 4V.
$$

Expanding the integral about the stationary points gives the leading term of the approximation as

$$
\eta_t(r, \xi, t) \approx - \frac{P_0}{\pi^2 \rho} \text{Re} \left( i \int_0^1 \int_0^1 \frac{H}{C(0)} k^2 e^{-ut(\nu_{kkk}^2 k^2 - \nu_{\theta\theta\theta}^2 k^2)} d\theta dk \right)
$$
$$
- \frac{P_o}{2\pi^2 \rho} \text{Re} \left( i \int_{k_1}^{k_2} \int_{k_1}^{k_2} \frac{\tanh k_1 H e^{ik_2 r \cos \xi} k_2 e^{-ut(\nu_{kkk}^2 (k-k_2)^2 - \nu_{\theta\theta\theta}^2)})} \right). \quad (4.35)
$$

The second integral in eq. (4.35) can be evaluated as in section 4.2.2. Hence, eq. (4.35) becomes

$$
\eta_t(r, \xi, t) \approx - \frac{P_0}{\pi^2 \rho} \text{Re} \left( i \int_0^\infty \int_0^\infty \frac{H}{C(0)} k^2 e^{-ut(2V^2 k \theta^2 - \frac{C_{gr}r^2}{\pi^2} k^2)} d\theta dk \right)
$$
$$
- \frac{P_o}{\pi \rho} \text{Re} \left( \frac{\tanh k_1 H e^{ik_2 r \cos \xi} k_2 e^{-ut\nu_{kkk}^2}}{C(k_2)} \frac{1}{\sqrt{\frac{\nu_{kkk}^2}{\nu_{\theta\theta}}}} \right). \quad (4.36)
$$

The variable substitution $k \theta^2 = L^3$ and $k = S$ with the corresponding Jacobian. $\frac{3}{2} \sqrt{\frac{r}{\pi}}$ is used to transform the integral in eq. (4.36) to give

\(^2\)The load speed $\sqrt{gH}$ is the critical speed $C_{gH}$ introduced in chapter 1. In subsequent discussions $C_{gH} = \sqrt{gH}$. 

\[ \eta_t(r, \xi, t) \approx \]
\[- \frac{P_0}{2\pi^2 \rho} \Re \left( 3i \int_0^\infty \int_0^\infty \sqrt{\xi} e^{-it2\xi t^3} d\xi \frac{H}{C(0)} S^{\frac{1}{2}} e^{i \frac{r}{\xi} \cdot \frac{\rho}{\xi} \cdot \frac{\rho}{\xi} \cdot \frac{t}{S} \cdot dS} \right) \]
\[- \frac{P_0}{\pi \rho} \Re \left( \tanh k_{x_2} H e^{ik_{x_2} r \cos \xi} k_{x_2} e^{-it\xi} \frac{1}{\sqrt{\xi_{kk} \xi_{\theta\theta}}} \right) \]
\[= - \frac{P_0}{2\pi^2 \rho} \Re \left( 3i \frac{H}{C(0)} \int_0^\infty \frac{\tau^{\frac{1}{2}} e^{\tau \frac{1}{6}}}{\sqrt{\xi_{kk} \xi_{\theta\theta}}} \frac{e^{-\frac{r^3}{\xi} \tau^3}}{\sqrt{21} t} \frac{e^{-\frac{\rho^3}{\xi} \tau^3}}{\sqrt{21} t} d\tau \right) \]
\[- \frac{P_0}{2\pi^2 \rho} \Re \left( \tanh k_{x_2} H e^{ik_{x_2} r \cos \xi} k_{x_2} e^{-it\xi} \frac{1}{\sqrt{\xi_{kk} \xi_{\theta\theta}}} \right). \quad (4.37) \]

Proceeding as in the previous section, the two integrals in eq. (4.37) and the error terms can be evaluated to yield

\[ \eta_t(r, \xi, t) = \]
\[- \frac{P_0}{6\pi^2 \rho} \Re \left( i \frac{H e^{\frac{\tau}{\xi}}}{C(0)} \left( \frac{6}{(C_{x_2})_{kk}(0)} \right)^{\frac{1}{2}} \frac{1}{\sqrt{21} t^{\frac{1}{3}}} \Gamma\left( \frac{5}{6} \right) \Gamma\left( \frac{1}{2} \right) \right) + O(t^{-2}) \]
\[- \frac{P_0}{\pi \rho} \Re \left( \tanh k_{x_2} H e^{ik_{x_2} r \cos \xi} k_{x_2} e^{-it\xi} \frac{1}{\sqrt{\xi_{kk} \xi_{\theta\theta}}} \right) + O(t^{-2} e^{-it\xi}). \quad (4.38) \]

For the stationary point at the origin, the error term given in the first part of eq. (4.38) does not contain an exponential term, since the phase function \( \psi = 0 \) at the origin.
4.2.5 Case 5: Load Speed $V > \sqrt{gH}$

When the load speed $V$ is greater than $\sqrt{gH}$, as discussed earlier in section 4.2, there are additional stationary points near the origin given by $(0, \pm \theta_s)$ where $\theta_s \neq 0$ and $|\theta_s| < \frac{\pi}{4}$. The third stationary point away from the origin is given by $(k_{s2}, 0)$. At the stationary point $(0, \theta_s)$ the non-zero derivatives up to the third order are

\[
\begin{align*}
\xi_{k\theta} &= \xi_{\theta k} = 2V \sin 2\theta_s \\
\xi_{kkk} &= (C_{gr})_{kk} < 0 \\
\xi_{"k\theta k} &= \xi_{k\theta k} = 4V \cos 2\theta_s
\end{align*}
\]

Expanding the integral about the stationary points gives the leading term of the approximation as

\[
\begin{align*}
\eta_l(r, \xi, t) \approx & \quad - \frac{P_n}{2\pi^2 \rho} R e \left( i \int_{0}^{\theta_s - \epsilon} \int_{-\theta_s - \epsilon}^{\theta_s - \epsilon} \frac{H}{C(0)} k^2 e^{-it(\xi_{kk} k^2 (\theta - \theta_s)^2 - \xi_{kkk} k^3)} d\theta dk \right) \\
& - \frac{P_n}{2\pi^2 \rho} R e \left( i \int_{0}^{\theta_s - \epsilon} \int_{-\theta_s - \epsilon}^{\theta_s - \epsilon} \frac{H}{C(0)} k^2 e^{-it(\xi_{kk} k^2 (\theta - \theta_s)^2 - \xi_{kkk} k^3)} d\theta dk \right) \\
& - \frac{P_n}{2\pi^2 \rho} R e \left( i \int_{k_{s2} - \epsilon}^{k_{s2} + \epsilon} \int_{-k_{s2} - \epsilon}^{k_{s2} + \epsilon} \frac{\text{tanh}(k_{s2} H e^{i k_{s2} r \cos \xi} \frac{k_{s2}}{C(k_{s2})})}{k_{s2}^2 e^{-it(\xi_{kk} k_{s2}^2 (k - k_{s2})^2 - \xi_{kkk} k_{s2}^3)}} d\theta dk \right) \\
& \quad \left( \frac{k_{s2}}{C(k_{s2})} \right).
\end{align*}
\]

(4.39)

In the first two integral in eq. (4.39), the exponent of the exponential terms can be simplified by completing the square on the variables $(\theta - \theta_s)$ and $(\theta + \theta_s)$. Rewriting $\theta - \theta_s$ and $\theta + \theta_s$ as $\theta$ and proceeding as in section 4.2.2 for the last integral in eq. (4.39) yield
\[ \eta(r, \xi, t) \approx -\frac{P_o}{\pi^2 \rho} \Re e \left( \int_0^\infty \frac{H}{C(0)} k^2 e^{it\frac{1}{2} \cos(2\theta_s) \tan^2(\theta_s) \left( k - \frac{C_{kr}}{n} k^2 \right)} \left( \int_{-\infty}^\infty e^{-it2V \cos(2\theta_s) k^2 \theta_s^2} d\theta \right) dk \right) \]

Evaluating the \( \theta \)-integral by rotating the integration path as in section 4.2.2 gives

\[ \eta(r, \xi, t) \approx -\frac{P_o}{\pi \rho} \Re e \left( \frac{\tanh k_s H e^{ik_s r \cos \xi} k_s^2 e^{-it\dot{\xi}}}{C(k_s^2)} \frac{1}{t \sqrt{\dot{c}_{kk} \sqrt{\dot{c}_{\theta \theta}}}} \right). \] (4.40)

The integral in eq. (4.41) can be estimated using the method of steepest descent. Following the steps given in the appendix (see section A.0.3 of the Appendix) for an integral such as the integral in eq. (4.41) yields

\[ \eta(r, \xi, t) \approx -\frac{P_o}{\pi^2 \rho} \Re e \left( \frac{H \tan(2\theta_s)}{C(0) |(C_{gr})_{kk}|} e^{-\frac{t}{2} \sqrt{\frac{\tan(2\theta_s)}{2} \sqrt{\frac{C_{rr}}{C_{rr} + C_{kk}}}}} \right) \]

\[ -\frac{P_o}{\pi \rho} \Re e \left( \frac{\tanh k_s H e^{ik_s r \cos \xi} k_s^2 e^{-it\dot{\xi}}}{C(k_s^2)} \frac{1}{t \sqrt{\dot{c}_{kk} \sqrt{\dot{c}_{\theta \theta}}}} \right) + O(t^{-2} e^{-it\dot{\xi}}). \] (4.41)

### 4.2.6 Case 6: Load Speed \( V \) approaches \( \sqrt{gH} \)

When the load speed \( V \) approaches \( \sqrt{gH} \) from below, the denominator of the first term of eq. (4.32) involving \( \dot{c}_{kk} = (C_{gr})_k(k_s^1, 0) \) and \( \dot{c}_{\theta \theta} = 4k_s \sqrt{\dot{c}_{kk}} \) approaches 0 since the stationary point \( (k_s^1, 0) \) also approaches \( (0, 0) \). When the load speed \( V \)
approaches \( \sqrt{gH} \) from above, eq. (4.42) suggests that the stationary points near the origin give much smaller contribution than the stationary point away from the origin. Hence, the estimate given by eq. (4.32) is not valid as \( V \to \sqrt{gH} \) and the estimate given by eq. (4.38) is only valid when \( V = \sqrt{gH} \).

Since for \( V \) near \( \sqrt{gH} \) there are stationary points near the origin, an estimate of the integral contribution from these stationary points can be obtained by expanding the integrand of eq. (4.16) about \( k = \theta = 0 \) to yield the leading order term (only the contribution from the stationary point near the point \((0,0)\) will be analyzed)

\[
(\eta_t(r,\xi,t))_1 \approx -\frac{P_o}{2\pi^2 \rho} \Re \left( \int_0^\infty \int_{-\infty}^\infty i \frac{H}{C(0)} k^2 e^{-it(\sqrt{gH} - V)k - \frac{C_{gV}^r kk}{\pi} k^3 - 2i k \theta^2} d\theta dk \right) 
\]

where \((\eta_t)_1\) denotes the contribution to \( \eta_t \) from the stationary point near the point \((0,0)\). When \( V = \sqrt{gH} \), eq. (4.43) is the same as the first term of eq. (4.35). Since away from the origin, the contribution is small, the integral can be written as

\[
(\eta_t(r,\xi,t))_1 \approx -\frac{P_o}{2\pi^2 \rho} \Re \left( \int_0^\infty \int_{-\infty}^\infty i \frac{H}{C(0)} k^2 e^{-it(\sqrt{gH} - V)k - \frac{C_{gV}^r kk}{\pi} k^3 - 2i k \theta^2} d\theta dk \right) 
\]

Note that \((C_{gV})_{kk} < 0\) at the stationary point \((0,0)\). Using the substitution

\[
\sqrt{t2Vk\theta} = u, \quad \text{yields}
\]

\[
(\eta_t(r,\xi,t))_1 \approx -\frac{P_o}{2\pi^2 \rho} \Re \left( \frac{H}{\sqrt{2V} C(0)} \frac{1}{\sqrt{t}} \int_0^\infty \int_{-\infty}^\infty i k^{\frac{3}{2}} e^{-it(\sqrt{gH} - V)k - \frac{C_{gV}^r kk}{\pi} k^3 - \frac{1}{2} \theta^2} d\theta dk \right) 
\]

which can be simplified to obtain

\[
(\eta_t(r,\xi,t))_1 \approx -\frac{P_o}{2\pi^2 \rho} \Re \left( \frac{H}{\sqrt{2V} C(0)} \frac{1}{\sqrt{t^{\frac{3}{2}}}} \int_0^\infty \frac{\sqrt{\pi}}{\sqrt{t^{\frac{3}{2}}}} e^{-it(\sqrt{gH} - V)k - \frac{C_{gV}^r kk}{\pi} k^3} dk \right) 
\]
Introducing a stretched variable \( k = \sqrt{\frac{\xi}{(C_{gr})_{kk}}} \), eq. (4.46) becomes

\[
(\eta(t, \xi, t))_1 \approx - \frac{P_o}{2\pi^2 \rho} \Re e \left( \frac{H}{\sqrt{2V C(0)}} \left( \frac{2}{|(C_{gr})_{kk}|} \right) \frac{\sqrt{\pi}}{t^2} e^{-\frac{\xi^2}{4t^2}} \int_0^\infty i_s \frac{1}{\pi} e^{-u(B_s - \frac{\xi^2}{4t^2} \frac{1}{\pi} ds} \right)
\]

(4.47)

where \( B = \sqrt{\frac{2}{|(C_{gr})_{kk}|}} (\sqrt{gH} - V) \). Note that \( B \) can be positive or negative depending on whether \( V \) approaches \( \sqrt{gH} \) from above or below. Further evaluation using the substitution \( s = \frac{\xi}{t^3} \) yields

\[
(\eta(t, \xi, t))_1 \approx - \frac{P_o}{2\pi^2 \rho} \Re e \left( \frac{H}{\sqrt{2V C(0)}} \left( \frac{2}{|(C_{gr})_{kk}|} \right) \frac{\sqrt{\pi}}{t^2} e^{-\frac{\xi^2}{4t^2}} \int_0^\infty i_s \frac{1}{\pi} e^{-u(B_s - \frac{\xi^2}{4t^2} \frac{1}{\pi} ds} \right)
\]

(4.48)

The integral in eq. (4.48) can be expressed in terms of the function \( D_{i}(\lambda) \) defined in Appendix A. Using that definition, the estimate to eq. (4.16) can now be written as

\[
\eta(t, \xi, t) = \eta(t, \xi, t) \approx - \frac{P_o}{2\pi^2 \rho} \Re e \left( \frac{H}{\sqrt{2V C(0)}} \left( \frac{2}{|(C_{gr})_{kk}|} \right) \frac{\sqrt{\pi}}{t^2} e^{-\frac{\xi^2}{4t^2}} \frac{\Gamma(\frac{3}{4})}{t^3} D_{i}(t^\frac{3}{4} B) \right)
\]

\[
\eta(t, \xi, t) \approx - \frac{P_o}{2\pi^2 \rho} \Re e \left( \frac{\text{tanh}(k_{xz}) H}{C(k_{xz})} e^{ik_{xz} \cos \xi} k_{xz} e^{-\eta; \frac{1}{t}} \frac{1}{t^{\frac{3}{4}}} e^{-\sqrt{\frac{\xi}{kkk} / \sqrt{\xi_{\eta\eta}}}} + O(t^{-2} e^{-\eta; \frac{3}{4}}) \right)
\]

(4.49)

The analysis of the ice deflection for large time \( t \) will be carried out in the following section.

### 4.3 The behaviour of the ice deflection for large time

In section 4.2, the asymptotic estimates for the time derivative \( \eta; \) of the ice deflection were derived for a number of ranges of load speeds. These estimates
are to be used to determine the behaviour of the ice deflection $\eta$ for large time. There are two steps in the analysis. First, it is shown that for all load speeds, the ice deflection is bounded for finite time. Then, the asymptotic estimate for the time derivative of the ice deflection is used to infer the large time behaviour of the ice deflection.

Recall the expression for the ice deflection given in eq. (4.12). It needs to be shown that the integral in eq. (4.12) is bounded for any given finite time $T > 0$. In what follows, the bounds

$$\left| \frac{e^{-\iota \gamma t} - 1}{\iota \gamma} \right| \leq \frac{|e^{-\iota \gamma t}| + 1}{|\iota \gamma|} \leq \frac{2}{|\iota \gamma|}$$

(4.50)

and

$$\left| \frac{e^{-\iota \gamma t} - 1}{\iota \gamma} \right| = \left| \cos(\iota \gamma t) - \iota \sin(\iota \gamma t) - 1 \right| \leq t \left| \frac{\sin^2(\frac{\iota \gamma t}{2})}{\iota \gamma t} \right| + t \left| \frac{\sin(\iota \gamma t)}{\iota \gamma t} \right| \leq 2t$$

(4.51)

where $j = 1, 2$ are used. Corresponding to these two bounds, the $(k_1, k_2)$-plane is divided into two regions $D_0$ and $\overline{D_0}$, the complement of $D_0$, where

$$D_0 = \{(k_1, k_2)\mid -A_1 < k_1 < A_2 \text{ and } -B_1 < k_2 < B_2\}$$

(4.52)

and the constants $A_1, A_2, B_1, B_2$ are taken so that $|\iota \gamma_1|$ and $|\iota \gamma_2|$ are greater than 1 in $\overline{D_0}$. Since $\iota \gamma = kC + k_1 \iota V = \pm 1$ implies that $kC < k_1 \iota V + 1$ and $C$ grows as $k^{\frac{1}{4}}$, it follows that $D_0$ is a bounded region.

Using the bounds given by eq. (4.50) and eq. (4.51), the integral in (4.12) can now be written as
4.3 The behaviour of the ice deflection for large time

\[ |\eta(X, z, t)| \leq \frac{P_o}{4\pi^2} \int \int_{D_0} \left( \left| e^{-i\omega_1 t} - 1 \right| + \left| e^{i\omega_2 t} - 1 \right| \right) \left| \frac{\tanh(kH)}{2\rho C} \right| dk_1 dk_2 \]

\[ + \frac{P_o}{4\pi^2} \int \int_{D_0} \left( \left| e^{-i\omega_1 t} - 1 \right| + \left| e^{i\omega_2 t} - 1 \right| \right) \left| \frac{\tanh(kH)}{2\rho C} \right| dk_1 dk_2 \]

\[ \leq \frac{tP_o}{2\pi^2} \int \int_{D_0} \left[ \left| \frac{1}{\rho C} \right| \right] dk_1 dk_2 + \frac{P_o}{4\pi^2} \int \int_{D_0} \left( \left| \frac{1}{\omega_1} \right| + \left| \frac{1}{\omega_2} \right| \right)\left| \frac{1}{\rho C} \right| dk_1 dk_2. \quad (4.53) \]

Since \( C \) is non-zero and \( \omg \approx k^{\frac{1}{2}}, \omega_j \approx k^{\frac{1}{2}} \) as \( k \to \infty \), it follows that each of the integrals in eq. (4.53) is bounded, and consequently, the bound of the ice deflection can be written as

\[ |\eta(X, z, t)| \leq tM_0 + M_1 \quad (4.54) \]

where \( M_0 \) and \( M_1 \) are the bounds on the integrals in eq. (4.53).

To determine the large time behaviour of \( \eta \), note that

\[ \eta(r, \xi, t) = \int_0^t \eta(r, \xi, \tau) d\tau = \int_0^T \eta(r, \xi, \tau) d\tau + \int_T^t \eta(r, \xi, \tau) d\tau \quad (4.55) \]

where \( t \) is taken so that \( t > T \). Let \( \eta_0 \) denote the large \( t \) estimate of \( \eta \). Then for sufficiently large \( T \), eq. (4.55) can be rewritten as

\[ \eta(r, \xi, t) = \eta(r, \xi, T) + \int_T^t \eta_0(r, \xi, \tau) d\tau + \int_T^t (\eta(r, \xi, \tau) - \eta_0(r, \xi, \tau)) d\tau \quad (4.56) \]

Note that the third term on the right hand side of eq. (4.56) is the error term given by eq. (4.20). In the next sections, the behaviour of the ice deflection \( \eta \) is analyzed at different load speeds by using the asymptotic estimates of \( \eta_0 \) obtained in section 4.2.
4.3 The behaviour of the ice deflection for large time

4.3.1 Case 1: Load speed \( \dot{V} = (C_{gr})_{\text{min}} \)

For this case, the asymptotic estimates of \( \eta \) given by eq. (4.30) is integrated by applying integration by parts. Substituting the result into eq. (4.56) yields

\[
\eta(r, \xi, t) \approx \eta(r, \xi, T) - \frac{P_o \tanh(k_sH)}{3\pi^2 \rho \tilde{C}(ks)} \frac{\sqrt{k_s}}{\sqrt{2V} \sqrt{(C_{gr})_{kk}/6}} \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right) \left( 1 + \frac{5}{6v^2} \left( e^{-iTV} - e^{-iT\frac{c}{v}} \right) \right) + O(t^{-\frac{2}{3}} e^{-iTV}).
\]

Taking only the leading term of the asymptotic estimates, the approximation to the ice deflection can be written as

\[
\eta(r, \xi, t) \approx \eta(r, \xi, T) - \frac{P_o \tanh(k_sH)}{3\pi^2 \rho \tilde{C}(ks)} \frac{\sqrt{k_s}}{\sqrt{2V} \sqrt{(C_{gr})_{kk}/6}} \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right) \left( 1 + \frac{5}{6v^2} \left( e^{-iTV} - e^{-iT\frac{c}{v}} \right) \right) + O(t^{-\frac{2}{3}} e^{-iTV}).
\]

For a line load moving with speed \( \dot{V} = (C_{gr})_{\text{min}} \). Schulkes and Sneyd [25] showed that the time-dependent ice displacement decays as \( t^{-\frac{1}{3}} \). For a point load, eq. (4.58) shows that the ice displacement decays more quickly, as \( t^{-\frac{3}{5}} \).

4.3.2 Case 2: Load speed \( (C_{gr})_{\text{min}} < \dot{V} < C_{\text{min}} \) and \( C_{\text{min}} < \dot{V} < \sqrt{gH} \)

For this case, the asymptotic estimate given by eq. (4.32) is used to evaluate the behaviour of the ice deflection \( \eta \). Using eq. (4.56) where the integral of \( \tilde{\eta} \) is evaluated by applying integration by parts to eq. (4.32), the ice deflection \( \eta \) can be shown to be
4.3 The behaviour of the ice deflection for large time

\[ \eta(r, \xi, t) \approx \eta(r, \xi, T) \]

\[ - \frac{P_o}{\pi \rho} \Re \left( \frac{\tanh k_{s1} H e^{i k_{s1} r} r \cos \xi k_{s1}}{C(k_{s1})} \frac{1}{\sqrt{\zeta_{kk}} \sqrt{\zeta_{\theta \theta}}} \left[ \frac{1}{i \zeta} \left( \frac{e^{-i \zeta \dot{\xi}}}{T} - \frac{e^{-i \zeta \dot{\xi}}}{t} \right) \right] \right) \]

\[ - \frac{P_o}{\pi \rho} \Re \left( \frac{\tanh k_{s2} H e^{i k_{s2} r} r \cos \xi k_{s2}}{C(k_{s2})} \frac{1}{\sqrt{\zeta_{kk}} \sqrt{\zeta_{\theta \theta}}} \left[ \frac{1}{i \zeta} \left( \frac{e^{-i \zeta \dot{\xi}}}{T} - \frac{e^{-i \zeta \dot{\xi}}}{t} \right) \right] \right) + O(t^{-2} e^{-it \zeta}). \] (4.59)

Eq. (4.59) shows that the ice displacement for a moving point load decays as \( t^{-1} \). This can be compared to a line load for the same range of load speeds where the ice displacement decays as \( t^{-\frac{1}{2}} \) (cf. Schukles and Sneyd [25]).

4.3.3 Case 3: Load Speed \( V = C_{\text{min}} \)

Proceeding as in sections 4.3.1 and 4.3.2 and applying the techniques to eq. (4.56) and eq. (4.34), the large time behaviour of the ice deflection for the load speed \( C_{\text{min}} \) is given by

\[ \eta(r, \xi, t) \approx \eta(r, \xi, T) \]

\[ - \frac{P_o}{\pi \rho} \Re \left( \frac{\tanh k_{s1} H e^{i k_{s1} r} r \cos \xi k_{s1}}{C(k_{s1})} \frac{1}{\sqrt{\zeta_{kk}} \sqrt{\zeta_{\theta \theta}}} \left[ \frac{1}{i \zeta} \left( \frac{e^{-i \zeta \dot{\xi}}}{T} - \frac{e^{-i \zeta \dot{\xi}}}{t} \right) \right] \right) + O(t^{-2} e^{-it \zeta}) \]

\[ - \frac{P_o}{\pi \rho} \Re \left( \frac{\tanh k_{\text{min}} H e^{i k_{\text{min}} r} r \cos \xi k_{\text{min}}}{C(k_{\text{min}})} \frac{1}{\sqrt{\zeta_{kk}(k_{\text{min}}, 0)} \sqrt{\zeta_{\theta \theta}(k_{\text{min}}, 0)}} \left( \ln t - \ln T \right) \right) + O(t^{-1}). \] (4.60)

Note that in the limit \( t \to \infty \) the ice deflection grows with time as \( \ln t \). In the
case of a line load. Kheisin [14] found that the ice deflection grows as $t^{\frac{1}{2}}$, which was also confirmed by Schulkes and Sneyd [25].

4.3.4 Case 4: Load Speed $V = \sqrt{gH}$

For this case, the asymptotic estimate given by eq. (4.38) is used to analyze the behaviour of the ice deflection $\eta$. Proceeding as in sections 4.3.1-4.3.3, the ice deflection can be shown to be

$$\eta(r, \xi, t) \approx \eta(r, \xi, T)$$

$$= \frac{-P_o}{2\pi^2\rho} \mathcal{R}e \left( i \frac{H e^{-t\xi}}{C(0)} \left( \frac{6}{|C_{yykk}(0)|} \right)^{\frac{3}{2}} \frac{1}{\sqrt{2V}} \left( \frac{1}{t^{1/3}} - \frac{1}{T^{1/3}} \right) \Gamma \left( \frac{5}{6} \right) \Gamma \left( \frac{1}{2} \right) \right) + O(t^{-1})$$

$$= \frac{-P_o}{\pi\rho} \mathcal{R}e \left( \frac{\tanh k_s \rho H e^{ik_s \rho \cos \xi \cdot k_s}}{C(k_s)} \left( \frac{1}{\sqrt{\varepsilon_{kk}} \sqrt{\varepsilon_{\theta\theta}}} \right) \left[ \frac{\epsilon^{-2i\zeta}}{t} - \frac{e^{-2\zeta}}{t} \right] \right) + O(t^{-2}e^{-2\zeta}). \quad (4.61)$$

This shows that $\eta$ decays with time. The contribution to the integral from the stationary point $(0, 0)$ decays as $t^{-\frac{1}{2}}$. The contribution to the integral from the stationary point $(k_s, 0)$ decays as $t^{-1}$. For a line load, Schulkes and Sneyd [25] showed that the ice displacement grows as $t^{\frac{1}{4}}$.

4.3.5 Case 5: Load Speed $V > \sqrt{gH}$

Proceeding as in section 4.3.1 and noting that the first integral in eq. (4.42) decays exponentially i.e. much faster than the second integral. the leading contribution to the ice deflection can be shown to be
\[
\begin{align*}
\eta(r, \xi, t) & \approx \eta(r, \xi, T) \\
& - \frac{P_0}{2\pi^2 \rho} \Re \left( \frac{\tanh k_{s_2} H e^{ik_{s_2}r \cos \xi} k_{s_2}}{C(k_{s_2})} \frac{1}{\sqrt{|c_{kk1}|} \sqrt{|c_{gg1}|}} \left[ \frac{1}{it} \left( e^{-i\frac{\tau \dot{\zeta}}{T}} - e^{-i\frac{\tau \dot{\zeta}}{T}} \right) \right] \right) + O(t^{-2} e^{-i\frac{\tau \dot{\zeta}}{T}}). \quad (4.62)
\end{align*}
\]

This indicates that the waves behind the load disappear much faster than the short waves in front of the load. The integral in eq. (4.62) shows that the contribution from the stationary point \((k_{s_2}, 0)\), which corresponds to the short waves, decays as \(t^{-1}\).

### 4.3.6 Case 6: Load Speed \(v^*\) approaches \(\sqrt{gh}\)

The contribution of the stationary point near the origin to the ice deflection \(\eta\) is obtained by integrating eq. (4.48).

\[
\begin{align*}
(\eta(r, \xi, t))_1 & \approx - \frac{P_0}{2\pi^2 \rho} \Re \left( \frac{H}{\sqrt{2V}} C(0) \left( \frac{2}{|(C_{gr})_{kk1}|} \right) \frac{3}{2} \sqrt{\pi} e^{i\frac{\tau \dot{\zeta}}{T}} i\frac{\Gamma(\frac{3}{2})}{\sqrt{3}} \int_{T}^{t} \left[ \frac{(Di(t \dot{\zeta} B) - 1)}{\tau^T} + \frac{1}{\tau^T} \right] d\tau \right) \\
& = - \frac{P_0}{2\pi^2 \rho} \Re \left( \frac{H}{\sqrt{2V}} C(0) \left( \frac{2}{|(C_{gr})_{kk1}|} \right) \frac{3}{2} \sqrt{\pi} e^{i\frac{\tau \dot{\zeta}}{T}} i\frac{\Gamma(\frac{3}{2})}{\sqrt{3}} \left[ \frac{3}{2} \sqrt{|B|} \int_{\frac{r \dot{\zeta} B}{|\lambda|}}^{t \dot{\zeta} B} \frac{(Di(\lambda) - 1)}{|\lambda|^T} d\lambda - 3 \left( \frac{1}{t^T} - \frac{1}{T^T} \right) \right] \right) \quad (4.63)
\end{align*}
\]

where a substitution \(\lambda = \tau^{\frac{3}{2}} B\) has been used and \((\eta)_1\) denotes the contribution to \(\eta\) from the stationary point near the point \((0, 0)\).

For \(B = 0\) (corresponding to \(v^* = \sqrt{gh}\)), the integral term in eq. (4.63) becomes zero and consequently, the transient part of eq. (4.63) decays with time.
as $t^{-1}$. Furthermore, the integral term in eq. (4.63) is uniformly bounded since the integral $\int_{-\infty}^{\infty} (D(t) - 1)/|\lambda|^{3/2} d\lambda$ is bounded.

The ice deflection for the load speed $V$ near $\sqrt{gH}$ can now be expressed as

$$
\eta(r, \xi, t) \approx \eta(r, \xi, T)
$$

$$
- \frac{P_o}{2\pi^2 \rho} \Re \left( \frac{H}{\sqrt{2V C(0)}} \left( \frac{2}{||C_{fr}||} \right) \frac{1}{\sqrt{\pi e^{i\varphi} \frac{i\Gamma(\frac{3}{2})}{\sqrt{3}}} \left[ 3 \frac{2}{V^3 B} t^{1/2} \frac{D(t) - 1}{|\lambda|^{3/2}} d\lambda - 3 \left( \frac{1}{t^{1/2}} - \frac{1}{T^{1/2}} \right) \right] \right)
$$

$$
- \frac{P_o}{\pi \rho} \Re \left( \frac{\tanh k_{s2} H e^{ik_{s2} r \cos \xi} k_{s2}}{C(k_{s2})} \frac{1}{\sqrt{u_{kk}^\ast \sqrt{u_{\theta\theta}^\ast}}} \left[ \frac{1}{iU^\ast} \left( \frac{e^{-iUT^\ast}}{T} - \frac{e^{-iT^\ast}}{t} \right) \right] + O(t^{-2} e^{-it^\ast}) \right). \quad (4.64)
$$

**4.4 Numerical Comparisons**

To support the results obtained in the previous section, the integral for the ice deflection $\eta$ in eq. (4.12) is calculated numerically using the Fast Fourier Transform, which will be referred to as the FFT method. The evaluation of a double integral is carried out using a 1024 by 1024, a 2048 by 2048, or a 4096 by 4096 grid. However, to achieve acceptable results, in most calculations it is necessary to use the finest grid in computing the integral given by eq. (4.12) with the FFT method.

The asymptotic estimates given in eqs. (4.59), (4.60), and (4.61) include a constant term which comes from the integration and corresponds to the contribution from the steady state term in the original integral given in eq. (4.12). Thus, this constant term needs to be taken into account before a comparison with the results from the FFT can be made. In plotting the results, the values obtained from the asymptotic formula are shifted so that they agree with the values from the FFT at a fixed point of time (in the caption of each figure, the value of the time where
agreement is forced is given). Several results are illustrated in the figures given in the following pages. The parameters which are taken from Takizawa [31] are $E = 5 \times 10^8$, $\nu = \frac{1}{3}$, $h = 0.175$ m, $g = 9.8$ m/sec$^2$. The critical speed $C_{\text{min}}$ is at 6.2 m/sec and the second critical speed $C_{gH}$ is at 8.2 m/sec. In all cases, the plots of the computed results from the FFT method and the result from the asymptotic formula agree very well at the point of observation.

The computations show that as the distance from the center of the load to the point of observation increases, it takes a longer time for the results obtained using the asymptotic formula to match the results obtained using the FFT method. For a load speed of 7 m/sec and an observation point 25 m behind the load, it takes about 85 seconds before the asymptotic solution agrees with that of the FFT method. As the distance is increased to 50 m, it takes about 120 to 130 seconds before the two solutions agree. Plotting the asymptotic estimate for the ice deflection for various locations around the point load also shows that for a given load speed, the rate of decay or growth of the wave amplitude is independent of the direction of the observation point relative to the source.

At the load speed $C_{gH}$, the plot for a point load shows that the time-dependent part of the ice displacement decays as $t \to \infty$. Consequently, the steady state solution found by Milinazzo et al. [19] for a two-dimensional load is attainable. In that paper, the steady solution is finite at the load speed $C_{gH}$. This is in contrast with a line load, where Schulkes and Sneyd [25] found that the ice displacement grows as $t^{\frac{1}{3}}$. 
Ice deflection at the load speed $V = 6.2\, \text{m/sec} = C_{\text{min}}$

Figure 4.2: The ice deflection versus time at the load speed $C_{\text{min}}$. The plot shows the ice displacement (vertical axis) against time (in seconds - horizontal axis) at the observation point $x = -18.0\, \text{m}$ and $z = 0.0\, \text{m}$ (behind the load). The result obtained using the Fast Fourier Transform is shown as the solid line, and the result using the asymptotic approximation is shown as the dashed line. The two plots are matched at $t = 40\, \text{sec}$. The parameters are $E = 5 \times 10^8$, $\nu = \frac{1}{3}$, $h = 0.175\, \text{m}$, $g = 9.8\, \text{m/sec}^2$. 
4.4 Numerical Comparisons

Ice deflection at the load speed $V = 8.2 \text{ m/sec} \left(= C g H \right)$

Figure 4.3: The ice deflection versus time at the load speed $C g H$. The plot illustrates the ice displacement (vertical axis) against time (in seconds - horizontal axis) at the observation point $x = 18.0 \text{ m}$ and $z = 0.0 \text{ m}$ (ahead of the load). The solid line represents the result obtained using the Fast Fourier Transform, and the dashed line represents the result obtained using the asymptotic approximation. The two plots are matched at $t = 90 \text{ sec}$. The parameters are $E = 5 \times 10^8$, $\nu = \frac{1}{3}$, $h = 0.175 \text{ m}$, $g = 9.8 \text{ m/sec}^2$. 
4.4 Numerical Comparisons

Ice deflection at the load speed \( v = 8.2 \text{ m/sec} \) (= \( C_g h \))

Figure 4.4: The ice deflection versus time at the load speed \( C_g h \). The plot illustrates the ice displacement (vertical axis) against time (in seconds - horizontal axis) at the observation point \( x = -18.0 \text{ m} \) and \( z = 0.0 \text{ m} \) (behind the load). The solid line represents the result obtained using the Fast Fourier Transform, and the dashed line represents the result obtained using the asymptotic approximation. The two plots are matched at \( t = 170 \text{ sec} \). The parameters are \( E = 5 \times 10^8, \nu = \frac{1}{3}, h = 0.175 \text{ m}, g = 9.8 \text{ m/sec}^2 \).
Ice deflection at the load speed $V = 5.5 \text{ m/sec}$

Figure 4.5: The ice deflection versus time at the load speed $(C_{yr})_{\text{min}} < V < C_{\text{min}}$. The plot shows the ice displacement (vertical axis) against time (in seconds - horizontal axis) at the observation point $r = 18.0 \text{ m}$ and $z = 18.0 \text{ m}$ (ahead of the load). The result obtained using the Fast Fourier Transform is shown as the solid line, whereas the result obtained using the asymptotic approximation is shown as the dashed line. The two plots are matched at $t = 90 \text{ sec}$. The parameters are $E = 5 \times 10^8$, $\nu = \frac{1}{3}$, $h = 0.175 \text{ m}$, $g = 9.8 \text{ m/sec}^2$. 
4.4 Numerical Comparisons

Ice deflection at the load speed $V = 7.0$ m/sec

Figure 4.6: The ice deflection versus time at the load speed $C_{min} < V < \sqrt{gH}$. The plot shows the ice displacement (vertical axis) against time (in seconds - horizontal axis) at the observation point $x = -18.0$ m and $z = 18.0$ m (behind the load). The solid line represents the result obtained using the Fast Fourier Transform (1024 by 1024 grid) and the dashed line represents the result obtained using the asymptotic approximation. The two plots are matched at $t = 85$ sec. The parameters are $E = 5 \times 10^8$, $\nu = \frac{1}{3}$, $h = 0.175$ m, $g = 9.8$ m/sec$^2$. 

\[ x \times 10^{-4} \]

\[ -10 \]

\[ 0 \]

\[ 5 \times 10^{-4} \]

\[ -5 \]

\[ 0 \]

\[ 5 \times 10^{-4} \]

\[ -5 \]

\[ 0 \]

\[ 5 \times 10^{-4} \]

\[ -5 \]
Ice deflection at the load speed \( V = 7.0 \text{ m/sec} \)

Figure 4.7: The ice deflection versus time at the load speed \( C_{\text{min}} < V < \sqrt{gH} \). The figure illustrates the ice displacement (vertical axis) against time (in seconds - horizontal axis) at the observation point \( x = -50.0 \text{ m} \) and \( z = 0.0 \text{ m} \) (behind the load). The result obtained using from the Fast Fourier Transform (2048 by 2048 grid) is shown as the solid line and the result obtained using the asymptotic approximation is shown as the dashed line. The two plots are matched at \( t = 125 \text{ sec} \). The parameters are \( E = 5 \times 10^5 \), \( \nu = \frac{1}{3} \), \( h = 0.175 \text{ m} \), \( g = 9.8 \text{ m/sec}^2 \).
Chapter 5

Summary

5.1 Summary

This research considered several aspects of the analysis of moving loads on floating ice sheets. The first two parts of the study considered steadily moving loads and examine the consequences of the approximations that are made about the ice water system. The third part dealt with the large time behaviour of the response of the ice sheet due to an impulsively started point load. The following is a summary of the results that were obtained.

In chapter 2. it was found that the assumption of infinite depth does not substantially simplify the analysis of the steady problem examined by Milinazzo et al. [19]. This is in spite of the fact that for the case of infinite depth the analysis of the dispersion relation reduces to the analysis of the roots of a polynomial (see eq. (2.19)), whereas for the case of finite depth the dispersion relation involves transcendental functions (see Milinazzo et al. [19], Davys et al. [5]). In fact, the presence of a branch cut that arises in the analysis of the case of infinite depth leads to a further complication. For both the case of finite depth and the case of infinite depth, the load speed $C_{\text{min}}$ represents a critical speed at which no steady solution is possible. However, the dispersion relations shown in Figures 2.1 and 2.2, indicate a different behaviour when the load speed tends to infinity. For the case of finite depth, a quiescent zone emerges behind the load at the load speed $C_{\text{gH}}$: but for the case of infinite depth, the waves behind the load persist for all load speeds greater than $C_{\text{min}}$. It is also found that the assumption
of infinite depth does not significantly influence the character and the form of the ice deflection as quantified by, for example, the wavelength of waves or the amplitude of the maximum ice deflection at a given speed (Figures 2.13 and 2.14). Moreover for each case, the ice deflection decays as the reciprocal of the square root of the distance from the load. It is concluded that there is no advantage in assuming that the depth is infinite.

In chapter 3, it was shown that taking the ice thickness into account by taking into consideration the variation of the pressure and applied stress vertically through the ice plate results in an equation that has the same form of an equation used by Kerr [11] for an ice plate that is subjected to in-plane forces. The relevant coefficient in the resulting differential equation depends on the square of the ice thickness. If the wave of interest has wavelength that is comparable to the ice thickness, then taking this coefficient into consideration might give a better approximation, but if the wave of interest has wavelength that is much larger than the ice thickness, then the effect of this coefficient on the characteristics of the wave is negligible.

The results of chapter 4 represent the most important contribution of this study. In that chapter, the time-dependent response to a moving point load for large time of the ice deflection was presented. The analysis given in Schulkes and Sneyd [25] for a line load could not be extended. Consequently, a new approach based on the first time-derivative of the ice deflection was developed. The result is a two-dimensional integral free of singularities to which the method of stationary phase can be applied. An approximation to the ice deflection is obtained by integrating the estimate for the time derivative. The Fast Fourier Transform was used to check the results for values of t where both the numerical and asymptotic estimates are valid. Most calculations were carried out on a 4096 by 4096 grid.

It was found that at all load speeds other than the load speed $C_{min}$ the transient part of the ice deflection decays with time. This includes the load speed
$C_{gH}$ where it was found that the transients decay with time as $t^{-\frac{1}{4}}$. Further analysis of the ice deflection near the load speed $C_{gH}$ also shows that the transient part of the ice deflection decays with time. This is in contrast to the case of a line load where the ice deflection grows with time as $t^{\frac{1}{3}}$ at the load speed $C_{gH}$ (cf. Schulkes et al. [25]). This implies that the steady solutions obtained by Milinazzo et al. [19] are attainable. At the load speed $C_{\text{min}}$, the ice deflection grows as $\ln t$ (see Figure 4.2) for large time. Again, this can be compared to the case of a line load where the ice deflection grows with time as $t^{\frac{1}{3}}$ at the critical speed $C_{\text{min}}$. All of these findings have important implications for the use of ice sheets for transportation.

The result in chapter 4 also shows that at a given load speed, the rate of decay or growth of the wave amplitude does not depend on the direction of the observation point relative to the source. However, the further away the observation point is from the source, the longer it takes for the asymptotic estimate to agree with the numerical result from the Fast Fourier Transform. This implies that the asymptotic estimates for large time found in this chapter are not uniform in terms of the distance from the source.
Bibliography


Appendix A

The function $Di(\lambda)$

Below, the properties of the integral

$$
\int_0^\infty s^{\frac{3}{2}} e^{-i\lambda s - \frac{c^2}{3}} ds
$$

(A.1)

where $\lambda$ is a real constant, are determined. The integral in eq. (A.1) can be rewritten by using contour integration along the closed path $I_0 = I_1 + I_R + I_2$ in the complex $s$-plane where $I_1 = \{s : s = r, 0 < r < R\}$, $I_R = \{s : s = Re^{i\theta}, 0 \leq \theta \leq \frac{\pi}{6}\}$, and $I_2 = \{s : s = re^{i\frac{\pi}{3}}, 0 < r < R\}$. It is straightforward to show that the contribution from the integration along $I_R$ goes to zero as $R \to \infty$. Consequently,

$$
\int_0^\infty s^{\frac{3}{2}} e^{-i\lambda s - \frac{c^2}{3}} ds = \int_0^\infty r^{\frac{3}{2}} e^{\frac{1}{2} r^2} e^{-i\lambda r e^{i\frac{\pi}{6}}} e^{i\frac{\pi}{6}} dr
$$

(A.2)

For $\lambda = 0$ the integral can be evaluated exactly using the substitution $\frac{r^2}{3} = z$.

The result is

$$
D_0 = \int_0^\infty s^{\frac{3}{2}} e^{\frac{1}{2} r^2} ds = \frac{e^{\frac{1}{12}}} {\sqrt{3}} \int_0^\infty z^{\frac{3}{2}} e^{-\frac{1}{6} z}dz = \frac{e^{\frac{1}{12}}} {\sqrt{3}} \Gamma(\frac{5}{6}).
$$

(A.3)

Let $Di(\lambda)$ be defined as

$$
Di(\lambda) = \frac{1}{D_0} \int_0^\infty s^{\frac{3}{2}} e^{-i\lambda s - \frac{c^2}{3}} ds.
$$

(A.4)

In the following sections, estimates of $Di(\lambda)$ will be obtained for both small and large values of $\lambda$. To obtain asymptotic estimates for large $\lambda$, the cases of positive and negative $\lambda$ are analyzed separately.
A.0.1 Case 1: Small values of $\lambda$.

To obtain an expansion for small $\lambda$, eq. (A.2) is used to rewrite $D_i(\lambda)$ in the form

$$D_i(\lambda) = \frac{\epsilon^{\frac{\lambda^2}{2}}}{D_o} \int_0^\infty r^\frac{1}{2} e^{-r} e^{-i\lambda r e^{i\pi}} dr.$$ (A.5)

Expanding the integrand for small values of $\lambda$ gives

$$D_i(\lambda) = \frac{\epsilon^{\frac{\lambda^2}{2}}}{D_o} \int_0^\infty r^\frac{1}{2} e^{-r} \left(1 - i\epsilon^{\frac{\lambda^2}{2}} r \lambda - \frac{\epsilon^{\frac{\lambda^2}{2}}}{2} r^2 \lambda^2 + \ldots\right) dr$$

$$= \frac{\epsilon^{\frac{\lambda^2}{2}}}{D_o} \left( \Gamma\left(\frac{3}{6}\right) - i\frac{\epsilon^{\frac{\lambda^2}{2}}}{\sqrt{3}} \frac{\lambda^2}{2} \lambda^2 + \ldots \right).$$

A.0.2 Case 2: Large values of $|\lambda|$, $\lambda > 0$.

Introducing the stretched variable $s = \sqrt{\lambda} z$, $D_i(\lambda)$ can be written as

$$D_i(\lambda) = \frac{\lambda^{\frac{3}{4}}}{D_o} \int_0^\infty z^\frac{1}{2} e^{i\lambda^{\frac{3}{4}} (1 - z - \frac{z^2}{2})} dz.$$ (A.6)

Using the method of stationary phase, eq. (A.6) can be evaluated to yield

$$D_i(\lambda) \approx \frac{\lambda^{\frac{3}{4}}}{D_o} \int_0^\infty e^{i\lambda^{\frac{3}{4}} (1 - z - \frac{z^2}{2})} dz = \frac{\sqrt{\pi \lambda}}{2D_o} e^{-t^2 \lambda^{\frac{1}{4}}} e^{i\pi \frac{1}{2}}.$$ (A.7)

A.0.3 Case 3: Large values of $|\lambda|$, $\lambda < 0$.

Using the stretched variable $s = \sqrt{|\lambda|} z$, the function $D_i(\lambda)$ can be written as

$$D_i(\lambda) = \frac{|\lambda|^{\frac{3}{4}}}{D_o} \int_0^\infty z^\frac{1}{2} e^{i\lambda^{\frac{3}{4}} (1 - z - \frac{z^2}{2})} dz.$$ (A.8)

The phase function of the integrand in eq. (A.8) has similar form to the Airy integral treated in Nayfeh [20] pp. 96-100. Nayfeh approximates the integral by deforming the path of integration to a path of steepest descent that passes through one of the saddle points $z = \pm i$. For the integral in eq. (A.8), this can be carried out by using contour integration along the closed path $I_C = I_{11} +$
Appendix A

\[ I_{R1} + I_{22} + I_{33} \] in the complex \( z \)-plane where \( I_{11} = \{ z : z = r, 0 < r < R \} \).
\( I_{R1} = \{ z : z = Re^{i\theta}, 0 \leq \theta \leq \frac{\pi}{6} \} \), \( I_{22} \dagger = \{ z : z = x + iy, 1 + \frac{1}{3}x^2 - y^2 = 0, 0 \leq x < \infty, y \geq 1 \} \), and \( I_{33} = \{ z : z = iy, 0 \leq y \leq 1 \} \). It is straightforward to show that the contribution from the integration along \( I_{R1} \) goes to zero as \( R \to \infty \).

Along the steepest descent path \( I_{22} \), the largest contribution comes from around the saddle point \( z = +i \). That contribution can be evaluated to yield
\[
\frac{|\lambda|^{\frac{3}{2}}}{D_o} \int_{I_{22}} \frac{z^{\frac{1}{2}} e^{i\lambda\frac{3}{4}(z^2 - \frac{1}{2})}}{dz} \approx \frac{i^{\frac{3}{2}}}{2D_o} \sqrt{\pi|\lambda|e^{-\frac{1}{4}|\lambda|^2}}. \tag{A.9}
\]

Along the path \( I_{33} \), the largest contribution comes from around the origin and that contribution is
\[
\frac{|\lambda|^{\frac{3}{2}}}{D_o} \int_{I_{33}} \frac{z^{\frac{1}{2}} e^{i\lambda\frac{3}{4}(z^2 - \frac{1}{2})}}{dz} \approx \frac{|\lambda|^{\frac{3}{2}}}{D_o} \int_0^1 s^{\frac{3}{4}} e^{-\lambda\frac{3}{8}(s - \frac{1}{2})} ds
\approx \frac{|\lambda|^{\frac{3}{2}}}{D_o} \int_0^\infty s^{\frac{3}{4}} e^{-\lambda\frac{3}{8}s} ds
\approx \frac{i^{\frac{3}{2}} \Gamma(\frac{3}{2})}{D_o |\lambda|^{\frac{3}{2}}}. \tag{A.10}
\]

Consequently, ignoring terms that are exponentially small, the leading term contribution of eqs. (A.9) and (A.10) to \( Di(\lambda) \) is
\[
Di(\lambda) \approx \frac{i^{\frac{3}{2}} \Gamma(\frac{3}{2})}{D_o |\lambda|^{\frac{3}{2}}}. \tag{A.11}
\]

Having estimated \( Di(\lambda) \) for both small and large values of \( \lambda \), it is clear that the integral
\[
\int_{-\infty}^\infty \left( \frac{Di(\lambda) - 1}{|\lambda|^{\frac{3}{2}}} \right) d\lambda \tag{A.12}
\]
is finite.

\(^1\) The path \( I_{22} \) is the steepest path taken along the branch of the hyperbola \( 1 + \frac{1}{3}x^2 - y^2 = 0 \) in the upper right half of the \( z \)-plane.