Mobile Guards' Strategies for Graph Surveillance and Protection

by

Virgélot Virgile
M.Sc., Université de Montréal, 2019
B.Sc., National Chiao Tung University, 2016

A Dissertation Submitted in Partial Fulfillment of the
Requirements for the Degree of

DOCTOR OF PHILOSOPHY

in the Department of Mathematics and Statistics

Mobile Guards' Strategies for Graph Surveillance and Protection

by

Virgélot Virgile
M.Sc., Université de Montréal, 2019
B.Sc., National Chiao Tung University, 2016

Supervisory Committee

Dr. Gary MacGillivray, Co-Supervisor
(Department of Mathematics and Statistics)

Dr. C. M. Mynhardt, Co-Supervisor
(Department of Mathematics and Statistics)

Dr. Sajin Koroth, Outside Member
(Department of Computer Science)

ABSTRACT

In this dissertation, we study the "one guard moves" model of both the eternal domination game and the eviction game.

We investigate the computational complexity of deciding whether k guards can respond to any sequence of attacks on an n-vertex graph G in both games. We show that this decision problem is EXPTIME-complete when neither G nor k is fixed, and when the initial configuration of the guards is given in both cases. We further show that in the case of the eternal domination game, if the guards can choose their initial configuration and the graph is directed, the decision problem remains EXPTIME-complete. We present an algorithm that decides the problem in time $O(kn^{k+2})$ for both games, marking a significant improvement over the previously fastest known algorithm which has time complexity $O(n^{2k+2})$. Our algorithm further determines the maximum number of attacks (potentially infinite) the guards can defend from each configuration.

We study the relationship between the eternal domination number of a graph and its clique covering number using both large-scale computation and analytic methods. In doing so, we answer two open questions of Klostermeyer and Mynhardt, and disprove a conjecture of Klostermeyer and MacGillivray (The Fundamental Conjecture [Eternal Domination: Criticality and Reachability, *Discuss. Math. Graph Theory* 37 (2017), no. 1, 63–77]). We prove that the smallest graph having its eternal domination number less than its clique covering number has ten vertices. We also demonstrate that for any integer $k \geq 2$, there exist infinitely many graphs having domination number and eternal domination number equal to k containing dominating sets which are not eternal dominating sets.

In addition, we show that there exists a function f such that for any integer $k \geq 1$, any graph with independence number k has eviction number at most $f(k)$. We further show that the eviction number of cographs can be computed in polynomial time.

Finally, we study the length of both games when played on an n-vertex graph on which are located k guards; that is, the maximum number of turns required before a winner can be decided.

Table of Contents

List of Tables

List of Figures

ACKNOWLEDGEMENTS

I am grateful to God for blessing me with health and strength, and for guiding me to encounter so many wonderful people during my academic journey.

I would like to thank my supervisors, Dr. Gary MacGillivray and Dr. Kieka Mynhardt for their support, guidance, understanding and patience throughout my studies. Their invaluable assistance during the various phases of my life has contributed significantly to my academic growth and to my personal development. I am profoundly grateful for their mentorship and could not have asked for more.

I would also like to thank Dr. Wendy Myrvold for her comments on the dissertation proposal and for her provision of computing power which aided significantly in the execution of my code for this research.

My gratitude extends to my supervisory committee members, Dr. Sajin Koroth and Dr. Jan van Vuuren, for dedicating their time to review the dissertation and for providing encouraging feedback during and after the defense.

I am deeply in debt to my family for their endless love, encouragement, and understanding during this demanding period of study. To my parents Jacob and Belone for putting education at the forefront of our family values and for their unwavering support throughout every step of this journey. To my sisters Kathiana and Randale for their ongoing support and encouragement. To my wife Rose Daphney for always being there whenever I needed her and for her support through the highs and lows of this academic pursuit. To my sons Yoshua and Nathan for being the source of joy and inspiration, and for helping me see the bigger picture beyond academia, reminding me of the importance of family and love.

Finally, I extend my heartfelt appreciation to everyone I have had the opportunity to meet and who have had a positive influence on my life during my studies.

DEDICATION

To my parents, whose selfless sacrifices have been essential to my achievements.

Chapter 1

Introduction

This chapter serves as the starting point of our study, beginning with an overview of the problem. Subsequently, we proceed with a thorough review of the relevant literature. To conclude, we outline the dissertation's structure and our significant contributions.

1.1 Problem Overview

Life often challenges us to make the most impact with the least resources. Consider the following example in our modern world.

> "Sensor networks play an indispensable role. As a matter of fact, they are pivotal for accurate climate predictions, critical in urban planning and crucial for early alerts related to forest fires, earthquakes or floods. Moreover, they ensure our infrastructure's safety by monitoring the health of bridges, tunnels and buildings. While all of these networks use a multitude of sensors, a pressing challenge remains: How can we cover a whole area without using an excessive number of sensors?"

This problem can be represented mathematically using a graph which we denote by $G = (V(G), E(G))$. In this context, $V(G)$ is a set of points, referred to as *vertices*, that represent the locations in need of monitoring. Further, $E(G)$ is a set of lines, referred to as *edges*, that connect some specific pairs of these points. These edges represent either direct communication links or some other potential connections between these locations. Thus, should there be a sensor (or a guard) located on a vertex v of G, that guard covers not only v but all the other vertices that are connected (or adjacent) to v. These vertices are also known as the neighbours of v. Using this model, the challenge is to find a subset S of vertices of G such that every vertex of G is either in S or adjacent to a vertex in S. In graph theory, such a subset S of a graph G is known as a *dominating set* of G. In this regard, the *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

However, real-world systems seldom remain static. As situations evolve, the once optimal "dominating set" may face unforseen challenges. Revisiting our sensor network example, imagine a situation where a critical sensor must be moved from its original position due to maintenance needs, adverse weather conditions, or other complications. The challenge then becomes: "How do we reposition this sensor without compromising the overall coverage?" This is the point where the eternal domination game becomes relevant, as it provides valuable insights into a way to tackle these challenges.

1.2 Literature Review

Eternal domination is based on a graph protection model which was introduced by Burger, Cockayne, Grundlingh, Mynhardt, Van Vuuren and Winterbach in [9]. In graph protection models, a set of guards must be maintained on the vertices or edges of a graph in order to defend attacks on the graph. Modern studies of these models were initiated in the 1990s [4, 47, 48, 53] and began with papers in which the authors studied strategies to defend the Roman Empire.

In the standard version of the eternal domination game, which we will simply refer to as *eternal domination* (or *the eternal domination game*), the guards stay on the vertices of the graph (at most one guard on each vertex), each attack occurs at discrete time on a vertex not containing a guard and only one guard (not always the same) is allowed to move in response to an attack. A guard may only respond to an attack on a neighbouring vertex and the guard does so by moving to that vertex. The game starts with the guards choosing their opening configuration. This is followed by a sequence of turns, with each turn starting as the attacker selects a vertex on which there is no guard, and is followed by a response from a guard in the neighbourhood of that vertex. The guards win if they are able to respond to the sequence of attacks; otherwise, the attacker wins, that is, if at some time $t = 1, 2, 3, \dots$ there is no guard in the neighbourhood of the selected vertex. Assuming the guards move optimally, an *eternal dominating set* of a graph G is an opening configuration (a dominating set of G) from which they can defend any sequence of attacks on G. The *eternal domination number* of a graph G, denoted by $\gamma^{\infty}(G)$, is the minimum cardinality of an eternal dominating set of G.

The *all guards move* model of the game, known as *the m-eternal domination game* (where the m stands for multiple guards move), is a variant in which we allow all the guards to move at the same time in response to a single attack. This model was introduced later by Goddard, Hedetniemi and Hedetniemi in [23] and was motivated by the idea that some of the remaining guards might be better positioned to defend the next attack if they move to a neighbour. There are also variations depending on the number of guards that can occupy a vertex. Burger, Cockayne, Gründlingh, Mynhardt, Van Vuuren and Winterbach [9] showed that it does not make any difference in the game if we allow many guards to occupy a single vertex in the eternal domination game. Finbow, Gaspers, Messinger and Ottaway [19] showed that it can make a big difference in the game if many guards are allowed to occupy the same vertex in the m-eternal domination game.

A more recent model of the game, known as *the fractional eternal domination game*, was introduced by Devvrit, Krim-Yee, Kumar, MacGillivray, Seamone, Virgile and Xu in [17]. This model, inspired from the area of fractional domination (in graph theory), is a natural relaxation of the m-eternal domination game and offers the flexibility for multiple guards to respond to a single attack by sharing the responsibilities among them. To respond to an attack, each guard is now allowed to send fractions of its total weight to any of its neighbours. The fractional eternal domination game was also studied in [55].

We may sometimes want the guards in our network to maintain additional configurations at all times. For example, we might want to have a guard in the open neighbourhood of each guard so that they can watch over each other. In addition, the guards may need to communicate to each other over a connected network, in which case the subgraph induced by the vertices on which they are located must be connected. This motivates other variants of eternal domination and m-eternal domination such as *eternal total domination*, *m-eternal total domination*, *eternal connected domination* and *m-eternal connected domination* introduced by Klostermeyer and Mynhardt in [34]. In the eternal (or m-eternal) total domination game, for each guard located on a vertex v, there must be another guard located on a neighbour of v. Such a configuration which induces a dominating set is known as a *total dominating set*. In the eternal (or m-eternal) connected domination game, the subgraph induced by the vertices on which the guards are located must be connected. Likewise, such a configuration which induces a dominating set is known as a *connected dominating set*.

Another variant which has received attention is the *eviction game*, where only vertices containing a guard may be attacked. This game was introduced by Klostermeyer and MacGillivray in [31] and is the most accurate representation of the sensor network example in the previous section. In the standard version of the eviction game, which we will simply refer to as *eviction* (or *the eviction game*), the guards start by choosing their opening configuration, which must induce a dominating set. We consider the case where at most one guard is located on each vertex. At each time $t = 1, 2, 3, \dots,$ the attacker selects a vertex v on which there is a guard and the guard on v responds by moving to an unoccupied neighbour; that is, a neighbour of v on which there is no guard. If each of the neighbours of v is occupied; that is, if there is a guard on each of the neighbours of v, then the guard on v must stay put and cannot contribute to the dominating set for a time unit. When each of the neighbours of a vertex on which there is a guard g is occupied, we say that g is *surrounded*. Only the guard that is attacked is allowed to move to a neighbour. The guards win the game if they are able to maintain a dominating set in the graph after responding to each attack; otherwise, the attacker wins. Assuming the guards move optimally, an *eternal dominating set* of a graph G (in the eviction game) is an opening configuration (a dominating set of G) from which they can defend any sequence of attacks on G. The *eviction number* of a graph G, denoted by $e^{\infty}(G)$, is the minimum cardinality of an eternal dominating set of G in the eviction game.

The *all guards move model of the eviction game* is a variant in which each guard is allowed to move to a neighbour in response to an attack on a guard, provided that each vertex contains at most one guard after the response to the attack. Klostermeyer and MacGillivray [31] studied the one guard moves and the all guards move models of the eviction game. The eviction game was further studied in [33, 35].

Bagan, Joffard and Kheddouci [5] studied the eternal domination game on digraphs. The main difference between a graph and a *digraph* is the assignment of a direction to each of the edges, now referred to as *arcs*. Consequently, the guards must move in the direction of the arcs when the game is played on a digraph. A *dominating set* of a digraph G is a set $S \subseteq V$ such that any vertex $v \in V - S$ has an in-neighbour in S, that is, a vertex $u \in S$ such that uv is an arc of G. In this setting, the *directed eternal domination number* of a digraph G is the minimum number of guards necessary to respond to any sequence of attacks on G. Bagan, Joffard and Kheddouci generalised to digraphs many results that were previously proved for undirected graphs. They further introduced the notion of the *oriented eternal domination number* of a graph where the goal is to find an orientation of the graph which minimizes its directed eternal domination number.

In this dissertation, our primary focus lies on the "one guard moves" model for both eternal domination and eviction. To provide a solid foundation for our exploration, we first define some key terminologies and notations, and introduce some preliminary results that will be used frequently. We will then present some structural results from the literature in the following two subsections before exploring the algorithmic and complexity approaches to the game in the subsequent subsection.

Background

For any positive integers n and k, let $[n]$ denote the set $\{1, 2, 3, \ldots, n\}$ and let $\binom{[n]}{k}$ denote the set of k-subsets of the set $[n]$.

The *open neighbourhood* of a vertex v, denoted by $N(v)$, is the set of vertices that are adjacent to v. These vertices are known as the *neighbours* of v. The *closed neighbourhood* of v is the set $N[v] = N(v) \cup \{v\}$.

A vertex v of a graph G is said to be a *universal vertex* of G if v has degree $|V(G)| - 1$; that is, v is adjacent to every vertex in $V(G) - \{v\}$. A vertex v of a graph G is said to be *isolated* if v has degree 0; that is, if v has no neighbour in G. Put differently, an isolated vertex is a vertex with empty open neighbourhood.

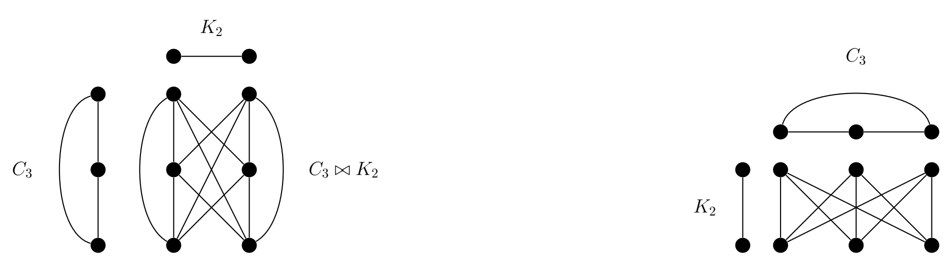
A *matching* M in a graph G is a set of edges of G, no two of which share a common vertex. A vertex v is said to be matched (or saturated, or covered) by M if v is an endpoint of an edge of the matching M. We say that M is a *perfect matching* of G if M saturates all the vertices of G.

A graph $H = (V(H), E(H))$ is said to be a *subgraph* of a graph $G = (V(G), E(G))$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The graph H is further said to be an *induced subgraph* of G if $E(H) = \{uv \in E(G) : u, v \in V(H)\}$ and a *spanning subgraph* of G if $V(H) = V(G)$.

The *disjoint union* of a graph G and a graph H, denoted by $G + H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. We denote the disjoint union of k disjoint copies of a graph G by kG. The *join* of a graph G and a graph H, denoted by $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

The *bow tie product* of a graph G with a graph H, denoted by $G \bowtie H$, is the graph with vertex set $\{(v_i, v_j) : v_i \in V(G), v_j \in V(H)\}$, where two vertices (v_i, v_j) and (v_i', v_j') are adjacent if and only if one of the following conditions holds:
(1) $v_i v_i' \in E(G)$ and $v_j = v_j'$,
(2) $v_i v_i' \in E(G)$ and $v_j v_j' \in E(H)$.
See Figure 1.1 for an example.

As shown in Figure 1.1, $G \bowtie H \ncong H \bowtie G$ in general.



(a) $C_3 \bowtie K_2$ (b) $K_2 \bowtie C_3$

Figure 1.1: Bow tie product of two graphs.

The *independence number* of a graph G, denoted by $\alpha(G)$, is the maximum cardinality of an independent set of G, that is, a set $S \subseteq V(G)$ such that for any $u, v \in S$, $uv \notin E(G)$. In contrast, the *clique number* of a graph G, denoted by $\omega(G)$, is the maximum cardinality of a clique of G, that is, a set $S \subseteq V(G)$ such that for any $u, v \in S$, $uv \in E(G)$.

It follows from the two preceding definitions that an independent set of a graph is, in fact, a clique of the complement graph. This relationship highlights the following connection between the independence number of a graph and the clique number of its complement.

Observation 1.1. For any graph G, $\alpha(G) = \omega(\overline{G})$.

The *clique covering number* of a graph G, denoted by $\theta(G)$, is the minimum cardinality of a clique covering of G, that is, a partition $\{V_1, V_2, V_3, \ldots, V_k\}$ of $V(G)$ such that each V_i induces a clique. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colours required to colour the vertices of G in a way such that no two adjacent vertices have the same colour. This type of colouring is commonly known as a *proper colouring* of G.

Observe that a proper colouring of G with k colours is obtained by partitionning $V(G)$ into $\{V_1, V_2, \ldots, V_k\}$ where each V_i induces an independent set. This relationship highlights the following connection between the clique covering number of a graph and the chromatic number of its complement.

Observation 1.2. For any graph G, $\theta(G) = \chi(\overline{G})$.

A vertex v of a graph G is said to be *critical*, with respect to the parameter θ, if $\theta(G - v) = \theta(G) - 1$. The graph G is said to be *vertex critical* (with respect to θ) if every vertex of G is a critical vertex (with respect to θ).

Let u, v be two vertices of a graph G such that uv is not an edge of G. Then, the edge uv is said to be *critical* (with respect to θ) if $\theta(G + \{uv\}) = \theta(G) - 1$. The graph G is said to be *edge critical* (with respect to θ) if for any $u, v \in V(G)$, the edge uv not being in G implies that uv is a critical edge (with respect to θ) of G.

A graph G is said to be *critical* (with respect to θ) if G is both vertex critical and edge critical (with respect to θ).

An *eternal dominating family* of a graph G (in the eternal domination game) is a collection of eternal dominating sets $D_1, D_2, D_3, \ldots, D_l$ of G that satisfy the following properties.

(1) For any $i, j \in [l]$, $|D_i| = |D_j|$.
(2) For any $i \in [l]$ and any $w \in V(G) - D_i$, there exist $v \in D_i$ and $j \in [l] - \{i\}$ such that $(D_i \cup \{w\}) - \{v\} = D_j$.

An *eternal dominating family* of a graph G (in the eviction game) is a collection of eternal dominating sets $D_1, D_2, D_3, \ldots, D_l$ of G that satisfy the following properties.

(1) For any $i, j \in [l]$, $|D_i| = |D_j|$.

(2) For any $i \in [l]$ and any $v \in D_i$, either $N[v] \subseteq D_i$, or there exist $w \in N[v] - D_i$ and $j \in [l] - \{i\}$ such that $D_i \cup \{w\} - \{v\} = D_j$.

A *minimal eternal dominating family* of a graph G (with respect to a specific model) is an eternal dominating family $\mathscr{F}$ of G such that $\mathscr{F} - \mathscr{S}$ is not an eternal dominating family for any $\mathscr{S} \subseteq \mathscr{F}, \mathscr{S} \neq \emptyset$. An *optimum eternal dominating family* of a graph G (with respect to a specific model) is an eternal dominating family of G of smallest cardinality in which every eternal dominating set is a minimum eternal dominating set of G.

The complete graph of order n, denoted by K_n, is the graph with vertex set $[n]$ and edge set $\binom{[n]}{2}$. The path of order n, denoted by P_n, is the graph with vertex set $\{v_1, v_2, v_3, \ldots, v_n\}$ and edge set $\{v_i v_j : i - j = \pm 1\}$. The cycle of order n (where $n \geq 3$), denoted by C_n, is the graph with vertex set $\{v_0, v_1, v_2, \ldots, v_{n-1}\}$ and edge set $\{v_i v_j : i - j \equiv \pm 1 \pmod{n}\}$.

A graph G is said to be *bipartite* if $V(G)$ can be partitionned into two parts X and Y in a way such that the subgraph induced by each part is empty; that is, it has no edge. The bipartite graph G is said to be complete if it contains every edge joining a vertex of X to a vertex of Y. The complete bipartite graph with partite sets of cardinality m and n is denoted by $K_{m,n}$.

A graph G is said to be *triangle-free* if its largest clique has cardinality at most two. The *circulant graph* $C_n[k_1, k_2, k_3, \ldots, k_l]$, where $\{k_1, k_2, k_3, \ldots, k_l\} \subseteq \mathbb{Z}^+$ and $1 \leq k_1 < k_2 < k_3 < \cdots < k_l \leq \lfloor \frac{n}{2} \rfloor$, is the graph with vertex set $\{v_0, v_1, v_2, \ldots, v_{n-1}\}$ such that two vertices v_i and v_j are adjacent if and only if $i - j \equiv \pm k_p \pmod{n}$ for some $p \in \{1, 2, 3, \ldots, l\}$. A *perfect graph* is a graph such that the chromatic number of any of its induced subgraphs is equal to the clique number of that subgraph.

A graph G is said to be *connected* if for every pair of vertices $u, v \in V(G)$, there is a path in G that connects u to v; otherwise, the graph G is said to be *disconnected*. A *component* of a graph G is a maximal connected induced subgraph of G; that is, a graph G_i such that G_i is a connected induced subgraph of G and for any $v \in G \backslash G_i$, v has no neighbour in G_i. The number of components of a graph G is denoted by $c(G)$.

We refer to a graph G as being a *smallest graph having some property* $\mathcal{P}$ if no graph H on fewer than $|V(G)|$ vertices has property $\mathcal{P}$.

To facilitate a clear understanding, terminologies that are specific to a particular section of the dissertation will be introduced in that section. Moreover, when a graph G is clear from context, we use $n, V, E, \omega, \chi, \gamma, \alpha, e^\infty, \gamma^\infty, \theta$ to denote respectively $|V(G)|, V(G), E(G), \omega(G), \chi(G), \gamma(G), \alpha(G), e^\infty(G), \gamma^\infty(G), \theta(G)$.

Eternal Domination

It is straightforward to see that the eternal domination number of a graph is the sum of the eternal domination number of its components; therefore, we restrict our attention to connected graphs.

Observe that we may obtain upper and lower bounds on $\gamma^\infty(G)$ by considering the problem on the induced and spanning subgraphs of G as follows.

Observation 1.3. Let G be a graph and let H be a subgraph of G.

1. [30] If H is an induced subgraph of G, then $\gamma^\infty(G) \geq \gamma^\infty(H)$.
2. [2] If H is a spanning subgraph of G, then $\gamma^\infty(G) \leq \gamma^\infty(H)$.

A direct consequence of Observation 1.3, stated more formally in Observation 1.4, is that increasing the number of guards in a graph never weakens their effectiveness in the eternal domination game.

Observation 1.4. If $k < |V(G)|$ guards can defend an arbitrarily long sequence of attacks on a graph G, then so can $k+1$ guards.

Observation 1.4 follows from the fact that one guard can stay put on any vertex of G while the k remaining guards defend the subgraph induced by the $|V(G)| - 1$ remaining vertices.

The reader can easily check that $\gamma^\infty(K_n) = 1$ and $\gamma^\infty(\overline{K_n}) = n$. Hence, by considering a minimum clique covering of G and playing the game independently on each clique of the covering we obtain an upper bound on $\gamma^\infty(G)$. Likewise, if we consider a maximum independent set S of G and play the game on the subgraph induced by S, we obtain a lower bound on $\gamma^\infty(G)$. In consequence, we have the following observation of Burger, Cockayne, Grundlingh, Mynhardt, Van Vuuren and Winterbach [9].

Observation 1.5 ([9])**.** For any graph G, $\alpha(G) \leq \gamma^\infty(G) \leq \theta(G)$.

The Weak Perfect Graph Theorem [40, 41] states that the complement of a perfect graph is perfect. As a direct consequence we have the following corollary.

Corollary 1.6 ([9])**.** *For any perfect graph G, $\alpha(G) = \gamma^\infty(G) = \theta(G)$.*

The exact values of γ^∞ for graphs that belong to some specific families such as paths, cycles and complete bipartite graphs are given in the following proposition.

Proposition 1.7 ([9])**.** *For any integers $n, m \geq 1$,* *(1)* $\gamma^\infty(P_n) = \lceil \frac{n}{2} \rceil$*,*

(2) $\gamma^\infty(C_3) = 1$ *and* $\gamma^\infty(C_n) = \lceil \frac{n}{2} \rceil$ *for any* $n \geq 4$*,*

(3) $\gamma^\infty(K_{m,n}) = \max\{m, n\}$*.*

The inequalities in Observation 1.5 can both be strict. Observe that the cycle of order 5 is a graph with $\alpha = 2 < 3 = \gamma^\infty$ and is the smallest graph for which $\alpha < \gamma^\infty$. As for the second inequality, the first proof of the existence of a graph with $\gamma^\infty < \theta$ is due to Goddard, Hedetniemi and Hedetniemi [23] and follows from Theorem 1.8 and its generalisation, Theorem 1.9, which shows that the eternal domination number of a graph is bounded above by a function of the independence number of the graph.

Theorem 1.8 ([23])**.** *If G is a graph such that $\alpha(G) = 2$, then $\gamma^\infty(G) \leq 3$.*

Theorem 1.9 ([29])**.** *For any graph G, $\gamma^\infty(G) \leq \binom{\alpha(G)+1}{2}$.*

It is known that for any integer $k \geq 2$ there exists a triangle-free graph G with chromatic number k. The first known construction of a family containing such graphs with arbitrarily large chromatic numbers is due to Blanche Descartes [15, 16]. Mycielski [44] described the construction of a family $\mathcal{F} = \{M_2, M_3, M_4, \ldots\}$ of triangle-free graphs starting with $M_2 = K_2$, where, for each $k \geq 3$, the graph M_k is obtained from the graph M_{k-1} and has chromatic number k (see M_4 in Figure 1.2a).

(a) M_4 (b) $\overline{M_4}$

Figure 1.2: The Grötzsch graph (a) and its complement (b).

Corollary 1.10 ([23])**.** *For any integer $k \geq 4$, $\gamma^\infty(\overline{M_k}) < \theta(\overline{M_k})$.*

Theorem 1.11 ([12])**.** *The Grötzsch graph (Figure 1.2a) is the unique smallest triangle-free graph with chromatic number at least* 4*.*

Corollary 1.12. *The complement of the Grötzsch graph (Figure 1.2b) is the unique smallest graph with independence number two and clique covering number at least four.*

Klostermeyer and Mynhardt [37] posed the following question.

Question 1.13 ([37])**.** Is the complement of the Grötzsch graph, a graph of order 11, the smallest graph with eternal domination number less than its clique covering number?

As we have seen before, the guards must always be located on the vertices of a dominating set of a graph G in order to defend a sequence of attacks on G. With this in mind, many researchers showed interest in characterizing graphs for which the domination number is equal to the eternal domination number. Klostermeyer and Mynhardt [36, 38] asked the following questions.

Question 1.14 ([38])**.** Let G be a graph with $\gamma(G) = \gamma^{\infty}(G)$. Is any minimum dominating set of G an eternal dominating set of G?

Question 1.15 ([36])**.** Does there exist a triangle-free graph G such that $\alpha(G) = \gamma^{\infty}(G) < \theta(G)$?

A question of Klostermeyer and MacGillivray, which was later stated as a conjecture by Klostermeyer and Mynhardt, is the following.

Conjecture 1.16 (The Gamma-Theta Conjecture [37, 38])**.** For any graph G, $\gamma(G) = \gamma^{\infty}(G)$ if and only if $\gamma(G) = \theta(G)$.

It is worth noting that while the guards may be located on the vertices of an eternal dominating set, it is not guaranteed that every guard can effectively respond to an attack on every vertex in its neighbourhood. This observation comes from the fact that using the wrong guard to defend an attack can render the resulting configuration no longer an eternal dominating set. This raises an intriguing question posed by Klostermeyer and MacGillivray in [32]:

> "Can every guard from any eternal dominating set, with a neighbouring vertex that does not contain a guard, always defend some attack?"

In their work, they affirmatively answer this question for graphs G such that $\alpha(G) = 2$ and $\gamma^{\infty}(G) = \theta(G)$. The Fundamental Conjecture, extending this question to general graphs, stating that the answer is positive in all cases, was first proposed by Klostermeyer and MacGillivray [32] and received further support from Klostermeyer and Mynhardt [38].

Conjecture 1.17 (The Fundamental Conjecture [32, 38])**.** Let G be a graph with $\delta \geq 1$ and minimum eternal dominating set D. For every vertex $v \in D$ with an unoccupied neighbour, there exists an eternal dominating set D' such that $D' = (D - \{v\}) \cup \{u\}$, where $u \in N(v) - D$.

Eviction

We now shift our focus to the eviction game. As in Section 1.2, unless otherwise specified, all graphs in this section are connected. We begin with the following proposition of Klostermeyer and MacGillivray, which serves as the analogue of Observation 1.4 in the eviction game.

Proposition 1.18 ([31]). *If $k < |V(G)|$ guards can defend an arbitrarily long sequence of attacks on a graph G, then so can $k+1$ guards.*

Let us now examine how the eviction number of a graph G relates to some other parameters such as the domination number, the independence number and the clique covering number of G.

Proposition 1.19 ([31]). *For any graph G, $\gamma(G) \leq e^{\infty}(G) \leq \theta(G)$.*

Since $\alpha(G)$ is a lower bound on $\gamma^{\infty}(G)$ and belongs to the interval $[\gamma(G), \theta(G)]$, it is reasonable to ask whether $\alpha(G)$ is also a lower bound on $e^{\infty}(G)$ but Observation 1.21 shows that this is not the case.

Proposition 1.20. *Let G be a graph with at least two universal vertices. Then $e^{\infty}(G) = 1$.*

Proof. Let G be a graph and let u, v be two universal vertices of G. By moving back and forth on the vertices u and v, one guard can dominate all of the vertices of G at each time $t = 1, 2, 3, \ldots$. □

Observation 1.21. Let G be the join of the graph K_2 with the graph $\overline{K_m}$ (see Figure 1.3). Then $\alpha(G) = m$ and $e^{\infty}(G) = 1$.

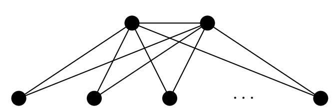


Figure 1.3: $K_2 \vee \overline{K_m}$

Observe that $\alpha(G)$ is a lower bound on $e^{\infty}(G)$ when G belongs to some specific class as shown below.

Proposition 1.22 ([31]). *For any triangle-free graph G, $e^{\infty}(G) \geq \alpha(G)$.*

Proposition 1.23 ([31]). *For any bipartite graph G, $e^{\infty}(G) = \alpha(G)$.*

We now explore the values of e^{∞} for graphs that belong to some specific families such as paths, cycles and complete bipartite graphs.

Proposition 1.24 ([31]). *For any integers $n, m \geq 1$,*

(1) $e^{\infty}(P_n) = \lceil \frac{n}{2} \rceil$,

(2) $e^{\infty}(C_3) = 1$, $e^{\infty}(C_5) = 2$ *and* $e^{\infty}(C_n) = \lceil \frac{n}{2} \rceil$ *for any* $n \neq 3, 5$,

(3) $e^{\infty}(K_{m,n}) = \max\{m, n\}$.

Note that Proposition 1.24 shows that $\alpha(G)$ is not an upper bound on $e^\infty(G)$, as $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor < \lceil \frac{n}{2} \rceil$ when n is an odd integer greater than 5.

Although $\alpha(G)$ is neither an upper bound nor a lower bound on $e^\infty(G)$, Klostermeyer and MacGillivray show that $e^\infty(G)$ is bounded for the first three values of α.

Observation 1.25 ([31])**.** Let G be a graph such that $\alpha(G) = 1$. Then $e^\infty(G) = 1$.

Proposition 1.26 ([31])**.** *Let G be a graph such that $\alpha(G) = 2$. Then $e^\infty(G) \leq 2$.*

Theorem 1.27 ([31])**.** *Let G be a graph such that $\alpha(G) = 3$. Then $e^\infty(G) \leq 5$.*

As we have seen in Theorem 1.9, the eternal domination number of a graph is bounded by a function of the independence number of the graph. As for the eviction number of the graph, no such bound is known in general. Motivated by the fact that the eviction number of the graphs that have been studied is relatively small, compared to their independence number and their eternal domination number, Klostermeyer and MacGillivray asked the following questions.

Question 1.28 ([31])**.** Does there exist a constant c such that $e^\infty(G) \leq c\alpha(G)$ for all graphs G?

Question 1.29 ([31])**.** Does there exist a graph G such that $\gamma^\infty(G) < e^\infty(G)$?

Algorithmic and Complexity Approaches

There is often interest in determining the number of resources (in most cases *time* and *space*) that is required to solve a problem. In the context of games, the central challenge usually revolves around demonstrating the presence or absence of a winning strategy for a specific player. It is established that the outcome of some games can be decided in polynomial time (assuming optimal play) while others necessitate exponential time. Notable examples of decision problems that fall into the second category include certain variations and generalisations of Chess [20], Go [51], Checkers [52], Cops and Robbers [25, 27]. In this section, we explore the algorithmic and complexity approaches that have been previously employed in addressing the eternal domination and the eviction games. We begin by introducing the specific decision problems we are interested in.

ETERNAL DOMINATION	
Instance:	A graph G and an integer k.
Question:	Is $\gamma^\infty(G) \leq k$?

ETERNAL DOMINATING SET	
Instance:	A graph G and a configuration D of guards.
Question:	Is D an eternal dominating set of G in the eternal domination game?

EVICTION	
Instance:	A graph G and an integer k.
Question:	Is $e^{\infty}(G) \leq k$?

EVICTION ETERNAL DOMINATING SET	
Instance:	A graph G and a configuration D of guards.
Question:	Is D an eternal dominating set of G in the eviction game?

A decision problem is said to fall into the class P if it can be solved by a deterministic algorithm in polynomial time. The class NP contains decision problems for which a proposed solution, although potentially hard to find, can be verified by a deterministic algorithm in polynomial time. A decision problem belongs to the class PSPACE if it can be solved by a deterministic algorithm in polynomial space. Finally, the class EXPTIME contains problems that can be solved by a deterministic algorithm in exponential time. For a deeper exploration of these fundamental concepts in computational complexity, the reader is encouraged to consult [3, 22].

Klostermeyer and MacGillivray [30, 31] presented an algorithm that decides in time $O(n^{2k+2})$ the set of configurations (which may be empty) from which k guards can respond to any sequence of attacks on an n-vertex graph in both eternal domination and eviction (also see [6] for a more general version). This shows that the four previous decision problems belong to the complexity class EXPTIME and can be decided in polynomial time when k is fixed. Furthermore, by Corollary 1.6, if the input is restricted to perfect graphs, ETERNAL DOMINATION is in P. Klostermeyer and MacGillivray showed that EVICTION is NP-hard, even when the input is restricted to graphs that belong to some specific classes. Klostermeyer [28] also studied the complexity of ETERNAL DOMINATING SET under narrow assumptions on the sequence of attacks. In particular, Klostermeyer showed that ETERNAL DOMINATING SET is hard for the second level of the polynomial hierarchy, a complexity class that lies between NP and PSPACE, and is NP-complete when the sequence of attacks is part of the input.

While this dissertation is not intended to provide an exhaustive analysis of all the results for every variant of the eternal domination game, readers seeking a broader exploration of the game and its various facets are encouraged to consult the surveys [37, 38], as well as additional references such as [9, 21, 23, 36, 39, 42, 55].

1.3 Outline of the Dissertation

The chapters of this dissertation are structured as follows.

In Chapter 2, we investigate the computational complexity of deciding whether k guards can respond to any sequence of attacks on a graph G of order n in both the eternal domination game and the eviction game. We begin by presenting, in Section 2.1, an algorithm that decides the problem in time $O(kn^{k+2})$ for both games. This improves on the fastest known algorithm so far which decides the problem in time $O(n^{2k+2})$, and is adapted from the main algorithm in [45] that determines whether k cops can capture a robber on a graph of order n in the standard game of cops and robbers. Following this, we show that when the initial configuration of the guards is given, the decision problem is EXPTIME-complete for both games, thereby conclusively determining the complexity of ETERNAL DOMINATING SET and EVICTION ETERNAL DOMINATING SET for graphs. Our next point in this chapter is that, when the input is a digraph, ETERNAL DOMINATION is EXPTIME-complete. This also conclusively determines the complexity of ETERNAL DOMINATION for digraphs. We conclude the chapter by describing the methods used in our large-scale computations.

Moving to Chapter 3, we study the standard eternal domination game from a structural point of view using both large-scale computation and analytic methods. We begin in Section 3.1 by considering the relationship between the eternal domination number of a graph and its clique covering number. In fact, we show in Section 3.1.1 that the smallest graph having its eternal domination number smaller than its clique covering number has ten vertices. In doing so, we answer Question 1.13 of Klostermeyer and Mynhardt. We further determine the complete set of 10-vertex and 11-vertex graphs having eternal domination numbers less than their clique covering numbers. Moreover, we show that the smallest triangle-free graph with this property has order 13, as does the smallest circulant graph. We also find the complete set of circulant graphs of order 20 or less with eternal domination numbers less than their clique covering numbers and describe a method to generate an infinite family of triangle-free graphs and an infinite family of circulant graphs with eternal domination numbers less than their clique covering numbers. In Section 3.2, we show that the answer to Question 1.14 is no by presenting a class of graphs with domination numbers equal to their eternal domination numbers which contains minimum dominating sets that are not eternal dominating sets. Our results in Section 3.1 and Section 3.2 are also published in [42]. Moving to Section 3.3, we provide an infinite family of counterexamples to the Fundamental Conjecture. We conclude the chapter by a study on the length of the game; that is, the minimum number of turns required in the game before a winner can be decided.

In Chapter 4, we turn our attention to the eviction game, which we also study from a structural perspective. In fact, we show in Section 4.1 that for any integer $k \geq 1$, there exists a constant, which we denote by $f(k)$, such that any graph with independence number k has eviction number at most $f(k)$. We further show in Section

4.2 that the eviction number of cographs can be computed in polynomial time. In line with the approach taken in Chapter 3, we conclude Chapter 4 with a study on the length of the game.

In Chapter 5, we conclude the dissertation by providing a summary of our significant results and a list of open questions for future research.

Chapter 2

Algorithms, Complexity and Computational Methods

This chapter is dedicated to a study of the games from algorithmic and complexity aspects. We begin by describing two algorithms, each of time complexity $O(kn^{k+2})$, that decide whether an n-vertex graph G has an eternal dominating set of size at most k respectively in the eternal domination game and the eviction game. Note that our algorithms are adapted from the main algorithm in [45] that determines whether k cops can capture a robber on a graph of order n in the standard game of cops and robbers. Then, we study the complexity of each of the decision problems that were introduced in Chapter 1. Finally, we describe our large-scale computations conducted for the purpose of verifying some propositions that relate the eternal domination number and the eviction number of a graph to the other well-studied parameters such as the domination, independence and clique covering numbers of the graph.

2.1 A Faster Algorithm

In this section, we describe our algorithms that determine whether an n-vertex graph G has an eternal dominating set of size at most k. One should note that our algorithms also determine whether a given configuration of guards is an eternal dominating set of the graph. In the negative case, it further determines the maximum number of attacks the guards can defend from the given configuration.

It should be emphasized that Algorithm 1 is adapted for the eternal domination game while Algorithm 2 is adapted for the eviction game. We have used bold text to draw attention to the distinction between the two algorithms.

Algorithm 1. Deciding whether $\gamma^\infty(G) \leq k$.

Input: A graph G of order n and an integer k, where $0 < k \leq n$.
Output: *True* if $\gamma^\infty(G) \leq k$ and *False* if $\gamma^\infty(G) > k$.

(0) In the initialization phase, create two arrays, A and T, that are indexed by the set $\binom{[n]}{k} = \{S_1, S_2, S_3, \ldots, S_{\binom{n}{k}}\}$ of all possible configurations of k guards in G. Additionally, create two lists L_1 and L_2, and a counter $t \leftarrow 1$.

(1) For each configuration S_i in $\binom{[n]}{k}$:
  (a) If S_i is a dominating set of G, then set $A[S_i]$ to 1.
  (b) If S_i is not a dominating set of G, then:
    (i) Set $A[S_i]$ to 0.
    (ii) Set $T[S_i]$ to t.
    (iii) Add S_i to L_1.

(2) Create a matrix $M = (m_{i,j})$ with $\binom{n}{k}$ rows and n columns. Then, iterate over each row i in $[\binom{n}{k}]$ and each column j in $[n]$:
  (a) If $j \in S_i$, then set $m_{i,j}$ to ∞.
  (b) If $j \notin S_i$, then set $m_{i,j}$ to $|N(j) \cap S_i|$.

(3) While L_1 is not empty:
  (a) Iterate through each configuration S_i in L_1.
    (i) Generate all the configurations of k guards from which S_i can be reached in one move. These are the configurations $S_{i'}$ such that $|S_i \Delta S_{i'}| = 2$ and there exist a vertex $v \in S_i \backslash S_{i'}$, and a vertex $v' \in S_{i'} \backslash S_i$, such that vv' is an edge in G.
    (ii) For each such $S_{i'}$ with $A[S_{i'}] = 1$, suppose S_i can be reached from $S_{i'}$ by moving a guard, **say to vertex** j. Then, decrement the value of $m_{i',j}$ by 1. If $m_{i',j}$ becomes 0:
      A. Update the value of $A[S_{i'}]$ to 0.
      B. Add $S_{i'}$ to L_2.
    (iii) Remove S_i from L_1 and update the value of $A[S_i]$ to -1.
  (b) If L_2 is not empty:
    (i) Increment the value of t by 1.
    (ii) For each configuration $S_{i'} \in L_2$:
      A. Update the value of $T[S_{i'}]$ to t.
      B. Add $S_{i'}$ to L_1.
      C. Remove $S_{i'}$ from L_2.

(4) If $A[S_i] = 1$ for some $i \in [\binom{n}{k}]$, then return *True*; otherwise, return *False*.

Algorithm 2. Deciding whether $e^{\infty}(G) \leq k$.

Input: A graph G of order n and an integer k, where $0 < k \leq n$.
Output: *True* if $e^{\infty}(G) \leq k$ and *False* if $e^{\infty}(G) > k$.

(0) In the initialization phase, create two arrays, A and T, that are indexed by the set $\binom{[n]}{k} = \{S_1, S_2, S_3, \ldots, S_{\binom{n}{k}}\}$ of all possible configurations of k guards in G. Additionally, create two lists L_1 and L_2, and a counter $t \leftarrow 0$.

(1) For each configuration S_i in $\binom{[n]}{k}$:
  - (a) If S_i is a dominating set of G, then set $A[S_i]$ to 1.
  - (b) If S_i is not a dominating set of G, then:
    - (i) Set $A[S_i]$ to 0.
    - (ii) Set $T[S_i]$ to t.
    - (iii) Add S_i to L_1.

(2) Create a matrix $M = (m_{i,j})$ with $\binom{n}{k}$ rows and n columns. Then, iterate over each row i in $[\binom{n}{k}]$ and each column j in $[n]$:
  - **(a) If $j \notin S_i$, then set $m_{i,j}$ to ∞.**
  - **(b) If $j \in S_i$, then:**
    - **(i) If $N(j) \subseteq S_i$, then set $m_{i,j}$ to ∞.**
    - **(ii) If $N(j) \nsubseteq S_i$, then set $m_{i,j}$ to $|N(j) - S_i|$.**

(3) While L_1 is not empty:
  - (a) Iterate through each configuration S_i in L_1.
    - (i) Generate all the configurations of k guards from which S_i can be reached in one move. These are the configurations $S_{i'}$ such that $|S_i \Delta S_{i'}| = 2$ and there exist a vertex $v \in S_i \backslash S_{i'}$, and a vertex $v' \in S_{i'} \backslash S_i$, such that vv' is an edge in G.
    - (ii) For each such $S_{i'}$ with $A[S_{i'}] = 1$, suppose S_i can be reached from $S_{i'}$ by moving a guard, **say from vertex** j. Then, decrement the value of $m_{i',j}$ by 1. If $m_{i',j}$ becomes 0, then:
      - A. Update the value of $A[S_{i'}]$ to 0.
      - B. Add $S_{i'}$ to L_2.
    - (iii) Remove S_i from L_1 and update the value of $A[S_i]$ to -1.
  - (b) If L_2 is not empty, then:
    - (i) Increment the value of t by 1.
    - (ii) For each configuration $S_{i'} \in L_2$:
      - A. Update the value of $T[S_{i'}]$ to t.
      - B. Add $S_{i'}$ to L_1.
      - C. Remove $S_{i'}$ from L_2.

(4) If $A[S_i] = 1$ for some $i \in [\binom{n}{k}]$, then return *True*; otherwise, return *False*.

i	S_i	$A[S_i]$	1	2	3	4	5	$T[S_i]$
1	$\{1,2\}$	0	∞	∞	1	0	1	1
2	$\{1,3\}$	1	∞	2	∞	1	1	1
3	$\{1,4\}$	1	∞	1	1	∞	2	1
4	$\{1,5\}$	0	∞	1	0	1	∞	1
5	$\{2,3\}$	0	1	∞	∞	1	0	1
6	$\{2,4\}$	1	1	∞	2	∞	1	1
7	$\{2,5\}$	1	2	∞	1	1	∞	1
8	$\{3,4\}$	0	0	1	∞	∞	1	1
9	$\{3,5\}$	1	1	1	∞	2	∞	1
10	$\{4,5\}$	0	1	0	1	∞	∞	1

(a) Array A and Matrix M after initialization.

i	S_i	$A[S_i]$	1	2	3	4	5	$T[S_i]$
1	$\{1,2\}$	-1	∞	∞	1	0	1	1
2	$\{1,3\}$	-1	∞	0	∞	1	1	2
3	$\{1,4\}$	-1	∞	1	1	∞	0	2
4	$\{1,5\}$	-1	∞	1	0	1	∞	1
5	$\{2,3\}$	-1	1	∞	∞	1	0	1
6	$\{2,4\}$	-1	1	∞	0	∞	1	2
7	$\{2,5\}$	-1	0	∞	1	1	∞	2
8	$\{3,4\}$	-1	0	1	∞	∞	1	1
9	$\{3,5\}$	-1	1	1	∞	0	∞	2
10	$\{4,5\}$	-1	1	0	1	∞	∞	1

(b) Array A and Matrix M after reduction.

Table 2.1: Array A and matrix M that correspond to the graph C_5 in Algorithm 1, where $k = 2$ (assuming the vertices of C_5 are $1, 2, 3, 4, 5$). The shaded cells represent the values that were modified during the reduction process.

i	S_i	$A[S_i]$	1	2	3	4	5	$T[S_i]$
1	$\{1,2\}$	0	1	1	∞	∞	∞	0
2	$\{1,3\}$	1	2	∞	2	∞	∞	0
3	$\{1,4\}$	1	2	∞	∞	2	∞	0
4	$\{1,5\}$	0	1	∞	∞	∞	1	0
5	$\{2,3\}$	0	∞	1	1	∞	∞	0
6	$\{2,4\}$	1	∞	2	∞	2	∞	0
7	$\{2,5\}$	1	∞	2	∞	∞	2	0
8	$\{3,4\}$	0	∞	∞	1	1	∞	0
9	$\{3,5\}$	1	∞	∞	2	∞	2	0
10	$\{4,5\}$	0	∞	∞	∞	1	1	0

(a) Array A and Matrix M after initialization.

i	S_i	$A[S_i]$	1	2	3	4	5	$T[S_i]$
1	$\{1,2\}$	0	1	1	∞	∞	∞	0
2	$\{1,3\}$	1	1	∞	1	∞	∞	1
3	$\{1,4\}$	1	1	∞	∞	1	∞	1
4	$\{1,5\}$	0	1	∞	∞	∞	1	0
5	$\{2,3\}$	0	∞	1	1	∞	∞	0
6	$\{2,4\}$	1	∞	1	∞	1	∞	1
7	$\{2,5\}$	1	∞	1	∞	∞	1	1
8	$\{3,4\}$	0	∞	∞	1	1	∞	0
9	$\{3,5\}$	1	∞	∞	1	∞	1	1
10	$\{4,5\}$	0	∞	∞	∞	1	1	0

(b) Array A and Matrix M after reduction.

Table 2.2: Array A and matrix M that correspond to the graph C_5 in Algorithm 2, where $k = 2$ (assuming the vertices of C_5 are $1, 2, 3, 4, 5$). The shaded cells represent the values that were modified during the reduction process.

Before getting to the proof of correctness of Algorithm 1 and Algorithm 2, we introduce a significant proposition about the length of the game, which will be studied thoroughly later in the following two chapters.

Proposition 2.1. *Let G be a graph such that $\gamma^\infty(G) > k$. Then there exists $t \geq 1$ such that $\binom{[n]}{k}$ can be uniquely partitionned into $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \dots, \mathcal{D}_t\}$ in a way such that for any $i \in [t]$, i turns are necessary and sufficient for the attacker to win in the eternal domination game played on G against k guards from an opening configuration $D \in \mathcal{D}_i$ (assuming optimal play).*

Proof. Since $\gamma^\infty(G) > k$, there are configurations in $\binom{[n]}{k}$ that are not dominating sets of G. Let $\mathcal{D}_1$ be the set of all such configurations. Then one turn is necessary (since the attacker must select a vertex that cannot be defended) and sufficient (since the attacker can select a vertex which is not dominated by any guard) to win against k guards from an opening configuration in $\mathcal{D}_1$. Suppose there exist i pairwise disjoint subsets $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \dots, \mathcal{D}_i$ of $\binom{[n]}{k}$, where $i \geq 1$, such that for any $j \in \{1, 2, 3, \dots, i\}$, j turns are necessary and sufficient for the attacker to win against k guards from an opening configuration in $\mathcal{D}_j$. If $\binom{[n]}{k} = \bigcup_{j=1}^{i} \mathcal{D}_j$, then we are done. Suppose this is not the case. Since the configurations in $\binom{[n]}{k} - \bigcup_{j=1}^{i} \mathcal{D}_j$ are not eternal dominating sets, the guards can be forced to move to a configuration in $\bigcup_{j=1}^{i} \mathcal{D}_j$ from each of them (after some sequence of attacks); otherwise, k guards can avoid losing on G by moving from configurations in $\binom{[n]}{k} - \bigcup_{j=1}^{i} \mathcal{D}_j$ to configurations in $\binom{[n]}{k} - \bigcup_{j=1}^{i} \mathcal{D}_j$ (which are all dominating sets) after each attack. Let $\mathcal{D}_{i+1}$ be the set of all the configurations in $\binom{[n]}{k} - \bigcup_{j=1}^{i} \mathcal{D}_j$ from which the guards can be forced to move to a configuration in $\bigcup_{j=1}^{i} \mathcal{D}_j$ on the next turn. Observe that since $\binom{[n]}{k} - \bigcup_{j=1}^{i} \mathcal{D}_j$ is not empty, so is $\mathcal{D}_{i+1}$; otherwise, k guards can still avoid losing on G by moving from a configuration in $\binom{[n]}{k} - \bigcup_{j=1}^{i} \mathcal{D}_j$ to a configuration in $\binom{[n]}{k} - \bigcup_{j=1}^{i} \mathcal{D}_j$ after each attack. Consider a configuration $D \in \mathcal{D}_{i+1}$. The attacker wins in at most $i+1$ turns when the guards start on D since they can be forced to move to a configuration in $\bigcup_{j=1}^{i} \mathcal{D}_j$ after the first turn and, by the induction hypothesis, the guards lose on a configuration in $\bigcup_{j=1}^{i} \mathcal{D}_j$ in at most i turns. Now, note that the guards can force the game to last for at least $i+1$ turns when they start from the configuration D; otherwise, by the induction hypothesis $D \in \bigcup_{j=1}^{i} \mathcal{D}_j$, which is a contradiction. Since $\binom{[n]}{k}$ is finite and $\mathcal{D}_i \neq \emptyset$ for each i, the proposition follows. □

Proposition 2.1 can be adapted for the eviction game as well. We formally state this in the following proposition, with the proof omitted.

Proposition 2.2. *Let G be a graph such that $e^\infty(G) > k$. Then there exists $t \geq 0$ such that $\binom{[n]}{k}$ can be uniquely partitionned into $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_t\}$ in a way such that for any $i \in [t]$, i turns are necessary and sufficient for the attacker to win the eternal domination game played on G against k guards from an opening configuration $D \in \mathcal{D}_i$ (assuming optimal play).*

We now prove the correctness of our algorithms.

Theorem 2.3. *Algorithm 1 correctly determines whether an n-vertex graph G has an eternal dominating set of size at most k in the eternal domination game.*

Proof. It suffices to show that Algorithm 1 finds the partition of $\binom{[n]}{k}$ into the set $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \dots, \mathcal{D}_t\}$ as described in Proposition 2.1 if and only if $\gamma^\infty(G) > k$. Observe that the algorithm starts by initializing the arrays A and T, and finding the set of all the configurations of k guards in the graph G that are not dominating sets. We call this set of configurations $\mathcal{D}_1$ to indicate that it contains all the opening configurations from which the guards lose on the first turn. The configurations in $\mathcal{D}_1$ are then added to a list L_1. Next, in Step 2, the algorithm initializes the matrix M by assigning each $m_{i,j}$ one of the possible values ∞ and $|N(j) \cap S_i|$. If j is a vertex in the configuration S_i, then the algorithm sets $m_{i,j}$ to ∞ to indicate that vertex j cannot be attacked when the guards are located on the configuration S_i. If j is not a vertex in the configuration S_i, then the algorithm sets $m_{i,j}$ to $|N(j) \cap S_i|$ to indicate that there are exactly $|N(j) \cap S_i|$ guards that can defend an attack on vertex j when the guards are located on the configuration S_i. Moving to Step 3, for any integer $i \geq 2$, the algorithm finds the set of all the configurations in $\binom{[n]}{k} - \bigcup_{j=1}^{i-1} \mathcal{D}_j$, denoted by $\mathcal{D}_i$, from which the guards can be forced to move to a configuration in $\bigcup_{j=1}^{i-1} \mathcal{D}_j$ on the next turn. These configurations are all generated at time $t = i$, from the configurations in $\mathcal{D}_{i-1}$, which are in L_1 at that time, and they are added to another list L_2. To see this, observe that a configuration $S_{i'} \in \binom{[n]}{k}$ is added to L_2 at time $t = i$ if and only if there exists $j \notin S_{i'}$ such that $m_{i',j} = 0$ at time $t = i$ and $m_{i',j} > 0$ at time $t < i$. In this case, there are $|N(j) \cap S_{i'}|$ configurations to which the guards can move from $S_{i'}$ (after an attack on vertex $j \notin S_{i'}$) which have already been added to L_1 at time $t = 1, 2, 3, \ldots, i-1$. Since the guards can only move to $|N(j) \cap S_{i'}|$ configurations from $S_{i'}$ when vertex j is attacked, and the guards lose from each of these configurations in at most $i-1$ turns, we conclude that the guards lose from $S_{i'}$ in at most i turns. Now, since each configuration in $\binom{[n]}{k}$ belongs to $\mathcal{D}_i$ for at most one i, each configuration is added to L_2 and hence L_1 at most once. Hence, the algorithm terminates after at most $\binom{n}{k}$ configurations have been considered (in L_1). Finally, in Step 4, if all the configurations in $\binom{[n]}{k}$ have been added to L_1, then k guards cannot defend the graph since the algorithm finds the partition; otherwise, the configurations S_i such that $A[S_i] = 1$ that have not been added to L_1 when the algorithm halts are precisely the eternal dominating sets of G. □

Theorem 2.4. *Algorithm 2 correctly determines whether an n-vertex graph G has an eternal dominating set of size at most k in the eviction game.*

It is worth noting that the technique used in the proof of Theorem 2.3 can be easily adapted to prove Theorem 2.4. Moreover, Algorithm 1 and Algorithm 2 can be readily modified to suit other variants of the game such as the total and connected versions of eternal domination and eviction.

Proposition 2.5. *Algorithm 1 has time complexity $O(kn^{k+2})$.*

Proof. Observe that Step 1 can be done in time $O(kn^{k+1})$ since it only consists in generating the set of all $\binom{n}{k} = O(n^k)$ possible configurations of the guards in the graph and checking in time $O(kn)$ whether each of these configurations is a dominating set. In addition to that, Step 2 can be done in time $O(kn^{k+1})$ since M has $\binom{n}{k} \times n = O(n^{k+1})$ entries and for any entry $m_{i,j}$ considered, vertex j is either one of the k vertices in S_i or has at most k neighbours in S_i. Moving on, Step 3 can be completed within $O(kn^{k+2})$ time for the following reasons. Each configuration S_i is added to L_1 at most once; moreover, the set of configurations $S_{i'}$ from which S_i can be reached in one move has cardinality at most nk. For each $S_{i'}$ generated in this step, we can find i' by using a ranking algorithm (see [49]) in time $O(n)$. Finally, Step 4 can be done in time $O(n^k)$ since it only consists of checking the values of the entries of the array A. As a consequence, Algorithm 1 has time complexity $O(kn^{k+2})$. □

Proposition 2.6. *Algorithm 2 has time complexity $O(kn^{k+2})$.*

The proof of Proposition 2.6 is similar to the proof of Proposition 2.5 with the only difference being in Step 2 which can now be completed in time $O(n^{k+2})$.

2.2 Computational Complexity

While Algorithm 1 and Algorithm 2 demonstrate that we can determine whether a given graph G has an eternal dominating set of size at most k, or whether a configuration of k guards is an eternal dominating set of G in time $O(kn^{k+2})$ for both the eternal domination game and the eviction game, it fails to provide any further insight into the computational complexity of these decision problems. In this section, we aim to address this limitation by establishing the EXPTIME-completeness of the problems when neither G nor k is fixed, and when the initial configuration of the guards is given. We begin with the following definition.

Definition 2.7. The *friendship graph F_k* (or the k-fan), where $k \geq 1$, is the graph obtained from the join of k disjoint copies of P_2 with K_1 (see Figure 2.1 for the case where $k = 5$).

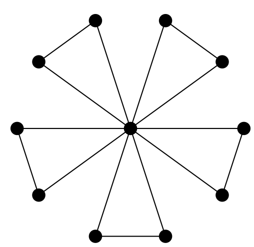


Figure 2.1: The friendship graph $F_5 = 5P_2 \vee K_1$.

Proposition 2.8. *For any integer $n \geq 1$, there exists a connected n-vertex graph G such that any optimum eternal dominating family of G (in the eternal domination game) contains at least $2^{\Omega(n)}$ dominating sets.*

Proof. Let $n \geq 1$ be a given integer. Suppose $n = 2k + 1$ for some integer $k \geq 1$. Consider the friendship graph F_k on $n = 2k + 1$ vertices. Since $\alpha(F_k) = \theta(F_k) = k$, Observation 1.5 implies that $\gamma^\infty(F_k) = k$. However, F_k has $2^k = 2^{(n-1)/2}$ different independent sets of size k. Since any optimum eternal dominating family contains all of these independent sets, it must be of size at least $2^{(n-1)/2}$.

Now, suppose $n = 2k$ for some integer $k \geq 1$. Let G be the graph obtained from the friendship graph F_k by removing a vertex of degree two. Again, since $\alpha(G) = \theta(G) = k$, Observation 1.5 implies that $\gamma^\infty(G) = k$. Note that G now has $2^{k-1} = 2^{(n/2)-1}$ independent sets of size k and any optimum eternal dominating family contains each of them. Therefore, any optimum eternal dominating family of G must be of size at least $2^{(n/2)-1}$. This completes the proof. □

Before presenting a similar result for the eviction game, we would like to draw attention to the following observation.

Observation 2.9. Let G be a graph and let v be a universal vertex of G. Then, $e^\infty(G) \geq c(G - \{v\})$.

(a) (b)

Figure 2.2: Cases considered in the proofs of Proposition 2.8 and Proposition 2.10.

Proposition 2.10. *For any integer $n \geq 1$, there exists a connected n-vertex graph G such that any optimum eternal dominating family of G (in the eviction game) contains at least $2^{\Omega(n)}$ dominating sets.*

Proof. Let $n \geq 1$ be a given integer. Suppose $n = 2k + 1$ for some integer $k \geq 1$. Consider the graph obtained from the friendship F_k on $n = 2k+1$ vertices by removing an edge joining two vertices of degree two (see Figure 2.2a). Since $\alpha(G) = \theta(G) = k+1$, by Proposition 1.19 and Observation 2.9, $e^\infty(G) = k + 1$. If the universal vertex of G is occupied, then evict the guard located on this vertex. In this case, there must be a guard on each leaf of G. By evicting any guard located on a leaf of G, the guard will move to the universal vertex of G. From this configuration, each guard located on a vertex of degree two can be evicted to its unique neighbour of degree two at any time during the game. Therefore, there are at least $3 \times 2^{k-1} = 3 \times 2^{(n-3)/2}$ configurations of the guards that can be forced (since the number of configurations of two guards on the subgraph induced by the leaves and the universal vertex is three).

Now, suppose $n = 2k$ for some integer $k \geq 1$. Let G be the graph obtained from the friendship graph F_k by removing a vertex of degree two (see Figure 2.2b). Again, since $\alpha(G) = \theta(G) = k$, by Proposition 1.19 and Observation 2.9, $e^\infty(G) = k$. Using a similar strategy as in the previous case, it can be shown that there are at least

$2 \times 2^{k-1} = 2 \times 2^{(n-2)/2}$ configurations of the guards that can be forced. This completes the proof. □

An immediate implication of Proposition 2.8 and Proposition 2.10 is that any algorithm that is required to store all the eternal dominating sets in an optimum eternal dominating family will inevitably need exponential space and hence exponential time.

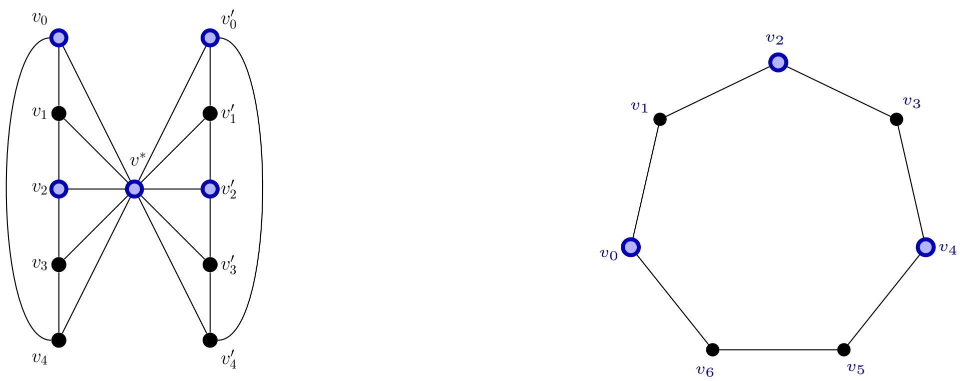
Proposition 2.11. *In the eternal domination game, if the sequence of attacks is known in advance, the number of guards required to defend a graph G can be less than $\gamma^\infty(G)$.*

Proof. Consider the graph $G \cong 2C_5 \vee K_1$ which is obtained from the join of two disjoint copies of the graph C_5 with the graph K_1 (see Figure 2.3a). Observe that $\gamma^\infty(G) = 6$. However, five guards can eternally defend this graph against any sequence of attacks that is known in advance to them. To see this, consider the opening configuration $\{v^*, v_0, v_2, v_0', v_2'\}$ of the guards. We will show that the two guards located on the subgraph induced by the vertices v_0, v_1, v_2, v_3, v_4 (respectively, $v_0', v_1', v_2', v_3', v_4'$) can defend any sequence of attacks on this subgraph. We may assume without loss of generality that the attacker only selects vertices from the set $\{v_0, v_1, v_2, v_3, v_4\}$; otherwise, apply the same strategy on the subgraph induced by the vertices $v_0', v_1', v_2', v_3', v_4'$ when any of these vertices is attacked. We consider three cases for the first attack.

(1) If v_3 is attacked, then move the guard on v_2 to v_3.
(2) If v_4 is attacked, then move the guard on v_0 to v_4.
(3) Suppose v_1 is attacked. We now consider four subcases, depending on the location of the second attack.
   (a) If the second attack is on v_0, then move the guard on v_0 to v_1 first and the same guard back to v_0 next.
   (b) If the second attack is on v_2, then move the guard on v_2 to v_1 first and the same guard back to v_2 next.
   (c) If the second attack is on v_3, then move the guard on v_0 to v_1 first and the guard on v_2 to v_3 next.
   (d) If the second attack is on v_4, then move the guard on v_2 to v_1 first and the guard on v_0 to v_4 next.

It is clear that the guards are always located on a dominating set of G after each response since the guard on v^* never moves. Observe also that in each of the previous cases, the guards are always located on a similar configuration to their opening configuration after the first attack, in case 1 and case 2, or after the first two attacks, in case 3. □

Proposition 2.12. *In the eviction game, if the sequence of attacks is known in advance, the number of guards required to defend a graph G can be less than $e^\infty(G)$.*



(a) $2C_5 \vee K_1$ (b) C_7

Figure 2.3: Examples from the proof of Proposition 2.11 and Proposition 2.12.

Proof. We will show that three guards can respond to any sequence of attack that is known in advance on the vertices of the graph C_7. Observe that, by Proposition 1.24, $e^\infty(C_7) = 4$. Consider the opening configuration $\{v_0, v_2, v_4\}$. We consider three cases for the first attack.

(1) If v_0 is attacked, then move the guard on v_0 to v_6.
(2) If v_4 is attacked, then move the guard on v_4 to v_5.
(3) Suppose v_2 is attacked. We now consider four subcases, depending on the location of the second attack.
   (a) If the second attack is on v_1, then move the guard on v_2 to v_1 first and the same guard back to v_2 next.
   (b) If the second attack is on v_3, then move the guard on v_2 to v_3 first and the same guard back to v_2 next.
   (c) If the second attack is on v_0, then move the guard on v_2 to v_1 first and the guard on v_0 to v_6 next.
   (d) If the second attack is on v_4, then move the guard on v_2 to v_3 first and the guard on v_4 to v_5 next.

Observe that the guards are always located on the vertices of a dominating set of G after each response. Moreover, in each of the previous cases, the guards are located on a similar configuration to their opening configuration after the first attack, in case 1 and case 2, or after the first two attacks, in case 3. □

We now consider the hardness of the decision problems.

Theorem 2.13. *ETERNAL DOMINATION is NP-hard.*

Proof. We give a standard polynomial-time reduction from the widely recognized NP-hard problem 3-SAT (see for example [22]) to ETERNAL DOMINATION. Let φ be an instance of 3-SAT with N variables and M clauses. We construct a graph G and show that φ is satisfiable if and only if $\gamma^\infty(G) \leq 2N$.

(1) The vertex set of G is constructed as follows:
   (a) For each $i \in [N]$, we introduce the vertices $x_i, \overline{x_i}$ and y_i. The vertices in the set $S = \bigcup_{i\in[N]} \{x_i, \overline{x_i}\}$ are called *literal vertices*.
   (b) For each $j \in [M]$, we introduce the vertex c_j. The vertices in the set $C = \{c_1, c_2, c_3, \ldots, c_M\}$ are called *clause vertices*.

(2) The edge set of G is constructed as follows:
   (a) For each $i \in [N]$, we add an edge between y_i and both of x_i and $\overline{x_i}$.
   (b) For each $i \in [N]$ and each $j \in [M]$, we add an edge between the literal vertex x_i (respectively $\overline{x_i}$) and the clause vertex c_j if and only if the literal vertex x_i (respectively $\overline{x_i}$) satisfies the clause c_j in φ.
   (c) For each $i, j \in [M]$ with $i \neq j$, we add an edge between the clause vertices c_i and c_j.

See Figure 2.4 for an illustration where the edges between the clause vertices are omitted.

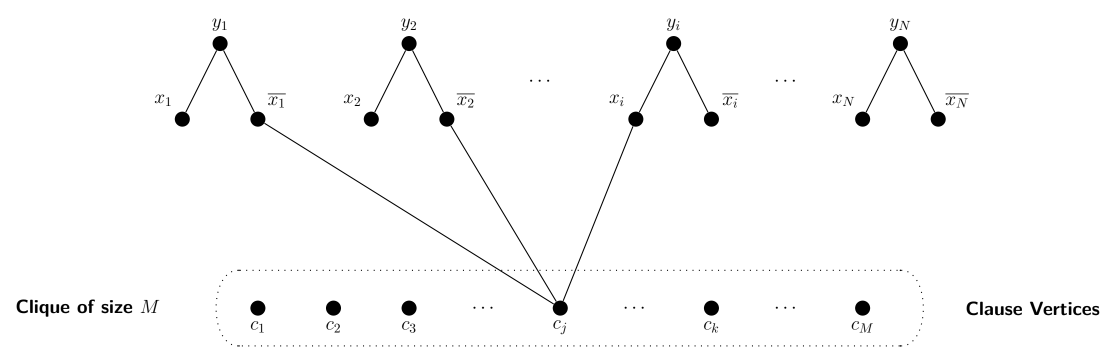


Figure 2.4: Reduction of 3-SAT to ETERNAL DOMINATION, where $c_j = (\overline{x_1} \vee \overline{x_2} \vee x_i)$.

Observe that G can be constructed in polynomial time. We now claim that φ is satisfiable if and only if $\gamma^\infty(G) \leq 2N$.

Suppose φ is satisfiable. To show that $\gamma^\infty(G) \leq 2N$, it suffices to show that $\theta(G) \leq 2N$. To this end, consider a satisfying truth assignment $\mathcal{A} = \{x_1^*, x_2^*, x_3^*, \ldots, x_N^*\}$ of φ, where $x_i^* \in \{x_i, \overline{x_i}\}$ for each $i \in [N]$. Let P_1 be the set of all the neighbours of x_1^* in C. In other words, P_1 is the set of all the clause vertices for which the corresponding clauses in φ are satisfied whenever x_i^* is *True*. For each $i > 1$, let P_i be the set of all the neighbours of x_i^* in $C - \bigcup_{j=1}^{i-1} P_j$. Put differently, P_i is the set of all the clause vertices for which the corresponding clauses in φ are satisfied whenever x_i^* is *True* but are not satisfied when x_i^* is *False* and $x_1^* \vee x_2^* \vee x_3^* \vee \cdots \vee x_{i-1}^*$ is *True*. Since $\mathcal{A}$ is a satisfying truth assignment of φ, $C = \bigcup_{i=1}^N P_i$. Thus, $\bigcup_{i=1}^N \{\{y_i, \overline{x_i^*}\}\} \cup \bigcup_{i=1}^N \{\{x_i^*\} \cup P_i\}$ is a partition of $V(G)$ into $2N$ cliques. Consequently, $\gamma^\infty(G) \leq 2N$.

Suppose $\gamma^\infty(G) \le 2N$. We need to show that φ is satisfiable. Since $S = \bigcup_{i=1}^{N}\{x_i, \overline{x_i}\}$ is an independent set of G of size $2N$, we may assume without loss of generality that S is an eternal dominating set of G and the guards are initially located on the vertices of S. After any response to the sequence of attacks on the vertices $y_1, y_2, y_3, \dots, y_N$ (in this order), there is a guard located on exactly one literal vertex in the set $\{x_i, \overline{x_i}\}$ for each $i \in [N]$ (say the guard is located on x_i^*, where $x_i^* \in \{x_i, \overline{x_i}\}$). Since S is an eternal dominating set and the guards can be assumed to play optimally, all the clause vertices in G are dominated. Then, for any $j \in [M]$, there exists $i \in [N]$ such that the variable vertex x_i^* dominates the clause vertex c_j. Equivalently, for any $j \in [M]$, there exists $i \in [N]$ such that the clause c_j is satisfied whenener x_i^* is *True* in φ. As a result, φ is satisfied if we assign all the variables in the set $\{x_i^*\}_{i\in[N]}$ to *True*. Therefore, φ is satisfiable if and only if $\gamma^\infty(G) \le 2N$. □

Before proceeding to a corollary that results from Theorem 2.13, we present the following definitions.

Definition 2.14. A graph G is said to be $(1,2)$-*split* if its vertex set can be partitionned into two sets A and B where $G[A]$ (the subgraph of G induced by the vertices of the set A) is a clique and $G[B]$ is a triangle-free graph.

Definition 2.15. The *house graph* is the complement of the graph P_5 (see Figure 2.5).

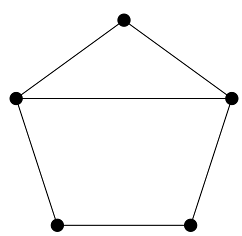


Figure 2.5: The house graph.

The graph constructed in the proof of Theorem 2.13 belongs to many graph classes. Hence, we have the following corollary.

Corollary 2.16. *ETERNAL DOMINATION is NP-hard even when the input is restricted to graphs belonging to the intersection of the following classes.*
(1) C_k*-free graphs* ($k \neq 3, 5$).
(2) $K_{2,3}$*-free graphs.*
(3) House-free graphs.
(4) $(1,2)$*-split graphs.*

Before progressing further, let us introduce two more definitions and revisit Klostermeyer and MacGillivray's theorem about the hardness of EVICTION.

Definition 2.17. A graph G is said to be a chordal graph if every induced cycle of G is a triangle.

Definition 2.18. A graph G is said to be an interval graph if each of its vertices can be associated with an interval of the real line in a way such that two vertices are adjacent if and only if the corresponding intervals have non-empty intersection.

Theorem 2.19 ([31])**.** *EVICTION is NP-hard, even when the input is restricted to chordal and interval graphs*

After establishing the NP-hardness of ETERNAL DOMINATION, we turn our attention to two other scenarios that offer further insight into the problem's complexity. We first examine the case where the initial configuration of the guards is given (in both games), and subsequently we explore the scenario involving directed input graphs (in the eternal domination game). We will prove three theorems that demonstrate the EXPTIME-completeness of the decision problems associated with each scenario.

Our next set of results make use of a game, played on a Boolean formula, that serves as a key tool for proving the EXPTIME-hardness of the decision problems we consider. The game, which we refer to as the Formula Game, was introduced by Robson [50] and is played as follows:

> "Two players (Player X and Player Y) are given two disjoint sets of variables $X = \{x_1, x_2, x_3, \ldots, x_N\}$ and $Y = \{y_1, y_2, y_3, \ldots, y_N\}$ together with an initial assignment of values to these variables and a Boolean formula ϕ in disjunctive normal form (with no variable occurring more than k times). At each time $t = 0, 1, 2, \ldots$, Player X assigns (changes) the value of x_i for some $i \in [N]$ and Player Y responds by assigning (changing or not) the value of y_i for the same i. Player X wins if ϕ is ever *True* after Player Y's turn. The problem we are interested in is to decide whether Player X can win the game if each player plays optimally."

Observe that the Formula Game can be reformulated by introducing the formula φ, which represents the negation of the original formula ϕ. In this perspective, φ is a Boolean formula in conjunctive normal form which is satisfied by the initial truth assignment to the variables and Player X wins if φ is ever *False* after Player Y's turn.

Theorem 2.20 ([50])**.** *The Formula Game is EXPTIME-complete.*

The Formula Game was used by Robson [50] to prove the EXPTIME-completeness of a generalisation of the game of Go under Japanese rules. It is also similar to another game that is known in the literature as the Alternating Boolean Formula Game (ABF or $G6$ [54]), which was used to prove the EXPTIME-completeness of various games including the game of Cops and Robbers [25, 27]. The main difference between the two games is that in the Alternating Boolean Formula Game (ABF), Player Y is allowed (but not required) to change the value of y_j for any $j \in [N]$ while in Robson's Formula Game, Player Y is allowed (but not required) to change the value of y_i for the same i as Player X.

We are now ready to prove the main theorems of this section.

Theorem 2.21. *ETERNAL DOMINATING SET is EXPTIME-complete.*

Proof. ETERNAL DOMINATING SET is clearly in EXPTIME since Algorithm 1 determines in time $O(kn^{k+2})$ the set of configurations (potentially empty) from which k guards can respond to any sequence of attacks on a graph of order n. So, it suffices to show that ETERNAL DOMINATING SET is EXPTIME-hard. To this end, we will show that the Formula Game introduced in Section 1.2 has a polynomial-time reduction to ETERNAL DOMINATING SET. That is, given an instance of the Formula Game, we will construct in polynomial time a graph G and we will show that Player Y can win in the Formula Game if and only if $2N+1$ guards can respond to any sequence of attacks on G from a particular initial configuration.

Let $\mathcal{I}$ be an instance of the Formula Game with variable sets $X = \{x_1, x_2, x_3, \ldots, x_N\}$, $Y = \{y_1, y_2, y_3, \ldots, y_N\}$, formula $\varphi = C_1 \wedge C_2 \wedge C_3 \wedge \cdots \wedge C_M$ in conjunctive normal form and initial satisfying truth assignment $\mathcal{A}$. We may assume without loss of generality that the variables $x_1, x_2, x_3, \ldots, x_N$ and $y_1, y_2, y_3, \ldots, y_N$ are all *True* in $\mathcal{A}$; otherwise, we can relabel the variables in X and Y in a way such that this condition is satisfied. In the corresponding instance of ETERNAL DOMINATING SET, the graph $G = (V, E)$ is constructed as follows (see Figure 2.6 for an illustration).

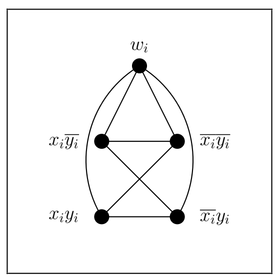
Firstly, for each $i \in [N]$, we introduce the vertices $x_iy_i, x_i\overline{y_i}, \overline{x_i}y_i$ and $\overline{x_iy_i}$, which we call *variable vertices*. Second, we introduce the vertices $z_1, z_2, z_3, \ldots, z_M$, which we call *clause vertices*. Note that for each $i \in [N]$, we aim to have each of the variable vertices $x_iy_i, x_i\overline{y_i}, \overline{x_i}y_i$ and $\overline{x_iy_i}$ correspond to a truth assignment to the variables x_i and y_i in the following sense: "If there is a guard on x_iy_i (respectively $x_i\overline{y_i}, \overline{x_i}y_i$ and $\overline{x_iy_i}$), that guard dominates the clause vertices for which the corresponding clauses are satisfied when x_i or y_i (respectively x_i or $\overline{y_i}, \overline{x_i}$ or y_i, $\overline{x_i}$ or $\overline{y_i}$) is *True*". Next, we introduce N new vertices $w_1, w_2, w_3, \ldots, w_N$. Finally, we introduce two special vertices z and z'.

To summarize, the vertex set of G is $V = U \cup C \cup W \cup Z$ where:
(1) $U = \bigcup_{i=1}^{N} U_i$, where $U_i = \{x_iy_i, x_i\overline{y_i}, \overline{x_i}y_i, \overline{x_iy_i}\}$ for each $i \in [N]$.
(2) $C = \{z_1, z_2, z_3, \ldots, z_M\}$.
(3) $W = \{w_1, w_2, w_3, \ldots, w_N\}$.
(4) $Z = \{z, z'\}$.

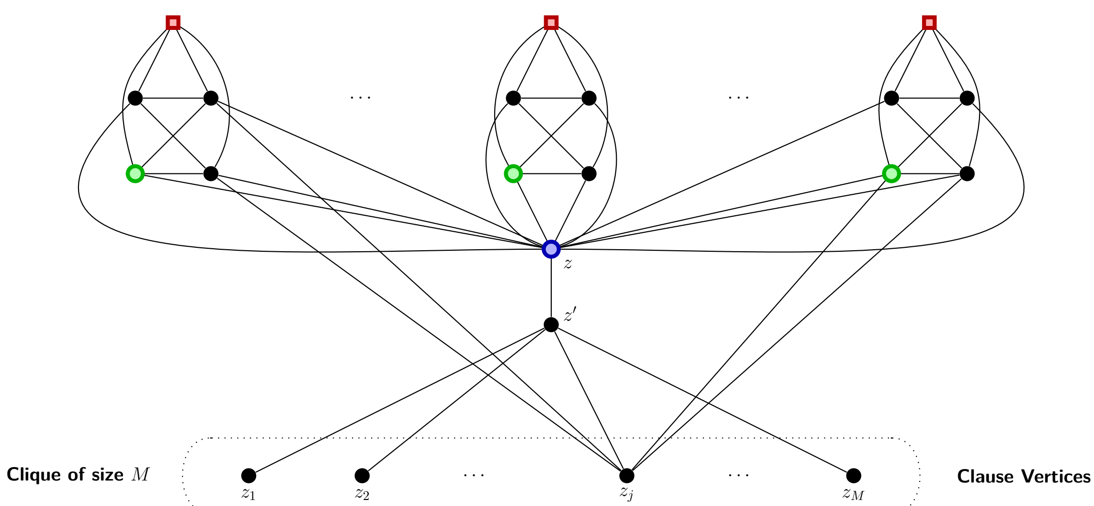
The edge set of G is $E = F \cup F' \cup F'' \cup F^{(3)} \cup F^{(4)} \cup F^{(5)} \cup \{zz'\}$ where:

(1) $F = \bigcup_{i=1}^{N} F_i$, where $F_i = \{\{x_iy_i, \overline{x_i}y_i\}, \{x_iy_i, \overline{x_iy_i}\}, \{x_i\overline{y_i}, \overline{x_i}y_i\}, \{x_i\overline{y_i}, \overline{x_iy_i}\}\}$ $\forall i \in [N]$.
(2) $F' = \bigcup_{i=1}^{N} F'_i$, where $F'_i = \{\{w_i, x_iy_i\}, \{w_i, x_i\overline{y_i}\}, \{w_i, \overline{x_i}y_i\}, \{w_i, \overline{x_iy_i}\}\}$ $\forall i \in [N]$.
(3) F'' contains an edge between any pair of clause vertices.
(4) $F^{(3)}$ contains an edge between each variable vertex and the vertex z.
(5) $F^{(4)}$ contains an edge between each clause vertex and the vertex z'.
(6) $F^{(5)}$ contains an edge between the variable vertex $x_i^*y_i^*$ and the clause vertex z_j if and only if the corresponding clause C_j is satisfied in the formula φ whenever x_i^* or y_i^* is *True*, where $x_i^* \in \{x_i, \overline{x_i}\}$ and $y_i^* \in \{y_i, \overline{y_i}\}$, for each $i \in [N]$ and each $j \in [M]$.

For the initial configuration, we place a guard on each vertex of the set $\{z\} \cup \bigcup_{i \in [N]} \{w_i, x_iy_i\}$. See Figure 2.6 for an illustration (the edges between the clause vertices are omitted). Note that the construction can be carried out in polynomial time.



(a) Subgraph induced by the vertices of $U_i \cup \{w_i\}$.



(b) Graph G.

Figure 2.6: Reduction of the Formula Game to ETERNAL DOMINATING SET, where $C_j = (\overline{x_1} \vee y_N)$.

In the rest of the proof, we will show that Player X can win in the Formula Game if and only if the attacker can win in the eternal domination game, played on G, when the guards start from the given initial configuration.

Claim 1. If Player X can win in the Formula Game, then the attacker can win in the eternal domination game, played on G, when the guards start from the given initial configuration.

Proof. We prove by induction on t that if Player X can win in the Formula Game in at most t turns, then the attacker can win in the eternal domination game, played on G, when the guards start from the given initial configuration in at most $3t$ turns. We first consider the case where $t = 1$. If Player X can win in the Formula Game in one turn, then there exists $i \in [N]$ such that Player X wins after changing the value of x_i (say from *True* to *False*) no matter what the response of Player Y is. In this case, the attacker selects the vertices $\overline{x_i}y_i, \overline{x_i y_i}, w_i$ in this order. After responding to these attacks, the guards will be located on a configuration which corresponds to an unsatisfying truth assignment to the variables in the Formula Game. So, there exists $j \in [M]$ such that all the literals in the clause C_j are *False*; therefore, the corresponding clause vertex z_j is not dominated by any guard in G. Suppose, for any truth assignment from which Player X can win in at most t turns ($t \geq 1$) in the Formula Game, the attacker can win in the eternal domination game in at most $3t$ turns from the corresponding initial configuration. Consider a satisfying truth assignment to the variables in the Formula Game from which Player X can win in at most $t + 1$ turns. Then, there exists $i \in [N]$ such that Player X can win in at most t turns after changing the value of x_i no matter what the response of Player Y is. In this case, the attacker selects the vertices $\overline{x_i}y_i, \overline{x_i y_i}, w_i$ (without loss of generality) in this order. After responding to these attacks, the guards will be located on a configuration which corresponds to a truth assignment to the variables in the Formula Game from which Player X can win in at most t turns. By the induction hypothesis, we know that the attacker can win from that configuration in at most $3t$ turn. As a result, the attacker can win from the initial configuration in at most $3(t + 1)$ turns. ♢

Claim 2. If Player Y can win in the Formula Game, then the guards can win in the eternal domination game, played on G, when they start from the given initial configuration.

Proof. Suppose Player Y has a winning strategy in the Formula Game. We will show that the guards can also win in the eternal domination game, played on G, when they start from the given initial configuration, by moving according to the same strategy. We may assume without loss of generality that the attacker plays optimally and hence never selects a clause vertex or the vertex z'; otherwise, the guards can guarantee a win after responding to such an attack since there would be a guard in each clique of a clique covering of G.

Now, we colour the guard located at vertex z blue, the guards located at vertex w_i red for each $i \in [N]$, and the remaining guards green. Observe that the green guards are located on a configuration which corresponds to the initial truth assignment in the following sense: "The clause vertex $x_i^* y_i^*$ contains a guard if and only if the corresponding literals x_i^* and y_i^* are both *True* in the truth assignment". Now, we will show that throughout the game, the green guards will always be located on a configuration which corresponds to a truth assignment from which Player Y can always win in the Formula Game (in short, a *good* configuration).

For the strategy of the guards, we consider the following cases:

(1) If a red guard can respond to an attack, then we move that guard to defend the attack. Since no green guard has been moved, the green guards are still located on a configuration which corresponds to a good assignment.

(2) Suppose there is an attack on a vertex that cannot be defended by a red guard. We may assume without loss of generality that this vertex is $\overline{x_i} y_i$ for some $i \in [N]$. In this case, there is a red guard located on $\overline{x_i}\overline{y_i}$ and a green guard on either $x_i y_i$ or $x_i \overline{y_i}$. Move the green guard to the attacked vertex. Now, we consider the response of Player Y to the move of Player X that consists of changing the value of x_i from *True* to *False* in the Formula Game. If player Y assigns the value *True* to y_i, then we know that the green guards are still located on a configuration which corresponds to a good assignment in the Formula Game; otherwise, we switch the colours of the guards located at $\overline{x_i} y_i$ and $\overline{x_i}\overline{y_i}$ so that the green guards are still located on a configuration which corresponds to a good assignment in the Formula Game. Since Player Y can win in the Formula Game, we conclude that the guards can defend the vertices in G against any sequence of attacks. ♢

From Claims 1 and 2, we know that Player X can win in the Formula Game if and only if the attacker can win in the eternal domination game, played on G, when the guards start from the given initial configuration. Therefore, the Formula Game is polynomial-time reducible to ETERNAL DOMINATING SET. As a result, ETERNAL DOMINATING SET is EXPTIME-complete. □

We now present a similar result for the eviction model of the game, without a detailed proof.

Theorem 2.22. *EVICTION ETERNAL DOMINATING SET is EXPTIME-complete.*

Proof. We only describe the graph that can be constructed in polynomial time to reduce an instance of the Formula game to an instance of EVICTION ETERNAL DOMINATING SET (see Figure 2.7). The proof follows from a similar idea as the proof of Theorem 2.21, so we only highlight the main differences between the two.

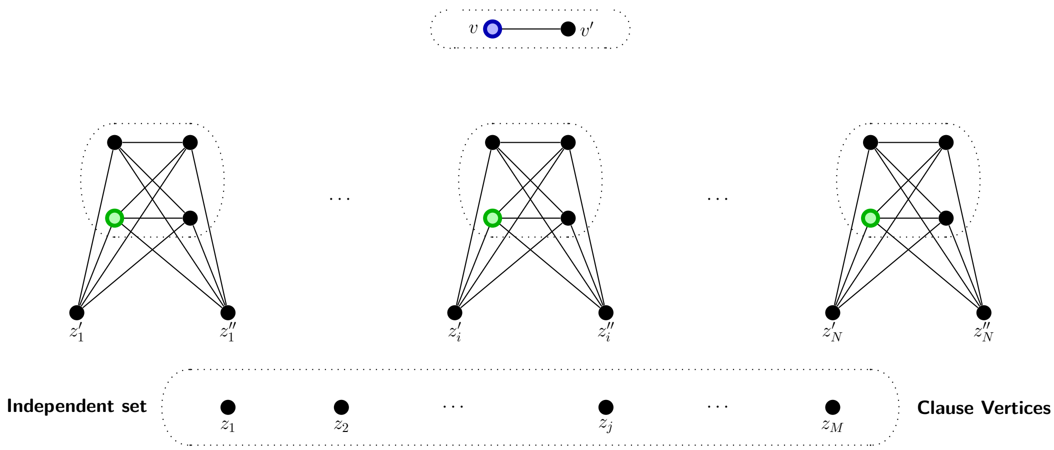


Figure 2.7: Reduction of the Formula Game to EVICTION ETERNAL DOMINATING SET.

Firstly, for each $i \in [N]$, we introduce the vertices $x_i y_i, x_i \overline{y_i}, \overline{x_i} y_i$ and $\overline{x_i y_i}$, which we call *variable vertices*, and we also add the edges $\{x_i y_i, \overline{x_i} y_i\}$, $\{x_i y_i, \overline{x_i y_i}\}$, $\{x_i \overline{y_i}, \overline{x_i} y_i\}$, $\{x_i \overline{y_i}, \overline{x_i y_i}\}$ the same way as in the proof of Theorem 2.21. Second, we introduce the vertices $z_1, z_2, z_3, \ldots, z_M$, which we call *clause vertices*. Note that for each $i \in [N]$, we aim to have each of the variable vertices $x_i y_i, x_i \overline{y_i}, \overline{x_i} y_i$ and $\overline{x_i y_i}$ correspond to a truth assignment to the variables x_i and y_i as in the proof of Theorem 2.21. Therefore, we add an edge between the variable vertex $x_i^* y_i^*$ and the clause vertex z_j if and only if the corresponding clause C_j is satisfied in the formula φ whenever x_i^* or y_i^* is *True*, where $x_i^* \in \{x_i, \overline{x_i}\}$ and $y_i^* \in \{y_i, \overline{y_i}\}$, for each $i \in [N]$ and each $j \in [M]$.

Our goal is to have exactly one guard located in the set $\{x_i y_i, x_i \overline{y_i}, \overline{x_i} y_i, \overline{x_i y_i}\}$ for each $i \in [N]$ at each time $t = 0, 1, 2, \ldots$ so that the configuration of the guards corresponds to a truth assignment to the variable in the Formula Game at each time $t = 0, 1, 2, \ldots$. However, if there is exactly one guard located in $\{x_i y_i, x_i \overline{y_i}, \overline{x_i} y_i, \overline{x_i y_i}\}$ for some i, then there is a variable vertex of $\{x_i y_i, x_i \overline{y_i}, \overline{x_i} y_i, \overline{x_i y_i}\}$ that is not dominated (since the subgraph induced by these vertices is not complete). To manage this concern, we add two new universal vertices ("universal" with respect to the variable vertices) v and v^* so that a new guard can move back and forth on these vertices and dominate all the variable vertices at all time.

To prevent a guard from leaving the region $\{x_i y_i, x_i \overline{y_i}, \overline{x_i} y_i, \overline{x_i y_i}\}$, we add two dummy clause vertices z_i' and z_i'', where $z_i' = z_i'' = (x_i \vee \overline{x_i} \vee y_i \vee \overline{y_i})$ for each $i \in [N]$. Each of these vertices is adjacent to each vertex of the set $\{x_iy_i, x_i\overline{y_i}, \overline{x_i}y_i, \overline{x_i}\overline{y_i}\}$ and to no other vertex in the graph. Note that the clause vertices induce an independent set of the graph, as opposed to the reduction in the proof of Theorem 2.21.

For the initial configuration, we place a guard on each vertex of the set $\{v\} \cup \bigcup_{i\in[N]}\{x_iy_i\}$ (see Figure 2.7 for an illustration). Note that the construction can be carried out in polynomial time and Player Y can win in the Formula Game if and only if $N+1$ guards can respond to any sequence of attacks on G from a particular initial configuration. □

Now, we extend our analysis of the decision problem ETERNAL DOMINATION by considering directed graphs. The corresponding decision problem will be referred to as (DIRECTED) ETERNAL DOMINATION. Note that the main difference between Theorem 2.21 and Theorem 2.23 is that the initial configuration of the guards does not really matter in (DIRECTED) ETERNAL DOMINATION (since the guards can choose their initial position) but is very important in ETERNAL DOMINATING SET (since the guards are given an initial position).

In many cases, this difference can have a noteworthy impact in the game. As we will see in Section 3.2, a graph may have an eternal dominating set of size k (in which case, k guards can win on the graph by playing optimally after choosing this eternal dominating set as their opening configuration) and a dominating set of size k that is not an eternal dominating set (in which case, k guards lose even by playing optimally after being forced to use this configuration as their opening configuration).

Theorem 2.23. *(DIRECTED) ETERNAL DOMINATION is EXPTIME-complete.*

Proof. Since Algorithm 1 determines in time $O(kn^{k+2})$ the set of configurations from which k guards can respond to any sequence of attacks on a digraph of order n, (DIRECTED) ETERNAL DOMINATION is in EXPTIME. To prove its completeness within this class, it suffices to prove that it is EXPTIME-hard. Again, we will prove this by showing that the Formula Game introduced in Section 1.2 has a polynomial-time reduction to (DIRECTED) ETERNAL DOMINATION . That is, for any instance of the Formula Game, we will construct in polynomial time a digraph D and we will show that Player Y can win in the Formula Game if and only if $2N+1$ guards can respond to any sequence of attacks on D.

So, we aim to construct a digraph based on the construction in Theorem 2.21 but with some extra vertices and arcs that allow the attacker to force the guards to move to a configuration which corresponds to the initial truth assignment in the Formula Game. This is the key point in the proof, especially when the attacker can win the game from this configuration in ETERNAL DOMINATING SET and there exists another configuration (with the same number of guards) from which the guards can win the same game.

We start by adding to φ the clauses $C_1', C_2', C_3', \ldots, C_N'$, where $C_i' = (x_iy_i \vee x_i\overline{y_i} \vee \overline{x_i}y_i \vee \overline{x_i y_i})$ for each $i \in [N]$, so that the new formula is $C_1 \wedge C_2 \wedge C_3 \wedge \cdots \wedge C_M \wedge C_1' \wedge C_2' \wedge C_3' \wedge \cdots \wedge C_N'$. Observe that we do not change the outcome of the Formula Game by adding these clauses since they are always satisfied. Now, construct the graph G the same way as in the proof of Theorem 2.21 (with the additional clause vertices $z_1', z_2', z_3', \ldots, z_N'$ which correspond to the clauses $C_1', C_2', C_3', \ldots, C_N'$), and modify it as follows to get the digraph D.

(1) Remove the vertices z and z', and all their incident edges from the graph.

(2) Replace each edge joining a variable vertex to a clause vertex by an arc from the variable vertex to the clause vertex (a sample is portrayed by the thin blue arcs in Figure 2.8), and all the other edges in the graph by an arc in each direction (a sample is indicated by the black edges in Figure 2.8).

(3) Add a new vertex z^*, an arc from z^* to each of the variable vertices $x_iy_i, \overline{x_i}y_i, \overline{x_i y_i}$ for each $i \in [N]$ (a sample is indicated by the dotted red arcs in Figure 2.8), and an arc from each clause vertex to the vertex z^* (a sample is indicated by the thick yellow arcs in Figure 2.8).

(4) Finally, for any $i, j \in [N]$ with $i \neq j$, add an arc from $\overline{x_i}y_i$, $\overline{x_i y_i}$ and $x_i\overline{y_i}$ to x_jy_j (a sample is indicated by the dashed green arcs in Figure 2.8).

For the initial configuration, the blue guard is now located on vertex z^* and each other guard is located on the same vertex as in Theorem 2.21.

Claim 3. If Player X can win in the Formula Game, then the attacker can win in the (directed) eternal domination game.

Proof. Suppose Player X can win in the Formula Game. We show that the attacker can always force the guards to be located on a configuration which corresponds to the initial truth assignment to the variables in the Formula Game. From there, the attacker can play the same way as in Theorem 2.21 and win. To see this, note that the attacker can start by selecting the vertices $x_iy_i, x_i\overline{y_i}$ for each $i \in [N]$, then vertex z^*. Since $I = \bigcup_{i=1}^{N}\{x_iy_i, x_i\overline{y_i}\}$ is an independent set of D and there is no arc from I to z^*, there will be a guard located on each vertex in I and a guard located on z^* after a successful response to the attack. Next, the attacker selects vertex w_i for each $i \in [N]$.
Suppose at some time t, there are N guards located in U, a guard located on w_i for each $i \in [N]$, and a guard located on z^*. Suppose further that at this time, the guards in U are located on (without loss of generality) $x_1y_1, x_2y_2, x_3y_3, \ldots, x_iy_i$

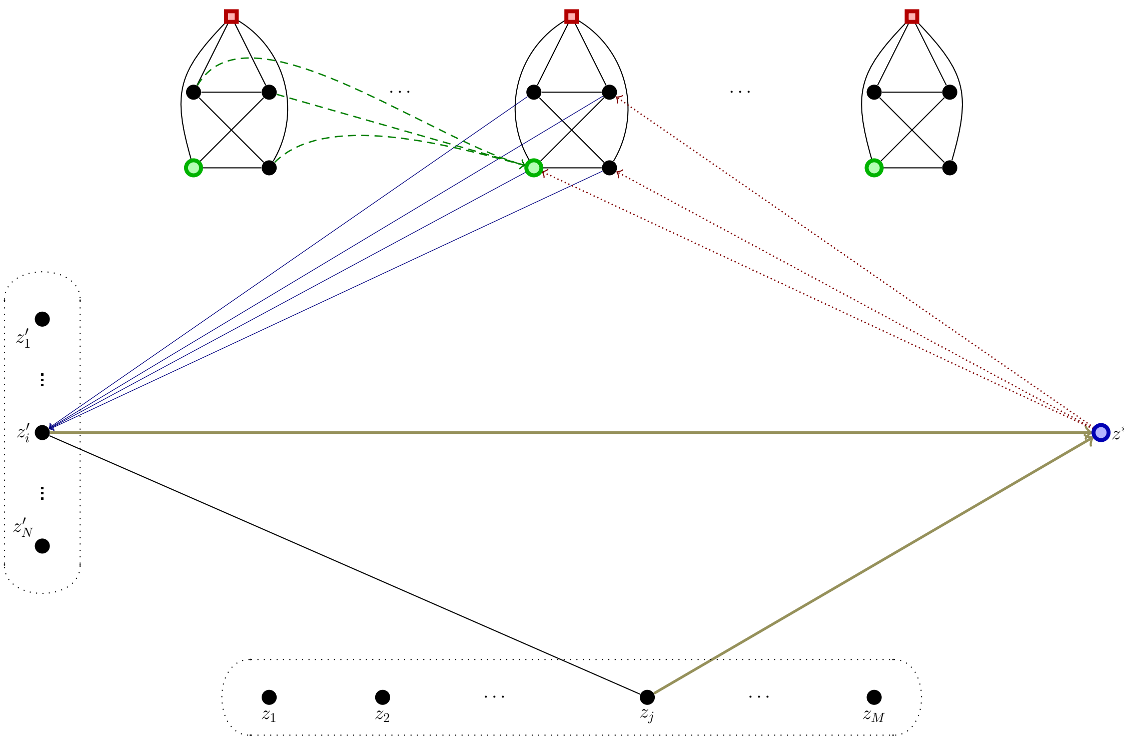


Figure 2.8: Reduction of the Formula Game to (DIRECTED) ETERNAL DOMINATION.

and $x_{i+1}\overline{y_{i+1}}, x_{i+2}\overline{y_{i+2}}, \ldots, x_N\overline{y_N}$ for some $i < N$. Then, repeat the following until the guards are located on the desired configuration.

First, the attacker selects vertex z'_{i+1}. Note that this forces the guard located on U_{i+1} (on $x_{i+1}\overline{y_{i+1}}$ to be more precise) to move there to defend the attack. Next, the attacker selects vertex $x_{i+1}y_{i+1}$. Since the guard on w_{i+1} has a private neighbour (vertex $x_{i+1}\overline{y_{i+1}}$) and there is no arc from x_jy_j to $x_{i+1}y_{i+1}$ for any $j \leq i$, either the guard on vertex z^* or a guard in U_k (more precisely on $x_k\overline{y_k}$, for some $k \geq i+2$) defends the attack. We consider each case separately:

(1) The guard on z^* defends the attack. In this case the attacker selects vertex z^* and the guard on z'_{i+1} must move to defend the attack.

(2) The guard on U_{i+2} (without loss of generality) defends the attack. In this case, the attacker selects vertex $x_{i+2}y_{i+2}$. Since the guard on w_{i+2} has a private neighbour (vertex $x_{i+2}\overline{y_{i+2}}$) and there is no arc from x_jy_j to $x_{i+2}y_{i+2}$ for any $j \leq i+1$, either the guard on vertex z^* or a guard in U_k (for some $k \geq i+3$) defends the attack. If the guard on z^* responds to the attack, then we repeat (1); otherwise, repeat (2).

Once Steps (1) and (2) have been completed, the guards will be located on the desired configuration. Now that the guards are located on a configuration that corresponds to the initial truth assignment to the variables in the Formula

Game, it remains to show that the attacker can win by following the same strategy as in the proof of Theorem 2.21. Note that the attacker's original strategy (in G) consists in selecting only vertices from the subgraph induced by the set $U \cup W$. Should the guards prevent the attacker from winning in D, they have to use the newly added arcs (from $\overline{x_i}y_i$, $\overline{x_i y_i}$ and $x_i\overline{y_i}$ to $x_j y_j$ where $i, j \in [N], i \neq j$) at some time t. However, if a guard in $U_i \cup W_i$ defends an attack in U_j, there will be exactly one guard in $U_i \cup W_i$ and that guard either does not dominate z_i' or has two private neighbours (w_i and z_i'). Hence, the attacker still wins. $\diamondsuit$

Claim 4. If Player Y can win in the Formula Game, then the guards can win in the (directed) eternal domination game.

Proof. Suppose Player Y can win in the Formula Game. Then the guards can start from a configuration in the digraph which corresponds to the given initial truth assignment to the variables in the Formula Game. If the attacker plays as "expected", that is, by never attacking a clause vertex, then the guards win by following the same strategy in the proof of Theorem 2.21. So, suppose at some time t in the game, that the attacker attacks a clause vertex. A green guard from a variable vertex (in U_j, without loss of generality) responds to the attack. In this case, there is a green guard on a clause vertex, a blue guard on the vertex z^*, exactly one (red) guard in the set $U_j \cup \{w_j\}$ and two guards in $U_i \cup \{w_i\}$ for each $i \neq j$. If there is an attack on a clause vertex, the guard on the other clause vertex responds to the attack. If there is an attack on a vertex in $U_i \cup \{w_i\}$, the two guards in there respond to the attack the same way they would do if there was a green guard on some vertex in U_j. Finally, the guard on the vertex z^* never moves to respond to an attack, except when both vertices in either the set $\{x_j y_j, x_j\overline{y_j}\}$ or the set $\{\overline{x_j y_j}, \overline{x_j}y_j\}$ are attacked, in which case, we colour these vertices red and green, and the guard on the clause vertex blue. If z^* is now attacked, the blue guard on a clause vertex moves to z^*. After responding to the attack, the green guards are located on the vertices of a configuration which corresponds to a good assignment in the Formula Game. $\diamondsuit$

This completes the proof. $\square$

2.3 Computational Methods

In the following chapter, we report on a large-scale computation performed on a PowerEdge R7425 server equipped with two AMD EPYC processors and totalling 64 threaded CPU cores for the purposes of verifying some of the propositions in the dissertation. The computations were done in Python using NetworkX, Algorithm 1, Algorithm 2 and the previously fastest known algorithms (see [30, 31, 42]) along with a few basic graph algorithms.

In our search, we often needed to generate the set of all graphs belonging to some particular class; to this end, we used NAUTY [43] (version 2.7001) and PLANTRI [7, 8] (version 5.2). On the one hand, using NAUTY, we were able to generate the set of all graphs, the set of all triangle-free graphs, and the set of all cubic graphs of order n for some given values of n. On the other hand, using PLANTRI, we were able to generate the set of all planar graphs of order n for some given values of n.

Since we studied the relationship between the eternal domination number and other graph parameters such as the domination number, the independence number and the clique covering number, we also found the value of each of these parameters for each graph in our computation.

We computed the independence number of each graph in our search by computing the clique number of its complement using the built-in functions `complement()` and `graph_clique_number()` from the Python package NetworkX. As for the clique covering number, we computed this value using three different approaches. For general graphs, we started by enumerating the set of maximal cliques of the graph using the built-in function `find_cliques()` from NetworkX. Using a naive method and an integer programming method, we found the size of the smallest subset of the set of maximal cliques which covers all the vertices of the graph. To solve the integer programs, we used the Python packages `PuLP` and `MIP` and they both agreed on the results. Our naive method is on average faster than the integer programming method on the set of graphs of up to order 11. When it is known in advance that the graph is triangle-free, we first found the size of a maximum matching in the graph using the function `max_weight_matching()` from NetworkX and then substracted this number from the order of the graph. As for the domination number of the graph, we found this value by first generating the k-vertex subsets of the n-vertex set. Then, using the built-in function `is_dominating_set` from NetworkX, we checked whether the graph had a dominating set of that size.

Now, to find the eternal domination number of the graph, we first compared its independence number to its clique covering number using the algorithms described above. If these parameters are equal, then they are also equal to the eternal domination number. Otherwise, we computed the eternal domination number using Algorithm 1.

The largest single computation took approximately 216 CPU days before the results (Table 3.2) were found. The class of graphs on which our computations were the slowest is the set of maximal triangle-free graphs with independence numbers less than their clique covering numbers. For example, it took about 40 CPU hours to output the eternal domination number of the circulant graph $C_{29}[10, 11, 12, 13, 14]$ from Section

3.4, (see Table 3.8).

Now, to find the eviction number of the graph, we first compared its domination number to its clique covering number using the algorithms described above. If these parameters are equal, then they are also equal to the eviction number. Otherwise, we computed the eternal domination number using Algorithm 2.

Chapter 3

Eternal Domination

Our analysis in this chapter is centered on the structural aspects of the eternal domination game. We initiate our investigation by studying the relationship between the eternal domination number of a graph and its clique covering number. Moving forward, we transition our attention to presenting an infinite class of counterexamples to the Fundamental Conjecture, and conclude by presenting findings that are related to the length of the game on general graphs. The results in the following two sections are also published in [42].

3.1 Eternal Domination and Clique Covering

3.1.1 Smallest Graphs with $\gamma^{\infty} < \theta$

We begin the section by showing that the graphs G_1 and G_2 depicted in Figure 3.1 are the smallest graphs having their eternal domination numbers less than their clique covering numbers. Observe that G_2 is obtained from G_1 by adding the edge (67).

(a) G_1 (b) G_2

Figure 3.1: Smallest graphs with $\gamma^{\infty} < \theta$.

We first show that these graphs have eternal domination numbers less than their clique covering numbers. To this end, we begin with the following observations; the

(a) $\overline{G_1}$ (b) $\overline{G_2}$

Figure 3.2: Complements of the graphs in Figure 3.1.

proofs are easy and left to the reader.

Observation 3.1. For the graphs $G_1, G_2, \overline{G_1}$ and $\overline{G_2}$ depicted in Figure 3.1 and in Figure 3.2,
(1) $\alpha(G_1) = \alpha(G_2) = \omega(\overline{G_1}) = \omega(\overline{G_2}) = 3$.
(2) $\theta(G_1) = \theta(G_2) = \chi(\overline{G_1}) = \chi(\overline{G_2}) = 4$.

Observation 3.2. The graph $\overline{G_1}$ contains exactly six triangles, namely $\{2, 3, 4\}$, $\{5, 6, 7\}$, $\{0, 2, 8\}$, $\{0, 1, 7\}$, $\{1, 3, 9\}$ and $\{6, 8, 9\}$, any two of which share at most one vertex.

Observation 3.3. The subgraph induced by the vertices in the open neighbourhood of each vertex of $\overline{G_1}$ is isomorphic to either $2K_2$ or to $K_1 \cup K_2$.

Observation 3.4. Suppose k guards are located on a graph G with independence number k. The attacker can force the guards to move to any given maximum independent set of G.

The following result can be verified by computer, but we include a proof as well.

Proposition 3.5. *For the graphs G_1 and G_2 depicted in Figure 3.1,*

$$\gamma^\infty(G_1) = \gamma^\infty(G_2) < \theta(G_1) = \theta(G_2).$$

Proof. Since G_1 is a spanning subgraph of G_2, it suffices to show that $\gamma^\infty(G_1) = 3$. Observation 1.3 will then imply that $\gamma^\infty(G_2) = 3$. We do this by contradiction and assume that $\gamma^\infty(G_1) \geq 4$. The proof is illustrated in Figures 3.3, 3.4, 3.5, 3.6 and 3.7, where a thick black edge corresponds to an edge in G_1 and a thin blue edge corresponds to an edge in $\overline{G_1}$. By Observation 3.4, we may assume that three guards are initially located on the vertices of an independent set of G_1. Let k be the smallest integer such that there exists a sequence of attacks of length k that the three guards cannot defend. Suppose the guards respond optimally to that sequence of attacks. At time $t = k - 1$, they will be located on the vertices b, c and d, none of which is adjacent to vertex a

(see Figure 3.3).

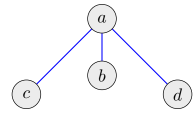


Figure 3.3: Configuration of the guards at time $t = k - 1$.

Following Observation 3.3, we may assume that vertex c is not adjacent to vertex d and vertex b is adjacent to both of c and d. So, we colour the edge cd blue and the edges bc and bd black (see Figure 3.4).

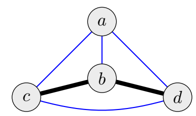
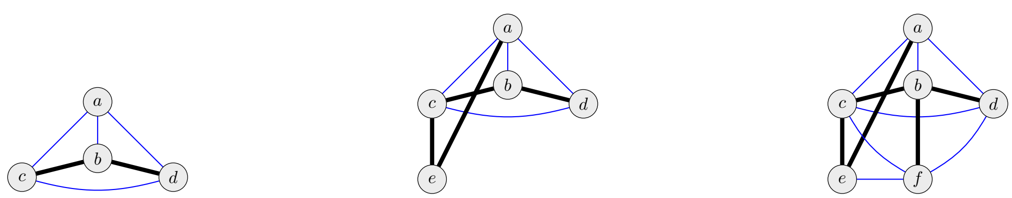


Figure 3.4: Configuration of the guards at time $t = k - 1$.

Note that in the previous time ($t = k - 2$), the guards dominated all the vertices of G, in particular vertex a was dominated. We now consider two cases depending on the location of the guards at time $t = k - 2$.

– **Case 1:** In the previous turn, a guard moved from a vertex e to vertex c (see Figure 3.5b). In this case, e is adjacent to a because a was dominated at time $t = k-2$ but is undominated at time $t = k-1$, and b has a neighbour $f \notin \{a, c, d\}$ which is non-adjacent to all of c, d and e (otherwise the guards would still be on a dominating set if the guard on b moved to c instead; see Figure 3.5c). This contradicts Observation 3.2 since the thin blue triangles $\{a, c, d\}$ and $\{f, c, d\}$ share two vertices (c and d; see Figure 3.5c).



(a) (b) (c)

Figure 3.5: Configuration of the guards in Case 1 of the proof of Proposition 3.5.

– **Case 2:** In the previous turn, a guard moved from a vertex e to vertex b (see Figure 3.6b). As before, e is adjacent to a; in this case, c has a neighbour f which is non-adjacent to b, d and e (otherwise the guards would still be on a dominating set if the guard on c moved to b instead). For the same reason, d has a neighbour g which is not adjacent to b, c and e, as shown in Figure 3.6c. Following Observation 3.2, f must be adjacent to g, otherwise $\{f, b, g\}$

and $\{f, e, g\}$ would be two thin blue triangles sharing two vertices. By the same observation, a must be adjacent to f as $\{a, c, d\}$ and $\{a, f, d\}$ would share two vertices otherwise. For a similar reason, a must be adjacent to g as $\{a, c, g\}$ and $\{a, c, d\}$ would be two thin blue triangles sharing two vertices otherwise (see Figure 3.6d).

Now, the edges ce and de cannot be both thin blue, otherwise the thin blue triangles $\{a, c, d\}$ and $\{e, c, d\}$ would share two vertices. We assume without loss of generality that the edge ce is black and we consider the two choices for the colour of the edge de.

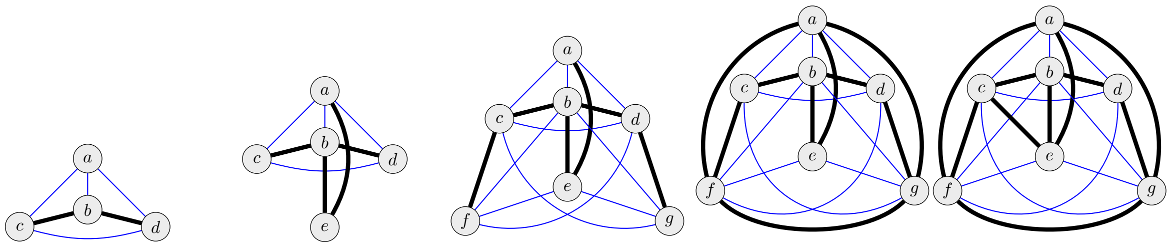


Figure 3.6: Configuration of the guards in Case 2 of the proof of Proposition 3.5.

So, we obtain two blue/black colourings of the edges of the complete graph on 7 vertices. One of the thin blue subgraphs obtained (Figure 3.7) must be an induced subgraph of $\overline{G_1}$.

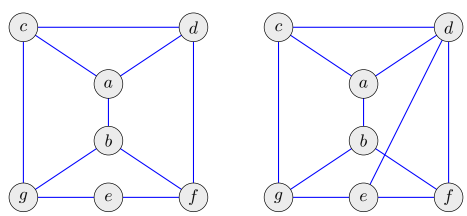


Figure 3.7: Induced subgraphs of $\overline{G_1}$.

The reader can check that $\overline{G_1}$ does not contain any of those graphs as an induced subgraph. Hence, we obtain a contradiction and conclude that $\gamma^\infty(G_1) = 3$. □

In the remaining part of this section, we verify that any graph with fewer than 10 vertices has eternal domination number equal to its clique covering number.

Proposition 3.6. *For any graph G of order 9 or less, $\gamma^\infty(G) = \theta(G)$.*

Proposition 3.6 was verified by computer; the search can be described as follows. Suppose G is a smallest graph such that $\gamma^\infty(G) < \theta(G)$ with the maximum number of edges. Observation 1.5 implies that $\alpha(G) < \theta(G)$; moreover, by the choice of G and Observation 1.3, deleting a vertex from the graph or adding a missing edge to the graph decreases its clique covering number by 1. Consequently, G is a critical (with respect to θ) graph.

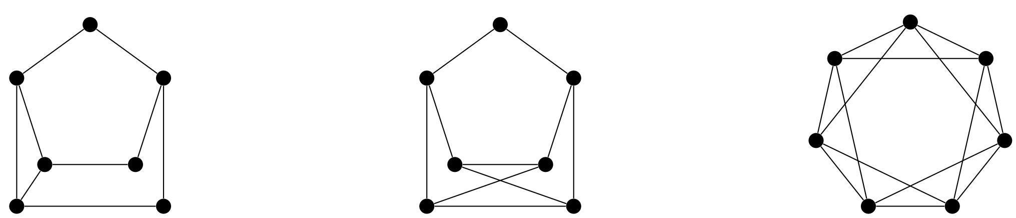
Table 3.1 shows the number of critical graphs of order 9 or less and each of these graphs is either drawn in Figures 3.8 or 3.9, or listed in Graph6 format in Table 5.1 in Appendix A. To find these graphs, we used NAUTY [43] to generate the set of connected graphs on fewer than 10 vertices. Then, using some basic graph algorithms, as described in Section 2.3, we computed the independence number and the clique covering number of each of these graphs. For those graphs with independence numbers less than their clique covering numbers, we checked whether they are critical graphs (with respect to θ) before computing their eternal domination numbers using Algorithm 1 and the previously fastest known algorithm. All the computations were done in Python.

Using a similar technique, without considering only critical graphs, we found the complete set of 10-vertex and 11-vertex graphs with $\gamma^\infty < \theta$. Those graphs are listed in Table 5.2 in the appendix in Graph6 format.

The graph C_5 is the only critical graph of order 5 with $\alpha < \theta$. We display the critical graphs of order 7 and 8 in Figures 3.8 and 3.9, and list the critical graphs (with respect to θ) of order 9 with $\alpha < \theta$ in Graph6 format in the appendix (Table 5.1).

n	Total	$\alpha < \theta$	Vertex-Critical & $\alpha < \theta$	Critical & $\alpha < \theta$	Critical & $\gamma^\infty < \theta$
5	21	1	1	1	0
6	112	3	0	0	0
7	853	33	8	3	0
8	11117	498	7	4	0
9	261080	16539	353	38	0
10	11716571	975676	5159	290	1

Table 3.1: Number of critical graphs (with respect to θ) on n vertices with $\gamma^\infty < \theta$.



(a) (b) (c)

Figure 3.8: Critical graphs (with respect to θ) of order 7 with $\alpha < \theta$.

3.1.2 Graph Classes

In this section, we focus on different classes of graphs, starting with triangle-free graphs, before moving on to circulant graphs and planar graphs. We briefly mention claw-free

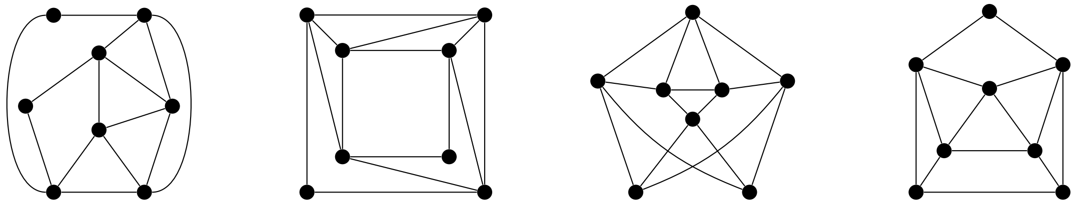


Figure 3.9: Critical graphs (with respect to θ) of order 8 with $\alpha < \theta$.

graphs and cubic graphs.

Triangle-Free Graphs

Erdős, Kleitman and Rothschild [18] showed that almost all triangle-free graphs are bipartite, therefore almost all triangle-free graphs are perfect and satisfy $\gamma^{\infty} = \theta$. However, Goddard, Hedetniemi and Hedetniemi [23] showed that there exist triangle-free graphs with $\gamma^{\infty} < \theta$. They gave the circulant graphs $C_{18}[1,3,8]$ and $C_{21}[1,3,8]$, which they found using computer assistance, as examples. Using Algorithm 1, we checked that any triangle-free graph of order 12 or less has eternal domination number equal to its clique covering number.

We found 13 triangle-free graphs of order 13 with $\gamma^{\infty} < \theta$ (see Table 5.3). The smallest previously known triangle-free graph with this property has 17 vertices and is a subgraph of the circulant graph $C_{18}[1,3,8]$. The following corollary describes a way to generate an infinite family of triangle-free graphs with $\gamma^{\infty} < \theta$.

Observation 3.7. Let G be a triangle-free graph. Then $G \bowtie K_2$ is triangle-free.

Corollary 3.8. *Let G be a triangle-free graph such that $\gamma^{\infty}(G) \leq \lceil \frac{n}{2} \rceil - 1$. Then, $\gamma^{\infty}(G \bowtie K_2) < \theta(G \bowtie K_2)$.*

Proof. Let G be a triangle-free graph such that $\gamma^{\infty}(G) \leq \lceil \frac{n}{2} \rceil - 1$. Observation 3.7 implies that $G \bowtie K_2$ is triangle-free; as a result, $\theta(G \bowtie K_2) \geq n$. Observe that the vertices of $G \bowtie K_2$ can be partitionned into two subsets, each of which induces a subgraph isomorphic to G. This means that $\gamma^{\infty}(G \bowtie K_2) \leq 2\gamma^{\infty}(G) \leq 2(\lceil \frac{n}{2} \rceil - 1)$; consequently, $\gamma^{\infty}(G \bowtie K_2) \leq n - 1$. □

Klostermeyer and Mynhardt [36] posed the following question.

Question 3.9 ([36])**.** Does there exist a triangle-free graph G such that $\alpha(G) = \gamma^{\infty}(G) < \theta(G)$?

Note that the 13 triangle-free graphs on 13 vertices with $\gamma^{\infty} < \theta$ we found also satisfy $\alpha < \gamma^{\infty} < \theta$. In Proposition 3.10, we describe a property of a smallest such a graph (if it exists); then we show by using computer assistance that no graph of order 14 or less satisfies this property.

Proposition 3.10. *Suppose there exist triangle-free graphs with* $\alpha = \gamma^\infty < \theta$ *and let* G *be such a graph with minimum order. Let* X *be a maximum independent set of* G *and let* $Y = V(G) - X$. *Then* $|X| = |Y| - 1$.

Proof. It is clear that $|X| \geq |Y| - 1$, otherwise, by Observation 1.3, $\alpha(G - \{v\}) = \gamma^\infty(G - \{v\}) < \theta(G - \{v\})$ for any $v \in Y$, and $G - \{v\}$ would be a smaller triangle-free graph with $\alpha = \gamma^\infty < \theta$. So, it remains to show that $|X| \leq |Y| - 1$. Suppose this is false, in other words $|X| \geq |Y|$. Let G' be the spanning bipartite subgraph of G obtained by deleting all edges having both endpoints in Y. Observe that the graph G' does not contain a matching that covers all of the vertices in Y, otherwise, G would contain a matching that matches each vertex in Y to a vertex in X, and this would imply that $\alpha(G) = \theta(G)$, which is a contradiction. Consequently, by Hall's matching condition, Y contains a subset S such that $|N_{G'}(S)| < |S|$. Now, consider the subgraph H of G induced by $S \cup N_{G'}(S)$. Since there is no edge between a vertex in H and a vertex in $X - V(H)$, we have $\alpha(H) = |N_{G'}(S)|$, otherwise G contains an independent set of size at least $|X| + 1$. Moreover, since $\gamma^\infty(G) = |X|$ we must have $\gamma^\infty(H) = |N_{G'}(S)|$: this is true because the attacker may force the $|X|$ guards to be located in X and from there only attacks the vertices in H. In this case, only the $|N_{G'}(S)|$ guards are able to respond to that sequence of attacks on H. Hence, H would be a smaller triangle-free graph with $\alpha = \gamma^\infty < \theta$. □

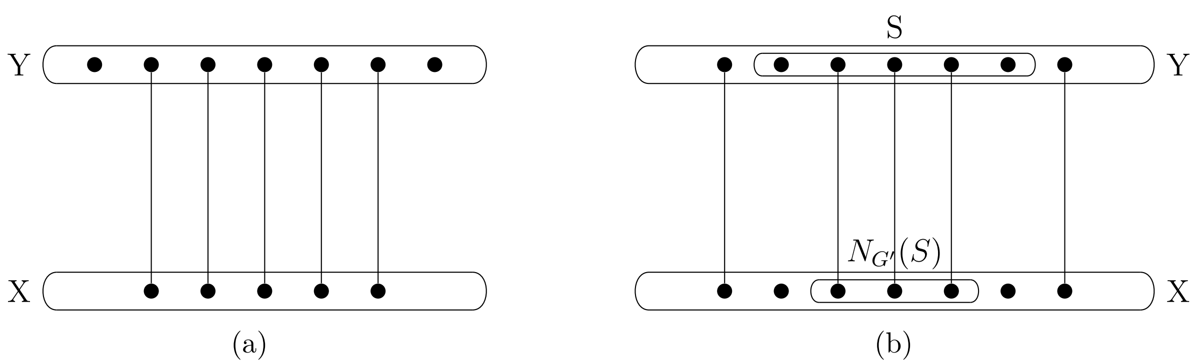


Figure 3.10: Cases considered in the proof of Proposition 3.10.

Using the property stated in Proposition 3.10, to find a smallest triangle-free graph G with $\alpha = \gamma^\infty < \theta$, we only need to check the triangle-free graphs of order $2k + 1$ with $\alpha = k$ and $\theta = k + 1$. Again, using NAUTY, we generated the set of triangle-free graphs of odd orders on fewer than 14 vertices; then, we computed the independence number and the clique covering number of each of these graphs using the algorithms described in Chapter 2. For graphs with $\alpha = \frac{n-1}{2}$ and $\theta = \frac{n+1}{2}$, we computed the eternal domination number of the graph. We performed the computation listed in Table 3.2 on a cluster which required approximately 218 CPU days.

Observation 3.11. There is no triangle-free graph on $n \leq 14$ vertices with $\alpha = \gamma^\infty < \theta$.

n	Total	$\alpha = \lfloor \frac{n}{2} \rfloor$	$\alpha = \lfloor \frac{n}{2} \rfloor$ & $\theta = \lceil \frac{n}{2} \rceil$	$\alpha = \gamma^\infty = \lfloor \frac{n}{2} \rfloor$ & $\theta = \lceil \frac{n}{2} \rceil$
5	6	1	1	0
7	59	8	8	0
9	1380	276	276	0
11	90842	29660	29660	0
13	19425052	9606337	9606334	0

Table 3.2: Number of connected triangle-free graphs on n vertices with $\alpha = \lfloor \frac{n}{2} \rfloor$ and $\theta = \lceil \frac{n}{2} \rceil$ for odd n.

Maximal Triangle-Free Graphs

A graph G is said to be *maximal triangle-free* if G is triangle-free and the insertion of any missing edge in G creates a triangle. The maximal triangle-free graphs are considered to be an interesting family of graphs since several problems on triangle-free graphs can be studied by restricting attention to maximal triangle-free graphs.

Observe that if G is a triangle-free graph with $\lfloor \frac{n}{2} \rfloor = \alpha(G) = \gamma^\infty(G) < \theta(G) = \lceil \frac{n}{2} \rceil$, as in Proposition 3.10, then G is a subgraph of a maximal triangle-free graph G^* with $\gamma^\infty(G^*) < \theta(G^*)$. This follows since the clique covering number of a triangle-free graph G is at least $\lceil \frac{n}{2} \rceil$ and adding any missing edge to G that does not create a triangle neither decreases its clique covering number nor increases its eternal domination number.

Our computer search yields the following result.

Observation 3.12. There is no maximal triangle-free graph on $n < 18$ vertices with $\alpha = \gamma^\infty < \theta$.

Circulant Graphs

Circulant graphs are another interesting family of graphs on which we study the problem. Using Algorithm 1, we found all the circulant graphs of order 20 or less with $\gamma^\infty < \theta$ (see Table 3.4).

The following proposition shows that the bow tie product of a circulant graph with K_2 is a circulant graph.

n	Total	$\alpha < \theta$	$\gamma^\infty < \theta$	$\alpha = \gamma^\infty < \theta$
10	31	7	0	0
11	61	28	0	0
12	147	51	0	0
13	392	247	5	0
14	1274	687	2	0
15	5036	4022	173	0
16	25617	19623	61	0
17	164796	153224	24910	0

Table 3.3: Number of maximal triangle-free graphs on n vertices with $\gamma^\infty < \theta$.

n	List of graphs
13	$C_{13}[1,3,4]$, $C_{13}[1,2,3,5]$.
14	None
15	$C_{15}[1,3,4]$.
16	$C_{16}[1,2,4,5]$, $C_{16}[1,2,3,4,6]$.
17	$C_{17}[1,2,4,8], C_{17}[1,2,3,5,6], C_{17}[1,2,3,5,8]$.
18	$C_{18}[1,3,8], C_{18}[1,2,4,5,6], C_{18}[1,2,4,5,6,9]$.
19	$C_{19}[1,4,6], C_{19}[1,3,5,6], C_{19}[1,2,3,4,5,7], C_{19}[1,2,3,5,7,8]$.
20	$C_{20}[1,5,8], C_{20}[2,5,6], C_{20}[1,6,8,9], C_{20}[1,2,4,5,6], C_{20}[1,2,4,5,7]$ $C_{20}[1,2,5,7,8], C_{20}[1,2,3,4,5,7,8], C_{20}[1,2,3,4,6,7,10], C_{20}[1,3,4,7,8,9,10]$.

Table 3.4: List of small circulant graphs with $\gamma^\infty < \theta$.

Proposition 3.13. *For any integer n,*

$$C_n[k_1, k_2, ..., k_l] \bowtie K_2 \cong C_{2n}[2k_1, 2k_2, \ldots, 2k_l, 2k_1+1, 2k_2+1, \ldots, 2k_l+1].$$

Proof. Let $G = C_n[k_1, k_2, ..., k_l]$ be a circulant graph and let $H = C_n[k_1, k_2, ..., k_l] \bowtie K_2$. Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(K_2) = \{0, 1\}$. For each $i \in \{1, 2, \ldots, n\}$, let $u_{2i} = (v_i, 0)$ and $u_{2i+1} = (v_i, 1)$. The definition of the bow tie product implies the following statements:

- The vertices u_{2i} and u_{2j} are adjacent in H if and only if the vertices v_i and v_j are adjacent in G; that is, if and only if $i - j \equiv \pm k \pmod n$ for some $k \in \{k_1, k_2, \ldots, k_l\}$. Equivalently, u_{2i} is adjacent to u_{2j} if and only if $2i - 2j \equiv \pm k \pmod{2n}$ for some $k \in \{2k_1, 2k_2, ..., 2k_l\}$.
- The vertices u_{2i+1} and u_{2j+1} are adjacent in H if and only if the vertices v_i and v_j are adjacent in G. Therefore, u_{2i+1} is adjacent to u_{2j+1} if and only if $2i-2j \equiv \pm k \pmod{2n}$ for some $k \in \{2k_1, 2k_2, ..., 2k_l\}$.
- The vertices u_{2i+1} and u_{2j} are adjacent in H if and only if the vertices v_i and v_j are adjacent in G; that is, if and only if $i - j \equiv \pm k \pmod n$ for some $k \in$

$\{k_1, k_2, \ldots, k_l\}$. Equivalently, u_{2i+1} is adjacent to u_{2j} if and only if $(2i+1)-2j \equiv \pm k \pmod{2n}$ for some $k \in \{2k_1+1, 2k_2+1, ..., 2k_l+1\}$.

Thus, for each $i, j \in \{0, 1, 2, \ldots, 2n-1\}$, the vertices u_i and u_j are adjacent if and only if $i-j \equiv \pm k \pmod{2n}$ for some $k \in \{2k_1, 2k_2, \ldots, 2k_l, 2k_1+1, 2k_2+1, \ldots, 2k_l+1\}$. □

Corollary 3.14. *There exist infinitely many circulant graphs with $\gamma^\infty < \theta$.*

Proof. This follows from Observation 3.7, Corollary 3.8 and Proposition 3.13. □

Planar Graphs

A graph is said to be *planar* if it can be drawn on a plane in a way such that no two of its edges intersect, except possibly at their common vertices. The graph is further said to be *outerplanar* if it has a planar drawing in which all of the vertices of the graph belong to the unbounded face of the drawing.

Anderson, Barrientos, Brigham, Carrington, Vitray and Yellen [2] showed that any outerplanar graph has eternal domination number equal to its clique covering number. Moreover, no planar graph where $\gamma^\infty < \theta$ is known. This suggest that it might be true for general planar graphs and motivates the following question of Klostermeyer and Mynhardt.

Question 3.15 ([37])**.** Is it true that $\gamma^\infty(G) = \theta(G)$ if G is planar?

We first introduce two more definitions, and then describe some properties of a smallest planar graph with $\gamma^\infty < \theta$, if it exists.

Definition 3.16. A graph G is said to be *k-connected* if G has more than k vertices and remains connected whenever fewer than k vertices are removed.

Definition 3.17. A vertex v of a connected graph G is said to be a *cut vertex* of G if $G - \{v\}$ has a least two components.

Proposition 3.18. *Let $\mathcal{F}$ be a family of graphs satisfying a hereditary property $\mathcal{P}$; i.e. if $G \in \mathcal{F}$ and G satisfies property $\mathcal{P}$, then H satisfies property $\mathcal{P}$ for any subgraph H of G. Suppose $\mathcal{F}$ contains graphs G such that $\gamma^\infty(G) < \theta(G)$. Then, the smallest such graph G is a 2-connected vertex critical (with respect to θ) graph.*

Proof. It is clear that G is a vertex-critical graph with respect to θ; otherwise, by Observation 1.3, G contains a proper induced subgraph which is also in $\mathcal{F}$ with $\gamma^\infty < \theta$, which is a contradiction. Suppose G is not 2-connected and let v be a cut vertex of G, that is a vertex such that $G - \{v\}$ has k components $\{G_1, G_2, \ldots, G_k\}$ where $k \geq 2$. By the minimality of G, $\gamma^\infty(G_i) = \theta(G_i)$ for each $i \in \{1, 2, \ldots, k\}$. Since $\gamma^\infty(G) \geq \sum_{i=1}^{k} \gamma^\infty(G_i)$, $\theta(G) \leq \sum_{i=1}^{k} \theta(G_i) + 1$ and $\gamma^\infty(G) < \theta(G)$, we conclude that

$\gamma^\infty(G) = \sum_{i=1}^{k} \gamma^\infty(G_i)$ and $\theta(G) = \sum_{i=1}^{k} \theta(G_i) + 1$. This implies that there exists $i \in \{1, 2, \ldots, k\}$ such that $\gamma^\infty(G_i) = \gamma^\infty(G_i + \{v\})$ and $\theta(G_i + \{v\}) = \theta(G_i) + 1$. In this case, $G_i + \{v\}$ is a smaller graph in $\mathcal{F}$ with $\gamma^\infty < \theta$, which is a contradiction. □

Since planarity is a hereditary property, the smallest planar graph with $\gamma^\infty < \theta$, if it exists, is a 2-connected vertex-critical (with respect to θ) graph. Observe that none of the graphs on 11 vertices or less satisfying $\gamma^\infty < \theta$ we found (Table 5.2) are planar, hence we have the following observation.

Observation 3.19. There is no planar graph on 11 vertices or less with $\gamma^\infty < \theta$.

We also considered planar graphs of higher orders (12 and 13) in our search and we used plantri [7, 8] (version 5.2) to generate them. Due to the limitations of plantri, we only considered 3-connected planar graphs and obtained the following observation.

Observation 3.20. There is no 3-connected planar graph on 13 vertices or less with $\gamma^\infty < \theta$.

n	Total	$\alpha < \theta$	Vertex-Critical & $\alpha < \theta$	Vertex-Critical & $\gamma^\infty < \theta$
10	32300	2773	14	0
11	440564	25771	74	0
12	6384634	745440	878	0
13	96262938	6774391	2475	0

Table 3.5: Number of 3-connected planar graphs on n vertices.

Claw-Free Graphs

A *claw-free graph* is a graph that does not contain the complete bipartite graph $K_{1,3}$, also known as the claw, as an induced subgraph. In particular, any graph with independence number 2 is claw-free. Observe that the complements of the Mycielski graphs M_k described in Section 1.2 are claw-free. These graphs have independence number 2, eternal domination number 3 and clique covering number k for $k \geq 4$. Consequently, there are infinitely many claw-free graphs with $\gamma^\infty < \theta$.

Cubic Graphs

A *cubic graph* is a graph in which all vertices have degree 3. Using NAUTY along with Algorithm 1, we generated the set of cubic graphs on fewer than 18 vertices and obtained the following observation.

Observation 3.21. There is no cubic graph on $n \leq 16$ vertices with $\gamma^\infty < \theta$.

n	Total	$\alpha < \theta$	$\gamma^\infty < \theta$
4	1	0	0
6	2	0	0
8	5	2	0
10	19	9	0
12	85	46	0
14	509	320	0
16	4060	2888	0

Table 3.6: Number of connected cubic graphs on n vertices.

3.1.3 The Gamma-Theta Conjecture

The primary objective of this section is to provide data that support the Gamma-Theta Conjecture, which states that the domination number of a graph is equal to the eternal domination number of the graph if and only if the domination number of the graph is equal to the clique covering number of the graph. Our computer search yields the following observation.

Observation 3.22. There is no counterexample to the Gamma-Theta Conjecture of order $n \leq 11$.

n	Total	$\gamma = \alpha$	$\gamma = \gamma^\infty$	$\gamma = \gamma^\infty = \theta$
5	21	6	5	5
6	112	24	22	22
7	853	88	67	67
8	11117	524	358	358
9	261080	4515	2265	2265
10	11716571	73515	23394	23394
11	1006700565	2324209	396755	396755

Table 3.7: Number of connected graphs on n vertices with $\gamma = \alpha$, $\gamma = \gamma^\infty$ and $\gamma = \theta$.

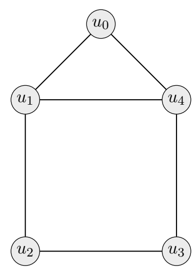
The computation in Table 3.7 was performed on a cluster which required approximately 85 CPU days. We found 54 graphs (listed in Table 5.2 in the appendix) with $\gamma^\infty < \theta$ among which there are 53 graphs with $\alpha = \gamma^\infty < \theta$ and none with $\gamma = \gamma^\infty < \theta$.

3.2 γ-Sets and γ^∞-Sets

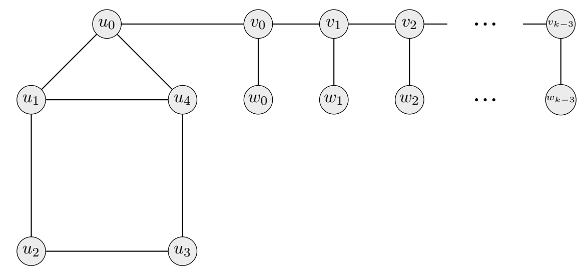
This section is dedicated to providing a negative answer to Question 1.14 of Klostermeyer and Mynhardt, as to whether every minimum dominating set, also known as a γ-set, is a minimum eternal dominating set, also known as a γ^∞-set, in a graph with domination number equal to the eternal domination number. We state this more formally in the following proposition.

Proposition 3.23. *For any integer $k \geq 2$, there exists a graph G such that $\gamma(G) = \gamma^{\infty}(G) = k$ having minimum dominating sets which are not eternal dominating sets.*

Proof. Consider the house graph G_1 in Figure 3.11a. It can be easily checked that $\gamma(G_1) = \gamma^{\infty}(G_1) = \theta(G_1) = 2$. Moreover, the set $\{u_1, u_4\}$ is a dominating set of G; however, if the guards are located on the vertices u_1 and u_4, then any response to an attack on the vertex u_0 leaves either the vertex u_2 or the vertex u_3 undominated on the next turn. Figure 3.11b shows a generalisation of the graph in Figure 3.11a where $\gamma(G_2) = \gamma^{\infty}(G_2) = k$. The set $\{u_1, u_4\} \cup \{v_0, v_1, \ldots, v_{k-3}\}$ is a dominating set of size k, but any response to an attack on vertex u_0 leaves one of the vertices u_2, u_3 or w_0 undominated on the next turn. □



(a) A graph G_1 with $\gamma(G_1) = \gamma^{\infty}(G_1) = 2$ having more minimum dominating sets than eternal dominating sets.



(b) A graph G_2 with $\gamma(G_2) = \gamma^{\infty}(G_2) = k$ having more minimum dominating sets than eternal dominating sets.

Figure 3.11: Two graphs with domination numbers equal to their eternal domination numbers having more minimum dominating sets than minimum eternal dominating sets.

3.3 The Fundamental Conjecture

In this section, our goal is to provide an infinite family of counterexamples to the Fundamental Conjecture. Before presenting our result, it is pertinent to remind the reader of the precise statement of the conjecture.

Conjecture (The Fundamental Conjecture [38])**.** *Let G be a graph with $\delta \geq 1$ and minimum eternal dominating set D. For every vertex $v \in D$ with an unoccupied neighbour, there exists an eternal dominating set D' such that $D' = (D - \{v\}) \cup \{u\}$, where $u \in N(v) - D$.*

We now proceed with the following lemmas.

Lemma 3.24. *The circulant graph $C_{11}[4, 5]$ satisfies $\gamma^{\infty}(C_{11}[4, 5]) = 6$.*

Proof. It is known from using computer assistance (Section 3.1.2), that any triangle-free graph G on fewer than 13 vertices satisfies $\gamma^\infty(G) = \theta(G)$. Since $C_{11}[4,5]$ is a triangle-free graph on 11 vertices with $\theta(C_{11}[4,5]) = 6$, we conclude that $\gamma^\infty(C_{11}[4,5]) = 6$. □

It is also worth noting that Burnside's Lemma [10] implies there are $\frac{\binom{11}{5}+11\times\binom{5}{2}}{22} = \frac{462+110}{22} = 26$ different ways (up to symmetry) that five guards can be located in G. Using Algorithm 1, we have verified that the guards can lose in at most 10 turns from any of these configurations. The maximum number of attacks the guards can defend from each of these configurations (assuming optimal play) is listed in Figure 5.1 and in Table 5.5 in Appendix A.

Before moving on to the next lemma, we present the following crucial definition.

Definition 3.25. A vertex v of a graph G is said to be *critical* (with respect to γ^∞) if $\gamma^\infty(G - \{v\}) = \gamma^\infty(G) - 1$.

It is important to note that a critical vertex with respect to γ^∞ is not necessarily a critical vertex with respect to θ, and vice versa.

Lemma 3.26. *Let G be a graph and let v be a non-critical vertex (with respect to γ^∞) of G. Suppose G has a minimum eternal dominating set D such that $N[v] \subseteq D$. Then, the graph G' obtained from G by adding the vertices of the set $\{u, u'\}$ and edges of the set $\{uv, uu'\}$ is a counterexample to the Fundamental Conjecture.*

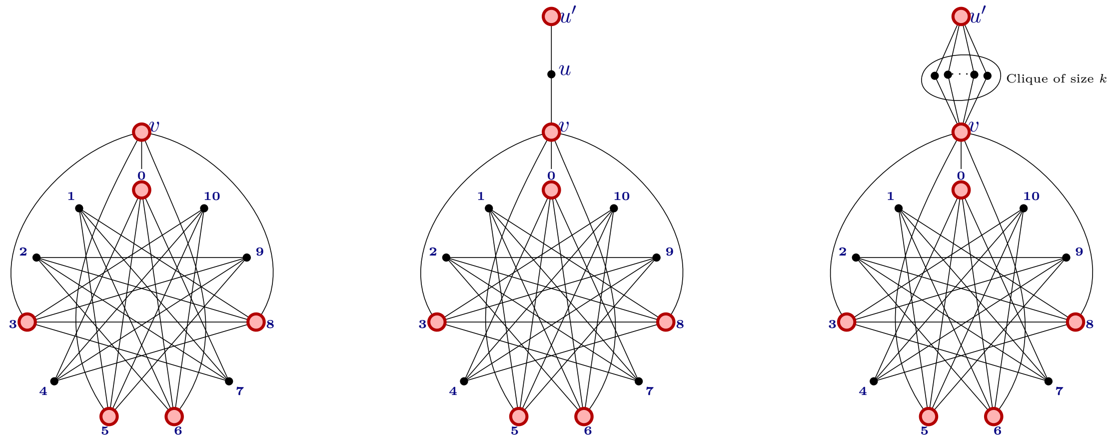
Proof. Since D is a minimum eternal dominating set of G, $D' = D \cup \{u'\}$ is a minimum eternal dominating set of G'. The vertex v has only one neighbour that does not contain a guard in D', which is vertex u. However, if the guard on vertex v moves to defend an attack on vertex u, the remaining guards in $G - \{v\}$ cannot defend that induced subgraph because v is not a critical vertex and the guard on u that comes from v is not adjacent to any vertex in $G - \{v\}$. □

We now introduce our main theorem of the section.

Theorem 3.27. *For any integer $k \geq 1$, there exist a graph G and a minimum eternal dominating set D of G such that $v \in D$, $|N(v) - D| = k$ and $(D - \{v\}) \cup \{w\}$ is not an eternal dominating set of G, for all $w \in N(v) - D$.*

Proof. We first show that the set of graphs that satisfy the hypothesis of Lemma 3.26 is not empty. To this end, consider the graph G which is obtained from the circulant graph $C_{11}[4,5]$ by adding an edge from a new vertex v to each of the vertices in the set $\{0, 3, 5, 6, 8\}$ (see Figure 3.12a).

We have seen in Lemma 3.24 that $\gamma^\infty(C_{11}[4,5]) = \theta(C_{11}[4,5]) = 6$. Since $\theta(G)$, and hence $\gamma^\infty(G)$, are also equal to 6, v is not a critical vertex of G. It now remains to show



(a) Graph G that satisfies the hypothesis of Lemma 3.26.

(b) G': A counterexample to the Fundamental Conjecture

(c) G'': A counterexample to the Fundamental Conjecture.

Figure 3.12: Counterexamples (b) and (c) to the Fundamental Conjecture.

that $\{0, 3, 5, 6, 8, v\}$ (indicated by red vertices in Figure 3.12) is an eternal dominating set of G. To this end, we will show that after responding to the first attack, there will be a guard located in each clique of a clique covering of G. By symmetry, we only need to consider the attacks on the vertices of the set $\{1, 2, 4\}$. We proceed with cases:

Case 1: Vertex 1 is attacked. The guard on vertex 6 defends the attack and our desired partition is $\{\{v, 6\}, \{0, 4\}, \{1, 7\}, \{2, 8\}, \{3, 10\}, \{5, 9\}\}$.

Case 2: Vertex 2 is attacked. The guard on vertex 6 defends the attack and our desired partition is $\{\{v, 6\}, \{0, 7\}, \{1, 5\}, \{4, 8\}, \{3, 10\}, \{2, 9\}\}$.

Case 3: Vertex 4 is attacked. The guard on vertex 8 defends the attack and our desired partition is $\{\{v, 8\}, \{1, 5\}, \{2, 6\}, \{4, 8\}, \{3, 10\}, \{4, 9\}\}$.

By Lemma 3.26, the graph G' obtained from G by adding the vertices u, u' and the edges uv, uu' is a counterexample to the Fundamental Conjecture. Now, if we replace vertex u by a clique $\{u_1, u_2, u_3, \ldots, u_k\}$ of size k and add an edge between each vertex of the clique and each vertex of the set $\{v, u'\}$, we obtain a graph G'' with minimum eternal dominating set $D'' = \{u', v, 0, 3, 5, 6, 8\}$ where the guard on v has k neighbours and cannot move to any of them. Consequently, there are infinitely many counterexamples to the Fundamental Conjecture. □

3.4 Length of the Game

As we conclude our study on the eternal domination game, an interesting question arises: "How long does the game actually last before a winner can be decided?"

To address this question more clearly, we introduce the following definitions.

Definition 3.28. Let $t_1(G)$ denote the maximum number of turns required for the attacker to win in the eternal domination game, played on a graph G, against fewer than $\gamma^\infty(G)$ guards, assuming optimal moves from both players. We think of $t_1(G)$ as the length of the game when played on a graph G.

The following proposition shows that $t_1(G)$ is always reached when playing with $\gamma^\infty(G) - 1$ guards.

Proposition 3.29. *If k guards can defend any sequence of attacks of length at most l on a graph G, then so can $k+1$ guards.*

Proof. This is a direct consequence of the fact that any superset of vertices of a dominating set of a graph G is a dominating set of G. Moreover, by having more guards located on G, the attacker can limit the sequence of attacks on G. □

Note, however, that there exist graphs G with arbitrarily large values of γ^∞ where $t_1(G)$ is reached for any integer k such that $\gamma(G) \leq k < \gamma^\infty(G)$ (see Figure 3.13 for an example).

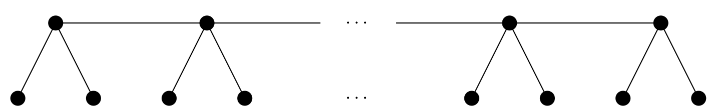


Figure 3.13: Graph G with $\gamma(G) = l$, $\gamma^\infty(G) = 2l$, where $t_1(G) = 2$ is reached for any integer k such that $l \leq k < 2l$.

Definition 3.30. Let $T_1(n, k)$ denote the maximum length of an eternal domination game played on an n-vertex graph G on which are located k guards (assuming $k < \gamma^\infty(G)$).

Definition 3.31. Let $T_1(n)$ denote the maximum value of $t_1(G)$ taken among all graphs G of order n. Equivalently, this is the maximum value of $T_1(n, k)$ taken among all $k \leq n$.

The idea behind $T_1(n)$ is simple: if k guards can effectively defend any sequence of attacks that lasts no more than $T_1(n)$ turns on an n-vertex graph G, then those k guards can defend G against any sequence of attacks.

The following observations show that $t_1(G)$, $T_1(n, k)$ and $T_1(n)$ are well defined.

Proposition 3.32. *For any $n, k \in \mathbb{N}$, $T_1(n, k) = O(n^k)$.*

Proof. Proposition 3.32 follows from the fact that there are $\binom{n}{k}$ configurations of k guards in a graph G of order n and no configuration is ever repeated at two different times when the game is played optimally on G. $\square$

Since $\binom{n}{k}$ is maximized when $k = \lfloor \frac{n}{2} \rfloor$, we have the following observation.

Observation 3.33. For any graph G of order n, $1 \leq t_1(G) \leq T_1(n) \leq \binom{n}{\lfloor n/2 \rfloor}$. Moreover, $t_1(G) = 1$ if and only if $\gamma(G) = \gamma^{\infty}(G)$.

Klostermeyer and Mynhardt [37] asked the following question.

Question 3.34 ([37])**.** Does there exist a constant c such that $T_1(n) = O(n^c)$ for all n?

We present some bounds and the exact values of $t_1(G)$ for graphs G that belong to some specific families.

Proposition 3.35. *Let G be a graph such that $\gamma^{\infty}(G) = \alpha(G)$. Then $t_1(G) \leq \min\{\alpha(G), n - \alpha(G)\}$.*

Proof. Suppose $\alpha \leq \frac{n}{2}$ and assume there are fewer than $\gamma^{\infty}(G)$ guards on G. The guards lose in at most α turns if the attacker successively selects the unoccupied vertices of a maximum independent set of G. Consider the case where $\alpha > \frac{n}{2}$ and let I be a maximum independent set of G. Observe that there are at most $n - \alpha(G)$ unoccupied vertices in I. By selecting those unoccupied vertices in I, the attacker wins the game in at most $n - \alpha(G)$ turns. $\square$

Proposition 3.36. *Let H be a spanning subgraph of a graph G such that $\gamma^{\infty}(G) = \gamma^{\infty}(H)$. Then $t_1(G) \geq t_1(H)$.*

Proof. Suppose $\gamma^{\infty}(G) - 1$ guards can respond to any sequence of attacks of length at most l on H. We will show that $\gamma^{\infty}(G) - 1$ guards can also respond to any sequence of attacks of length at most l on G. To see this, observe that since G and H have the same vertex set, any sequence of attacks on the vertices of G is also a sequence of attacks on the vertices of H. Moreover, since G has all the edges of H, the guards in G can always respond to a sequence of attacks according to the strategy of the guards in H. Since the guards can always win in H against any sequence of attacks of length at most l, so can the guards in G. $\square$

3.4.1 Paths and Cycles

In this section, we study the values of $t_1(G)$ when G is a path, a cycle or the complement of a cycle.

Proposition 3.37. *For any integer $n \geq 3$,*

$$t_1(P_n) = \begin{cases} \lfloor \log_2(n+1) \rfloor & \text{if } n \text{ is odd,} \\ \lfloor \log_2(n+2) \rfloor - 1 & \text{if } n \text{ is even.} \end{cases}$$

Proof. We prove this by induction on n. The cases $n = 3$ and $n = 4$ are clear. Let $n \geq 5$ and suppose the proposition holds for all paths on fewer than n vertices. We now proceed by cases (see Figure 3.14).

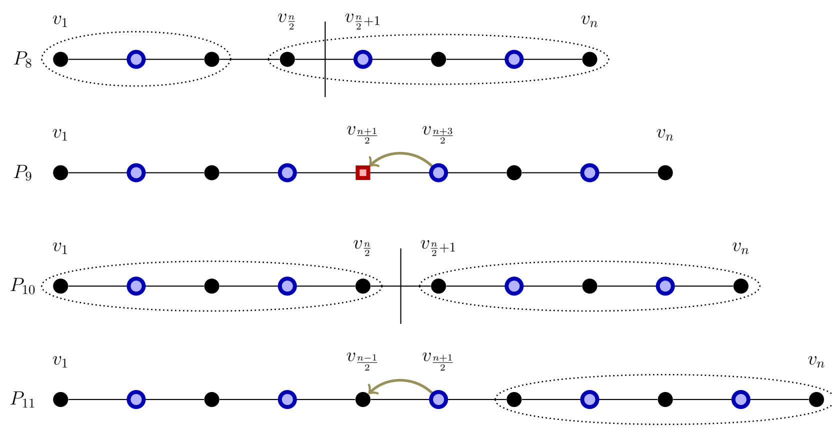


Figure 3.14: Optimal strategies of the players on the path P_n in the eternal domination game.

Case 1: $n \equiv 0 \pmod 4$

Since $\gamma^\infty(P_n) = \frac{n}{2}$, we may assume that at most $\frac{n}{2} - 1$ guards are located on P_n. We first prove that the attacker wins in at most $\lfloor \log_2(n+2) \rfloor - 1$ turns. We may assume without loss of generality that at most $\frac{n}{4} - 1$ guards are located on the vertices of the set $\{v_1, v_2, v_3, \ldots, v_{\frac{n}{2}}\}$. Let i be the largest integer smaller than $\frac{n}{2} + 1$ such that v_i is unoccupied. Then there are at most $\lfloor \frac{(i-1)-1}{2} \rfloor$ guards located on the path $P = v_1 - v_2 - \ldots - v_{i-1}$. By the induction hypothesis, the attacker wins in at most $\lfloor \log_2(i-1) + 1 \rfloor = \lfloor \log_2 i \rfloor \leq \lfloor \log_2 \frac{n}{2} \rfloor = \lfloor \log_2 n \rfloor - 1 = \lfloor \log_2(n+2) \rfloor - 1$ turns by playing on the path $P = v_1 - v_2 - \ldots - v_{i-1}$.

We now prove that the attacker needs at least $\lfloor \log_2(n-2) \rfloor - 1$ turns to win. Place a guard on each of the vertices of the set $\{v_2, v_4, v_6, \ldots, v_{\frac{n}{2}-2}\} \cup \{v_{\frac{n}{2}+1}, v_{\frac{n}{2}+3}, \ldots, v_{n-1}\}$. Now defend the paths $P = v_1 - v_2 - \ldots - v_{\frac{n}{2}-1}$ and $P' = v_{\frac{n}{2}} - v_{\frac{n}{2}+1} - \ldots - v_n$ independently with the guards that are currently located on each of them. Note that the attacker cannot win on the subgraph P' and by the induction hypothesis, needs at least $\lfloor \log_2((\frac{n}{2} - 1) + 1) \rfloor = \lfloor \log_2 n \rfloor - 1 = \lfloor \log_2(n+2) \rfloor - 1$ turns to win on P.

Case 2: $n \equiv 1 \pmod 4$

Since $\gamma^\infty(P_n) = \frac{n+1}{2}$, we may assume that at most $\frac{n-1}{2}$ guards are located on P_n. We first prove that the attacker wins in at most $\lfloor \log_2(n+1) \rfloor$ turns. Suppose there exists $i \in [n]$ such that v_i and v_{i+1} are unoccupied. Then there are either fewer than $\gamma^\infty(P)$ guards located on the path $P = v_1 - v_2 - v_3 - \ldots - v_i$ or fewer than $\gamma^\infty(P')$ guards located on the path $P' = v_{i+1} - v_{i+2} - v_{i+3} - \ldots - v_n$. Since $|V(P)| < n$ and $|V(P')| < n$, by the induction hypothesis, the attacker wins in at most $\lfloor \log_2(n+1) \rfloor$ turns by playing optimally on one of these subgraphs. Therefore, we may further assume that the guards are located on the vertices of the set $\{v_2, v_4, v_6, \ldots, v_{n-1}\}$. Now, attack vertex $v_{\frac{n+1}{2}}$. We may assume without loss of generality that the guard on $v_{\frac{n+3}{2}}$ defends the attack. So, the subgraph induced by the vertices of the set $\{v_{\frac{n+5}{2}}, v_{\frac{n+7}{2}}, v_{\frac{n+9}{2}}, \ldots, v_n\}$, which is a path of order $\frac{n-3}{2}$, can be defended by the only $\frac{n-5}{4}$ guards that are located in there. By the induction hypothesis, the attacker wins in at most $1 + t_1(P_{\frac{n-3}{2}}) = 1 + \lfloor \log_2(\frac{n-3}{2} + 1) \rfloor = \lfloor \log_2(n-1) \rfloor = \lfloor \log_2(n+1) \rfloor$ by playing optimally on this subgraph after the first attack.

We now prove that the attacker needs at least $\lfloor \log_2(n+1) \rfloor$ turns to win. Let the initial configuration of the guards be $\{v_2, v_4, v_6, \ldots, v_{n-1}\}$ and suppose there is an attack on a vertex v_i. For the sake of simplicity, let us assume that $i \leq \frac{n+1}{2}$. Move the guard on v_{i+1} to v_i. From there, use the $\frac{i+1}{2}$ guards that are located on $P = v_1 - v_2 - \ldots - v_{i+1}$ to defend P and the remaining $\frac{n-1}{2} - \frac{i+1}{2} = \frac{n-i-2}{2}$ guards that are located on $P' = v_{i+2} - v_{i+3} - \ldots - v_n$ to defend P'. Note that the attacker cannot win on P and, in this case, needs at least $t_1(P_{\frac{n-3}{2}})$ turns to win on P'. Therefore, the total number of turns required is $1 + t_1(P_{\frac{n-3}{2}}) = \lfloor \log_2(n+1) \rfloor$.

Case 3: $\boxed{n \equiv 2 \pmod 4}$

Since $\gamma^\infty(P_n) = \frac{n}{2}$, we may assume that at most $\frac{n}{2} - 1$ guards are located on P_n. We first prove that the attacker wins in at most $\lfloor \log_2(n+2) \rfloor - 1$ turns. We may assume without loss of generality that at most $\frac{n-2}{4}$ guards are located on the vertices of the set $\{v_1, v_2, v_3, \ldots, v_{\frac{n}{2}}\}$. Let i be the largest integer smaller than $\frac{n}{2} + 2$ such that v_i is unoccupied. Then there are at most $\lfloor \frac{(i-1)-1}{2} \rfloor$ guards located on the path $P = v_1 - v_2 - \ldots - v_{i-1}$. By the induction hypothesis, the attacker wins in at most $\lfloor \log_2(i-1) + 1 \rfloor = \lfloor \log_2 i \rfloor \leq \lfloor \log_2(\frac{n}{2} + 1) \rfloor = \lfloor \log_2(n+2) \rfloor - 1$ turns by playing on the path $P = v_1 - v_2 - \ldots - v_{i-1}$.

We now prove that the attacker needs at least $\lfloor \log_2(n+2) \rfloor - 1$ turns to win. Place a guard on each of the vertices of the set $\{v_2, v_4, v_6, \ldots, v_{\frac{n}{2}-1}\} \cup \{v_{\frac{n}{2}+2}, v_{\frac{n}{2}+4}, \ldots, v_{n-1}\}$. Now defend the paths $P = v_1 - v_2 - \ldots - v_{\frac{n}{2}}$ and $P' = v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2} - \ldots - v_n$ independently with the guards that are currently located in each of them. By the induction hypothesis, the attacker needs at least $\lfloor \log_2(\frac{n}{2} + 1) \rfloor = \lfloor \log_2(n+2) \rfloor - 1$ turns to win on either P or P'.

Case 4: $\boxed{n \equiv 3 \pmod 4}$

Since $\gamma^\infty(P_n) = \frac{n+1}{2}$, we may assume that at most $\frac{n-1}{2}$ guards are located on P_n. We first prove that the attacker wins in at most $\lfloor \log_2(n+1) \rfloor$ turns. For the same reason as in Case 2, we may further assume that the guards are located on the vertices of the set $\{v_2, v_4, v_6, \ldots, v_{n-1}\}$. Now, attack vertex $v_{\frac{n-1}{2}}$. If the guard on $v_{\frac{n-3}{2}}$ moves to $v_{\frac{n-1}{2}}$, then by the induction hypothesis, the attacker wins in the subgraph $v_1 - v_2 - v_3 - \ldots - v_{\frac{n-5}{2}}$ in at most $t_1(P_{\frac{n-5}{2}})$ turns and consequently, in at most $1 + t_1(P_{\frac{n-5}{2}}) = 1 + \lfloor \log_2(\frac{n-5}{2} + 1) \rfloor = \lfloor \log_2(n-3) \rfloor \leq \lfloor \log_2(n+1) \rfloor$ turns in total. If the guard on $v_{\frac{n+1}{2}}$ moves to $v_{\frac{n-1}{2}}$, then by the induction hypothesis, the attacker wins in the subgraph $v_{\frac{n+3}{2}} - v_{\frac{n+5}{2}} - v_{\frac{n+7}{2}} - \ldots - v_n$ in at most $t_1(P_{\frac{n-1}{2}})$ turns and consequently in at most $1 + t_1(P_{\frac{n-1}{2}}) = 1 + \lfloor \log_2(\frac{n-1}{2} + 1) \rfloor = \lfloor \log_2(n+1) \rfloor$ turns in total.

We now prove that the attacker needs at least $\lfloor \log_2(n+1) \rfloor$ turns to win. Let the initial configuration of the guards be $\{v_2, v_4, v_6, \ldots, v_{n-1}\}$ and suppose there is an attack on a vertex v_i. For the sake of simplicity, let us assume that $i < \frac{n+1}{2}$. Move the guard on v_{i+1} to v_i. From there, use the $\frac{i+1}{2}$ guards that are located on $P = v_1 - v_2 - \ldots - v_{i+1}$ to defend P and the remaining $\frac{n-i-2}{2}$ guards that are located on $P' = v_{i+2} - v_{i+3} - \ldots - v_n$ to defend P'. Note that the attacker cannot win on P and, in this case, needs at least $t_1(P_{\frac{n-1}{2}})$ turns to win on P'. Therefore, the total number of turns required is $1 + t_1(P_{\frac{n-1}{2}}) = \lfloor \log_2(n+1) \rfloor$ turns to win on P'. □

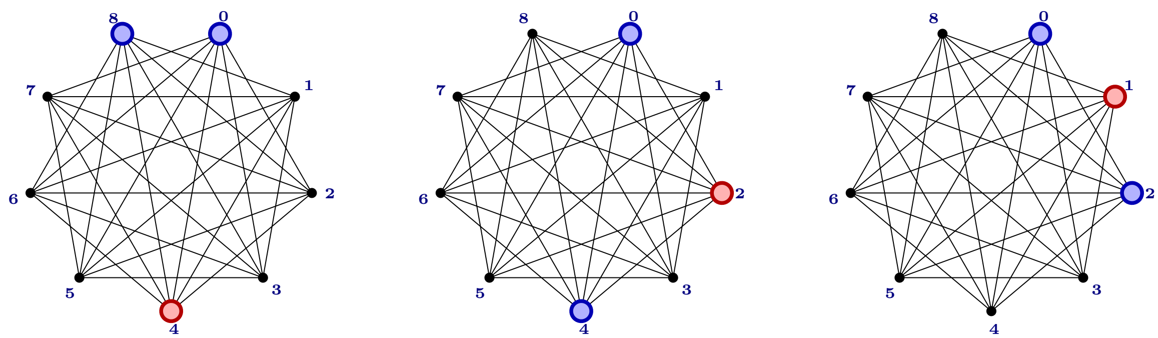
Corollary 3.38. *For any integer $n \geq 6$,*

$$t_1(C_n) = \begin{cases} \lfloor \log_2(n+1) \rfloor & \text{if } n \text{ is odd,} \\ \lfloor \log_2(n+2) \rfloor - 1 & \text{if } n \text{ is even.} \end{cases}$$

Proof. The fact that these values are lower bounds on $t_1(C_n)$ follows from Proposition 3.36 since P_n is a spanning subgraph of C_n and $\gamma^\infty(P_n) = \gamma^\infty(C_n) = \lceil \frac{n}{2} \rceil$. To see that these values are also upper bounds on $t_1(C_n)$, we first observe that since $\gamma^\infty(C_n) = \lceil \frac{n}{2} \rceil$, we may assume without loss of generality that there are fewer than $\lceil \frac{n}{2} \rceil$ guards located on C_n and that the vertices v_0 and v_{n-1} are unoccupied. Now, attack the vertices of the path $v_0 - v_1 - v_2 - \ldots - v_{n-1}$ the same way as in the proof of Proposition 4.29. The guards lose after at most $\lfloor \log_2(n+1) \rfloor$ turns if n is odd and $\lfloor \log_2(n+2) \rfloor - 1$ turns if n is even. □

Proposition 3.39. *For any integer $n \geq 5$,*

$$\begin{cases} t_1(\overline{C_n}) = 1 & \text{if } n \text{ is even,} \\ t_1(\overline{C_n}) = \lceil \log_2(n-1) \rceil & \text{if } n \text{ is odd.} \end{cases}$$



(a) (b) (c)

Figure 3.15: Optimal strategies of the players in the graph $\overline{C_9}$ in the eternal domination game.

Proof. If n is even, then $\gamma(\overline{C_n}) = \alpha(\overline{C_n}) = \gamma^\infty(\overline{C_n}) = \theta(\overline{C_n}) = 2$. So, the attacker wins against one guard located in $\overline{C_n}$ on the first turn.

Suppose n is odd. Then $\alpha(\overline{C_n}) = 2$ and $\gamma^\infty(\overline{C_n}) = \theta(\overline{C_n}) = 3$. We will show that $\lceil \log_2(n-1) \rceil$ turns are necessary and sufficient for the attacker to win against two guards. Let d_1 and d_2 be respectively the length (with respect to C_n, the complement of $\overline{C_n}$) of the odd and the even paths between the vertices on which the guards are located in their opening configuration in $\overline{C_n}$. Assuming that, at each time t, the two guards are located on v_i and v_j in $\overline{C_n}$, let $d_1(v_i, v_j)$ and $d_2(v_i, v_j)$ be respectively the length of the odd and the even paths between v_i and v_j with respect to C_n. Since we can always relabel the vertices of the graph, we may assume without loss of generality that $i < j < n$ and $j = i + d_2(v_i, v_j)$ at any time during the game.

We first prove that $t_1(\overline{C_n}) \leq \lceil \log_2(n-1) \rceil$. Observe that if $d_2(v_i, v_j)$ equals two, then the attacker wins on the next turn (by attacking the only vertex v_{i+1} which is at distance one from each of v_i and v_j, with respect to C_n) and this is the only case where the attacker can win on the next turn. Therefore, it suffices to show that the attacker can always force $d_2(v_i, v_j)$ to decrease to two in at most $\lceil \log_2(n-1) \rceil - 1$ turns. We prove this by induction on d_2, the length of the even path (with respect to C_n) between the vertices of which the guards are located in their opening configuration in $\overline{C_n}$, which is at most $n - 1$.

If $d_2 = d_2(v_i, v_j) = 4$, then after responding to an attack on vertex v_{i+2}, the guards will move to a configuration $\{v_{i'}, v_{j'}\}$ where $d_2(v_{i'}, v_{j'}) = 2$ in one turn (where $1 = \lceil \log_2 4 \rceil - 1 = \lceil \log_2 d_2 \rceil - 1$).

If $d_2 = d_2(v_i, v_j) = 6$, then after responding to an attack on vertex v_{i+4}, the guards will move to a configuration $\{v_{i'}, v_{j'}\}$ where $d_2(v_{i'}, v_{j'}) \leq 4$ in one turn. Consequently, the attacker can force the guards to move to a configuration $\{v_{i''}, v_{j''}\}$ where $d_2(v_{i''}, v_{j''}) = 2$ in at most two turns (where $2 = \lceil \log_2 6 \rceil - 1 = \lceil \log_2 d_2 \rceil - 1$).

Suppose this is true for any configuration $\{v_i, v_j\}$ such that $6 \leq d_2 = d_2(v_i, v_j) \leq 2(l-1)$; that is, if the guards are located on the vertices of a configuration $\{v_i, v_j\}$ where $d_2(v_i, v_j) \leq 2(l-1)$, then the attacker can force them to move to a configuration $\{v_{i''}, v_{j''}\}$ where $d_2(v_{i''}, v_{j''}) = 2$ in at most $\lceil \log_2(2(l-1)) \rceil - 1 = \lceil \log_2 d_2 \rceil - 1$ turns. Consider a configuration $\{v_i, v_j\}$ such that $d_2 = d_2(v_i, v_j) = 2l$. We now proceed with two subcases.

(1) l is even.

After responding to an attack on vertex $v_{\frac{i+j}{2}}$, the guards will be located on a configuration $\{v_{i'}, v_{j'}\}$ such that $d_2(v_{i'}, v_{j'}) = l$. By the induction hypothesis, the attacker can force the guards to move from $\{v_{i'}, v_{j'}\}$ to a configuration $\{v_{i''}, v_{j''}\}$ such that $d_2(v_{i''}, v_{j''}) = 2$ in at most $\lceil \log_2 l \rceil - 1$ turns. Consequently, the attacker can force the guards to move from $\{v_i, v_j\}$ to $\{v_{i''}, v_{j''}\}$ in at most $1 + \lceil \log_2 l \rceil - 1 = \lceil \log_2(2l) \rceil - 1 = \lceil \log_2 d_2 \rceil - 1$ turns.

(2) l is odd.

After responding to an attack on vertex $v_{\frac{i+j+2}{2}}$, the guards will be located on a configuration $\{v_{i'}, v_{j'}\}$ such that $d_2(v_{i'}, v_{j'}) \leq l + 1$. By the induction hypothesis, the attacker can force the guards to move from $\{v_{i'}, v_{j'}\}$ to a configuration $\{v_{i''}, v_{j''}\}$ such that $d_2(v_{i''}, v_{j''}) = 2$ in at most $\lceil \log_2(l+1) \rceil - 1$ turns. Consequently, the attacker can force the guards to move from $\{v_i, v_j\}$ to $\{v_{i''}, v_{j''}\}$ in at most $\lceil \log_2(2(l+1)) \rceil - 1 = \lceil \log_2(2l) \rceil - 1 = \lceil \log_2 d_2 \rceil - 1$ turns.

We now prove that $t_1(\overline{C_n}) \geq \lceil \log_2 n \rceil$ by giving a strategy for the guards that ensures the game will last for at least $\lceil \log_2(n-1) \rceil$ turns. According to our strategy, the guards are initially located on the vertices v_i and v_j such that $d_2(v_i, v_j) = n - 1$. Now, it suffices to show that the guards can always move at any time during the game from any dominating set $\{v_i, v_j\}$ (where $d_2(v_i, v_j) \geq 4$) to a dominating set $\{v_{i'}, v_{j'}\}$ such that $d_2(v_{i'}, v_{j'}) \geq \frac{d_2(v_i, v_j)}{2}$. In this case, the guards can avoid being located on the vertices of a configuration $\{v_{i''}, v_{j''}\}$ where $d_2(v_{i''}, v_{j''}) = 2$ in at least $\lceil \log_2(n-1) \rceil - 1$ turns. To this end, suppose a vertex v_k is attacked. If v_k belongs to the path of odd length (with respect to C_n) that connects v_i to v_j, then either $d_2(v_i, v_k) > d_2(v_i, v_j)$ (in this case move the guard on v_j to v_k) or $d_2(v_j, v_k) > d_2(v_i, v_j)$ (in this case move the guard on v_i to v_k). So, suppose v_k belongs to the path of even length that connects v_i to v_j (with respect to C_n). If v_k separates the even path from v_i to v_j in two smaller even paths where $v_i - \ldots - v_k$ is the longest one (without loss of generality), then move the guard from v_j to v_k. In this case, $d_2(v_i, v_k) \geq \frac{d_2(v_i, v_j)}{2}$. Otherwise, v_k separates the even path from v_i to v_j (with respect to C_n) in two smaller odd paths where $v_i - v_{i+1} - \ldots - v_k$ is the longest one (without loss of generality). In this case, move the guard from v_i to v_k. As a result, we obtain a configuration $\{v_j, v_k\}$ where $d_2(v_j, v_k) \geq \frac{d_2(v_i, v_j)}{2} + d_1(v_i, v_j)$. □

We now present some tight asymptotic bounds on $T_1(n, k)$ for the first two values of k.

Proposition 3.40. $T_1(n, 1) = 2$.

Proof. Consider a graph G with $\gamma^\infty(G) \geq 2$ having one universal vertex v on which there is a guard. Two turns are necessary and sufficient for the attacker to win on G against that guard. □

Proposition 3.41. $T_1(n, 2) = \theta(\log n)$.

Proof. The fact that $T_1(n, 2) = \Omega(\log n)$ follows from Proposition 3.39. So, it suffices to show that $T(n, 2) = O(\log n)$. Let G be a graph of order n such that $\gamma^\infty(G) \geq 3$. Suppose two guards are located in G. We may assume without loss of generality that $\alpha(G) = 2$; otherwise, the guards lose in at most three turns if the attacker selects the vertices of a maximum independent set. Since $\alpha(G) < \theta(G)$, G is not a perfect graph. As a result, G has an odd hole or an odd antihole. If G has an odd hole, then that odd hole is an induced C_5; otherwise, $\alpha(G) > 2$ (contradiction). In this case, the guards lose in at most four turns since it takes at most two turns to force two guards to move to a maximum independent set of C_5 and two guards lose in C_5 in at most two turns. If G has an odd antihole $\overline{C_{2k+1}}$, then the guards lose in that antihole in at most $\lceil \log_2(2k) \rceil \leq \lceil \log_2(n-1) \rceil$ turns. Consequently, the guards lose in G is at most $\lceil \log_2(n-1) \rceil$ turns. This completes the proof. □

In the following, we introduce a set of circulant graphs that highlight significant observations concerning both the eternal domination number and the length of an eternal domination game.

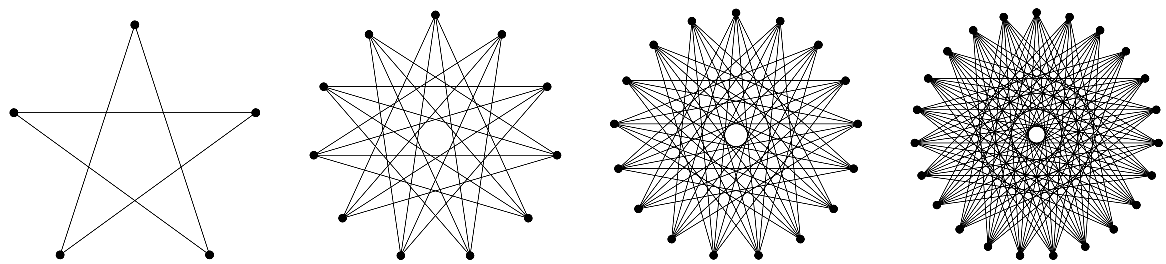
3.4.2 A Family of Circulant Graphs

Consider the family $\mathcal{F} = \{\mathcal{G}_k\}_{k \geq 0}$ in which each $\mathcal{G}_k$ is the circulant graph $C_{6k+5}[2k+2, 2k+3, \ldots, 3k+2]$ on $n = 6k+5$ vertices (see Figure 3.16 for the first four graphs of the family). We have the following proposition.

Proposition 3.42. *For any $k \geq 0$, $\omega(\mathcal{G}_k) = 2$, $\alpha(\mathcal{G}_k) = 2k+2$ and $\theta(\mathcal{G}_k) = 3k+3$.*

Proof. Let $k \geq 1$ and consider the graph $\mathcal{G}_k$. We first show that $\mathcal{G}_k$ is triangle-free. Let v_i, v_j be two neighbours of a vertex v_0 (without loss of generality) of $\mathcal{G}_k$. By definition of $\mathcal{G}_k$, $i, j \in [2k+2, 4k+3]$, and hence, $|i-j| \leq 2k+1$. Consequently, v_i is not adjacent to v_j and $\{v_0, v_i, v_j\}$ is not a triangle of $\mathcal{G}_k$.

We now show that $\alpha(\mathcal{G}_k) = 2k+2$. To this end, let S be a maximum independent set of $\mathcal{G}_k$ and suppose $v_0 \in S$ (without loss of generality). Then $v_i \notin S$ for any



(a) $C_5[2]$ (b) $C_{11}[4,5]$ (c) $C_{17}[6,7,8]$ (d) $C_{23}[8,9,10,11]$

Figure 3.16: The first four graphs of the family $\mathcal{F}$.

$i \in \{2k+2, 2k+3, 2k+4, \ldots, 4k+3\}$. Since the subgraph induced by $S - (\{v_0\} \cup \{v_{2k+2}, v_{2k+3}, v_{2k+4}, \ldots, v_{4k+3}\})$ has $4n+2$ vertices and a perfect matching, it contains at most $2n+1$ vertices in S. Hence, $|S| \leq 2n+2$. The fact that $|S| \geq 2n+2$ follows since the set $\{v_0, v_1, v_2, \ldots, v_{2k+1}\}$ is an independent set of $\mathcal{G}_k$.

Finally, we show that $\theta(\mathcal{G}_k) = 3k+3$. The lower bound $\theta(\mathcal{G}_k) \geq 3k+3$ follows since $\mathcal{G}_k$ is a triangle-free graph on $6k+5$ vertices. The upper bound $\theta(\mathcal{G}_k) \leq 3k+3$ follows since C_{6k+5} is a subgraph of $\mathcal{G}_k$ and satisfies $\theta(C_{6k+5}) = 3k+3$. This completes the proof. □

The following table shows the values of $\alpha, \theta, \gamma^{\infty}$ and $t_1(G)$ for the first five graphs in the family $\mathcal{F}$.

	α	θ	γ^{∞}	t_1
$C_5[2]$	2	3	3	2
$C_{11}[4,5]$	4	6	6	10
$C_{17}[6,7,8]$	6	9	9	31
$C_{23}[8,9,10,11]$	8	12	12	130
$C_{29}[10,11,12,13,14]$	10	15	15	546

Table 3.8: Values of $\alpha, \gamma^{\infty}, \theta$ and t_1 for the first five graphs in $\mathcal{F}$.

The values presented in the last two columns of Table 3.8 were computed through the implementation of Algorithm 1 in Python. Note that these values suggest that $\mathcal{F}$ is a class of graphs for which the length of the game is exponential with respect to the order of the graphs. We state this conjecture in Chapter 5.

Chapter 4

Eviction

In this chapter, we study the eviction game from a structural point of view. Our exploration begins with graphs of fixed independence number. We then shift our focus to describing a polynomial time algorithm for determining the eviction number of cographs, and conclude by presenting findings that are related to the length of the game on general graphs.

4.1 Graphs with Fixed Independence Number

It is easy to see that the ratio $\frac{\alpha(G)}{e^{\infty}(G)}$ is unbounded: the graph $G \cong K_2 \vee \overline{K_t}$ (see Figure 1.3) has two universal vertices, hence $e^{\infty}(G) = 1$. Since $\alpha(G) = t$, the statement follows.

It is not so easy to determine whether the ratio $\frac{e^{\infty}(G)}{\alpha(G)}$ is bounded or not. If $\alpha(G) = 1$, then G is a complete graph and $e^{\infty}(G) = 1$. Klostermeyer and MacGillivray [31] showed that $e^{\infty}(G) \leq 2$ when $\alpha(G) = 2$, and $e^{\infty}(G) \leq 5$ when $\alpha(G) = 3$. The cycle C_7 is an example of a graph with $\alpha = 3$ and $e^{\infty} = 4$. Therefore, disjoint unions of C_7 provide infinitely many (disconnected) graphs G for which $\frac{e^{\infty}(G)}{\alpha(G)} = \frac{4}{3}$. To see that a similar result holds for connected graphs, consider the graph G_k obtained by joining a vertex v to the disjoint union of k copies of C_7, and a new vertex w to v (see Figure 4.1 for the case where $k = 2$). The graph G_k satisfies the properties described in the following proposition.

Proposition 4.1. *For any $k \geq 1$, $\alpha(G_k) = 3k + 1$ and $e^{\infty}(G_k) = \theta(G_k) = 4k + 1$.*

Proof. The reader can easily verify that $\alpha(G_k) = 3k + 1$ and $\theta(G_k) = 4k + 1$. So, by Proposition 1.19, we only need to show that $e^{\infty}(G_k) \geq 4k+1$. Suppose the assumption is false. Then, G_k can be defended by $4k$ guards (by Proposition 1.18). Consider a configuration of these $4k$ guards on G_k. If w is occupied and v is unoccupied, then

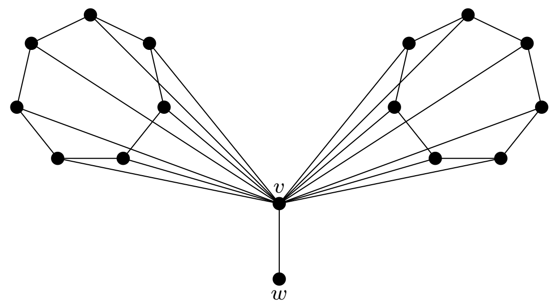


Figure 4.1: Graph G_k for the case where $k = 2$.

evict the guard on w to v. Otherwise, v and w are both occupied; in this case, evict the guard on v to an unoccupied vertex of one of the copies of C_7 and then evict the guard on w to v. So, we may assume without loss of generality that v is occupied and w is unoccupied. Since there are $4k$ guards located on G_k, there is a copy of C_7, which we will refer to as C_7^*, on which are located fewer than four guards. Evict the guards located on C_7^* in a way such that a vertex of C_7^* is not dominated by any guard located on a vertex of this subgraph. Now, evict the guard located on v. If the guard moves to w, then a vertex of C_7^* is not dominated. If the guard moves somewhere else, then w is not dominated. This contradicts our assumption that $4k$ guards can defend the graph. □

It is unknown whether there exists a graph G such that $\alpha(G) = 3$ and $e^\infty(G) = 5$. There are also no results in the literature bounding the eviction number in terms of the independence number when the latter is at least 4.

The results in this section are motivated by Question 1.28 of Klostermeyer and MacGillivray [31], which still remains unanswered. We aim to show the existence of a function f such that any graph with independence number k has eviction number at most $f(k)$. We begin by illustrating one of the difficulties we encountered when trying to determine the eviction number of a graph by considering its subgraphs.

4.1.1 Eviction is Different

For almost any domination-type parameter π, adding an edge to a graph G can only result in a graph G' with $\pi(G') \leq \pi(G)$, and usually $\pi(G') \in \{\pi(G), \pi(G) - 1\}$. This is not the case with the eviction number.

Consider the graph $G \cong K_1 + (K_2 \vee \overline{K_t})$. Observe that G has eviction number two and the graph G' obtained from G by adding an edge from the isolated vertex of G to one of the vertices of degree $t + 1$ of G has eviction number $t + 1$. The problem occurs when the attacker can force a guard to be surrounded. This is trivially the case for K_1, but there are infinitely many graphs with this property. For example, the spider $\mathrm{Sp}(2;k)$, which is obtained from the star $K_{1,k}$ by subdividing each edge exactly once (that is inserting a new vertex in the middle of each edge of $K_{1,k}$), has eviction number $k+1$ and the attacker can force guards to be on the central vertex c and all its neighbours. Now, when c is joined to a vertex of another graph, for example $K_2 \vee \overline{K_t}$ (and again there are infinitely many examples), the eviction number of the resulting graph can be arbitrarily higher than the eviction number of $\mathrm{Sp}(2;k) + (K_2 \vee \overline{K_t})$ (see Figure 4.2a).

(a) $\mathrm{Sp}(2;k) + (K_2 \vee \overline{K_{2,t}})$ (b) G_2

Figure 4.2: Examples of graphs whose eviction number increases by the addition of one edge.

The graph G_2 in Figure 4.2b is an example of a connected graph for which adding an edge increases the eviction number. In fact, three guards can defend the subgraph induced by the vertices $v_0, v_2, v_2', v_2'', v_2'''$ using the same strategy from the previous paragraph while one guard defends the subgraph induced by the vertices v_1, v_1', v_1'', v_1'''. However, the graph G_2' obtained from G_2 by the addition of the edge v_0v_1 cannot be defended by four guards. To see this, suppose four guards are initially located on the vertices of G_2'. We may assume without loss of generality that the vertex v_0 is occupied since its neighbourhood is an independent set. We may further assume without loss of generality that there are two guards located on v_2 and v_2', and one guard located on v_1'. After the sequence of attacks $v_0, v_1', v_1, v_2', v_2$, a vertex of G_2' is not dominated.

A graph may have several vertices on which guards are surrounded at the same time. By Proposition 1.23 and Proposition 1.24, the complete bipartite graph $K_{r,r+s}$, where $s > 0$, has eviction number $r+s$. The attacker can force r guards to be located on the vertices of degree $r+s$ and s guards to be located on the vertices of degree r; these s guards are all surrounded.

The above paragraphs illustrate that, when determining the eviction number of a graph G by considering how guards can defend various subgraphs of G, it may be important to establish that the guards can always move within these specific subgraphs.

4.1.2 The Function f

We begin our result on the function f with the following definition.

Definition 4.2. The *Ramsey number* of k and l, denoted by $r(k,l)$, is the minimum integer n such that any graph on at least n vertices contains either an independent set of size k or a clique of size l.

Lemma 4.3. *For any integer $k \geq 1$, there exists a constant $c(k) = c_k$ such that if G is a graph with independence number k and has at least c_k disjoint maximum independent sets, then G can be defended by k^2 guards. Moreover, no guard is ever prevented from moving by having its neighbours occupied by other guards.*

Proof. Let $k \geq 1$ and let $l_0, l_1, l_2, l_3, \ldots, l_k$ be sufficiently large positive integers (which will be chosen later). Let G be a graph with independence number k. Suppose G has at least $c_k = l_0$ disjoint maximum independent sets $R_1, R_2, R_3, \ldots, R_{c_k}$, where $R_i = \{v_{i,1}, v_{i,2}, v_{i,3}, \ldots, v_{i,k}\}$ for each $i = 1, 2, 3, \ldots, c_k$. If $l_0 \geq r(k+1, l_1)$, Ramsey's Theorem [46] guarantees that there exist l_1 vertices in the set $\bigcup_{i \in [l_0]} \{v_{i,1}\}$ which induce a complete subgraph of G. Without loss of generality, let $C_1 = \{v_{1,1}, v_{2,1}, v_{3,1}, \ldots, v_{l_1,1}\}$ be such a set. If $l_1 \geq r(k+1, l_2)$, Ramsey's Theorem guarantees that there exist l_2 vertices in the set $\bigcup_{i \in [l_1]} \{v_{i,2}\}$ which induce a complete subgraph of G. Without loss of generality, let $C_2 = \{v_{1,2}, v_{2,2}, v_{3,2}, \ldots, v_{l_2,2}\}$ be such a set. Likewise, for each $j \in \{3, 4, \ldots, k-1\}$, if $l_{j-1} \geq r(k+1, l_j)$, Ramsey's Theorem guarantees that there exist l_j vertices in the set $\bigcup_{i \in [l_{j-1}]} \{v_{i,j}\}$ which induce a complete subgraph of G. Finally, if $l_{k-1} \geq r(k+1, k+1)$, then there exist $l_k = k+1$ vertices in the set $\bigcup_{i \in [l_{k-1}]} \{v_{i,k}\}$ which induce a complete subgraph of G. To summarize, let:

$$
\begin{aligned}
l_k &= k+1 \\
l_{k-1} &= r(k+1, l_k) &&= r(k+1, k+1) \\
l_{k-2} &= r(k+1, l_{k-1}) &&= r(k+1, r(k+1, k+1)) \\
&\vdots \\
l_0 &= r(k+1, l_1) &&= \underbrace{r(k+1, r(k+1, r(k+1, \ldots)))}_{k \text{ times}}.
\end{aligned}
$$

As explained above, if G has at least $c_k = l_0$ disjoint independent sets of size k, then G has the subset of vertices $S^* = \bigcup_{i \in [k+1], j \in [k]} \{v_{i,j}\}$, where:

(i) $R_i = \bigcup_{j \in [k]} \{v_{i,j}\}$ is a maximum independent set for each $i \in [k+1]$,

(ii) $C'_j = \bigcup_{i \in [k+1]} \{v_{i,j}\}$ is a clique of size $k+1$ for each $j \in [k]$.

We can represent S^* as an array with rows $R_i, i \in [k+1]$, and columns $C'_j, j \in [k]$, as below:

$$\begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} & \cdots & v_{1,k} \\ v_{2,1} & v_{2,2} & v_{2,3} & \cdots & v_{2,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{k+1,1} & v_{k+1,2} & v_{k+1,3} & \cdots & v_{k+1,k} \end{bmatrix}.$$

In this case, place k guards in C'_j for each $j \in [k]$ so that the subset S^* contains exactly k^2 vertices. For any $j \in [k]$, if a guard in C'_j is attacked, the guard can always relocate to the only unoccupied vertex in C'_j. Since there are exactly $k+1$ rows $R_1, R_2, R_3, \ldots, R_{k+1}$ and there are exactly k^2 guards located in $S^* = \bigcup_{i \in [k+1]} R_i$ at each time $t = 1, 2, 3, \ldots$, by the Generalized Pigeonhole Principle, there is an integer $m \in [k+1]$ such that row m contains at least $\lceil \frac{k^2}{k+1} \rceil = k$ guards; that is, all the vertices in R_m are occupied. Since R_m is a maximum independent set of G, all the vertices in G are dominated by R_m. This completes the proof. □

Observe that our proof of Lemma 4.3 shows that $c_1 \leq r(2,2) = 2$ and $c_2 \leq r(3, r(3,3)) = 18$. We are now ready to prove the main theorem of the section.

Theorem 4.4. *There exists a function f such that if G is a graph with independence number $k \geq 1$, then $e^\infty(G) \leq f(k)$. In particular, $f(1) = 1$ and $f(k) \leq \frac{2kc_k(k^{k-1}-1)}{k-1}$ when $k \geq 2$, where c_k is as in Lemma 4.3.*

Proof. The cases $k = 1, 2, 3$ are clear (see [31]). Let G be a graph such that $\alpha(G) = k \geq 4$. We may assume that $|V| > f(k)$; otherwise, the theorem clearly holds for G. If G has at least c_k disjoint independent sets of size k, then, by Lemma 4.3, G can be defended by k^2 guards. So, we may further assume that G has fewer than c_k disjoint maximum independent sets. We first prove the following claim:

> **Claim 5.** There exist a positive integer $l < k$ and a subset S of vertices of G such that $\alpha(G - S) = l$ and $G - S$ has at least $c_l + |S|$ disjoint independent sets of size l.
>
> *Proof.* Let $G_0, G_1, G_2, \ldots, G_{k-1}$ be a sequence of subgraphs of G (where $G = G_0$) that satisfy the following conditions for each $i \in \{0, 1, 2, \ldots, k-2\}$.
> (i) $\alpha(G_i) = k - i$.
> (ii) $G_{i+1} = G_i - S_i$, where S_i is a smallest subset of vertices of G_i such that $\alpha(G_i - S_i) = k - i - 1$.
>
> Since G_0 has fewer than c_k disjoint independent sets of cardinality k, we have $|S_0| < kc_k$.
> If G_1 has at least $c_{k-1} + |S_0|$ disjoint independent sets of cardinality $k - 1$, then we are done, hence suppose this is not the case. Then, we have $|S_1| <$

$(k-1)(c_{k-1}+|S_0|) < (k-1)(c_{k-1}+kc_k) < k^2c_k$.
If G_2 has at least $c_{k-2}+|S_0|+|S_1|$ disjoint independent sets of cardinality $k-2$, then we are done, hence suppose this is not the case. Then, we have $|S_2| < (k-2)(c_{k-2}+|S_0|+|S_1|) < (k-2)(c_{k-2}+kc_k+k^2c_k) < k^3c_k$.
Likewise, for each $i \in \{3,4,5,\ldots,k-2\}$, if G_i has at least $c_{k-i}+|S_0|+|S_1|+\cdots+|S_{i-1}|$ disjoint independent sets of cardinality $k-i$, then we are done, hence suppose this is not the case. Then $|S_i| < (k-i)(c_{k-i}+|S_0|+|S_1|+\cdots+|S_{i-1}|) < (k-i)(c_{k-i}+kc_k+k^2c_k+\cdots+k^ic_k) < k^{i+1}c_k$.
Since G is a graph of order at least $1+\frac{2kc_k(k^{k-1}-1)}{(k-1)}$, G has a subset $S=\bigcup_{i=0}^{k-2} S_i$ of cardinality at most $|S_0|+|S_1|+|S_2|+\cdots+|S_{k-2}| < c_k(k+k^2+k^3+\ldots+k^{k-2}+k^{k-1}) = \frac{kc_k(k^{k-1}-1)}{k-1}$ such that $G-S$ is a graph on at least $1+|S|$ vertices with independence number 1. This completes the proof. $\diamondsuit$

Now, consider a smallest such subset of vertices S such that $\alpha(G-S)=l<k$ and $G-S$ has at least $c_l+|S|$ disjoint independent sets of size l. Let M be a smallest matching in G that covers the largest number of vertices in S. Let S' be the set of vertices in $G-S$ that belong to an edge of the matching M. Since $|S'| \leq |S|$, $\alpha(G-(S\cup S'))=l$ and $G-(S\cup S')$ has at least c_l disjoint independent sets of size l. As a consequence of Lemma 4.3, l^2 guards (and therefore at most k^2 guards) can defend $G-(S\cup S')$ and any guard that is evicted can always move to an unoccupied vertex in that subgraph.

Since $|S\cup S'| < \frac{2kc_k(k^{k-1}-1)}{k-1}$, in the rest of the proof, we will show that there exists a strategy with no more than $\frac{kc_k(k^{k-1}-1)}{k-1}$ guards to defend G', the subgraph of G induced by $S\cup S'$, where it is always possible for any guard located on G' to move at each step of the game to a vertex of G'. Let the initial configuration of the guards be such that there is exactly one guard on each edge of M and one guard on each vertex that is not covered by M, where such a vertex necessarily belongs to S. We will maintain the invariant that there is a guard on exactly one vertex of each edge of a matching M that covers the largest number of vertices of G' and one guard on each of the vertices that are not covered by the matching. The invariant is clearly initially satisfied. Suppose at some time $t=1,2,3,\ldots$ a guard located on x_i, a vertex that is covered by M, is attacked. Observe that the guard located on x_i has at least one unoccupied neighbour y_i, which is matched to x_i by M, since there is only one guard on each edge of the matching. Move the guard to its neighbour y_i. The invariant obviously still holds. Now, suppose a guard located on a vertex z_i in G' that is not covered by M is attacked. Note that our choice of M implies that $z_i \in S$. We consider two cases:

Case 1: If z_i has no unoccupied neighbour in G, then there is nothing to do.

Case 2: If z_i has an unoccupied neighbour x_i, then x_i is covered by M and therefore belongs to G'; otherwise we could find a matching that covers more vertices of S. Then, there exists $y_i \in S$ such that x_i is matched to y_i by M and there is a guard on y_i. In this case, move the guard on z_i to x_i and consider the new matching $M' = (M \cup \{x_i z_i\}) \backslash \{x_i y_i\}$. The invariant clearly still holds.

Since G is a graph on at least $1 + \frac{2kc_k(k^{k-1}-1)}{k-1}$ vertices, $V(G)$ can be partitionned into two sets V_1 and V_2 in a way such that at most $k^2 < \frac{kc_k(k^{k-1}-1)}{k-1}$ guards can effectively defend the subgraph of G induced by V_1 and at most $\frac{kc_k(k^{k-1}-1)}{k-1}$ guards can effectively defend the subgraph induced by V_2. This completes the proof. □

4.2 Cographs

A *cograph* (or *complement reducible graph*) is a graph that can be generated from the trivial graph K_1 by complementation and disjoint union. These graphs are also known under various characterizations, among which the following.

Proposition 4.5 ([13, 14])**.** *A cograph is a graph that does not contain P_4 as an induced subgraph.*

The following characterization follows easily from a result of Corneil, Lerchs and Burlingham.

Proposition 4.6 ([13])**.** *A cograph is a graph that can be generated from the following operations:*
(1) K_1 is a cograph.
(2) If G_1 and G_2 are cographs, then so is $G_1 + G_2$.
(3) If G_1 and G_2 are cographs, then so is $G_1 \vee G_2$.

It is worth noting that every cograph can be represented by a *cotree*, which is a rooted tree where:

- each leaf corresponds to a vertex of the graph;
- the other vertices are labelled 0 or 1, where 0 and 1 indicate respectively the disjoint union and the join applied to the graphs represented by its children.

Given a cograph G, the cotree of G can be constructed in linear time (see [14]).

Definition 4.7. A dominating set D satisfies Property $*$ if there exists $v \in D$ such that $N[v] \subseteq D$ (see Figure 4.3 for an example).

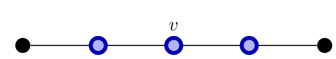


Figure 4.3: A dominating set that satisfies Property $*$.

Definition 4.8. A graph G is said to be of Type I if G has an optimum eternal dominating family in which no eternal dominating set satisfies Property $*$.

Definition 4.9. A graph G is said to be of Type II if every optimum eternal dominating family of G contains an eternal dominating set that satisfies Property $*$.

The goal of this section is to show, through a series of propositions, that the eviction number of cographs can be computed in polynomial time. We know from Proposition 4.6 that if $\mathcal{G}$ is a non trivial cograph, then there exist two disjoint graphs G and H such that either $\mathcal{G} = G + H$ or $\mathcal{G} = G \vee H$. If $\mathcal{G} = G + H$, then $e^{\infty}(\mathcal{G}) = e^{\infty}(G) + e^{\infty}(H)$. For this reason, we will be focusing on the case where $\mathcal{G} = G \vee H$. Note that G and H are not necessarily connected. We start by introducing some notations.

We have introduced the notation $c(G)$ to denote the number of components of a graph G in Chapter 1. We now introduce two more related notations.

Let $c_1(G)$ denote the number of components of G with a universal vertex and let $c_2(G)$ denote the number of components of G without a universal vertex.

Let G' be the subgraph of G induced by the components that are cographs of Type I and let G'' be the subgraph of G induced by the components that are cographs of Type II. The components of a cograph G are therefore denoted by $G_1, G_2, G_3, \ldots, G_{c(G)}$; but when more precision is needed, we denote the components of G by $G'_1, G'_2, \ldots, G'_{c(G')}$ and $G''_1, G''_2, \ldots, G''_{c(G'')}$.

Observation 4.10 ([13])**.** Let G be a connected cograph. Then G has diameter at most 2.

Proof. We prove the contrapositive. Let G be a graph such that $\operatorname{diam}(G) \geq 3$. Then, G has four vertices v_1, v_2, v_3, v_4 such that $v_1v_2, v_2v_3, v_3v_4 \in E(G)$ and $v_1v_3, v_2v_4, v_1v_4 \notin E(G)$. In this case, $v_1 - v_2 - v_3 - v_4$ is an induced P_4 and consequently, by Proposition 4.5, G is not a cograph. □

Lemma 4.11 ([11], Theorem 4)**.** *Let G be a connected cograph. Then $\gamma(G) = \gamma_c(G) \leq 2$.*

Proof. Let G be a connected cograph. If G has a universal vertex, then $\gamma(G) = \gamma_c(G) = 1$. So, we may assume that G has no universal vertex. Now, it suffices to show that $\gamma_c(G) \leq 2$. Since G is connected, there exist G_1 and G_2 such that $G = G_1 \vee G_2$. Let $u \in V(G_1)$ and $v \in V(G_2)$. Then $uv \in E(G)$, u dominates all the vertices in G_2 and v dominates all the vertices in G_1. This completes the proof. □

Corollary 4.12. *Let G be a connected cograph. Then $e^{\infty}(G \vee K_1) \in \{1, 2\}$ and $G \vee K_1$ is a cograph of Type I.*

Proof. Let G be a connected cograph. If G has a universal vertex, then $G \vee K_1$ has at least two universal vertices; therefore, $e^{\infty}(G \vee K_1) = 1$. Suppose G has no universal vertex. Then G has a dominating set $\{u, v\}$ such that $uv \in E(G)$. In this case, $G \vee K_1$ has a triangle uvw, where $w \in V(K_1)$, such that any two of its vertices form an eternal dominating set. The fact that G is a cograph of Type I follows since no guard is ever surrounded in both cases. □

Observation 4.13. Let G be a graph and let S be an independent set of G. Suppose there are $k \leq |S|$ guards located on the vertices of S. Then, the attacker can force the guards to leave S until each guard (if any) remaining on S is surrounded.

Observation 4.14. Let G and H be two disjoint empty graphs of order n and m. Then $e^{\infty}(G \vee H) = \max\{n, m\}$. Moreover, $G \vee H$ is a graph of Type I if and only if $n = m$.

Proof. If G and H are two empty graphs of order n and m, then $G \vee H$ is the complete bipartite graph $K_{n,m}$. Therefore, $e^{\infty}(G \vee H) = \max\{n, m\}$. If $n = m$, then a guard can move back and forth on the vertices of each edge of a perfect matching of G. Consequently, the graph is of Type I. If $n \neq m$, a guard located on the largest partite set cannot move when all the vertices of the smaller partite set are occupied. Consequently, the graph is of Type II. □

Proposition 4.15. *Let G and H be two disjoint non-empty graphs. If $G \vee H$ has at least two universal vertices, then $e^{\infty}(G \vee H) = 1$; otherwise, $e^{\infty}(G \vee H) = 2$. Moreover, $G \vee H$ is a graph of Type I in both cases.*

Proof. If $G \vee H$ has at least two universal vertices, then $e^{\infty}(G \vee H) = 1$. The fact that $G \vee H$ is a graph of Type I follows since a guard can always move back and forth on any two universal vertices. Suppose $G \vee H$ has at most one universal vertex. It is clear that $e^{\infty}(G \vee H) > 1$. So, it suffices to show that $e^{\infty}(G \vee H) \leq 2$. Let uv be an edge of G and let wx be an edge of H. Move a guard back and forth on the vertices u, v and another guard back and forth on the vertices w, x. The guard on uv dominates all the vertices in H and the guard on wx dominates all the vertices of G. Therefore, $e^{\infty}(G \vee H) \leq 2$. As a result, $e^{\infty}(G \vee H) = 2$. Since the guard on uv and the guard on wx are never surrounded, $G \vee H$ is a graph of Type I. □

In the rest of this section, we consider the case when G is a non-empty graph and $H \cong \overline{K_m}$ is the empty graph on m vertices.

Let $u_1, u_2, u_3, \ldots, u_m$ be the vertices of $\overline{K_m}$. Observe that if a guard located on u_i for some $i \in [m]$ is attacked, the guard must move to an unoccupied vertex of G (if such a vertex exists). For this reason, we may assume without loss of generality that the guards are initially located on the vertices of G and if there is a guard located on a vertex of $\overline{K_m}$, that guard is surrounded, in which case all the vertices of G are occupied.

The case $c(G) < m$

We begin with the following definition.

Definition 4.16. Let $\mathcal{F}$ be the family of non-empty graphs G such that each component of G has exactly one universal vertex.

Proposition 4.17. *For any non-empty cograph $G \in \mathcal{F}$ and any integer $m > c(G)$, $e^\infty(G \vee \overline{K_m}) = c(G) + 1$ and $G \vee \overline{K_m}$ is a cograph of Type I.*

Proof. We first prove the lower bound. Suppose $c(G)$ guards can eternally defend $G \vee \overline{K_m}$. If at some time $t = 1, 2, 3, \ldots$ there is a guard that is not located on a universal vertex of a component of G, then attack each vertex of $\overline{K_m}$ on which there is a guard until all the guards eventually move to G. Since there are $c(G)$ guards located in G and each component must have at least one guard, we may assume that each component has exactly one guard. In this case, a component in which the unique universal vertex is unoccupied has at least one undominated vertex. So, we may assume that at any time $t = 1, 2, 3, \ldots$ each occupied vertex of a component of G is a universal vertex (with respect to the component). Attack each vertex of G on which there is a guard. Since a guard located on a component of G cannot move to a non-universal vertex (with respect to the component), each guard must move to $\overline{K_m}$. Since $\overline{K_m}$ has $m > c(G)$ components, some vertex in this induced subgraph is not dominated after any response to the sequence of attacks.

We now prove the upper bound by providing a strategy for the guards. Without loss of generality, let G_1 be a nontrivial component of G. Place two guards on the triangle induced by the vertices of a connected dominating set of G_1 and the vertex u_1. For any integer $i \in \{2, 3, 4, \ldots, c(G)\}$, move one guard back and forth on vertex u_i and the unique universal vertex of G_i. The $c(G) + 1$ guards can eternally defend $G \vee \overline{K_m}$ this way and no guard is ever prevented from moving by having its neighbours occupied by other guards. □

Proposition 4.18. *For any non-empty cograph $G \notin \mathcal{F}$ and any integer $m \geq c(G)$, $e^\infty(G \vee \overline{K_m}) = c_1(G) + 2c_2(G)$ and $G \vee \overline{K_m}$ is a cograph of Type I.*

Proof. We first prove the lower bound. By Observation 4.13, we may assume that any guard located on a vertex of $\overline{K_m}$ is surrounded. Suppose there is no guard on a vertex of $\overline{K_m}$. Since each component of G with a universal vertex must contain a guard and each component of G without a universal vertex must contain at least two guards, there must be at least $c_1(G) + 2c_2(G)$ guards located in G. Suppose, on the other hand, there is a guard on a vertex of $\overline{K_m}$. Since each component of G without a universal vertex has more than one vertex, and all vertices in G are occupied (because the guard on $\overline{K_m}$ is surrounded), there are also at least $c_1(G) + 2c_2(G)$ guards located on G.

We now prove the upper bound by providing a strategy for the guards. For any $i \in [c(G)]$, if G_i has a universal vertex, then move a guard back and forth on u_i and the universal vertex in G_i; otherwise, G_i has a connected dominating set of size 2. In this case, place two guards on the triangle induced by u_i and the vertices of the connected dominating set of G_i. Therefore, the $c_1(G) + 2c_2(G)$ guards can eternally defend $G \vee \overline{K_m}$ and no guard is ever surrounded. □

Corollary 4.19. *Let G be a connected cograph of Type II. Then, there exist a cograph H and a positive integer $m < c(H)$ such that $G = H \vee \overline{K_m}$.*

The case $c(G) \geq m$

In this section, we assume that $G'_1, G'_2, G'_3, \ldots, G'_{c(G')}$ is a labelling of the components of Type I of G in a way such that $e^\infty(G'_1) \leq e^\infty(G'_2) \leq \cdots \leq e^\infty(G'_{c(G')})$. We now present our main theorem.

Theorem 4.20. *For any non-empty cograph G and any positive integer m,*

A. *If $c(G'') \leq m \leq c(G)$, then $e^\infty(G \vee \overline{K_m}) = f(G, m)$ and $G \vee \overline{K_m}$ is a cograph of Type I, where:*

$$f(G,m) = c_1(G'') + 2c_2(G'') + \sum_{i=1}^{c(G)-m} e^\infty(G'_i) + c_1\left(\underset{i=c(G)-m+1}{\overset{c(G')}{+}} G'_i\right) + 2c_2\left(\underset{i=c(G)-m+1}{\overset{c(G')}{+}} G'_i\right).$$

B. *If $m < c(G'') \leq c(G)$, then $e^\infty(G \vee \overline{K_m}) = e^\infty(G)$ and $G \vee \overline{K_m}$ is a cograph of Type II in which every eternal dominating family contains an eternal dominating set that satisfies Property $*$.*

Before proving our main theorem of the section, we introduce the following lemma.

Lemma 4.21. *Suppose guards placed on the vertices of $G \vee \overline{K_m}$ can defend the graph against any sequence of attacks. Consider a configuration of the guards such that each vertex of $\overline{K_m}$ is unoccupied and there are fewer than $e^\infty(G_i)$ guards located on each component $G_1, G_2, \ldots, G_r$ of G, where $r < c(G)$. Then at least r guards can be evicted from $G_1 + G_2 + \cdots + G_r$ to $\overline{K_m}$. In particular, $r \leq m$.*

Proof. We prove this by contradiction. Consider such a configuration as described in the lemma and suppose the attacker is unable to force at least r guards to move to

$\overline{K_m}$. Attack the guards in each component of the set $\{G_1, G_2, G_3, \ldots, G_r\}$ (with not enough guards) in a way such that there is at least one vertex that is not dominated by any guard located on the component; possibly some guards move to vertices of $\overline{K_m}$. Observe that such a sequence of attacks always exists since the components do not have enough guards to defend them. Since fewer than r guards move to $\overline{K_m}$, by attacking those guards and forcing them back to G, the attacker can force at least one G_i to have an undominated vertex. □

Proof of Theorem 4.20. Suppose the statement of the theorem is false and let G be a smallest counterexample (with respect to order). We begin with the following claim.

Claim 6. Suppose there are at least $e^\infty(G''_i)$ guards located on a component G''_i that is a proper subgraph of Type II of G. There exists a strategy that forces a guard to be surrounded in G''_i at some time during the game.

Proof. We prove this by induction on the number of join operations that are needed to generate G''_i. For the base case, it is easy to verify that the claim holds when G''_i is a complete bipartite graph. Suppose this is true for all connected subgraphs of Type II of G that can be generated by fewer than k join operations. Consider a connected proper subgraph G''_i of G that can be generated using k join operations. Since G''_i is a cograph of Type II on fewer vertices than G, by Corollary 4.19 and the hypothesis of Theorem 4.20, $G''_i = H' \vee \overline{K_{m'}}$ where the number of components of Type II of H' is greater than m'. Consider the guards located on G''_i. By Observation 4.13, we may assume that either all these guards are located on the components of H', or there is a guard on $\overline{K_{m'}}$ that is surrounded. In the latter case, we are done, hence assume that all the guards are located on the components of H'. Consider the Type II components of H'. Say they are $H'_1, H'_2, H'_3, \ldots, H'_r,\ H'_{r+1}, \ldots, H'_{r+s}$, where each $H'_j,\ j = 1, 2, 3, \ldots, r$, contains fewer than $e^\infty(H'_j)$ guards, and each $H'_j,\ j = r+1, \ldots, r+s$, contains at least $e^\infty(H'_j)$ guards. The attacker now plays as follows. By attacking guards in each $H'_j,\ j = 1, 2, 3, \ldots, r$, Lemma 4.21 guarantees that at least r guards can be forced to move to vertices of $\overline{K_{m'}}$. In particular, $s > 0$, otherwise $r > m'$ and fewer than r guards can move to $\overline{K_{m'}}$. By the induction hypothesis, the attacker can also attack the guards in each $H'_j,\ j = r+1, \ldots, r+s$, to ensure that a guard on some vertex in each H'_j, say x_j, is surrounded. By now attacking the guards on x_j, $j = r+1, \ldots, m'$, the attacker can force these guards to move to $\overline{K_{m'}}$ until each vertex of $\overline{K_{m'}}$ contains a guard. Since $m' < r + s$, the guard on x_{r+s} does not move and is now surrounded in H'_{r+s} as well as in G''_i. This completes the proof of our claim.

◇

Observe that the definition of the Type II cographs that was introduced earlier requires an optimum number of guards on the graph for the attacker to be able to force a guard to be surrounded, whereas in the proof we may have more than the optimum number of guards. It is an open question, in general, as to whether the attacker can force a guard to be surrounded in any graph G on which are located more than $e^{\infty}(G)$ guards when this is possible with exactly $e^{\infty}(G)$ guards. Claim 6 is there to ensure that the attacker can always do this when G is a cograph.

We now proceed by cases.

Case 1. $\boxed{c(G'') \leq m \leq c(G)}$

Observe that G has at least $f(G,m)$ vertices since each component G_i of G has at least $e^{\infty}(G_i)$ vertices. Therefore, we may assume, by Observation 4.13, that all the guards are located on the vertices of G. We first prove that $e^{\infty}(G \vee \overline{K_m}) = f(G,m)$. Suppose that $e^{\infty}(G \vee \overline{K_m}) < f(G,m)$. We may assume without loss of generality that there are $f(G,m)-1$ guards located in G since by Proposition 1.18, if fewer than $f(G,m)-1$ guards can defend $G \vee \overline{K_m}$, then so can $f(G,m)-1$ guards. We now consider two subcases:

Case 1.1: $c(G'') \geq 1$.

Observe that there must be at least $c_1(G'') + 2c_2(G'')$ guards located on G'' since there must be at least one guard on each component with a universal vertex and at least two guards on each component without a universal vertex. Consider a configuration such that the number of guards located on G'' is minimized (with respect to the eternal dominating family).

Attack the guards located on G'' as described in the proof of Lemma 4.21: If a component G''_i has fewer than $e^{\infty}(G''_i)$ guards, attack the guards so that some vertex of G''_i is not dominated by any guard in G''_i; as shown, the same number of guards (as such components) can be forced to move to $\overline{K_m}$. By Claim 6, if a component G''_i has at least $e^{\infty}(G''_i)$ guards, the attacker can force a guard to be surrounded and then to move to $\overline{K_m}$. Hence we may assume that there are exactly $c(G'')$ guards in $\overline{K_m}$.

By our choice of the initial configuration, these $c(G'')$ guards located on $\overline{K_m}$ cannot move to G'. Therefore, by our hypothesis that G is a smallest counterexample to the theorem, there must be at least $\sum_{i=1}^{c(G)-m} e^{\infty}(G'_i) + c_1\left(+_{i=c(G)-m+1}^{c(G')} G'_i\right) + 2c_2\left(+_{i=c(G)-m+1}^{c(G')} G'_i\right)$ guards located on G' to defend the subgraph $G' \vee \overline{K_{m-c(G'')}}$. This contradicts our assumption that fewer than $f(G,m)$ guards can defend the graph $G \vee \overline{K_m}$.

Case 1.2: $c(G'') = 0$.

Observe that there must be at least one guard in each component of G' with a universal vertex and at least two guards in each component of G' without a universal vertex. If there are at least $c(G) - m$ components G'_i on which are located at least $e^\infty(G'_i)$ guards, then there are at least $f(G,m)$ guards located on the graph. This contradicts our assumption that there were at most $f(G,m)-1$ guards located on the graph. So, we may assume that there are fewer than $c(G) - m$ components G'_i, on which are located at least $e^\infty(G'_i)$ guards. Then, there are at least $m+1$ components G'_i of G' on which are located fewer than $e^\infty(G'_i)$ guards. Attack the guards on these $m+1$ components in a way such that each of them has a vertex that is not dominated by any guard in the component. Now, attack all of the guards located in $\overline{K_m}$. Since there are more components G'_i that contain an undominated vertex than there are guards in $\overline{K_m}$, a component G'_i contains a vertex that is not dominated by any guard after any response to the sequence of attacks. This contradicts our assumption that $e^\infty(G \vee \overline{K_m}) < f(G,m)$.

We now provide a strategy that shows $f(G,m)$ guards can respond to any sequence of attacks on $G \vee \overline{K_m}$. Defend the subgraph induced by G'' and the first $c(G'')$ vertices of $\overline{K_m}$ with $c_1(G'') + 2c_2(G'')$ guards according to the strategy in the proof of Proposition 4.18. Then, defend the subgraph induced by the last $m - c(G'')$ components of G' and the last $m - c(G'')$ vertices of $\overline{K_m}$ with $c_1(+_{i=c(G)-m+1}^{c(G')} G'_i) + 2c_2(+_{i=c(G)-m+1}^{c(G')} G'_i)$ guards according to the same strategy in the proof of Proposition 4.18. Finally, defend the subgraph induced by the first $c(G) - m$ components of G' with $\sum_{i=1}^{c(G)-m} e^\infty(G'_i)$ guards. Note that no guard is ever prevented from moving by having its neighbours occupied by other guards according to this strategy; consequently, $G \vee \overline{K_m}$ is a graph of Type I.

So, we obtain a contradiction since we assumed that either $e^\infty(G \vee \overline{K_m}) \neq f(G,m)$ or $G \vee \overline{K_m}$ is a graph of Type II.

It is worth noting that $f(G,m)$ is minimized by our choice of the labelling of the components of G'.

Case 2. $\boxed{m < c(G'') \leq c(G)}$

We first prove that $e^\infty(G \vee \overline{K_m}) = e^\infty(G)$.

Suppose there are fewer than $e^\infty(G)$ guards located in $G \vee \overline{K_m}$. We may assume without loss of generality that there are exactly $e^\infty(G) - 1$ guards located in $G \vee \overline{K_m}$ since by Proposition 1.18, if fewer than $e^\infty(G) - 1$ guards can defend $G \vee \overline{K_m}$, then so can $e^\infty(G) - 1$. Since $e^\infty(G) < |V(G)|$, we may further assume that the guards are all located on G. We consider two subcases.

Case 2.1: $c(G') \geq 1$.

Suppose there are fewer than $e^\infty(G'')$ guards located on the vertices of G'' at some time $t = 1, 2, 3, \ldots$. Since G'' has fewer vertices than G, by the hypothesis of Theorem 4.20, and the choice of G as a smallest counterexample, the attacker has a winning strategy by playing on the subgraph induced by $G'' \vee \overline{K_m}$. So, we may assume without loss of generality that at any time $t = 1, 2, 3, \ldots$ there are at least $e^\infty(G'')$ guards located on the vertices of G''. Consider such a configuration where the number of guards in G'' is minimized with respect to the eternal dominating family. There exists a component G'_i on which are located fewer than $e^\infty(G'_i)$ guards. By Claim 6 and Lemma 4.21, there is a sequence of attacks on $G'' \vee \overline{K_m}$ that forces m guards to move from G'' to $\overline{K_m}$. Now, attack the guards in G'' in a way such that all the vertices of $\overline{K_m}$ are occupied. Next attack the guards in G'_i until a vertex in this subgraph is not dominated by any guard on the subgraph. Then attack each vertex in $\overline{K_m}$. After any response to the attack, either some vertex of G remains undominated or a guard moves to G'_i. This either contradicts our assumption on the number of guards in G'' as stated, or results in a set of guards that is not dominating.

Case 2.2: $c(G') = 0$.

Assume that there are $k''_1, k''_2, k''_3, \ldots, k''_{c(G'')}$ guards located respectively on the components $G''_1, G''_2, G''_3, \ldots, G''_{c(G'')}$ of G. Let $c(G'') = r + s + t$, where:

(a) $G''_1, G''_2, \ldots, G''_r$ are the components of G'' such that $k''_i - e^\infty(G''_i) < 0$,

(b) $G''_{r+1}, G''_{r+2}, \ldots, G''_{r+s}$ are the components of G'' such that $k''_i - e^\infty(G''_i) = 0$,

(c) $G''_{r+s+1}, G''_{r+s+2}, \ldots, G''_{r+s+t}$ are the components of G'' such that $k''_i - e^\infty(G''_i) > 0$.

Since there are $e^\infty(G'') - 1$ guards located on G'', $r > 0$, and since $c(G'') \geq 2$, $s + t > 0$. We may further assume that the guards are located in such a way that $\mathcal{S} = \sum_{i=1}^{t} [k''_{r+s+i} - e^\infty(G''_{r+s+i})]$ is minimized (and equals 0 if $t = 0$). Note that $\mathcal{S} \geq r - 1$. Each of the s components in (b) has a guard that can be forced to move to $\overline{K_m}$, and the t components in (c) together have at least $t + r - 1$ guards, each of which can be forced to move to $\overline{K_m}$ (by Claim 6). Since $m \leq c(G'') - 1 = r + s + t - 1$, the attacker can force a guard onto each vertex of $\overline{K_m}$. Next, the attacker attacks all components G''_i of G'' on which are located fewer than $e^\infty(G''_i)$ guards until a vertex is not dominated by any guard located in G''_i. Finally, the attacker attacks the guards in $\overline{K_m}$. If a guard moved to $\overline{K_m}$ from a component in (c), it must move back to one of those components, by the choice of $\mathcal{S}$. Any other guard on $\overline{K_m}$ moves to a component in (a) or (b). However, since $r > 0$, some component in (a) or (b) now contains a configuration of guards that is not dominating.

To see that $e^\infty(G)$ guards can defend the graph, consider the following strategy. Place $e^\infty(G_i)$ guards on a minimum eternal dominating set of each component G_i of G. If G_i is a graph of Type I, then those $e^\infty(G_i)$ guards can defend G_i without ever having to leave this subgraph. Suppose G_i is a graph of Type II and let us denote

$e^\infty(G_i)$ by r. Suppose further that the guards on G_i are labelled $g_1, g_2, g_3, \ldots, g_r$ and are respectively initially located on the vertices $v_1, v_2, v_3, \ldots, v_r$ of G_i. Suppose, without loss of generality, that the guard g_1 is attacked. If $\overline{K_m}$ has an unoccupied vertex, then move g_1 to $\overline{K_m}$. Otherwise, move g_1 according to a winning strategy in G_i and switch the labelling of v_1 and the new location of g_1. Now, suppose that the guards $g_1, g_2, g_3, \ldots, g_p$ are located on $\overline{K_m}$ and the guards $g_{p+1}, g_{p+2}, g_{p+3}, \ldots, g_r$ are located on $v_{p+1}, v_{p+2}, v_{p+3}, \ldots, v_r$. Note that when there is a guard located on $\overline{K_m}$, the configuration of guards in $G \vee \overline{K_m}$ is dominating. If the attacker attacks the guards $g_p, g_{p-1}, g_{p-2}, \ldots, g_1$, they simply move back to $v_p, v_{p-1}, v_{p-2}, \ldots, v_1$, hence suppose, without loss of generality, that the next attack is on g_{p+1}. If $\overline{K_m}$ has an unoccupied vertex, then move g_{p+1} to $\overline{K_m}$. In doing so, we obtain a similar configuration as the preceding. Suppose this is not the case. Then all the vertices of $\overline{K_m}$ are occupied. Consider the following subcases:

- v_{p+1} is adjacent to v_j for some $j \in [p]$. Then move g_{p+1} to v_j and switch the labelling of v_{p+1} and v_j.
- v_{p+1} is not adjacent to v_j for any $j \in [p]$. Then move g_{p+1} according to a winning strategy in G_i when the guards are located on $\{v_1, v_2, v_3, \ldots, v_r\}$ then switch the labelling of the vertices v_{p+1} and the new location of g_{p+1}.

In both subcases, the guards are located on a similar configuration after their response to the attack.

To complete the proof of Case 2, observe that if there are at least $e^\infty(G'')$ guards located on G'', as in the proof of Claim 6, m guards can be evicted from G'' to $\overline{K_m}$. Moreover, since $m < c(G'')$, there will be at least one of the components G''_i of G'' on which are located at least $e^\infty(G''_i)$ guards after all the vertices of $\overline{K_m}$ are occupied. Since G''_i is a graph of Type II, there is a sequence of attacks on G''_i which will result in a guard being surrounded in G''_i and therefore in $G'' \vee \overline{K_m}$. As a result, $G \vee \overline{K_m}$ is a graph of Type II in which every eternal dominating family contains an eternal dominating set that satisfies Property $*$. □

Corollary 4.22. *The eviction number of cographs can be computed in polynomial time.*

4.3 Length of the Game

To conclude this chapter, we also study the length of the game when played on a graph G. Again, we assume that all players are making their best possible moves and we consider the case where the number of guards located on G is less than $e^\infty(G)$. We begin with the following definitions, which are analogues of $t_1(G)$, $T_1(n,k)$ and $T_1(n)$ for the eviction game.

Definition 4.23. Let $t_2(G)$ denote the maximum number of turns required for the attacker to win in the eviction game, played on a graph G, against fewer than $e^\infty(G)$ guards.

Note that there exist graphs G with arbitrarily large values of $e^\infty(G)$ such that $t_2(G)$ is reached using k guards for any integer k such that $3 \leq k < e^\infty(G)$. An illustrative example can be found in Figure 4.4. This observation leads us to pose the subsequent question.

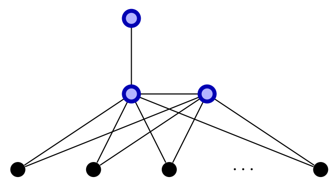


Figure 4.4: A graph G such that $t_2(G) = 4$ is reached using k guards for any integer k such that $3 \leq k < e^\infty(G)$. The initial configuration of the guards is indicated by the big blue vertices.

Question 4.24. Is it true that $t_2(G)$ is always reached when playing with $e^\infty(G) - 1$ guards?

We now proceed with two other definitions.

Definition 4.25. Let $T_2(n, k)$ denote the maximum length of an eviction game played on an n-vertex graph G on which are located k guards (assuming $k < e^\infty(G)$).

Definition 4.26. Let $T_2(n)$ denote the maximum value of $t_2(G)$ taken among all graphs G of order n. Equivalently, this is the maximum value of $T_2(n, k)$ taken among all $k \leq n$.

The following proposition and observation, which serve as analogues of Proposition 3.32 and Observation 3.33, show that $t_2(G)$, $T_2(n, k)$ and $T_2(n)$ are well defined and follow from a similar argument as in Proposition 3.32 and Observation 3.33.

Proposition 4.27. *For any $n, k \in \mathbb{N}$, $T_2(n, k) = O(n^k)$.*

Proof. The proof is similar to the proof of Proposition 3.32. □

Observation 4.28. For any graph G of order n, $0 \leq t_2(G) \leq T_2(n) \leq \binom{n}{\lfloor n/2 \rfloor}$. Moreover, $t_2(G) = 0$ if and only if $\gamma(G) = e^\infty(G)$.

4.3.1 Paths and Cycles

We begin our study of $t_2(G)$ on paths and cycles.

Proposition 4.29. *For any integer $n \geq 3$,*

$$t_2(P_n) = \begin{cases} \lfloor \log_2 n \rfloor & \text{if } n \text{ is odd,} \\ \lfloor \log_2 (n-2) \rfloor - 1 & \text{if } n \text{ is even.} \end{cases}$$

Proof. We prove this by induction on n. The cases $n = 3$ and $n = 4$ are clear. Let $n \geq 5$ and suppose the proposition holds for all paths on fewer than n vertices. We now proceed by cases (see Figure 4.5).

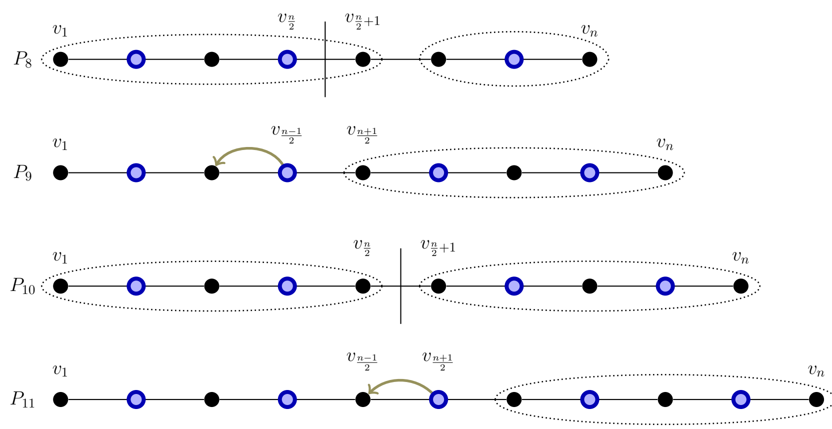


Figure 4.5: Optimal strategies of the players on the path P_n in the eviction game.

Case 1: $\boxed{n \equiv 0 \pmod 4}$

Since $e^{\infty}(P_n) = \frac{n}{2}$, we may assume that at most $\frac{n}{2} - 1$ guards are located on P_n. We first prove that the attacker wins in at most $\lfloor \log_2(n-2) \rfloor - 1$ turns. We may assume without loss of generality that at most $\frac{n}{4} - 1$ guards are located on the vertices of the set $\{v_1, v_2, v_3, \ldots, v_{\frac{n}{2}}\}$. Let i be the largest integer smaller than $\frac{n}{2} + 1$ such that v_i is unoccupied. Then there are at most $\lfloor \frac{(i-1)-1}{2} \rfloor$ guards located on the path $P = v_1 - v_2 - \ldots - v_{i-1}$. By the induction hypothesis, the attacker wins in at most $\lfloor \log_2(\frac{n}{2} - 1) \rfloor = \lfloor \log_2(n-2) \rfloor - 1$ turns by playing on the path $P = v_1 - v_2 - \ldots - v_{i-1}$.

We now prove that the attacker needs at least $\lfloor \log_2(n-2) \rfloor - 1$ turns to win. Place a guard on each of the vertices of the set $\{v_2, v_4, v_6, \ldots, v_{\frac{n}{2}-2}\} \cup \{v_{\frac{n}{2}+1}, v_{\frac{n}{2}+3}, \ldots, v_{n-1}\}$. Now defend the paths $P = v_1 - v_2 - \ldots - v_{\frac{n}{2}-1}$ and $P' = v_{\frac{n}{2}} - v_{\frac{n}{2}+1} - \ldots - v_n$ independently with the guards that are currently located in each of them. Note that the attacker cannot win on the subgraph P' and by the induction hypothesis, needs at least $\lfloor \log_2(\frac{n}{2} - 1) \rfloor = \lfloor \log_2(n-2) \rfloor - 1$ turns to win on P.

Case 2: $\boxed{n \equiv 1 \pmod 4}$

Since $e^{\infty}(P_n) = \frac{n+1}{2}$, we may assume that at most $\frac{n-1}{2}$ guards are located on P_n. We first prove that the attacker wins in at most $\lfloor \log_2 n \rfloor$ turns. Suppose there exists $i \in [n]$

such that v_i and v_{i+1} are unoccupied. Then there are either fewer than $e^\infty(P)$ guards located on the path $P = v_1 - v_2 - v_3 - \ldots - v_i$ or fewer than $e^\infty(P')$ guards located on the path $P' = v_{i+1} - v_{i+2} - v_{i+3} - \ldots - v_n$. Since $|V(P)| < n$ and $|V(P')| < n$, by the induction hypothesis, the attacker wins in at most $\lfloor \log_2 n \rfloor$ turns by playing optimally on one of these subgraphs. Therefore, we may further assume that the guards are located on the vertices of the set $\{v_2, v_4, v_6, \ldots, v_{n-1}\}$. Now, attack vertex $v_{\frac{n-1}{2}}$. If the guard moves to $v_{\frac{n-3}{2}}$, by the induction hypothesis, the attacker wins in the subgraph $v_{\frac{n+1}{2}} - v_{\frac{n+3}{2}} - \ldots - v_n$ in at most $t(P_{\frac{n+1}{2}}) = \lfloor \log_2 n \rfloor - 1$ turns. If the guard moves to $v_{\frac{n+1}{2}}$, by the induction hypothesis, the attacker wins in the subgraph $v_1 - v_2 - \ldots - v_{\frac{n-3}{2}}$ in at most $t(P_{\frac{n-3}{2}})$ turns.

We now prove that the attacker needs at least $\lfloor \log_2 n \rfloor$ turns to win. Let the initial configuration of the guards be $\{v_2, v_4, v_6, \ldots, v_{n-1}\}$ and suppose there is an attack on a vertex v_i. For the sake of simplicity, let us assume that $i < \frac{n+1}{2}$. Move the guard to v_{i-1}. From there, use the $\frac{i}{2}$ guards that are located on $P = v_1 - v_2 - \ldots - v_i$ to defend P and the remaining $\frac{n-i-1}{2}$ guards that are located on $P' = v_{i+1} - v_{i+2} - \ldots - v_n$ to defend P'. In this case, the attacker needs at least $1 + t(P_{\frac{n+1}{2}}) = 1 + \lfloor \log_2(\frac{n+1}{2}) \rfloor = \lfloor \log_2 n \rfloor$ turns to win.

Case 3: $\boxed{n \equiv 2 \pmod 4}$

Since $e^\infty(P_n) = \frac{n}{2}$, we may assume that at most $\frac{n}{2} - 1$ guards are located on P_n. We first prove that the attacker wins in at most $\lfloor \log_2(n-2) \rfloor - 1$ turns. We may assume without loss of generality that at most $\frac{n-2}{4}$ guards are located on the vertices of the set $\{v_1, v_2, v_3, \ldots, v_{\frac{n}{2}}\}$. Let i be the largest integer smaller than $\frac{n}{2} + 2$ such that v_i is unoccupied. Then there are at most $\lfloor \frac{(i-1)-1}{2} \rfloor$ guards located on the path $P = v_1 - v_2 - \ldots - v_{i-1}$. By the induction hypothesis, the attacker wins in at most $\lfloor \log_2(\frac{n}{2}) \rfloor = \lfloor \log_2 n \rfloor - 1 = \lfloor \log_2(n-2) \rfloor - 1$ turns by playing on the path $P = v_1 - v_2 - \ldots - v_{i-1}$.

We now prove that the attacker needs at least $\lfloor \log_2(n-2) \rfloor - 1$ turns to win. Place a guard on each of the vertices of the set $\{v_2, v_4, v_6, \ldots, v_{\frac{n}{2}-1}\} \cup \{v_{\frac{n}{2}+2}, v_{\frac{n}{2}+4}, \ldots, v_{n-1}\}$. Now defend the paths $P = v_1 - v_2 - \ldots - v_{\frac{n}{2}}$ and $P' = v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2} - \ldots - v_n$ independently with the guards that are currently located in each of them. By the induction hypothesis, the attacker needs at least $\lfloor \log_2(\frac{n}{2}) \rfloor = \lfloor \log_2(n-2) \rfloor - 1$ turns to win.

Case 4: $\boxed{n \equiv 3 \pmod 4}$

Since $e^\infty(P_n) = \frac{n+1}{2}$, we may assume that at most $\frac{n-1}{2}$ guards are located on P_n. We first prove that the attacker wins in at most $\lfloor \log_2 n \rfloor$ turns. For the same reason as in Case 2, we may further assume that the guards are located on the vertices of the set $\{v_2, v_4, v_6, \ldots, v_{n-1}\}$. Now, attack vertex $v_{\frac{n+1}{2}}$. Without loss of generality, suppose the guard moves to $v_{\frac{n-1}{2}}$. By the induction hypothesis, the attacker wins in the subgraph $v_{\frac{n+3}{2}} - v_{\frac{n+5}{2}} - \ldots - v_n$ in at most $t(P_{\frac{n-1}{2}}) = \lfloor \log_2 n \rfloor - 1$ turns.

We now prove that the attacker needs at least $\lfloor \log_2 n \rfloor$ turns to win. Let the initial configuration of the guards be $\{v_2, v_4, v_6, \ldots, v_{n-1}\}$ and suppose there is an attack on a vertex v_i. For the sake of simplicity, let us assume that $i < \frac{n+1}{2}$. Move the guard to v_{i-1}. From there, use the $\frac{i}{2}$ guards that are located on $P = v_1 - v_2 - \ldots - v_i$ to defend P and the remaining $\frac{n-i-1}{2}$ guards that are located on $P' = v_{i+1} - v_{i+2} - \ldots - v_n$ to defend P'. In this case, the attacker needs at least $1 + t(P_{\frac{n-1}{2}}) = 1 + \lfloor \log_2(\frac{n-1}{2}) \rfloor = \lfloor \log_2 n \rfloor$ turns to win.

This completes the proof. □

Corollary 4.30. *For any integer $n \geq 6$,*

$$t_2(C_n) = \begin{cases} \lfloor \log_2 n \rfloor & \text{if } n \text{ is odd,} \\ \lfloor \log_2(n-2) \rfloor - 1 & \text{if } n \text{ is even.} \end{cases}$$

Proof. Since $e^\infty(C_n) = \lceil \frac{n}{2} \rceil$, we may assume without loss of generality that there are fewer than $\lceil \frac{n}{2} \rceil$ guards located on C_n and that the vertices v_0 and v_{n-1} are unoccupied.

To see that these values are upper bounds on $t_2(C_n)$, attack the guards located on the vertices of the path $v_0 - v_1 - v_2 - \ldots - v_{n-1}$ the same way as in the proof of Proposition 4.29. The guards lose after at most $\lfloor \log_2 n \rfloor$ turns if n is odd and $\lfloor \log_2(n-2) \rfloor - 1$ turns if n is even.

To see that these values are lower bounds on $t_2(C_n)$, observe that the guards can always move (on the path $v_0 - v_1 - v_2 - \ldots - v_{n-1}$) according to the same strategy that was introduced in the proof of Proposition 4.29 whenever they are attacked. □

4.3.2 Asymptotic Bounds on $T_2(n, k)$ and $T_2(n)$

We now consider $T_2(n, k)$ for the first few values of k.

Proposition 4.31. $T_2(n, 1) = 1$.

Proof. Consider a graph G such that $e^\infty(G) > 1$. Then G has at most one universal vertex. If G has exactly one universal vertex, then the guard located on this vertex loses on the first turn. □

Definition 4.32. The configuration graph generated by k guards on a graph G, denoted by $\mathcal{C}(G, k)$, is the graph with vertex set the dominating sets of size k of G such that two vertices X and Y are adjacent if and only if there exist $x \in X \backslash Y$ and $y \in Y \backslash X$ such that $xy \in E(G)$ and $X - \{x\} = Y - \{y\}$.

Proposition 4.33. *$T_2(n, 2) = \Omega(n)$.*

Proof. We will show that there exist arbitrarily large graphs G of order n such that $e^\infty(G) = 3 > 2$ and $t_2(G) = \frac{n}{2} - 1$. To this end, let $k \geq 1$ and let H be the circulant graph $C_{4k+1}[1, 2, 3, \ldots, k]$. Let G be the n-vertex graph (where $n = 4k + 2$) obtained from H by adding a new vertex v and an edge between v and each vertex of H but v_k and v_{3k+1}.

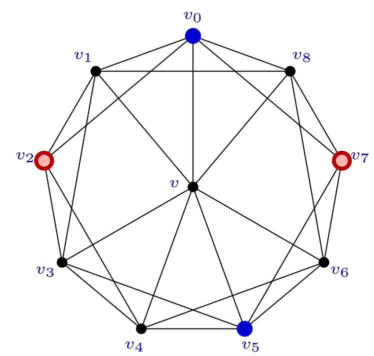


Figure 4.6: Graph G from the proof of Proposition 4.33 where $k = 2$.

Suppose there are two guards located on the vertices of G. Observe that there is exactly one vertex in H that dominates both of the vertices v_k and v_{3k+1}, the vertex v_0. So, if at some time t, a guard moves to v, we may assume that the other guard is located on v_0 and the guards lose on the next turn (after an attack on v_0). Now, observe that $\{v_i, v_j\}$ is a dominating set of H if and only if $i \equiv \pm j \pmod{2k}$; moreover, for any given dominating set $\{v_i, v_j\}$ of H, there exists only one k (respectively k') such that $\{v_i, v_k\}$ (respectively $\{v_j, v'_k\}$) is a dominating set of H. The configuration graph generated by two guards on G is the graph obtained by joining a vertex to the two central vertices of a path on $4k$ vertices (see Figure 4.7 for the case where $k = 2$).

The guards can be forced to move towards any leaf of the path on $4k$ vertices. After that, the guards lose on the next turn since they can be forced to move to $\{v_k, v_{3k+1}\}$ from each leaf. If the guards start on $\{v_0, v_{2k+1}\}$, the game lasts for at least $2k = \frac{n}{2} - 1$ turns.

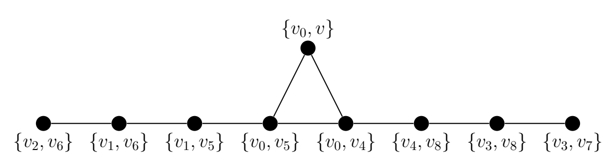


Figure 4.7: Configuration graph from the proof of Proposition 4.33 where $k = 2$.

This completes the proof. □

Observation 4.34. Suppose there are at least k guards located on the vertices of P_{2k-1}. The attacker can always force any leaf and its unique neighbour to be occupied at some time t during the game.

Theorem 4.35. $T_2(n) = \Omega(n^2)$.

Proof. It suffices to show that there exist arbitrarily large graphs G of order n such that $t_2(G) \geq cn^2$ for some $c > 0$. Let k be a positive integer and let $n = 3k + 2$. Let G be the graph such that:

$$V(G) = \bigcup_{i=1}^{2k-1} \{u_i\} \cup \{u^*, v^*\} \cup \bigcup_{i=1}^{k+1} \{v_i\}$$

and

$$E(G) = \bigcup_{i=1}^{2k-2} \{u_i u_{i+1}\} \cup \{u^* u_1, u^* v^*\} \cup \bigcup_{i=1}^{k+1} \{v^* v_i, u^* v_i\}$$

(see Figure 4.8 for the case where $k = 4$ and $n = 14$).

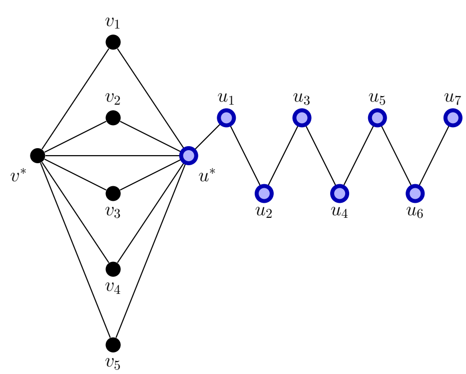


Figure 4.8: A graph G on 14 vertices with $e^\infty(G) = 9$, where the attacker needs at least 21 turns to win against 8 guards.

It can be easily verified that $\alpha(G) = \theta(G) = 2k+1$. We claim that $e^\infty(G) = 2k+1$ and that $t_2(G) \geq \binom{k+1}{2}$. Suppose $2k$ guards are located on the vertices of G. We first show that the attacker can always force k guards to be located on $S = \bigcup_{i=1}^{k+1} \{v_i\}$.

To see this, repeat the following until the guards are located on the desired configuration. Suppose there are fewer than k guards located in S (without loss of generality only the vertices $v_1, v_2, v_3, \ldots, v_l$ are occupied, where $l < k$). Since v_{l+1} must be dominated, there is at least one guard located in the set $\{u^*, v^*\}$. If there are two guards in $\{u^*, v^*\}$, then attack the guard located on vertex v^*. The guard must move to v_{l+1}. Suppose there is only one guard located in $\{u^*, v^*\}$. If the guard is on u^*, attack u^*. The guard must move to v^*. Note that there are at least $k + 1$ guards located on the subgraph induced by $T = \bigcup_{i=1}^{2k-1}\{u_i\}$. By Observation 4.34, a guard can always be forced to move from a vertex of T to u^*. In this case, attack the guard on v^*. The guard must move to u_{l+1}.

After k guards are located on the vertices of S, there is a guard located on $\{u^*, v^*\}$. If the guard is located on u^*, attack u^* so that the guard moves to either v^* or v_{k+1}. In this case, there are fewer than k guards located on the vertices of T. Since $e^\infty(G[T]) = k$ and none of the $k+1$ other guards can defend a vertex in $G[T]$, the guards eventually lose on this subgraph.

Now, suppose there are $2k$ guards located on the vertices of the set $\{u^*\} \cup T$. We will show that the guards can play in a way such that the game last for at least $\binom{k+1}{2}$ turns. Observe that if there is a guard on either u^* or v^*, all the vertices in S are dominated; moreover, if there are at least k guards on T, the guards can defend all the vertices of $G[T]$. To see that the attacker needs at least $\binom{k+1}{2}$ turns to win, note that the attacker needs at least $1+2+3+\ldots+k = \binom{k+1}{2}$ turns to force the guards located on $u_1, u_2, u_3, \ldots, u_k$ to leave $G[T]$. This completes the proof. □

Chapter 5
Conclusion

As we complete our dissertation, it is important to highlight that our exploration of the eternal domination game and the eviction game has not only enriched our understanding of some aspects of the game but also presented a gallery of open questions, each of which provides a path for future research.

Complexity

To begin with, our research revealed in Chapter 2 that the problem of deciding whether k guards can respond to any sequence of attacks on an n-vertex graph G from a given initial configuration is EXPTIME-complete in both the eternal domination game and the eviction game. However, when the guards can choose their initial configuration, does the decision problem still retain its EXPTIME-completetess? We conjecture so.

Conjecture 5.1. ETERNAL DOMINATION is EXPTIME-complete.

Conjecture 5.2. EVICTION is EXPTIME-complete.

Algorithms

As we have seen in Section 1.2, the eternal domination number of a perfect graph can be determined in polynomial time. We have also shown in Section 4.2 that the eviction number of a cograph can be determined in polynomial time. It would be helpful to know more classes of graphs for which these parameters can be determined in polynomial time. This gives rise to the following problem.

Problem 5.3. Find further classes that contain graphs for which the eternal domination number can be computed in polynomial time.

Problem 5.4. Find further classes that contain graphs for which the eviction number can be computed in polynomial time.

Bounds

As shown by Klostermeyer and MacGillivray in [31], if G is a graph with independence number 3, then G has eviction number at most 5. However, it is unknown whether there exists a graph G such that $\alpha(G) = 3$ and $e^\infty(G) = 5$. This naturally gives rise to the following question.

Question 5.5. Does there exist a graph G such that $\alpha(G) = 3$ and $e^\infty(G) = 5$?

We have shown in Section 4.2 that for any integer $k \geq 1$, there exists a value $f(k)$ such that any graph with independence number at most k has eviction number at most $f(k)$. However, our value for $f(k)$ does not appear to be tight even for the first few values of k. This strengthen the relevance of the question posed by Klostermeyer and MacGillivray which we revisit next.

Question 5.6 ([31])**.** Does there exist a constant c such that $e^\infty(G) \leq c\alpha(G)$?

Since no graph with ratio $\frac{e^\infty}{\alpha} > \frac{4}{3}$ is known, $\frac{4}{3}$ is perhaps a constant that satisfies the condition in the preceding question. This motivates the following question.

Question 5.7. Does there exist a graph G such that $\frac{e^\infty(G)}{\alpha(G)} > \frac{4}{3}$?

We base the following conjecture on the observation that the eternal domination number of the first four graphs in the infinite family $\mathcal{F}$ of circulant graphs that was introduced in Section 3.4 is equal to their clique covering number.

Conjecture 5.8. For any graph $\mathcal{G}_k \in \mathcal{F}$, $\gamma^\infty(\mathcal{G}_k) = \theta(\mathcal{G}_k) = 3k + 3$.

Length

The data presented in Table 3.8 illustrates that the length of the eternal domination game appears to be growing quickly. This raises the question of whether this growth is exponential.

Question 5.9. Is it true that there exists $\delta > 0$ such that $T_1(n) \geq 2^{\delta n}$ for any integer $n \geq 3$?

We also raise a similar question for the eviction game as well as follows.

Question 5.10. Is it true that there exists $\delta > 0$ such that $T_2(n) \geq 2^{\delta n}$ for any integer $n \geq 3$?

Propositions 3.32 and 4.27 show that n^k is an asymptotic upper bound on $T_1(n,k)$ and $T_2(n,k)$. We were wondering whether this is also an asymptotic lower bound on $T(n,k)$ for some k.

Question 5.11. Does there exist $k > 2$ such that $T_1(n,k) = \theta(n^k)$?

Question 5.12. Does there exist $k > 2$ such that $T_2(n,k) = \theta(n^k)$?

Eternal Dominating Families

Question 5.13. Does there exist a graph G of Type II such that G has an eternal dominating family, where the dominating sets have cardinality exceeding $e^\infty(G)$, that does not satisfy Property $*$?

Bibliography

[1] N. Alon and J. H. Spencer, The probabilistic method. With an appendix by Paul Erdős, *Wiley-Interscience Series in Discrete Mathematics and Optimization*, John Wiley & Sons, (1992).

[2] M. Anderson, C. Barrientos, R. C. Brigham, J. R. Carrington, R. P. Vitray and J. Yellen, Maximum-demand graphs for eternal security, *J. Combin. Math. Combin. Comput.* 61 (2007), 111–127.

[3] S. Arora and B. Barak, Computational Complexity: A Modern Approach, *Cambridge University Press* (2009).

[4] J. Arquilla and H. Fredricksen, "Graphing" an Optimal Grand Strategy, *Military Operations Research* (1995), 3–17.

[5] G. Bagan, A. Joffard and H. Kheddouci, Eternal dominating sets on digraphs and orientations of graphs, *Discrete Appl. Math.* **291** (2021), 99–115.

[6] A. Bonato and G. MacGillivray, Characterizations and algorithms for generalized cops and robbers games, *Contrib. Discrete Math.*, 12 (2017), no. 1, 110–122.

[7] G. Brinkmann and B. D. McKay, Fast generation of planar graphs, *MATCH Commun. Math. Comput. Chem.* 58 (2007), no. 2, 323–357.

[8] G. Brinkmann. and B. D. McKay, The program **plantri**, Available at `https://users.cecs.anu.edu.au/~bdm/plantri/`

[9] A. P. Burger, E. J. Cockayne, W. R. Grundlingh, C. M. Mynhardt, J. H. van Vuuren and W. Winterbach, Infinite order domination in graphs, *J. Combin. Math. Combin. Comput.* 50 (2004), 179–194.

[10] M. Bóna, A Walk through Combinatorics: An introduction to Enumeration and Graph Theory, Fourth Edition, 2017, *World Scientific Publishing Co. Pte. Ltd., New Jersey.*

[11] E. Camby and O. Schaudt, A New Characterization of P_k-free Graphs, In: D. Kratsch, I. Todinca, (eds), Graph-Theoretic Concepts in Computer Science. WG 2014. Lecture Notes in Computer Science, vol 8747 (2014). Springer, Cham.

[12] V. Chvátal, The minimality of the Mycielski graph, *Graphs and Combinatorics (Proc. Capital Conf., George Washington Univ., Washington, D.C., 1973), Lecture Notes in Math.* 406 (1974), 243–246, Springer, Berlin.

[13] D. G. Corneil, H. Lerchs, L. S. Burlingham, Complement reducible graphs, *Discrete Appl. Math.* **3** (1981), no. 3, 163–174.

[14] D. G. Corneil, Y. Perl and L. K. Stewart, A linear recognition algorithm for cographs, *SIAM J. Comput.* **14** (1985), no. 4, 926–934.

[15] B. Descartes, A three colour problem, *Eureka* 9 (1947), 21.

[16] B. Descartes, Solution to advanced problem no. 4526, *Amer. Math. Monthly* 61 (1954), 352.

[17] F. Devvrit, A. Krim-Yee, N. Kumar, G. MacGillivray, B. Seamone, V. Virgile and A. Xu, Fractional eternal domination: securely distributing resources across a network, *Discuss. Math. Graph Theory, to appear* (2023).

[18] P. Erdős, D. J. Kleitman and B. L. Rothschild, Asymptotic enumeration of K_n-free graphs, *Colloquio Internazionale sulle Teorie Combinatorie (Rome, 1973)*, 2 (1976), 19–27.

[19] S. Finbow, S. Gaspers, M. E. Messinger and P. Ottaway, A note on the eternal dominating set problem, *Internat. J. Game Theory* **47** (2018), 543–555.

[20] A. S. Fraenkel and D. Lichtenstein, Computing a perfect strategy for $n \times n$ chess requires time exponential in n, *J. Combin. Theory Ser. A*, **31** (1981), no. 2, 199–214.

[21] A. Gagnon, A. Hassler, J. Huang, A. Krim-Yee, F. Mc Inerney, A. Mejía Zacarías, B. Seamone and V. Virgile, A method for eternally dominating strong grids, *Discrete Math. Theor. Comput. Sci.* **22** (2020), no. 1, Paper No. 8, 11 pp.

[22] M. R. Garey and D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, *Freeman San Francisco* **174** (1979).

[23] W. Goddard, S. M. Hedetniemi and S. T. Hedetniemi, Eternal security in graphs, *J. Combin. Math. Combin. Comput.* **52** (2005), 169–180.

[24] J. L. Goldwasser and W. F. Klostermeyer, Tight bounds for eternal dominating sets in graphs, *Discrete Math.*, **308** (2008), no. 12, 2589–2593.

[25] A. S. Goldstein and E. M. Reingold, The complexity of pursuit on a graph, *Theoret. Comput. Sci.*, **143** (1995), no. 1, 93–112.

[26] M. Grötschel, L. Lovász and A. Schrijver, Geometric algorithms and combinatorial optimization, Springer-Verlag, (1988).

[27] W. B. Kinnersley, Cops and robbers is EXPTIME-complete, *J. Combin. Theory Ser. B*, **111** (2015), 201–220.

[28] W. F. Klostermeyer, Complexity of eternal security, *J. Combin. Math. Combin. Comput.* **61** (2007), 135–140.

[29] W. F. Klostermeyer and G. MacGillivray, Eternal security in graphs of fixed independence number, *J. Combin. Math. Combin. Comput.* 63 (2007), 97–101.

[30] W. F. Klostermeyer and G. MacGillivray, Eternal dominating sets in graphs, *J. Combin. Math. Combin. Comput.* 68 (2009), 97–111.

[31] W. F. Klostermeyer, M. Lawrence and G. MacGillivray, Dynamic Dominating Sets: the Eviction Model for Eternal Domination, *J. Combin. Math. Combin. Comput.* **97** (2016), 247–269.

[32] W. F. Klostermeyer and G. MacGillivray, Eternal domination: criticality and reachability, *Discuss. Math. Graph Theory* **37** (2017), no. 1, 63–77.

[33] W. F. Klostermeyer, M. E. Messinger and A. Angeli Ayello, An eternal domination problem in grids, *Theory and Applications of Graphs*, 4 (2017), no. 1, Art. 2, 23 pp.

[34] W. F. Klostermeyer and C. M. Mynhardt, Eternal Total Domination in Graphs, *Ars Comb.* 107 (2012), 473–492.

[35] W. F. Klostermeyer and C. M. Mynhardt, A dynamic domination problem in trees, *Trans. Comb.*, 4 (2015), no. 4, 15–31.

[36] W. F. Klostermeyer and C. M. Mynhardt, Domination, eternal domination and clique covering, *Discuss. Math. Graph Theory* 35 (2015), no. 2, 283–300.

[37] W. F. Klostermeyer and C. M. Mynhardt, Protecting a graph with mobile guards, *Appl. Anal. Discrete Math.*, 10 (2016), no. 1, 1–29.

[38] W. F. Klostermeyer and C. M. Mynhardt, Eternal and secure domination in graphs, *Topics in domination in graphs, Dev. Math.* 64 (2020), 445–478, Springer, Cham.

[39] A. Krim-Yee, B. Seamone and V. Virgile, Eternal domination on prisms of graphs, *Discrete Appl. Math.* 283 (2020) 734–736.

[40] L. Lovász, A characterization of perfect graphs, *J. Combin. Theory Ser. B* 13 (1972), 95–98.

[41] L. Lovász, Normal hypergraphs and the perfect graph conjecture, *Discrete Math.* 2 (1972), no. 3, 253–267.

[42] G. MacGillivray, C. M. Mynhardt and V. Virgile, Eternal domination and clique covering, *Electron. J. Graph Theory Appl. (EJGTA)* 10 (2022), no. 2, 603–624.

[43] B. D. McKay and A. Piperno, Practical graph isomorphism, II., *J. Symbolic Comput.* 60 (2014), 94–112.

[44] J. Mycielski, Sur le coloriage des graphes, *Colloq. Math.* 3 (1955), 161–162.

[45] J. Petr, J. Portier and L. Versteegen, A faster algorithm for Cops and Robbers, *Discrete Appl. Math.* **320** (2022), 11–14.

[46] F. P. Ramsey, On a Problem of Formal Logic, *Proc. London Math. Soc.* (2) 30 (1929), no. 4, 264–286.

[47] C. S. ReVelle, Can you protect the Roman Empire?, *Johns Hopkins Magazine*, 49 (1997) no. 2, 40.

[48] C. S. ReVelle and K. E. Rosing, Defendens imperium romanum: a classical problem in military strategy, *Amer. Math. Monthly*, 107 (2000), no. 7, 585–594.

[49] E. M. Reingold, J. Nievergelt and N. Deo, Combinatorial Algorithms: Theory and Practice, (1977).

[50] J. M. Robson, Another intractable game on Boolean expressions, *Technical Report, Department of Computer Science, The Australian National University*, (1981).

[51] J. M. Robson, The complexity of Go, *Proc. IFIP, 1983* (1983).

[52] J. M. Robson, N by N checkers is Exptime complete, *SIAM J. Comput.*, **13** no. 2 (1984), 252–267.

[53] I. Stewart, Defend the Roman Empire!, *Scientific American*, 281 (1999), no. 6, 136–138.

[54] L. J. Stockmeyer and A. K. Chandra, Provably difficult combinatorial games, *SIAM J. Comput.*, **8** no. 2 (1979), 151–174.

[55] V. Virgile, Domination éternelle dans les graphes, Master's Thesis, Université de Montréal, Montréal, QC, Canada (2019), `https://papyrus.bib.umontreal.ca/xmlui/handle/1866/22167`.

Appendix A

All the graphs in this appendix are listed in Graph6 format.

<table>
<tr><th>n</th><th>List of graphs</th></tr>
<tr><td>5</td><td>DUW</td></tr>
<tr><td>7</td><td>FCptO FCxv? FUzro</td></tr>
<tr><td>8</td><td>GCrbuW GCpveg GCxvcw GEhuSw</td></tr>
<tr><td>9</td><td>H?bBVbQ H?bBTjQ H?bBThY H?bBTiU H?bDJqU H?bF`xw H?bDjpw H?bfBqU H?bbVaU<br>H?bbV_] H?bvbro H?rF`zo H?q`qjo H?o~fbo HCQ`faw HCQeJaL HCrfRym HCrbvW}
HCrUrze HCpvfim HCpvexy HCpvVW} HCpvUzq HCpvUzU HCpvRym HCpunrM HCpunp]
HCrJvi] HCzbvZe HCxvfri HCxvfpy HCxvez[HCxvezM HCvdjrM HEjfaxu HEhuTxm
HEhvTy{ HUzvvx}</td></tr>
</table>

Table 5.1: List of critical graphs (with respect to θ) with $\alpha < \theta$.

<table>
<tr><th>n</th><th>List of graphs</th></tr>
<tr><td>10</td><td>IEhbtj{ro IEhbtn{ro</td></tr>
<tr><td>11</td><td>JQyurj]yt|? JEhbtj{rvu? JEhbtj{rvx? JEhbtj{rvf? JEhbtj{ruv? JEhbtj{rvT_
JEhbtj{rtt_ JEhbtj{rrt_ JEhbtj{rv}? JEhbtj{rv|? JEhbtj{rvv? JEhbtj{rv^?
JEhbtj{rvx_ JEhbtj{rvt_ JEhbtj{rvl_ JEhbtj{rv_ JEhbtj{rvf_ JEhbtj{rtv_
JEhbtj{rv~? JEhbtj{rv|_ JEhbtj{rvv_ JEhbtj{rv~_ JEhbtnm~E|? JEhbtnm~FZ?
JEhbtnm~D]_ JEhbtnm~@}_ JEhbtnN~Fu? JEhbtnN~Bv? JEhbtnN~Fw_ JEhbtnN~Fe_
JEhbtnN~Eu_ JEhbtnN~Bu_ JEhbtnN~Ef_ JEhbtnN~F}? JEhbtnN~Fv? JEhbtnN~F{_
JEhbtnN~Fu_ JEhbtnN~F]_ JEhbtnN~Ff_ JEhbtnN~Ev_ JEhbtnN~Bv_ JEhbtnN~F~?
JEhbtnN~F}_ JEhbtnN~Fv_ JEhbtnN~F~_ JEhvUtn~D{_ JEhutz{xr_ JEhru^u~Fj?
JEhru]v~Et_ JEjbtnN~Dm_ JEnfbz\zbv? JCXetqu|uz? JCXetq}vVm? JCXetq}|uz?</td></tr>
</table>

Table 5.2: List of connected graphs with $\gamma^{\infty} < \theta$.

<table>
<tr><th>n</th><th>List of graphs</th></tr>
<tr><td>13</td><td>L?`@F?M]DgYOFo L?`DAboU`w@{hS L?`@?boNAsLGBw L?`@?boNAsOwYO L?`@?boNAs@{os
L?`@Cbod`w@{YS L?BDB?{{AsRGBs L?`@C`wl?{DgQs L?BDB?{Ucq^?Fo L?`@F@wlEcBoBo<br>L?`@F@wlEcBoFo L?`@F@wlEcBgFo L?`@F?kQ_}YSlC</td></tr>
</table>

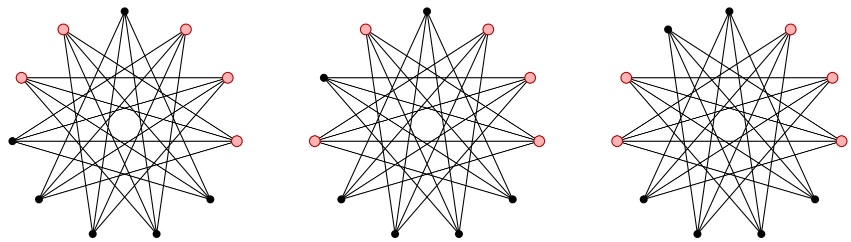
Table 5.3: List of triangle-free graphs with $\gamma^{\infty} < \theta$.

<table>
<tr><th>n</th><th>List of graphs</th></tr>
<tr><td>13</td><td>L?`DAboU`w@{hS L?BDB?{{AsRGBs L?BDB?{Ucq^?Fo L?`@F?kQ_}YSlC L?`@?boNAsOwYO</td></tr>
<tr><td>14</td><td>M?`@F?kQckBwsglC? M?`@?boNAs@{oshQ?</td></tr>
<tr><td>15</td><td>N??EFBOK_}DsrC~?^_? N??EFBOK_ZbwJgrC^_? N?AAF@SBo}TSm_{CN_? N?AA@aoqTsVO^?BwGl_
N?AADB_HdsN_VOE[B{? N?AADB_HeobMm_VOB{? N?AADB_HaqbM^?m_B{? N?ABB?WwGNN_eoFs^@?
N?ABAaQFbgTG}?JWPX? N??CBBOk@\HqfO^_@}? N?`@F?kQcKBwsglCN@? N??ED?qw@{JgRW~?By?
N??ED?qwAaHkjA~?By? N??ED?qwAaHkJa~?By? N??ED?qwAaPdRW~?By? N??ED?qwAyJgRW~?By?
N??ED?qwEXHkN_~?By? N??ED?qwEXHkVOFs^_? N??ED?qwAZHkN_~?By? N??CB@OeOmCuiI~?No?
N?AAD@WUDD_}N_VQ^?_ N??CFBOBw}Ds^_~?No? N??EDBo{@{JgFoB{^_? N?AA@BOHn_N_m_uS@|?
N??CFBOK_}BwJgrCNo? N??CB@OKCyTcrC^_@}? N??ED@_MdoRC~?FwGj_ N??EDbGl?]EsLg~?HIo
N?AA@B_@zwVOuOFw^A? N?AAF?e{EpN_m_Fo@y? N?AA@Bo{@{JgeoB{@}? N??ED@_Nfo^?VOfG?|_
N??ED@_NayRc~?Fw^_? N??ED@_NayRcN_~?B{? N?AA@Boy?^FoVOeoAy? N?AADb_RO^Bs^?m_UQ?
N?AADb_RTc~?N_BwBs? N?AA@b_HaYPWFguCN_? N??CB?Xs?]cufOxGNo? N?AA@BoZ?^QYm_VOFo?
N??CABoyBKRHBwDsVg? N?AA@?O{BwVOuOyG@}? N?AA@?O{BwRWigRW@}? N?AA@?O{@kJgZGaw@N?
N?AA@?O{@kJgigB{XL? N?AA@?O{@{JgeoyG@}? N?AA@?O{@{JgZGaw@}? N?AA@?O{@{XKVOig@}?
N?AA@?O{?}DsRWaw@}? N??E@b_sEYBsJc~?@}? N?AAD@O{Ai^?N_ii@}? N?AAD@O{AiN_N_B{TT?
N?AAD@O{AyN_m_ig@}? N??ED?WHiJJIMaxG^_? N?ABA_oebgTGqGBwP\? N?ABA_oedQXC^?BwGf?
N?AA@b_qQYAu^?m_@{? N?AA@b_qO^Bs^?m_Eq? N?AA@b_qOl`{^?m_^?? N?AA@b_qOl`{^?m_N_?
N??EDaKIaYMCN_tCET? N?AA@BoJ_^YI^?eoVO? N?AA@?WwGNN_m_uON_? N?AA@?WwGNN_m_uOBy?
N?AA@?WwGNN_m_uO^@? N?AA@b_RRgBsN_aqUQ? N?AA@b_RSUEoFg}?XG_ N?AA@b_RSUUQ\AFg^??
N??CFBOKeWbwJg^_^_? N??CFBOKeW`mFoJgNo? N??ED?qN_}HkrG~?^_? N??ED?qN_}HkrGVO^_?<br>N??ED?qSROQ`FobK^_? N??ED?qSO}HkJc~?^_? N??ED?qSO}HkJcrG^_? N??ED?qSO}HkJcbK^_?
N??ED@OAw^FoVOjG]a? N?AA@B_{BwVOuOFw@}? N?AA@B_{@{JgeoB{B{? N??EDaKRR`]?N_Br^_?
N??CB@OyDPRgBwDuNo? N?AAF@Sy?mtSm_FoN_? N?AA@AOwFoRWJgRWBy? N?AA@AOwDsZGZGFs@}?
N?AA@ao{@kGkqQB{VO_ N?AAD@Wgno^?N_VS?|_ N??CEBoy?^Ay~?NoNo? N??CBBOyBKRHDsJg^_?
N?AADBOQRwDsm_FoEq? N?AAD?oqN_VOLcVO@|? N?AAD?oqJgVOBwVOPT? N??CFBOQ_}BwJglC^_?
N??CFBOQ_}DslC^_^_? N??ED?qEqyHkxCN_^_? N??ED?qEo}HkxC~?^_? N??ED?qEo}WeRW~?^_?
N??ED?qEqZHkN_xC^_? N?AA@B_}BwFoVOeoB{? N?AA@B_}@{FoVOeoB{? N?AA@B_}@{JgeoFwB{?
N?AA@AO{BwRWJgRW@}? N?AA@AO{DsZGFoZG@}? N?AA@AO{CuFoVOZG@}? N?AA@AO{BHVOVOeq@}?
N?AA@AO{BHFoVOB{RX? N?AA@BO{FoJgeoJg@}? N?AA@BO{@{JgeoJg@}? N??CFBO{?}DsB{~?No?
N?`@?boNAs[G`oBwBo_ N?`@?boNAsLG`o`o@{? N?`@?boNAsLG`o`oN?W N?`@?boNAsSgooBo?Fw
N?AA@B_\BwVOuOA{B{? N?AA@B_\BwVOasuOB{? N?AA@B_\DSPYTQuOB{? N?AA@B_\@[YH}?asB{?
N?AA@B_\@[YHasVOB{? N?AA@B_\CU~?N_VOB{? N?AA@B_\CUvON_VOB{? N?AA@B_\CUvOuON_B{?
N?AA@B_\CUvOuOFwB{? N?AA@B_\CUfoVOeoB{? N??ED@o]@sJGBw~?Of_ N?AA@_owBwVOFcqW@}?
N?AA@BO}FoJgeoJg?~? N?AA@_goW]As}?^?VO? N?AA@_goW]As^?m_N_? N?AA@_goW]As^?m_Fq?
N?AA@_goW]As^?m_]@? N?AA@_goW]As^?m_YD? N?AA@_goW]As^?m_[@_ N?AA@b@BpLZ@^?m_Fs?
N?AA@b@Bo\N_m_{EFs? N?AA@b@Bv_n_m_NgFs? N?AA@BOX@\@]VO}?RY? N?AAF@ScfoTSZCFoBw?
N?AAF@Sc`kBwigZC@N? N??ED?ZDp{[cVOyC^_? N?AA@_o{FoFoRWaw@}? N?AA@_o{@kGkB{qYVO_
N?AA@_o{AJN_qWB{VP? N??ED?ZFp{[cVO~?^_? N??FEaK[N_BoJO~?@z? N?AA@?WeF@Cueo^?Gm?
N??CB@OeRcCuFobY^_? N?AADB_\FoVOVOA{B{? N?AADB_\BwVOVOA{B{? N?AA@B_HbwVOVOFwZI?
N?AA@B_HbwVOFwFwZI? N?AA@B_HbwVOE[FwZI? N?AA@B_HaI^?uSmaB{? N?AA@B_HaIN_FwuSVP?
N?AA@B_HaIN_EMeoN_? N?AA@B_HaIN_EMeoRW? N?AA@B_HaIN_EMeoB{? N?AA@B_HaIOW}?FwBF_
N?AA@B_HaIRWFwuSVP? N?AA@B_HaIB{uSmaB{? N?AA@B_HaiTP}?uSB{? N?AA@B_HaiTP^?FwZI?
N?AA@B_HaiTPm_FwZI? N?AA@B_HayRW}?FwZI? N?AA@B_HayRW^?FwZI? N?AA@B_H_rjgm_FwZI?
N?AADAo{AJFoVOB{VP? N??ED@_{@{FoVOfG@}? N??ED@_{@{JgfGB{^_? N??ED@_{@{JgfGB{B{?
N?AA@_o}@{^?RWaw?}_ N??CFBOQdguaFoJg^_? N?AADB_kdsN_RSFwB{? N?AADB_k_NHi^?Fw^@?
N?ABAaQT?]N_}?iSDM?</td></tr>
</table>

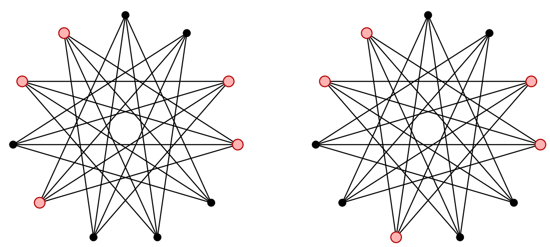
Table 5.4: List of maximal triangle-free graphs with $\gamma^\infty < \theta$.

Appendix B

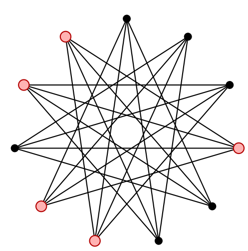
This appendix contains additional material that support the proof of Lemma 3.24.



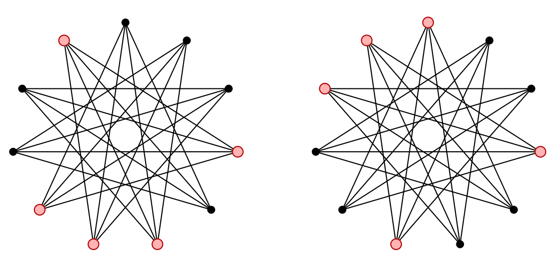
(a) D_1: Configurations from which the guards can lose in one move.



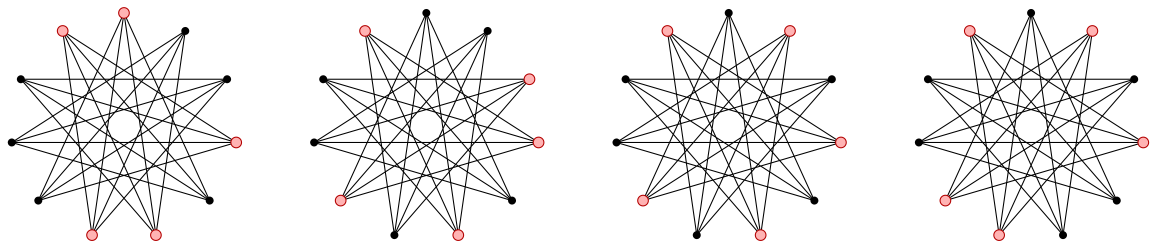
(b) D_2: Configurations from which the guards lose in two moves.



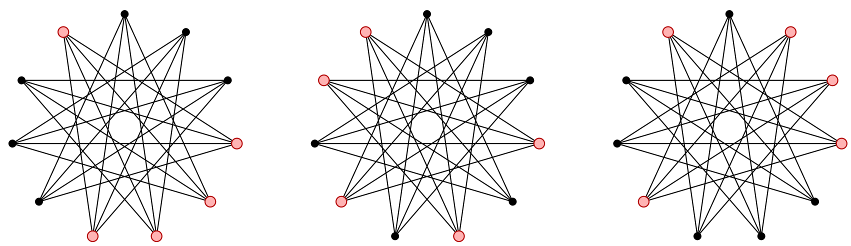
(c) D_3: Configurations from which the guards lose in three moves.



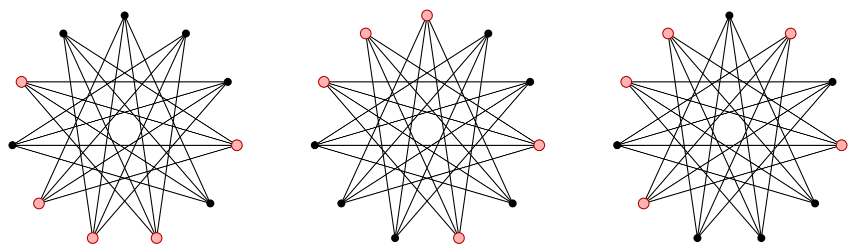
(d) D_4: Configurations from which the guards lose in four moves.



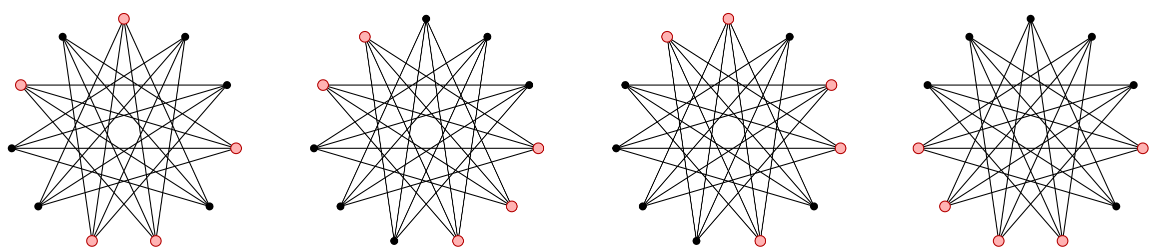
(e) D_5: Configurations from which the guards lose in five moves.



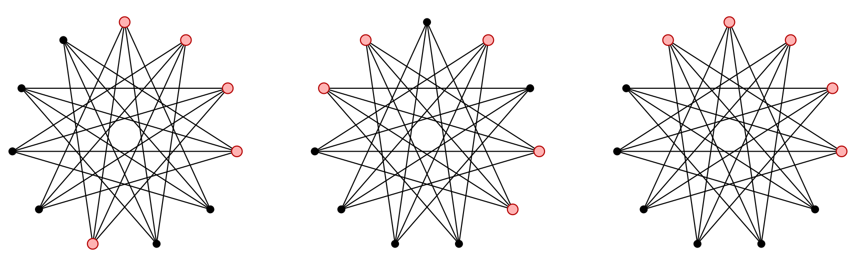
(f) D_6: Configurations from which the guards lose in six moves.



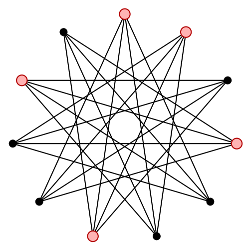
(g) D_7: Configurations from which the guards lose in seven moves.



(h) D_8: Configurations from which the guards lose in eight moves.



(i) D_9: Configurations from which the guards lose in nine moves.



(j) D_{10}: Configurations from which the guards lose in ten moves.

Figure 5.1: List of configurations of five guards in $C_{11}[4, 5]$.

	List of configurations (up to symmetry)	Total
D_1	$\{0,1,2,4,5\}$, $\{0,1,2,4,6\}$, $\{0,1,2,5,6\}$	66
D_2	$\{0,1,4,5,7\}$, $\{0,1,4,5,8\}$, $\{0,3,4,7,10\}$	33
D_3	$\{0,4,5,7,8\}$	22
D_4	$\{0,4,7,8,9\}$, $\{0,3,4,5,8\}$	33
D_5	$\{0,3,4,8,9\}$, $\{0,1,4,7,9\}$, $\{0,2,4,7,9\}$, $\{0,2,4,7,8\}$	66
D_6	$\{0,4,8,9,10\}$, $\{0,4,5,7,9\}$, $\{0,1,2,4,7\}$	55
D_7	$\{0,5,7,8,9\}$, $\{0,3,4,5,9\}$, $\{0,2,4,5,7\}$	55
D_8	$\{0,3,5,8,9\}$, $\{0,4,5,9,10\}$, $\{0,1,3,4,9\}$, $\{0,6,7,8,9\}$	77
D_9	$\{0,1,2,3,8\}$, $\{0,2,4,5,10\}$, $\{0,1,2,3,4\}$	44
D_{10}	$\{0,2,3,5,8\}$	11

Table 5.5: List of configurations of five guards in the circulant graph $C_{11}[4, 5]$ (up to symmetry).