Algebraic Cycles on Products of Generically Smooth Quadrics

by

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ABSTRACT

In this thesis, we study rationality questions for algebraic cycles (modulo 2) on products of generically smooth projective quadrics over fields of any characteristic. More specifically, let $X_1, \cdots, X_n$ be generically smooth projective quadrics over a field F with anisotropic totally singular part. We study the image $\overline{Ch}(X_1 \times \cdots \times X_n)$ of the scalar-extension homomorphism $Ch(X_1 \times \cdots \times X_n) \to Ch((X_1 \times \cdots \times X_n)_{\overline{F}})$, where $\overline{F}$ denotes a fixed algebraic closure of F and Ch denotes the total Chow group modulo 2. This has been studied extensively in the case where $X_1, \cdots, X_n$ are smooth by A.Vishik, N.Karpenko, A.Merkurjev and others. Our goal is to extend the existing theory to include the case of singular but generically smooth quadrics in characteristic 2. Here we follow recent work of Karpenko, who has considered the special case where $X_1 = \cdots = X_n$.

First, we show that the image $\overline{Ch}(X_1 \times \cdots \times X_n)$ inherits a ring structure and an action of Steenrod operations from the mod-2 Chow ring of the smooth locus of $X_1 \times \cdots \times X_n$. Using the ring structure, we then introduce and study a composition of rational correspondences (modulo 2) for products of generically smooth projective quadrics, laying foundations for an investigation of non-totally singular quadratic forms in any characteristic by algebraic-geometric methods. In this direction, we introduce a new discrete invariant for such forms, which we call the *rational correspondence type*. This extends the *motivic decomposition type* previously defined by Vishik for non-degenerate forms. We extend several well-known results on Vishik's invariant to our more general setting. These include a number of restrictions imposed by splitting pattern invariants, as well as results that relate the rational correspondence types of different forms in situations where their associated quadrics can be related via suitable Chow correspondences. Using these results, we compute the rational correspondence type for certain families of forms, including generic forms of even dimension, and so-called quasi-strongly excellent forms. In the final part of the thesis, we show that the deepest result of Vishik on the motivic decomposition type, the so-called *excellent connections theorem*, remains valid for arbitrary non-totally singular forms of dimension at most 9. We also apply our methods to the study of a conjecture of Hoffmann and Laghribi on the classification of singular Pfister neighbours, and to the study of the isotropy behaviour of quadratic forms over function fields of quadrics.

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Victoria is the start of my journey. I would like to conclude by the following sentences of Dickens:

"It was the best of times, it was the worst of times; it was the age of wisdom, it was the age of foolishness; it was the epoch of belief, it was the epoch of incredulity; it was the season of Light, it was the season of Darkness; it was the spring of hope, it was the winter of despair; we had everything before us, we had nothing before us."

DEDICATION

In memory of Alexander Grothendieck (1928-2014).

In memory of people sacrificing their lives in pursuit of FREEDOM (1989-2022).

Notations and Terminology

Throughout the thesis, *rings* will be commutative rings with unity. A vector space means a finite-dimensional vector space. By a *scheme*, we mean a separated scheme of finite type over some field, and a *variety* means an integral scheme. If X is a scheme, and x is a point of X, then we say X is *regular* at x if the local ring at x is a regular local ring. The set of all smooth points of X is called the *smooth locus* of X. Also, the set of all non-smooth points of X is called the *singular locus* of X. A *flat morphism* between two schemes is always assumed to have relative dimension n for some integer n. Finally, we give a brief list of some of the basic notations that will be used in the thesis.

$F \subset K$	— the inclusion of a field F as a subfield of a larger field K.
$char(F)$	— the characteristic of a field F.
$\overline{F}$	— a field extension of F containing an algebraic closure of F.
$F^{\times}$	— the multiplicative group of a field F.
$\mathcal{O}_{X,x}$	— the local ring of a scheme X at a point $x \in X$.
$\mathcal{m}_{X,x}$	— the maximal ideal of the local ring of a scheme X at a point $x \in X$.
$l(M)$	— the length of a module M.
$Frac(R)$	— the fraction field of an integral domain R.
$F(X)$	— the function field of a variety X over a field F.
$\kappa(x)$	— the residue field of the local ring of a scheme X at a point $x \in X$.
$X(K)$	— the set of all morphisms from $Spec(K)$ to a scheme X.
X_{sm}	— the smooth locus of a scheme X.

Chapter 1

Introduction

1.1 Overview of the research topic

A quadratic form is a homogeneous polynomial of degree 2 in a finite set of variables with coefficients in a ring. While the study of these objects can be dated back to Babylonian times, it was in the work of the great number theorists of the 18th and 19th centuries that a systematic theory of quadratic forms began to emerge. Indeed, much of the best work carried out in arithmetic during this period dealt with understanding the structure of quadratic forms over algebraic number fields (like the rational numbers) and their rings of integral elements (like the ordinary ring of integers). Intrinsic interest aside, quadratic forms have since emerged as objects of importance in numerous subfields of mathematics, where they often arise as invariants of interest in certain classification problems. For instance, we can mention noncommutative ring theory (norm forms of composition algebras), Lie theory (Killing forms), topology (cohomology intersection forms of manifolds), as well as more applied areas such as coding theory (construction of space-time block codes). The algebraic theory of quadratic forms, initiated by E. Witt in 1937, seeks to understand the classification of quadratic forms with coefficients in an arbitrary field up to isomorphism (i.e., up to invertible linear change of variables).

From a modern perspective, the "non-degenerate" side of this theory finds its most natural generalization in the study of linear algebraic groups over general fields. Indeed, the automorphism groups of non-degenerate quadratic forms over fields provide fundamental examples of reductive group schemes (namely, *orthogonal groups*), and the study of noncommutative variants of quadratic-form structures (namely, hermitian forms over division algebras with involution) leads to a complete classification of all such group schemes over any given field. This perspective has brought the subject into close contact with many parts of algebra, including the Galois cohomology of fields, algebraic K-theory, as well as developing branches of modern algebraic geometry related to the theory of algebraic cycles and motives. In particular, the study of algebraic cycles on the projective homogeneous varieties naturally associated to non-degenerate quadratic forms (namely, quadrics and higher orthogonal Grassmannians) has, over the past three decades, led to major progress on some of the central problems in the theory of quadratic forms that seemed inaccessible by the elegant, but less sophisticated algebraic techniques that were established by the 1960s and 70s. This places the subject at a vibrant intersection of algebra, algebraic geometry and homotopy theory.

1.2 History of the research topic

The study of quadratic forms origins from finding solutions to certain quadratic equations. With the development of abstract algebra in the 20th century, people are trying to classify quadratic forms over an arbitrary field F. The problem of classifying quadratic forms over fields may be seen as an algebraic analogue of the topological problem of classifying principal bundles for the action of a topological group over a given base space.

A classical approach to the latter problem is the construction of characteristic classes, i.e., invariants of principal bundles taking values in certain cohomology groups of the base space. Guided by this idea, A.Delzant introduced in the 1960s the Stiefel-Whitney invariants of a non-degenerate quadratic form over a field F, a family of invariants taking values in the Galois cohomology ring of F with $\mathbb{Z}/2\mathbb{Z}$-coefficients.

Remarkably, these invariants were already known on some level to H.Minkowski, who in the late 19th century showed that they are sufficient to classify non-denegerate quadratic forms over the field of rational numbers (a fact later extended to all global fields by H.Hasse in the early 20th century). While the Stiefel-Whitney invariants classify nonsingular quadratic forms over global fields, it was noted in the 1960s that they are far from being sufficient over general fields. This insufficiency was investigated by A.Pfister [15], who integrated and expanded the theory of Witt rings of quadratic forms initiated by Witt [22] in the 1930s. From Pfister's work, J.Milnor [14] ultimately stated a fundamental conjecture relating the Witt ring of a field to the Galois cohomology ring of that field with $\mathbb{Z}/2\mathbb{Z}$-coefficients in 1969. From that point on, Milnor's conjecture became the central problem in the area, and attempts to understand it gradually brought the subject into closer and closer contact with algebraic K-theory and other developing branches of algebraic geometry.

Independently of Milnor's work, a nascent algebraic-geometric approach to the theory of quadratic forms involving the use of function fields of quadrics and other homogeneous varieties was already initiated by M.Knebusch in the early 1970s. As part of this work, Knebusch [12] introduced new discrete invariants of non-degenerate quadratic forms which, though not unrelated, are on some level more coarse and accessible than the Stiefel-Whitney invariants and other Galois-cohomological constructions tied up in the statement of the Milnor Conjecture. Most notable here is the *splitting pattern invariant*, which has played a central role in much of the theory developed since Knebusch's work opened up a new chapter in the theory of quadratic forms, and set the stage for an influx of algebraic-geometric ideas in the subsequent decades.

Using the algebraic-geometric methods, more progress had been made in the 1970s. For example, J.K.Arason, A.Pfister, T.Y.Lam and R.Elman derived some results about Milnor's Conjecture from using function fields of Pfister forms. Independently, D.Quillen computed the regular K-theory of conics in the early 1970s while Swan computed the higher K-theory of an arbitrary smooth quadric. These results were used in the 1980s by Merkurjev-Suslin, M.Rost and Rost-Jacob to prove the Milnor Conjecture in low degrees. However, knowing of K-theory of quadrics was insufficient to treat the general case, which required the understanding of invariants of Pfister quadrics, more directly related to algebraic cycles. In the late 1980s, Rost made the first steps in this direction by computing the Chow motives and Chow groups of arbitrary Pfister and even excellent quadric. Later, V.Voevodsky determined the mod-2 motivic cohomology of Pfister quadric, leading to a complete proof of the Milnor Conjecture (for which he earned a Fields medal in 2002).

Following Voevodsky's proof of the Milnor Conjecture, it became clear that developing methods from the theory of algebraic cycles and motives could be used to attack other problems from the algebraic theory of quadratic forms that had remained wide open since the 1960s and 70s. The first steps were made here by A.Vishik [21], who studied the Chow motives of arbitrary smooth quadrics, and introduced important new discrete algebraic-geometric invariants of non-degenerate quadratic forms, including the so-called motivic decomposition type. In the early 2000s, Vishik's ideas were developed further by Vishik himself, N.Karpenko and A. Merkurjev, leading to substantial progress on many of the central open problems in the subject (an overview of this progress is given in the book by Elman, Karpenko and Merkurjev [2]). A high point of this work is Vishik's *Excellent Connections theorem* [20] which loosely states that excellent quadrics are the "most decomposable" among all anisotropic quadrics of given dimension over a field of characteristic different from 2. This leads, in this setting, to a collection of severe restrictions on Vishik's motivic decomposition type invariant coming from Knebusch's splitting pattern invariant, and has many important applications.

A key driver behind all of this work was the construction [1] in the late 1990s of Steenrod squaring operations on the mod-2 Chow groups of smooth algebraic varieties over fields of characteristic different from 2 by P.Brosnan. It should be noted here that Steenrod operations were independently constructed in the wider context of motivic cohomology by Vovoedsky, and played an important role in the proof of the Milnor Conjecture.

Historically, people mainly considered characteristic not 2 fields. In 2018, Primozic [16] constructed Steenrod Operations on mod 2 Chow groups for smooth varieties over characteristic 2 fields. This development allows people to extend many results from characteristic not 2 fields to characteristic 2 fields for the study of non-degenerate forms.

1.3 Description of the thesis

While the advent of the algebraic-geometric machinery discussed above led to major progress in the theory of quadratic forms over the past few decades, this progress was hitherto largely limited to the case where the field of coefficients has characteristic different from 2. The two central reasons for this restriction were the following:

(1) Over fields of characteristic 2, an anisotropic quadratic form need not be non-degenerate. As such, a complete algebraic-geometric approach to the theory of quadratic forms in characteristic 2 must involve the study of singular quadrics and their higher orthogonal Grassmannians, and the theory of algebraic cycles on singular varieties is much less well developed than for smooth varieties. For example, we have no suitable "motivic category" in which to study singular varieties.

(2) Even if we restrict our considerations to non-degenerate quadratic forms, the theory established by Vishik and others still relies on non-trivial algebraic-geometric constructions, including the construction of certain cohomological operations and algebraic cobordism. Traditionally, these constructions have come with characteristic restrictions. In particular, Brosnan's construction of Steenrod squares on mod-2 Chow groups was initially restricted to fields of characteristic not 2.

Though both problems persist to some extent, recent developments have gone some way to remove the barriers to progress. More specifically, E.Primozic [16] has recently succeeded in extending the construction of Steenrod squares on the mod-2 Chow groups of smooth varieties to fields of characteristic 2, allowing for the extension of certain results on discrete invariants of non-degenerate quadratic forms over fields of characteristic not 2 to be extended to characteristic 2. At the same time, work of N.Karpenko [11] also suggests that it should be possible to extend many of these results to singular quadratic forms by an effective reduction to the non-degenerate case. In particular, Karpenko has used this idea (together with Primozic's work) to prove the singular case of an old conjecture of D.Hoffmann on the splitting patterns of quadratic forms (the non-degenerate case had previously been proved in characteristic not 2 by Karpenko in 2003, with only limited progress being made on the singular case in the interim).

The aim of this thesis is to give a systematic study of the quadratic forms over fields of any characteristic using the algebraic-geometric ideas alluded to above. We briefly introduce the content of each Chapter here.

In Chapter 2, we review some of the theory of Chow groups, including functorial properties of Chow groups. As examples, we will compute certain Chow groups which will be used later. We also review the Steenrod operations on the mod-2 Chow groups of smooth varieties constructed by Brosnan for the characteristic not 2 fields [1] and recently extended to characteristic 2 fields by Primozic [16].

In Chapter 3, we review some basic knowledge of quadratic forms and their associated quadrics. Not only non-degenerate quadratic forms are studied, we also study the singular forms. We introduce several splitting pattern invariants for quadratic forms and explain certain relationships between them. We then discuss Pfister neighbours and introduce the notion of *strongly excellent forms*. We establish an explicit characterization of these forms, which extends a well-known characterization of so-called excellent forms in the non-degenerate theory.

In Chapter 4, we begin our study of algebraic cycles (mod-2) on products of generically smooth quadrics extending recent work of Karpenko [11]. Let F be a field and $X_1, \cdots, X_n$ be generically smooth quadrics over F which are given by quadratic forms with anisotropic totally singular parts. Let $\overline{Ch}(X_1 \times \cdots \times \cdots X_n)$ be the image of $Ch(X_1 \times \cdots \times X_n)$ under the change of field homomorphism $Ch(X_1 \times \cdots \times X_n) \to Ch((X_1 \times \cdots \times X_n)_{\overline{F}})$ where $\overline{F}$ is an algebraic closure of F. We show that $\overline{Ch}(X_1 \times \cdots \times X_n)$ inherits a ring structure from the mod-2 Chow ring of the smooth locus of $X_1 \times \cdots \times X_n$. We then use this to define a composition law for rational mod-2 algebraic cycles between products of generically smooth quadrics. We establish basic properties of the composition law analogue to known properties of the composition of correspondences for smooth varieties. In the case where $X_1, \cdots, X_n$ have maximal Witt index, we describe $\overline{Ch}(X_1 \times \cdots \times X_n)$ and its ring structure explicitly. We also describe the composition law for rational cycles explicitly. This allows us to interpret the main problems of intersection as explicit rationality problems for cycles over a separable closure of the ground field. Again, following Karpenko, we show also that $\overline{Ch}(X_1 \times \cdots \times X_n)$ admits an action of mod-2 Steenrod operations, and we compute the actions in the case $\overline{Ch}((X_1 \times \cdots \times X_n)_{\overline{F}})$ explicitly. In the final part of this Chapter, we derive the relationship between $\overline{Ch}(X_1 \times \cdots \times X_n)$ and $\overline{Ch}(Y_1 \times \cdots \times Y_n)$, where each Y_i is given by the ansiotropic part of X_i. This allows us to study $\overline{Ch}(X_1 \times \cdots \times X_n)$ using scalar-extension arguments.

In Chapter 5, we use the results in Chapter 4 to introduce a new discrete invariant for forms which are not totally singular but have anisotropic totally singular part. We call the invariant *rational correspondence type*. For non-degenerate quadratic forms, this coincides with the *motivic decomposition type* studied by A.Vishik [21]. We show that many of the restrictions on Vishik's motivic decomposition type imposed by the Knebusch splitting pattern are also valid for the rational correspondence type of singular forms. In the last part of this chapter, we prove certain results relating the rational correspondence type of different quadrics. These extend results of Vishik on the motivic decomposition type of non-degenerate forms to singular forms. Using these results, we compute the rational correspondence types of any strongly excellent form or even quasi-strongly excellent form, which extends a result of M.Rost on non-degenerate excellent forms. We also describe the rational correspondence type of a Pfister neighbour in terms of the rational correspondence type of its complementary form. In conclusion, we show that generic forms of even dimension are indecomposable, meaning that the rational correspondence type consists of a single component.

Finally, we consider the concept of virtual Pfister neighbours i.e., forms becoming Pfister neighbours over field extensions. As part of this, we extend Izhboldin's classification of virtual Pfister neighbours of dimension $2^n + 2$ to singular forms.

In Chapter 6, we review a fundamental result of Vishik asserting the existence of so-called excellent connections in the motivic decomposition type of an arbitrary anisotropic smooth quadric, as well as some of its corollaries. We raise the question of whether it is possible to extend Vishik's results to singular but generically smooth quadrics when the characteristic of the base field is 2 and show that the answer is yes for quadrics of dimension ≤ 7. In this chapter, we also apply the methods developed in Chapter 4 and Chapter 5 to obtain partial results on a conjecture of Hoffmann and Laghribi, concerning the classification of Pfister neighbours over the base field of characteristic 2. Finally, we also apply the method to the study of a conjecture of S.Scully on the isotropy behavior of quadratic forms over function fields.

Chapter 2

Chow Groups

2.1 Chow groups

Fix a field F.

Definition 2.1.1. Let X be a scheme over F. A *k-cycle* is a formal sum of k-dimensional subvarieties of X, i.e., $\sum n_\alpha [V_\alpha]$ where $n_\alpha \in \mathbb{Z}$ and V_α are k-dimensional subvarieties. The group of k-cycles on X is the free abelian group on k-dimensional subvarieties of X and is denoted by $Z_k X$. The class of a k-dimensional subvariety V in $Z_k X$ is denoted $[V]$.

Under the same setting as above, we let W be a $(k+1)$-dimensional subvariety of X and $F(W)$ be the function field of W. Now we take any k-dimensional subvariety V of W with the generic point η. Then there is a unique group homomorphism

$$Ord_V : F(W)^\times \to \mathbb{Z}$$

that sends each $r \in \mathcal{O}_{W,\eta} \setminus \{0\}$ to the length of the Artinian ring $\mathcal{O}_{W,\eta}/(r)$.

Let V_α be any codimension 1 subvariety of W with the generic point η_α. We write $Ord_\alpha := Ord_{V_\alpha}$ for short. For any $f \in F(W)^\times$, we define $[div(f)] := \sum_\alpha Ord_\alpha(f)[V_\alpha]$ to be the *divisor* of f, where the sum is taken over all codimension 1 subvarieties V_α of W. This is also well-defined since $Ord_\alpha(f)$ are zero for all but finitely many V_α.

When W is regular at each η_α, we know that each $\mathcal{O}_{W,\eta_\alpha}$ is a DVR, i.e., a discrete valuation ring. Let $Frac(\mathcal{O}_{W,\eta_\alpha})$ be the fraction field of $\mathcal{O}_{W,\eta_\alpha}$. Then each DVR $\mathcal{O}_{W,\eta_\alpha}$ determines a unique valuation $Val_\alpha : Frac(\mathcal{O}_{W,\eta_\alpha})^\times \to \mathbb{Z}$. In this case, we have $Val_\alpha = Ord_\alpha$ and for any $f \in F(W)^\times$, the divisor of f is just the divisor of poles and zeros given by $[div(f)] := \sum_\alpha Val_\alpha(f)[V_\alpha]$.

The definition of divisors gives rise to the concept of rational equivalence.

Definition 2.1.2. Let X be a scheme over F and Z_kX be the group of k-cycles on X. We say a k-cycle α is *rationally equivalent to zero* if there is a finite number of $(k+1)$-dimensional subvarieties W_i of X and $f_i \in F(W_i)^\times$ such that $\alpha = \sum[div(f_i)]$. Then we denote it by $\alpha \sim 0$.

Since Ord_V is a group homomorphism, we know that $[div(f)] = -[div(1/f)]$. Thus the cycles rationally equivalent to zero form a subgroup of Z_kX and we write it as Rat_kX. Finally, we come to the definition of Chow groups.

Definition 2.1.3. Let X be a scheme over F and $Rat_k(X)$ be defined as above. The *Chow group* on X of k-dimensional cycles is the quotient group of Z_kX by Rat_kX, i.e.,

$$CH_k(X) := Z_kX/Rat_kX.$$

Also, we define $CH_*(X)$ to be the direct sum of $CH_i(X)$ for $i \in [0, dim(X)]$. Sometimes, we will just write $CH(X)$ instead of $CH_*(X)$ for convenience.

Remark 2.1.4. Let X be any scheme over F and X_{red} be the associated reduced scheme. From the definition, we know that $CH(X) = CH(X_{red})$.

$CH(X)$ has two natural grading structures. For the lower grading $CH_i(X)$, it is just what we have defined above. For the upper grading, we define the group $CH^i(X)$ to be the subgroup of $CH(X)$ generated by cycles corresponding to subvarities of codimension i. $CH^*(X)$ is the direct sum of $CH^i(X)$. Suppose $X_1, \cdots, X_r$ are irreducible components of X. Then the relation between the two grading structures is given by

$$CH_i(X) = \oplus_{j=1}^r CH^{dim(X_j)-i}(X_j)$$

for $i \in [0, dim(X)]$ from the definition.

Example 2.1.5. We give a trivial example here. Let $X = Spec(F)$. Since X consists of only one point $\{x\}$, X doesn't have any one dimension subvariety, hence $Rat_k(X) = 0$. Now $Z_kX = \mathbb{Z} \cdot [\{x\}] \cong \mathbb{Z}$, and we have that $CH_0(X) \cong \mathbb{Z}$ while $CH_i(X) = 0$ for $i > 0$.

Let X be a scheme over F and W be a subscheme of X. We suppose $W_1, \cdots, W_r$ are irreducible components of X. Also, we let the generic point of each W_i be η_i. Then $\mathcal{O}_{W,\eta_i}$ is a zero-dimensional Noetherian local ring, hence an Aritinian ring. Now we let $m_i = l(\mathcal{O}_{W,\eta_i})$ and define the *fundamental cycle* $[W]$ by

$$[W] := \sum_i m_i [W_i].$$

If W is a reduced scheme, we have $m_i = 1$ for each $i \in [1, r]$.

When X is a smooth scheme, the group $CH^*(X)$ can be equipped with a grading-preserving multiplication, thereby giving it the structure of a commutative graded ring. The identity element is $[X]$. Suppose Y, Z are two subvarieties of X satisfying $[Y] \in CH^p(X)$ and $[Z] \in CH^q(X)$. Also, we suppose $W := Y \cap V$ (scheme-theoretical intersection) is a reduced scheme with irreducible components $W_1, \cdots, W_m$ of the same codimension $p+q$. Then the product of $[Y]$ and $[Z]$ is $[W] = \sum_{i=1}^m [W_i]$ in Chow groups.

More generally, we have the following proposition.

Proposition 2.1.6. [2, Proposition 57.21] *Let Y and Z be two closed subvarieties of a smooth scheme X. Also, we suppose that Y, Z are of codimension p,q respectively. Let $W := Y \cap Z$ be the scheme-theoretical intersection of Y and Z. Let $V_1, V_2, \cdots, V_s$ be all irreducible components of W and $[W] = \sum m_i [V_i]$ for some $m_i > 0$. If every irreducible component of W has codimension $(p+q)$, then*

$$[Y] \cdot [Z] = \sum_{i=1}^{s} n_i [V_i]$$

for some integers n_i with $1 \leq n_i \leq m_i$.

Remark 2.1.7. The integers n_i above are called *intersection multiplicities*. Explicit formulas are given in [3]. With these formulas, this proposition completely determines the multiplication on $CH^*(X)$.

Example 2.1.8. Let V be a $n+1$-dimensional vector space over F. Then $\mathbb{P}(V)$ is a n-dimensional projective space. For any $i+1$-dimensional subspace V_i where $i \in [0, n]$, we have an i-dimensional projective subspace $\mathbb{P}(V_i)$. In the Chow group of $\mathbb{P}(V) = \mathbb{P}^n$, it can be shown that the class of $\mathbb{P}(V_i)$ doesn't depend on the vector space V_i for $i \in [0, n]$. As a result, we denote the class by l_i. Moreover, it's known that l_i is a basis of $CH_i(\mathbb{P}^n)$, i.e., $CH_i(\mathbb{P}^n) = \mathbb{Z} \cdot l_i \cong \mathbb{Z}$ for any $i \in [0, n]$.

Let $H := \{x_0 = 0\}$ be a hyperplane in $\mathbb{P}^n$. Notice that all hyperplanes have the same class in $CH(\mathbb{P}^n)$: actually, we suppose $H^{'}$ is any hyperplane given by a degree 1 homogeneous polynomial f. Then $[div(f/x_0)] = [H^{'}] - [H]$ in $Z_k\mathbb{P}^n$. Hence $[H^{'}] = [H]$ in $CH^1(\mathbb{P}^n)$ and we denote the class as h. We claim that $h \cdot l_i = l_{i-1}$ for $i \in [1, n]$. Since different projective subspaces have the same class in $CH(\mathbb{P}^n)$, we can choose one $\mathbb{P}^i$ such that $\mathbb{P}^i \cap H = \mathbb{P}^{i-1}$. Then the result follows from the discussion before Proposition 2.1.6.

Since $[\mathbb{P}^n] = l_n$ is the identity, we know that $l_{n-1} = h$. By induction, we have $h^i = l_{n-i}$ for $i \in [1, n]$. By setting $h^0 = l_n$, we conclude that $CH^*(\mathbb{P}^n)$ is a ring which is isomorphic to $\mathbb{Z}[h]/(h^{n+1})$.

2.2 Functorial properties

Fix a field F.

Let X and Y be two schemes over F and $f : X \to Y$ be a proper morphism (e.g., a closed embedding) between them. Then there exists a unique homomorphism $f_* : CH_k(X) \to CH_k(Y)$ such that

$$f_*([V]) = \begin{cases} [F(V) : F(f(V))] \cdot [f(V)] & \text{if } dim(f(V)) = dim(V) \\ 0 & \text{if } dim(f(V)) \neq dim(V) \end{cases}$$

for any subvariety $V \subset X$. It's called the *push-forward* induced by f. Push-forwards are functorial: suppose Z is a scheme over F and $g : Y \to Z$ is a proper morphism. Then we have $(g \circ f)_* = g_* \circ f_*$.

Recall that by a flat morphism, we mean a flat morphism of constant relative dimension. Let X and Y be two schemes over F and $g : Y \to X$ be a flat morphism of relative dimension n. Then there exists a unique homomorphism $g^* : CH_k(X) \to CH_{k+n}(Y)$ such that

$$g^*([Z]) = [g^{-1}(Z)]$$

for any k-dimensional subvariety Z of X and $[g^{-1}(Z)]$ is the fundamental cycle of $g^{-1}(Z)$ in $CH_{k+n}(Y)$. It's called the *pull-back* homomorphism induced by g. Pull-backs are functorial: suppose that W is a scheme over F and $f : W \to Y$ is a flat morphism. Then we have $(g \circ f)^*([V]) = f^*(g^*[V])$ for any subvariety $V \subset X$.

Let X be a scheme over F. Suppose U is an open subscheme of X. Then the inclusion of U into X is a flat morphism of dimension 0. In this case, the pull-back homomorphism on Chow groups is surjective:

Proposition 2.2.1. [3, Proposition 1.8] *Let Y be a closed subscheme of X, and let $U = X - Y$. Let $i : Y \to X$ be the closed embedding and $j : U \to X$ be the open embedding. Then the sequence*

$$CH_k(Y) \xrightarrow{i_*} CH_k(X) \xrightarrow{j^*} CH_k(U) \to 0$$

is exact for any k. The exact sequence is called the localization sequence.

Now we let X be a variety of dimension n and Y be a scheme over F. Suppose the morphism $f : Y \to X$ is a dominant morphism and $x \in X$ is the generic point. We define the *generic fiber* of f to be the scheme-theoretical fiber of f at the generic point x, i.e., $Y_x := f^{-1}(x)$. For any nonempty open subset $U \subset X$, we have a natural flat morphism: $g_U : Y_x \to f^{-1}(U)$ of relative dimension $(-n)$. So this induces a pull-back homomorphism: $g_U^* : CH_*(f^{-1}(U)) \to CH_{*-n}(Y_x)$.

By [2, Corollary 52.14], the colimit of the system $\{CH_*(f^{-1}(U_\alpha))\}$ is equal to $CH_{*-n}(Y_x)$. Thus for any cycle α in $CH_{*-n}(Y_x)$, α can be lifted to a cycle $\alpha' \in CH_*(f^{-1}(U_\alpha))$ for some nonempty open subscheme U_α. Also, we know that $CH_*(Y) \to CH_*(f^{-1}(U_\alpha))$ is a surjective group homomorphism. This gives the following proposition.

Proposition 2.2.2. [2, Proposition 57.10] *Let X be a variety of dimension n and $f : Y \to X$ a dominant morphism. Let x denote the generic point of X and Y_x the generic fiber of f. Then the pull-back homomorphism $CH_k(Y) \to CH_{k-n}(Y_x)$ is surjective for all k.*

Let X be a variety of dimension n over F and Y be a scheme over F. The Corollary 2.2.3 is obtained by applying Proposition 2.2.2 to the projection morphism $X \times_F Y \to X$.

Corollary 2.2.3. [2, Corollary 57.11] *For every variety X of dimension n and scheme Y over F, the pull-back homomorphism $CH_k(X \times Y) \to CH_{k-n}(Y_{F(X)})$ is surjective for all k.*

In the discussion above, we know that pull-back homomorphisms between Chow groups are induced by flat morphisms. However, we can also define the pull-back homomorphism induced by any morphism if the target scheme of a morphism is smooth.

Let Z and W be two schemes and $g : Z \to W$ be a regular closed embedding of codimension r. Then there is a homomorphism $g^* : CH_k(W) \to CH_{k-r}(Z)$. This homomorphism is induced by the regular closed embedding g and called *Gysin homomorphism*. Gysin homomorphism has a strong relation with intersections. Actually, by [2, Corollary 57.20], if V is a k-dimensional variety in W satisfying that $g^{-1}(V)$ is reduced and the irreducible components of $g^{-1}(V)$ have the same dimension $k-r$, then we have

$$g^*([V]) = [g^{-1}(V)].$$

Let X and Y be two schemes over F. There is a well-defined external product map

$$f : CH_p(X) \otimes CH_q(Y) \to CH_{p+q}(X \times Y); [V] \otimes [W] \mapsto [V \times W]$$

for any p-dimensional subvariety V of X and q-dimensional subvariety W of Y. Let $\alpha \in CH(X)$ and $\beta \in CH(Y)$ be two cycles. We define $\alpha \times \beta := f(\alpha \otimes \beta)$. Also, we should remind that push-forwards and pull-backs are compatible with external products. Let X', Y' be two schemes over F and $g_1 : X \to X'$ and $g_2 : Y \to Y'$ are two morphisms. If g_1, g_2 are proper, then we have $(g_1 \times g_2)_*(\alpha \times \beta) = (g_1)_*(\alpha) \times (g_2)_*(\beta)$; if g_1, g_2 are flat, we have $(g_1 \times g_2)^*(\alpha \times \beta) = (g_1)^*(\alpha) \times (g_2)^*(\beta)$ [3, Proposition 1.10].

Now we suppose that X is a smooth scheme over F and $i : X \to X \times X$ is the diagonal morphism. Notice that i is a regular closed embedding. For cycles α and β in $CH(X)$, we know that $\alpha \times \beta$ is a cycle in $CH(X \times X)$. Then we have

$$\alpha \cdot \beta = i^*(\alpha \times \beta).$$

This describes the relation between the Gysin homomorphism and the product of cycles.

Now for any morphism $f : Y \to X$ with X smooth, the morphism $i = (id, f) : Y \to Y \times X$ is a regular closed embedding of codimension $dim(X)$ and the projection $p : Y \times X \to X$ is a flat morphism of relative dimension $dim(Y)$. By setting $d = dim(X) - dim(Y)$, we define the *pull-back homomorphism* to be

$$f^* := (i^* \circ p^*) : CH_k(X) \to CH_{k-d}(Y).$$

When f is in addition a flat morphism of equidimensional schemes, then the pull-back f^* defined above coincides with the pull-back homomorphism induced by a flat morphism. Also, if f is a regular closed embedding, then the pull-back defined above is just the Gysin homomorphism.

The following proposition says that the pull-back homomorphism is compatible with the push-forward homomorphism in a fibre product diagram.

Proposition 2.2.4. [3, Proposition 1.7.] *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a fibre product diagram of schemes. If f', f are proper morphisms and g, g' are flat morphisms, then for any cycle $\alpha \in CH(X)$, we have $(f'_* \circ g'^*)(\alpha) = (g^* \circ f_*)(\alpha)$.

Proposition 2.2.5 (Projection Formula)**.** [2, Proposition 56.9] *Suppose X and Y are two smooth schemes over F. Let $f : Y \to X$ be a proper morphism of smooth schemes. Then*

$$f_*(\alpha \cdot f^*(\beta)) = f_*(\alpha) \cdot \beta$$

for every $\alpha \in CH^*(Y)$ and $\beta \in CH^*(X)$

Obviously, we can view $CH(Y)$ as a $CH(X)$-module via f^*. Then the projection formula just means that the push-forward f_* is a $CH(X)$-module homomorphism.

The projection formula requires schemes X, Y to be smooth schemes. Sometimes we need the following weaker version of projection formula, which only requires X to be smooth.

Proposition 2.2.6. [2, Proposition 56.11] *Let $f : Y \to X$ be a morphism of equidimensional schemes with X smooth. Then*

$$f_*(f^*(\beta)) = f_*([Y]) \cdot \beta$$

for every $\beta \in CH^*(X)$.

The discussion above describes some important results of integral Chow groups, i.e., Chow groups with integer coefficients. Let Λ be a ring and X be a scheme over F. We define the *Chow group on X with coefficients in* Λ by $CH(X;\Lambda) := CH(X) \otimes_{\mathbb{Z}} \Lambda$.

One should notice that the theory we described above can be developed for $CH(X;\Lambda)$ with the same results.

In this paper, we have particular interests in $CH(X;\mathbb{Z}/2\mathbb{Z})$, i.e., the Chow group on X with coefficients in $\mathbb{Z}/2\mathbb{Z}$. For convenience, we denote it by $Ch(X)$.

2.3 Chow correspondences

Let Λ be a commutative ring and F be a field. In this section, we will consider Chow groups with coefficients in Λ.

Definition 2.3.1. Let X and Y be two schemes over F and $X_1, \cdots, X_r$ be the irreducible components of X with dimension $n_1, \cdots, n_r$ respectively. For every $i \in \mathbb{Z}$, we set

$$Corr_i(X, Y; \Lambda) := \coprod_{k=1}^{r} CH_{i+n_k}(X_k \times Y; \Lambda).$$

We define a *correspondence between* X *and* Y *of degree* i *with coefficients in* Λ to be an element $\alpha \in Corr_i(X \times Y; \Lambda)$.

Let X and Y be two schemes over F. For $X \times Y$, we have the *exchange isomorphism* $X \times Y \cong Y \times X$, which sends (x, y) to (y, x) for points $x \in X, y \in Y$. We denote the image of $\alpha \in CH(X \times Y)$ under the exchange isomorphism by α^t. The cycle α^t is a cycle in $CH(Y \times X)$, and we call it the *transpose of* α.

For any scheme X over F, we have two canonical morphisms, i.e., the structure morphism $p_X : X \to Spec(F)$ and the diagonal morphism $d_X : X \to X \times X$. If p_X is a proper morphism, we say X is a *complete scheme*. Also, if X is smooth, then the diagonal morphism is regular closed embedding of codimension $dim(X)$.

We let X, Y and Z be schemes over F with Y smooth and complete. Then we have a proper morphism

$$p_2 := 1_X \times p_Y \times 1_Z : X \times Y \times Z \to X \times Z$$

and a regular closed embedding

$$d_2 := 1_X \times d_Y \times 1_Z : X \times Y \times Z \to X \times Y \times Y \times Z.$$

Definition 2.3.2. Let X, Y and Z be schemes over F with Y smooth and complete.

Then the *composition of correspondences* is a bilinear pairing

$$Corr_i(Y,Z;\Lambda)\times Corr_j(X,Y;\Lambda)\to Corr_{i+j}(X,Z;\Lambda)$$

defined by

$$(\beta,\alpha)\mapsto \beta\circ\alpha := (p_2)_*\circ(d_2)^*(\alpha\times\beta)$$

where $(d_2)^*$ is the Gysin homomorphism.

The pairing above is associative, i.e., for any four schemes X, Y, Z, T with Y and Z smooth and complete and any $\alpha\in Corr_*(X\times Y,\Lambda)$, $\beta\in Corr_*(Y\times Z,\Lambda)$, $\gamma\in Corr_*(Z\times T,\Lambda)$, we have $(\gamma\circ\beta)\circ\alpha=\gamma\circ(\beta\circ\alpha)$.

Under the same setting as Definition 2.3.2, we have flat projection morphisms

$$p_3 := 1_X\times 1_Y\times p_Z : X\times Y\times Z\to X\times Y.$$

and

$$p_1 := p_X\times 1_Y\times 1_Z\to Y\times Z.$$

If X, Y and Z are all smooth and complete schemes, then we have an alternative formula for the composition of correspondences which only involves with projection morphisms.

Proposition 2.3.3. [2, Proposition 63.2] *Let X, Y, Z be complete smooth schemes over F. Suppose $\alpha\in CH(X\times Y)$ and $\beta\in CH(Y\times Z)$ are two correspondences. Then we have $\beta\circ\alpha=(p_2)_*((p_3)^*(\alpha)\cdot(p_1)^*(\beta))$.*

Let $f: X\to Y$ be a morphism of schemes. We denote the graph of f by Γ_f. Since Γ_f is a closed subscheme of $X\times Y$, Then $[\Gamma_f]$ is a cycle in $CH(X\times Y)$.

Proposition 2.3.4. [2, Proposition 62.4] *Let X, Y, Z be schemes over F with Y smooth and complete.*

Then for any morphism $g: Y\to Z$, we have $[\Gamma_g]\circ\alpha=(1_X\times g)_*(\alpha)$ where α is any cycle in $CH(X\times Y)$.

For any morphism $f: X\to Y$, we have $\beta\circ[\Gamma_f]=(f\times 1_Z)^*(\beta)$ where β is any cycle in $CH(Y\times Z)$.

From Definition 2.3.2, we can define group homomorphisms induced by Chow correspondences.

Definitions 2.3.5. Let X, Y and Z be schemes over F.

When X is smooth and complete, then for any cycle $\alpha \in CH_*(X \times Y)$, we define the *push-forward homomorphism* induced by α to be

$$\alpha_* : CH_*(Z \times X) \to CH_*(Z \times Y); \beta \mapsto \alpha \circ \beta$$

for any $\beta \in CH(Z \times X)$. If $\alpha = [\Gamma_f]$ for some morphism $f : X \to Y$, by Proposition 2.3.3, we know that $\alpha_* = (1_Z \times f)_*$

When Y is smooth and complete, then for any cycle $\alpha \in CH_*(X \times Y)$, we define the *pull-back homomorphism* induced by α to be

$$\alpha^* : CH_*(Y \times Z) \to CH_*(X \times Z); \beta \mapsto \beta \circ \alpha.$$

for any $\beta \in CH(Y \times Z)$. If $\alpha = [\Gamma_f]$ for some morphism $f : X \to Y$, by Proposition 2.3.3, then $\alpha^* = (f \times 1_Z)^*$.

From the discussion above, one can realize that Chow correspondences give us a way to constitute a new category: the objects in the new category are smooth complete schemes over F and the morphisms are given by Chow correspondences. Also, there is a natural functor from the category of smooth complete schemes to the new category.

2.4 Relative cellular decomposition

Fix a field F.

Definition 2.4.1. Let X and Y be two schemes over F. A morphism $g : Y \to X$ is called an *affine bundle of rank d* if g is flat and the fiber of g over any point $x \in X$ is isomorphic to the affine space $\mathbb{A}^d_{F(x)}$.

Remark 2.4.2. An affine bundle is stable under base-change. Let X and Y be two schemes over F and $g : Y \to X$ be an affine bundle of rank d. Let Z be any scheme over F. Then we have the base-change morphism $g' := g \times id : Y \times Z \to X \times Z$. Since flat morphisms are stable under base-change, g' is also flat of relative dimension d. Let $s = (x, z)$ be a point in $X \times Z$. Then $g'^{-1}(x) = g^{-1}(x) \times Spec(F(z)) \cong \mathbb{A}^d_{F(x)} \times_F Spec(F(z)) \cong \mathbb{A}^d_{F(s)}$.

Example 2.4.3. Let $\pi : V \to M$ be a vector bundle of rank d. Let $y \in V$ be any point and $\pi(y) = x \in M$. Then we have $\pi^{-1}(x) \cong Spec(\kappa(x)) \times \mathbb{A}^d$. Also, by taking an open subscheme U containing x such that $\pi^{-1}(U) \cong U \times \mathbb{A}^d$, we know that $\mathcal{O}_V(U)$ is a free $\mathcal{O}_M(U)$-module, hence $\mathcal{O}_{V,y}$ is a free $\mathcal{O}_{M,x}$-module by localization. Thus we conclude that a vector bundle of rank d is an affine bundle of rank d.

An affine bundle induces an isomorphism between Chow groups.

Theorem 2.4.4 (Homotopy Invariance)**.** [2, Theorem 52.13] *Let Y and X be two schemes over F and $g : Y \to X$ be an affine bundle of rank d. Then the pull-back homomorphism*

$$g^* : CH_k(X) \to CH_{k+d}(Y)$$

is an isomorphism for any k.

Next result generalizes the homotopy invariance. We give a definition first. Let X be a scheme over F. By a *relative cellular structure* on X, we mean a filtration of closed subschemes

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$$

together with affine bundles $p_i : U_i := X_i/X_{i-1} \to Y_i$ of rank d_i for all $i \in [0, n]$, in which $Y_i, i \in [1, n]$ is a smooth complete scheme.

Let $\alpha_i \in CH(X \times Y_i)$ be the class of the closure of the graph of p_i in $X \times Y_i$. By definition, α_i is a correspondence between X and Y_i of degree $dim(X) - dim(X_i)$. Taking the transpose of α_i, we know that α_i^t is a correspondence between Y_i and X of degree d_i.

Let Z be any scheme over F. Since Y_i is smooth and complete for $i \in [1, n]$, we have group homomorphisms $a_i := (\alpha_i^t)_*$. When $i = 0$, we let $f_0 : X_0 \to X$ be the closed embedding and define $a_0 := (1_Z \times f_0)_* \circ (p_0^*)^{-1}$. Now we have defined homomorphisms

$$a_i : CH_{*-d_i}(Z \times Y_i) \to CH_*(Z \times X).$$

for $i = 0, \cdots, n$.

Theorem 2.4.5. [2, Theorem 66.2] *Let X be a scheme over F with relative cellular structure as above. For any scheme Z over F, we have an isomorphism*

$$\sum_{i=0}^{n} a_i : \bigoplus_{i=0}^{n} CH_{*-d_i}(Z \times Y_i) \to CH_*(Z \times X)$$

Example 2.4.6. Let Z be any scheme over F and $X = \mathbb{P}^n$. By considering $X_i = \mathbb{P}^i$ for $i \in [0, n]$, we have $U_i = X_i \setminus X_{i-1} \cong \mathbb{A}^i$ and $Y_i = Spec(F)$ for $i \in [0, n]$. Since $U_i \to Y_i$ is an affine bundle of rank i, by Theorem 2.4.5 we have

$$CH_k(Z) \oplus CH_{k-1}(Z) \oplus \cdots \oplus CH_{k-n}(Z) \cong CH_k(Z \times \mathbb{P}^n).$$

Explicitly, the isomorphism above is given by

$$\sum_{i=0}^{n} \alpha_i \to \sum_{i=0}^{n} \alpha_i \times l_i \in CH_k(Z \times \mathbb{P}^n)$$

for $\alpha_i \in CH_{k-i}(Z)$ and $l_i \in CH_i(\mathbb{P}^n)$.

In particular, when $Z = Spec(F)$, we know that $CH_k(\mathbb{P}^n) \cong \mathbb{Z}$ for any $k \in [0, n]$.

2.5 Projective bundle theorem

Fix a field F. Let E be a vector bundle of rank r on an F-scheme X with projection $p : E \to X$. We know that p is a flat morphism of relative dimension r. Suppose $s : X \to E$ is the zero section. Then s is a closed embedding.

Definition 2.5.1. Under the same setting as above, we define the *Euler class* of the vector bundle $p : E \to X$ to be

$$e(E) := (p^*)^{-1} \circ s_* : CH_*(X) \to CH_{*-r}(X).$$

Let E be a vector bundle of rank r on a F-scheme X with projection $p : E \to X$. We consider the associated projective bundle $q : \mathbb{P}(E) \to X$. Note that q is a flat morphism of relative dimension $r - 1$. Let $L \to \mathbb{P}(E)$ be the tautological line bundle over $\mathbb{P}(E)$ and e the Euler class of L. The following theorem can be viewed as a generalization of the isomorphism we described in Example 2.4.6.

Theorem 2.5.2 (Projective bundle theorem)**.** [2, Theorem 53.10] *The homomorphism*

$$\phi(E) := \bigoplus_{i=1}^{r} e^{r-i} \circ q^* : \bigoplus_{i=1}^{r} CH_{*-(i-1)}(X) \to CH_*(\mathbb{P}(E))$$

is an isomorphism. For every $\alpha \in CH_*(\mathbb{P}(E))$, *it can be written in the form*

$$\alpha = \sum_{i=1}^{r} e^{r-i}(q^*(\alpha_i))$$

for uniquely determined elements $\alpha_i \in CH_{*-(i-1)}(X)$.

The Projective Bundle Theorem gives rise to the definition of Chern class. In the same setting as above, we let $\alpha \in CH_k(X)$ be any cycle. Consider the cycle $e^r \circ q^*(\alpha)$, this is a cycle in $CH_{k-1}(\mathbb{P}(E))$. By the Projective Bundle Theorem, there exists unique cycle $\alpha_i \in CH_{k-i}(X)$ for $i \in 1, \cdots, r$ such that $\sum_{i=1}^{r} \alpha_i = e^r \circ q^*(\alpha)$. Moreover, we set $\alpha := \alpha_0$.

Definitions 2.5.3. The *i-th Chern class of E* is defined to be the group homomorphism given by

$$c_i(E) : CH_k(X) \to CH_{k-i}(X), \alpha \mapsto \alpha_i = c_i(E)(\alpha).$$

for every $i = 0, \cdots, r$ and we set $c_i = 0$ if $i > r$ or $i < 0$. The *total Chern class of E* is defined to be $c(E) := c_0(E) + c_1(E) \cdots + c_r(E)$.

The following proposition gives the functorial properties of the total Chern class.

Proposition 2.5.4. [2, Proposition 54.5] *Let* $f : Y \to X$ *be a morphism and* E *a vector bundle over* X. *Set* $E^{'} = f^*(E)$. *Then*

(1) If f *is proper, then* $c(E) \circ f_* = f_* \circ c(E^{'})$

(2) If f *is flat, then* $f^* \circ c(E) = c(E^{'}) \circ f^*$.

For smooth schemes, the action of Chern classes can be described as follows:

Proposition 2.5.5. [2, Proposition 58.15] *Let* X *be a smooth scheme over* F *and* $p : E \to X$ *be a vector bundle. Then for any* $\alpha \in CH(X)$, *we have that* $c(E)(\alpha) = c(E)([X]) \cdot \alpha$.

The following proposition is called the *Whitney sum formula.*

Proposition 2.5.6. [2, Proposition 54.7] *Let* X *be a scheme over* F *and* $0 \to E^{'} \to E \to E^{''} \to 0$ *be an exact sequence of vector bundles over* X. *Then* $c(E) = c(E^{'}) \circ c(E^{''})$*,i.e,*

$$c_n(E) = \sum_{i+j=n} c_i(E^{'}) \circ c_j(E^{''})$$

for every n.

2.6 Steenrod operations

Fix a field F. Through this section, a scheme will mean a quasi-projective scheme. Let X be a scheme over F. Recall that we use $Ch(X)$ to denote the Chow group of X with coefficients in $\mathbb{Z}/2\mathbb{Z}$. In paper [1], Brosnan constructed the Steenrod operations for Chow groups modulo 2 when the characteristic of F is not 2. When the characteristic of F is 2, the Steenrod operations were constructed by Primozic in [16].

Let X be a smooth scheme over F. There is a group homomorphism $Sq_X : Ch(X) \to Ch(X)$ such that for any proper morphism $j : V \to X$ of smooth schemes, we have

$$Sq(j_*([V])) = j_*(c(N_j)[V])$$

where $c(N_j)$ is the Chern class of the virtual normal bundle of the proper morphism.

The group homomorphism is called the *cohomological Steenrod operation* on X. and the kth homogeneous part of the cohomological Steenrod operation on X is a map increasing the codimension by k, i.e.,

$$Sq_X^k : Ch^*(X) \to Ch^{*+k}(X).$$

To simplify, we also denote the cohomological Steenrod operation by Sq^*.

When the characteristic of F is zero, by the resolution of singularities, the property above completely determines the cohomological Steenrod operation and it's still an open question in other characteristics. But in general, we know that the cohomological Steenrod operation commutes with pull-back homomorphisms.

Theorem 2.6.1. [16, Corollary 3.4][2, Theorem 61.9] *Let $f : Y \to X$ be a morphism of smooth schemes. Then the diagram*

$$\begin{array}{ccc} Ch_*(X) & \xrightarrow{Sq_X} & Ch_*(X) \\ {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow \\ Ch_*(Y) & \xrightarrow{Sq_Y} & Ch_*(Y) \end{array}$$

commutes.

For the computation of homogeneous parts of the cohomological Steenrod operation, we have a theorem below.

Theorem 2.6.2. [16, Corollary 7.3, Proposition 7.4][2, Theorem 61.13] *Let X be a smooth scheme. Then for any $\beta \in Ch^k(X)$,*

$$Sq_X^r(\beta) = \begin{cases} \beta & \text{if } r = 0 \\ \beta^2 & \text{if } r = k \\ 0 & \text{if } r < 0 \text{ or } r > 0. \end{cases}$$

Theorem 2.6.3. [16, Proposition 5.1][2, Theorem 61.14]

Let X and Y be two smooth schemes. Then

$$Sq_{X\times Y} = Sq_X \times Sq_Y$$

We also have the *Cartan Formula*, which computes the cohomological Steenrod operation of products of cycles.

Proposition 2.6.4. [16, Proposition 6.1][2, Corollary 61.15] *Let X be a smooth scheme. Then*

$$Sq_X(\alpha \cdot \beta) = Sq_X(\alpha) \cdot Sq_X(\beta)$$

for all $\alpha, \beta \in Ch(X)$. Equivalently,

$$Sq_X^n(\alpha \cdot \beta) = \sum_{k+m=n} Sq_X^k(\alpha) \cdot Sq_X^m(\beta)$$

for all n.

Definition 2.6.5. Let X be a smooth scheme over F. We define the *homological Steenrod operation* by $Sq_X := c(-T_X) \circ Sq^X$, where $c(-T_X)$ is the inverse of the total Chern class of the tangent bundle of X. To simplify, we also denote the homological Steenrod operation by Sq_*.

Under the same setting as Definition 2.6.5, $Sq^X : Ch(X) \to Ch(X)$ is a map such that for any proper morphism $j : Z \to X$ of smooth schemes, we have

$$Sq^X(j_*([Z])) = j_*(c(-T_Z)[Z])$$

where T_Z is the tangent bundle over Z. The kth homogeneous part of the homological

Steenrod opertion on X is a map decreasing the dimension by k

$$Sq_k^X : Ch_*(X) \to Ch_{*-k}(X).$$

The homological Steenrod operation commutes with push-forward homomorphisms.

Proposition 2.6.6. [16, Proposition 8.1][2, Theorem 60.6] *Let $f : X \to Y$ be a proper morphism of smooth projective varieties. Then the following diagram*

$$\begin{array}{ccc} Ch_*(X) & \xrightarrow{Sq^X} & Ch_*(X) \\ {\scriptstyle f_*}\downarrow & & {\scriptstyle f_*}\downarrow \\ Ch_*(Y) & \xrightarrow{Sq^Y} & Ch_*(Y) \end{array}$$

commutes.

The following theorem comes directly from Definition 2.6.5 and Theorem 2.6.2.

Theorem 2.6.7. $Sq_k^X = 0$ *if* $k < 0$ *and* Sq_0^X *is the identity.*

Remark 2.6.8. Let X be a scheme(not necessarily smooth) over F. When the characteristic of F is not 2, we can still construct the homological Steenrod operation on $Ch(X)$ with the same results as above.

Example 2.6.9. Let $X = \mathbb{P}^d$ be a projective space and $h \in Ch^1(X)$ the class of a hyperplane. Recall Example 2.1.8, we know that $Ch(X) \cong \mathbb{F}_2[h]/(h^{d+1})$. Here we compute the Steenrod operations on the generator h.

By Theorem 2.6.2, we have $Sq_X(h) = h + h^2 = h(1+h)$. Also, by Propsoition 2.6.4, we have $Sq_X(h^i) = h^i(1+h)^i$. Using the Cartan formula, we obtain that

$$Sq_X^r(h^i) = \tbinom{i}{r} h^{i+r}.$$

Let L be the canonical line bundle over X. We have an exact sequence of vector bundles: $0 \to \mathcal{O}_X \to L^{d+1} \to T_X \to 0$, in which T_X is the tangent bundle over X.

By Proposition 2.5.6 and [2, Proposition 54.3], we have $c(T_X) = c(L)^{d+1} = (1+h)^{d+1}$. Hence

$$Sq^X(h^i) = c(T_X)^{-1} \circ Sq_X(h^i) = h^i(1+h)^{i-d-1}.$$

Chapter 3

Quadratic Forms

In this chapter, we will review important results on quadratic forms and quadrics. For results we mention without proof, one can find detailed discussion in references [2], [4] and [7].

3.1 Symmetric bilinear forms

Fix a field F.

Definition 3.1.1. A *symmetric bilinear form* on an F-vector space V is a map $b : V \times V \to F$ satisfying

(1) $b(v + v^{'}, w) = b(v, w) + b(v^{'}, w)$

(2) $b(cv, w) = cb(v, w)$

(3) $b(v, w) = b(w, v)$

for all $v, v^{'}, w, w^{'} \in V$, $c \in F$.

By a *symmetric bilinear form over F*, we will mean a symmetric bilinear form b on some F-vector space. The F-vector space is called the *underlying vector space* of b and the dimension of the vector space will be called the *dimension of* b and denoted $dim(b)$.

Notation 3.1.2. We use $D(b) := \{b(v, v) \mid v \in V \text{ with } b(v, v) \neq 0\}$ to denote the set of nonzero values of b.

Let b be a symmetric bilinear form on an F-vector space V. Suppose that K/F is a field extension. We define the symmetric bilinear form b_K on $V_K := K \otimes_F V$ by setting

$$b_K(a \otimes v, c \otimes w) := acb(v, w)$$

for any $a, c \in K$ and $v, w \in V$.

We have two binary operations for symmetric bilinear forms. Let b_1, b_2 be two symmetric bilinear forms over F with underlying vector spaces V_1, V_2 respectively. The *external orthogonal sum* of b_1 and b_2, denoted by $b_1 \perp b_2$, is the symmetric bilinear form on $V_1 \oplus V_2$, defined by setting

$$(b_1 \perp b_2)(v_1 \oplus v_2, w_1 \oplus w_2) := b_1(v_1, w_2) + b_2(v_2, w_2)$$

for all vectors $v_i, w_i \in V_i$ where $i = 1, 2$.

The *tensor product* of b_1 and b_2, denoted by $b_1 \otimes b_2$, is the symmetric bilinear form on $V_1 \otimes_F V_2$, defined by setting

$$(b_1 \otimes b_2)(v_1 \otimes v_2, w_1 \otimes w_2) := b_1(v_1, w_1) \cdot b_2(v_2, w_2)$$

for all vectors $v_i, w_i \in V_i$ where $i = 1, 2$.

Definitions 3.1.3. Let b_1, b_2 be two symmetric bilinear forms with underlying F-vector spaces V_1 and V_2 respectively. An *isometry* $f : b_1 \to b_2$ is a linear map $f : V_1 \to V_2$ such that for any vectors $v, w \in V_1$, we have: $b_2(f(v), f(w)) = b_1(v, w)$. If in addition, f is a linear isomorphism between V_1 and V_2, we say that b_1, b_2 are *isometric* and write $b_1 \simeq b_2$.

Also, if b_1 is isometric to b_2 multiplied by a nonzero element of F, i.e., $b_1 \simeq ab_2$ for some $a \in F^\times$, we say b_1, b_2 are *similar.*

Let b be a symmetric bilinear form on V over F . We say two vectors $v, w \in V$ are *orthogonal* if $b(v, w) = 0$. Similarly, if $U \subset V$ is a subspace, we define the *orthogonal complement* of U to be

$$U^\perp := \{v \in V \mid b(v, w) = 0 \text{ for any } w \in W\}.$$

If $V = U \oplus W$ where $U, W \subset V$ and $U \subset W^\perp$, then $b = b_{|U} \perp b_{|W}$ and we call b the *internal orthogonal sum* of $b_{|U}$ and $b_{|W}$.

Examples 3.1.4. (1) Let $a \in F$. The symmetric bilinear form on F given by $b(x,y) = axy$ for some $x, y \in F$ is denoted $\langle a \rangle_b$.

(2) Let $a_1, \cdots, a_n \in F$. The symmetric bilinear form on F^n given by $\langle a_1 \rangle_b \perp \langle a_2 \rangle \perp \cdots \perp \langle a_n \rangle_b$ is denoted $\langle a_1, \cdots, a_n \rangle_b$.

Remark 3.1.5. The symmetric bilinear form $\langle a_1, \cdots, a_n \rangle_b$ is called a *diagonal form*. A symmetric bilinear form that is isometric to a diagonal form is called *diagonalizable*.

Definitions 3.1.6. Let b be a symmetric bilinear form on an F-vector space V. We define the *radical* of b by $rad(b) := V^{\perp}$. If $rad(b) = 0$, then we say that the form b is *non-degenerate*.

In the definition above, one should notice that the radical of a bilinear symmetric form over F commutes with base-extension, i.e., $rad(b_K) = rad(b) \otimes K$ for any field extension K/F.

Example 3.1.7. Let V be a vector space and $V^{\vee}$ be its dual space. The *hyperbolic symmetric bilinear form* $\mathbb{H}(V)$ is the symmetric bilinear form on $V \oplus V^{\vee}$ defined by setting

$$b(v_1 + f_1, v_2 + f_2) := f_1(v_2) + f_2(v_1).$$

for any $v_1, v_2 \in V$ and $f_1, f_2 \in V^{\vee}$. This form is non-degenerate. $\mathbb{H}(F)$ is called the *hyperbolic plane* and denoted $\mathbb{H}$.

Remark 3.1.8. If $dim(V) = n$, then $\mathbb{H}(V) \simeq n\mathbb{H}$, where by an abuse of notation, $n\mathbb{H}$ means the external orthogonal sum of n hyperbolic planes.

Definitions 3.1.9. Let b be a symmetric bilinear form on an F-vector space V. A vector $v \in V$ is called *anisotropic* if $b(v,v) \neq 0$ or *isotropic* if $v \neq 0$ and $b(v,v) = 0$. We say the form b is *anisotropic* if $b(v,v) \neq 0$ for any nonzero vector $v \in V$. A subspace $U \subset V$ is called *totally isotropic* if $b_{|U} = 0$. If b is non-degenerate and has a totally isotropic subspace of dimension $\frac{1}{2}dim(b)$, then we say that b is *metabolic*.

Theorem 3.1.10 (Bilinear Witt Decomposition Theorem)**.** [2, Theorem 1.2.7] *A non-degenerate symmetric bilinear form is an orthogonal sum of an anisotropic form and a metabolic form. Also, the anisotropic form is unique up to isometry.*

Recall that we have the orthogonal sum and tensor product of symmetric bilinear forms over F. These two operations induce an addition and multiplication on the set of isometry classes of non-degenerate symmetric bilinear forms, giving it the structure of a commutative semi-ring.

The identity elements of addition and multiplication are the zero form and $\langle 1\rangle_b$ respectively.

We define the *Witt-Grothendieck* ring of F to be the Grothendieck ring of this semi-ring and denote it by $\widehat{W}(F)$.

The quotient ring $W(F) := \widehat{W}(F)/([\mathbb{H}])$ is called the *Witt ring* of non-degenerate symmetric bilinear forms over F. Elements in $W(F)$ are called *Witt classes*.

In $W(F)$, all Witt classes of even-dimensional non-degenerate symmetric bilinear forms constitute an ideal I, which is called the *fundamental ideal*.

Definition 3.1.11. Let $a \in F^\times$. The form $\langle 1, -a\rangle_b$ is called a *1-fold Pfister form* and we denote it by $\langle\langle a\rangle\rangle_b$. Let $a_1, \cdots, a_n \in F^\times$. The tensor product of 1-fold Pfister forms $\langle\langle a_1\rangle\rangle_b \otimes \langle\langle a_2\rangle\rangle_b \cdots \otimes \langle\langle a_n\rangle\rangle_b$ is called an *n-fold Pfister form* and we denote it by $\langle\langle a_1, \cdots, a_n\rangle\rangle_b$.

Bilinear Pfister forms have two important properties as below.

Proposition 3.1.12. [2, Corollary 6.2] *Let b be an n-fold Pfister form. Then for any $a \in D(b)$, we have $ab \simeq b$.*

Proposition 3.1.13. [2, Corollary 6.3] *A bilinear Pfister form is either anisotropic or metabolic.*

It can be shown that $\langle 1, -1\rangle_b = 0$ in $W(F)$ [2, Page 20]. We suppose that $\langle a, b\rangle$ is any 2-dimensional non-degenerate symmetric bilinear form, then we have $\langle a, b\rangle = \langle 1, -(-a)\rangle_b - \langle 1, -b\rangle_b$ in $W(F)$. This gives rise to the following proposition.

Proposition 3.1.14. [2, Page 24] *The fundamental ideal I is generated by Witt classes of 1-fold Pfister forms. In particular, I^n is generated by Witt classes of n-fold Pfister forms.*

3.2 Quadratic forms

Fix a field F.

Definition 3.2.1. A *quadratic form* on an F-vector space V is a map $\phi : V \to F$ satisfying

(1) $\phi(cv) = c^2\phi(v)$ for all $c \in F, v \in V$.

(2) The map $b_\phi : V \times V \to F$ given by

$$b_\phi(v, w) := \phi(v + w) - \phi(v) - \phi(w)$$

is a symmetric bilinear form.

for any $v, w \in V$ and $c \in F$

The associated bilinear form b_ϕ of a quadratic form ϕ is called the *polar form* of ϕ. By a quadratic form over F, we will mean a quadratic form ϕ on an F-vector space. This vector space will be called the *underlying vector space* of ϕ and denoted V_ϕ. Its dimension will be called the *dimension of* ϕ and denoted $dim(\phi)$.

Notation 3.2.2. We use $D(\phi) := \{\phi(v) \mid v \in V, \phi(v) \neq 0\}$ to denote the set of nonzero values of ϕ.

Definitions 3.2.3. Let ϕ_1 and ϕ_2 be two quadratic forms with underlying spaces V_1 and V_2 respectively. An *isometry* $f : \phi_1 \to \phi_2$ is a linear map $f : V_1 \to V_2$ such that $\phi_2(f(v)) = \phi_1(v)$ for any vector $v \in V_1$. If in addition, f is an isomorphism, we say the two forms are *isometric* and write $\phi_1 \simeq \phi_2$. Also, if ϕ_1 is isometric to ϕ_2 multiplied by a nonzero element of F, i.e., $\phi_1 \simeq a\phi_2$ for some $a \in F^\times$, we say that ϕ_1, ϕ_2 are *similar*.

Let K/F be a field extension and ϕ be a quadratic form over F. We define the quadratic form ϕ_K on $V_K := K \otimes_F V$ with the polar form $(b_\phi)_K$ by setting

$$\phi_K(a \otimes v) := a^2\phi(v)$$

for any $a \in K$ and $v \in V$.

Let ϕ_1 and ϕ_2 be two quadratic forms with underlying F-vector spaces V_1 and V_2 respectively. The *external orthogonal sum* of ϕ_1 and ϕ_2, denoted by $\phi_1 \perp \phi_2$, is the quadratic form on $V_1 \oplus V_2$, given by

$$(\phi_1 \perp \phi_2)(v \oplus w) := \phi_1(v) + \phi_2(w)$$

for any $v \in V_1$, $w \in V_2$. The polar form of $\phi_1 \perp \phi_2$ is $b_{\phi_1} \perp b_{\phi_2}$.

Let ϕ be a quadratic form on V and b be a symmetric bilinear form on W. The *tensor product* of b and ϕ , denoted by $b \otimes \phi$, is the unique quadratic form on $W \otimes_F V$ with polar form $b \otimes b_\phi$ satisfying

$$(b \otimes \phi)(w \otimes v) := b(w, w) \cdot \phi(v)$$

for any $v \in V$, $w \in W$.

Notation 3.2.4. Let n be a non-negative integer and ϕ be a quadratic form over F. By an abuse of notation, we let

$$n\phi := \underbrace{\phi \perp \cdots \perp \phi}_{n}$$

Without specific explanation, if n is a non-negative integer, we will view $n\phi$ as an orthogonal sum of n's ϕ rather than n multiplying ϕ.

Examples 3.2.5. (1) Let ϕ be the quadratic form on F given by $\phi(x) = ax^2$ for all $x \in F$, where $a \in F$ is fixed. We denote this form by $\langle a \rangle$. Let $a_1, \cdots, a_n \in F$. The quadratic form $\langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$, which can be written for short as $\langle a_1, \cdots, a_n \rangle$, is called a *diagonal form*. A quadratic form that is isometric to a diagonal form is called *diagonalizable*. When the characteristic of F is not 2, every quadratic form over F is diagonalizable.

(2) Let ϕ be the quadratic form on F^2 given by $\phi(x, y) = ax^2 + xy + by^2$ for any $(x, y) \in F^2$, where $a, b \in F$ are fixed. We denote this form by $[a, b]$.

(3) The form $\mathbb{H} := [0, 0]$ is called the *hyperbolic plane* and it is isometric to $[0, a]$ for any $a \in F$. In particular, any quadratic form which is isometric to an orthogonal sum of hyperbolic planes, is called a *hyperbolic form*.

Definitions 3.2.6. Let ϕ be a quadratic form on an F-vector space V over F. A vector $v \in V$ is called *anisotropic* if $\phi(v) \neq 0$ or isotropic if $v \neq 0$ and $\phi(v) = 0$. If $\phi(v) \neq 0$ for any nonzero vector $v \in V$, we say ϕ is *anisotropic*. Suppose $W \subset V$ is a subspace. Then W is called a *totally isotropic space* if every nonzero vector in W is isotropic.

Let ϕ be a quadratic form on an F-vector space V. Suppose $U, W \subset V$ are two subspaces such that $V = U \oplus W$ and $U \subset W^\perp$. Then $\phi = \phi_{|U} \perp \phi_{|W}$ and we call ϕ the *internal orthogonal sum* of $\phi_{|U}$ and $\phi_{|W}$.

Definitions 3.2.7. Let ϕ be a quadratic form on an F-vector space V. We say ϕ is *totally singular* if $V^{\perp} = V$.

Definition 3.2.8. Let ϕ be a quadratic form on an F-vector space V. We define the *totally singular part* of ϕ by $\phi_{ts} := \phi \mid_{rad(b_\phi)}$. ϕ_{ts} is a totally singular form on $rad(b_\phi)$.

Definitions 3.2.9. Let ϕ be a quadratic form on an F-vector space V. We define the *quadratic radical* of ϕ by

$$rad(\phi) := \{v \in rad(b_\phi) \mid \phi(v) = 0\}$$

Also, we say ϕ is *non-degenerate* if $rad(\phi) = 0$ and $dim(rad(b_\phi)) \leq 1$.

It's obvious that the quadratic radical of a quadratic form ϕ is a subspace of V_ϕ. Notice that although $rad(b_{\phi_K}) = rad(b_\phi) \otimes K$, we only have $rad(\phi) \otimes K \subset rad(\phi_K)$.

Theorem 3.2.10 (Structure Theorem)**.** [2, Chapter II, 7.B] *Let ϕ be a quadratic form on an F-vector space V.*

When the characteristic of F is not 2, then $\phi_{ts} \simeq \langle 0, \cdots, 0\rangle$ and $\phi \simeq \phi' \perp \phi_{ts}$ where ϕ' is non-degenerate and diagonalizable.

When the characteristic of F is 2, $\phi_{ts} \simeq \langle c_1, \cdots, c_s\rangle$ for some $c_1, \cdots, c_s \in F$ and $\phi \simeq \phi' \perp \phi_{ts}$ where ϕ' is an even dimensional non-degenerate quadratic form. ϕ' is not diagonalizable, but $\phi' \simeq [a_1, b_1] \perp \cdots \perp [a_r, b_r]$ for some $a_1, b_1, \cdots, a_r, b_r \in F$.

Since $V^{\perp}$ is determined uniquely, $rad(b_\phi)$ is uniquely determined up to isometry. So when the characteristic of F is 2, the numbers r and s in the structure theorem are determined uniquely.

Definitions 3.2.11. Let ϕ be a quadratic form over F. When the characteristic of F is 2, we suppose that

$$\phi \cong [a_1, b_1] \perp \cdots [a_r, b_r] \perp \langle c_1, \cdots, c_s\rangle$$

for some $a_1, b_1, \cdots, a_r, b_r, c_1, \cdots, c_s \in F$. Then we say ϕ *is of type* (r, s). In particular, if $s = 0$, we say the form ϕ is *nonsingular.*

Theorem 3.2.12 (Witt decomposition theorem)**.** [2, Theorem 8.5]

Let ϕ be a quadratic form on an F-vector space V. Then there exists a unique integer $i_W(\phi)$ and an anisotropic form ϕ_{an} such that

$$\phi \simeq i_W(\phi)\mathbb{H} \perp \phi_{an} \perp d\langle 0\rangle.$$

Also, the form ϕ_{an} is unique up to isometry.

Definitions 3.2.13. Let ϕ be a quadratic form and $\phi \simeq i_W(\phi)\mathbb{H} \perp \phi_{an} \perp d\langle 0\rangle$. We call ϕ_{an} the *anisotropic part* of ϕ and $i_W(\phi)$ the *Witt index* of ϕ. Also, the integer $(i_W(\phi)+d)$ is called the *total index* of ϕ and is denoted $i_t(\phi)$.

Remark 3.2.14. We can extend Definition 3.2.11 when the characteristic of F is not 2. Let F_{sep} be a separable closure of F. For a quadratic form ϕ over F, we suppose that $i_W(\phi_{F_{sep}}) = r$ and set $s := dim(\phi) - 2r$. Then we say the form ϕ is *of type* (r, s).

Definition 3.2.15. Let ϕ and ψ be two quadratic forms over F. We say ϕ and ψ are Witt equivalent if $\phi_{an} \simeq \psi_{an}$ and denote it by $\phi \sim \psi$.

Recall that isometry classes of anisotropic non-degenerate symmetric bilinear forms over F form the Witt ring $W(F)$ under the operations induced by orthogonal sums and tensor products. Since an orthogonal sum of even-dimensional non-degenerate quadratic forms is still an even-dimensional non-degenerate quadratic form, isometry classes of even-dimensional non-degenerate quadratic forms constitute a monoid under the addition induced by orthogonal sums of quadratic forms. We define the quotient of the Grothendieck group of the monoid by the subgroup generated by the class of the hyperbolic plane to be *quadratic Witt group* of F and denote it by $W_q(F)$. The tensor product induces a natural $W(F)$-module structure on $W_q(F)$.

Definitions 3.2.16. Let $a \in F^\times$. We set

$$\langle\langle a]] := \begin{cases} \langle\langle a\rangle\rangle_b \otimes \langle 1\rangle & \text{if } char(F) \neq 2 \\ [1, a] & \text{if } char(F) = 2. \end{cases}$$

and call this *1-fold Pfister form.* Let $n \geq 1$ and $a_1, \cdots, a_{n-1} \in F^\times, a_n \in F$. We set

$$\langle\langle a_1, \cdots, a_n]] := \langle\langle a_1, \cdots, a_{n-1}\rangle\rangle_b \otimes \langle\langle a_n]]$$

and call it an *n-fold quadratic Pfister form.* A quadratic form which is similar to a quadratic Pfister form is called a *general quadratic Pfister form.*

We denote the $W(F)$-submodule of $W_q(F)$ generated by classes of n-fold quadratic Pfister forms by I_q^n. For quadratic forms in I_q^n, we have the following fundamental fact, which is known as *Arason-Pfister Hauptsatz.*

Theorem 3.2.17. [2, Theorem 23.7] *Let ϕ be a non-zero anisotropic quadratic form over F. If $\phi \in I_q^n$, then $dim(\phi) \geq 2^n$.*

For quadratic Pfister forms, we have the following main results:

Proposition 3.2.18. [2, Corollary 9.9] *Let π be a quadratic Pfister form. Then for any $x \in D(\pi)$, we have that $x\pi \simeq \pi$.*

Theorem 3.2.19. [2, Corollary 9.10] *A quadratic Pfister form is either anisotropic or hyperbolic.*

Definition 3.2.20. Let ϕ and ψ be quadratic forms over F. We say that ψ is a *subform* of ϕ if ϕ is isometric to an internal orthogonal summand of ψ, i.e., $\phi \simeq \psi \perp \phi'$, where ϕ' is a quadratic form over F.

Let ϕ be a non-degenerate quadratic form on an F-vector space V and $U \subset V$ be a subspace. When the characteristic of F is not 2, $\phi |_U$ is a subform of ϕ. But when the characteristic is 2, this needn't to be true in general. So we introduce the concept of *domination*.

Definitions 3.2.21. Let ϕ be a quadratic form on an F-vector space V and ψ be a quadratic form on an F-vector space W. We say ϕ *dominates* ψ if there exists an injective isometry $f : W \to V$ and we denote this by $\phi \succ \psi$.

Accompanying the concept of domination, we have the following definition.

Definition 3.2.22. Let ϕ be a quadratic form on an F-vector space V and ψ be a quadratic form on an F-vector space W. For an injective isometry $f : W \to V$, we define the *complement form of ψ in ϕ relative to f* to be the form $\phi |_{f(W)^{\perp}}$ and denote it by $\psi^C_{f,\phi}$.

Theorem 3.2.23. [7, Lemma 3.1] *Let ϕ be a non-degenerate quadratic form on an F-vector space V and ψ be a quadratic form on an F-vector space W.*

When the characteristic of F is not 2, then $\phi \succ \psi$ if and only if ψ is isometric to a subform of ϕ, i.e, $\phi \simeq \psi \perp \psi'$ for some quadratic form ψ'. Moreover, if $\phi \cong \psi \perp \psi'$ and $f : V_\psi \to V_\phi$ is an injective isometry, then $\psi^C_{f,\phi} \cong \psi'$.

When the characteristic of F is 2, we suppose $\psi \simeq \psi_r \perp \psi_{ts}$, in which ψ_r is an even-dimensional non-degenerate form and $\psi_{ts} \simeq \langle c_1, \cdots, c_s\rangle$ for $c_1, \cdots, c_s \in F$.

Suppose ϕ has an odd dimension, then $\phi \succ \psi$ if and only if there exists an even-dimensional non-degenerate quadratic form τ, non-negative integers $s^{'} \leq s \leq s^{'} + 1$, $c_{s'+1} \in F$ and $d_j \in F$ with $1 \leq j \leq s^{'}$ such that

$$\phi \simeq \phi_r \perp \tau \perp [c_1, d_1] \perp \cdots \perp [c_{s'}, d_{s'}] \perp \langle c_{s'+1} \rangle.$$

Moreover, if $\phi \simeq \phi_r \perp \tau \perp [c_1, d_1] \perp \cdots \perp [c_{s'}, d_{s'}] \perp \langle c_{s'+1} \rangle$ and $f : V_\psi \to V_\phi$ is an injective isometry, then $\psi^C_{f,\phi} \cong \tau \perp \langle c_1, \cdots, c_{s'} \rangle \perp \langle c_{s'+1} \rangle$.

Suppose ϕ has an even dimension, then $\phi \succ \psi$ if and only if there exists an even-dimensional non-degenerate quadratic form τ and $d_j \in F$ with $1 \leq j \leq s$ such that

$$\phi \simeq \phi_r \perp \tau \perp [c_1, d_1] \perp \cdots \perp [c_s, d_s].$$

Moreover, if $\phi \simeq \phi_r \perp \tau \perp [c_1, d_1] \perp \cdots \perp [c_s, d_s]$ and $f : V_\psi \to V_\phi$ is an injective isometry, then $\psi^C_{f,\phi} \cong \tau \perp \langle c_1, \cdots, c_s \rangle$.

Let ϕ be a non-degenerate quadratic form on an F-vector space V and ψ be a quadratic form on an F-vector space W. If $\phi \succ \psi$, by Theorem 3.2.23, the complement of ψ in ϕ is independent of injective isometries. Let $f : W \to V$ be any injective isometry. We then define $\psi^C_\phi := \psi^C_{f,\phi}$.

Corollary 3.2.24. *Suppose the characteristic of F is 2. Let ϕ be an even-dimensional non-degenerate form over F and ψ be a quadratic form of type (r, s) over F. If $\phi \succ \psi$, then $dim(\phi) \geq 2r + 2s$.*

Proof. Since ϕ is an even-dimensional non-degenerate form, then $\phi_{ts} = 0$. Suppose $\psi \simeq \psi_r \perp \langle c_1, \cdots, c_s \rangle$ where ψ_r is an even-dimensional non-degenerate subform and $\langle c_1, \cdots, c_s \rangle$ is the totally singular part of ψ.

When $\phi \succ \psi$, from Theorem 3.2.23, we have $\phi \simeq \psi_r \perp \tau \perp [c_1, d_1] \perp \cdots \perp [c_s, d_s]$ for some even-dimensional non-degenerate form τ and $d_1, \cdots, d_s \in F$. Thus ϕ is dominated by a subform of ϕ with dimension $2r + 2s$. Thus we conclude that $dim(\phi) \geq 2r + 2s$. □

Definition 3.2.25. Suppose the characteristic of F is 2. Let ϕ and ψ be two quadratic forms over F and $\phi \succ \psi$. If ψ is a quadratic form of type (r, s) and ϕ is an even-dimensional non-degenerate quadratic form of dimension $2r + 2s$, then we say ϕ is a *nonsingular completion*(n.s.c) of ψ.

Corollary 3.2.26. [7, Lemma 3.7] *Let ϕ be a non-degenerate quadratic form over F and q be a quadratic form over F. If $\phi \succ q$, then $-q \perp \phi \sim q_{\phi}^{C}$. In particular, if σ is a totally singular form over F and ρ is a nonsingular completion of σ, then $\rho \perp \sigma \sim \sigma$.*

3.3 Associated varieties

Fix a field F. In this section, we will associate schemes to quadratic forms. As a result, properties of quadratic forms will be identified as properties of schemes.

Definitions 3.3.1. Let ϕ be a quadratic form on an F-vector space V. We define the associated quadric of ϕ to be the closed subscheme of $\mathbb{P}(V)$, which is defined by the vanishing of ϕ. The associated quadric is denoted X_{ϕ}. Moreover, if ϕ is anisotropic, then we say that X_{ϕ} is an anisotropic quadric.

Suppose that K/F is any field extension and ϕ is a quadratic form over F. We have $X_{\phi_K} = (X_{\phi})_K := X_{\phi} \times_F Spec(K)$. The construction above allows us to identify isotropy property of ϕ with the existence of rational points in X_{ϕ}. More generally, K-points $X(K)$ are identified with isotropic lines in V_K.

Definition 3.3.2. Let ϕ be a quadratic form on an F-vector space V. For any $0 \leq i \leq dim(\phi)$, we can consider the Grassmannian on totally isotropic i-dimensional subspaces of V. This is a projective variety and we denote it by $Gr(i, \phi)$.

The definition above is a generalization of the concept of the associated quadric. Actually, $X_{\phi} = Gr(1, \phi)$ since each point of X_{ϕ} is determined by an isotropic vector of V_{ϕ}.

The following result describes the singular locus of a quadric.

Proposition 3.3.3. *Let ϕ be a quadratic form over F with dimension ≥ 2. Then the singular locus of X_{ϕ} is $X_{\phi} \cap \mathbb{P}(V_{rad(b_{\phi})}) = X_{\phi_{ts}}$.*

Proof. Let $\overline{F}$ be an algebraic closed field extension of F , and we denote $\phi_{\overline{F}}$ by $\overline{\phi}$ for short. As mentioned before, $rad(b_{\phi})$ commutes with field extensions. So we have $rad(b_{\overline{\phi}}) = rad(b_{\phi}) \otimes \overline{F}$. It turns out that

$$(X_{\phi} \cap \mathbb{P}(V_{rad(b_{\phi})}))_{\overline{F}} = X_{\overline{\phi}} \cap \mathbb{P}(V_{rad(b_{\overline{\phi}})}).$$

If we can show that $(X_\phi \cap \mathbb{P}(V_{rad(b_\phi)}))_{\overline{F}}$ is the singular locus of $(X_\phi)_{\overline{F}}$, then since the smooth locus commutes with any base-extension, i.e., $(X_{sm})_{\overline{F}} = (X_{\overline{F}})_{sm}$, we can conclude $X_\phi \cap \mathbb{P}(V_{rad(b_\phi)})$ is the singular locus of X_ϕ.

As a result, we reduce the problem to $\overline{\phi}$. Since $\overline{F}$ is an algebraically closed field, we know that $\overline{\phi} \simeq r\mathbb{H} \perp s\langle 0\rangle$ or $\overline{\phi} \simeq r\mathbb{H} \perp \langle 1\rangle \perp (s-1)\langle 0, \cdots, 0\rangle$ for some non-negative integers s, r. By choosing a suitable basis of $V_{\overline{\phi}}$, we can identify $X_{\overline{\phi}}$ with the projective hypersurface $\{f = 0\}$ in $\mathbb{P}^{2r+s-1}$ where $f = x_1y_1 + \cdots + x_ry_r + 0 \cdot z_1^2 + \cdots + 0 \cdot z_s^2$ or $f = x_1y_1 + \cdots + x_ry_r + z_1^2 + 0 \cdot z_2^2 + \cdots + 0 \cdot z_s^2$ respectively.

Consequently, the Jacobian matrix of the polynomial is

$$\begin{bmatrix} y_1 & x_1 & \cdots & y_r & x_r & 0 & \cdots & 0 \end{bmatrix}.$$

or the Jacobian matrix is

$$\begin{bmatrix} y_1 & x_1 & \cdots & y_r & x_r & 2z_1 & \cdots & 0 \end{bmatrix}.$$

When the characteristic of F is 2, the latter is the same as the former. When the characteristic of F is not 2, in the latter Matrix, z_1 must be zero if we have $x_1 = y_1 = \cdots = x_r = y_r = 0$.

Finally, by the Jacobian criterion of projective hypersurface, the singular locus of $X_{\overline{\phi}}$ is given by $x_1 = y_1 = \cdots = x_r = y_r = 0$. So the singular locus is determined by $rad(b_{\overline{\phi}})$. This finishes our proof. □

Corollary 3.3.4. *Let ϕ be a quadratic form over F with dimension ≥ 2. Then X_ϕ is smooth if and only if ϕ is non-degenerate.*

Proof. Since the smooth locus of a variety commutes with base-extension, we can reduce the case to an algebraically closed field. So X_ϕ is smooth if and only if $(X_{\overline{\phi}})_{ts}$ is empty. Notice that we have

$$(X_{\overline{\phi}})_{ts} = X_{\overline{\phi}} \cap \mathbb{P}(V_{rad(b_{\overline{\phi}})}) = \mathbb{P}(V_{rad(\overline{\phi})}),$$

we know that $(X_{\overline{\phi}})_{ts}$ is empty if and only if $rad(\overline{\phi}) = 0$. But $rad(\overline{\phi}) = 0$ is equivalent to that $dim(rad(b_\phi)) \leq 1$ and $rad(\phi) = 0$, which means that ϕ is non-degenerate. □

Corollary 3.3.5. *Let ϕ be a quadratic form over F with dimension ≥ 2. Suppose the characteristic of F is not 2. When ϕ is anisotropic, X_ϕ is smooth.*

Proof. In this case, $\phi_{ts} = 0$. Thus ϕ is non-degenerate and X_ϕ is smooth by Corollary 3.3.4. □

Remark 3.3.6. When the characteristic of F is 2, an anisotropic quadratic form can have nontrivial totally singular part. So the associated quadric needn't to be smooth in general.

In the rest of this section, we will consider *places*.

Definition 3.3.7. Let L and K be two fields. A *place* $\pi : L \rightharpoonup K$ is a local ring homomorphism $f : R \to K$ where R is a valuation subring of L. We say the place π is defined on R. In particular, if L and K are extensions of F and a place $L \to K$ is defined and the identity on F, we say the place is an *F-place*.

Let $\pi_1 : K \rightharpoonup L$ be a place defined on R and $\pi_2 : L \rightharpoonup E$ be a place defined on S. Then the *composition of places* $\pi_2 \circ \pi_1 : K \rightharpoonup E$ is given by the composition of local ring homomorphisms $g \circ f \mid_{f^{-1}(S)}$ where $g : S \to E$ is the local ring homomorphism of π_2 and $f : R \to L$ is the local ring homomorphism of π_1.

Proposition 3.3.8. *Let X be a complete scheme over F and $\pi : E \to K$ be an F-place. If $X(E) \neq \emptyset$, then $X(K) \neq \emptyset$.*

Proof. Let $f : Spec(E) \to X$ be an E-point of X. Since X is complete, there exists a unique point $x \in X$ dominates the valuation ring $R \subset E$ [4, Theorem 4.7]. That is to say, $\mathcal{O}_{X,x} \subset R$ and the maximal ideal M of R contains the maximal ideal $\mathcal{M}_{X,x}$ of $\mathcal{O}_{X,x}$. Thus we have a ring homomorphism $F(x) = \mathcal{O}_{X,x}/\mathcal{M}_{X,x} \to R/M \to K$, which gives rise to a K-point of X. □

Corollary 3.3.9. *Let ϕ be a quadratic form on V over F and $\pi : E \to K$ be an F-place. Then we have $i_t(\phi_K) \geq i_t(\phi_E)$.*

Proof. Suppose $i_t(\phi_E) = n$. Then $Gr(n, \phi)(E)$ is not empty. From Proposition 3.3.8, since $\pi : E \to K$ is an F-place, we know that $Gr(n, \phi)(K)$ is also not empty. Hence $i_t(\phi_K) \geq n = i_t(\phi_E)$. □

3.4 Function fields of quadrics

Definition 3.4.1. Let ϕ be a quadratic form on an F-vector space V and X_ϕ be the associated quadric. If X_ϕ is a variety, we define the *function field* of ϕ to be the function field $F(X_\phi)$ and denote it by $F(\phi)$. If X_ϕ is not a variety, we set $F(\phi) = F$.

Remark 3.4.2. Explicitly, $X_\phi = Proj(Sym^*(V^\vee)/(\phi))$, where $Sym^*(V^\vee)$ is the symmetric algebra operation on the dual space $V^\vee$. We view ϕ as a polynomial in $Sym^*(V^\vee)$, then X_ϕ is integral if and only if ϕ is irreducible. When ϕ is irreducible, $F(\phi)$ is nothing but the subfield of all degree zero elements of $Frac(Sym^*(V^\vee)/(\phi))$. Obviously, ϕ is isotropic over $F(\phi)$.

Lemma 3.4.3. *An anisotropic quadratic form stays anisotropic under purely transcendental extension.*

Proof. Let ϕ be an anisotropic quadratic form over F. It suffices to prove the case that $F(t)/F$ is a purely transcendental extension with transcendence degree 1.

We suppose $\phi_{F(t)}$ is isotropic, and assume v is an isotropic vector of $V_{F(t)}$ such that $\phi_{F(t)}(v) = 0$. Clear all denominators, we assume that $v \in F[t] \otimes V$ and write $v = v_0 + t \otimes v_1 + \cdots + t^n \otimes v_n$ with $v_n \neq 0$. Since ϕ is anisotropic, n is a positive integer.

Now we have $\phi_{F(t)}(v) = 0$ while

$$\phi_{F(t)}(v) = t^{2n}\phi(v) + \phi_{F(t)}(v_0 + \cdots + t^{n-1} \otimes v_{n-1}) + t^n b_{\phi_{F(t)}}(v_0 + \cdots + t^{n-1}v_{n-1}, v_n)$$

Since the degrees of $t^n b_{\phi_{F(t)}}(v_0 + \cdots + t^{n-1}v_{n-1}, v_n)$ and $\phi_{F(t)}(v_0 + \cdots + t^{n-1} \otimes v_{n-1})$ are at most $2n-1$, so $t^{2n}\phi(v_n)$ mush vanish and hence $\phi(v_n) = 0$, which contradicts our assumption that ϕ is anisotropic. □

Proposition 3.4.4. *Let ϕ be an irreducible quadratic form over F. Then $F(\phi)/F$ is a purely transcendental extension if and only if $i_W(\phi) > 0$.*

Proof. Suppose $i_W(\phi) > 0$. Thus $\phi \simeq \mathbb{H} \perp \psi'$ for some quadratic form ψ'. By choosing a suitable basis, we suppose ϕ is represented by a degree 2 homogeneous polynomial $f = xy + g(x_2, x_3, \cdots, x_n)$. Then the function field of ϕ is $F(x_2, \cdots, x_n)[y]/(y + g(x_2, x_3, \cdots, x_n)) \cong F(x_2, \cdots, x_n)$, which is a purely transcendental extension of F.

For the other direction, we assume that $F(\phi)/F$ is a purely transcendental extension. We can write $\phi \simeq \phi' \perp \langle 0, \cdots, 0\rangle$ where $rad(\phi') = 0$.

Suppose $i_W(\phi') = 0$, and it just means ϕ' is anisotropic. Since $F(\phi)/F$ is a purely transcendental extension , we have that $\phi'_{F(\phi)}$ stays anisotropic by Lemma 3.4.3. Moreover, since $F(\phi') \subset F(\phi)$, we know that $\phi'_{F(\phi')}$ is also anisotropic. But we know $\phi'_{F(\phi')}$ is isotropic, which is a contradiction. Thus we conclude that $i_W(\phi') > 0$ and of course, $i_W(\phi) > 0$. □

Let K and F be two fields and K/F be a field extension. We say that K/F is a *separable extension* if there exists an intermediate field E such that K/E is an algebraic separable extension and E/F is a purely transcendental extension.

Proposition 3.4.5. *Let ϕ be a quadratic form over F of dimension larger than or equal to 2. We suppose ϕ is not totally singular. Then $F(\phi)/F$ is a separable extension.*

Proof. When ϕ is reducible, then the function field of ϕ is F as we defined, which is trivially a separable extension of F.

So we assume ϕ is irreducible. Since $dim(\phi) \geq 2$, there exists a 2-dimensional subspace U of V_ϕ such that $\phi \mid_U \simeq [a, b]$ for $a, b \in F$. Let Z be a set of variables. Choosing a suitable basis, we use the degree 2 homogeneous polynomial $f = ax^2 + xy + by^2 + g(Z)$ to represent ϕ. As a result, $F(\phi) = F(y, Z)[x]/(ax^2 + x + b + g(Z))$.

An application of separability test shows that $F(\phi)$ is a separable extension over $F(y, Z)$. Thus we conclude that $F(\phi)$ is a separable extension. □

Let K/F be a separable field extension and $char(F) = 2$. A basic algebra fact says that, if $a_1, \cdots, a_n \in F$ are F^2-linear independent, then they are also K^2-linear independent[13, Proposition 4.1]. This gives the following lemma.

Lemma 3.4.6. [19, Lemma 2.1] *Suppose the characteristic of F is 2. Let ϕ be an anisotropic totally singular quadratic form over F. Then ϕ stays anisotropic over any separable field extension K/F.*

We have defined the tensor product of a symmetric bilinear form and a quadratic form. When the characteristic of F is 2, we know that totally singular forms are the diagonal parts of symmetric bilinear forms. Let ϕ and ψ be two totally singular forms over F. The *tensor product of totally singular forms* is the quadratic form on the F-vector space $V_\phi \otimes V_\psi$ given by

$$(\phi \otimes \psi)(v \otimes w) := \phi(v) \cdot \psi(w)$$

for any $v \in V_\phi$ and $w \in V_\psi$.

Lemma 3.4.7. *Suppose the characteristic of F is 2. Let ϕ be an anisotropic totally singular form on an F-vector space V and $L = F(\sqrt{a})$ where $a \in F^\times \setminus F^{\times 2}$. If $i_t(\phi_L) = n$, then $\phi \simeq \langle 1, a \rangle \otimes \tau \perp \tau'$ for some totally singular forms τ' and τ with $dim(\tau) = n$.*

Proof. We prove the statement by inducting on $dim(\phi)$. When $dim(\phi) \leq 1$, the case is trivial. So we suppose $dim(\phi) \geq 2$. When $i_t(\phi_L) = n$, we let U be a maximal totally isotropic subspace of V_{ϕ_L} with basis $\{u_1, \cdots, u_n\}$. Here we write $u_i = v_i + \sqrt{a}w_i, i = 1, \cdots, n$ for vectors $v_i, w_i \in V_\phi$. Set $V^{'} = span\{v_1, w_1\}$ and $V^{''} = span\{v_2, w_2, \cdots, v_n, w_n\}$.

At first, we notice that v_1, w_1 are linear independent. Actually, suppose $v_1 = \lambda w_1$ for some $\lambda \in F^\times$. Since $\phi_L(u_1) = 0$, we have that $\lambda^2\phi(w_1) = a\phi(w_1)$. Since ϕ is anisotropic, we obtain $a = \lambda^2$, which is a contradiction. Thus $\phi\mid_{V'} \simeq b\langle 1, a\rangle$ where $b = \phi(w_1)$. For the form $\phi\mid_{V''}$, by induction, it admits a subform of the shape $\langle 1, a\rangle \otimes \sigma$ for some totally singular form σ of dimension $n-1$ over F. Since $dim(V^{''}) \leq 2(n-1)$, we conclude that $\phi\mid_{V''} = \langle 1, a\rangle \otimes \sigma$.

Now we want to show $V^{'} \cap V^{''} = 0$. Let x be any nonzero vector in $V^{'} \cap V^{''}$. Since $x \in V^{'}$ and ϕ is anisotropic, we have that $\phi(x) = b \cdot \alpha$ for some nonzero element $\alpha \in D(\langle 1, a\rangle)$. Since $x \in V^{''}$, we know that $b \cdot \alpha \in D(\phi\mid_{V''})$. By Proposition 3.1.12, viewing $\langle 1, a\rangle$ as the diagonal part of the bilinear Pfister form $\langle\langle a\rangle\rangle_b$, we know that $b \cdot \alpha^2 \in D(\phi\mid_{V''})$. Hence we have $b \in D(\phi\mid_{V''})$. Notice that $a \in D(\langle 1, a\rangle)$, by another application of Proposition 3.1.12, we have $a \cdot b \in D(\phi\mid_{V''})$. Recall that $b = \phi(w_1)$ and $\phi(v_1) = a\phi(w_1)$. Since ϕ is anisotropic over F, we have that $w_1, v_1 \in V^{''}$ and then $u_1 \in V_L^{''}$, which is a contradiction. As a result, we have that $V^{'} \cap V^{''} = 0$.

Finally, we can identify $V^{'} \oplus V^{''}$ with a subspace of V_ϕ. But then

$$\phi\mid_{V'\oplus V''} \simeq \langle 1, a\rangle\phi(w_1) \perp \langle 1, a\rangle\sigma \simeq \langle 1, a\rangle \otimes (\sigma \perp \langle\phi(w_1)\rangle)$$

where $\sigma \perp \langle\phi(w_1)\rangle$ is a totally singular form of dimension n. This finishes our proof. □

Proposition 3.4.8. *Let ϕ be an irreducible quadratic form on an F-vector space V. We assume ϕ_{ts} is anisotropic. Then for any field extension E/F such that $i_t(\phi_E) > 0$, there exists an F-place $\pi : F(\phi) \rightharpoonup E$.*

Proof. Let X be the associated quadric to ϕ. Since $i_t(\phi_E) > 0$, $X(E) \neq \emptyset$. Then we prove the proposition by considering two cases:

Case 1: The E-point lies in the smooth locus of X

Let's denote the image of $f : Spec(E) \to X$ by x. In this case, x is a smooth point, hence a regular point. Whenever x is a regular point, one can always construct an F-place $F(\phi) \rightharpoonup E$. This is shown in [2, Appendix 103].

Case 2: The E-point lies in the singular locus of X

Under our assumption, this case only happens when the characteristic of F is 2. Since the E-point lies in the singular locus of X, then E/F is an inseparable extension by Lemma 3.4.6. This means there exist fields $F \subset E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_s = E$ such that E_0/F is a separable extension, and each E_i is a degree 2 purely inseparable extension of E_{i-1}.

Since E_0/F is separable, ϕ_{ts} stays anisotropic over E_0 by Lemma 3.4.6. Let $i \geq 1$ be minimal such that $(\phi_{ts})_{E_i}$ is isotropic. Replacing F by E_{i-1} and E by E_i, we reduce the case to $E = F(\sqrt{a})$ for some $a \in F^\times$. By Lemma 3.4.7, we obtain that $\phi_{ts} \simeq b\langle 1, a\rangle \perp \phi^{'}$ for some $b \in F^\times$. Let x be a degree 2 point of X defined by the vanishing of $b\langle 1, a\rangle$.

Now we show the point x is a regular point. We assume $dim(\phi^{'}) = 2r + s - 2$. Then $dim(\phi) = 2r + s$ and $dim(X) = 2r + s - 2$. Let Z be a set of $(2r + s - 2)$ variables, i.e., $Z = \{z_3, \cdots, z_s, x_1, y_1, \cdots, x_r, y_r\}$. Written in coordinates, ϕ can be represented by a degree-2 homogeneous polynomial $f = bz_1^2 + abz_2^2 + g(Z)$. Now x is a point of the affine variety

$$Spec(F[z_2, Z]/(b + abz_2^2 + g(Z)).$$

Now we write $(Z) := (z_3, \cdots, z_s, x_1, y_1, \cdots, x_r, y_r)$ as an ideal in $F[z_2, Z]$. Localizing the ring $F[z_2, Z]/(b + abz_2^2 + g(Z))$ at the prime ideal (Z), we obtain the local ring of x, say $\mathcal{O}_x$. The maximal ideal $\mathcal{M}_x$ of $\mathcal{O}_x$ is $(Z)/(b + abz_2^2 + g(Z))$. Notice that $\mathcal{M}_x/\mathcal{M}_x^2 = (Z)/(Z)^2$ has $(2r+s-2)$ generators, while $dim(\mathcal{M}_x/\mathcal{M}_x^2) \geq dim(X)$. This shows that $dim(\mathcal{M}_x/\mathcal{M}_x^2) = dim(X)$, hence x is a regular point. Then the F-place $F(\phi) \rightharpoonup E$ is constructed in [2, Appendix 103].

Combining the two cases, we finish our proof. □

We are interested in the isotropy behaviour of a quadratic form over the function field of another form. The following theorems are well-known.

Theorem 3.4.9 (Domination Theorem)**.** [7, Theorem 4.2] *Let ϕ be an anisotropic quadratic form over F. Let ψ be a quadratic form which is not totally singular and ψ_{ts} is anisotropic. If the form $\phi_{F(\psi)}$ is hyperbolic. Then ϕ is an orthogonal summand of $ab\psi$ for any $a \in D(\phi)$ and $b \in D(\psi)$.*

Theorem 3.4.10 (Separation Theorem)**.** [7, Theorem 4.4] *Let ϕ and ψ be anisotropic forms over F such that $dim(\phi) < 2^n \leq dim(\psi)$ for some positive integer n. Then*

$\phi_{F(\psi)}$ *is anisotropic.*

The following theorem gives an equivalent condition under which an anisotropic quadratic form is dominated by another one. We will use the theorem below in Chapter 5.

Theorem 3.4.11. [7, Proposition 3.11]

Let ϕ and ψ be two anisotropic quadratic forms over F with $dim(\phi) \geq \dim(\psi)$. Also, we define $n(\phi, \psi) := i_t(\phi \perp -\psi)$. Then $\phi \succ \psi$ if and only if $n(\phi, \psi) \geq dim(\psi)$.

3.5 Splitting patterns

Fix a field F.

Let ϕ be a quadratic form on an F-vector space V. We define discrete invariants of ϕ as below:

Definition 3.5.1. The *full splitting pattern* of the quadratic form ϕ is the set of integers $i_t(\phi_K)$ where K goes through all field extensions of F i.e., the set

$$FSP(\phi) := \{i_t(\phi_K) \mid K/F \text{ is any field extension}\}.$$

Definitions 3.5.2. The *separable splitting pattern* of the quadratic form ϕ is the set of integers $i_t(\phi_K)$ where K goes through all separable field extensions of F i.e., the set

$$SSP(\phi) := \{i_t(\phi_K) \mid \ K/F \text{ is a separable extension}\}.$$

Also, we define the *separable height* $h_{sep}(\phi)$ to be the number of elements in $SSP(\phi)$ minus 1, i.e., $h_{sep}(\phi) :=\mid SSP(\phi) \mid -1$.

Let ϕ be any quadratic form on an F-vector space V. We define the *standard splitting tower* of ϕ through a series of constructions. We set $F_0 = F$ and $\phi_0 = \phi_{an}$. For $i \geq 1$, we set $F_i = F_{i-1}(\phi_{i-1})$ and $\phi_i = ((\phi_{i-1})_{F_i})_{an}$. We define the *Knebusch height* $h(\phi)$ of ϕ to be the smallest integer n with $dim(\phi_n) \leq 1$.

The form ϕ_i is called the i-th *kernel form* of ϕ for $i \in [0, h(\phi)]$. The *standard splitting tower* of ϕ is the tower $F_0 \subset \cdots \subset F_{h_\phi}$ for $i \in [0, h(\phi)]$.

Definitions 3.5.3. Let ϕ be a quadratic form on an F-vector space V. Let F_i be the standard splitting tower of ϕ and ϕ_i be the i-th kernel form for $i \in [0, h(\phi)]$.

The *Knebusch splitting pattern* of ϕ is the set of integers $i_t(\phi_{F_i})$ where i goes through $[0, h(\phi)]$, i.e.,

$$KSP(\phi) := \{i_t(\phi_{F_i}), i \in [0, h(\phi)]\}$$

We call $i_t(\phi_{F_i})$ the *i-th absolute higher Knebusch index* and denote it by $\mathfrak{j}_i(\phi)$. Also, we define the *i-th relative higher Knesbusch index* to be $\mathfrak{i}_i(\phi) := \mathfrak{j}_i(\phi) - \mathfrak{j}_{i-1}(\phi)$ where $\mathfrak{j}_i(\phi)$ is the i-th absolute higher Knebusch index.

The following example shows that Knebusch splitting pattern and the full splitting pattern are not the same.

Example 3.5.4. Let $\phi = \langle 1, a, b, ab, c\rangle$ be a 5-dimensional anisotropic totally singular quadratic form over the field of rational functions $F(a, b, c)$ with variables a, b, c. Later, by Corollary 3.5.10, We have $\mathfrak{i}_1(\phi) = 1$ and $\phi_1 \simeq \langle 1, a, b, ab\rangle$. Since ϕ_1 is the diagonal part of a bilinear Pfister form, by Proposition 3.1.13, ϕ_1 is either anisotropic or metabolic. So $\mathfrak{i}_2(\phi) = 2$. As a result, The Knebusch splitting pattern is $(0, 1, 3, 4)$.

But if we take the field extension $K = F(\sqrt{a})$, then $\phi_K \simeq \langle 1, 0, 0, b, c\rangle$. Lemma 3.4.7 shows that $i_t(\phi_K) \leq \frac{1}{2}dim(\phi)$, thus $i_t(\phi_K) = 2$. Thus the full splitting pattern of ϕ contains the integer 2.

However, we have the following relation:

Proposition 3.5.5. *Let ϕ be an anisotropic quadratic form with dimension at least 2 and K be an extension of F such that ϕ_K is isotropic. Then $i_t(\phi_K) \geq \mathfrak{i}_1(\phi)$. In other words, $\mathfrak{i}_1(\phi)$ is the smallest nonzero element of $FSP(\phi)$.*

Proof. By Proposition 3.4.8, we have an F-place $\pi : F(\phi) \rightharpoonup K$. Thus by Corollary 3.3.9, we have that $i_t(\phi_K) \geq i_t(\phi_{F(\phi)}) = \mathfrak{i}_1(\phi)$. □

More generally,

Lemma 3.5.6. *Let ϕ be a quadratic form over F and K be an extension of F. Let $r \leq h(\phi)$ be the smallest positive integer for which $i_t(\phi_K) \leq \mathfrak{j}_r(\phi)$. If $i_W(\phi_K) > \mathfrak{j}_{r-2}(\phi)$, then $i_t(\phi_K) = \mathfrak{j}_r(\phi)$.*

Proof. Let (F_i, ϕ_i) be the standard splitting tower and ith kernel forms of ϕ. We write $K \cdot F_i$ for the composite of fields. By assumption, since $i_W(\phi_K) > \mathfrak{j}_{r-2}(\phi)$, we know that $i_W(\phi_{K \cdot F_{r-2}}) > i_W(\phi_{F_{r-2}})$, hence $i_W((\phi_{r-2})_K) > 0$. By Proposition 3.4.4, $F((\phi_{r-2})_K) = K \cdot F_{r-1}$ is a purely transcendental extension of $K \cdot F_{r-2}$. Thus there exists an F-place $K \cdot F_{r-1} \rightharpoonup K \cdot F_{r-2}$. Also, notice that $\mathfrak{j}_{r-2}(\phi) > \mathfrak{j}_{r-3}(\phi)$, we have $i_W(\phi_{K\cdot F_{r-3}}) > i_W(\phi_{F_{r-3}})$. Hence $i_W((\phi_{r-3})_K) > 0$. Another application of Proposition 3.4.4 shows that $F((\phi_{r-3})_K) = K \cdot F_{r-2}$ is a purely transcendental extension of $K \cdot F_{r-3}$. Thus we have an F-place $K \cdot F_{r-2} \rightharpoonup K \cdot F_{r-3}$. Inductively, we have an F-place $K \cdot F_{r-1} \rightharpoonup K$.

Since $i_W(\phi_K) > \mathfrak{j}_{r-2}(\phi)$, we know $\phi_{F_{r-1}} \simeq \mathfrak{j}_{r-1}(\phi)\mathbb{H} \perp \phi_{r-1}$. By assumption, we have that $i_t(\phi_K) > \mathfrak{j}_{r-1}(\phi)$. This means $(\phi_{r-1})_{K\cdot F_{r-1}}$ is isotropic. Thus we have an F_{r-1}-place

$$F_r = F(\phi_{r-1}) \to K \cdot F_{r-1}$$

by Proposition 3.4.8. Finally, we have an F-place from F_r to K. By Corollary 3.3.9, we conclude that $i_t(\phi_K) = \mathfrak{j}_r(\phi)$. □

Corollary 3.5.7. *Let ϕ be a quadratic form over F. Then $h_{sep}(\phi) \leq h(\phi)$, and $SSP(\phi) = \{\mathfrak{j}_i(\phi) \mid 0 \leq i \leq h_{sep}(\phi)\}$.*

Proof. Without loss of generality, we assume ϕ is an anisotropic quadratic form. Let K/F be any separable field extension such that $i_t(\phi_K) > 0$. By Lemma 3.4.6, we have $i_W(\phi_K) = i_t(\phi_K)$. So there exists an integer m such that $i_t(\phi_K) \leq \mathfrak{j}_m(\phi)$. Now we choose m as the smallest integer satisfying the property. Since $i_W(\phi_K) = i_t(\phi_K) > \mathfrak{j}_{m-1}(\phi)$, by Lemma 3.5.6, we have $i_t(\phi_K) = \mathfrak{j}_m(\phi)$. This finishes our proof. □

Corollary 3.5.8. *Let ϕ be a quadratic form over F. If ϕ is non-degenerate, then $FSP(\phi) = SSP(\phi) = KSP(\phi)$.*

Proof. Without loss of generality, we assume ϕ is anisotropic.

Since ϕ is non-degenerate, for any field extension K/F, we have $i_t(\phi_K) = i_W(\phi_K)$. By the same argument in the proof of Corollary 3.5.7, we know that $FSP(\phi) = KSP(\phi)$. Also, let $F_i, i \in [1, h(\phi)]$ be the standard splitting tower. Then each F_i is also a separable extension of F. By Corollary 3.5.7, we conclude $FSP(\phi) = SSP(\phi) = KSP(\phi)$. □

Theorem 3.5.9. [2, Karpenko, Primozic, Scully, Proposition 79.4] *Let ϕ be an anisotropic quadratic form of dimension larger than or equal to 2, and s be the smallest non-negative integer such that $\mathfrak{i}_1(\phi) \leq 2^s$. Then $dim(\phi) - \mathfrak{i}_1(\phi)$ is divisible by 2^s.*

Corollary 3.5.10. [6, Lemma 4.1] *Let ϕ be an anisotropic quadratic form of dimension ≥ 2 over F. If $dim(\phi) = 2^n + m$ for some non-negative integer n and integer $1 \leq m \leq 2^n$, then $\mathfrak{i}_1(\phi) \leq m$.*

Proof. Let $\phi \succ \psi$ with $dim(\psi) = dim(\phi) - \mathfrak{i}_1(\phi) + 1$. Notice that ψ is anisotropic over F since ϕ is anisotropic. Now we consider the vector space $V_{\phi_{F(\phi)}}$. Since $dim(\psi) + \mathfrak{i}_1(\phi) > dim(\phi)$, we know that $V_{\psi_{F(\phi)}}$ contains at least one isotropic vector. Hence $\psi_{F(\phi)}$ is isotropic. By the separation theorem, we know that $dim(\psi) \geq 2^n$. However, when $dim(\psi) = 2^n$, we have that $dim(\phi) - \mathfrak{i}_1(\phi) = 2^n - 1$. By Theorem 3.5.9, if s is the smallest non-negative integer such that $\mathfrak{i}_1(\phi) \leq 2^s$, then we conclude that $s = 0$ and $dim(\phi) = 2^n$, which contradicts our assumption. Finally, we have $dim(\psi) > 2^n$ and $\mathfrak{i}_1(\phi) \leq m$. □

Definition 3.5.11. Let ϕ be an anisotropic quadratic form of dimension ≥ 2 over F and $dim(\phi) = 2^n + m$ for some non-negative integer n and integer $1 \leq m \leq 2^n$. We say ϕ has *maximal splitting* if $\mathfrak{i}_1(\phi) = m$.

Example 3.5.12. Here we give an example to illustrate Theorem 3.5.9 in determining the first index of anisotropic forms. Let ϕ be an anisotropic quadratic form over F. We let $dim(\phi) = 2^n + m$ for non-negative integers n and $1 \leq m \leq 2^n$. By Corollary 3.5.10, we have that $\mathfrak{i}_1(\phi) \leq m$. Now we assume $\mathfrak{i}_1(\phi) > m/2$. Then we claim that $\mathfrak{i}_1(\phi) = m$.

To see this, notice that $dim(\phi) - \mathfrak{i}_1(\phi) < 2^n + m - m/2 = 2^n + m/2$. We take the smallest integer s such that $\mathfrak{i}_1(\phi) \leq 2^s$. By Theorem 3.5.9, we have that $2^s \mid dim(\phi) - \mathfrak{i}_1(\phi)$. Of course, $2^s \mid 2^n$. However, 2^s cannot divide any integer less than $m/2$ since $m/2 < \mathfrak{i}_1(\phi) \leq 2^s$. Thus it forces $\mathfrak{i}_1(\phi) = m$, i.e., ϕ has maximal splitting.

We give examples of splitting patterns of some special quadratic forms.

Example 3.5.13. Let $a\pi$ be an anisotropic general Pfister form over F and $a \in F^\times$. We suppose $dim(a\pi) = 2^n$ for some positive integer n. Since $a\pi$ is a non-degenerate quadratic form, the three splitting patterns are the same by Corollary 3.5.8. Since $F(a\pi) = F(\pi)$ and $\pi_{F(\pi)}$ is hyperbolic by Theorem 3.2.18, the splitting pattern of $a\pi$ is $(0, 2^{n-1})$.

Example 3.5.14. Let $\phi \simeq x_1[1, y_1] \perp \cdots \perp x_r[1, y_r] \perp \langle z_1, \cdots, z_s \rangle$ be an anisotropic quadratic form over $F(x_1, y_1, \cdots, x_r, y_r, z_1, \cdots, z_s)$ and the characteristic of F be 2. Quadratic forms of this kind are called *generic forms*.

Now we consider the full splitting pattern of ϕ. Let $K_1 = F(x_1[1, y_1])$. Actually, K_1 is the function field of the subform $x_1[1, y_1]$. Then $i_t(\phi_K) = 1$. Then we let $K_2 = K_1 \cdot F(x_2[1, y_2])$, and we have $i_t(\phi_{K_1}) = 2$. Similarly, we set $K_i = K_{i-1} \cdot F(x_i[1, y_i])$ for $i = 1, \cdots, r$. When $r < i \leq s + r - 1$, we set $K_i = K_r(\sqrt{z_1}, \sqrt{z_2}, \cdots, \sqrt{z_{i-r+1}})$.

As a result, we have that $i_t(\phi_{K_i}) = i$ for $1 \leq r \leq r+s-1$. Since this goes through all possible integers of the total index of ϕ under any field extension, we conclude that the full spitting pattern of ϕ is $(0, 1, 2, \cdots, r+s-1)$.

3.6 Stably birational equivalence

Fix a field F.

Definition 3.6.1. Let ϕ and ψ be two anisotropic quadratic forms over F with dimension larger than or equal to 2. We say ϕ and ψ are *stably birational equivalent* if $\phi_{F(\psi)}$ and $\psi_{F(\phi)}$ are isotropic forms and denote it by $\phi \overset{st}{\sim} \psi$.

Definition 3.6.2. Let X and Y be two varieties over F. We say X and Y are *stably birational* if there exist integers m, n such that $X \times \mathbb{A}^n$ is birational to $Y \times \mathbb{A}^m$.

Proposition 3.6.3. *Let ϕ and ψ be anisotropic quadratic forms of dimension ≥ 2 over F. Then the following statements are equivalent:*

(1) $\phi \overset{st}{\sim} \psi$;

(2) There exist F-places $F(\phi) \rightharpoonup F(\psi)$ and $F(\psi) \rightharpoonup F(\phi)$.

(3) For any field extension K/F, ϕ_K is isotropic if and only if ψ_K is isotropic.

(4) X_ϕ and X_ψ are stably birational equivalent as varieties over F.

Proof. (1) $\Rightarrow$ (2): Since ϕ is isotropic over $F(\psi)$, by Proposition 3.4.8, there exists an F-place $F(\phi) \rightharpoonup F(\psi)$. Similarly, since ψ is isotropic over $F(\phi)$, we also have an F-place $F(\psi) \rightharpoonup F(\phi)$.

(2) $\Rightarrow$ (3): Suppose ϕ_K is isotropic, then we have an F-place $F(\phi) \rightharpoonup K$ by Proposition 3.4.8. Since we have an F-place $F(\psi) \rightharpoonup F(\phi)$, we then obtain an F-place $F(\psi) \rightharpoonup K$ by composing them. Thus ψ_K is isotropic by Corollary 3.3.9. The proof is same when ψ_K is isotropic.

(3) $\Rightarrow$ (1): This is trivially true by taking $K = F(\phi)$ and $K = F(\psi)$

(4) $\Rightarrow$ (1): Since X_ϕ and X_ψ are stably birational equivalent, there exists a field E such that $E/F(\phi)$ and $E/F(\psi)$ are purely transcendental extensions. Since $\phi_{F(\phi)}$ is isotropic and $F(\phi) \subset E$, ϕ_E is also isotropic. By Lemma 3.4.3, we conclude that ϕ is also isotropic over $F(\psi)$. Similarly, we also have $\psi_{F(\phi)}$ is isotropic. This means $\phi \overset{st}{\sim} \psi$.

(1) $\Rightarrow$ (4): The equivalence of (1) and (4) for totally singular forms is proved by Totaro [19, Theorem 6.5]. We can exclude the case that one form is totally singular while another form is not totally singular by Proposition 3.4.5 and Lemma 3.4.6. For anisotropic quadratic forms which are not totally singular, we prove as follows.

Notice that under our assumption, ϕ and ψ are irreducible quadratic form. Also, since ϕ and ψ are not totally singular forms, $F(\phi)$ and $F(\psi)$ are separable extensions over F by Proposition 3.4.5. Thus ϕ_{ts} and ψ_{ts} stay anisotropic over $F(\phi)$ and $F(\psi)$.

Since $\phi_{F(\psi)}$ is isotropic, then the function field $F(\phi_\psi) = F(\phi) \cdot F(\psi)$ is a purely transcendental extension of $F(\psi)$ by Proposition 3.4.4. Also, $F(\psi_{F(\phi)}) = F(\phi) \cdot F(\psi)$ is a purely transcendental extension of $F(\phi)$ by the same argument. Thus we have: $F(X_\phi \times \mathbb{A}^n) = F(\phi) \cdot F(\psi) = F(X_\psi \times \mathbb{A}^m)$ for some positive integers m, n. Hence $X_\phi \times \mathbb{A}^n$ is birational with $X_\psi \times \mathbb{A}^m$ and we conclude that X_ϕ and X_ψ are stably birational equivalent. □

In the last of this section, we state two important results. The non-degenerate case of Theorem 3.6.4 is due to Vishik while the singular case is due to Totaro. For Theorem 3.6.5, Karpenko-Merkurjev did the non-degenerate case and Totaro did the singular case in [19].

Theorem 3.6.4. [2, Theorem 76.5] *Let ϕ and ψ be two anisotropic quadratic forms of dimension larger than or equal to 2. If $\phi \overset{st}{\sim} \psi$, then $dim(\phi) - \mathfrak{i}_1(\phi) = dim(\psi) - \mathfrak{i}_1(\psi)$.*

Theorem 3.6.5. [2, Theorem 76.5] *Let ϕ and ψ be two anisotropic quadratic forms of dimension larger than or equal to 2. If $\phi_{F(\psi)}$ is isotropic, then $dim(\psi) - \mathfrak{i}_1(\psi) \leq dim(\phi) - \mathfrak{i}_1(\phi)$. Moreover, if $dim(\psi) - i_1(\psi) = dim(\phi) - i_1(\phi)$, then $\psi \overset{st}{\sim} \phi$.*

3.7 Pfister neighbours

Fix a field F.

Definition 3.7.1. Let ϕ be an anisotropic quadratic form of dimension ≥ 2 over F. If ϕ is stably birational equivalent to a Pfister form, then we say that ϕ is a *Pfister neighbour*.

The following proposition gives equivalent definitions of Pfister neighbour.

Proposition 3.7.2. *Let ϕ be an anisotropic quadratic form of dimension ≥ 2 over F. Then the following are equivalent:*

(1) ϕ is a Pfister neighbour

(2) There exists a quadratic Pfister form π over F and a scalar $a \in F^{\times}$ such that $\phi \prec a\pi$ and $dim(\phi) > \frac{1}{2}dim(\pi)$.

Moreover, when this holds, the forms π and $a\pi$ in (2) are unique up to isometry.

Proof. (1) $\Rightarrow$ (2): Suppose ϕ is a Pfister neighbour stably birational to a quadratic Pfister form π. Since $\pi_{F(\phi)}$ is hyperbolic, by the domination theorem, $a\phi \prec \pi$ for some $a \in F^{\times}$. Multiplying the scalar a on both sides, we know that $\phi \prec a\pi$. Here we suppose $dim(a\pi) = 2^n$ for some positive integer n. If $dim(\phi) \leq \frac{1}{2}dim(\pi)$, then $dim(\phi) \leq 2^{n-1} < 2^n = dim(a\pi)$. By the Separation Theorem (Theorem 3.4.10), we have that $\phi_{F(\pi)}$ stays anisotropic, which is a contradiction.

(2) $\Rightarrow$ (1): Since $\phi \prec a\pi$, then $a\pi_{F(\phi)}$ is isotropic since $V_{\phi_{F(\phi)}}$ contains an isotropic vector. As a result, $\pi_{F(\phi)}$ is isotropic. Also, since $dim(\phi) > \frac{1}{2}dim(\pi)$ and any maximal totally isotropic space of $a\pi_{F(\pi)}$ has dimension $\frac{1}{2}dim(\pi)$, then $V_{\phi_{F(\pi)}}$ must contain at least one isotropic vector of $V_{a\pi_{F(\pi)}}$. Thus $\phi_{F(\pi)}$ is also isotropic. This proves that $\phi \overset{st}{\sim} \pi$.

When this holds, we suppose that ϕ is dominated by two general Pfister forms, i.e., $a_1\pi_1 \succ \phi$ and $a_2\pi_2 \succ \phi$. Since $\phi_{F(\pi_2)}$ is isotropic and $a_1\pi_1 \succ \phi$, then $(a_1\pi_1)_{F(\pi_2)}$ is isotropic. Notice that $F(a_2\pi_2) = F(\pi_2)$ and they all dominate ϕ, then we have that $a_1\pi_1 \succ a_2\pi_2$ by the Domination Theorem (Theorem 3.4.9). This implies that $a_1\pi_1 \simeq a_2\pi_2$ since they have the same dimension.

In addition, by multiplying scalars, we have that $a_1a_2\pi_1 \simeq \pi_2$ and $\pi_1 \simeq a_1a_2\pi_2$. Since $\pi_1 \simeq a_1a_2\pi_2$, for any $x \in D(\pi_1)$, there exists a $y \in D(\pi_2)$ such that $x = a_1a_2y$. Now we fix a $x_0 \in D(\pi_1)$ and $x_0 = a_1a_2y_0$ for some $y_0 \in D(\pi_2)$. Then we have that $a_1a_2\pi_1 \simeq a_1a_2x_0\pi_1 \simeq y_0\pi_1$ by Proposition 3.2.18. Now we have that $\pi_1 \simeq y_0\pi_2$. Since $y_0 \in D(\pi_2)$, we still use Proposition 3.2.18 and conclude $\pi_1 \simeq \pi_2$. □

Proposition 3.7.3. *Pfister neighbours have maximal splitting.*

Proof. Let ϕ be a Pfister neighbour with dimension $2^n + m$ for some positive integer n and integer $1 \leq m \leq 2^n$. Let π be the quadratic Pfister form stably birational to ϕ. Then by Proposition 3.7.2, we have that $dim(\pi) = 2^{n+1}$. By Theorem 3.6.4, we have that $dim(\phi) - \mathfrak{i}_1(\phi) = dim(\pi) - \mathfrak{i}_1(\pi) = 2^n$. Hence $\mathfrak{i}_1(\phi) = m$. This shows that Pfister neighbours have maximal splitting. □

Let ϕ be a Pfister neighbour over F and π be the general quadratic form dominating ϕ. Recall that by Corollary 3.2.26, we always have $-\phi \perp \pi \sim \phi^C_\pi$.

Since π is uniquely determined by ϕ up to isometry by Proposition 3.7.2, then the complement of ϕ in π is also uniquely determined by ϕ. We call ϕ_π^C the *complement* of a Pfister neighbour and denote it by ϕ^C for short.

Definition 3.7.4. Let ϕ be a quadratic form over K and K/F be a field extension. We say the quadratic form ϕ is *defined over* F if there exists a quadratic form ψ over F such that $\phi \simeq \psi_K$.

Proposition 3.7.5. *Let ϕ be a Pfister neighbour. Then the anisotropic part of ϕ over $F(\phi)$ is defined over F. Also, we have $(\phi_{F(\phi)})_{an} \simeq -\phi^C_{F(\phi)}$.*

Proof. Let π be the unique general quadratic form which dominates ϕ. Thus we know $\pi \perp -\phi \sim \phi^C$ from our discussion before. Under the field extension $F(\phi)$, π becomes hyperbolic. Thus we have $(\phi_{F(\phi)})_{an} \sim -\phi^C_{F(\phi)}$. But one should notice that $dim(\phi^C) = dim(\pi) - dim(\phi) < 2^{n-1} < dim(\phi)$ for some integer n. Thus by the separation theorem, ϕ^C stays anisotropic over $F(\phi)$ and we conclude that $(\phi_{F(\phi)})_{an}$ is defined over F. □

Lemma 3.7.6. *Suppose the characteristic of F is 2. Let σ be a totally singular form over F and E/F be a field extension. Then the anisotropic part of σ_E is defined over F.*

Proof. Without loss of generality, we assume that σ is anisotropic and σ_E is isotropic. Let σ be represented by the polynomial $f = c_1x_1^2 + \cdots + c_sx_s^2$ for $c_1, \cdots, c_s \in F$.

There exist elements $m_1^{'}, \cdots, m_s^{'} \in E$ such that $m_1^{'2}c_1 + \cdots + m_s^{'2}c_s = 0$. Since an isotropic vector is not a zero vector, we assume $m_s^{'} \neq 0$. Finally, there exist elements $m_1, \cdots, m_{s-1} \in E$ such that $c_s = c_1m_1^2 + \cdots + c_{s-1}m_{s-1}^2$. Notice that $m_1, \cdots, m_{s-1}$ cannot be all zero since σ is anisotropic over F. Now we have that $\sigma_E \simeq \langle c_1, \cdots, c_{s-1}, c_1m_1^2 + \cdots + c_{s-1}m_{s-1}^2\rangle$. Thus by setting $z_i = x_i + m_ix_s$ for $i = 1, \cdots, s-1$, we have that $\sigma_E \simeq \langle c_1, \cdots, c_{s-1}, 0\rangle$, which is defined over F. Inductively, we conclude that $(\sigma_E)_{an}$ is defined over F. □

When the characteristic of F is not 2, it's well-known that the converse of Proposition 3.7.5 holds for any anisotropic quadratic form over F [2, Theorem 28.3]. But when the characteristic of F is 2, the converse statement fails in general for anisotropic quadratic forms. To illustrate this, Hoffmann and Laghribi gave the following examples. We give a definition first.

Definition 3.7.7. Suppose the characteristic of F is 2. Let ϕ be a Pfister neighbour of (r,s)-type and π be the general Pfister form dominating ϕ of dimension 2^n for some positive integer n. We define ϕ to be a *close Pfister neighbour* if $2r+2s=2^n$.

In the following examples, we always assume the characteristic of F is 2.

Example 3.7.8. [7, Example 6.2] Let q be an anisotropic totally singular form over F. Then $(q_{F(q)})_{an}$ is defined over F by Lemma 3.7.6. But q cannot be a Pfister neighbour. Actually, suppose q is a Pfister neighbor of type (r,s) and q is dominated by the general Pfister form π. Suppose $dim(\pi)=2^n$ for some positive integer n. By Corollary 3.2.23, we have $2^n \geq 2r+2s$. Also, since q is a Pfister neighbour, then $dim(q)=2r+s>2^{n-1}$. Notice that q is a totally singular form, which means $r=0$. Hence we have $s>2^{n-1}$ and $s\leq 2^{n-1}$, which is a contradiction.

Example 3.7.9. [7, Example 6.3] Let q be an anisotropic quadratic form of type $(1,s)$. Then $F(q)$ is a separable extension and $(q_{F(q)})_{an} \simeq (q_{ts})_{F(q)}$ by Lemma 3.4.3 and Proposition 3.4.5. Notice that $(q_{ts})_{F(q)}$ is a totally singular form, hence it is defined over F by Lemma 3.4.6.

Now we suppose that q is a Pfister neighbour and q is dominated by the general Pfister form π. Let $dim(\pi)=2^n$, then $s+1=2^{n-1}$ by Corollary 3.2.24. Thus if q is an anisotropic quadratic form of type $(1,s)$ and $s+1$ is not a 2-power, then q is not a Pfister neighbour.

Example 3.7.10. [7, Example 6.4] Let $\pi=\langle\langle a_1,\cdots,a_n;b]]$ be an $(n+1)$-fold quadratic Pfister form and $\tau=\langle\langle a_1,\cdots,a_m\rangle\rangle_b$ be a m-fold bilinear Pfister form on V with $m>n$. We can view τ as a totally singular form over F by taking the diagonal part of $V\times V$. So we assume that τ is an anisotropic quadratic form and just write $\tau=\langle\langle a_1,\cdots,a_m\rangle\rangle$.

Suppose $\phi=\pi\perp x\tau$ for some $x\in F^\times$. Let E/F be any field extension such that τ_E stays ansiotropic and ϕ_E is isotropic. Since ϕ_E is isotropic, there exists a $y\in D(\pi_E)$ such that $y=xz$ for some $z\in D(x\tau)$. Since $y\pi_E\simeq\pi_E$ and $xz\tau_E\simeq y\tau_E\simeq x\tau_E$ by Proposition 3.1.12 and Proposition 3.2.18, we have that $\phi_E\simeq y(\pi_E\perp\tau_E)$.

We expand π as an orthogonal sum of forms $(\prod_i a_i)[1,b]$ and we expand τ as an orthogonal sum of the forms $\langle\prod_j a_j\rangle$. Notice that $[u,v]\perp\langle u\rangle\simeq\mathbb{H}\perp\langle u\rangle$, we then have that $\pi\perp\tau\sim\tau$, thus $y(\pi_E\perp\tau_E)\sim y\tau_E\simeq x\tau_E$.

Now we let ψ be a Pfister neighbour of π and $\eta=\psi\perp x\tau$ form some $x\in F^\times$. If E is a field extension such that τ_E is anisotropic and η_E is isotropic, we have that:

$\pi_E \perp \psi_E \perp x\tau_E \sim \psi_E \perp (\pi_E \perp x\tau_E) \sim \psi_E \perp x\tau_E$ by discussion above. Also, we have that $(\pi_E \perp \psi_E) \perp x\tau_E \sim \psi_E^C \perp x\tau_E$ by Corollary 3.2.26. Thus we have: $\eta_E \sim \psi_E^C \perp x\tau_E$.

Now we set $E = F(\eta)$. By our discussion, $\eta_E \sim (\psi^C \perp x\tau)_E$. When ψ is a close Pfister neighbour, $(\psi^C \perp x\tau)_E$ is a totally singular form. Since ψ is not a totally singular form, η is either not a totally singular form. Thus $(\psi^C \perp x\tau)$ stays anisotropic by Lemma 3.4.3. Thus η is defined over $F(\eta)$.

However, η cannot be a Pfister neighbour. Actually, we suppose η is a Pfister neighbour. Since $2^m + 2^n < dim(\eta) \leq 2^m + 2^{n+1} \leq 2^{m+1}$, so η is dominated by a $(m+1)$-fold general Pfister form. Suppose η is of (r,s)-type, then $r + 2s \leq 2^{m+1}$. But this is a contradiction as $r \geq 1$ and $s \geq 2^m$.

However, Hoffmann and Laghribi proved the following result:

Theorem 3.7.11. [7, Theorem 6.6] *Suppose the characteristic of F is 2. Let q be an anisortopic quadratic form such that the anisotropic part of q over $F(q)$ is defined over F. Also, we suppose that q is of type (r,s) with $s < 2r$. When $s \leq 5$ or $s = 5$ the totally singular part of q is similar to $\langle\langle a,b\rangle\rangle \perp \langle c\rangle$ for some $a,b,c \in F$, q is a Pfister neighbour.*

Conjecture 3.7.12. [7, Conjecture 6.5] *Suppose the characteristic of F is 2. Let q be an anisotropic form such that the anisotropic part of q over $F(q)$ is defined over F. If q is of type (r,s) with $s < 2r$, then q is a Pfister neighbour.*

3.8 Excellent and strongly excellent forms

Fix a field F.

Definition 3.8.1. Let ϕ be a quadratic form over F. We say that ϕ is an *excellent form* if $(\phi_E)_{an}$ is defined over F for any field extension E/F.

Definition 3.8.2. Let ϕ be a quadratic form over F. We say that ϕ is *strongly excellent* if there exists Pfister neighbours $q_0, q_1, \cdots, q_{h_{sep}(\phi)-1}$ such that:

(1) $q_0 = \phi_{an}$.

(2) $q_i^C = q_{i+1}$ for all $0 \leq i \leq h_{sep}(\phi) - 2$

(3) $q_{h_{sep}(\phi)-1}^C = 0$ or $q_{h_{sep}(\phi)-1}^C = \langle c\rangle, c \in F^\times$ if the characteristic of F is not 2, and equal to $(\phi_{ts})_{an}$ if the characteristic of F is 2.

Proposition 3.8.3. *Let ϕ be a strongly excellent quadratic form over F and a sequence of forms $q_0, \cdots, q_{h_{sep}(\phi)-1}$ be Pfister neighbours as in the definition above, then*

(1) If $1 \leq r \leq h_{sep}(\phi) - 1$, then $(-1)^r \phi_r \simeq (q_r)_{F_r}$ where $F_0 \subset \cdots \subset F_r \subset \cdots$ is the Knebusch splitting tower of ϕ, and ϕ_r is the rth higher kernel form of ϕ.

(2) ϕ is an excellent form.

Proof. (1): For Pfister neighbours $q_0, \cdots, q_{h_{sep}(\phi)-1}$, we suppose the associated general Pfister forms are $\pi_0, \cdots, \pi_{h_{sep}(\phi)-1}$ respectively. Obviously, $q_0 = \phi_{an} = \phi_1$ and $F(q_0) = F_1$. Notice that $\pi_0 \perp (-\phi_0) \sim q_0^C = q_1$, we have that $(\pi_0 \perp (-q_0))_{F_1} \sim (q_0^C)_{F_1}$. Taking anisotropic parts on both sides, we have that $(q_1)_{F_1} \simeq ((-q_0)_{F_1})_{an} \simeq -\phi_1$ by Proposition 3.7.5. Now we consider $(-q_1)_{F_1} \perp (\pi_1)_{F_1} \sim (q_2)_{F_1}$. Under the field extension $F((-q_1)_{F_1}) = F(\phi_1) = F_2$, we have $((\phi_1)_{F_2})_{an} \simeq ((q_2)_{F_1})_{F_2}$, which is $\phi_2 \simeq (q_2)_{F_2}$ since $F_1 \subset F_2$. The proposition is proved by continuing the process finite times.

(2): When $char(F) \neq 2$, since $\pi_i, i \in [0, h_{sep}(\phi) - 1]$ is non-degenerate, we have $\pi_i = q_i \perp q_{i+1}$ for $i \in [0, \cdots, h_{sep}(\phi) - 1]$ and $q_{h_{sep}(\phi)} := q^C_{h_{sep}(\phi)-1}$. Eliminate the common terms between π_i and π_{i+1}, we have that

$$q_0 \sim \pi_0 \perp (-1)\pi_1 \perp \cdots \perp (-1)^{h_{sep}(\phi)-1}\pi_{h_{sep}(\phi)-1} \perp (-1)^{h_{sep}(\phi)} q_{h_{sep}(\phi)}.$$

When the $char(F) = 2$, using the relation $q_i \perp \pi_i \sim q_{i+1}$, we have that :

$$q_0 \sim \pi_0 \perp \pi_1 \perp \cdots \perp \pi_{h_{sep}(\phi)-1} \perp q_{h_{sep}(\phi)}.$$

Notice the definition of $q^C_{h_{sep}-1}$, we have the following relations in general:

$$\begin{aligned} q_0 &\sim \pi_0 \perp -\pi_1 \perp \cdots \perp (-1)^{h_{sep}(\phi)-1}\pi_{h_{sep}(\phi)-1} \perp (-1)^{h_{sep}} q_{h_{sep}(\phi)} && char(F) \neq 2 \\ q_0 &\sim \pi_0 \perp -\pi_1 \perp \cdots \perp (-1)^{h_{sep}(\phi)-1}\pi_{h_{sep}(\phi)-1} \perp (-1)^{h_{sep}(\phi)} (\phi_{ts})_{an} && char(F) = 2 \end{aligned}$$

Let K/F be any field extension. If π_i becomes hyperbolic for any $i \in [0, h_{sep}(\phi) - 1]$. When the characteristic of F is not 2, then $(\phi_K)_{an}$ is a zero form or $(\phi_K)_{an} \simeq \langle c \rangle$ for some $c \in F^\times$, hence is defined over F. When the characteristic of F is 2, then (ϕ_K) is Witt equivalent to a totally singular form. Then $(\phi_K)_{an}$ is defined over F by Lemma 3.7.6. As a result, ϕ is an excellent form.

If there exists π_i such that $(\pi_i)_K$ is not hyperbolic, we take s to be the smallest integer with this property. We still use the relation $\pi_i \perp -q_i \sim q_{i+1}$ and obtain that $\phi_K, (q_0)_K, (q_1)_K, \cdots, (q_s)_K$ are Witt-equivalent. Since $(q_s)_K$ is a Pfister neighbour of an anisotropic general Pfister form $(\pi_s)_K$, so $(q_s)_K$ is also anisotropic. As a result, $(\phi_K)_{an} \simeq (q_s)_K$. This showed that $(\phi_K)_{an}$ is defined over F.

Combining these two cases, we claim that ϕ is an excellent form. □

Proposition 3.8.4. *Let ϕ be an anisotropic quadratic form over F. Then the following are equivalent:*

(1) ϕ is strongly excellent.

(2) There exist anisotropic Pfister forms $\pi_0, \cdots, \pi_{h_{sep}(\phi)-1}$ over F, a form $q_{h_{sep}(\phi)}$ over F and a scalar $a \in F^\times$ such that :

1.$\phi_{an} \sim a(\pi_0 \perp -\pi_1 \perp \cdots \perp (-1)^{h_{sep}(\phi)-1}\pi_{h_{sep}(\phi)-1} \perp (-1)^{h_{sep}(\phi)}q_{h_{sep}(\phi)})$

2.$\pi_r \subset \pi_{r-1}$ *for all* $r \in [0, h_{sep}(\phi) - 1]$

3.$q_{h_{sep}(\phi)} \prec \pi_{h_{sep}(\phi)-1}$ *with* $dim(q_{h_{sep}(\phi)}) < (1/2)\pi_{h_{sep}(\phi)-1}$

4.*if* $char(F) = 2$, $aq_{h_{sep}(\phi)} \simeq \phi_{ts}$.

if $char(F) \neq 2$, $aq_{h_{sep}(\phi)} = 0$ *or* $aq_{h_{sep}(\phi)} \simeq \langle c\rangle, c \in F^\times$

5.$dim(\pi_0) > dim(\pi_1) > \cdots > dim(\pi_{h_{sep}(\phi)-1})$.

Proof. (1) ⇒ (2): Suppose ϕ is a strongly excellent form and $\phi_{an} = q_0, \cdots, q_{h_{sep}(\phi)-1}$ are the Pfister neighbours as in definition. To simplify our notations, we denote $s := h_{sep}(\phi) - 1$. In the proof of Proposition 3.8.3, we know there exist general Pfister forms $x_0\pi_0, \cdots, x_s\pi_s$ such that:

$$q_0 \sim x_0\pi_1 \perp -x_1\pi_1 \perp x_2\pi_2 \perp \cdots \perp (-1)^s x_s\pi_s \perp (-1)^{s+1}q_s^C.$$

for $x_1, \cdots, x_s \in F^\times$. At first, since $q_{i+1} = q_i^C$ and q_{i+1} and q_i are all Pfister neighbours, we have that $dim(q_{i+1}) > \frac{1}{2}dim(x_{i+1}\pi_{i+1}) = \frac{1}{2}dim(\pi_{i+1})$. Also, since $q_{i+1} = q_i^C$ and $dim(q_i) > \frac{1}{2}dim(x_i\pi_i) = \frac{1}{2}dim(\pi_i)$, we conclude that $dim(\pi_i) > dim(\pi_{i+1})$ for $i \in [0, s-1]$.

By definition, notice that $q_i^C = q_{i+1}$ for $1 \leq i \leq s-1$, so $x_i\pi_i \succ q_{i+1}$. Since q_{i+1} is a Pfister neighbour of $x_{i+1}\pi_{i+1}$, then q_{i+1} is isotropic over the field extension $F(x_{i+1}\pi_{i+1}) = F(\pi_{i+1})$. Thus $(x_i\pi_i)_{F(\pi_{i+1})}$ is isotropic, hence hyperbolic. But $x_i\pi_i$ is anisotropic, thus $(\pi_i)_{F(\pi_{i+1})}$ must be hyperbolic. Since $1 \in D(\pi_i) \cap D(\pi_{i+1})$, by the Domination Theorem (Theorem 3.4.9), we conclude that $\pi_{i+1} \subset \pi_i$ for $i \in [0, s-1]$.

Now we take an element $a \in D(q_s)$. Since $q_{s-1}^C = q_s$ and $x_s\pi_s \succ q_s$, we know that q_s is dominated by $x_{s-1}\pi_{s-1}$ and $x_s\pi_s$. Thus there exist $y_{s-1} \in D(\pi_{s-1})$ and

$y_s \in D(\pi_s)$ such that $x_{s-1}y_{s-1} = a = x_s y_s$. By Proposition 3.2.18, we have that $x_{s-1}\pi_{s-1} \simeq x_{s-1}y_{s-1}\pi_{s-1} \simeq a\pi_{s-1}$. Similarly, we also have: $x_s\pi_s \simeq a\pi_s$. Now we use induction. Suppose that $x_{s-i+1}\pi_{s-i+1} \simeq a\pi_{s-i+1}$ for some i. Consider $x_{s-i}\pi_{s-i}$, we take a value $b \in D(q_{s-i})$. By the same argument, we have that $x_{s-i}\pi_{s-i} \succ q_{s-i}$ and $a\pi_{s-i+1} \succ q_{s-i}$. So there exist $y_{s-i} \in D(\pi_{s-i})$ and $y_{s-i+1} \in D(\pi_{s-i+1})$ such that $x_{s-i}y_{s-i} = b = ay_{s-i+1}$. Hence $x_{s-i}\pi_{s-i} \simeq ay_{s-i+1}\pi_{s-i}$. But since $\pi_{s-i+1} \subset \pi_{s-i}$, we have that $D(\pi_{s-i+1}) \subset D(\pi_{s-i})$, hence $ay_{s-i+1}\pi_{s-i} \simeq a\pi_{s-i}$. Thus we have conclude that

$$q_0 \sim a(\pi_0 \perp -\pi_1 \perp \pi_2 \perp \cdots \perp (-1)^s\pi_s \perp aq_s^C).$$

Now we take $q_{h_{sep}(\phi)} := aq_s^C$. Then $a \cdot aq_s^C$ obviously satisfies the condition 4. Notice that $q_s^C \prec a\pi_s$, hence we multiply $a \in F^\times$ on both sides and obtain $q_{h_{sep}(\phi)} \prec \pi_s$. Also, $dim(q_{h_{sep}(\phi)}) = dim(\pi_s) - dim(q_s)$, and we have $dim(q_{h_{sep}(\phi)}) < (1/2)\pi_s$.

(2) $\Rightarrow$ (1): Still , to simplify notations, we denote $s := h_{sep}(\phi) - 1$. Then we have

$$\phi_{an} \sim a\pi_0 \perp -a\pi_1 \perp a\pi_2 \perp \cdots \perp (-1)^s a\pi_s \perp aq_{s+1}.$$

Since $dim(q_{s+1}) < (1/2)\pi_s$ and $\pi_s \succ q_{s+1}$, we have that $(aq_{s+1})^C$ is a Pfister neighbour of $a\pi_s$, which we denote as aq_s. Since $\pi_s \subset \pi_{s-1}$, we have that $\pi_{s-1} \succ q_s$. Notice that $dim(\pi_{s-1}) > dim(\pi_s)$, hence $(aq_s)^C$ is a Pfister neighbour of $a\pi_{s-1}$, which we denote as aq_{s-1}. Inductively, we have Pfister neighbours aq_i of $a\pi_i$ for $i = 0, \cdots, s$. Also, we have that $(aq_i)^C = aq_{i+1}$ for $i \in [0, s-1]$.

Now we denote $a\pi_i$ as π_i' for $i \in [0, s]$ and aq_i as q_i' for $i \in [0, s+1]$, we have:

$$\phi_{an} \sim \pi_0' \perp -\pi_1' \perp \pi_2' \perp \cdots \perp (-1)^s\pi_s' \perp q_{s+1}'$$

Using the relation $\pi_i' \perp -q_i' \sim q_{i+1}'$, we consider $\phi_{an} \perp (-q_0')$ as follows:

$$\begin{aligned}
\phi_{an} \perp (-q_0') &\sim \pi_0' \perp (-q_0') \perp \cdots \perp (-1)^{s+1}q_{s+1}' \\
&\sim (-1)(\pi_1 - q_1') \perp \pi_2 \perp \cdots \perp (-1)^s\pi_s' \perp (-1)^{s+1}q_{s+1}' \\
&\sim (-1)^s(\pi_s' - q_s') \perp (-1)^{s+1}q_{s+1}' \\
&\sim (-1)^s q_{s+1}' \perp (-1)^{s+1}q_{s+1}' \\
&\sim 0.
\end{aligned}$$

Thus we have that $\phi_{an} \simeq q_0'$ since π_0' is anisotropic. Finally, by condition (4), we conclude that ϕ is a strongly excellent form. □

Remark 3.8.5. For non-degenerate quadratic forms, an excellent form is equivalent to a strongly excellent form, which is proved in [2, Theorem 28.3]. But it fails for general quadratic forms. For example, a totally singular form is an excellent form, but it's not strongly excellent since a totally singular form cannot be a Pfister neighbour.

Let ϕ be a quadratic form over F and $char(F) = 2$. When the separable height of ϕ is 1, we have $\phi \simeq [a, b] \perp \phi_{ts}$ for some $a, b \in F$. Hence over any field extension such that $i_t(\phi_K) > 0$, $(\phi_K)_{an}$ is either a totally singular form or $(\phi_K)_{an} \simeq [a, b]_K \perp ((\phi_{ts})_K)_{an}$. In both cases, $(\phi_K)_{an}$ is defined over F, hence is an excellent form.

Now we assume ϕ is of type $(1, s)$. When $s+1$ is not a 2-power, by Example 3.7.9, ϕ is not a Pfister neighbour, hence not a strongly excellent form.

Definition 3.8.6. Let ϕ be a quadratic form over F and $\phi_0, \cdots, \cdots \phi_{h_{sep}(\phi)-1}$ be its kernel forms. If $\phi_i, i \in [0, h_{sep}(\phi) - 1]$ are all Pfister neighbours, we say ϕ is a *quasi-strongly excellent form.*

By Proposition 3.8.3, we know that any strongly excellent form is quasi-strongly excellent. But even for non-degenerate quadratic forms, we don't know whether the converse is true. Here we conjecture as below.

Conjecture 3.8.7. *All quasi-strongly excellent forms are strongly excellent.*

The separable splitting pattern of an anisotropic quasi-strongly excellent form is determined by its dimension. We illustrate it as below.

Lemma 3.8.8. [8, Lemma 2.6] *Every natural number n can be written uniquely as an alternating sum of 2-powers*

$$n = 2^{a_m} - 2^{a_{m-1}} + \cdots + (-1)^{m-1}2^{a_1} + (-1)^m\epsilon$$

with $\epsilon, m, a_1, \cdots, a_m \in \mathbb{N} \cup \{0\}$ satisfying $0 < a_1 < a_2 < \cdots < a_{m-1} < a_m$ and

$$\epsilon = \begin{cases} 0 \text{ and } a_1 < a_2 - 1 & \text{if } n \text{ is even} \\ 1 \text{ and } 1 < a_1 & \text{if } n \text{ is odd.} \end{cases}$$

For $n \in \mathbb{N}$, the expansion above is called the *alternating 2-expansion* of n. For $j = 0, \cdots, m$, the number $2^{a_j} - 2^{a_{j-1}} + \cdots \pm 2^{a_1} \mp \epsilon$ is called the *jth alternating 2-partial sums* of n and denoted by $n^{(j)}$.

Proposition 3.8.9. *Let ϕ be an anisotropic quasi-strongly excellent form over F and $\phi_1, \cdots, \phi_{h_{sep}(\phi)-1}$ be the kernel forms of ϕ. We suppose that $dim(\phi) = n$ and $2^{a_m} - 2^{a_{m-1}} + \cdots + (-1)^{m-1}2^{a_1} + (-1)^m\epsilon$ is the alternating 2-expansion of n, then the the separable splitting pattern of ϕ is given by the following formula*

$$\mathfrak{j}_k(\phi) = (n - n^{(m-k)})/2.$$

for $k \in [0, h_{sep}(\phi)]$.

Proof. We use induction to prove the proposition. Since ϕ is an anisotropic quadratic form, $\mathfrak{j}_0(\phi) = (n - n(m-0))/2 = (n-n)/2 = 0$. So the statement is true for $k = 0$. Now we suppose the statement is true when $i = k$, $k \in [0, \cdots, h_{sep(\phi)} - 1]$. So $\mathfrak{j}_k(\phi) = (n - n^{(m-k)})/2$. Since ϕ_k is a Pfister neighbour, by Proposition 3.7.5, we have that $((\phi_k)_{F_{k+1}})_{an} = \phi_k^C$. Notice that we have $dim(\phi_k) = n - 2\mathfrak{j}_k(\phi) = n^{(m-k)}$, thus the associated general Pfister form of ϕ_k has dimension $2^{a_{m-n}}$ and $dim(\phi_k^C) = n^{m-(k+1)}$. Thus we have:

$$\begin{aligned}\mathfrak{j}_{k+1}(\phi) &= \mathfrak{j}_k(\phi) + (dim(\phi_k) - dim(\phi_k^C))/2 \\ &= (n - n^{(m-k)} + n^{(m-k)} - n^{(m-k-1)}))/2 \\ &= (n - n^{(m-(k+1))})/2.\end{aligned}$$

This proves our formula. □

Chapter 4

Rational Cycles on Products of Generically Smooth Quadrics

Let F be a field and $\overline{F}$ be an algebraic closure of F. We suppose that $X_1, \cdots, X_r$ are generically smooth projective quadrics over F, and set $X := X_1 \times \cdots \times X_r$. In this chapter, we want to study the image $\overline{Ch}(X)$ of the change of field homomorphism $Ch(X) \to Ch(X_{\overline{F}})$. In the first three sections, we will adapt a method of Karpenko [11] to show that $\overline{Ch}(X)$ inherits a ring structure and the action of Steenrod operations from $Ch(X_{sm})$, where X_{sm} denote the smooth locus of X(Karpenko treated the special case where $X_1 = \cdots = X_r$).

In the fourth section, we will use the aforementioned ring structure to define a composition $\overline{Ch}(X \times Y) \times \overline{Ch}(Y \times Z) \to \overline{Ch}(X \times Z)$ for products of generically smooth quadrics X, Y, Z over F. This extends the usual results for smooth quadrics. In the final section, we will relate $\overline{Ch}(X)$ to $\overline{Ch}(X_{an})$, where X_{an} is the product of the anisotropic parts of $X_1, \cdots, X_r$.

Throughout this chapter, since we have particular interests in Chow groups with coefficients in $\mathbb{F}_2$, we will only consider this kind of Chow groups. But one should note that this condition is not necessary for our discussion in some results. For example, the results in the first two sections are the same if we replace Chow groups with coefficients in $\mathbb{F}_2$ with integral Chow groups.

4.1 Forms with maximal Witt index

Fix a field F. Let ϕ be a quadratic form over F with maximal Witt index r and $dim(\phi) \geq 2$. This means $\phi \simeq r\mathbb{H} \perp \phi'$ for some subform ϕ', where $\phi' \cong \langle c \rangle \perp \phi_{ts}$ for $c \in F^{\times}$ or $\phi' \cong \phi_{ts}$. Now we choose V' to be a maximal totally isotropic subspace of $V_{r\mathbb{H}}$ and V_{ts} to be the underlying vector space of ϕ_{ts}, where ϕ_{ts} is the totally singular part of ϕ. We set $W := V' + V_{ts}$ and notice that it is a direct sum.

Let X be the associated quadric of ϕ. For the form $\psi := \phi \mid_W$, we denote the associated quadric by Y. Here Y is a closed subvariety of X. Note that $X \cap \mathbb{P}(W) = Y$. Hence the canonical projection $\mathbb{P}(V) \setminus \mathbb{P}(W) \to \mathbb{P}(V/W)$ restricts to a morphism $f : X \setminus Y \to \mathbb{P}(V/W)$:

Lemma 4.1.1. *f is an affine bundle of rank $dim(\phi) - r - 1$.*

Proof. Let $Z = \{z_1, \cdots, z_{dim(\phi)-2r}\}$ be a set of variables. By choosing a suitable basis, we represent ϕ by a polynomial $x_1 y_1 + \cdots + x_r y_r + g(Z)$ where $g(Z)$ represents the subform ϕ' and the quadric Y is given by $x_1 = \cdots = x_r = 0 = g(Z)$.

Now for $i \in [1, r]$, we let $U_i \subset \mathbb{P}(V/W)$ be the affine open subscheme given by $\{x_i \neq 0\}$. Consider the scheme-theoretic fiber $f^{-1}(U_i)$, it is a subvariety of $X \setminus Y$. We consider the morphism

$$U_i \times \mathbb{A}^{dim(\phi)-r-1} \to f^{-1}(U_i)$$

which is given by sending

$$([b_1 : \cdots : b_{i-1} : 1 : b_{i+1} : \cdots : b_r], (c_1, \cdots, c_{i-1}, c_{i+1}, \cdots, c_r, d_1, \cdots : d_{dim(\phi)-2r}))$$

to

$$[b_1 : \cdots : b_{i-1} : 1 : b_{i+1} : \cdots : b_r : c_1 : \cdots : c_{i-1} : c_i : c_{i+1} : \cdots : c_r : d_1 : \cdots : d_{dim(\phi)-2r}]$$

where $c_i = -(b_1 c_1 + \cdots + b_{i-1} c_{i-1} + b_{i+1} c_{i+1} + \cdots + b_r c_r + g(d_1, \cdots, d_{dim(\phi)-2r}))$. This morphism is an isomorphism which makes the following diagram commute

$$\begin{array}{ccc} U_i \times \mathbb{A}^{dim(\phi)-r-1} & \xrightarrow{\sim} & f^{-1}(U_i) \\ \downarrow p & \swarrow f & \\ U_i & & \end{array}$$

(where $p : U_i \times \mathbb{A}^{dim(\phi)-r-1} \to U_i$ is the projection onto the first factor). Since the projection p is a flat morphism of rank $dim(\phi) - r - 1$, then f is also flat of rank $dim(\phi) - r - 1$. Obviously, the fiber of each point in U_i is isomorphic to an affine space. Hence we conclude that f is an affine bundle of rank $dim(\phi) - r - 1$. □

The lemma above shows that X admits a relative cellular structure $\emptyset \subset Y \subset X$ with $f : X \setminus Y \to \mathbb{P}(V/W)$ an affine bundle of rank $dim(\phi) - r - 1$. By Theorem 2.4.5, we have the following proposition

Proposition 4.1.2. *For any F-variety Z, we have*

$$Ch_*(X \times Z) \simeq Ch_*(Y \times Z) \oplus Ch_{*-(dim(\phi)-r-1)}(\mathbb{P}(V/W) \times Z).$$

Recall that we define $\psi := \phi \mid_W$. We denote the subspace $rad(\psi) \subset W$ as W_{rad}. Notice that $\mathbb{P}(W_{rad}) \subset Y$ is a closed subvariety of Y. We define $W_{an} := W/W_{rad}$. $\psi \mid_{W_{an}}$ is anisotropic and $\psi \mid_{W_{an}} = \psi_{an}$. Let Y_{an} be the associated quadric of ψ_{an} (it is possible that Y_{an} is empty).

Since $\psi \mid_{W_{rad}} = 0$, the canonical projection $\mathbb{P}(W) \setminus \mathbb{P}(W_{rad}) \to \mathbb{P}(W_{an})$ induces a morphism $s : Y \setminus \mathbb{P}(W_{rad}) \to Y_{an}$.

Lemma 4.1.3. *$s : Y \setminus \mathbb{P}(W_{rad}) \to Y_{an}$ is an affine bundle of rank $dim(W_{rad})$.*

Proof. Consider the following base-change diagram

$$\begin{array}{ccc} Y \setminus \mathbb{P}(W_{rad}) & \longrightarrow & Y_{an} \\ \big\downarrow & & \big\downarrow \\ \mathbb{P}(W) \setminus \mathbb{P}(W_{rad}) & \longrightarrow & \mathbb{P}(W_{an}) \end{array}$$

Since $\mathbb{P}(W) \setminus \mathbb{P}(W_{rad}) \to \mathbb{P}(W_{an})$ is a vector bundle, by Example 2.4.3, it is also an affine bundle. Thus $s : Y \setminus \mathbb{P}(W_{rad}) \to Y_{an}$ is also an affine bundle by base-change (Remark 2.4.2). The rank is given by $dim(Y) - dim(Y_{an}) = dim(W_{rad})$. □

Recall that we assume $\phi \cong r\mathbb{H} \perp \phi'$ with maximal Witt index r, then $\phi' \cong \langle c \rangle \perp \phi_{ts}$ for $c \in F^\times$ or $\phi' \cong \phi_{ts}$. Let $dim(\phi_{ts}) = s$. Then $dim(\mathbb{P}(W)) = r + s - 1$.

Lemma 4.1.4. *For any F-variety Z, let $(h \times id) : \mathbb{P}(W_{rad}) \times Z \to Y \times Z$ be the closed embedding. Then there exists a homomorphism*

$$\pi_Z : Ch_*(Y \times Z) \to Ch_*(\mathbb{P}(W_{rad}) \times Z)$$

such that $\pi_Z \circ (h \times id)_* = id$. In particular, $(h \times id)_*$ is injective.

Proof. We assume that $\mathbb{P}(W_{rad})$ has dimension $r^{'}$, and denote the only non-zero element of $Ch_i(\mathbb{P}(W_{rad}))$ as $l_i^{'}$ for $0 \leq i \leq r^{'}$. Similarly, we denote the only non-zero element of $Ch_i(\mathbb{P}(W))$ as l_i for $0 \leq i \leq r+s-1$ (Example 2.1.8).

By the Projective Bundle Theorem (Theorem 2.5.2), we have an isomorphism

$$\phi : \oplus_{j=0}^{r+s-1} Ch_{*-j}(Z) \cong Ch_*(\mathbb{P}(W) \times Z), \sum_{j=0}^{r+s-1} \alpha_j \mapsto \sum_{j=0}^{r+s-1} l_j \times \alpha_j.$$

Similarly, we have an isomorphism

$$\psi : \oplus_{j=0}^{r^{'}} Ch_{*-j}(Z) \cong Ch_*(\mathbb{P}(W_{rad}) \times Z), \sum_{j=0}^{r^{'}} \alpha_j \mapsto \sum_{j=0}^{r^{'}} l_j^{'} \times \alpha_j.$$

Consider the projection homomorphism

$$pr : \oplus_{j=0}^{r+s-1} Ch_{*-j}(Z) \to \oplus_{j=0}^{r^{'}} Ch_{*-j}(Z).$$

which projects onto the first $r^{'}$ factors. Set $f := \psi \circ pr \circ \phi^{-1}$. Since $h_*(l_j^{'}) = l_j$ for any $0 \leq j \leq r^{'}$, we know that $f \circ (h \times id)_* = id_{Ch(\mathbb{P}(W_{rad}) \times Z)}$.

Now we take the composition

$$\pi_Z : Ch(Y \times Z) \to Ch(\mathbb{P}(W) \times Z) \xrightarrow{f} Ch(\mathbb{P}(W_{rad}) \times Z).$$

where $Ch(Y \times Z) \to Ch(\mathbb{P}(W) \times Z)$ is the push-forward induced by the closed embedding $Y \times Z \hookrightarrow \mathbb{P}(W) \times Z$.

Finally, we obtain that $\pi_Z \circ (h \times id)_* = id$. □

Remark 4.1.5. The retraction we constructed above is compatible with push-forwards. Under the same setting, we assume $Z^{'}$ is an F-variety and there exists a proper morphism $f : Z \to Z^{'}$. Then the following diagram commutes

$$\begin{array}{ccc} Ch(Y \times Z) & \xrightarrow{\pi_Z} & Ch(\mathbb{P}(W_{rad}) \times Z) \\ \downarrow{\scriptstyle (id \times f)_*} & & \downarrow{\scriptstyle (id \times f)_*} \\ Ch(Y \times Z^{'}) & \xrightarrow{\pi_{Z^{'}}} & Ch(\mathbb{P}(W_{rad}) \times Z^{'}) \end{array}$$

This comes directly from the construction of π_Z and $\pi_{Z^{'}}$.

Let ϕ_{an} be the anisotropic part of ϕ and X_{an} be the associated quadric of ϕ_{an}. Since ϕ is with maximal Witt index, we know that $X_{an} = Y_{an}$ (When $char(F) \neq 2$, if $\phi \simeq r\mathbb{H} \perp \langle 1 \rangle \perp \langle 0, \cdots, 0 \rangle$, we know that $\phi_{an} \neq \psi_{an}$, but we still have $X_{an} = Y_{an} = \emptyset$).

Consider the localization sequence

$$Ch_*(\mathbb{P}(W_{rad}) \times Z) \to Ch_*(Y \times Z) \to Ch_*(Y \setminus \mathbb{P}(W_{rad}) \times Z) \to 0$$

By Lemma 4.1.3, the canonical projection $s : Y \setminus \mathbb{P}(W_{rad}) \to Y_{an}$ is an affine bundle of rank $dim(W_{rad})$. Then by the Homotopy Invariance (Theorem 2.4.4), we have an isomorphism

$$(s \times id)^* : Ch_*(Y_{an} \times Z) \to Ch_{*+dim(W_{rad})}(Y \setminus \mathbb{P}(W_{rad}) \times Z).$$

Let $(r \times id) : Y \setminus \mathbb{P}(W_{rad}) \times Z \hookrightarrow Y \times Z$ be the open embedding. Then we define the homomorphism $g := (s \times id)^{*-1} \circ (r \times id)^* : Ch_*(Y \times Z) \to Ch_{*-dim(W_{rad})}(Y_{an} \times Z)$.

By Lemma 4.1.4, the push-forward

$$(h \times id)_* : Ch_*(\mathbb{P}(W_{rad}) \times Z) \to Ch_*(Y \times Z)$$

is split injective. Hence we have the following split exact sequence

$$\begin{aligned} 0 \longrightarrow Ch_*(\mathbb{P}(W_{rad}) \times Z) &\xrightarrow{(h\times id)_*} Ch_*(Y \times Z) \\ &\xrightarrow{\quad g \quad} Ch_{*-dim(W_{rad})}(Y_{an} \times Z) \longrightarrow 0 \end{aligned} \tag{4.1}$$

Before we give a full description of $Ch_*(X \times Z)$, we want to make an additional observation about the image of $Ch_{*-dim(W_{rad})}(Y_{an} \times Z)$ in $Ch_*(Y \times Z)$ (under the splitting of g given by the construction above). More precisely, we will show that it can be identified as a subgroup of the image of the push-forward induced by the projection $(Y \times Z) \times Y_{an} \to Y \times Z$.

We give a lemma first. Let V_1, V_2 be two varieties over F and U_1 be an open subscheme of V_1. We assume $t : U_1 \to V_2$ is an affine bundle of rank n and $r : U_1 \hookrightarrow V_1$ is the open embedding. We have a surjective homomorphism $Ch_*(V_1) \to Ch_*(V_2)$ by composing $r^* : Ch(V_1) \to Ch(U_1)$ with $(t^*)^{-1} : Ch_*(U_1) \cong Ch_{*-n}(V_2)$, where the latter isomorphism is obtained by the Homotopy Invariance (Theorem 2.4.4).

Let $pr : V_1 \times U_1 \to V_1$ be the projection onto the first factor. Notice that the composition

$$Ch_*(V_1 \times U_1) \xrightarrow{pr_*} Ch(V_1) \xrightarrow{r^*} Ch(U_1)$$

is surjective: actually, let $\Delta : U_1 \to U_1 \times U_1$ be the diagonal morphism. Then for any closed subvariety $Z \subset U_1$, we take the closure of $\Delta(Z)$ in $V_1 \times U_1$. This gives a cycle $[\overline{\Delta(Z)}] \in Ch(V_1 \times U_1)$ such that the image of $[\overline{\Delta(Z)}]$ under the composition is $[Z]$.

From the commutative diagram

$$\begin{array}{ccccc} Ch(V_1 \times U_1) & \xrightarrow{pr_*} & Ch(V_1) & \xrightarrow{r^*} & Ch(U_1) \\ \downarrow{\scriptstyle (id\times t)^{*-1}} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle (t^*)^{-1}} \\ Ch(V_1 \times V_2) & \xrightarrow{pr_*} & Ch(V_1) & \xrightarrow{(t^*)^{-1}\circ r^*} & Ch(V_2) \end{array}$$

we obtain the following lemma.

Lemma 4.1.6. [11, Lemma B.1] *The composition*

$$Ch_*(V_1 \times V_2) \xrightarrow{pr_*} Ch_*(V_1) \xrightarrow{(t^*)^{-1}\circ r^*} Ch_{*-n}(V_2)$$

is surjective.

Now we let π_Z be the retraction of $(h \times id)_*$ constructed in Lemma 4.1.4. Then there exists a unique splitting $\mu : Ch_{*-dim(W_{rad})}(Y_{an} \times Z) \to Ch_*(Y \times Z)$ of g such that $Im(\mu) = Ker(\pi_Z)$.

Proposition 4.1.7. *Let $pr : (Y \times Z) \times Y_{an} \to Y \times Z$ be the projection morphism onto the first factor. Then $Ker(\pi_Z) \subset Im((pr)_*)$.*

Proof. To simplify notations, we set $dim(W_{rad}) = n'$. Since $Y \setminus \mathbb{P}(W_{rad})$ is an open subscheme of Y while $s : Y \setminus \mathbb{P}(W_{rad}) \to Y_{an}$ is an affine bundle of rank n', by Lemma 4.1.6, the composition

$$Ch_*((Y \times Z) \times (Y_{an} \times Z)) \xrightarrow{pr_*} Ch_*(Y \times Z) \xrightarrow{g} Ch_{*-n'}(Y_{an} \times Z)$$

is surjective. In particular, the composition

$$Ch_*((Y \times Z) \times Y_{an}) \xrightarrow{pr_*} Ch_*(Y \times Z) \xrightarrow{g} Ch_{*-n'}(Y_{an} \times Z)$$

is also surjective.

We identify $Ch_{*-n'}(Y_{an}\times Z)$ with $Ker(\pi_Z)$ through the splitting μ. Since $Ch((Y\times Z)\times Y_{an})\to Ch_{*-n'}(Y_{an}\times Z)$ is surjective, then for any $\beta\in Ker(\pi_Z)$, there exists an $\alpha\in Ch((Y\times Z)\times Y_{an})$ such that $g\circ(pr)_*(\alpha)=g(\beta)$. Thus $pr_*(\alpha)-\beta\in Ker(g)$ where $Ker(g)=Im((h\times id)_*)$ from the exactness of (3.1). Hence there exists a $\gamma\in Ch(\mathbb{P}(W)\times Z)$ such that $(h\times id)_*(\gamma)=pr_*(\alpha)-\beta$. Now we compose the retraction π_Z on both sides, we have $\gamma=\pi_Z\circ(pr)_*(\alpha)$, which gives

$$(pr)_*(\alpha)-(h\times id)_*(\pi_Z)(pr)_*(\alpha)=\beta.$$

We only need to show that $(h\times id)_*(\pi_Z)(pr)_*(\alpha)$ lies in $Im((pr)_*)$. The proposition follows from the commutative diagram

$$\begin{array}{ccccc}
Ch((Y\times Z)\times Y_{an}) & \xrightarrow{\pi_{Z\times Y_{an}}} & Ch((\mathbb{P}(W_{rad})\times Z)\times Y_{an}) & \xrightarrow{(h\times id)_*\times id} & Ch((Y\times Z)\times Y_{an})\\
\downarrow{\scriptstyle (pr)_*} & & \downarrow{\scriptstyle (p_1)_*} & & \downarrow{\scriptstyle (pr)_*}\\
Ch(Y\times Z) & \xrightarrow[\pi_Z]{} & Ch(\mathbb{P}(W_{rad})\times Z) & \xrightarrow[(h\times id)_*]{} & Ch(Y\times Z)
\end{array}$$

where $p_1:(\mathbb{P}(W_{rad})\times Z)\times Y_{an}\to\mathbb{P}(W_{rad})\times Z$ is the projection onto the first factor (Remark 4.1.5). □

Now we can give a full description of $Ch_*(X\times Z)$ by Proposition 4.1.2 and the split exact sequence (3.1). In the following theorem, we use A_1, A_2 and A_3 to denote $\mathbb{P}(W_{rad})$, X_{an} and $\mathbb{P}(V/W)$ respectively. Also, we set $s^{'}=dim(\mathbb{P}(W_{rad}))$ and $l=dim(\phi)-r-1$.

Theorem 4.1.8. *We have an isomorphism*

$$f:Ch_*(X)\to Ch_*(A_1)\oplus Ch_{*-(s'+1)}(A_2)\oplus Ch_{*-l}(A_3).$$

More generally, for any F-variety Z, we have an isomorphism

$$f_Z:Ch_*(X\times Z)\to Ch_*(A_1\times Z)\oplus Ch_{*-(s'+1)}(A_2\times Z)\oplus Ch_{*-l}(A_3\times Z).$$

which makes the following diagram commute

$$\begin{array}{ccccc}
Ch(X)\otimes Ch(Z) & \xrightarrow{f\otimes id} & (\oplus_{i=1}^3 Ch(A_i))\otimes Ch(Z) & \xrightarrow{\sim} & \oplus_{i=1}^3(Ch(A_i)\otimes Ch(Z))\\
\downarrow{\scriptstyle\times} & & & & \downarrow{\scriptstyle\times}\\
Ch(X\times Z) & & \xrightarrow{f_Z} & & \oplus_{i=1}^3(Ch(A_i\times Z))
\end{array}$$

where the external product maps are denoted by $\times$ *and the isomorphism*

$$(\oplus_{i=1}^{3} Ch(A_i)) \otimes Ch(Z) \cong \oplus_{i=1}^{3}(Ch(A_i) \otimes Ch(Z))$$

is the canonical isomorphism.

Remark 4.1.9. Let's identify $Ch_{*-(s'+1)}(Y_{an} \times Z)$ with $Ker(\pi_Z)$ under the splitting $\mu : Ch_{*-(s'+1)}(Y_{an} \times Z) \to Ker(\pi_Z)$ we introduced right after the split exact sequence (3.1). Since $X_{an} = Y_{an}$, by Proposition 4.1.7, we know that $Ch(X_{an} \times Z) \subset Im((pr)_*)$ where $pr : (Y \times Z) \times X_{an} \to Y \times Z$ is the projection. Let $p : (X \times Z) \times X_{an} \to X \times Z$ be the projection onto the first factor. Since $Y \times Z$ is a closed subvariety of $X \times Z$, we know that $Im((pr)_*) \subset Im(p_*)$. Hence we have $Ch(X_{an} \times Z) \subset Im(p_*)$.

Recall that there's no restriction on the F-variety Z. So for Chow groups of the product of quadrics, we could use the decomposition above iteratively to figure out a suitable representation. For example, we let q_1, q_2 be two quadratic forms with maximal Witt index and Q_1, Q_2 be the associated quadrics.

In Theorem 4.1.8, we can substitute Z with Q_2, and the decomposition becomes

$$Ch_*(Q_1 \times Q_2) \simeq Ch_*(\mathbb{P}^{s_1'} \times Q_2) \oplus Ch_{*-(s_1'+1)}(Q_{1,an} \times Q_2) \oplus Ch_{*-l_1}(\mathbb{P}^{r_1-1} \times Q_2).$$

for some integers r_1, l_1, s_1'. Also, $Ch(Q_{1,an} \times Q_2)$ lies in the image of $Ch((Q_1 \times Q_2) \times Q_{1,an}) \to Ch(Q_1 \times Q_2)$ which is induced by the projection $p : (Q_1 \times Q_2) \times Q_{1,an} \to Q_1 \times Q_2$.

For $Ch_*(\mathbb{P}^{s_1'} \times Q_2)$, it can be decomposed further as:

$$Ch_*(\mathbb{P}^{s_1'} \times \mathbb{P}^{s_2'}) \oplus Ch_{*-(s_2'+1)}(\mathbb{P}^{s_1'} \times Q_{2,an}) \oplus Ch_{*-l_2}(\mathbb{P}^{s_1'} \times \mathbb{P}^{r_2-1}).$$

for some integers s_2', l_2, s_2. Also, $Ch(\mathbb{P}^{s_1'} \times Q_{2,an})$ lies in the image of $Ch((Q_1 \times Q_2) \times Q_{2,an}) \to Ch(Q_1 \times Q_2)$ which is induced by the projection $(Q_1 \times Q_2) \times Q_{an} \to Q_1 \times Q_2$ since $\mathbb{P}^{s_1'} \times Q_2$ is a closed subvariety of $Q_1 \times Q_2$.

Similarly, we can decompose $Ch_{*-(s_1'+1)}(Q_{1,an} \times Q_2)$ and $Ch_{*-l_1}(\mathbb{P}^{r_1-1} \times Q_2)$ and conclude that in these decompositions, Chow groups having the form $Ch(Q_{i,an} \times Z)$ for some F-variety Z lie in the image of push-forwards induced by certain projections.

More generally, we consider a product of quadrics $Q := Q_1 \times Q_2 \times \cdots \times Q_n$, where each quadric Q_i is given by a quadratic form ϕ_i with maximal Witt index r_i. Let $V_i := V_{\phi_i}$ and W_i be the subspace we considered in the beginning of this section.

Also, we denote $rad(\phi_i\mid_{W_i})$ by $W_{i,rad}$ and we let $Q_{i,an}$ be the quadric given by the anisotropic part of $\phi_i\mid_{W_i}$. We set that $dim(\mathbb{P}(W_{i,rad}))=s_i^{'}$, $dim(\mathbb{P}(V_i/W_i))=h_i$ and $dim(\phi_i)-r_i-1=l_i$.

Repeating the process above finitely many times, we have the following proposition.

Proposition 4.1.10. *$Ch_*(Q_1\times\cdots\times Q_n)\cong A\oplus B$, where:*

(1) A equals

$$\{\oplus_{I=1}^{n}[\bigoplus_{J\subset[1,n],|J|=I} Ch_{*-\sum_{k=1}^{I}(l_{i_k})}(\mathbb{P}^{(I,J,1)}\times\cdots\times\mathbb{P}^{(I,J,n)})]\}\oplus Ch_*(\mathbb{P}^{s_1^{'}}\times\cdots\times\mathbb{P}^{s_n^{'}}).$$

where $(I,J,i)=h_i$ if $i\in J$ and $(I,J,i)=s_i^{'}$ otherwise.

(2) B is a direct sum taken over Chow groups having the form

$$Ch_m(Z_1\times\cdots\times Q_{k,an}\times\cdots\times Z_n)$$

for some $k=1,\cdots,n$ and integer m, where $Z_i\in\{\mathbb{P}^{s_i^{'}},Q_{i,an},\mathbb{P}^{h_i}\}$ for each $i\neq k$.

Remark 4.1.11. (1) In view of Theorem 4.1.8, for each Q_i, we have an isomorphism

$$f_i: Ch_*(Q_i)\to Ch_*(\mathbb{P}^{s_i^{'}})\oplus Ch_{*-(s_i^{'}+1)}(Q_{i,an})\oplus Ch_{*-(dim(\phi_i)-r_i-1)}(\mathbb{P}^{h_i}).$$

Let $A_i:=Ch_*(\mathbb{P}^{s_i^{'}})\oplus Ch_{*-(dim(\phi_i)-r_i-1)}(\mathbb{P}^{h_i})$ and $B_i:=Ch_{*-(s_i^{'}+1)}(Q_{i,an})$. Then we have an isomorphism

$$f^{'}:=\otimes_{i=1}^{n}f_i: Ch(Q_1)\otimes\cdots\otimes Ch(Q_n)\to\otimes_{i=1}^{n}(A_i\oplus B_i).$$

we also have a canonical isomorphism, $\otimes_{i=1}^{n}(A_i\oplus B_i)\cong(\otimes_{i=1}^{n}A_i)\oplus B^{'}$, where $B^{'}$ is a direct sum taken over all terms having the form

$$Ch_{m_1}(Z)\otimes\cdots\otimes Ch_{m_k}(Q_{k,an})\otimes\cdots\otimes Ch_{m_n}(Z_n)$$

for integers $m_1,\cdots,m_n$ and $Z_i\in\{\mathbb{P}^{s_i^{'}},Q_{i,an},\mathbb{P}^{h_i}\}$ for each $i\neq k$. By Proposition

4.1.10, $Ch_*(Q_1 \times \cdots \times Q_n) \cong A \oplus B$. We have the following commutative diagram

$$\begin{array}{ccc} Ch(Q_1) \otimes \cdots \otimes Ch(Q_n) & \xrightarrow{f'} & (\otimes_{i=1}^n A_i) \oplus B' \\ \downarrow \times & & \downarrow \times \\ Ch(Q_1 \times \cdots \times Q_n) & \xrightarrow{\sim} & A \oplus B \end{array}$$

where $\times$ is the external product map.

(2) B is contained in the subgroup of $Ch(Q_1 \times \cdots \times Q_n)$ generated by the images of push-forwards

$$Ch((Q_1 \times \cdots \times Q_k \times \cdots \times Q_n) \times Q_{k,an}) \to Ch(Q_1 \times \cdots \times Q_n)$$

which is induced by the projection $(Q_1 \times \cdots \times Q_k \times \cdots \times Q_n) \times Q_{k,an} \to Q_1 \times \cdots \times Q_k \times \cdots \times Q_n$ for $k = 1, \cdots, n$.

4.2 Split forms

Fix a field F. Let ϕ be a quadratic form over F with maximal Witt index $r \neq 0$. We say the form ϕ is a *split form* if $\phi \simeq r\mathbb{H} \perp \phi'$ for some subform ϕ' where $\phi' \simeq \langle c, 0, \cdots, 0 \rangle$ for $c \in F$.

Now we assume that ϕ is a split form and let X be the associated quadric of ϕ. Let W be the subspace of V_ϕ as we considered in Section 1 and Y be the closed subvariety of X given by $\phi\,|_W$.

When $\phi_{ts} = \phi' \simeq \langle c, 0, \cdots, 0 \rangle$ with $c \in F^\times$, Y is a double hyperplane in the projective space $\mathbb{P}(W)$ and the corresponding reduced variety $Y_{red} \cong \mathbb{P}^{dim(\phi)-r-2}$. As mentioned in Remark 2.1.4, we have $Ch(Y) \cong Ch(Y_{red}) = Ch(\mathbb{P}^{dim(\phi)-r-2})$.

When $\phi_{ts} \simeq \langle 0, \cdots, 0 \rangle$, $Y \cong \mathbb{P}^{r+dim(\phi_{ts})-1}$. In this case, $\phi_{ts} = \phi'$ or $\phi' = \langle c \rangle \perp \phi_{ts}$ with $c \in F^\times$. By Theorem 4.1.8, we have the following proposition.

Proposition 4.2.1. *For any F-variety Z, when $\phi_{ts} = \phi' \simeq \langle 1, 0, \cdots, 0 \rangle$, we have*

$$Ch_*(X \times Z) \simeq Ch_*(\mathbb{P}^{dim(\phi)-r-2} \times Z) \oplus Ch_{*-(dim(\phi)-r-1)}(\mathbb{P}(V/W) \times Z).$$

When $\phi_{ts} = \langle 0, \cdots, 0 \rangle$, we have

$$Ch_*(X \times Z) \simeq Ch_*(\mathbb{P}^{r+dim(\phi_{ts})-1} \times Z) \oplus Ch_{*-(dim(\phi)-r-1)}(\mathbb{P}(V/W) \times Z).$$

More generally, we consider the product of quadrics $X_1 \times \cdots \times X_n$. We assume that each X_i is given by a split quadratic form ϕ_i with non-zero maximal Witt index r_i. Let $\phi_{i,ts}$ be the totally singular part of ϕ and $dim(\phi_{i,ts}) = s_i$. By Proposition 4.2.1, for each $Ch_*(X_i)$, we let $A_i := Ch_*(\mathbb{P}^{k_i}) \oplus Ch_{dim(\phi_i)-r_i-1}(\mathbb{P}(V_i/W_i))$ where $k_i = r_i + s_i - 2$ or $k_i = r_i + s_i - 1$ (depends on the split form ϕ_i). Then $Ch_*(X_i) \cong A_i$.

In view of Proposition 4.1.10, $Ch(X_1 \times \cdots \times X_n) \cong A$. By Remark 4.1.11, we have the following commutative diagram

$$\begin{array}{ccc} Ch(X_1) \otimes \cdots \otimes Ch(X_n) & \xrightarrow{\sim} & \otimes_{i=1}^n A_i \\ \downarrow \times & & \downarrow \times \\ Ch(X_1 \times \cdots \times X_n) & \xrightarrow{\sim} & A \end{array}$$

By the Projective Bundle Theorem (Theorem 2.5.2), the external product map

$$\otimes_{i=1}^n A_i \to A$$

is an isomorphism. Hence we have the following proposition.

Proposition 4.2.2. *Let $X_1 \times \cdots \times X_n$ be the product of quadrics and we assume that each X_i is given by a split form. Then the external product map*

$$Ch(X_1) \otimes \cdots \otimes Ch(X_n) \to Ch(X_1 \times \cdots \times X_n)$$

is an isomorphism.

Now we want to give an explicit description for elements in $Ch(X_1 \times \cdots \times X_n)$. From the external product isomorphism, it suffices to consider the Chow group of one quadric X which is given by a split form ϕ. We assume that the Witt index of ϕ is r and $dim(\phi_{ts}) = s$. By setting $Z = Spec(F)$, Proposition 4.2.1 says that

$$Ch_*(X) \simeq Ch_*(\mathbb{P}^k) \oplus Ch_{*-(dim(\phi)-r-1)}(\mathbb{P}(V/W)).$$

where $k = r + s - 1$ or $k = r + s - 2$.

From the decomposition, we denote the only non-zero element in $Ch_i(X)$ for $0 \leq i \leq r + s - 2$ by l_i. Also, we denote the only non-zero element in $Ch^j(X)$ for $0 \leq j < r-1$ by h^j. Notice that $Ch_{dim(\phi)-r-1}(X) \simeq Ch_{dim(\phi)-r-1}(\mathbb{P}^k) \oplus Ch_0(\mathbb{P}(V/W))$. We denote the only non-zero element in $Ch_0(\mathbb{P}(V/W))$ by h^{r-1}.

Actually, the elements $h^j, j = 0, \cdots, r-1$ are given by the pull-back of classes of codimension j planes in $Ch(\mathbb{P}(V_\phi))$ under the closed embedding $i : X \hookrightarrow \mathbb{P}(V_\phi)$. The affine bundle $X \setminus Y \to \mathbb{P}(V/W)$ that we used to construct the decomposition of $Ch_*(X)$ is given by the composition $X \setminus Y \hookrightarrow \mathbb{P}(V) \setminus \mathbb{P}(W) \to \mathbb{P}(V/W)$ where $\mathbb{P}(V) \setminus \mathbb{P}(W) \to \mathbb{P}(V/W)$ is the canonical projection. Hence for any $j \leq r-1$, the image of $Ch^j(\mathbb{P}(V/W))$ in $Ch^j(X)$ is given by the pull-back $i^* : Ch^j(\mathbb{P}(V_\phi)) \to Ch^j(X)$.

For $i \in [0, r+s-2]$, the element l_i is given by the class of any dimension i projective linear space. To see this, let W be any totally isotropic subspace of dimension $i+1$ in V_ϕ. To prove that $l_i = [\mathbb{P}(W)]$, it suffices to show that $[\mathbb{P}(W)] \neq 0$ in $Ch(X)$. But if $i : X \hookrightarrow \mathbb{P}(V_\phi)$ is the closed embedding, then $i_*([\mathbb{P}(W)]) = [\mathbb{P}(W)]$ in $Ch(\mathbb{P}(V_\phi))$ and this element is not zero by Example 2.1.8. When $k = r+s-1$ but $dim(\phi) = 2r+s+1$, there's still only one nonzero element $l_{r+s-1} \in Ch_{r+s-1}(X)$. By a similar argument as above, l_{r+s-1} is given by the class of any dimension $r+s-1$ projective linear space.

Now we suppose that $k = r+s-1$ and $dim(\phi) = 2r+s$. In the decomposition above, we consider the injection $Ch_*(\mathbb{P}^k) \hookrightarrow Ch_*(X)$. Let α be the only non-zero element in $Ch_{r+s-1}(\mathbb{P}^k)$. Let $W \subset \mathbb{P}(V_\phi)$ be any $(r+s)$-dimensional totally isotropic subspace. Then the class $[\mathbb{P}(W)] \in Ch_{r+s-1}(X)$ equals $a\alpha + bh^{r-1}$ for some elements $a, b \in \mathbb{F}_2$. We have $a\alpha = i_*(a\alpha + bh^{r-1}) = \alpha$ where $i : X \hookrightarrow \mathbb{P}(V_\phi)$ is the closed embedding. Thus $a = 1$. As a result, in $Ch_{r+s-1}(X)$, we know that $[\mathbb{P}(W)] = \alpha$ or $[\mathbb{P}(W)] = l'_{r+s-1} := \alpha + h^{r-1}$. It can be shown that the latter one is also possible [2, Proposition 68.2]. So we have the following definition.

Definitions 4.2.3. When $k = r+s-1$ and $dim(\phi) = 2r+s$, we define an *orientation* of X to be a choice of one of the two classes of $(r+s)$-dimensional linear subspaces in $Ch(X)$. We denote this class by l_{r+s-1}. Quadrics with an orientation are called *oriented* quadric.

In conclusion, $Ch_*(X)$ is free with basis elements $\{l_0, \cdots, l_{r+s-1}, h^0, \cdots, h^{r-1}\}$ if $k = r+s-1$ and $Ch_*(X)$ is free with basis elements $\{l_0, \cdots, l_{r+s-2}, h^0, \cdots, h^{r-1}\}$ if $k = r+s-2$.

More generally, we consider the product of quadrics $X_1 \times \cdots \times X_n$. Suppose that each X_i is given by a split form. Then by Proposition 4.2.2, we know that $Ch(X_1 \times \cdots \times X_n) \simeq Ch(X_1) \otimes \cdots \otimes Ch(X_n)$. Since each $Ch(X_i)$ has basis $\{h^*, l_*\}$, $Ch(X_1 \times \cdots \times X_n)$ is an $\mathbb{F}_2$-vector space with basis elements $\{a_1 \times \cdots \times a_n\}$ where each $a_i \in \{l_*, h^*\}(l_*, h^* \in Ch(X_i))$.

4.3 Rational cycles

Fix a field F. We let $\overline{F}$ be an algebraic closure of F. Let X be a scheme over F. Since the change of field morphism $\overline{X} := X \times Spec(\overline{F}) \to X$ is a flat morphism of rank 0, it induces a pull-back homomorphism $Ch(X) \to Ch(\overline{X})$. The following theorem is important for our discussion on the image of the change of field homomorphism.

Theorem 4.3.1 (Springer's Theorem)**.** [2, Corollary 71.3] *Let ϕ be an anisotropic quadratic form over F and X be the associated quadric of ϕ. Then the image of the degree homomorphism $deg : Ch_0(X) \to \mathbb{Z}/2\mathbb{Z}$ is equal to 0.*

Now we let $\phi_i, i \in [1, n]$ be anisotropic totally singular forms over F with underlying vector space V_i. Suppose X_i is the associated quadric of ϕ_i. We set $X := X_1 \times \cdots \times X_n$. In the beginning, we want to discuss the image of the change of field homomorphism

$$Ch(X) \to Ch(\overline{X}).$$

Lemma 4.3.2. *Consider the closed embedding*

$$i : X \to \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_n).$$

The induced push-forward homomorphism

$$i_* : Ch_*(X) \to Ch_*(\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_n))$$

is zero.

Proof. Since $X_i, i = 1, \cdots, r$ are given by anisotropic quadratic forms, thus the Springer's Theorem says that every closed point in X has even degree. Hence the degree homomorphism $deg : Ch(X) \to \mathbb{Z}/2\mathbb{Z}$ is zero. Let Z be a closed subvariety of X. We let $h : Z \hookrightarrow X \xrightarrow{i} \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_n)$ be the composition of closed embeddings. Then

$$h_*([Z]) = \sum_{i_1+\cdots+i_n=dim(Z)} a_{i_1,\cdots,i_n} l_{i_1} \times \cdots \times l_{i_n}$$

for some $a_{i_1,\cdots,i_n} \in \mathbb{F}_2$ by the Projective Bundle Theorem (Theorem 2.5.2)

Now we suppose that $(i_1, \cdots, i_n)$ is any set of numbers such that $a_{i_1,\cdots,i_n} \neq 0$.

Since Z is a closed subvariety of X, we have the following commutative diagram.

$$\begin{array}{ccc} Ch(Z) & \xrightarrow{h_*} & Ch(\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_n)) \\ & \searrow^{deg} & \downarrow{\scriptstyle deg} \\ & & \mathbb{Z}/2\mathbb{Z} \end{array}$$

By Proposition 2.2.6, we have that

$$\begin{aligned} a_{i_1,\cdots,i_n} &= deg(h_*([Z]) \cdot (h^{i_1} \times \cdots \times h^{i_n})) \\ &= deg(h_*(h^*(h^{i_1} \times \cdots \times h^{i_n}))) \\ &= deg(h^*(h^{i_1} \times \cdots \times h^{i_n})) = 0 \end{aligned}$$

As a result, we know that $h_*([Z]) = 0$ for any closed subvariety Z of X. □

Corollary 4.3.3. *The change of field homomorhism*

$$Ch(X) \to Ch(\overline{X})$$

is zero.

Proof. For any $i \in [1, n]$, the variety $\overline{X_i}$ is a double plane in $\overline{\mathbb{P}(V_i)}$. Thus for each $i \in [1, n]$, the associated reduced variety of $\overline{X_i}$ is a projective subspace of $\overline{\mathbb{P}(V_i)}$. By the Projective Bundle Theorem (Theorem 2.5.2),

$$Ch(\overline{X_1} \times \cdots \times \overline{X_r}) \to Ch(\overline{\mathbb{P}(V_1)} \times \cdots \times \overline{\mathbb{P}(V_r)})$$

is an injective homomorphism.

By Lemma 4.3.2, the composition of homomorphisms

$$Ch(X_1 \times \cdots \times X_n) \to Ch(\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_n)) \to Ch(\overline{\mathbb{P}(V_1)} \times \cdots \times \overline{\mathbb{P}(V_n)})$$

is zero. Since this coincides with the composition of homomorphisms

$$Ch(X_1 \times \cdots \times X_n) \to Ch(\overline{X_1} \times \cdots \times \overline{X_n}) \to Ch(\overline{\mathbb{P}(V_1)} \times \cdots \times \overline{\mathbb{P}(V_n)}).$$

$Ch(X) \to Ch(\overline{X})$ is the zero homomorphism. □

In general, we have the following proposition.

Proposition 4.3.4. *Let $\phi_i, i = 1, \cdots, n$ be anisotropic totally singular quadratic forms over F and X_i be the associated quadric of ϕ_i. We set $X := X_1 \times \cdots \times X_r$. Then for integers $m_1, \cdots, m_n \geq 0$, the change of field homomorphism*

$$Ch(\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_n} \times X) \to Ch(\overline{\mathbb{P}^{m_1}} \times \cdots \times \overline{\mathbb{P}^{m_n}} \times \overline{X})$$

is zero.

Proof. For $i = 1, \cdots, n$, we let l_{j_i} denote the only nonzero element of $Ch_{j_i}(\mathbb{P}^{m_i})$ and $\overline{l_{j_i}}$ denote the only nonzero element of $Ch_{j_i}(\overline{\mathbb{P}^{m_i}})$.

For any cycle $\alpha \in Ch(\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_n} \times X)$, by the Projective Bundle Theorem (Theorem 2.5.2), we have

$$\alpha = \sum_{j_1, \cdots, j_n, i} l_{j_1} \times \cdots \times l_{j_n} \times \alpha_i$$

with cycles $\alpha_i \in Ch(X)$. Let $- : Ch(X) \to Ch(\overline{X})$ be the change of field homomorphism. By Corollary 4.3.3, $\overline{\alpha_i} = 0$ for each i. Since the change of field homomorphism

$$Ch(\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_n} \times X) \to Ch(\overline{\mathbb{P}^{m_1}} \times \cdots \times \overline{\mathbb{P}^{m_n}} \times \overline{X}).$$

is given by

$$\sum_{j_1, \cdots, j_n, i} l_{j_1} \times \cdots \times l_{j_n} \times \alpha_i \mapsto \sum_{j_1, \cdots, j_n, i} \overline{l_{j_1}} \times \cdots \times \overline{l_{j_n}} \times \overline{\alpha_i} = 0.$$

the result follows. □

4.3.1 The case of forms having maximal Witt index

Let ϕ be a quadratic form over F with nonzero maximal Witt index r and anisotropic totally singular part ϕ_{ts}. Suppose $dim(\phi_{ts}) = s$. We let X_{an} be the quadric given by ϕ_{ts} (When $dim(\phi_{ts}) \leq 1$, X_{an} is empty). We set $V := V_\phi$ and let W be the subspace as we considered in Section 1.

By Theorem 4.1.8, we have

$$Ch_*(X) \cong Ch_*(\mathbb{P}^{r-1}) \oplus Ch_{*-r}(X_{an}) \oplus Ch_{*-(dim(\phi)-r-1)}(\mathbb{P}(V/W)).$$

By Proposition 4.2.1, we have

$$Ch_*(\overline{X}) \simeq Ch_*(\mathbb{P}^k) \oplus Ch_{*-(dim(\phi)-r-1)}(\mathbb{P}(V/W))$$

where $k = r+s-1$ or $k = r+s-2$ depending on $\overline{\phi}$.

We consider the change of field homomorphism

$$Ch(X) \to Ch(\overline{X}).$$

$Ch_{*-r}(X_{an})$ will disappear under this homomorphsm by Proposition 4.3.4. Hence the image of $Ch(X) \to Ch(\overline{X})$ is an $\mathbb{F}_2$-vector space spanned by $\{l_0, \cdots, l_{r-1}\}$ and $\{h^0, \cdots, h^{r-1}\}$. This gives the following theorem.

Theorem 4.3.5. *Let ϕ be a quadratic form over F with non-zero maximal Witt index r and anisotropic totally singular part ϕ_{ts}. Let X be the associated quadric of ϕ. Then image of the change of field homomorphism $Ch(X) \to Ch(\overline{X})$ is an $\mathbb{F}_2$-vector space with basis elements $\{l_0, \cdots, l_{r-1}, h^0, \cdots, h^{r-1}\}$ (take a suitable orientation if needed).*

Definitions 4.3.6. Let X be a product of quadrics, i.e., $X := X_1 \times \cdots \times X_n$ for $n \geq 1$ and $X_1, \cdots, X_n$ are quadrics given by some quadratic forms over F. Then we say a cycle in $Ch(\overline{X})$ is *rational* if the cycle lies in the image of the change of field homomorphism. We denote the image of change of field homomorphism by $\overline{Ch}(X)$. Also, $\overline{Ch}^i(X)$ with *upper grading* i is defined by the image of $Ch^i(X)$ under the change of field homomorphism. Similarly, $\overline{Ch}_j(X)$ with *lower grading* j is defined by the image of $Ch_j(X)$ under the change of field homomorphism

For any $i = 1, \cdots, n$, we suppose that ϕ_i is a quadratic form over F with non-zero maximal Witt index r and anisotropic totally singular part. Let $X_i, i = 1, \cdots, n$ be the quadrics associated with ϕ_i and $X := X_1 \times \cdots \times X_n$.

By proposition 4.2.2, we have the external product isomorphism

$$Ch(\overline{X_1}) \otimes \cdots \otimes Ch(\overline{X_n}) \cong Ch(\overline{X_1} \times \cdots \times \overline{X_n}).$$

In the decomposition of $Ch_*(X)$, Chow groups of products of varieties involving at least one $X_{i,an}$(given by $\phi_{i,ts}$ for some i) will disappear in $Ch(\overline{X})$ by Proposition 4.3.4. Combining Theorem 4.3.5, we have the external product isomorphism

$$\overline{Ch}(X_1) \otimes \cdots \otimes \overline{Ch}(X_n) \cong \overline{Ch}(X_1 \times \cdots \times X_n)$$

This gives the following theorem.

Theorem 4.3.7. *Let $X_1, \cdots, X_r$ and X be under the same setting as above. Then the image of $Ch(X) \to Ch(\overline{X})$ is an $\mathbb{F}_2$-vector space with basis elements $\{a_1 \times \cdots \times a_r\}$ where $a_i \in \{l_*, h^*\}$ ($\{l_*, h^*\}$ are basis elements of $\overline{Ch}(X_i)$).*

4.3.2 The case of forms with anisotropic totally singular part

We let F_{sep} be the separable closure of F in $\overline{F}$. In the beginning, we will introduce the following notations, which will be used in the rest of this thesis.

Notation 4.3.8. For any F-variety Z, we set $\overline{Z} := Z_{\overline{F}}$ and $\widetilde{Z} := Z_{F_{sep}}$. For any quadratic form ϕ over F, we write $\widetilde{\phi} := \phi_{F_{sep}}$ and $\overline{\phi} := \phi_{\overline{F}}$.

Let $\phi_1, \cdots, \phi_n$ be quadratic forms over F. We assume that the totally singular part $\phi_{i,ts}$ of ϕ_i is anisotropic. Here we shall remind that $\widetilde{\phi}$ has the maximal Witt index while $\phi_{i,ts}$ stays anisotropic and $\overline{\phi_{i,ts}} \simeq \langle 1, \cdots, 0 \rangle$ or $\phi_{i,ts} = 0$ if $char(F) \neq 2$.

Let $X_1, \cdots, X_n$ be the associated quadrics of $\phi_1, \cdots, \phi_n$ respectively. We denote the singular locus of each X_i by $X_{i,ts}$ ($X_{i,ts} = \emptyset$ when $dim(\phi_{i,ts}) \leq 1$). For each $i \in [1, \cdots, n]$, we define the smooth locus of X_i to be $U_i := X_i \setminus X_{i,ts}$. We set $X := X_1 \times \cdots \times X_n$ and $U := U_1 \times \cdots \times U_n$ in this section. Notice that $Ch(U)$ has a natural ring structure since U is a smooth scheme.

Lemma 4.3.9. *The degree homomorphism $deg : Ch(\widetilde{U}) \to \mathbb{Z}/2\mathbb{Z}$ is well-defined.*

Proof. We have the following localization sequence

$$Ch(\widetilde{X_{1,ts}} \times \widetilde{X_2} \times \cdots \times \widetilde{X_n}) \to Ch(\widetilde{X_1} \times \widetilde{X_2} \times \cdots \times \widetilde{X_n}) \twoheadrightarrow Ch(\widetilde{U_1} \times \widetilde{X_2} \times \cdots \times \widetilde{X_n}).$$

By Lemma 3.4.6, $\phi_{1,ts}$ stays anisotropic over any separable field extension. Hence $\widetilde{X_{1,ts}}$ is given by an anisotropic quadratic form and by Springer's Theorem(Theorem 4.3.1), each point in $\widetilde{X_{1,ts}} \times \widetilde{X_2} \times \cdots \times \widetilde{X_n}$ has even degree. We have $deg(\alpha) = 0$ for any $\alpha \in Ch_0(\widetilde{X_{1,ts}} \times \widetilde{X_2} \times \cdots \times \widetilde{X_n})$. So this induces well-defined degree homomorphism

$$deg : Ch(\widetilde{U_1} \times \widetilde{X_2} \times \cdots \times \widetilde{X_n}) \to \mathbb{Z}/2\mathbb{Z}.$$

Now we consider the localization sequence

$$Ch(\widetilde{U_1} \times \widetilde{X_{2,ts}} \times \cdots \times \widetilde{X_n}) \to Ch(\widetilde{U_1} \times \widetilde{X_2} \times \cdots \times \widetilde{X_n}) \twoheadrightarrow Ch(\widetilde{U_1} \times \widetilde{U_2} \times \widetilde{X_r} \times \cdots \times \widetilde{X_n}).$$

The same argument shows that we have a well-define degree homomorphism

$$deg : Ch(\widetilde{U_1} \times \widetilde{U_2} \times \widetilde{X_3} \cdots \times \widetilde{X_n}) \to \mathbb{Z}/2\mathbb{Z}.$$

Continuing in this way, we prove the result. $\square$

The degree homomorphism gives rise to the definition of *numerically trivial* elements.

Definition 4.3.10. For any smooth F-variety V, an element $\alpha \in Ch_i(V)$ is called *numerically trivial* if for any $\beta \in Ch^i(V)$, we have $deg(\alpha \cdot \beta) = 0$.

By Lemma 4.3.9, similarly, we can consider numercially trivial elements in $Ch(\widetilde{U})$. Let N be the set of all numerically trivial elements in $Ch(\widetilde{U})$. It's obvious that N is an ideal of the ring $Ch(\widetilde{U})$.

Proposition 4.3.11. *Let $r : \widetilde{U} \hookrightarrow \widetilde{X}$ be the open embedding and $- : Ch(\widetilde{X}) \to Ch(\overline{X})$ be the change of field homomorphism. Then there exists a unique group homomorphism f such that the following diagram commutes*

$$\begin{array}{ccc} Ch(\widetilde{X}) & \xrightarrow{\ -\ } & Ch(\overline{X}) \\ \big\downarrow{\scriptstyle r^*} & \nearrow{\scriptstyle f} & \\ Ch(\widetilde{U}) & & \end{array} .$$

Proof. Since $\widetilde{U_i} = \widetilde{X_i} \setminus \widetilde{X_{i,ts}}$, by the localization sequence, we have

$$Ch(\widetilde{X_{1,ts}} \times \widetilde{X_2} \times \cdots \times \widetilde{X_n}) \to Ch(\widetilde{X_1} \times \widetilde{X_2} \times \cdots \times \widetilde{X_n}) \twoheadrightarrow Ch(\widetilde{U_1} \times \widetilde{X_2} \times \cdots \times \widetilde{X_n})$$

By Proposition 4.3.4, the homomorphism

$$Ch(\widetilde{X_{1,ts}} \times \widetilde{X_2} \times \cdots \times \widetilde{X_n}) \to Ch(\overline{X_{1,ts}} \times \overline{X_2} \times \cdots \times \overline{X_n})$$

is a zero homomorphism. Thus the homomorphism

$$Ch(\widetilde{X_1} \times \widetilde{X_2} \times \cdots \times \widetilde{X_n}) \to Ch(\overline{X_{1,ts}} \times \overline{X_2} \times \cdots \times \overline{X_n})$$

factors through

$$f_1 : Ch(\widetilde{U_1} \times \widetilde{X_2} \times \cdots \times \widetilde{X_n}) \to Ch(\overline{X_1} \times \overline{X_2} \times \cdots \times \overline{X_n}).$$

Under the same argument, f_1 factors through

$$f_2 : Ch(\widetilde{U_1} \times \widetilde{U_2} \times \widetilde{X_3} \times \cdots \times \widetilde{X_n}) \to Ch(\overline{X_1} \times \overline{X_2} \times \cdots \times \overline{X_n}).$$

Repeating the process finite times, we have a group homomorphism

$$f : Ch(\widetilde{U}) \to Ch(\overline{X}).$$

The uniqueness of f follows from the surjectivity of $Ch(\widetilde{X}) \to Ch(\widetilde{U})$. □

The following lemma shows that $Ker(f) \subset Ch(\widetilde{U})$ is an ideal.

Lemma 4.3.12. *$Ker(f) = N$.*

Proof. We'll show $Ker(f) \subset N$ first. Let's consider the kernel of the group homomorphism

$$f^{'} : Ch(\widetilde{X}) \to Ch(\overline{X}).$$

Combining Proposition 4.1.10 and Proposition 4.3.4, we know that $Ker(f^{'})$ consists of summands having the form $Ch(\widetilde{\mathbb{P}^{k_1} \times \cdots \times X_{j,ts} \times \cdots \times \mathbb{P}^{k_n}})$. By Remark 4.1.11, it follows that

$$Ker(f^{'}) \subset \oplus_{j=1}^{n} Im((pr_j)_*)$$

where $(pr_j) : \widetilde{X} \times \widetilde{X_{j,ts}} \to \widetilde{X}$ is the projection.

We consider the commutative diagram

$$\begin{array}{ccc} Ch(\widetilde{X} \times \widetilde{X_{j,ts}}) & \xrightarrow{(pr_j)_*} & Ch(\widetilde{X}) \\ \downarrow{f''} & & \downarrow{f'} \\ Ch(\overline{X} \times \overline{X_{j,ts}}) & \xrightarrow{(\overline{pr_j})_*} & Ch(\overline{X}) \end{array}.$$

where $f^{''}$ is the change of field homomorphism and $\overline{pr_j} : \overline{X} \times \overline{X_{j,an}} \to \overline{X}$ is the projection. By Proposition 4.3.4 and the decomposition of $Ch(\widetilde{X} \times \widetilde{X_{j,ts}})$, we know that $f^{'} \circ (pr_j)_* = (\overline{pr_j})_* \circ f^{''} = 0$ for any $j \in [1, n]$. Hence

$$Ker(f^{'}) = \oplus_{j=1}^{n} Im((pr_j)_*).$$

Now we let $r_j : \widetilde{U} \times \widetilde{X_{j,an}} \hookrightarrow \widetilde{X} \times \widetilde{X_{j,an}}$ and $r : \widetilde{U} \hookrightarrow \widetilde{X}$ be the open embeddings.

Then the following commutative diagram

$$\begin{array}{ccc} \oplus_{j=1}^{n} Ch(\widetilde{X} \times \widetilde{X_{j,ts}}) & \xrightarrow{\oplus (r_j)^*} & \oplus_{j=1}^{n} Ch(\widetilde{U} \times \widetilde{X_{j,ts}}) \\ \downarrow{\scriptstyle \oplus (pr_j)_*} & & \downarrow{\scriptstyle \oplus (pr_j^{'})_*} \\ Ch(\widetilde{X}) & \xrightarrow{\quad r^* \quad} & Ch(\widetilde{U}) \\ \downarrow{\scriptstyle f^{'}} & \swarrow{\scriptstyle f} & \\ Ch(\overline{X}) & & \end{array}$$

shows that $Ker(f) = \oplus_{j=1}^{n} Im((pr_j^{'})_*)$ where

$$pr_j^{'} : Ch(\widetilde{U} \times \widetilde{X_{j,ts}}) \to Ch(\widetilde{U})$$

is the projection

For any $j \in [1, n]$, we set $Z_j = \widetilde{U} \times \widetilde{X_{j,ts}}$. Because $\widetilde{X_{j,ts}}$ is given by an anisotropic quadratic form, so all closed points of Z_j have even degree. Consider the composition of proper morphisms

$$h : Z \hookrightarrow Z_j \overset{pr_j^{'}}{\to} \widetilde{U}$$

where Z is any subvariety of Z_j of dimension i. Now we let β be any cycle in $Ch^i(\widetilde{U})$. Then the cycle $h_*([Z]) \cdot \beta \in Ch_0(\widetilde{U})$. By Proposition 2.2.6, we know that $h_*([Z]) \cdot \beta = h_*(\gamma)$ for a cycle $\gamma \in Ch_0(Z_j)$. But $deg(h_*(\gamma)) = deg(\gamma) = 0$. Hence $[Z]$ is a numerically trivial element. As a result, we conclude that $Ker(f) \subset N$.

Now we prove $N \subset Ker(f)$ by induction on n. When $n = 0$, the statement is trivial. when $n \geq 1$, for $i = 0, \cdots, r_j - 1$, by an abuse of notations, we write $h^i \in Ch^i(\widetilde{U_j})$ and $l_i \in Ch_i(\widetilde{U_j})$ for elements whose images are h^i and l_i under the change of field homomorphism $Ch(\widetilde{U_j}) \to Ch(\overline{X_j})$. These elements do exist. For $l_i \in Ch(\widetilde{U_j})$, it is the class of a totally isotropic subspace of dimension $i+1$ in $V_{r_j \mathbb{H}}$. Also, h^i is the pull-back of the class of codimension i planes under the closed embedding $\widetilde{U_j} \hookrightarrow \mathbb{P}(V_{\widetilde{\phi_j}}) \setminus \widetilde{X_{j,ts}}$.

Now we consider any numerical trivial element $\alpha \in Ch(\widetilde{U})$. By adding suitable elements of $Ker(f)$ to α, we may assume that

$$\alpha = h^{i_1} \times \square_{i_1} + \cdots + h^{i_m} \times \square_{i_m} + l_{j_1} \times \square_{j_1} + \cdots + l_{j_k} \times \square_{j_k}$$

where $\square_s \in Ch(\widetilde{U_2} \times \cdots \times \widetilde{U_n})$ and $\square_s$ is the external product with each factor being

h^* or l_* for any $s \in \{i_1, \cdots, i_m, j_1, \cdots, j_k\}$.

We consider the products $(l_i \times [\widetilde{U_2} \times \cdots \times \widetilde{U_n}])\alpha$ for $i = i_1, \cdots, i_m$ and the products are $l_0 \times \Box_{i_1}, \cdots, l_0 \times \Box_{i_m}$ respectively. Similarly, we consider the products $(h^j \times [\widetilde{U_2} \times \cdots \times \widetilde{U_n}])\alpha$ for $j = j_1, \cdots, j_k$ and the products are $l_0 \times \Box_{j_1}, \cdots, l_0 \times \Box_{j_k}$ respectively.

For any $s \in \{i_1, \cdots, i_m, j_1, \cdots, j_k\}$, here we claim that $\Box_s$ is a numerically trivial element in $Ch(\widetilde{U_2} \times \cdots \times \widetilde{U_n})$. If not, there will exist a $\beta \in Ch(\widetilde{U_2} \times \cdots \times \widetilde{U_n})$ such that $deg(\beta \cdot \Box_s) \neq 0$. Then we have: $deg((l_0 \times \beta) \cdot (h^0 \times \Box_s)) = deg(l_0 \times (\beta \cdot \Box_s)) = deg(\beta \cdot \Box_s) \neq 0$. But it's impossible since $l_0 \times \Box_s$ is numerically trivial. Let $f_1 : Ch(\widetilde{U_1}) \to Ch(\overline{X_1})$ and $g : Ch(\widetilde{U_2} \times \cdots \times \widetilde{U_n}) \to Ch(\overline{X_2} \times \cdots \times \overline{X_n})$ be the group homomorphisms similarly constructed as in Proposition 4.3.11. From the induction hypothesis, we know $\Box_s \in Ker(g)$ for any $s \in \{i_1, \cdots, i_m, j_1, \cdots, j_k\}$. Hence

$$
\begin{aligned}
f(\alpha) &= f(h^{i_1} \times \Box_{i_1} + \cdots + h^{i_m} \times \Box_{i_m} + l_{j_1} \times \Box_{j_1} + \cdots + l_{j_k} \times \Box_{j_k}) \\
&= f_1(h^{i_1}) \times g(\Box_{i_1}) + \cdots + f_1(h^{i_m}) \times g(\Box_{i_m}) \\
&+ f_1(l_{j_1}) \times g(\Box_{j_1}) + \cdots + f_1(l_{j_k}) \times g(\Box_{j_k}) \\
&= f_1(h^{i_1}) \times 0 + \cdots + f_1(l_{j_k}) \times 0 = 0.
\end{aligned}
$$

Thus we conclude that $\alpha \in Ker(f)$. □

Since $Ker(f)$ is an ideal in $Ch(\widetilde{U})$, then the image of f has a natural ring structure induced by that of $Ch(\widetilde{U})$, i.e., $\overline{Ch}(\widetilde{X}) \cong Ch(\widetilde{U})/(Ker(f))$. We conclude the discussion above in the following theorem.

Theorem 4.3.13. *The image of the change of field homomorphism*

$$Ch(\widetilde{X}) \to Ch(\overline{X})$$

has a ring structure induced by the ring structure on $Ch(\widetilde{U})$.

Proposition 4.3.14. *The image of the change of field homomorphism*

$$- : Ch(X) \to Ch(\overline{X})$$

is a subring of $\overline{Ch}(\widetilde{X})$.

Proof. Let $r : U \hookrightarrow X$ be the open embedding and $s : Ch(U) \to Ch(\widetilde{U})$ be the change of field homomorphism. By setting $h := f \circ s$, we have the following commutative diagram

$$\begin{array}{ccc} Ch(X) & \xrightarrow{\ -\ } & \overline{Ch}(\widetilde{X}) \\ \Big\downarrow{\scriptstyle r^*} & \overset{h}{\nearrow} & {\scriptstyle f}\Big\uparrow \\ Ch(U) & \xrightarrow{\ s\ } & Ch(\widetilde{U}) \end{array}$$

Notice that s is a ring homomorphism, hence h is also a ring homomorphism. Since r^* is surjective, $\overline{Ch}(X) = Im(h \circ r^*) = Im(h)$. □

Remark 4.3.15. At this point, we shall explain the meaning of products in $\overline{Ch}(X)$ clearly. Let $r : U \hookrightarrow X$ be the open embedding and $- : Ch(X) \to Ch(\overline{X})$ be the change of field homomorphism. Recall that we have a unique ring homomorphism $h : Ch(U) \to \overline{Ch}(\widetilde{X})$ making the following diagram commute:

$$\begin{array}{ccc} Ch(X) & \xrightarrow{\ -\ } & \overline{Ch}(\widetilde{X}) \\ \Big\downarrow{\scriptstyle r^*} & \underset{h}{\nearrow} & \\ Ch(U) & & \end{array}$$

For cycles $\alpha, \beta \in Ch(X)$, we denote their images under the change of field homomorphism by $\overline{\alpha}$ and $\overline{\beta}$ respectively. Then $\overline{\alpha} \cdot \overline{\beta} = h(r^*(\alpha) \cdot r^*(\beta))$.

Now we want to describe the ring structure of $\overline{Ch}(\widetilde{X})$. By Theorem 4.3.7, we have the external product isomorphism

$$\overline{Ch}(\widetilde{X_1} \times \cdots \times \widetilde{X_n}) \cong \overline{Ch}(\widetilde{X_1}) \otimes \cdots \otimes \overline{Ch}(\widetilde{X_n}).$$

Then the multiplication on $\overline{Ch}(\widetilde{X_1} \times \cdots \times \widetilde{X_n})$ is obtained by multiplication on each $\overline{Ch}(\widetilde{X_i})$, i.e., $(a_1 \times \cdots \times a_n) \cdot (b_1 \times \cdots \times b_n) = (a_1 \cdot b_1) \times \cdots \times (a_n \cdot b_n)$ where a_i, b_i are the standard basis elements $\{l_*, h^*\}$ in $\overline{Ch}(\widetilde{X_i})$. As a result, we only need to describe the multiplication on $\overline{Ch}(\widetilde{X_i})$.

Proposition 4.3.16. *Let ϕ be a quadratic form over F with anisotropic totally singular part. Suppose that $\widetilde{\phi}$ has nonzero maximal Witt index r. We let X be the associated quadric of ϕ and $\{l_0, \cdots, l_{r-1}, h^0, \cdots, h^{r-1}\}$ be the standard basis elements in $\overline{Ch}(\widetilde{X})$. Then we have*

(1) $h^i \cdot h^j = h^{i+j}$ for $i, j \in [0, r-1]$;

(2) $l_i \cdot h^j = l_{i-j}$ for $i, j \in [0, r-1]$;

(3) $l_i \cdot l_i = 0$ *for any* $i \in [0, r-1]$ *except the case that* $\phi_{ts} = 0$ *and* D *is divisible by* 4*. In this case,* $l_{r-1} \cdot l_{r-1} = l_0$ *while* $l_i \cdot l_i = 0$ *for any* $i \neq r-1$*.*

Proof. To simplify notations, we can assume that $F = \widetilde{F}$ (so $\widetilde{X} = X$).

We set $U := X \setminus X_{ts}$ as the smooth locus of X. Remark 4.3.15 says that we have a surjective ring homomorphism $h : Ch(U) \to \overline{Ch}(X)$. To avoid symbol conflicts, we replace the symbol h with $-$. Let $g : U \hookrightarrow \mathbb{P}(V) \setminus X_{ts}$ be the closed embedding. Since $- : Ch(U) \to \overline{Ch}(X)$ is a surjective ring homomorphism, then the map $\psi := - \circ g^* : Ch(\mathbb{P}(V) \setminus X_{ts}) \to \overline{Ch}(X)$ is still a ring homomorphism.

Consider the localization sequence

$$Ch(X_{ts}) \to Ch(\mathbb{P}(V)) \to Ch(\mathbb{P}(V) \setminus X_{ts}) \to 0,$$

we have $Ch(\mathbb{P}(V) \setminus X_{ts}) \cong Ch(\mathbb{P}(V))/Ch(X_{ts})$. Since the term $Ch(X_{ts})$ disappears under the change of field homomorphism $- : Ch(\mathbb{P}(V)) \to Ch(\overline{\mathbb{P}(V)})$, thus we have a well-define ring homomorphism $- : Ch(\mathbb{P}(V) \setminus X_{ts}) \to \overline{Ch}(\mathbb{P}(V))$ (actually, it's an isomorphism).

Also, let $\overline{g} : \overline{X} \to \overline{\mathbb{P}(V)}$ be the closed embedding. We have the following commutative diagram.

$$\begin{array}{ccccc} Ch(\mathbb{P}(V) \setminus X_{ts}) & \xrightarrow{g^*} & Ch(U) & \xrightarrow{g_*} & Ch(\mathbb{P}(V) \setminus X_{ts}) \\ {\scriptstyle -}\downarrow & \searrow^{\psi} & \downarrow{\scriptstyle -} & & \downarrow{\scriptstyle -} \\ \overline{Ch}(\mathbb{P}(V)) & \xrightarrow[\overline{g}^*]{} & \overline{Ch}(X) & \xrightarrow[\overline{g}_*]{} & \overline{Ch}(\mathbb{P}(V)) \end{array}$$

(1): Recall that any class $h^i \in \overline{Ch}^i(X)$ is obtained by the pull-back of the class of codimension-i planes in $Ch(\overline{\mathbb{P}(V)})$ under the closed embedding $\overline{g}$. Firstly, we will show that $h^i = \psi([H]^i)$, where $[H]$ is the hyperplane class in $Ch(\mathbb{P}(V) \setminus X_{ts})$. It follows from the diagram above that $h^i = \overline{g}^*(\overline{[H]^i}) = \overline{g^*([H]^i)} = \psi([H]^i)$. Then the formula comes directly from $h^i \cdot h^j = \psi([H]^i) \cdot \psi([H]^j) == \psi([H]^i \cdot [H]^j) = \psi([H]^{i+j}) = h^{i+j}$.

(2): It suffices to prove that $h \cdot l_i = l_{i-1}$ for any $i \in [0, r-1]$. Let $W \subset V_{r\mathbb{H}}$ be any totally isotropic subspace of dimension $i+1$, where $i \in [0, r-1]$. Then we have $l_i = \overline{[\mathbb{P}(W)]}$ (taking an orientation if needed). Notice that $[\mathbb{P}(W)]$ is a class in both $Ch(\mathbb{P}(V) \setminus X_{ts})$ and $Ch(U)$. By using the Projection Formula (Theorem 2.2.5), we have

$$\overline{g}_*(h \cdot l_i) = \overline{g}_*(\overline{g^*([H]) \cdot [\mathbb{P}(W)]}) = \overline{g_*([\mathbb{P}(W)] \cdot [H])} = \overline{[\mathbb{P}(W')]} \neq 0$$

where $\mathbb{P}(W')$ is a projective linear space of dimension $i-1$. Since $\overline{g}_*(h \cdot l_{i-1})$ is a nonzero cycle in $\overline{Ch}_{i-1}(\mathbb{P}(V))$ and l_{i-1} is the only nonzero element in $\overline{Ch}_i(X)$, we conclude that $h \cdot l_i = l_{i-1}$.

(3): Suppose that $dim(\phi_{ts}) = s$. For the cycle $l_i \cdot l_i$ with $i \leq r-1$, the codimension counting shows that $(D-i)+(D-i) \geq D+s$. Thus $l_i \cdot l_i$ is always zero except the case that $s=0, i=r-1$ and $D=2(r-1)$. In this case, we have $U=X$. Let W be any totally isotropic subspace of dimension r. Thus we have $l_{r-1} = [\mathbb{P}(W)]$ after taking an orientation. Since $D=2(r-1)$, when D is divisible by 4, we know that $r-1$ is divisible by 2. For a vector space W', recall that if W' is obtained by applying even number reflections, then $[\mathbb{P}(W)] = [\mathbb{P}(W')]$. It's easy to see that there's a sequence of $(r-1)$ reflections under which the image of W (i.e., W') intersects W is a line. Hence $[\mathbb{P}(W)] \cdot [\mathbb{P}(W')] = l_{r-1} \cdot l_{r-1} = l_0$.

When D is not divisible by 4, we know that $r-1$ is an odd number. Recall that if a vector space W' is obtained by applying odd number reflections, then $[\mathbb{P}(W)] + [\mathbb{P}(W')] = h^{r-1}$. Now there's a sequence of $(r-1)$-reflections under which the image of W (i.e., W') intersects W is a line. As a result, we have $l_{r-1} \cdot (l_{r-1} + h^{r-1}) = l_{r-1} \cdot l_{r-1} + l_0 = l_0$. So $l_{r-1} \cdot l_{r-1} = 0$. $\square$

4.3.3 Steenrod operations

In this subsection, we will introduce the Steenrod operations on $\overline{Ch}(X)$.

Let $\phi_i, i = 1, \cdots, n$ be quadratic forms on F-vector spaces V_i with anisotropic totally singular part. Suppose $X_1, \cdots, X_n$ are the associated quadrics of $\phi_1, \cdots, \phi_n$ respectively. Also, let U_i be the smooth locus of X_i. We set $X := X_1 \times \cdots \times X_n$ and $U := U_1 \times \cdots \times U_n$.

Let f be the unique ring homomorphism which makes the following diagram commute:

$$\begin{array}{ccc} Ch(\widetilde{X}) & \xrightarrow{\ -\ } & \overline{Ch}(\widetilde{X}) \\ \downarrow{\scriptstyle r^*} & \nearrow{\scriptstyle f} & \\ Ch(\widetilde{U}) & & \end{array}$$

where $r : \widetilde{U} \hookrightarrow \widetilde{X}$ is the open embedding and $- : Ch(\widetilde{X}) \to Ch(\overline{X})$ is the change of field homomorphism. Recall that we have the cohomological Steenrod operation $Sq^*_{\widetilde{U}} : Ch(\widetilde{U}) \to Ch(\widetilde{U})$ and that f is surjective.

Theorem 4.3.17. *$Ker(f)$ is stable under the cohomological Steenrod operation $Sq^*_{\widetilde{U}}$: $Ch(\widetilde{U}) \to Ch(\widetilde{U})$, i.e., $Sq^*_{\widetilde{U}}(Ker(f)) \subset Ker(f)$. Hence $Sq^*_{\widetilde{U}}$ induces a well-defined ring endmorphism $Sq^* : \overline{Ch}(\widetilde{X}) \to \overline{Ch}(\widetilde{X})$, which is called the cohomological Steenrod operation on $\overline{Ch}(\widetilde{X})$.*

Proof. For any smooth projective variety Z over F, we take any cycle $\gamma \in Ch_i(Z)$ with $i > 0$. Recall that the homological Steenrod operation Sq_* over Z is defined to be $Sq_* = c(-T_Z) \circ Sq^*_Z$. Since Sq_* commutes with push-forwards (Proposition 2.6.6), one has

$$deg(c(-T_Z) \circ Sq^*_Z(\gamma)) = Sq_*(deg(\gamma)) = deg(\gamma) = 0. \tag{4.2}$$

Recall that since Z is smooth, the Chern class $c(-T_Z) = c(-T_Z)[Z]$ (Proposition 2.5.5). Thus we have $Sq_* = c(-T_Z) \circ Sq^*_Z = c(-T_Z) \cdot Sq^*_Z$.

Now we show the equation (3.2) is also true for the open subvariety $\widetilde{U}$. Consider the closed embedding

$$i : \widetilde{U} \hookrightarrow (\widetilde{\mathbb{P}(V_1)} \setminus \widetilde{X_{1,ts}}) \times \cdots \times (\widetilde{\mathbb{P}(V_n)} \setminus \widetilde{X_{n,ts}}).$$

and the open embedding

$$j : (\widetilde{\mathbb{P}(V_1)} \setminus \widetilde{X_{1,ts}}) \times \cdots \times (\widetilde{\mathbb{P}(V_n)} \setminus \widetilde{X_{n,ts}}) \hookrightarrow (\widetilde{\mathbb{P}(V_1)}) \times \cdots \times (\widetilde{\mathbb{P}(V_n)}).$$

To simplify notations, we set $V := (\widetilde{\mathbb{P}(V_1)} \setminus \widetilde{X_{1,ts}}) \times \cdots \times (\widetilde{\mathbb{P}(V_n)} \setminus \widetilde{X_{n,ts}})$ and $Y := \widetilde{\mathbb{P}(V_1)} \times \widetilde{\mathbb{P}(V_2)} \times \cdots \times \widetilde{\mathbb{P}(V_n)}$.

By Proposition 2.5.4 and Proposition 2.1.6, for any $\alpha \in Ch_i(\widetilde{U})$ with $i > 0$, we have

$$deg(Sq_*(\alpha)) = deg(i_*(Sq_*(\alpha))) = deg(Sq_*(i_*(\alpha))) = deg(c(-T_V) \cdot Sq^*(i_*(\alpha))).$$

Note that $j^* : Ch(Y) \to Ch(V)$ is surjective, so there exists a cycle $\beta \in Ch(Y)$ such that $i_*(\alpha) = j^*(\beta)$. Since Y is a projective smooth variety and $j : V \hookrightarrow Y$ is an open embedding, by Proposition 2.5.4 and Theorem 2.6.1, we have

$$deg(c(-T_Z)Sq^*(i_*(\alpha))) = deg(c(-T_Z)Sq^*(j^*(\beta))) = j^*deg(c(-T_Y)Sq^*(\beta)) = 0$$

Thus we showed for any $\alpha \in Ch_i(\widetilde{U})$ with $i > 0$, $deg(c(-T_{\widetilde{U}})Sq^*(\alpha)) = deg(\alpha) = 0$.

Recall that we use N to denote the ideal of numerically trivial elements in $Ch(\widetilde{U})$.

Lemma 4.3.12 says that $Ker(f) = N$. So finally, we need to show that N is stable under the total cohomological Steenrod operation. Let $\alpha \in N$ be a homogeneous numerically trivial element. We use induction on Sq^i. When $i = 0$, the statement is trivially true. Assume the statement is true for all $i < k$ for some $k \geq 1$. Now we take any homogeneous $\beta \in Ch(\widetilde{U})$ such that $\alpha \cdot \beta \in Ch_m(\widetilde{U})$ with $m > 0$. By the previous paragraph,

$$deg(c(-T_{\widetilde{U}})Sq^*(\alpha \cdot \beta)) = deg(\alpha \cdot \beta) = 0.$$

From the Cartan Formula(Proposition 2.6.4) and the induction hypothesis, the term $c(-T_{\widetilde{U}})Sq^*(\alpha \cdot \beta)$ becomes a sum of $Sq^k(\alpha) \cdot \beta$ and some numerically trivial elements. Hence $deg(Sq^k(\alpha) \cdot \beta) = 0$ and we prove the theorem. □

The homological Steenrod operation over $\widetilde{U}$ is defined to be $Sq_* = c(-T_{\widetilde{U}}) \circ Sq^*_{\widetilde{U}}$. Recall again that we have $c(-T_{\widetilde{U}}) = c(-T_{\widetilde{U}})[\widetilde{U}]$. Since the numerically trivial ideal N in $Ch(\widetilde{U})$ is stable under $Sq^*_{\widetilde{U}}$, it is also stable under Sq_*. So we have the following corollary.

Corollary 4.3.18. *$Ker(f)$ is stable under the homological Steenrod operation $Sq_* : Ch(\widetilde{U}) \to Ch(\widetilde{U})$, i.e., $Sq_*(Ker(f)) \subset Ker(f)$. Hence Sq_* induces a well-defined group homomorphism $Sq_* : \overline{Ch}(\widetilde{X}) \to \overline{Ch}(\widetilde{X})$, which is called the homological Steenrod operation on $\overline{Ch}(\widetilde{X})$.*

We let $h : Ch(U) \to \overline{Ch}(\widetilde{U})$ be the ring homomorphism which makes the following diagram commute

$$\begin{array}{ccc} Ch(X) & \xrightarrow{\ -\ } & \overline{Ch}(\widetilde{X}) \\ \downarrow{\scriptstyle r^*} & \overset{h}{\nearrow} & {\scriptstyle f}\uparrow \\ Ch(U) & \xrightarrow{\ s\ } & Ch(\widetilde{U}) \end{array}$$

where $-$ is the change of field homomorphism, $r : U \hookrightarrow X$ is the open embedding and $s : Ch(U) \to Ch(\widetilde{U})$ is the change of field homomorphism (Proposition 4.3.14). Then $Ker(h) = \{\alpha \in Ch(U) \mid s(\alpha) \in Ker(f)\}$. Since the cohomological Steenrod operation $Sq^*_U : Ch(U) \to Ch(U)$ commutes with pull-backs (Proposition 2.6.1), thus for any element $\alpha \in Ker(h)$, we have $s(Sq^*_U(\alpha)) = Sq^*_{\widetilde{U}}(s(\alpha)) \in Ker(f)$ by Theorem 4.3.17. As a result, we have the following Proposition.

Proposition 4.3.19. *The cohomological Steenrod Operation Sq^* over $Ch(U)$ induces a well-defined ring endomorphism on $\overline{Ch}(X)$, which is also called the cohomological*

Steenrod operation on $\overline{Ch}(X)$. The homological Steenrod Operation Sq_* over $Ch(U)$ induces a well-defined group homomorphism on $\overline{Ch}(X)$, which is also called the homological Steenrod operation on $\overline{Ch}(X)$.

In the last of this subsection, we will describe the cohomological Steenrod Operation $Sq^* : \overline{Ch}(\widetilde{X}) \to \overline{Ch}(\widetilde{X})$ explicitly. By Theorem 2.6.3, we know that

$$Sq_{(\widetilde{U}_1 \times \cdots \times \widetilde{U}_n)} = Sq_{\widetilde{U}_1} \times \cdots \times Sq_{\widetilde{U}_n}.$$

Thus we know that $Sq^* = Sq_1^* \times \cdots \times Sq_n^*$ where $Sq_i^* : \overline{Ch}(\widetilde{X}_i) \to \overline{Ch}(\widetilde{X}_i)$ is the cohomological Steenrod operation. As a result, it suffices to consider the case of one quadric. The following Proposition is a composition of results in [11] and [2].

Proposition 4.3.20. [11, Page 9] [2, Corollary 78.5] *Let ϕ be a quadratic form over F with anisotropic totally singular part ϕ_{ts}. We assume that X is the associated quadric of ϕ. Suppose that $\widetilde{\phi}$ has nonzero maximal Witt index r. Then the cohomological Steenrod operation $Sq^* : \overline{Ch}(\widetilde{X}) \to \overline{Ch}(\widetilde{X})$ is given by $Sq^*(h^i) = (1+h)^i h^i$ and $Sq^*(l_i) = (1+h)^{dim(\phi)-i-1} l_i$ for $i \in [0, r-1]$. In particular, for any $j \geq 0$,*

$$Sq^j(h^i) = \binom{i}{j} h^{i+j}, Sq^j(l_i) = \binom{dim(\phi)-i-1}{j} l_{i-j}.$$

Proof. To simplify notations, we set that $F = \widetilde{F}$ (so $X = \widetilde{X}$).

We let X_{ts} be the associated quadric of ϕ_{ts} and set $U := X \setminus X_{ts}$ as the smooth locus. Let $\{l_0, \cdots, l_{r-1}, h^0, \cdots, h^{r-1}\}$ be the standard basis elements of $\overline{Ch}(X)$. By Remark 4.3.15, for any $i \in [0, r-1]$, we represent the element l_i by the class of the projective space of any $i+1$-dimensional totally isotropic subspace $W \subset V_{r\mathbb{H}}$. Also, the element h^i can be represented as the pull-back of the class of any codimension i plane in $\mathbb{P}(V_\phi) \setminus X_{ts}$ under the closed embedding $g : U \hookrightarrow \mathbb{P}(V_\phi) \setminus X_{ts}$. To simplify notations, we set $V := \mathbb{P}(V_\phi) \setminus X_{ts}$.

Since $Sq^* : \overline{Ch}(X) \to \overline{Ch}(X)$ is induced by the cohomological Steenrod operation Sq_U^*, we only need to compute $Sq_U^*(h^i)$ and $Sq_U^*(l_i)$ in $Ch(U)$. Let $[H]$ be the class of a hyperplane in $Ch(V)$. Note that the cohomological Steenrod operation commutes with pull-back (Theorem 2.6.1), we have

$$Sq_U^*(h^i) = Sq_U^*(g^*([H]^i)) = g^*(Sq_V^*([H]^i)) = g^*([H]^i(1+[H])^i) = h^i(1+h)^i.$$

where $Sq_{\widetilde{V}}^*([H]^i) = [H](1+[H])^i$ by Example 2.6.9.

We let W be any $(i+1)$-dimensional totally isotropic subspace of $V_{r\mathbb{H}}$. Then $[\mathbb{P}(W)] \in Ch(U)$. Let $i : \mathbb{P}(W) \hookrightarrow U$ be the closed embedding. Then

$$Sq_U([\mathbb{P}(W)]) = Sq_U(i_*(\mathbb{P}(W))) = i_*(c(N)[\mathbb{P}(W)])$$

where $c(N)$ is the totally Chern class of the normal bundle of the embedding [2, Proposition 68.1].

Since we have the exact sequence of vector bundles [2, Proposition 104.15]

$$0 \to T_{\mathbb{P}(W)} \to i^*(T_U) \to N \to 0,$$

we only need to compute $c(T_{\mathbb{P}(W)})$ and $c(i^*T_U)$ (Proposition 2.5.6).

However, $c(T_{\mathbb{P}(W)})$ is computed as $(1+h^{'})^{i+1}$, where $h^{'}$ is the hyperplane class in $\mathbb{P}(W)$ (Example 2.6.9). Also, $c(i^*T_U)$ is also computed as the pull-back of $(1+h)^{dim(\phi)}$ along the closed embedding i [2, Lemma 78.1].

Proposition 2.5.6 then gives that

$$c(N)([\mathbb{P}(W)]) = (1+h^{'})^{dim(\phi)-i-1}[\mathbb{P}(W)].$$

Thus we have

$$Sq^*_U([\mathbb{P}(W)]) = i_*((1+h^{'})^{dim(\phi)-i-1}[\mathbb{P}(W)]) = (1+h)^{dim(\phi)-i-1}([\mathbb{P}(W)]),$$

which gives the result. □

4.4 Rational correspondences

Fix a field F. Let $\overline{F}$ be an algebraic closure of F and $\widetilde{F}$ be a separable closure of F in $\overline{F}$. Let ϕ be a quadratic form over F and Q be the associated quadric of ϕ. Recall that Q is a generically smooth quadric if and only if ϕ is not totally singular (Proposition 3.3.3).

Now we let X, Y, Z be products of generically smooth quadrics over F, i.e.,

$$X = X_1 \times \cdots \times X_r, Y = Y_1 \times \cdots \times Y_s, Z = Z_1 \times \cdots \times Z_t.$$

where X_i, Y_j, Z_k are generically smooth quadrics over F given by quadratic forms with anisotropic totally singular parts. We define morphisms

$$p_3 := 1_X \times 1_Y \times p_Z : X \times Y \times Z \to X \times Y$$

where $p_Z : Z \to Spec(F)$ and

$$p_1 := p_X \times 1_Y \times 1_Z : X \times Y \times Z \to Y \times Z$$

where $p_X : X \to Spec(F)$.

We also define $p := 1_X \times p_Y \times 1_Z : X \times Y \times Z \to X \times Z$ where $p_Y : Y \to Spec(F)$. Then we have the following definition.

Definition 4.4.1. By the previous sections, we have an $\mathbb{F}_2$-bilinear pairing

$$\overline{Ch}(Y \times Z) \times \overline{Ch}(X \times Y) \to Ch(\overline{X} \times \overline{Z})$$

defined by

$$(\overline{\beta}, \overline{\alpha}) \mapsto \overline{\beta} \circ \overline{\alpha} := p_*(p_3^*(\overline{\alpha}) \cdot p_1^*(\overline{\beta}))$$

where $\overline{\alpha} \in \overline{Ch}(X \times Y)$ and $\overline{\beta} \in \overline{Ch}(Y \times Z)$. The cycle $\overline{\beta} \circ \overline{\alpha}$ is called the *composition of rational correspondences.*

Lemma 4.4.2. *The composition of rational cycles is still rational. Thus the bilinear pairing above becomes*

$$\overline{Ch}(Y \times Z) \times \overline{Ch}(X \times Y) \to \overline{Ch}(X \times Z).$$

Proof. Let $\alpha \in Ch(X \times Y)$. We denote the image of α under the change of homomorphism as $\overline{\alpha} \in \overline{Ch}(X \times Y)$. We consider the commutative diagram which only involves with projections.

$$\begin{array}{ccc} \overline{X} \times \overline{Y} \times \overline{Z} & \xrightarrow{\overline{p_3}} & \overline{X} \times \overline{Y} \\ \downarrow & & \downarrow \\ X \times Y \times Z & \xrightarrow{p_3} & X \times Y \end{array}$$

This induces the following commutative diagram of Chow groups

$$\begin{array}{ccc} Ch(X \times Y \times Z) & \xleftarrow{(p_3)^*} & Ch(X \times Y) \\ \big\downarrow{-} & & \big\downarrow{-} \\ Ch(\overline{X} \times \overline{Y} \times \overline{Z}) & \xleftarrow[(\overline{p_3})^*]{} & Ch(\overline{X} \times \overline{Y}) \end{array}$$

where we use $-$ to denote the change of field homomorphism.

Since $\overline{\alpha}$ is a rational cycle, then diagram shows that the pull-back of $\overline{\alpha}$ is still rational, i.e., $(\overline{p_3})^*(\overline{\alpha}) := \overline{\alpha}' \in \overline{Ch}(X \times Y \times Z)$.

Let $\beta \in Ch(Y \times Z)$. We denote the image of β under the change of homomorphism as $\overline{\beta} \in \overline{Ch}(Y \times Z)$. Let $\overline{p_1} : X \times (Y \times Z) \to Y \times Z$ be the projection onto the second factor. By the same argument, we have $(\overline{p_1})^*(\overline{\beta}) := \overline{\beta}' \in \overline{Ch}(X \times Y \times Z)$.

Notice that $\overline{Ch}(X \times Y \times Z)$ is a ring, so $\overline{\alpha}' \cdot \overline{\beta}'$ is still rational. By the following diagram

$$\begin{array}{ccc} Ch(X \times Z) & \xleftarrow[p_*]{} & Ch(X \times Y \times Z) \\ \big\downarrow{-} & & \big\downarrow{-} \\ Ch(\overline{X} \times \overline{Z}) & \xleftarrow[\overline{p_*}]{} & Ch(\overline{X} \times \overline{Y} \times \overline{Z}) \end{array}$$

we conclude that $\overline{p_*}(\overline{\alpha}' \cdot \overline{\beta}')$ is a rational cycle in $Ch(\overline{X} \times \overline{Z})$, which gives the lemma. □

The following lemma describes the composition of elements in $\overline{Ch}(\widetilde{Y} \times \widetilde{Z})$ and $\overline{Ch}(\widetilde{X} \times \widetilde{Y})$ explicitly. The proof of the lemma below comes directly from definition of the composition of rational correspondences.

Lemma 4.4.3. *Let $\overline{\alpha}$ be a cycle in $\overline{Ch}(\widetilde{X})$ and $\overline{\beta}$ be a cycle in $\overline{Ch}(\widetilde{Y})$. By the external product map, we have a cycle $\overline{\alpha} \times \overline{\beta} \in \overline{Ch}(\widetilde{X} \times \widetilde{Y})$. Similarly, we also have a cycle $\overline{\gamma} \times \overline{\mu} \in \overline{Ch}(\widetilde{Y} \times \widetilde{Z})$. Then*

$$(\overline{\gamma} \times \overline{\mu}) \circ (\overline{\alpha} \times \overline{\beta}) = deg(\overline{\beta} \cdot \overline{\gamma})(\overline{\alpha} \times \overline{\mu}).$$

Moreover,if $\overline{\beta} = a_1 \times \cdots \times a_s$ and $\overline{\gamma} = b_1 \times \cdots \times b_s$ where each factor of $\overline{\beta}$ and $\overline{\gamma}$ is an element in $\{l_*, h^*\}$. Then

$$deg(\overline{\beta} \cdot \overline{\gamma}) = \begin{cases} 1 & \textit{if } a_i \cdot b_i = l_0 \textit{ for all } i \in [1, s] \\ 0 & \textit{otherwise.} \end{cases}$$

By Lemma 4.4.2, for any correspondence $\overline{\alpha} \in \overline{Ch}(X \times Y)$, we have a well-defined homomorphism

$$(\overline{\alpha})_* : \overline{Ch}(Z \times X) \to \overline{Ch}(Z \times Y), \overline{\beta} \mapsto \overline{\alpha} \circ \overline{\beta}.$$

which is called the *push-forward* homomorphism induced by $\overline{\alpha}$. Similarly, we also have a well-defined homomorphism

$$(\overline{\alpha})^* : \overline{Ch}(Y \times Z) \to \overline{Ch}(X \times Z), \overline{\beta} \mapsto \overline{\beta} \circ \overline{\alpha}.$$

which is called the *pull-back* homomorphism induced by $\overline{\alpha}$.

Proposition 4.4.4. *Let X and Y be products of generically smooth quadrics given by quadratic forms with anisotropic totally singular parts. Also, we let U_X and U_Y be the smooth loci of X and Y respectively. Suppose that we have a morphism $f : X \to Y$ such that $f(U_X) \subset U_Y$. We denote the graph of f by Γ_f, which gives rise to a cycle in $Ch(X \times Y)$. Then*

(1) $\overline{[\Gamma_f]}_* : \overline{Ch}(X) \to \overline{Ch}(Y)$ coincides with $\overline{f}_*$ where $\overline{f} : \overline{X} \to \overline{Y}$.

(2) If f is flat or a regular closed embedding or Y is smooth, then $\overline{[\Gamma_f]}^* : \overline{Ch}(Y) \to \overline{Ch}(X)$ coincides with $\overline{f}^*$ where $\overline{f} : \overline{X} \to \overline{Y}$.

Proof. By [4, Corollary 4.8, Chapter II], we know that f is a proper morphism. We denote the morphism $f\mid_{U_X}$ by $f^{'}$. By our assumption $f(U_X) \subset f(U_Y)$, we have a fiber product diagram

$$\begin{array}{ccc} U_X & \xrightarrow{1 \times f^{'}} & U_X \times U_Y \\ \downarrow g_1 & & \downarrow g_2 \\ X & \xrightarrow{1 \times f} & X \times Y \end{array}$$

where g_1 and g_2 are the open embeddings. Also, by Proposition 4.3.11, we also have the following commutative diagram

$$\begin{array}{ccc} Ch(X \times Y) & \xrightarrow{\ \overline{\ \ }\ } & \overline{Ch}(X \times Y) \\ \downarrow g_2^* & \nearrow h & \\ Ch(U_X \times U_Y) & & \end{array}$$

Let $p_1 : X \times Y \to Y$ and $p_2 : X \times Y \to X$ be two projections. Let $p_1^{'} : U_X \times U_Y \to U_Y$ and $p_2^{'} : U_X \times U_Y \to U_X$. For any cycle $\alpha \in Ch(X)$, we denote its image under the change of field homomoprhism by $\overline{\alpha}$.

(1): By definition, we have $\overline{[\Gamma_f]}_*(\overline{\alpha}) = \overline{[\Gamma_f]} \circ \overline{\alpha} = (p_1)_*(p_2^*(\overline{\alpha}) \cdot \overline{[\Gamma_f]})$. By Proposition 2.2.4 and the Projection Formula(Proposition 2.2.5), we have

$$\begin{aligned}
(p_1)_*(p_2^*(\overline{\alpha}) \cdot \overline{[\Gamma_f]}) &= (p_1)_* h(g_2^*(\alpha \times [Y]) \cdot g_2^*(1 \times f)_*([X])) \\
&= (p_1)_* h((g_1^*(\alpha) \times [U_Y]) \cdot (1 \times f^{'})_*([U_X])) \\
&= (p_1)_* h((1 \times f^{'})_*((1 \times f^{'})^*(g_1^*(\alpha) \times [U_Y]) \cdot [U_X])) \\
&= (p_1)_* h((1 \times f^{'})_*(1 \times f^{'})^*(p_2^{'*})(g_1^*(\alpha))) \\
&= (p_1)_* h((1 \times f^{'})_*(g_1^*(\alpha))) \\
&= (p_1)_* h g_2^*((1 \times f)_*(\alpha)) \\
&= \overline{f}_*(\overline{\alpha}).
\end{aligned}$$

(2): Let $g_3 : U_Y \hookrightarrow Y$ be the open embedding. By definition, for any cycle $\overline{\beta} \in Ch(Y)$, we have $\overline{[\Gamma_f]}^*(\overline{\beta}) = \overline{\beta} \circ \overline{[\Gamma_f]} = (p_2)_*((p_1)^*(\overline{\beta}) \cdot \overline{[\Gamma_f]})$. By Proposition 2.2.4 and the Projection Formula(Proposition 1.2.5), we have

$$\begin{aligned}
(p_2)_*(p_1^*(\overline{\beta}) \cdot \overline{[\Gamma_f]}) &= (p_2)_* h(g_2^*([X] \times [\beta]) \cdot g_2^*(1 \times f)_*([X])) \\
&= (p_2)_* h(([U_X] \times g_3^*(\beta)) \cdot (1 \times f^{'})_*([U_X])) \\
&= (p_2)_* h((1 \times f^{'})_*((1 \times f^{'})^*(([U_X] \times g_3^*(\beta)) \cdot [U_X])) \\
&= (p_2)_* h((1 \times f^{'})_*(1 \times f^{'})^*(p_1^{'*})(g_3^*(\beta))) \\
&= (p_2)_* h((1 \times f^{'})_* f^{'*} g_3^*(\beta)) \\
&= (p_2)_* h(1 \times f^{'})_*(g_1^*(f^*(\beta)) \\
&= (p_2)_* h g_2^*(1 \times f)_*(f^*(\beta)) \\
&= \overline{f}^*(\overline{\beta}).
\end{aligned}$$

This finishes our proof. □

Corollary 4.4.5. *Let X be a generically smooth quadric given by a quadratic form with anisotropic totally singular part. We let Δ be the class of the diagonal in $Ch(X \times X)$ and $\overline{\Delta}$ the image of Δ under the change of field homomorphism. Then the push-forward*

$$\overline{\Delta}_* : \overline{Ch}(X \times X) \to \overline{Ch}(X \times X)$$

is the identity endomorphism.

Proof. This corollary comes directly from Proposition 4.4.4 once we notice that Δ is the class of the graph of the identity morphism on X. $\square$

4.5 Products of quadrics with nonzero Witt index

Let ϕ' be a quadratic form on an F-vector space V' and W be an F-vector space of dimension r over F with the dual space $W^\vee$. Notice that we have the quadratic form $\mathbb{H}(W)$ defined on the vector space $W \oplus W^\vee$. Then we let $\phi = \mathbb{H}(W) \perp \phi'$ be the orthogonal sum of ϕ' and $\mathbb{H}(W)$. So ϕ is a quadratic form on the F-vector space $V := V' \oplus W \oplus W^\vee$.

Suppose X is the associated quadric of ϕ and X' is the associated quadric of ϕ'. We set $m = r + dim(X') + 1$.

Lemma 4.5.1. *We have an isomorphism*

$$f : Ch_*(X) \to Ch_*(\mathbb{P}(W)) \oplus Ch_{*-r}(X') \oplus Ch_{*-m}(\mathbb{P}(W^\vee)).$$

We use C_1, C_2, C_3 to denote $\mathbb{P}(W), X'$ and $\mathbb{P}(W^\vee)$ respectively. More generally, for any F-variety Z, we have an isomorphism

$$f_Z : Ch_*(X \times Z) \to Ch_*(C_1 \times Z) \oplus Ch_{*-r}(C_2 \times Z) \oplus Ch_{*-m}(C_3 \times Z).$$

which makes the following diagram commute

$$\begin{array}{ccccc} Ch(X) \otimes Ch(Z) & \xrightarrow{f \otimes id} & (\oplus_{i=1}^3 Ch(C_i)) \otimes Ch(Z) & \xrightarrow{\sim} & \oplus_{i=1}^3 (Ch(C_i) \otimes Ch(Z)) \\ \downarrow \times & & & & \downarrow \times \\ Ch(X \times Z) & & \xrightarrow{f_Z} & & \oplus_{i=1}^3 (Ch(C_i \times Z)) \end{array}$$

where the external product maps are denoted by $\times$ and the isomorphism

$$(\oplus_{i=1}^3 Ch(C_i)) \otimes Ch(Z) \cong \oplus_{i=1}^3 (Ch(C_i) \otimes Ch(Z))$$

is the canonical isomorphism from the tensor product.

Proof. Let $\psi := \phi \mid_{V' \oplus W}$ and Y be the associated quadric of ψ. Then Y is a closed subvariety in X of dimension $m - 1$. By the same argument as Lemma 4.1.1 shows that $X \setminus Y$ is an affine bundle over the smooth variety $\mathbb{P}(W^\vee)$ with rank $dim(Y) + 1$. Thus X has a relative cellular structure $\emptyset \subset Y \subset X$. By Theorem 2.4.5, for any

F-variety Z, we obtain

$$Ch_*(X \times Z) \simeq Ch_*(Y \times Z) \oplus Ch_{*-m}(\mathbb{P}(W^\vee) \times Z).$$

Note that $\mathbb{P}(W)$ is a closed subvariety of Y. Let $h \times id : \mathbb{P}(W) \times Z \hookrightarrow Y \times Z$ be the closed embedding and $r : Y \setminus \mathbb{P}(W) \hookrightarrow Y$ be the open embedding. We have the localization sequence

$$Ch_*(\mathbb{P}(W) \times Z) \xrightarrow{(h\times id)_*} Ch_*(Y \times Z) \xrightarrow{(r\times id)^*} Ch_*(Y \setminus \mathbb{P}(W)) \to 0$$

A similar argument as Lemma 4.1.3 shows that the canonical projection $p : Y \setminus \mathbb{P}(W) \to X'$ is an affine bundle of rank r. Thus we have the following exact sequence

$$Ch_*(\mathbb{P}(W) \times Z) \xrightarrow{(h\times id)_*} Ch_*(Y \times Z) \xrightarrow{(p\times id)^{*-1}\circ(r\times id)^*} Ch_{*-r}(X' \times Z) \longrightarrow 0.$$

A similar argument as Lemma 4.1.4 shows that we have a homomorphism $\pi_Z' : Ch_*(Y \times Z) \to Ch_*(\mathbb{P}(W) \times Z)$ such that $\pi_Z' \circ (h \times id)_* = id$ and π_Z' is compatible with push-forwards. In particular, $(h \times id)_*$ is split injective. This finishes our proof. □

In the decomposition of $Ch_*(X)$ from Lemma 4.5.1, we consider the projection

$$p : Ch_*(\mathbb{P}(W)) \oplus Ch_{*-r}(X') \oplus Ch_{*-m}(\mathbb{P}(W^\vee)) \twoheadrightarrow Ch_{*-r}(X')$$

and the inclusion

$$i : Ch_{*-r}(X') \hookrightarrow Ch_*(\mathbb{P}(W)) \oplus Ch_{*-r}(X') \oplus Ch_{*-m}(\mathbb{P}(W^\vee)).$$

We define the group homomorphism $pr_* := p \circ f : Ch_*(X) \to Ch_{*-r}(X')$ and the group homomorphism $in_* := f^{-1} \circ i : Ch_{*-r}(X') \to Ch_*(X)$.

Similarly, from the decomposition of $Ch_*(\overline{X})$, we still have group homomorphisms $pr_* : Ch_*(\overline{X}) \to Ch_{*-r}(\overline{X'})$ and $in_* : Ch_{*-r}(\overline{X'}) \to Ch_*(\overline{X})$. We use $-$ denote the change of field homomorphism. Obviously, we have the following commutative diagram.

$$\begin{array}{ccc} Ch(X) & \xrightarrow{pr_*} & Ch(X') \\ \downarrow{\scriptstyle -} & & \downarrow{\scriptstyle -} \\ Ch(\overline{X}) & \xrightarrow{pr_*} & Ch(\overline{X'}) \end{array}$$

Also, we have the following commutative diagram.

$$\begin{array}{ccc} Ch(X') & \xrightarrow{in_*} & Ch(X) \\ \big\downarrow{\scriptstyle -} & & \big\downarrow{\scriptstyle -} \\ Ch(\overline{X'}) & \xrightarrow{in_*} & Ch(\overline{X}) \end{array}$$

It follows that $pr_* : Ch_*(\overline{X}) \to Ch_{*-r}(\overline{X'})$ and $in_* : Ch_{*-r}(\overline{X'}) \to Ch_*(\overline{X})$ preserve rational cycles.

Proposition 4.5.2. *By taking suitable orientations, the group homomorphism* $pr_* : Ch_*(\overline{X}) \to Ch_{*-r}(\overline{X'})$ *is given by* $l_i \mapsto l_{i-r}$ *and* $h^j \mapsto h^{j-r}$ *for the standard basis elements* $\{l_*, h^*\}$ *in* $Ch(\overline{X})$*. Symmetrically, the group homomorphism* $in_* : Ch_{*-r}(\overline{X'}) \to Ch_*(\overline{X})$ *is given by* $l_{i-r} \mapsto l_i$ *and* $h^{j-r} \mapsto h^j$ *for the standard basis elements* $\{l_*, h^*\}$ *in* $Ch(\overline{X'})$

Proof. Note that $\mathbb{H}(W) \simeq r\mathbb{H}$. We set the maximal Witt index of $\overline{\phi'}$ to be $r^{'}$ and $dim(\phi^{'}_{ts}) = s$. When $\overline{\phi} \simeq (r + r^{'})\mathbb{H} \perp \overline{\phi^{'}_{ts}}$ and $\overline{\phi^{'}_{ts}} \simeq \langle 0, \cdots, 0\rangle$, recall that $Ch(\overline{X})$ is an $\mathbb{F}_2$-vector space spanned by the standard basis elements

$$\{l_0, \cdots, l_{r+r'+s-1}, h^0, \cdots, h^{r+r^{'}-1}\}.$$

This is the only case that we have the problem of orientations.

By Proposition 4.2.1, when $r \leq i < r+r^{'}+s-1$, there's only one non-zero element in $Ch_i(\overline{X})$, i.e., l_i. Also, there's only one non-zero element in $Ch_{i-r}(\overline{X'})$, i.e., l_{i-r}. Hence we have $l_i \mapsto l_{i-r}$ or $l_{i-r} \mapsto l_i$ when $r \leq i < r + r^{'} + s - 1$. Similarly, when $r \leq j \leq r + r^{'} + s - 1$, there's only one nonzero element $h^j \in Ch^j(\overline{X})$. Also, there's only one nonzero element $h^{j-r} \in Ch^{j-r}(\overline{X'})$. Hence we have $h^j \mapsto h^{j-r}$ or $h^{j-r} \mapsto h^j$.

Recall that

$$Ch_{r+r'+s-1}(\overline{X}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \cong Ch_{r'+s-1}(\overline{X}^{'})$$

(Proposition 4.2.1). Thus we have a problem of orientations. By explicitly calculating the images of $l_{r+r'+s-1}$ and $h^{r+r^{'}-1}$ under the homomorphisms pr_* and in_*, it can be shown that $h^{r+r^{'}-1} \in Ch^{r+r^{'}-1}(\overline{X})$ maps to $h^{r^{'}-1} \in Ch^{r^{'}-1}(\overline{X'})$ (vice versa) and $l_{r+r'+s-1}$ maps to the class of a projective linear subspace. Thus, for a suitable choice of orientation on $X^{'}$, we can say that $l_{r+r'+s-1}$ maps to $l_{r'+s-1}$ (vice verse) [2, Lemma 73.2].

We have the other two different cases. When $\overline{\phi'_{ts}} \simeq \langle 1, 0, \cdots, 0 \rangle$, $Ch(\overline{X})$ is an $\mathbb{F}_2$-vector space spanned by the standard basis elements

$$\{l_0, \cdots, l_{r+r'+s-2}, h^0, \cdots, h^{r+r'-1}\}.$$

Also, when $\overline{\phi'} \simeq r'\mathbb{H} \perp \langle 1 \rangle \perp \overline{\phi'_{ts}}$, $Ch(\overline{X})$ is an $\mathbb{F}_2$-vector space spanned by the standard basis elements $\{l_0, \cdots, l_{r+r'+s-1}, h^0, \cdots, h^{r+r'-1}\}$.

In these two cases, we don't have the problem of orientations. This means that there's only one nonzero element in $Ch_i(\overline{X})$ and $Ch^j(\overline{X'})$ respectively. Also, there's only one nonzero element in $Ch_{i-r}(\overline{X'})$ and $Ch^{j-r}(\overline{X'})$ respectively. Then the proposition follows directly. $\square$

Corollary 4.5.3. *Let ϕ be a quadratic form over F with totally singular part anisotropic. Suppose X is the associated quadric of ϕ. If ϕ is not totally singular, then $l_i \in \overline{Ch}(X)$ if and only if $i_t(\phi) > i$*

Proof. The first direction is trivial. For the second direction, we use induction on i. When $i = 0$, the statement follows from the Springer's Theorem (Theorem 4.3.1). Now we assume $i > 0$ and l_i is rational. Since $l_i \cdot h = l_{i-1}$ is also rational, by induction, we have $i_t(X) \geq i$. Now we assume $i_t(X) = i$ for contradiction. In this case, $\phi \simeq i\mathbb{H} \perp \phi_{an}$. Then by Proposition 4.5.2, we have $pr_* : \overline{Ch}(X) \to \overline{Ch}(X_{an})$ which sends l_i to l_0. But we know $\overline{Ch}_0(X_{an}) = 0$. This proves the statement. $\square$

Notice that we can identify the form $\mathbb{H}(W)$ with $r\mathbb{H}$. So the quadratic form ϕ can be assumed to have the form $\phi = r\mathbb{H} \perp \phi'$.

We now consider more general cases. Let $\phi_i, i = 1, \cdots, n$ be quadratic forms over F. For each ϕ_i, we assume that $\phi_i \simeq r_i\mathbb{H} \perp \phi'_i$ for integer r_i and subform ϕ'. Suppose X'_i is the associated quadric of ϕ'_i and X_i is the associated quadric of ϕ_i. We write $X := X_1 \times \cdots \times X_n$ and $X' := X'_1 \times \cdots \times X'_n$.

For each X_i, we let $(pr_i)_* : Ch_*(\overline{X_i}) \to Ch_{*-r_i}(\overline{X'_i})$ and $pr_* : Ch(\overline{X}) \to Ch(\overline{X'})$ be the group homomorphisms as we considered before. Using the compatibility of the decomposition of $Ch(\overline{X})$ with external product maps, by the same process of decomposition as stated before Proposition 4.1.10, we have a group homomorphism

$$pr := (pr_1)_* \otimes \cdots \otimes (pr_n)_* : Ch(\overline{X_1}) \otimes \cdots \otimes Ch(\overline{X_n}) \to Ch(\overline{X'_1}) \otimes \cdots \otimes Ch(\overline{X'_n}).$$

which makes the following diagram commute

$$\begin{array}{ccc} Ch(\overline{X_1}) \otimes \cdots \otimes Ch(\overline{X_n}) & \xrightarrow{pr} & Ch(\overline{X_1'}) \otimes \cdots \otimes Ch(\overline{X_n'}) \\ \downarrow \times & & \downarrow \times \\ Ch(\overline{X}) & \xrightarrow{pr_*} & Ch(\overline{X'}) \end{array}$$

where $\times$ is the external product isomorphism for $Ch(\overline{X})$ and $Ch(\overline{X'})$.

Similarly, for each $i \in [1, n]$, we let $(in_i)_* : Ch(\overline{X_i'}) \to Ch(\overline{X_i})$ and $in_* : Ch(\overline{X'}) \to Ch(\overline{X})$ be the group homomorphisms as we considered before. Then we have a group homomorphism

$$in := (in_1)_* \otimes \cdots \otimes (in_n)_* : Ch(\overline{X_1'}) \otimes \cdots \otimes Ch(\overline{X_n'}) \to Ch(\overline{X_1}) \otimes \cdots \otimes Ch(\overline{X_n}).$$

which makes the following diagram commute.

$$\begin{array}{ccc} Ch(\overline{X_1'}) \otimes \cdots \otimes Ch(\overline{X_n'}) & \xrightarrow{in} & Ch(\overline{X_1}) \otimes \cdots \otimes Ch(\overline{X_n}) \\ \downarrow \times & & \downarrow \times \\ Ch(\overline{X'}) & \xrightarrow{in_*} & Ch(\overline{X}) \end{array}$$

where $\times$ is the external product isomorphism for $Ch(\overline{X})$ and $Ch(\overline{X'})$.

We summarize our discussion above as the following proposition.

Proposition 4.5.4. *Let $X_1, \cdots, X_n$ be generically smooth projective quadrics over F. We assume that each X_i is given by a quadratic form $\phi_i \simeq r_i\mathbb{H} \perp \phi_i'$ for integer r_i and subform ϕ'. For each $i \in [1, n]$, we let X_i be the quadric given by ϕ_i'. We set $X := X_1 \times \cdots \times X_n$ and $X' := X_1' \times \cdots \times X_n'$. Then the group homomorphism $pr_* : Ch(\overline{X}) \to Ch(\overline{X'})$ preserve rationality and equals $(pr_1)_* \otimes \cdots \otimes (pr_n)_*$. Also, the group homomorphism $in_* : Ch(\overline{X'}) \to Ch(\overline{X})$ preserves rationality and equals $(in_1)_* \otimes \cdots \otimes (in_n)_*$.*

Chapter 5

Rational Correspondence Type

Fix a field F and let $\overline{F}$ be an algebraic closure of F. Let X be a generically smooth quadric given by an anisotropic quadratic form ϕ over F. Suppose that $dim(X) = D$.

In this Chapter, we study the group $\overline{Ch}_D(X \times X)$ by introducing an equivalence relation on $\overline{Ch}_D(X \times X)$. The set of equivalence classes is called the *rational correspondence type*. This extends the *Motivic Decomposition Type* of A.Vishik for smooth quadrics [21].

We extend several known results on the Motivic Decomposition Type. First, we establish restrictions on the rational correspondence type coming from the separable splitting pattern. Then we compare rational correspondence type of quadrics related by Chow correspondences (e.g. stably birational quadrics). We also compute rational correspondence types of certain families of quadrics, e.g. quadrics given by even-dimensional generic quadratic forms and quadrics given by quasi-strongly excellent forms(e.g. Pfister forms).

5.1 Essential cycles

Fix a field F. Let $\overline{F}$ be an algebraic closure of F and F_{sep} be the separable closure of F in $\overline{F}$. In this Chapter, we will closely follow R.Elman, N.Karpenko and A.Merkurjev's arguments [2, Section 73, Chapter XIII] and extend the theory to generically smooth quadrics. At first, we adopt the following terminologies as stated in [2, Section 72, Chapter XIII].

Let ϕ be a quadratic form of type $(n+1, s)$ over F and $dim(\phi) = D+2$. Suppose that ϕ_{ts} is anisotropic and X is the associated quadric of ϕ. We set $\widetilde{X} := X_{F_{sep}}$ and

$\overline{X} := X_{\overline{F}}$. Consider the image of the change of field homomorphism

$$Ch(\widetilde{X} \times \widetilde{X}) \to Ch(\overline{X} \times \overline{X}).$$

By Theorem 4.3.7, $\overline{Ch}(\widetilde{X} \times \widetilde{X})$ is an $\mathbb{F}_2$-vector space with standard basis elements having the forms $l_* \times l_*$, $l_* \times h^*$, $h^* \times l_*$ and $h^* \times h^*$. We say these basis elements are *basic cycles*. In addition, we say the basic cycles having the form $h^* \times h^*$ are *nonessential* and basic cycles having the forms $l_* \times l_*$, $h^* \times l_*$ and $l_* \times h^*$ are *essential*. One should note that all cycles having the form $h^* \times h^* \in Ch(\overline{X} \times \overline{X})$ are rational, since h is rational.

For any cycle $\alpha \in \overline{Ch}(\widetilde{X} \times \widetilde{X})$, we define the *decomposition* of α to be the representation of α as a sum of basic cycles. The *essense* of α is the sum of all essential basic cycles in the decomposition of α. If α equals the essense of α, we say α is *essential*. If the decomposition of α only contains nonessential basic cycles, we say α is *nonessential*. For a basic cycle β, we say β is *contained* in α if β is in the decomposition of α. For two cycles $\alpha, \alpha' \in \overline{Ch}(\widetilde{X} \times \widetilde{X})$, we say α' is a *subcycle* of α if every basic cycle contained in α' is also contained in α.

In this Chapter, we will focus on $\overline{Ch}_D(X \times X)$ because it contains important information we need. However, for the sake of convenience in our discussion, we will also consider $\oplus_{i \geq D} \overline{Ch}_i(X \times X)$.

Definition 5.1.1. Let α be an element of $\overline{Ch}(\widetilde{X} \times \widetilde{X})$. For every $i \in [0, n]$, the products $\alpha \cdot (h^0 \times h^i)$, $\alpha \cdot (h^1 \times h^{i-1})$,$\cdots$,$\alpha \cdot (h^i \times h^0)$ will be called the *ith order derivatives* of α.

Proposition 5.1.2. *(1) A derivative of an essential basic cycle in $\overline{Ch}_{D+r}(\widetilde{X} \times \widetilde{X})$ is still an essential basic cycle.*

(2) For any $r \geq 1$, non-negative integers i_1, j_1, i_2, j_2 satisfying $i_1 + j_i \leq r$, $i_2 + j_2 \leq r$, and nonzero essential cycle $\beta \in \overline{Ch}_{D+r}(\widetilde{X} \times \widetilde{X})$, the two derivatives $\beta \cdot (h^{i_1} \times h^{j_1})$ and $\beta \cdot (h^{i_2} \times h^{j_2})$ of β coincide only if $i_1 = i_2$ and $j_1 = j_2$.

(3) For any $r \geq 0$, nonzero integer i, j with $i + j \leq r$, and nonzero essential cycles $\beta_1, \beta_2 \in \overline{Ch}_{D+r}(\widetilde{X} \times \widetilde{X})$, the derivatives $\beta_1 \cdot (h^i \times h^j)$ and $\beta_2 \cdot (h^i \times h^j)$ coincide only if $\beta_1 = \beta_2$.

Proof. (1): Let $\beta = h^i \times l_{i+r}$ for some $i \in [0, n-r]$. By taking the derivative of β, we mean a multiplication of β by $h^{j_1} \times h^{j_2}$ for some integers j_1, j_2 with $j_1 + j_2 \leq r$.

Thus the derivative of β equals $h^{i+j_1} \times l_{i+r-j_2}$. Notice that $0 \leq i + j_1 \leq n$ and $0 \leq i + r - j_2 \leq n$, thus $h^{i+j_1} \times l_{i+r-j_2}$ is still an essential basic cycle. The case that $\beta = l_{i+r} \times h^i$ is similar.

(2): Suppose that $\beta = h^i \times l_{i+r}$ for some $i \in [0, n-r]$ (the case where $\beta = l_{i+r} \times h^r$ is similar). Then we have that $\beta \cdot (h^{i_1} \times h^{j_1}) = h^{i+i_1} \times l_{i+r-j_1} = h^{i+i_2} \times l_{i+r-j_2} = \beta \cdot (h^{i_2} \times h^{j_2})$ if and only if $i_1 = i_2$ and $j_1 = j_2$.

(3): Let $\beta_1 = h^{i_1} \times l_{i_1+r}$ and $\beta_2 = h^{i_2} \times l_{i_2+r}$. Then $\beta_1 \cdot (h^i \times h^j) = h^{i_1+i} \times l_{i_1+r-j} = h^{i_2+i} \times l_{i_1+r-j} = \beta_2 \cdot (h^i \times h^j)$ if and only if $i = j$. The other case that $\beta = l_{i+r} \times h^i$ can be discussed similarly. $\square$

Lemma 5.1.3. *Suppose for some $i, j \in [1, n]$, the basic cycle $l_i \times l_j$ is contained in a rational cycle. Then $i_W(\phi) > max\{i, j\}$.*

Proof. Let α be a cycle in $\overline{Ch}_{i+j}(X \times X)$ containing $l_i \times l_n$. For any basic cycle β in $\overline{Ch}_{i+j}(X \times X)$, we have

$$\beta_*(h^i) = \beta \circ h^i = \begin{cases} l_j & \text{if } \beta = l_i \times l_j \\ h^{D-j} & \text{if } \beta = l_i \times h^{D-j} \\ 0 & \text{otherwise} \end{cases}$$

Also, we have

$$\beta^*(h^j) = h^j \circ \beta = \begin{cases} l_i & \text{if } \beta = l_i \times l_j \\ h^{D-i} & \text{if } \beta = h^{D-i} \times l_j \\ 0 & \text{otherwise} \end{cases}$$

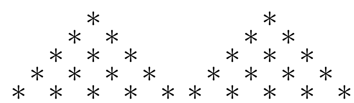
Thus when α only contains $l_i \times l_j$, we have $\alpha \circ h^i = l_j$ and $h^j \circ \alpha = l_i$. When α contains both $l_i \times l_j$ and $l_i \times h^{D-j}$, we have $\alpha \circ h^i = h^{D-j} + l_j$ and $h^j \circ \alpha = l_j$. When α contains both $l_i \times l_j$ and $h^{D-i} \times l_j$, we have $h^j \circ \alpha = l_i + h^{D-i}$ and $\alpha \circ h^i = l_j$.

Since the composition of rational cycles is still rational and h^k is always a rational cycle for $k \in [0, n]$, we know that l_i, l_j are in $\overline{Ch}(X)$. By Corollary 4.5.3, it means $i_t(X) > max\{i, j\}$. Since ϕ_{ts} is anisotropic, we conclude that $i_t = i_W > max\{i, j\}$. This finishes our proof. $\square$

When ϕ is anisotropic, by Lemma 5.1.3, the elements of $\oplus_{i \geq D} \overline{Ch}_i(X \times X)$ only involves essential basic cycles having forms $h^* \times l_*$ or $l_* \times h^*$. Adopting [2, Remark 73.8], we can visualize the essential basic cycles as follows.

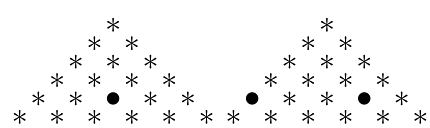
All the essential basic cycles can be expressed in two 'pyramids'. The rows of the pyramids from the bottom starting with 0, the top row has number n, and for every $r = 0, 1, \cdots, n$, the rth row of the left pyramid represents the essential basic cycles $h^i \times l_{r+i}, i = 0, \cdots, n-r$ of $\overline{Ch}_{D+r}(\widetilde{X} \times \widetilde{X})$, while the rth row of the right pyramid represents the essential basic cycles $l_{r+i} \times h^i, i = n-r, n-r-1, \cdots, 0$.

Example 5.1.4. when ϕ is given by a $(5+1, s)$-type anisotropic quadratic form, we write

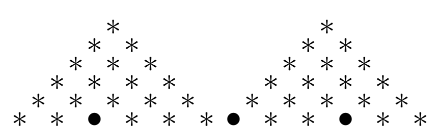


For any $\alpha \in \overline{Ch}(\widetilde{X} \times \widetilde{X})$, we fill in the pyramids by putting a *mark* in the points representing basic cycles contained in the decomposition of α; the picture thus obtained is the *diagram* of α. If α is homogeneous, the *marked points* lie in the same row. If α is homogeneous of dimension $\geq dim(X)$, then we can interpret the ith derivatives of α as a translation of the marked points of the diagram of α of moving them i rows lower.

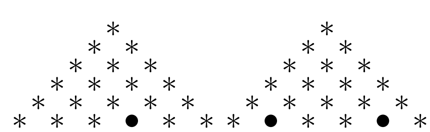
Example 5.1.5. Let ϕ be an anisotropic quadratic form of type $(6+1, s)$ and X be the associated quadric of ϕ. Let $\alpha \in \overline{Ch}(\widetilde{X} \times \widetilde{X})$ be the essential cycle $\alpha = h^2 \times l_3 + l_2 \times h^1 + l_5 \times h^4$. Then the diagram of α is



There are precisely two first-order derivatives of α. The first one is given by $\alpha \cdot (h^0 \times h^1) = h^0 \times l_0 + h^2 \times l_2 + l_3 \times h^3$ and the diagram is as follow.



The second one is given by $\alpha \cdot (h^1 \times h^0) = h^3 \times l_3 + l_1 \times h^1 + l_4 \times h^4$ and the diagram is as follow.



5.2 Rational correspondence type

Fix a field F. Let $\overline{F}$ be an algebraic closure of F and $\widetilde{F}$ be the separable closure of F in $\overline{F}$.

We assume that ϕ be is anisotropic quadratic form of type $(n+1, s)$ over F. Let X be the associated quadric of ϕ and $dim(X) = D$. Now we want to study the image $\overline{Ch}(X \times X)$ of the change of field homomorhism

$$Ch(X \times X) \to Ch(\overline{X} \times \overline{X}).$$

Recall that $\overline{Ch}(X \times X)$ is a subring of $\overline{Ch}(\widetilde{X} \times \widetilde{X})$.

Lemma 5.2.1. *The sum*

$$\sum_{i=0}^{n} (h^i \times l_i + l_i \times h^i) \in Ch(\overline{X} \times \overline{X})$$

is a rational cycle.

Proof. We let $\Delta \in Ch_D(X \times X)$ be the diagonal class and $\overline{\Delta}$ be the image of Δ under the change of field homomorphism $Ch_D(X \times X) \to \overline{Ch}_D(X \times X)$. Since any rational cycle in $\overline{Ch}_D(X \times X)$ is a sum of homogeneous basic cycles of dimension D, we have that

$$\overline{\Delta} = \sum_{i=0}^{n} a_i(h^i \times l_i) + \sum_{j=0}^{n} b_j(l_j \times h^j) + c(h^n \times h^n)$$

for some coefficients $a_i, b_j, c \in \mathbb{F}_2$. Here we shall remind that c will always be zero except for the case that the dimension of the totally singular part ϕ_{ts} of ϕ is zero and D is even.

Recall that we have the push-forward homomorphism induced by $\overline{\Delta}$, i.e., $\overline{\Delta}_* : \overline{Ch}(X \times X) \to \overline{Ch}(X \times X)$ which sends any cycle $\alpha \in \overline{Ch}(X \times X)$ to $\overline{\Delta} \circ \alpha$. By Corollary 4.4.5, we have that $\overline{\Delta}_*$ is an identity endomorphism. Thus for any $j \in [0, n]$, we have $b_j h^j = \overline{\Delta}_*(h^j) = h_j$, hence $b_j = 1$. For any $i \in [0, n-1]$, we have that $a_i l_i = \overline{\Delta}_*(l_i) = l_i$, thus $a_i = 1$.

When $i = n$, we have that $l_n = \overline{\Delta}_*(l_n) = a_n l_n + dh^n$, where $d \in \{b_n + c, c\}$. Thus we can conclude that $a_n = 1$ while $c = 0$ or $c = 1$. Finally, we conclude that the sum $\sum_{i=0}^{n}(h^i \times l_i + l_i \times h^i) = \overline{\Delta}$ or $\overline{\Delta} - h^n \times h^n$. This shows that $\sum_{i=0}^{n}(h^i \times l_i + l_i \times h^i)$ is always a rational cycle. □

Definition 5.2.2. Let α_1, α_2 be cycles in $\overline{Ch}(X \times X)$. The *intersection* of α_1 and α_2 is the sum of all basic cycles contained simultaneously in α_1 and α_2 and is denoted by $\alpha_1 \cap \alpha_2$.

Lemma 5.2.3. *If $\alpha_1, \alpha_2 \in \oplus_{i\geq 0}\overline{Ch}_{D+i}(X \times X)$, then the cycle $\alpha_1 \cap \alpha_2$ is rational.*

Proof. Without loss of generality, we can assume that α_1 and α_2 are homogeneous cycles of the same dimension $D + r$ and don't contain any nonessential basic cycle. The intersection is the essence of the composite of rational correspondence $\alpha_2 \circ (\alpha_1 \cdot (h^0 \times h^r))$. To see this, let $h^j \times l_{r+j}$ be any essential basic cycle contained in both α_1 and α_2. Then we have that $(h^j \times l_{r+j}) \circ [(h^j \times l_{r+j}) \cdot (h^0 \times h^r)] = h^j \times l_{r+j}$. Similarly, we have $(l_{r+j} \times h^j) \circ [(l_{r+j} \times h^j) \cdot (h^0 \times h^r)] = l_{r+j} \times h^j$.

Since the essence of a rational cycle is always rational, we conclude that $\alpha_1 \cap \alpha_2$ is rational. □

Definition 5.2.4. A nonzero cycle in $\oplus_{i\geq D}\overline{Ch}_i(X \times X)$ is called *minimal* if it decomposes as a sum of essential basic cycles and doesn't contain any proper rational subcycle.

Proposition 5.2.5. *(1) Two different minimal cycles intersect trivially.*

(2) Minimal cycles form a basis of the vector space of all essential cycles in $\oplus_{i\geq D}\overline{Ch}_i(X\times X)$.

(3) The sum of the minimal cycles of dimension D is equal to $\sum_{i=0}^{n}(h^i \times l_i + l_i \times h^i)$.

Proof. (1): Let α_1 and α_2 be two different minimal cycle. If $\alpha_1 \cap \alpha_2 \neq \emptyset$, then by Lemma 5.2.3, we obtain a proper rational subcycle of α_1 and α_2, which is a contradiction.

(2): Let α be any rational essential cycle with dimension lager than or equal to D. Then we can decompose α as a sum of minimal cycles. Since different minimal cycles intersect trivially, thus these minimal cycles give a basis for α.

(3): This is a direct result of part (2) and Lemma 5.2.1. □

The following Proposition states that derivatives of minimal cycles are still minimal cycles.

Lemma 5.2.6. *Let α be an essential rational cycle in $\overline{Ch}(X \times X)$ with dimension larger than or equal to D. Then α is a minimal cycle if and only if any derivative of α is a minimal cycle.*

Proof. For the first direction, we assume that α is a minimal cycle. Then it suffices to show that the two first order derivatives $\alpha\cdot(h^0\times h^1)$ and $\alpha\cdot(h^1\times h^0)$ are also minimal. If not, without loss of generality, we assume that $\alpha\cdot(h^0\times h^1)$ is not minimal while α is a minimal cycle. Since derivatives of a proper subcycle are proper subcycles of derivatives, we know that the cycle $\alpha\cdot(h^0\times h^i)$ with $i = dim(\alpha) - D$ is also not minimal. Let $\alpha^{'}$ be its proper subcycle. Then $\alpha^{'}$ has the following form.

$$\alpha^{'} = \sum_{j=0}^{n} a_j(h^j \times l_j) + \sum_{j=0}^{n} b_j(l_j \times h^j).$$

for some coefficients $a_j, b_j \in \mathbb{F}_2$.

In the same time, since $\alpha^{'}$ is a proper subcycle of $\alpha\cdot(h^0\times h^i)$, α must contain the subcycle

$$\sum_{j=0}^{n} a_j(h^j \times l_{j+i}) + \sum_{j=0}^{n} b_j(l_j \times h^{j-i}).$$

As a result, by taking the essence of the composite $\alpha\circ\alpha^{'}$, we obtain a rational subcycle β of α having the following form.

$$\sum_{j=0}^{n} a_j(h^j \times l_{j+i}) + \sum_{j=0}^{n} b_j(l_j \times h^{j-i}).$$

Hence the number of essential basic cycles contained in β is less than or equal to that of $\alpha^{'}$. This shows that α contains a rational proper subcycle, which contradicts our assumption.

For the second direction, If α is not minimal, then α contains a proper rational subcycle. So any derivative of α also contains a proper rational subcycle, which contradicts our assumption. □

Corollary 5.2.7. *The derivatives of a minimal cycle are disjoint.*

Proof. By Lemma 5.2.6, the derivatives of a minimal cycle are minimal. They are pairwise different by Proposition 5.1.2. Hence they intersect trivially by Proposition 5.2.5. □

Notation 5.2.8. Consider the essential basic cycles $\{h^i \times l_i\}, i \in [0, n]$. We label them as $0, \cdots, i, \cdots, n$. For essential basic cycles $\{l_i \times h^i\}, i \in [0, n]$, we label them as $D-0, \cdots, D-i, \cdots, D-n$. We define $\Lambda(X)$ to be the set of essential basic cycles

$\{h^i \times l_i, l_i \times h^i\}, i \in [0, n]$. Under the labelling above, we have

$$\Lambda(X) = \{i \mid 0 \le i \le n\} \sqcup \{D - i \mid 0 \le i \le n\}.$$

We denote the elements $\{i \mid 0 \le i \le n\}$ of $\Lambda(X)$ as i_{lo}, and the elements $\{D - i \mid 0 \le i \le n\}$ as i^{up}. The set of all i_{lo} elements is denoted as $\Lambda(X)_{lo}$ and the set of all i^{up} elements is denoted as $\Lambda(X)^{up}$. For any rational cycle $\alpha \in \overline{Ch}_D(X \times X)$, we define $\Lambda(\alpha)$ to be the set of all essential basic cycles contained in α under the labelling above and similarly we can define $\Lambda(\alpha)_{lo}$ and $\Lambda(\alpha)^{up}$.

We say $\lambda, \mu \in \Lambda(X)$ are *connected* if there exists a minimal cycle $\beta \in \overline{Ch}_D(X \times X)$ such that $\lambda, \mu \in \Lambda(\beta)$. This gives an equivalence relation on $\Lambda(X)$. The equivalence classes of $\Lambda(X)$ under this relation are called *connected components.* If $\Lambda(X)$ only has only one connected component, then we say $\Lambda(X)$ is *indecomposable.*

Definition 5.2.9. The *Rational Correspondence Type of* X is the set of all connected components of $\Lambda(X)$ and is denoted by $RCT(X)$.

Remark 5.2.10. When X is a smooth quadric, then $RCT(X)$ is exactly the 'Motivic Decomposition Type' of X developed by A.Vishik [21] ([2, Chapter XVII]).

For any minimal cycle $\alpha \in \overline{Ch}_D(X \times X)$, we define the *rank* of α to be the cardinality of $\Lambda(\alpha)$, i.e., $rank(\alpha) :=\mid \Lambda(\alpha) \mid$. Also, we define $a(\alpha)$ and $b(\alpha)$ to be the minimal and maximal elements in $\Lambda(\alpha)$ respectively. The *dimension of a minimal cycle* α is defined by $dim(\alpha) := b(\alpha) - a(\alpha)$.

Moreover, if α contains the essential basic cycle $h^0 \times l_0$, we say α is the *upper summand* of X and denoted by $U(X)$. When X is a smooth quadric, then the minimal cycle α with $a(\alpha) = 0$ is a rational projector defining the so-called upper summand in the motive of X (see Vishik [21]).

Lemma 5.2.11. *There exists a cycle in* $\overline{Ch}_{D+\mathfrak{i}_1(\phi)-1}(X \times X)$ *containing the essential basic cycle* $h^0 \times l_{\mathfrak{i}_1(\phi)-1}$.

Proof. If $D = 0$, this follows from the rationality of the cycle $\sum_{i=0}^{n}(h^i \times l_i + l_i \times h^i)$. So we assume $D > 0$. Consider the pull-back homomorphism $\overline{Ch}(X \times X) \to \overline{Ch}(X_{F(X)})$ induced by the morphism $X_{F(X)} \to X \times X$. By Corollary 2.2.2, it's surjective. It is also a restriction of the homomorphism $Ch(\widetilde{X} \times \widetilde{X}) \to Ch(\widetilde{X})$ mapping each basic cycle of type $h^0 \times l_i$ to l_i and vanishing all the other basic cycles. Therefore, an arbitrary preimage of $l_{\mathfrak{i}_1(\phi)-1} \in \overline{Ch}(X_{F(X)})$ under the surjection $\overline{Ch}(X \times X) \to \overline{Ch}(X_{F(X)})$ contains $h^0 \times l_{i_1-1}$. □

Recall that for a minimal cycle $\alpha \in \overline{Ch}_D(X \times X)$, we define $a(\alpha)$ to be the minimal element in $\Lambda(\alpha)$. We can generalize the definition. Let $\beta \in \overline{Ch}_{D+i}(X \times X)$ be a minimal cycle for some $i \in [0, n]$. Then we say that β *starts with* $h^j \times l_{i+j}$ if $h^j \times l_{i+j}$ is contained in β and there is no essential basic cycle $h^k \times l_{k+j}$ contained in β for any $k < j$.

Now we can state the following proposition as a generalisation of Lemma 5.2.11. It is proved when ϕ is an anisotropic non-degenerate quadratic form. One can prove similarly when ϕ is anisotropic and not totally singular.

Proposition 5.2.12. [2, Proposition 73.23] *Let $\alpha \in \overline{Ch}_D(X \times X)$ be a minimal cycle with $\mathfrak{j}_{k-1}(\phi) \leq a(\alpha) \leq \mathfrak{j}_k(\phi) - 1$ for some integer k. Then there exists a minimal cycle $\beta \in \overline{Ch}_{D+\mathfrak{i}_k(\phi)-1}(X \times X)$ starting with $h^{\mathfrak{j}_{k-1}(\phi)} \times l_{\mathfrak{j}_k(\phi)-1}$ such that α is a derivative of β.*

Proof. We assume that $a(\alpha) = \mathfrak{j}_{k-1}(\phi)$ and induct on the integer k. For the cases that $a(\alpha) \neq \mathfrak{j}_{k-1}(\phi)$, we can reduce to the case that $a(\alpha) = \mathfrak{j}_{k-1}(\phi)$ by using results from the next section (duality of cycles, Corollary 5.3.9).

When $k = 1$, the proposition comes directly from Lemma 5.2.11. Assume that $k > 1$. Let ϕ_1 be the first kernel form of ϕ, i.e., $\phi_1 = (\phi_{F(\phi)})_{an}$ and X_1 be the associated quadric of ϕ_1. We set that $D_1 = dim(X_1)$. Now we consider the group homomorphism $pr_* : \overline{Ch}((X \times X)_{F(X)}) \to \overline{Ch}(X_1 \times X_1)$ as in Proposition 4.5.4 and obtain that $a(pr_*(\alpha)) = \mathfrak{j}_{k-1}(\phi) - \mathfrak{i}_1(\phi)$.

By the induction hypothesis, we then have a cycle $\beta_1 \in \overline{Ch}_{D_1+\mathfrak{i}_k(\phi)-1}(X_1 \times X_1)$ starting with $h^{\mathfrak{j}_{k-1}(\phi)-\mathfrak{i}_1(\phi)} \times l_{\mathfrak{j}_{k-1}(\phi)-\mathfrak{i}_1(\phi)-1}$. Consider the group homomorphism $in_* : \overline{Ch}(X_1 \times X_1) \to \overline{Ch}((X \times X)_{F(X)})$ as in Proposition 4.5.4, we have a cycle $in_*(\beta_1) \in \overline{Ch}((X \times X)_{F(X)})$ starting with $h^{\mathfrak{j}_{k-1}(\phi)} \times l_{\mathfrak{j}_k(\phi)-1}$. Let $\overline{Ch}(X \times X \times X) \to \overline{Ch}((X \times X)_{F(X)})$ be the surjective homomorphism induced by the morphism $(X \times X)_{F(X)} \to X \times X \times X$ (Corollary 2.2.3). By taking the preimage of $in_*(\beta_1)$ under the surjective homomorphism, we obtain a cycle β' containing $h^0 \times h^{\mathfrak{j}_{k-1}(\phi)} \times l_{\mathfrak{j}_k(\phi)-1}$.

Now we let $p : (X \times X) \times X \to X \times X$ be the projection onto the first factor. Consider the preimage of α under the pull-back $p^* : \overline{Ch}(X \times X)$ induced by p, we have that $p^*(\alpha) = \alpha \times h^0 \in \overline{Ch}(X \times X \times X)$ which contains $h^{\mathfrak{j}_{k-1}(\phi)} \times l_{\mathfrak{j}_{k-1}(\phi)} \times h^0$. Let $p_2 : X \times X \times X \to X \times X$ be the projection which disappears the first and the third factors. Then $(p_2)_*(p^*(\alpha) \cdot \beta') \in \overline{Ch}(X \times X)$ is a cycle containing $a(h^{\mathfrak{j}_{k-1}(\phi)} \times l_{\mathfrak{j}_k(\phi)-1})$

with coefficient $a \in \mathbb{F}_2$. However, by a direct calculation of $(p_2)_*(p^*(\alpha)\cdot\beta')$, we obtain that $a = 1$.

We let β be the subcycle of $(p_2)_*(p^*(\alpha)\cdot\beta')$ which is minimal and contains $h^{\mathrm{j}_{k-1}(\phi)} \times l_{\mathrm{j}_k(\phi)-1}$. Then α is a derivative of β trivially from the minimality of α and β. This finishes our proof. $\square$

In the rest of this section, we will give some terminologies which will be used later. Suppose that $\Delta = \{i_1, \cdots, i_n\}$ is a set of integers. For any integer i, we say another set of integers Δ' is *a shift of* Δ *by* i if $\Delta' = \{i_1 + i, \cdots, i_n + i\}$, which is denoted by $\Delta' = \Delta[i]$. Moreover, if $\Delta := \{\Delta_1, \cdots, \Delta_s\}$ where $\Delta_i, i = 1, \cdots, s$ are set of integers, we define $\Delta[i] := \{\Delta_1[i], \cdots, \Delta_s[i]\}$.

Under the same notation above, we can define that $\Lambda(X)[i]_{lo} := \Lambda(X)_{lo}[i]$ and $\Lambda(X)[i]^{up} := \Lambda(X)^{up}[i]$. Finally, $\Lambda(X)[i] = \Lambda(X)[i]_{lo} \sqcup \Lambda(X)[i]^{up}$. Similarly, for any minimal cycle $\alpha \in \overline{Ch}_D(X \times X)$, we can define $\Lambda(\alpha)[i]_{lo}$ and $\Lambda(\alpha)[i]^{lo}$. Then $\Lambda(\alpha)[i] = \Lambda(\alpha)[i]_{lo} \sqcup \Lambda(\alpha)[i]^{up}$.

5.3 Restrictions from separable splitting pattern

Fix a field F. Let $\overline{F}$ be an algebraic closure of F and $\overline{F}$ be the separable closure of F in $\overline{F}$.

Proposition 5.3.1. *Let $\phi_1, \cdots, \phi_m$ be quadratic forms over F with anisotropic totally singular part and $X_1, \cdots, X_m$ be the associated quadrics respectively. Suppose there exists a rational cycle $\alpha \in \overline{Ch}((X_1 \times \cdots \times X_m)_K)$ containing the cycle $l_a \times h^{b_1} \times \cdots \times h^{b_{m-1}}$ for some integers $a, b_1, \cdots, b_{m-1}$. Then for any separable field extension K/F such that the Witt indices $i_W((\phi_2)_K), \cdots, i_W((\phi_m)_K)$ are larger than $max\{b_1, \cdots, b_{n-1}\}$, we have $i_W((\phi_1)_K) > a$.*

Proof. Without loss of generality, we may assume that α is homogeneous and K/F is such a separable field extension. We assume that $i_W((\phi_i)_K) = n_i$ for $i \in [2, m]$.

Since the totally singular part of ϕ_1 is anisotropic, we have that $i_W((\phi_1)_K) = i_t((\phi_1)_K)$ for any separable field extension K/F (Lemma 3.4.6). To show that $i_W((\phi_1)_K) > a$, it suffices to show that l_a is a rational cycle in $\overline{Ch}((X_1)_K)$ by Corollary 4.5.3. Recall that any rational cycle $\beta \in \overline{Ch}((X_2 \times \cdots \times X_m)_K)$ induces a push-forward homomorphism

$$\beta_* : \overline{Ch}((X_1 \times X_2 \times \cdots \times X_m)_K) \to \overline{Ch}((X_1)_K).$$

which sends $\alpha \in \overline{Ch}((X_1 \times \cdots \times X_m)_K)$ to $(\beta \circ \alpha) \in \overline{Ch}((X_1)_K)$. If we can find a β such that $\beta_*(\alpha) = l_a + ch^{dim(X_1)-a}$, then we are done since h^* is always a rational cycle in $\overline{Ch}((X_1)_K)$.

We take

$$\beta := l_{b_1} \times \cdots \times l_{b_{m-1}} \in Ch((X_2 \times \cdots X_m)_{\overline{F}}).$$

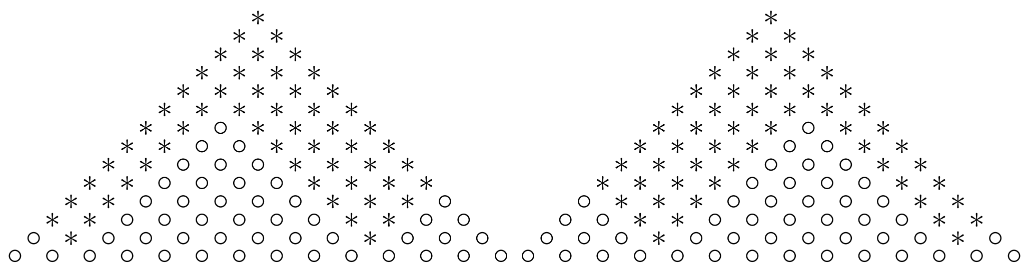
From our assumption, we know that $i_W((\phi_2)_K), \cdots, i_W((\phi_m)_K$ are all larger than $max\{b_1, \cdots, b_{m-1}\}$. Thus β is actually a rational cycle, i.e., $\beta \in \overline{Ch}((X_2 \times \cdots \times X_m)_K)$. Then a short calculation shows that the β has the desired property. □

Now we let ϕ be an anisotropic quadratic form of type $(n+1, s)$ over F and $dim(\phi) = D + 2$. Recall that the separable splitting pattern of ϕ is given by the Knebusch splitting pattern up to the separable height (Corollary 3.5.7). Let $F = F_0 \subset F_1 \subset \cdots \subset F_h$ be the Knebusch splitting tower up to the separable height, i.e., $h = h_{sep}(\phi)$ and $\phi = \phi_0, \phi_1, \cdots, \phi_h$ be the corresponding kernel forms. We set that $X_i := X_{\phi_i}$ which is the quadric given by ϕ_i for $i = 0, \cdots, h$.

Corollary 5.3.2. *If i, j are integers in the interval $[0, n]$ satisfying $i < \mathfrak{j}_q(\phi) \leq j$ for some $q \in [0, h)$, then no element in $\overline{Ch}(X \times X)$ contains either $h^i \times l_j$ or $l_j \times h^i$.*

Proof. Let i, j be integers in the interval $[0, n]$ such that $h^i \times l_j$ or $l_j \times h^i$ appears in the decomposition of some $\alpha \in \overline{Ch}(X \times X)$. Without loss of generality, we may assume α is homogeneous and $l_j \times h^i$ is contained in α. Now we suppose $i < \mathfrak{j}_q(\phi)$ for some $q \in [1, k)$. Let K/F be a separable field extension such that $i_W(\phi_L) = \mathfrak{j}_q(\phi)$ (for example, we can just take $K = F_q$). Then by Proposition 5.3.1, it implies that $\mathfrak{j}_q(\phi) > j$. This gives the result. □

Remark 5.3.3. When we draw the pyramids of essential basic cycles as stated in the first section, Corollary 5.3.2 gives a restriction on the pyramids. If we mark the essential basic cycles not restricted from Corollary 5.3.2 by $\circ$ and mark the others by $*$, we will get isosceles triangles based on the lower row of these pyramids. For example, we suppose Y is given by a $(14, 8)$-type anisotropic quadratic form ϕ with relative higher Knebusch index $2, 8, 4$ up to the separable height.

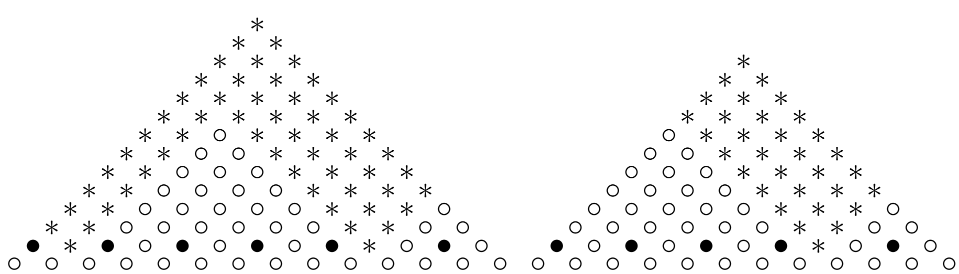


The triangles in the pyramids are called *shell triangles*. The shell triangles in the left pyramid are numbered form the left starting with 1 and the shell triangles in the right pyramid are numbered from the right starting with 1. For any $q \in [1, s]$, the bases of the qth triangles have $\mathfrak{i}_q(\phi)$ points.

It's a good point to explain the homomorphism pr_* as considered in Proposition 4.5.4 explicitly. We let ϕ_1 be the first kernel form of ϕ and Y_1 be the associated quadric of ϕ_1. Consider the composition of homomorphisms:

$$\overline{Ch}(Y \times Y) \hookrightarrow \overline{Ch}((Y \times Y)_{F(\phi)}) \xrightarrow{pr_*} \overline{Ch}(Y_1 \times Y_1)$$

We let α be a cycle in $\overline{Ch}(Y \times Y)$. Then the diagram $pr_*(\alpha)$ is obtained by erasing the first shell triangle in the diagram of α. We illustrate it in the following picture. The left part shows the diagram of α (in one pyramid) while the right one shows the diagram of $pr_*(\alpha)$ (in one pyramid).



Corollary 5.3.4. *Let α be a minimal cycle in $\overline{Ch}_D(X \times X)$ with $\mathfrak{j}_{k-1}(\phi) \leq a(\alpha) < \mathfrak{j}_k(\phi)$ for some integer $k \in [1, h]$.*

(1) There exists a minimal cycle $\beta \in \overline{Ch}_D(X \times X)$ such that $a(\beta) = \mathfrak{j}_{k-1}(\phi)$ and $\Lambda(\alpha) = \Lambda(\beta)[a(\alpha) - \mathfrak{j}_{k-1}(\phi)]$.

(2) *For integer $l \in [1, h_{sep}(\phi)]$ with $l > k$, if $\mathfrak{i}_l(\phi) < \mathfrak{i}_k(\phi)$, then α doesn't contain any essential basic cycle $h^i \times l_i$ and $l_i \times h^i$ with $i \in [\mathfrak{j}_{l-1}(\phi), \mathfrak{j}_l(\phi) - 1]$.*

Proof. (1): By Proposition 5.2.12, we have a minimal cycle $\gamma \in \overline{Ch}_{D+\mathfrak{i}_k(\phi)-1}(X \times X)$ starting with $h^{\mathfrak{j}_{k-1}(\phi)} \times l_{\mathfrak{j}_k(\phi)-1}$ such that

$$\alpha = \gamma \cdot (h^{a(\alpha)-\mathfrak{j}_{k-1}(\phi)} \times h^{\mathfrak{j}_k(\phi)-a(\alpha)-1}).$$

Similarly, the minimal cycle β is obtained by

$$\beta = \gamma \cdot (h^0 \times h^{\mathfrak{i}_k(\phi)-1}).$$

This establishes the statement.

(2): We assume that α contains the essential basic cycle $h^i \times l_i$, where $i \in [\mathfrak{j}_{l-1}(\phi), \mathfrak{j}_l(\phi) - 1]$. Then there exists a minimal cycle $\gamma \in \overline{Ch}_{D+\mathfrak{i}_k(\phi)-1}(X \times X)$ starting with $h^{\mathfrak{j}_{k-1}(\phi)} \times l_{\mathfrak{j}_k(\phi)-1}$ such that

$$\alpha = \gamma \cdot (h^{a(\alpha)-\mathfrak{j}_{k-1}(\phi)} \times h^{\mathfrak{j}_k(\phi)-a(\alpha)-1}).$$

by Proposition 5.2.12. So γ must contain the essential basic cycle $h^{i+\mathfrak{j}_{k-1}(\phi)-a(\alpha)} \times l_{i+\mathfrak{j}_k(\phi)-a(\alpha)-1}$. However, from the restriction from 'shell triangles' (Corollary 5.3.2), this is impossible. □

In the rest of this section, we will show that the separable splitting pattern imposes basic restrictions on the rational correspondence type of anisotropic quadrics.

Lemma 5.3.5. *For any cycle $\alpha \in \oplus_{i \geq D} \overline{Ch}_i(X \times X)$, the number of essential basic cycles contained in the decomposition of α is even.*

Proof. Let U be the smooth locus of X. Then the pull-back of diagonal morphism $\Delta : U \to U \times U$ induces a ring homomorphism $\Delta^* : Ch(U \times U) \to Ch(U)$. Let $f : Ch(U \times U) \to \overline{Ch}(X \times X)$ and $f' : Ch(U) \to \overline{Ch}(X)$ be the ring homomorphisms as we considered in Remark 4.3.15. Now we want to show that there exists a well-defined ring homomorphism ψ which makes the following diagram commute.

$$\begin{array}{ccc} Ch(U \times U) & \xrightarrow{\Delta^*} & Ch(U) \\ f \downarrow & & \downarrow f' \\ \overline{Ch}(X \times X) & \xrightarrow{\psi} & \overline{Ch}(X) \end{array}$$

Since f is a surjective ring homomorphism, it suffices to show that $\Delta^*(Ker(f)) \subset Ker(f^{'})$.

Let N be the numerically trivial ideal in $Ch(\widetilde{U} \times \widetilde{U})$. We consider the change of field homomorphism $s_1 : Ch(U \times U) \to Ch(\widetilde{U} \times \widetilde{U})$. Now we take any element $\beta \in Ker(f)$. Recall that $Ker(f) = \{\beta \mid s_1(\beta) \in N\}$. Let $N^{'}$ be the numerically trivial ideal in $Ch(\widetilde{U})$ and $s_2 : Ch(U) \to Ch(\widetilde{U})$ be the change of field homomorphism. Now we need to show that $\Delta^*(\beta) \in Ker(f^{'})$, where $Ker(f^{'}) = \{\beta \mid s_2(\beta) \in N^{'}\}$.

Consider the fiber product diagram:

$$\begin{array}{ccc} U & \xrightarrow{\Delta} & U \times U \\ {\scriptstyle s}\uparrow & & {\scriptstyle h}\uparrow \\ \widetilde{U} & \xrightarrow{\widetilde{\Delta}} & \widetilde{U} \times \widetilde{U} \end{array}$$

where s, h are the flat morphisms with $s^* = s_1, h^* = s_2$ and $\widetilde{\Delta}$ is the diagonal morphism.

For any element $\gamma \in Ch(\widetilde{U} \times \widetilde{U})$, by Proposition 2.2.4 and the Projection Formula (Proposition 2.2.5), we have

$$\begin{aligned} deg(s_2(\Delta^*(\beta)) \cdot \gamma) &= deg(\widetilde{\Delta}^*(s_1(\beta)) \cdot \gamma) \\ &= deg(\widetilde{\Delta}_*(\widetilde{\Delta}^*(s_1(\beta)) \cdot \gamma)) \\ &= deg(s_1(\beta) \cdot \widetilde{\Delta}_*(\gamma)) \\ &= 0. \end{aligned}$$

Thus we show that $s_2(\Delta^*(\beta)) \in N^{'}$, hence $\Delta^*(Ker(f)) \subset Ker(f^{'})$.

For any rational cycle α in $\overline{Ch}(X \times X)$, without loss of generality, we may assume that α is homogeneous of dimension $D+k$ for some $k \in [0, n]$ and α can be decomposed by n_0 essential basic cycles. We claim that $\psi(\alpha) = n_0 \cdot l_k \in \overline{Ch}(X)$. To see this, we can replace F with $\widetilde{F}$ and assume that ϕ has maximal Witt index and $\alpha = h^i \times l_{i+k}$ or $l_{i+k} \times h^i$. Consider the following commutative diagram:

$$\begin{array}{ccc} Ch(U \times U) & \xrightarrow{\Delta^*} & Ch(U) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{'}} \\ \overline{Ch}(X \times X) & \xrightarrow{\psi} & \overline{Ch}(X) \end{array}$$

We can identify α as an element in $Ch(U \times U)$. Then $\Delta^*(\alpha) = l_{i+k} \cdot h^i = l_k$, which shows that $\psi(\alpha) = l_k$.

Finally, since X is given by an anisotropic quadratic form ϕ, by Corollary 4.5.3, we conclude that n_0 is even. □

Theorem 5.3.6. *Let $\alpha \in \overline{Ch}(X \times X)$ be a cycle containing the top of a qth shell triangle for some $q \in [1, h_{sep}(\phi)]$. Then α also contains the top of the other qth shell triangle.*

Proof. We may assume that α contains the top of the left qth shell triangle. From the group homomorphism $pr_* : \overline{Ch}((X \times X)_{F_{q-1}}) \to \overline{Ch}(X_{q-1} \times X_{q-1})$ as we considered in Proposition 4.5.4, we may assume that $q = 1$ by replacing F by the field F_{q-1} and X by X_{q-1}.

Replacing α by its homogeneous component containing the top of the left shell triangle $h^0 \times l_{\mathfrak{j}_1(\phi)-1}$, we may assume that α is homogeneous and has the form

$$\alpha = h^0 \times l_{\mathfrak{i}_1(\phi)-1} + \beta + al_{\mathfrak{i}_1(\phi)-1} \times h^0.$$

for some cycle β and coefficient $a \in \mathbb{F}_2$. Now we use the group homomorphism $pr_* : \overline{Ch}((X \times X)_{F_1}) \to \overline{Ch}(X_1 \times X_1)$, we have that $pr_*(\alpha) = pr_*(\beta)$. Lemma 5.3.5 says that $pr_*(\beta)$ contains even number of essential basic cycles. Here we shall remind that β contains the same number of essential basic cycles of that of $pr_*(\beta)$ since $pr_*(\beta)$ only disappear the essential basic cycles marked in the first shell triangles as illustrated in Remark 5.3.3.

As a result, if $a = 0$, then the number of essential basis elements contained in α and the number of essential basis elements contained in $pr_*(\alpha)$ differ by 1. But it's a contradiction since the number of essential basis elements contained in α and $pr_*(\alpha)$ should always be even. □

Lemma 5.3.7. *Let $\rho \in \overline{Ch}_D(X \times X)$, $q \in [1, h_{sep}(\phi)]$ and $i \in [1, \mathfrak{i}_q(\phi)]$. Then the element $h^{\mathfrak{j}_{q-1}(\phi)+i-1} \times l_{\mathfrak{j}_{q-1}(\phi)+i-1}$ is contained in ρ if and only if the element $l_{\mathfrak{j}_q(\phi)-i} \times h^{\mathfrak{j}_q(\phi)-i}$ is contained in ρ*

Proof. By a similar argument as in the proof of Theorem 5.3.6, it suffices to prove this lemma for $q = 1$. Suppose that the basic cycle $h^{i-1} \times l_{i-1}$ is contained in a rational cycle ρ. By taking the minimal subcycle of ρ containing $h^{i-1} \times l_{i-1}$, we may assume that ρ is minimal. Let α be the minimal cycle containing $h^0 \times l_{\mathfrak{i}_1(\phi)-1}$. By Theorem 5.3.6, the cycle α also contains $l_{\mathfrak{i}_1(\phi)-1} \times h^0$. Therefore, the derivative $\alpha \cdot (h^{i-1} \times h^{\mathfrak{i}_1(\phi)-i})$

contains both $h^{i-1} \times l_{i-1}$ and $l_{\mathfrak{j}_1(\phi)-i} \times h^{\mathfrak{j}_1(\phi)-i}$. Since the derivative of a minimal cycle is minimal, the lemma follows directly from the minimality of ρ. The other direction can be proved similarly. □

Definition 5.3.8. The symmetric shell triangles are called *dual*. Two points are called *dual* if one represents the essential basic cycle $h^{\mathfrak{j}_{k-1}(\phi)+i} \times l_{\mathfrak{j}_k(\phi)-1}$ while another point represents the essential basic cycle $l_{\mathfrak{j}_k(\phi)-i-1} \times h^{\mathfrak{j}_{k-1}(\phi)+j}$ for some $k \in [1, h_{sep}(\phi)]$ and $i, j \in [0, \mathfrak{i}_k(\phi) - 1]$.

Corollary 5.3.9. *In the diagram of an element of* $\overline{Ch}(X \times X)$*, any two dual points are simultaneously marked or not marked.*

Proof. Let k be the number of the row containing two given dual points. The case $k = 0$ is treated in Lemma 5.3.7. The case of an arbitrary k can be reduced to the case of $k = 0$ by taking a kth order derivative. □

5.4 Restrictions from Steenrod operations

In this section, we will give some restrictions of the rational correspondence type from the cohomological Steenrod operation. The following results are proved by R.Elman, N.Karpenko and A.Merkurjev in [2] for smooth quadrics. Now we extend them to the generically smooth quadrics.

Fix a field F. Let $\overline{F}$ be an algebraic closure of F and $\widetilde{F}$ be the separable closure of F in $\overline{F}$.

Let ϕ be an anisotropic quadratic form of type $(n+1, s)$ over F and $dim(\phi) = D+2$. Now we let X be the associated quadric of ϕ. Of course we have $dim(X) = D$. Also, we let $F = F_0 \subset F_1 \subset \cdots \subset F_h$ be the Knebusch splitting tower up to the separable height, i.e., $h = h_{sep}(\phi)$. To simplify notations, we use $\mathfrak{i}_k$ and $\mathfrak{j}_k$ to denote the kth higher relative and absolute Witt index of ϕ for any $k \in [0, h]$ respectively.

Recall that we have the cohomological Steenrod operation $Sq^*_X : \overline{Ch}(\widetilde{X}) \to \overline{Ch}(\widetilde{X})$ satisfying $Sq^*_X(h^i) = h^i(1+h)^i$ and $Sq^*_X(l_i) = l_i \cdot (1+h)^{D+1-i}$. For any integer $j \geq 0$, we have

$$Sq^j(h^i) = \binom{i}{j} h^{i+j}, Sq^j(l_i) = \binom{D+1-i}{j} l_{i-j}.$$

(Proposition 4.3.20). We also have the cohomological Steenrod operation Sq^* : $\overline{Ch}(\widetilde{X} \times \widetilde{X}) \to \overline{Ch}(\widetilde{X} \times \widetilde{X})$ which is obtained by $Sq^* = Sq^*_X \times Sq^*_X$.

For any essential basic cycle $h^n \times l_m \in \overline{Ch}(\widetilde{X} \times \widetilde{X})$, we know that $Sq^*(h^n \times l_m) = Sq^*_{\widetilde{X}}(h^n) \times Sq^*_{\widetilde{X}}(l_m)$. This means that $Sq^*(h^n \times l_m)$ is a linear combination of the elements $h^i \times l_j$ with $i \geq n$ and $j \leq m$. Especially, when $n = 0$, $Sq^*(h^0 \times l_n)$ is a linear combination of the elements $h^0 \times l_j$ with $j \leq m$.

Similarly, $Sq^*(l_m \times h^n)$ is a linear combination of the elements $l_j \times h^i$ with $j \leq m$ and $i \geq n$. When $n = 0$, $Sq^*(l_m \times h^0)$ is a linear combination of the elements $l_j \times h^0$ with $j \leq m$.

Lemma 5.4.1. *Let $\pi \in \overline{Ch}_{D+\mathfrak{i}_1-1}(X \times X)$ be the minimal cycle starting with $h^0 \times l_{\mathfrak{i}_1-1}$. For any $j \geq 1$, the element $Sq^j(\pi)$ doesn't contain any essential basic cycle in the first shell triangle.*

Proof. From our discussion above, the cycle $Sq^j(h^0 \times l_{\mathfrak{i}_1-1})$ can only take values in $h^0 \times l_k$ with $k \leq \mathfrak{i}_1 - 1$. By Lemma 5.3.7, π also contains the essential basic cycle $l_0 \times h^{\mathfrak{i}_1-1}$. Thus the cycle $Sq^j(l_0 \times h^{\mathfrak{i}_1-1})$ can only take values in $l_0 \times h^{k'}$ with $k' \leq \mathfrak{i}_1 - 1$. Since $Sq^j(\pi)$ is a rational cycle, if $Sq^j(\pi)$ contains any essential basic cycle in the first shell triangle, it will contradict Corollary 5.3.9. □

Corollary 5.4.2. *$Sq^j(\pi) = 0$ for all $j \in [1, \mathfrak{i}_1 - 1]$.*

Proof. We assume that $Sq^j(\pi) \neq 0$ for some $j \in [1, \mathfrak{i}_1 - 1]$. From our discussion above, there exists a jth derivative of π which intersects $Sq^j(\pi)$ nontrivially. Since minimal cycles stays minimal after taking derivatives, we know that $Sq^j(\pi)$ contains the jth derivative of π. But it is impossible since $Sq^j(X)$ doesn't contain any essential basic cycle in the first shell triangle by Lemma 5.4.1. □

Proposition 5.4.3. *If $i \in \Lambda(U(X))_{lo}$, then $v_2(i) \geq n + 1$ for any n satisfying $\mathfrak{i}_1 > 2^n$ where $v_2(i)$ is the 2-adic valuation of i.*

Proof. We assume the statement is false. Let $i \in \Lambda(U(X))_{lo}$ be the smallest integer such that i is not divisible by 2^{n+1}. Then $h^i \times l_i$ is an essential basic cycle contained in $U(X)$. Let $\pi \in \overline{Ch}_{D+\mathfrak{i}_1-1}(X \times X)$ be the minimal cycle starting with $h^0 \times l_{\mathfrak{i}_1-1}$. By Proposition 5.2.12, we know that $U(X) = \pi \cdot (h^0 \times h^{\mathfrak{i}_1-1})$. Hence the essential basic cycle $h^i \times l_{i+\mathfrak{i}_1-1}$ is contained in π. Actually, π has the following form.

$$\sum_{m \in \Lambda(U(X))_{lo}, m<i} h^m \times l_{m+\mathfrak{i}_1-1} + h^i \times l_{i+\mathfrak{i}_1-1} + \sum_{s \in \Lambda(U(X))_{lo}, s>i} h^s \times l_{s+\mathfrak{i}_1-1} + \sum_{j \in \Lambda(U(X))^{up}} l_j \times h^{j-\mathfrak{i}_1-1}.$$

Now we want to show that, for any integer j, $Sq^j(\pi)$ always contain the cycle $Sq^0_X(h^i) \times Sq^j_X(l_{i+\mathfrak{i}_1-1}) = h^i \times Sq^j_X(l_{i+\mathfrak{i}_1-1})$. By the discussion before Proposition

4.3.20 and the Cartan Formula (Proposition 2.6.4), we have that $Sq^j(h^i \times l_{i+\mathfrak{i}_1-1}) = \sum_{k=0}^{j} Sq^k(h^i) \times Sq_X^{j-k}(l_{i+\mathfrak{i}-1})$. Thus to show the existence of $Sq_X^0(h^i) \times Sq_X^j(l_{i+\mathfrak{i}_1-1})$, we need to show that there is no integer $m < i$ with $h^m \times l_{m+\mathfrak{i}_1-1}$ contained in π such that $Sq^j(h^m \times l_{m+\mathfrak{i}_1-1}) = h^i \times Sq_X^j(l_{i+\mathfrak{i}_1-1})$.

Note that for any integers $t, l \in \mathbb{N}$, the binomial coefficient $\binom{t}{l}$ is odd if and only if the binary digits in the binary expansion of l is a subset of that of t ([2, Lemma 78.6]). So if t is divisible by 2^{n+1}, then the necessary condition for $\binom{t}{l}$ to be odd is that l is also divisible by 2^{n+1}. Now we assume such m exists and consider

$$Sq^j(h^m \times l_{m+\mathfrak{i}_1-1}) = \sum_{k=0}^{j} Sq_X^k(h^m) \times Sq^{j-k}(l_{m+\mathfrak{i}_1-1}).$$

By our assumption, such m will be divisible by 2^{n+1}. Also, there exists an integer $k_1 \in [0, j]$ such that $Sq_X^{k_1}(h^m) = \binom{m}{k_1} h^{m+k_1} = h^i$. Since $\binom{m}{k_1} \equiv 1 \mod 2$, we know k_1 must also be divisible by 2^{n+1}. Finally, we conclude that $i = m + k_1$ is divisible by 2^{n+1}, which is a contradiction.

As a result, for any integer j, $Sq^j(\pi)$ always contains the element $Sq_X^0(h^i) \times Sq_X^j(l_{i+\mathfrak{i}_1-1}) = h^i \times Sq_X^j(l_{i+\mathfrak{i}_1-1})$. Since $Sq^j(\pi) = 0$ for $j \in (0, \mathfrak{i}_1)$ (Corollary 5.4.2), we have that

$$Sq_X^j(l_{i+\mathfrak{i}_1-1}) = 0$$

where $j \in (0, \mathfrak{i}_1 - 1)$.

On the other hand, since $\mathfrak{i}_1 > 2^n$ and i is not divisible by 2^{n+1}, we obtain that $2^{v_2(i)} \in (0, i_1)$. Consider $Sq_X^{2^{v_2(i)}}(l_{i+\mathfrak{i}_1-1-2^{v_2(i)}})$, we have that

$$Sq_X^{2^{v_2(i)}}(l_{i+\mathfrak{i}_1-1-2^{v_2(i)}}) = \binom{D - i - \mathfrak{i}_1 + 2}{2^{v_2(i)}} l_{i+\mathfrak{i}_1-1-2^{v_2(i)}}.$$

(Proposition 4.3.20). Note that $D - \mathfrak{i}_1 + 2 = dim(\phi) - \mathfrak{i}_1$. By Theorem 3.5.9, we know that $\mathfrak{i}_1 \leq 2^{v_2(dim(\phi)-\mathfrak{i}_1)}$. Since $\mathfrak{i}_1 > 2^n$, we conclude that $dim(\phi) - \mathfrak{i}_1$ is divisible by 2^{n+1}. Thus $\binom{D-\mathfrak{i}_1+2}{2^{v_2(i)}} \equiv 1 \mod 2$ and $Sq_X^{2^{v_2(i)}}(l_{i+\mathfrak{i}_1-1}) \neq 0$, which is a contradiction. □

5.5 Restrictions from rational correspondences

Fix a field F. In this section, we will generalize some well-known results on smooth quadrics given by Vishik [10] to generically smooth quadrics. In the next few sections, we will use the results to study the rational correspondence type of quadrics given by

Pfister neighbours, even-dimensional generic forms and strongly excellent forms.

Lemma 5.5.1. *Let p, q be the two anisotropic quadratic forms over F which are not totally singular. We let P and Q be the associated quadrics of p, q respectively. Suppose that for any separable field extension E/F, $i_W(p_E) > 0$ is equivalent to $i_W(q_E) > \mathfrak{j}_{k-1}(q)$ for some $k \in [1, h_{sep}(q)]$. Then for any cycle $\alpha \in \overline{Ch}(P \times Q)$ containing $h^0 \times l_{\mathfrak{j}_k(q)-1}$, α must also contain $l_t \times h^{\mathfrak{j}_{k-1}(q)}$ where $t = dim(P) - dim(Q) + \mathfrak{j}_k(q) + \mathfrak{j}_{k-1}(q) - 1$*

Proof. We set $dim(p) = D_1 + 2$ and $dim(q) = D_2 + 2$. Without loss of generality, we may assume that α is a homogeneous cycle containing $h^0 \times l_{\mathfrak{j}_k(q)-1}$. Then $\alpha \in \overline{Ch}_{D_1+\mathfrak{j}_k(q)-1}(P \times Q)$ has the following form

$$\alpha = h^0 \times l_{\mathfrak{j}_k(q)-1} + \sum_{i=1}^{n_1} a_i(h^i \times l_{i-1+\mathfrak{j}_k(q)}) + \sum_{j=0}^{n_2} b_j(l_j \times h^{D_2-D_1-\mathfrak{j}_k(q)+j+1}).$$

for integers $a_i, b_j \in \mathbb{F}_2$ and $n_1, n_2 \in \mathbb{N}$.

Now we let q_{k-1} be the $(k-1)$th kernel form of q and Q_{k-1} be the associated quadric of q_{k-1}. We set that $dim(q_{k-1}) = D_2^{'} + 2$. Notice that $q_{k-1} = (q \mid_{F_{k-1}})_{an}$, where F_{k-1} is the field in the Knebusch splitting tower of q. Then we let $pr_* : \overline{Ch}(Q) \to \overline{Ch}(Q_{k-1})$ be the homomorphism as we considered in Proposition 4.5.4. Then we have a group homomorphism $p := id \otimes pr_* : \overline{Ch}(P \times Q) \to \overline{Ch}(P \times Q_{k-1})$. Then $\alpha_1 := p(\alpha) \in \overline{Ch}(P \times Q_{k-1})$ has the following form.

$$\alpha_1 = h^0 \times l_{\mathfrak{i}_k(q)-1} + \sum_{i=1}^{n_1} a_i(h^i \times l_{i-1+\mathfrak{i}_k(q)}) + \sum_{j=0}^{n_2} b_j(l_j \times h^{D_2-D_1-\mathfrak{j}_k(q)-\mathfrak{j}_{k-1}(q)+j+1}).$$

Notice that our assumption implies that for any field extension L/F containing F_{k-1}, $i_W(p_L) > 0$ is equivalent to $i_W((q_{k-1})_L) > 0$. We consider the surjective group homomorphism $\overline{Ch}(Q_{k-1} \times P) \to \overline{Ch}(P_{F(Q_{k-1})})$ induced by $P_{F(Q_{k-1})} \to Q_{k-1} \times P$ (Corollary 2.2.3). Since $i_W(p_{F(Q_{k-1})}) > 0$, we know that $l_0 \in \overline{Ch}(P_{F(Q_{k-1})})$. Hence we have a cycle $\beta \in \overline{Ch}(Q_{k-1} \times P)$ containing the cycle $h^0 \times l_0$. Without loss of generality, we may assume that β is homogeneous. Then β has the following form

$$\beta = h^0 \times l_0 + \sum_{i=1}^{n_1^{'}} a_i^{'}(h^i \times l_i) + \sum_{j=0}^{n_2^{'}} b_j^{'}(l_j \times h^{D_1-D_2^{'}+j}) \in \overline{Ch}_{D_2^{'}}(Q_{k-1} \times P).$$

for some integers $a_i^{'}, b_j^{'} \in \mathbb{F}_2$ and $n_1^{'}, n_2^{'} \in \mathbb{N}$. Now we consider the composition $\alpha_1 \circ \beta$, this gives a cycle $\beta^{'}$ in $\overline{Ch}(Q_{k-1} \times Q_{k-1})$ containing $h^0 \times l_{\mathfrak{i}_k(q)-1}$. By Corollary 5.3.9, $\beta^{'}$ must also contain the essential basic cycle $l_{\mathfrak{i}_k(q)-1} \times h^0$.

Note that the essential basic cycle $l_{\mathfrak{i}_k(q)-1} \times h^0$ is obtained by

$$(b_{\mathfrak{i}_k(q)-1}^{'} b_{D_1-D_2^{'}+\mathfrak{i}_k(q)-1}) deg(l_{D_1-D_2^{'}+\mathfrak{i}_k(q)-1} \cdot h^{D_1-D_2^{'}+\mathfrak{i}_k(q)-1})(l_{\mathfrak{i}_k(q)-1} \times h^0)$$

from our composition.

As a result, $b_{D_1-D_2^{'}+\mathfrak{i}_k(q)-1} = 1$ and α_1 contains the cycle $l_{D_1-D_2^{'}+\mathfrak{i}_k(q)-1} \times h^0$. Finally, we conclude that α contains the essential basic cycle $l_{D_1-D_2+\mathfrak{j}_k(q)+\mathfrak{j}_{k-1}(q)-1} \times h^{\mathfrak{j}_{k-1}(q)}$. This finishes our proof. □

The following theorem is a variant of Vishik's theorem [10] for generically smooth quadrics.

Theorem 5.5.2. *Let p, q be the two anisotropic quadratic forms over F which are not totally singular. We let P and Q be the associated quadrics of p, q respectively. For any separable field extension E/F, $i_W(p_E) > m$ is equivalent to $i_W(q_E) > n$ for some non-negative integers m, n. If there exists a minimal cycle $\gamma \in \overline{Ch}_{dim(P)}(P \times P)$ such that $a(\gamma) = m$, then there exists a minimal cycle $\beta \in Ch_{dim(Q)}(Q \times Q)$ such that $\Lambda(\beta) = \Lambda(\gamma)[n-m]$.*

Proof. To simplify notations, we set $dim(p) = D_1 + 2$ and $dim(q) = D_2 + 2$. We prove the theorem by several steps. In the first step, we will show that there exists a rational cycle in $\overline{Ch}(P \times Q)$ starting with $h^m \times l_n$. To see this, we induct on m.

When $m = 0$, consider the surjective homomorphism $\overline{Ch}(P \times Q) \to \overline{Ch}(Q_{F(p)})$ induced by $Q_{F(p)} \to P \times Q$ (Corollary 2.2.3). Since $i_W(p_{F(p)}) > 0$ is equivalent to $i_W(q_{F(p)}) > n$, we conclude that $l_n \in \overline{Ch}(Q_{F(p)})$. Thus the preimage of l_n under the surjective homomorphism starts with the cycle $h^0 \times l_n$.

Let p_1 be the first kernel form of p and P_1 be the associated quadric of p_1. Then the assumption that $i_W(p_E) > m$ is equivalent to $i_W(q_E) > n$ for any separable extension E/F implies the assumption that $i_W((p_1)_L) > m - \mathfrak{i}_1(p)$ is equivalent to $i_W(q_L)$ for any separable extension L/F containing $F(p)$. Since we have a minimal cycle $\gamma \in \overline{Ch}_{D_1}(P \times P)$ such that $a(\gamma) = m$. Let $pr_* : \overline{Ch}((P \times P)_{F(p)}) \to \overline{Ch}(P_1 \times P_1)$ be the group homomorphism as we considered in Proposition 4.5.4. Then $pr_*(\gamma) \in \overline{Ch}_{dim(P_1)}(P_1 \times P_1)$ is a cycle such that $a(pr_*(\gamma)) = m - \mathfrak{i}_1(p)$. By the induction hypothesis, we have a rational cycle $\alpha^{'} \in \overline{Ch}(P_1 \times Q)$ starting with the essential basic cycle $h^{m-\mathrm{i}_1(p)} \times l_n$.

Notice that $p_1 = (p_{F(p)})_{an}$. We let $in_* : \overline{Ch}(P_1) \to \overline{Ch}(P_{F(p)})$ be the group homomorphism we considered in Proposition 3.5.5. Then we have the group homomorphism $p := in_* \otimes id : \overline{Ch}(P_1 \times Q) \to \overline{Ch}(P_{F(P)} \times Q)$. We let $\alpha = p(\alpha^{'})$. Without loss of generality, we may assume that α is homogeneous. Then α has the following form.

$$\alpha = h^m \times l_n + \sum_{i=0}^{n_1} a_i(h^i \times l_{i+n-m}) + \sum_{j=0}^{n_2} b_j(l_j \times h^{D_2-D_1+j+m-n}) \in \overline{Ch}_{D_1+n-m}((P)_{F(p)} \times Q)$$

for some integers $n_1, n_2 \in \mathbb{N}$ and $a_i, b_j \in \mathbb{F}_2$.

Now we consider the surjective homomorphism $\overline{Ch}(P \times P \times Q) \to \overline{Ch}(P_{F(P)} \times Q)$ induced by $P_{F(P)} \times Q \to P \times P \times Q$ (Corollary 2.2.3), this will give us a rational cycle $\alpha_1 \in \overline{Ch}(P \times P \times Q)$ has the following form.

$$\alpha_1 = h^0 \times \alpha + \sum_i a_i^{'} h^i \times \alpha_i$$

for coefficients $a_i \in \mathbb{F}_2$ and cycles $\alpha_i \in \overline{Ch}(P \times Q)$. Also, we shall emphasize that α_1 starts with the cycle

$$h^0 \times \alpha = h^0 \times h^m \times l_n + \sum_{i=0}^{n_1} a_i(h^0 \times h^i \times l_{i+n-m}) + \sum_{j=0}^{n_2} b_j(h^0 \times l_j \times h^{D_2-D_1+j+m-n}).$$

By taking the homogeneous part of α_1 containing $h^0 \times h^m \times l_n$, we may assume that α_1 is homogeneous.

Recall that we have a cycle $\gamma \in \overline{Ch}_{D_1}(P \times P)$ which starts with $h^m \times l_m$. Then we consider the image of γ under the pull-back $\overline{Ch}(P \times P) \to \overline{Ch}(P \times P \times Q)$ induced by the projection onto the first factor $P \times P \times Q \to P \times P$, this gives us a cycle $\alpha_2 \in \overline{Ch}(P \times P \times Q)$ starting with the cycle $h^m \times l_m \times h^0$. Actually, α_2 has the following form.

$$\alpha_2 = h^m \times l_m \times h^0 + \sum_{(m+i) \in \Lambda(\gamma)_{lo}, i \neq 0} h^i \times l_i \times h^0 + \sum_{j \in \Lambda(\gamma)^{up}} l_j \times h^j \times h^0.$$

Now we consider the product $\alpha_1 \cdot \alpha_2$, the product has the following form.

$$\alpha_1 \cdot \alpha_2 = h^m \times l_0 \times l_n + \sum_{j>m} h^j \times ? \times ? + \sum_k l_k \times ? \times ?.$$

Let $g : \overline{Ch}(P \times P \times Q) \to \overline{Ch}(P \times Q)$ be the push-forward induced by the projection $P \times P \times Q \to P \times Q$ which disappears the middle factor. Then the cycle $g(\alpha_1 \cdot \alpha_2)$ has the following form.

$$g(\alpha_1 \cdot \alpha_2) = h^m \times l_n + \sum_{j>m} h^j \times ? + \sum_k l_k \times ?.$$

This gives the desired cycle in $\overline{Ch}(P \times Q)$ starting with $h^m \times l_n$, which finishes the induction.

Now we take $\gamma' := g(\alpha_1 \cdot \alpha_2) \circ \gamma$. Then $\gamma' \in \overline{Ch}(P \times Q)$ has the following form

$$\gamma' = h^m \times l_n + \sum_{i \in I} h^i \times l_? + \sum_{j \in J} l_j \times h^?$$

where $I \subset \Lambda(\gamma)_{lo} \setminus \{m\}$ and $J \subset \Lambda(\gamma)^{up}$

For integers m, n, by Corollary 3.5.7, we know that $\mathfrak{j}_{r-1}(p) \leq m < \mathfrak{j}_r(p)$ and $\mathfrak{j}_{k-1}(q) \leq n < \mathfrak{j}_k(q)$ for some integers $r \in [1, h_{sep}(p)]$ and $k \in [1, h_{sep}(q)]$. Then our assumption that $i_W(p_E) > m$ is equivalent to $i_W(q_E) > n$ for any separable extension E/F implies that $i_W(p_L) > \mathfrak{j}_{r-1}(p)$ is equivalent to $i_W(q_L) > \mathfrak{j}_k(q) - 1$ for any separable extension L/F by Corollary 3.5.7. We let $\beta_1 \in \overline{Ch}(P \times Q)$ be the cycle starting with $h^{\mathfrak{j}_{r-1}(p)} \times l_{\mathfrak{j}_k(q)-1}$ which is constructed the same as γ' as above.

Let p_{r-1} be the $(r-1)$th kernel form of p and F_{r-1} be the field in the Knebuch splitting tower. Also, we let P_{r-1} be the associated quadric of p_{r-1} and $pr_* : \overline{Ch}(P) \to \overline{Ch}(P_{r-1})$ be the group homomorphism we considered in Proposition 4.5.4. Consider the group homomorphism $p' := pr_* \otimes id : \overline{Ch}(P \times Q) \to \overline{Ch}(P_{r-1} \times Q)$, we have the cycle $\beta' := p'(\beta_1) \in \overline{Ch}(P_{r-1})$ containing $h^0 \times l_{\mathfrak{j}_k(q)-1}$. Note that our assumption implies that for any separable field extension L'/F containing F_{r-1}, $i_W((p_{r-1})_{L'}) > 0$ is equivalent to $i_W(q_{L'}) > \mathfrak{j}_k(q) - 1$. Now we set that $dim(P_{r-1}) = D_1'$. Hence by Lemma 5.5.2, β' also contains $l_t \times h^{\mathfrak{j}_{k-1}(q)}$ where $t = D_1' - D_2 + \mathfrak{j}_k(q) + \mathfrak{j}_{k-1}(q) - 1$. As a result, the cycle β_1 contains $l_{t+\mathfrak{j}_{r-1}(p)} \times h^{\mathfrak{j}_{k-1}(q)}$ and has the following form.

$$\beta_1 = h^{\mathfrak{j}_{r-1}(p)} \times l_{\mathfrak{j}_k(q)-1} + \cdots + l_{t+\mathfrak{j}_{r-1}(p)} \times h^{\mathfrak{j}_{k-1}(q)}.$$

Also, the cycle $\beta_1^t \in \overline{Ch}(Q \times P)$ has the following form.

$$\beta_1^t = h^{\mathfrak{j}_{k-1}(q)} \times l_{t+\mathfrak{j}_{r-1}(p)} + \cdots + l_{\mathfrak{j}_k(q)-1} \times h^{\mathfrak{j}_{r-1}(p)}.$$

We consider the cycle $\gamma_1 := \beta_1 \circ (\beta_1^t \cdot (h^0 \times h^t) \in \overline{Ch}(Q \times Q)$, which contains the essential basic cycle $h^{\mathfrak{j}_{k-1}(q)} \times l_{\mathfrak{j}_k(q)-1}$. By taking the minimal cycle in γ_1 which contains $h^{\mathfrak{j}_{k-1}(q)} \times l_{\mathfrak{j}_k(q)-1}$, we may assume that γ_1 is minimal. Let $\beta = \gamma_1 \cdot (h^{n-\mathfrak{j}_{k-1}(q)} \times h^{\mathfrak{j}_k(q)-1-n})$. Also, notice that $\gamma_2 := (\beta_1^t \cdot (h^{\mathfrak{i}_k(q)-1} \times h^t)) \circ \beta_1$ starts with the essential basic cycle $h^{\mathfrak{j}_{r-1}(p)} \times l_{\mathfrak{j}_{r-1}}(p)$, which lies in the same shell triangle as $h^m \times l_m$. By taking the minimal cycle in γ_1 which contains $h^{\mathfrak{j}_{r-1}(p)} \times l_{\mathfrak{j}_{r-1}(p)}$, we may also assume that γ_2 is minimal. From the composition law (Lemma 4.4.3), we have that $\Lambda(\beta) = \Lambda(\gamma_2)[n - \mathfrak{j}_{r-1}(p)]$. Since γ and γ_2 starts in the same shell triangle, by Corollary 5.3.4 and the construction of β_1, we know $\Lambda(\gamma) = \Lambda(\gamma_2)[m - \mathfrak{j}_{r-1}(p)]$. Then

$$\Lambda(\beta) = \Lambda(\gamma)[n - \mathfrak{j}_{r-1}(p) - (m - \mathfrak{j}_{r-1}(p))] = \Lambda(\gamma)[n - m].$$

gives the result. □

Corollary 5.5.3. *Let p, q be the two anisotropic quadratic forms over F which are not totally singular. We let P and Q be the associated quadrics of p, q respectively. We suppose that for any separable field extension E/F, $i_t(p_E) > 0$ is equivalent to $i_t(q_E) > n$ where n is a non-negative integer. Then there exist minimal cycles $\alpha \in \overline{Ch}_{dim(Q)}(Q \times Q)$ and $\beta \in \overline{Ch}_{dim(P)}(P \times P)$ such that $\Lambda(\beta) = \Lambda(\alpha)[n]$.*

Proof. There always exists a rational cycle in $\overline{Ch}_{dim(P)}(P \times P)$ starting with $h^0 \times l_0$, i.e., the upper summand $U(P)$ (Lemma 5.2.11). Thus the corollary follows directly from Theorem 5.5.2. □

5.6 Generic forms

In this section, we will compute the rational correspondence type on even-dimensional generic form which is not totally singular.

Fix a field F. Let $\overline{F}$ be an algebraic closure of F.

Definition 5.6.1. When $char(F) = 2$, we define the *generic form of type* (r, s) to be a quadratic form which is isometric to $x_1[1, y_1] \perp \cdots \perp x_r[1, y_r] \perp \langle z_1, \cdots, z_s \rangle$ over $F(x_1, \cdots, z_s)$ with algebraically independent variables $x_1, y_1, \cdots, x_r, y_r, z_1, \cdots, z_s$.

When $char(F) \neq 2$, we define the *generic form of dimension* k to be a quadratic form which is isometric to $\langle x_1, \cdots, x_k \rangle$ over $F(x_1, \cdots, x_k)$ with algebraically independent variables $x_1, \cdots, x_k$.

Lemma 5.6.2. *Let $F(t)$ be a purely transcendental extension of F with variable t. Let $q_1 \simeq \langle 1 \rangle \perp c\langle t \rangle$ and $q_2 \simeq c[1, t]$ for some $c \in F^\times$. Then the function fields $K(q_1)$ and K_{q_2} are all purely transcendental extensions of F.*

Proof. We compute $F(t)(q_1)$ and $F(t)(q_2)$ explicitly. q_1 is represented by the polynomial $f_1(x, y) = x^2 + cty^2$ and q_2 is represented by the polynomial $f_2(x, y) = cx + cxy + cty^2$. Then $F(t)(q_1) \cong F(t)[y]/(cty^2 + 1)$ and $F(t)(q_2) \cong F(t)[y]/(cty^2 + cy + c)$.

Notice that $F(t)(q_1) \cong F(\sqrt{-ct})$. To show $F(t)(q_1)$ is a purely transcendental extension over F, it suffices to show that $\sqrt{-ct}$ is transcendental over F. We suppose $\sqrt{-ct}$ is algebraic over F. Then $t = (\sqrt{-ct})^2/(-c)$ is also algebraic over F, which is a contradiction.

Notice that $F(t)(q_2) \cong F(\alpha)$, where α is a root of the polynomial $ty^2 + y + 1 = 0$. To show $F(t)(q_2)$ is a purely transcendental extension over F, it suffices to show that α is transcendental over F. We suppose α is algebraic over F. Then $t = -(\alpha + 1)/\alpha^2$ is also algebraic over F, which is a contradiction.

Thus $F(t)(q_1)$ and $F(t)(q_2)$ are purely transcendental extensions of F. □

Proposition 5.6.3. *Let ψ be a quadratic form over K where K/F is a field extension. Suppose that ϕ is a generic form over $F(x_1, \cdots, z_s)$ which has the same type as q. Then there exists a field extension $M/K(x_1, \cdots, z_s)$ such that $\phi_M \simeq \psi_M$ and M is purely transcendental over K.*

Proof. To simplify, we will only prove the statement when $char(F) = 2$. The proof for the case that $char(F) \neq 2$ is similar.

Suppose ψ has type (r, s). Let $\phi \simeq x_1[1, y_1] \perp \cdots \perp x_r[1, y_r] \perp \langle z_1, \cdots, z_s \rangle$ be the generic form of type (r, s) over $F(x_1, \cdots, z_s)$ where $x_1, \cdots, y_r, z_1, \cdots, z_s$ are algebraically independent variables. We suppose that $\psi \simeq a_1[1, b_1] \perp \cdots \perp a_r[1, b_r] \perp \langle c_1, \cdots, c_s \rangle$.

Now we let $\widetilde{K} := K(x_1, \cdots, z_s)$. Under the field extension $\widetilde{K}(\sqrt{z_1 c_1})$, we have $\langle z_1 \rangle \simeq \langle c_1 \rangle$. By Lemma 5.6.2, $\widetilde{K}(\sqrt{z_1 c_1})$ is a purely transcendental extension over $K(x_1, y_1, \cdots, x_r, y_r, z_2, \cdots, z_s)$. Inductively, we have a purely transcendental field extension $\widetilde{K}(\sqrt{z_1 c_1}, \cdots, \sqrt{z_s c_s})$ over $K(x_1, y_1, \cdots, x_r, y_r)$ such that

$$\psi_{\widetilde{K}(\sqrt{z_1 c_1}, \cdots, \sqrt{z_s c_s})} \simeq a_1[1, b_1] \perp \cdots \perp a_r[1, b_r] \perp \langle z_1, \cdots, z_s \rangle.$$

Let α_i be a root of the equation $t^2+t+(b_i-y_i)=0$. Then we have $[1,b_i]\simeq[1,y_i]$ under the field extension $\widetilde{K}(\alpha_i)$, which is a purely transcendental extension over $K(x_1,\cdots,x_i,x_{i+1},y_{i+1},\cdots,z_s)$ by Lemma 5.6.2 (to see this, in Lemma 5.6.2, we can set $c=1$ and $t^{'}=c^{'}+t$ for some $c^{'}\in F^{\times}$. Then the form $q_2\simeq[1,c^{'}-t^{'}]$ while $t^{'}$ is still purely transcendental over F). Inductively, $\widetilde{K}(\alpha_1,\cdots,\alpha_r)$ is a purely transcendental extension over $K(x_1,x_2,\cdots,x_r,z_1,\cdots,z_s)$ such that

$$\psi_{\widetilde{K}(\alpha_1,\cdots,\alpha_r)}\simeq a_1[1,y_1]\perp\cdots\perp a_r[1,y_i]\perp\langle z_1,\cdots,z_s\rangle$$

Finally, we consider the field extension $\widetilde{K}(\sqrt{a_ix_i})$. Under this extension, we have $\langle a_i\rangle\simeq\langle x_i\rangle$. By Lemma 5.6.2, this is a purely transcendental extension over $K(x_1,y_1,\cdots,x_{i-1},y_{i-1},y_{i+1},\cdots,x_r,y_r,z_1,\cdots,z_s)$. Inductively, we conclude that

$$\widetilde{K}(\sqrt{a_1x_1},\cdots,\sqrt{a_rx_r})$$

is a purely transcendental extension over $K(y_1,\cdots,y_r,z_1,\cdots,z_s)$ such that

$$\psi_{\widetilde{K}(\sqrt{a_1x_1},\cdots,\sqrt{a_rx_r})}\simeq x_1[1,b_1]\perp\cdots\perp x_r[1,b_r]\perp\langle c_1,\cdots,c_s\rangle.$$

By setting $M=\widetilde{K}(\sqrt{c_1z_1},\cdots,\sqrt{c_sz_s},\sqrt{a_1x_1},\cdots,\sqrt{a_rx_r},\alpha_1,\cdots,\alpha_r)$, we have that

$$\phi_M\simeq a_1[1,b_1]\perp\cdots\perp a_r[1,b_r]\perp\langle c_1,\cdots,c_s\rangle\simeq\psi_M$$

and M/K is a purely transcendental extension. This finishes our proof. □

Let $a\in F^{\times}$ and q be any quadratic form of dimension ≥ 2 over F. We define $\langle 1,-a\rangle\otimes q$ to be $q\perp(-a)q$. Then we have the following results.

Lemma 5.6.4. *Let ϕ be an anisotropic quadratic form over F. Suppose that $a\in F^{\times}\setminus F^{\times 2}$. If $q:=\langle 1,-a\rangle\otimes\phi$ is isotropic, then $i_t(q)\geq 2$.*

Proof. We assume $v+w$ is an isotropic vector in $V_q=V_\phi\oplus V_{(-a)\phi}$ such that $q(v+w)=\phi(v)-a\phi(w)=0$. At first, we will show that v and w are linear independent. To see this, we assume that $v=kw$ for some scalar $k\in F$. Then $k^2\phi(w)=\phi(kw)=a\phi(w)$. Since ϕ is anisotropic, we have $a=k^2$, which contradicts our assumption. As a result, we know that $W:=Span\{v,w\}$ is a two-dimensional subspace of V_ϕ.

Now we consider the form $\phi^{'}:=\phi\mid_W$. Notice that $\langle 1,-a\rangle\otimes\phi^{'}$ is dominated by $\langle 1,-a\rangle\otimes\phi$. If $\phi^{'}$ is non-degenerate, by the Structure Theorem (Theorem 3.2.10),

$\phi^{'} \simeq [b, c]$ for some $b, c \in F$. Then $\langle 1, -a\rangle \otimes \phi^{'}$ is an isotropic quadratic Pfister form. By Theorem 3.2.19, we know that $i_W(\langle 1, -a\rangle \otimes \phi^{'}) = 2$. Hence $i_t(\langle 1, -a\rangle \otimes \phi^{'}) \geq 2$.

If $\phi^{'}$ is degenerate, then $\phi^{'} \simeq \langle b^{'}, c^{'}\rangle$ for some $b^{'}, c^{'} \in F$. In this case, $\langle 1, -a\rangle \otimes \phi^{'}$ is the diagonal of an isotropic bilinear Pfister form. As a result, we have $i_t(\langle 1, -a\rangle \otimes \phi^{'}) = 2$ by Proposition 3.1.13. This also shows $i_t(\langle 1, -a\rangle \otimes \phi) \geq 2$. □

Corollary 5.6.5. *Let ϕ be an anisotropic quadratic form over F. Suppose that $q := \langle 1, -a\rangle \otimes \phi$ is anisotropic and not totally singular for some $a \in F^{\times} \setminus F^{\times 2}$. Then $\mathfrak{i}_1(q) \geq 2$.*

Proof. Let $L = F(q)$. Then $q_{F(q)}$ is isotropic. Since q is not totally singular, the field extenison $F(q)$ is separable by Proposition 3.4.5. Hence $i_t(q_F(q)) = i_W(q_{F(q)}) > 0$ by Lemma 3.4.6. From Lemma 5.6.4, the only thing we need to show is that $a \in L^{\times} \setminus L^{\times 2}$, which means $\langle 1, -a\rangle_L$ stays anisotropic. But since $dim(\langle 1, -a\rangle) < 2^2$ and $dim(q) \geq 2^2$, by the Separation Theorem (Theorem 3.4.10), we know that $\langle 1, -a\rangle_L$ is anisotropic. □

Proposition 5.6.6. *Let ϕ be an even-dimensional generic form which is not totally singular and X be the associated quadric of ϕ. Then $\Lambda(X)$ is indecomposable.*

Proof. To simplify, we will only prove the proposition when $char(F) = 2$. The other case can be proved similarly. Suppose ϕ is of type (r, s) where $r \geq 1$ and X is the associated quadric of ϕ. We prove it by inducting on the number of shell triangles of ϕ. At first, recall that the full splitting pattern of ϕ is $(0, 1, 2, \cdots, r+s-1)$ (Example 3.5.14).

When $r = 1$, the statement is trivially true. Now we assume $r > 1$. Let $\phi \simeq x_1[1, y_1] \perp \cdots \perp x_r[1, y_r] \perp \langle z_1, \cdots, z_s\rangle$ over $F(x_1, \cdots, z_s)$. Consider the field extension $L = F(x_1[1, y_1])$ and set $\psi := (\phi_L)_{an}$ with the associated quadric $X^{'}$. Notice that ψ is still a generic quadratic form of even-dimension. Since $r > 1$, ψ is not totally singular. Actually, we have

$$\psi \simeq x_2[1, y_2] \perp \cdots x_r[1, y_r] \perp \langle z_1, \cdots, z_s\rangle.$$

By the induction hypothesis, we know that $\Lambda(X^{'})$ is indecomposable. Let $pr_* : \overline{Ch}(X \times X) \to \overline{Ch}(X^{'} \times X^{'})$ be the group homomorphism as considered in Proposition 4.5.4. Let $\alpha \in \overline{Ch}(X \times X)$ be the minimal cycle starting with $h^0 \times l_0$. If $pr_*(\alpha) \neq \emptyset$, then $\Lambda(pr_*(\alpha)) = \Lambda(X^{'})$, which implies that $\alpha = h^0 \times l_0 + l_0 \times h^0 + \sum_{i=1}^{r-1}(h^i \times l_i + l_i \times h^i)$. This shows that $\Lambda(X)$ is indecomposable.

Now we will show that $pr_*(\alpha) \neq \emptyset$. Let p be an anisotropic quadratic form over K where K/F is a field extension and $q := \langle 1, -a \rangle \otimes p$ be anisotropic quadratic form of the same type as ϕ with $a \in K^\times \setminus K^{\times 2}$. These forms do exist. For example, we can take $K = F_1(t)$ where F_1 is a field extension of F and t is transcendental over F_1. Then we set $a = t$ while p is some anisotropic form over F_1 of dimension $r + s$ which is not totally singular.

By Proposition 5.6.3, there exists a purely transcendental field extension M/K such that $\phi_M \simeq q_M$. By Corollary 5.6.5, we know that $\mathfrak{i}_1(q_M) \geq 2$. As a result, under the field extension M, α contains the essential basic cycle $l_{\mathfrak{i}_1(q_M)-1} \times h^{\mathfrak{i}_1(q_M)-1}$. Hence α must also contain the essential basic cycle over the field F. This shows that $pr_*(\alpha) \neq \emptyset$, which finishes our proof. □

5.7 Pfister neighbours and excellent forms

In this section, we will discuss the rational correspondence type of quadrics given by certain kinds of quadratic forms. Fix a field F and let $\overline{F}$ be the algebraic closure of F. Let ϕ be an anisotropic quadratic form over F of type $(n+1, s)$ which is not totally singular. We suppose that X is the associated quadric of ϕ.

Let $\alpha \in \overline{Ch}_D(X \times X)$ be any minimal cycle. Recall that by Corollary 5.3.9, for any essential basic cycle contained in α, their dual cycles are also contained in α. As a result, $rank(\alpha) \geq 2$. If $rank(\alpha) = 2$, then we say that α is a *binary correspondence* and $\Lambda(\alpha)$ is a *binary component*.

The following theorem computes the rational correspondence type of any quadric given by a Pfister form, which is due to M.Rost [17].

Theorem 5.7.1. *Let q be an anisotropic quadratic Pfister form over F and Q be the associated quadric of q. Suppose $dim(q) = 2^n$ for some positive integer n. Then $RCT(Q)$ consists of binary components. More precisely, $\Lambda(U(Q)) = \{0, 2^{n-1} - 1\}$ and*

$$RCT(Q) = \bigsqcup_{i=0}^{2^{n-1}-1} \Lambda(U(Q))[i].$$

Proof. We know that $q_{F(q)}$ is hyperbolic by Theorem 3.2.19, hence $\mathfrak{i}_1(q) = 2^{n-1}$. Let $\alpha \in \overline{Ch}_{dim(Q)+\mathfrak{i}_1(q)-1}(Q \times Q)$ be the minimal cycle starting with $h^0 \times l_{\mathfrak{i}_1(\mathfrak{q})-1}$. Since $\oplus_{i \geq dim(Q)} \overline{Ch}_i(Q \times Q)$ only has one left (or right) shell triangle, we know that $\alpha = h^0 \times l_{\mathfrak{i}_1(q)-1} + l_{\mathfrak{i}_1(q)-1} \times h^0$ by Theorem 5.3.6. Note that $U(Q) = \alpha \cdot (h^0 \times h^{\mathfrak{i}_1(q)-1})$. Thus

$U(Q) = h^0 \times l_0 + l_{\mathfrak{i}_1(q)-1} \times h^{\mathfrak{i}_1(q)-1}$. The cycle $h^0 \times l_0$ corresponds to $0 \in \Lambda(U(Q))$ while $l_{\mathfrak{i}_1(q)-1} \times h^{\mathfrak{i}_1(q)-1}$ corresponds to $dim(Q) - (\mathfrak{i}_1(q) - 1) = 2^n - 2 - (2^{n-1} - 1) = 2^{n-1} - 1$. Let $\beta \in \overline{Ch}_{dim(Q)}(Q \times Q)$ be any minimal cycle with $a(\beta) \leq \mathfrak{i}_1(q) - 1$. Notice that $\Lambda(\beta)$ is a shift of $\Lambda(U(Q))$. We conclude that $RCT(Q) = \bigsqcup_{i=0}^{2^{n-1}-1} \Lambda(U(Q))[i]$. □

The following theorem is given by A.Vishik for quadrics given by Pfister neighbours when $char(F) \neq 2$ [21].

Theorem 5.7.2. *Let q be a Pfister neighbour over F and π be the associated anisotropic general Pfister form of q. We assume that $dim(\pi) = 2^{n+1}$ for some integer n and $dim(q) = 2^n + m$ with integer $1 \leq m \leq 2^n$. Let q^c be the complementary form of q in π. Let Q be the associated quadric of q and Q^c be the associated quadric of q^c. Then we have $\Lambda(U(Q)) = \{0, 2^{n-1} - 1\}$. Moreover, we have*

$$\begin{aligned} RCT(Q) &= (\bigsqcup_{i=0}^{m-1} \Lambda(U(Q)[i]) \bigsqcup (RCT(Q^c)[m]) \\ &= \bigsqcup_{i=0}^{m-1} \{i, 2^n - 1 + i\} \bigsqcup (RCT(Q^c)[m]). \end{aligned}$$

Proof. The case when $char(F) \neq 2$ is done by A.Vishik. So we will only prove the statement for the case when $char(F) = 2$.

We let X be the associated quadric of π. Since q and π are stably birational equivalent, we know that for any separable field extension E/F, $i_t(\pi_E) > 0$ is equivalent to $i_t(q_E) > 0$ (Proposition 3.6.3). By Corollary 5.5.3 and Theorem 5.7.1, we then have that $\Lambda(U(Q)) = \Lambda(U(X))[0] = \{0, 2^n - 1\}$.

Since $dim(q) = 2^n + m$, by Proposition 3.7.3, we know that $\mathfrak{i}_1(q) = m$. Let $\alpha \in \overline{Ch}_{dim(Q)}(Q \times Q)$ be any minimal cycle with $a(\alpha) \leq \mathfrak{i}_1(q) - 1 = m - 1$. Notice that $\Lambda(\alpha)$ is a shift of $\Lambda(U(Q))$. As a result, we obtain all connected component of $\Lambda(X)$ in the first shell, that is, $\bigsqcup_{i=0}^{m-1}(\Lambda(U(Q))[i])$.

Recall that $q^c_{F(q)} \simeq (q_{F(q)})_{an}$ (Proposition 3.7.5). This means, for any integer $k \geq m$, for any separable field extension L/F, $i_t(q^c_L) > k - m$ is equivalent to $i_t(q_L) > k$. Let β be any minimal cycle in $\overline{Ch}_{dim(Q^c)}(Q^c \times Q^c)$. Then by Theorem 5.5.2, there exists a minimal cycle $\gamma \in \overline{Ch}_{dim(Q)}(Q \times Q)$ such that $\Lambda(\gamma) = \Lambda(\beta)[k - (k - m)] = \Lambda(\beta)[m]$. As a result, we have

$$\bigsqcup_{\beta} (\Lambda(\beta)[m]) = RCT(Q^c)[m].$$

where the disjoint union is taken over all minimal cycles $\beta \in \overline{Ch}_{dim(Q^c)}(Q^c \times Q^c)$.

Let $\alpha^{'}$ be any minimal cycle in $\overline{Ch}_{dim(Q)}(Q \times Q)$. Then $a(\gamma^{'})$ is either $\leq m-1$ or $\geq m$. In the former case, we know $\alpha^{'}$ is a binary correspondence directly. In the latter case, we consider the group homomorphism $pr_* : \overline{Ch}((Q \times Q)_{F(q)}) \to \overline{Ch}(Q^c \times Q^c)$. As a result, $pr_*(\alpha^{'}) \in \overline{Ch}_{dim(Q^c)}(Q^c \times Q^c)$ contains a minimal subcycle starting with $a(\alpha^{'}) - m$. This means there exists a minimal cycle $\beta^{'} \in \overline{Ch}_{dim(Q^c)}(Q^c \times Q^c)$ such that $\Lambda(\alpha^{'}) = \Lambda(\beta^{'})[m]$ by Theorem 5.5.2.

Finally, we conclude that $RCT(Q) = (\bigsqcup_{i=0}^{m-1} \Lambda(U(Q)[i]) \bigsqcup (RCT(Q^c)[m])$ □

Let ψ be an anisotropic quadratic form over F and $F_0 = F \subset F_1 \subset \cdots \subset F_{h_{sep}(\psi)}$ be the Knebusch splitting tower up to the separable height $h_{sep}(\psi)$. For any $i \in [0, h_{sep}(\psi)]$, we have ith kernel form ψ_i. We let ψ_{ts} be the totally singular part of ψ.

Recall that we say ψ is a quasi-strongly excellent form if $\psi_i, i \in [0, h_{sep}(\psi) - 1]$ are all Pfister neighbours. Moreover, if each ψ_i is defined over F and $\psi_i^c \simeq \psi_{i+1}$ for any $i \in [0, h_{sep}(\psi) - 2]$ while $\psi^c_{h_{sep}(\psi)-1} = 0$ or $\psi^c_{h_{sep}(\psi)-1} = c$ for some $c \in F^{\times}$ or $\psi^c_{h_{sep}(\psi)-1}$ is isometric to the $(\psi_{ts})_{an}$ of ψ, we say ψ is a strongly excellent form.

By definitions, a strongly excellent form must be a quasi-strongly excellent form. So it suffices to compute the rational correspondence type of quadrics given by quasi-strongly excellent forms as in Corollary 5.7.5. We need the following two lemmas first.

Lemma 5.7.3. *Let ϕ be an anisotropic quadratic form over F and $F^{'}/F$ be a purely transcendental field extension. Let X be the associated quadric of ϕ. Then $RCT(X) = RCT(X_{F^{'}})$.*

Proof. We assume that $F^{'}/F$ is of finite transcendental degree k. Consider the projection $p : X \times \mathbb{A}^k \to X$ onto the first factor. We know that p is a vector bundle, hence an affine bundle. By the Homotopy Invariance (Theorem 2.4.4), we know that the pull-back $p^* : Ch(X) \to Ch(X \times \mathbb{A}^k)$ is an isomorphism. Note that $F(\mathbb{A}^k) = F^{'}$. Hence we have a surjective homomorphism $s : Ch(X \times \mathbb{A}^k) \to Ch(X_{F^{'}})$ induced by $X_{F^{'}} \to X \times \mathbb{A}^k$ (Corollary 2.2.3). Finally, we have a surjective homomorphism $s^{'} := s \circ p^* : Ch(X) \to Ch(X_{F^{'}})$.

Obviously, $s^{'}$ induces a surjective homomorphism $\overline{s} : \overline{Ch}(X) \to \overline{Ch}(X_{F^{'}})$. Thus we conclude $\overline{Ch}(X_{F^{'}}) = \overline{Ch}(X)$, hence $RCT(X) = RCT(X_{F^{'}})$. □

Lemma 5.7.4. [2, Lemma 88.5] *Let X be a variety, Y be a scheme and n be an integer such that the natural homomorphism $CH_i(X) \to CH_i(X_{F(y)})$ is surjective for* *every point $y \in Y$ and $i \geq dim(X) - n$. Then $CH_j(Y) \to CH_j(Y_{F(X)})$ is surjective for every $j \geq dim(Y) - n$.*

Notice that Lemma 5.7.4 is for integral Chow groups. But the proof still works for Chow groups with coefficients in $\mathbb{F}_2$. Now we come to the following main result.

Corollary 5.7.5. *Let ψ be an anisotropic quasi-strongly excellent form. Let X be the associated quadric of ψ. Then $RCT(X)$ consists of binary components. More precisely,*

$$RCT(X) = \bigsqcup_{k=1}^{h_{sep}(\psi)} \Big(\bigsqcup_{i=0}^{\mathfrak{i}_k(\psi)-1} \{\mathfrak{j}_{k-1}(\psi) + i, dim(X) - \mathfrak{j}_k(\psi) + 1 + i\}\Big).$$

where $h_{sep}(\psi)$ is the separable height of ψ.

Since the separable splitting pattern of a strongly excellent form is determined by its dimension (Proposition 3.8.9), we have completely determined the rational correspondence type of X.

Proof. Let $\psi_0, \cdots, \psi_{h_{sep}(\psi)-1}$ be the kernel forms of ψ. We let X_i be the associated quadric of ψ_i for each $i \in [1, h_{sep}(\psi)-1]$. To simplify notations, we set $s := h_{sep}(\psi)-1$.

By the definition, $\psi_0, \cdots, \psi_s$ are all Pfister neighbours. We let X_i^c be the associated quadric of ψ_i^c. For any $i \in [0, s-1]$, by Lemma 3.7.5, we have that $(\psi_i^c)_{F_{i+1}} = \psi_{i+1}$.

As a result, for $\psi_0 = \psi$, by Theorem 5.7.2, there exists an integer $m_0 = \mathfrak{i}_1(\psi)$ such that

$$RCT(X) = \bigsqcup_{i_0=0}^{m_0-1} (\Lambda(U(X))[i_0]) \bigsqcup (RCT(X^c)[m_0])$$

while $U(X)$ is a binary correspondence.

Now we want to show that $RCT(X^c) = RCT(X_1)$. To see this, let π be the associated Pfister form of $\psi = \psi_0$ and Y be the associated quadric of π.

For any closed point $x \in X^c \times X^c$, we have rational points on $(X^c \times X^c)_{F(x)}$. Hence there exists a rational point on X^c. Since $\psi^c \prec \pi$, we know that $\pi_{F(x)}$ is isotropic. Thus $Y_{F(x)}$ is a split quadric. Notice that $dim(X^c) < 1/2(dim(Y))$. We consider the natural homomorphism $Ch^i(Y) \to Ch^i(Y_{F(x)})$ induced by the projection $Y_{F(x)} \to Y$ for every $i < dim(X^c)$. Since $Y_{F(x)}$ is a split quadric, $Ch^i(Y_{F(x)})$ is generated by the pull-back of classes of planes in $(\mathbb{P}(V_\pi)_{F(x)}$ induced by the closed embedding $Y_{F(x)} \hookrightarrow \mathbb{P}(V_\pi)_{F(x)}$ (Proposition 4.2.1). These pull-back of classes of planes do exist in $Ch^i(Y)$, hence $Ch^i(Y) \to Ch^i(Y_{F(x)})$ is surjective for every $i < dim(X^c)$. By Lemma 5.7.4, we conclude that

$$s : Ch^{dim(X^c)}(X^c \times X^c) \to Ch^{dim(X^c)}((X^c \times X^c)_{F(\pi)})$$

is surjective.

Obviously, the surjective homomorphism s induces the surjective homomorphism

$$\overline{s} : \overline{Ch}_{dim(X_c)}(X_c \times X_c) \to \overline{Ch}_{dim(X_c)}((X_c \times X_c)_{F(\pi)}).$$

Since ψ is a Pfister neighbour of π, $\psi_{F(\pi)}$ is isotropic by Proposition 3.6.3. Thus $F(\pi)(\psi) = F(\psi_{F(\pi)})$ is a purely transcendental extension of $F(\pi)$. Similarly, $F(\pi)(\psi)$ is a purely transcendental extension of $F(\psi)$. By Lemma 5.7.3, we then obtain that

$$\overline{Ch}_{dim(X^c)}((X^c \times X^c)_{F(\pi)}) = \overline{Ch}_{dim(X^c)}((X^c \times X^c)_{F(\pi)(\psi)}) = \overline{Ch}_{dim(X^c)}((X^c \times X^c)_{F(\psi)}).$$

where $(X^c \times X^c)_{F(\psi)} = X_1 \times X_1$.

Finally, we have a surjective homomorphism

$$\overline{s} : \overline{Ch}_{dim(X_c)}(X_c \times X_c) \to \overline{Ch}_{dim(X_c)}(X_1 \times X_1).$$

This shows $\overline{Ch}_{dim(X_c)}(X_c \times X_c) = \overline{Ch}_{dim(X_c)}(X_1 \times X_1)$. We conclude that $RCT(X_1) = RCT(X_c)$ and

$$RCT(X) = \bigsqcup_{i_0=0}^{m_0-1} (\Lambda(U(X))[i_0]) \bigsqcup (RCT(X_1)[m_0]).$$

For ψ_1, the same argument shows that there exists an integer $m_1 = \mathfrak{i}_1(\psi_1)$ such that

$$RCT(X_1) = \bigsqcup_{i_1=0}^{m_1-1} (\Lambda(U(X_1))[i_1]) \bigsqcup (RCT(X_2)[m_1]).$$

while $U(X_1)$ is a binary correspondence.

Continuing in this way, notice that $\overline{Ch}(X_s^c \times X_s^c)$ is 0 and let $m_i = \mathfrak{i}_1(\psi_i)$ for

$i \in [2, s]$ we have that

$$\begin{aligned} RCT(X) &= \bigsqcup_{i_0=0}^{m_0-1} (\Lambda(U(X))[i_0]) \bigsqcup (RCT(X_1)[m_0]) \\ &= \bigsqcup_{i_0=0}^{m_0-1} (\Lambda(U(X))[i_0]) \bigsqcup \bigsqcup_{i_1=0}^{m_1-1} (\Lambda(U(X_1))[i_1 + m_0]) \bigsqcup (RCT(X_2)[m_1 + m_0]) \\ &= \cdots \\ &= \bigsqcup_{i_0=0}^{m_0-1} (\Lambda(U(X))[i_0]) \bigsqcup \{ \bigsqcup_{k=1}^{s} (\bigsqcup_{i_k=0}^{m_k-1} (\Lambda(U(X_k))[i_k + m_0 + \cdots + m_k])) \}. \end{aligned}$$

Since each $U(X_i), i \in [0, s]$ is a binary correspondence, we conclude that $RCT(X)$ consists of binary components. By Corollary 5.3.9, we have that

$$RCT(X) = \bigsqcup_{k=1}^{s+1} (\bigsqcup_{i=0}^{\mathfrak{i}_k(\psi)-1} \{ \mathfrak{j}_{k-1}(\psi) + i, dim(X) - \mathfrak{j}_k(\psi) + 1 + i \})$$

which finishes our proof. □

Remark 5.7.6. Recall that a strongly excellent form is an excellent form (Proposition 3.8.3) and for non-degenerate quadratic forms, strongly excellent is equivalent to excellent (Remark 3.8.5). So Corollary 5.7.5 also describes the rational correspondence type of quadrics given by non-degenerate excellent forms. When $char(F) \neq 2$, the rational correspondence type of quadrics given by anisotropic excellent forms is due to M.Rost [17].

For the rational correspondence type of quadrics given by nondegenerate anisotropic excellent forms, we have the following conjecture.

Conjecture 5.7.7. [21, Conjecture 4.12] *Let Q be a smooth anisotropic quadric. Then Q is given by an excellent form if and only if $RCT(Q)$ consists of binary components.*

More generally, we have the following conjecture.

Conjecture 5.7.8. [21, Conjecture 4.21] *Let Q be a smooth anisotropic quadric. Then Q is given by a Pfister neighbour if and only if $\Lambda(U(Q))$ is a binary component.*

5.8 Virtual Pfister neighbours

Fix a field F. Let $\overline{F}$ be an algebraic closure of F. In this section, we will discuss a large family of quadratic forms over F. In general, they are not Pfister neighbours over F. However, they will become Pfister neighbours over certain field extension of F.

Definition 5.8.1. Let q be an anisotropic quadratic form over F. We say the form q is a *virtual Pfister neighbour* if there exists a field extension L/F such that q_L is a Pfister neighbour.

The following lemma gives a result which is called "outer excellent connection" (see next Chapter for excellent connections).

Lemma 5.8.2. *Let q be a virtual Pfister neighbour of dimension $2^n + m$ for some positive integer n and integer $1 \leq m \leq 2^n$. We suppose that Q is the associated quadric of q. Recall that $U(Q)$ is the upper summand of Q. Then $\{2^n - 1\} \in \Lambda(U(Q))$.*

Proof. Let M/F be the field extension over which q becomes a Pfister neighbour. Since $\mathfrak{i}_1(q_M) = m$ (Proposition 3.7.3), we know that $0_{lo} \in \Lambda(U(X))$ is connected with $(m-1)^{up}$ by Corollary 5.3.9. Note that $dim(Q) = 2^n + m - 2$. Then $(m-1)^{up}$ corresponds to the integer $D - (m-1) = 2^n - 1$ in $\Lambda(U(X))$. □

The results below will give us the first example of virtual Pfister neighbours.

Lemma 5.8.3. *Fix a non-negative integer n. Let r, s be two integers satisfying $2^n < 2r + s \leq 2^{n+1}$ and $r + s \leq 2^n$. Then there exists an (r, s)-type Pfister neighbour over field extension K/F.*

Proof. Let $K = F(t_1, \cdots, t_{n+1})$ where $t_1, \cdots, t_{n+1}$ are algebraically independent variables. We have an $(n+1)$-fold anisotropic Pfister form $\pi := \langle\langle t_1, \cdots, t_{n+1}]]$. Then any (r, s)-type quadratic form dominated by π gives the result. □

Proposition 5.8.4. *Fix an integer n. Let r, s be two integers satisfying $2^n < 2r+s \leq 2^{n+1}$ and $r + s \leq 2^n$. Then generic forms of type (r, s) are virtual Pfister neighbours.*

Proof. Let ϕ be any generic form of type (r, s). By Lemma 5.8.3, we have a Pfister q neighbour of type (r, s) over the field $F(t_1, \cdots, t_{n+1})$. By Proposition 5.6.3, there exists a purely transcendental field extension $M/F(t_1, \cdots, t_{n+1})$ such that $\phi_M \simeq q_M$. Since q_M stays anisotropic (Lemma 3.4.3), we conclude that ϕ is a virtual Pfister neighbour. □

Now we consider virtual Pfister neighbours in a more general context. Let q be an (r,s)-type anisotropic quadratic form over F with dimension $\leq 2^{n+1}$ for some non-negative integer n. Also, we assume that $s \leq 2^n - r$. Let $K = F(t_1, \cdots, t_{n+1})$ where $t_1, \cdots, t_{n+1}$ are algebraically independent variables. Then we have an $(n+1)$-fold anisotropic Pfister form $\pi := \langle\langle t_1, \cdots, t_{n+1}]]$. Also, since K/F is a purely transcendental extension, q_K stays anisotropic. Consider the following quadratic form

$$\psi := q_K \perp -\pi. \tag{5.1}$$

Let $K_0 = K \subset K_1 \subset \cdots \subset K_h$ be the Knebusch splitting tower up to the separable height, i.e., $h = h_{sep}$ and $\psi_0, \cdots, \psi_h$ be the kernel forms.

Lemma 5.8.5. *Let $1 \leq i \leq h$. Suppose that $\pi_{K_{i-1}}$ is anisotropic, but that π_{K_i} is isotropic. Then the anisotropic dimension of $q_{K_{i-1}}$ is at most $2^{n+1} - dim(\psi_{i-1})$.*

Proof. At first, we have $q_{K_{i-1}} \sim \psi_{i-1} \perp \pi_{K_{i-1}}$. Since π_{i-1} is anisotropic, we know that $dim(\psi_{i-1} \perp \pi_{K_{i-1}}) > 2^{n+1} \geq dim(q_{K_{i-1}})$. This means that $\psi_{i-1} \perp \pi_{K_{i-1}}$ is isotropic. Because ψ_{i-1} and $\pi_{K_{i-1}}$ are all anisotropic forms, the isotropic vector of $\psi_{i-1} \perp \pi_{K_{i-1}}$ has the form $x + y$ for non-zero vectors x of ψ_{i-1} and y of $\pi_{K_{i-1}}$. This means that there exists a non-zero element $a \in D(\psi_{i-1})$ such that $-a \in D(\pi_{K_{i-1}})$. Now since π_{K_i} is isotropic and $K_i = F(\psi_{i-1})$, by the Domination Theorem(Theorem 3.4.9), we know that $\psi_{i-1} \prec -\pi_{K_{i-1}}$.

From the Witt equivalent relation $q_{K_{i-1}} \sim \psi_{i-1} \perp \pi_{K_{i-1}})$, we have

$$dim((q_{K_{i-1}})_{an}) = dim((\psi_{i-1} \perp \pi_{K_{i-1}})_{an}).$$

Recall that $\psi_{i-1} \prec -\pi_{K_{i-1}}$. By Theorem 3.4.11, we already know that $i_W(\psi_{i-1} \perp -(-\pi_{K_{i-1}})) \geq dim(\psi_{i-1})$. Thus we have

$$\begin{aligned} dim((q_{K_{i-1}})_{an}) &\leq dim(\psi_{i-1}) + 2^{n+1} - 2dim(\psi_{i-1}) \\ &= 2^{n+1} - dim(\psi_{i-1}). \end{aligned}$$

which proves the lemma □

Proposition 5.8.6. *When $dim(q) \leq 2^n$, there always exists a purely transcendental extension L/F such that π_L is anisotropic and $\pi_L \succ q_L$.*

Proof. We suppose that $dim(q) \leq 2^n$. Since $s \leq 2^n - r$, we know that $i_W(\psi_{K_h}) \geq$

$dim(q) = 2r + s$. Then there exists a smallest integer j such that $i_W(\psi_{K_j}) \geq dim(q)$. By Theorem 3.4.11, we know that $\pi_{K_j} \succ q_{K_j}$.

At first, we show that q_{K_j} is anisotropic. Let $K(\pi)$ be the function field of π. Since π splits over the field $K(\pi)$, we have $i_W(\psi_{K(\pi)}) \geq i_W(\pi_{K(\pi)}) = 2^n \geq dim(q)$. K_j is the smallest field in the Knebusch tower such that $i_W(\psi_{K_j}) \geq dim(q)$. Hence there exists an K-place $K_j \rightharpoonup K(\pi)$ (Proposition 3.4.8). The same argument as Lemma 5.6.2 shows that $K(\pi)$ is a purely transcendental extension of F. Thus $q_{K(\pi)}$ stays anisotropic. By Corollary 3.3.9, $0 = i_t(q_{K(\pi)}) \geq i_t(q_{K_j})$. Thus q_{K_j} is anisotropic. Now we use contradiction to show that π_{K_j} is also anisotropic. We assume that π_{K_j} is isotropic while $\pi_{K_{j-1}}$ anisotropic. By Lemma 5.8.5, we know that $dim(q_{K_{j-1}}) \leq 2^{n+1} - dim(\psi_{j-1})$. This shows that $i_W(\psi_{K_{j-1}}) \geq dim(q_{K_{j-1}})$, which contradicts our choice of j. Hence π_{K_j} is anisotropic.

Finally, we will show that K_j is a purely transcendental extension of F. To see this, we consider the composite of fields $K(\pi) \cdot K_j$. Since $i_W(\psi_{K(\pi)}) \geq 2^n$ and the choice of j, $K(\pi) \cdot K_j$ is then a tower of function fields of isotropic quadratic forms. Thus $K(\pi) \cdot K_j / K(\pi)$ is a purely transcendental extension (Proposition 3.4.4). Since $K(\pi)$ is a purely transcendental extension of F, we conclude that $K_j \cdot K(\pi)$ is also a purely transcendental extension of F.

Notice that q stays anisotropic over the field $K_j \cdot K(\pi)$. By setting $L = K_j \cdot K(\pi)$, this establishes our statement. □

Now we assume that $dim(q) > 2^n$. Over the field $K(\pi)$, we know that $q_{K(\pi)}$ is anisotropic since $K(\pi)/F$ is a purely transcendental extension. Thus $(\psi_{K(\pi)})_{an} = q_{K(\pi)}$. Notice that $K(\pi)$ is a separable field extension over K, then by Corollary 3.5.7, there exists an integer $1 \leq i \leq h$ such that $i_W(\psi_{K_i}) = i_W(\psi_{K(\pi)})$, hence $dim((\psi_{K_i})_{an}) = dim(q)$. Also, since there exists a K-place $K_i \to K(\pi)$ and $q_{K(\pi)}$ is anisotropic, we know q_{K_i} stays anisotropic by Corollary 3.3.9. Now we want to show that $\pi_{K_{i+1}}$ is anisotropic. To see this, we assume that $\pi_{K_{i+1}}$ is isotropic while π_{K_i} is anisotropic. Since q_{K_i} is anisotropic, by Lemma 5.8.5, we have that $dim((q_{K_i})_{an}) = dim(q_{K_i}) \leq 2^{n+1} - dim(\psi_i)$. Since $dim(\psi_i) = dim((\psi_{K_i})_{an}) = dim(q)$, we obtain that $dim(q) \leq 2^n$, which contradicts our assumption. Thus $\pi_{K_{i+1}}$ is anisotropic.

The following theorem is proved by Hoffmann [5] for the case $char(F) \neq 2$ and is proved by Hoffmann and Laghribi [6] for the case $char(F) = 2$.

Theorem 5.8.7. *When $dim(q) = 2^n + 1$, q is a virtual Pfister neighbour.*

Proof. From our discussion above, $dim(\psi_i) = dim(q) = 2^n + 1$. Then Theorem 3.5.9 says that $\mathfrak{i}_1(\psi_i) = 1$. Hence $i_W(\psi_{K_{i+1}}) = 2^n + 1 = dim(q)$. By Theorem 3.4.11, we have that $\pi_{K_{i+1}} \succ q_{K_{i+1}}$. Since $\pi_{K_{i+1}}$ is anisotropic, we conclude that q is a virtual Pfister neighbour. $\square$

More generally, we have the following statement.

Proposition 5.8.8. *If q has maximal splitting, then q is a virtual Pfister neighbour.*

Proof. We let $dim(q) = 2^n + m$ for integer $1 \leq m \leq 2^n$. As discussed above, there exists an integer i such that $dim(\psi_i) = dim(q)$ and q_{K_i}, π_{K_i}, $\pi_{K_{i+1}}$ stay anisotropic. Choosing a minimal integer $j \in [i, h]$ such that $q_{K_{j+1}}$ is isotropic. Since q has maximal splitting, we obtain that $dim((q_{K_{j+1}})_{an}) \leq 2^n - m$.

Since $\psi = q_K \perp -\pi$, we have that $\pi_{K_{j+1}} \sim \psi_{j+1} \perp -q_{K_{j+1}}$. Note that $dim(\psi_{j+1}) < dim(\psi_i) = 2^n + m$. By dimension counting, we know that $\pi_{K_{j+1}}$ is isotropic, hence $\pi_{K_{j+1}} \sim 0$. By the Domination Theorem (Theorem 3.4.9), we have $\pi_{K_j} \succ -\psi_j$. From Corollary 3.2.26, we have that $\pi_{K_j} \perp -(-\psi_j) \sim -\psi_j^c$. Also, we have that $\pi_{K_j} \perp \psi_j \sim q_{K_j}$. Hence $q_{K_j} \sim -\psi_j^c$.

Now we want to show that π_{K_j} is anisotropic. Since $\pi_{K_i}, \pi_{K_{i+1}}$ are anisotropic, we know that $j \geq i + 1$. If π_{K_j} is isotropic, we have that $\psi_j \sim q_{K_j} \perp -\pi_{K_j} \sim q_{K_j}$. Thus $q_{K_j} \simeq \psi_j$. However, since $dim(\psi_j) < 2^n + m$ and $dim(q_{K_j}) = dim(q) = 2^n + m$, this is impossible. We then conclude that π_{K_j} is anisotropic.

Finally, we have that $q_{K_j} \simeq -\psi_j^c$. Thus $q_{K_j} \prec \pi_{K_j}$ while π_{K_j} is anisotropic. This shows that q is a virtual Pfister neighbour. $\square$

The following theorem is proved by Izhboldin [9] for the case $char(F) \neq 2$.

Theorem 5.8.9. *when $dim(q) = 2^n + 2$, then q is a virtual Pfister neighour if and only if $2 \in SSP(q)$.*

Proof. The first direction is trivial from Proposition 3.7.3. Now we assume that $2 \in SSP(q)$ and consider the second direction. If $\mathfrak{i}_1(\psi_i) = 2$, then we are done since $i_W(\psi_{K_{i+1}}) = dim(q)$ and $\pi_{K_{i+1}}$ is anisotropic. So we assume that $\mathfrak{i}_1(\psi_i) = 1$. It suffices to show that $q_{K_{i+1}}$ is anisotropic. Since if $q_{K_{i+1}}$ is anisotropic, by Lemma 5.8.5, we have $\pi_{K_{i+2}}$ is anisotropic and $i_W(\psi_{K_{i+2}}) \geq dim(q)$, which shows that q is a virtual Pfister neighour.

Now we suppose that $q_{K_{i+1}}$ is isotropic. Notice that $K_{i+1} = K_i(\psi_i)$. Since $dim(\psi_i) = dim(q)$ and $\mathfrak{i}_1(\psi_i) = 1$, we have $\psi_i \overset{st}{\sim} q_{K_i}$ (Theorem 3.6.5). Since

$2 \in SSP(q)$, there exists a field extension L/F such that $i_W(q_L) = 2$. Let $\widetilde{K} := L(t_1, \cdots, t_{n+1})$. By Proposition 5.8.6, there exists a field extension $M/\widetilde{K}$ such that $(q_L)_{an} \prec \pi_M$ and M/L is a purely transcendental extension. Also, we know that $dim((\psi_M)_{an}) = dim(\pi) - dim((q_L)_{an}) = 2^n + 2 = dim(\psi_i) = dim(q)$. Let $\widetilde{M} := M \cdot K_i$ be the composite of fields. By our choice of i, $\widetilde{M}$ is a tower of function fields of isotropic forms, hence $\widetilde{M}/M$ is a purely transcendental extension (Proposition 3.4.4).

Since $q_{\widetilde{M}}$ is isotropic, then $(\psi_i)_{\widetilde{M}}$ is also isotropic from the stably birational equivalence of ψ_i and q_{K_i}. As a result, the dimension of the anisotropic part of $(\psi_i)_{\widetilde{M}}$ is strictly less than $dim(\psi_i)$. But it is impossible since the dimension of the anisotropic part of $(\psi_i)_{\widetilde{M}}$ equals that of $(\psi_i)_M$, which is exactly $dim(\psi_i)$. □

Remark 5.8.10. The classification of virtual Pfister neighbours up to dimension 10 is known. For the case of $char(F) \neq 2$, it is done by Izhboldin [9]. For the case of $char(F) = 2$, it's done by K.Quigley (UVic summer undergraduate research project, 2022).

Chapter 6

Excellent Connections for Smooth Quadrics and Related Problems

6.1 Excellent connections for smooth quadrics

Fix a field F.

Let ϕ be a non-degenerate anisotropic excellent quadratic form over F. We suppose that $dim(\phi) = n$. Recall that by Lemma 3.8.8, the integer n can be written uniquely as an alternating sum of 2-powers

$$n = 2^{a_m} - 2^{a_{m-1}} + \cdots + (-1)^{m-1}2^{a_1} + (-1)^m\epsilon$$

with $\epsilon, m, a_1, \cdots, a_m \in \mathbb{N} \cup \{0\}$ satisfying $0 < a_1 < a_2 < \cdots < a_{m-1} < a_m$ and

$$\epsilon = \begin{cases} 0 \text{ and } a_1 < a_2 - 1 & \text{if } n \text{ is even} \\ 1 \text{ and } 1 < a_1 & \text{if } n \text{ is odd.} \end{cases}$$

Also, we have the *jth alternating 2-partial sum* of n, which is given by

$$(n)^{(j)} = 2^{a_j} - 2^{a_{j-1}} + \cdots \pm 2^{a_1} \mp \epsilon.$$

Now we let X be the associated quadric of ϕ and $dim(X) = D$. By Corollary 5.7.5, the rational correspondence type of X is determined, i.e.,

$$RCT(X) = \bigsqcup_{k=1}^{h(\psi)} (\bigsqcup_{i=0}^{\mathfrak{i}_k(\psi)-1} \{\mathfrak{j}_{k-1}(\psi) + i, dim(X) - \mathfrak{j}_k(\psi) + 1 + i\}).$$

where $h_{sep}(\phi) = h(\phi)$ is the Knebusch height of ϕ since ϕ is non-degenerate (Corollary 3.5.8) and the splitting pattern of ϕ is given by

$$\mathfrak{j}_k(\phi) = (n - n^{(m-k)})/2$$

for $k \in [0, h(\phi)]$ (Corollary 3.8.9) This means that in $\Lambda(X)$, for any $k \in [1, h(\phi)]$ and any $s \in [1, \mathfrak{i}_k(\phi)]$, the element $(\mathfrak{j}_{k-1}(\phi) + s)_{lo}$ is connected and only connected with $(\mathfrak{j}_k(\phi) - s - 1)^{up}$. Note that the pairs $(\mathfrak{j}_{k-1}(\phi) + s, \mathfrak{j}_k(\phi) - s - 1)$ are only determined by the dimension n. Hence we call them *n-excellent pairs*.

The following theorem was proved by Vishik [20] when $char(F) \neq 2$. Using the Steenrod operations constructed by Primozic in [16], the same argument still works when $char(F) = 2$.

Theorem 6.1.1. [20, Theorem 3.3] *Let ψ be a non-degenerate anisotropic form over F with dimension n and Y be the associated quadric of ψ. For any rational cycle $\overline{v} \in \overline{Ch}_D(Y \times Y)$, we decompose v as*

$$\overline{v} = \sum_{i=1}^{[n/2]} (\alpha_i \cdot (h^i \times l_i) + \beta_i \cdot (l_i \times h^i)).$$

for elements $\alpha_i, \beta_i \in \mathbb{F}_2$. Then $\alpha_b = \beta_a$ for all n-excellent pairs (a, b).

Combining Theorem 6.1.1 and Corollary 5.3.9, we have the following theorem which is due to Vishik [20] in the case $char(F) \neq 2$. The proof of the theorem for $char(F) = 2$ is identical.

Theorem 6.1.2. [20, Theorem 2.1] *Let q be any non-degenerate anisotropic quadratic form of dimension ≥ 2 over F with associated projective quadric Q. We write $dim(q) - \mathfrak{i}_1(q) = 2^{r_1} - 2^{r_2} + \cdots + (-1)^{s-1}2^{r_s}$ for uniquely determined integers $s_1 > s_2 > \cdots > r_{s-1} > r_s + 1 \geq 1$. For any $k \in [1, s]$, we set*

$$d_k = \sum_{i=1}^{k-1} (-1)^{i-1} 2^{r_i - 1} + \epsilon(k) \sum_{j=k}^{s} (-1)^{j-1} 2^{r_j}$$

where $\epsilon(k) = 1$ *if* k *is even and* $\epsilon(k) = 0$ *if* k *is odd. Then for any* $k \in [1, s]$, $d_k \in \Lambda(U(Q))$.

The following result is a particular case of Theorem 6.1.2, which is called the "Binary Motive Theorem".

Corollary 6.1.3. [20, Theorem 2.1] *Under the same setting as above, we assume that* $dim(q) = 2^n + m$ *for some non-negative integers* n *and* $1 \leq m \leq 2^n$. *If* $rank(U(Q)) = 2$, *then* $s = 1$ *and* $dim(q) - \mathfrak{i}_1(q) = 2^n$, *which means that* q *has the maximal splitting.*

6.2 Singular forms over fields of characteristic 2

Fix a field F and we assume that $char(F) = 2$. In the last section, we introduced the excellent connections and related theorems for non-degenerate anisotropic quadratic forms. However, recall that when $char(F) = 2$, there do exist anisotropic singular quadratic forms. So it is natural to ask the following question.

Question 6.2.1. *Let* ψ *be an anisotropic quadratic form of dimension* n *which is not totally singular. Let* Y *be the associated quadric of* ψ. *We consider any* n-*excellent pair* (a, b). *Then if* $a \in \Lambda(Y)_{lo}$ *and* $b \in \Lambda(Y)^{up}$, *must* $a \in \Lambda(Y)_{lo}$ *connect to* $b \in \Lambda(Y)^{up}$*?*

The arguments in Vishik's proof of Theorem 6.1.1 reply on the existence of Steenrod oprations on $Ch(Y)$ and $Ch(Y \times Y)$. However, when Y is a singular quadric, we only have Steenrod operations on $\overline{Ch}(Y)$ and $\overline{Ch}(Y \times Y)$. So the proof doesn't carry over. Nevertheless, we can show that Question 6.2.1 has a positive answer for anisotropic singular forms up to dimension 9. Before our discussion on Question 6.2.1, we will give a lemma first.

Lemma 6.2.2. *Let* q *be an anisotropic quadratic form over* F *of type* (r, s). *We assume that* $dim(q) = 2^n + m$ *for non-negative integers* n *and* $1 \leq m \leq 2^n$. *If* $r < m$, *then* $\mathfrak{i}_1(q) < s$.

Proof. We assume that $\mathfrak{i}_1(q) \geq s$. Let Q be the associated quadric of q. Consider a subform p of q which is of type $(r, 1)$. As a result, we know that the codimension of p in q, i.e., $dim(p) - dim(q) = s - 1$. We want to show that $p \overset{st}{\sim} q$.

Since $p \prec q$, we know that $q_{F(p)}$ is isotropic. Conversely, we consider $p_{F(q)}$. Let V' be maximal totally isotropic subspace of $q_{F(q)}$. Since $\mathfrak{i}_1(q) \geq s$, we know that

$dim(V^{'}) \geq s$. However, note that $dim(p_{F(q)}) + dim(V^{'}) > dim(q)$, we conclude that $V_{p_{F(q)}} \cap V^{'} \neq \emptyset$. Thus $p_{F(q)}$ is isotropic. This shows that $p \overset{st}{\sim} q$.

Now we let P be the associated quadric of p. Notice that p is a non-degenerate quadratic form, thus $0 \in \Lambda(U(P))$ and $2^n - 1 \in \Lambda(U(P))$ by the excellent connections (Theorem 6.1.1). Since $p \overset{st}{\sim} q$, we know that $0 \in \Lambda(U(Q))$ and $2^n - 1 \in \Lambda(U(Q))$ by Corollary 5.5.3. This forces $dim(Q) - (r-1) = dim(q) - 2 - r + 1 \geq 2^n - 1$. Since $dim(q) = 2^n + m$, we obtain that $r \geq m$, which is a contradiction. Hence we conclude that $\mathfrak{i}_1(q) < s$. □

Corollary 6.2.3. *Let ϕ be an anisotropic form of type $(3,2)$ over F. Then $\mathfrak{i}_1(\phi) = 1$.*

Proposition 6.2.4. *Question 6.2.1 has a positive answer for anisotropic quadratic forms (not totally singular) of dimension up to 9.*

Proof. It suffices to consider singular forms. Let ϕ be an anisotropic form of type (r,s) with $s \geq 2$. We suppose that X is the associated quadric of ϕ and $dim(X) = D$

$dim(\phi) \leq 5$: When $dim(\phi) \leq 5$, we have $r \leq 1$. Hence $\Lambda(X) = \{0, D\}$, which is indecomposable.

$dim(\phi) = 6$: If ϕ is of type $(1,4)$, then the $\Lambda(X) = \{0, D\}$ is indecomposable. So we assume that ϕ is of type $(2,2)$. For 6-dimensional quadratic forms, we have excellent pairs $(0,1), (1,0)$ and $(2,2)$. We may exclude $(2,2)$ since $2 \notin \Lambda(X)_{lo}$.

(i) We assume that $\mathfrak{i}_1(\phi) = \mathfrak{i}_2(\phi) = 1$. By Corollary 5.3.9, we know that 0_{lo} is connected to 0^{up} while 1_{lo} is connected to 1^{up} in $\Lambda(X)$. Note that $2 \in SSP(\phi)$. By Theorem 5.8.9, we know that ϕ is a virtual Pfister neighbour. By Lemma 5.8.2, we have that 0_{lo} is also connected to 1^{up}. Thus we know that $\Lambda(X)$ is indecomposable. This shows the existence of excellent connections for ϕ.

(ii) We assume that $\mathfrak{i}_1(\phi) = 2$. In this case, the only connected components in $\Lambda(X)$ are $\{0_{lo}, 1^{up}\}$ and $\{1_{lo}, 0^{up}\}$, which shows the existence of excellent connections.

$dim(\phi) = 7$: For 7-dimensional quadratic forms, we have excellent pairs $(0,2)$, $(1,1)$ and $(2,0)$. When ϕ is of type $(1,5)$, we know that $\Lambda(X) = \{0, D\}$, which is indecomposable.

Now we assume that ϕ is of type $(2,3)$. Since $2 \notin \Lambda(X)$, we may exclude the excellent pairs $(0,2)$ and $(2,0)$. By Theorem 3.5.9 and the type of ϕ, we know that $\mathfrak{i}_1(\phi) = \mathfrak{i}_2(\phi) = 1$. Thus 1_{lo} is connected to 1^{up} by Corollary 5.3.9. This shows the existence of the excellent connection.

$dim(\phi) = 8$: For 8-dimensional quadratic forms, we have the excellent pairs $(0,3), (1,2), (2,1)$ and $(3,0)$. When ϕ is of type $(1,6)$ or ϕ is of type $(2,4)$, we obtain that $2 \notin \Lambda(X)$ and $3 \notin \Lambda(X)$. So there is nothing to show.

Now the only case we need to consider is that ϕ is of type $(3,2)$. Since $3 \notin \Lambda(X)$, we may exclude the excellent pairs $(0,3)$ and $(3,0)$. By Corollary 6.2.3, $\mathfrak{i}_1(\phi) \neq 2$. Combining the type of ϕ and Theorem 3.5.9, we have the following two cases.

(i) $\mathfrak{i}_1(\phi) = 1$, $\mathfrak{i}_2(\phi) = 2$. Then we have 1_{lo} is connected to 2^{up} and 2_{lo} is connected to 1^{up} by Corollary 5.3.9. Thus the excellent connections exist.

(ii) $\mathfrak{i}_1(\phi) = \mathfrak{i}_2(\phi) = \mathfrak{i}_3(\phi) = 1$. We let ϕ_1 be the first kernel form of ϕ. We have that $dim(\phi_1) = 6$ with $\mathfrak{i}_1(\phi_1) = \mathfrak{i}_2(\phi_1) = 1$. By our discussion before, $\Lambda(X_1)$ is indecomposable.

Let $\alpha \in \overline{Ch}_D(X \times X)$ be a minimal cycle with $a(\alpha) = 2$ or $a(\alpha) = 3$ and let $pr_* : \overline{Ch}(X \times X) \to \overline{Ch}(X_1 \times X_1)$ be the group homomorphism as considered in Proposition 4.5.4. By our discussion above, we have that $pr_*(\alpha) = \Lambda(X_1)$. Hence $\alpha = h^1 \times l_1 + h^2 \times l_2 + l_1 \times h^1 + l_2 \times h^2$. As a result, we know that 1_{lo} is connected to 2^{up} and 2_{lo} is connected to 1^{up}. The excellent connects also exist.

$dim(\phi) = 9$: For 9-dimensional quadratic forms, we have the excellent pairs $(0,0), (1,3), (2,2)$ and $(3,1)$. When ϕ is of type $(1,7)$, $\Lambda(X) = \{0, D\}$ is indecomposable.

(i) We suppose that ϕ is of type $(2,5)$. Then the only excellent pair we need to consider is $(0,0)$. By Theorem 3.5.9 and the type of ϕ, we know that $\mathfrak{i}_1(\phi) = 1 = \mathfrak{i}_2(\phi)$. Hence 0_{lo} is connected to 0^{up} in $\Lambda(X)$ by Corollary 5.3.9. The excellent connection exists.

(ii) We suppose that ϕ is of type $(3,3)$. We need to consider excellent pairs $(0,0)$ and $(2,2)$. By Theorem 3.5.9 and the type of ϕ, we have that $\mathfrak{i}_1(\phi) = \mathfrak{i}_2(\phi) = \mathfrak{i}_3(\phi) = 1$. Then 0_{lo} is connected to 0^{up} and 2_{lo} is connected to 2^{up} by Corollary 5.3.9. Hence the excellent pairs also exist. □

Under the same argument, it's not hard to show that excellent connections hold for all 10-dimensional quadratic forms with possible exception of forms of type $(3,4)$ and separable splitting pattern $(1,3)$. By Theorem 5.8.9, these forms are not virtual Pfister neighbours. They have been classified by K.Quigley (UVic undergraduate research project, 2022): all such forms are isometric to $\lambda(\langle\langle a,b,c]] \perp \langle\langle d,e\rangle\rangle)_{an}$ for some $\lambda, a, b, c, d, e \in F$. One can show if q is such form, there's no non-degenerate form ϕ and non-negative integer m such that $i_W(q_E) \geq 1$ is equivalent to $i_W(\phi_E) \geq m$ for any separable field extension E/F. Thus the methods we used to treat lower dimensional forms are not applicable. For forms of dimension larger than or equal to

11, little is known. Thus we cannot say much on these forms.

Since the proof of Theorem 6.1.2 is a combination of Theorem 6.1.1 and Corollary 5.3.9, the only ingredient left is the excellent connections for generically smooth quadrics. If Question 6.2.1 has a positive answer, we can immediately get a positive answer to the following question, which is a modified version of Theorem 6.1.2 for generically smooth quadrics.

Question 6.2.5. *Let q be an anisotropic quadratic form of dimension ≥ 2 over F which is not totally singular. Let Q be the associated projective quadric of q. We write $dim(q) - \mathfrak{i}_1(q) = 2^{r_1} - 2^{r_2} + \cdots + (-1)^{s-1}2^{r_s}$ for uniquely determined integers $s_1 > s_2 > \cdots > r_{s-1} > r_s + 1 \geq 1$. For any $k \in [1, s]$, we set*

$$d_k = \sum_{i=1}^{k-1}(-1)^{i-1}2^{r_i-1} + \epsilon(k)\sum_{j=k}^{s}(-1)^{j-1}2^{r_j}$$

where $\epsilon(k) = 1$ if k is even and $\epsilon(k) = 0$ if k is odd. Then for any $k \in [1, s]$, if $d(k) \in \Lambda(X)$, then $d_k \in \Lambda(U(Q))$.

Here we shall explain the difference between Theorem 6.1.2 for smooth quadrics and Question 6.2.5 for generically smooth quadrics.

Consider two quadrics X and Y. We assume that $dim(X) = dim(Y)$ and X is smooth while Y is generically smooth. Then from our discussion in Chapter 4, $\Lambda(Y) \subset \Lambda(X)$. Thus when we take the integer d_k of Y as in Theorem 6.1.2, for generically smooth quadrics, we need to make sure that d_k really exists in $\Lambda(Y)$.

The following gives a counterexample which shows that the Binary Motive Theorem (Corollary 6.1.3) is not true for generically smooth quadrics. Since the Binary Motive Theorem is a particular case of Theorem 6.1.2. Thus we can conclude that the Theorem 6.1.2 is not true any more for generically smooth quadrics.

Example 6.2.6. Let $q = \langle\langle a\rangle\rangle \otimes ([b, c] \perp \langle d\rangle) \perp \langle e\rangle$ be an anisotropic form over F which is of type $(2, 3)$. Note that $[b, c] \perp \langle d\rangle$ is a Pfister neighbour of a 2-fold Pfister form. We let $p = \langle\langle a\rangle\rangle \otimes ([b, c] \perp \langle d\rangle)$ be the subform of q, which is of type $(2, 2)$.

Notice that p is a Pfister neighbour of a 3-fold Pfister form. By Proposition 3.7.3, we then obtain that $\mathfrak{i}_1(p) = 2$. Thus for any separable field extension E/F, if $i_W(p_E) > 0$, we have that $i_W(p_E) \geq 2$. Since $p \prec q$. we also have that $i_W(q_E) \geq 2$. Conversely, note that the codimension of p in q is 1, i.e., $dim(q) - dim(p) = 1$. For any separable field extension E/F such that $i_W(q_E) \geq 2$, we let V' be any maximal totally isotropic subspace of V_{q_E}. Then $dim(V') + dim(p) > dim(q)$, which shows that $V' \cap V_{p_E} \neq \emptyset$. Thus p_E is isotropic. Finally, we conclude that for any separable field extension E/F, $i_W(p_E) > 0$ is equivalent to $i_W(q_E) > 1$.

Now we let P and Q be the associated quadrics of p, q respectively. By Theorem 3.5.9 and the type of q, we know that $\mathfrak{i}_1(q) = \mathfrak{i}_2(q) = 1$. Hence 0_{lo} is connected to 0^{up} and 1_{lo} is connected to 1^{up} by Corollary 5.3.9. We will show that $\Lambda(Q)$ is not indecomposable.

Since P is a Pfister neighbour, we have that $U(P) = h^0 \times l_0 + l_1 \times h^1$. By Corollary 5.5.3, we have the binary component $\{1_{lo}, 1^{up}\}$ in $\Lambda(Q)$. Thus $\Lambda(Q)$ is not indecomposable and $\Lambda(U(Q)) = \{0, 5\}$. This means $dim(U(Q)) = 5 - 0 = 5$. Notice that $dim(U(Q)) = dim(q) - \mathfrak{i}_1(q) - 1 = 5$. By the Binary Motive Theorem (Corollary 6.1.3), we should have that $dim(q) - \mathfrak{i}_1(q) = 2^2$. However, we actually have that $dim(q) - \mathfrak{i}_1(q) = 6$. Thus the Binary Motive Theorem is not true for generically smooth quadrics.

However, we still have the following proposition, which comes directly from Corollary 5.5.3 and Theorem 6.1.2.

Proposition 6.2.7. *Let p be an anisotropic quadratic form over F which is not totally singular. Let q be a non-degenerate quadratic form of dimension ≥ 2 over F. Let P, Q be the associated projective quadrics of p, q respectively. We write $dim(q) - \mathfrak{i}_1(q) = 2^{r_1} - 2^{r_2} + \cdots + (-1)^{s-1}2^{r_s}$ for uniquely determined integers $s_1 > s_2 > \cdots > r_{s-1} > r_s + 1 \geq 1$. For any $k \in [1, s]$, we set*

$$d_k = \sum_{i=1}^{k-1} (-1)^{i-1} 2^{r_i - 1} + \epsilon(k) \sum_{j=k}^{s} (-1)^{j-1} 2^{r_j}$$

where $\epsilon(k) = 1$ if k is even and $\epsilon(k) = 0$ if k is odd. If $q \overset{st}{\sim} p$, then for any $k \in [1, s]$, we have that $d_k \in \Lambda(U(P))$.

6.3 Singular Pfister neighbours

Fix a field F. Let q be an anisotropic quadratic form over F which is not totally singular. Recall that in Chapter 2, we proved that if q is a Pfister neighbour, then $(q_{F(q)})_{an}$ is defined over F. The question is that, if we know $(q_{F(q)})_{an}$ is defined over F, can we ensure that q is a Pfister neighbour?

This question has a positive answer if we assume $char(F) \neq 2$ [2, Theorem 28.3]. However, it fails in general when $char(F) = 2$ as discussed in Chapter 2. For example, the basic restriction for such q to be a Pfister neighbour is that $r + s \leq 2^n$ and $2r + s > 2^n$ for some non-negative integer n. Even if such q satisfies the basic restriction on its type, we can still not ensure that q is a Pfister neighbour (Example 3.7.9).

In the rest of this section, we will always assume that $char(F) = 2$ and q is of type (r, s) such that $r + s \leq 2^n$ and $2r + s > 2^n$. Also, we let $dim(q) = 2^n + m$ for non-negative integers n and $1 \leq m \leq 2^n$.

We have already stated the following conjecture which is due to Hoffmann and Laghribi ([6]) in Chapter 2. For the sake of convenience, we restate it here.

Conjecture 6.3.1. *If the anisotropic part of q over $F(q)$ is defined over F and $s < 2r$, then q is a Pfister neighbour.*

We make a stronger claim as below.

Conjecture 6.3.2. *If $(q_{F(q)})_{an}$ is defined over F and $s < 2^{n+1}/3$, then q is a Pfister neighbour.*

Under any field extension E/F, we know that the totally singular part $(q_{ts})_E$ of q_E is always defined over F (Lemma 3.7.6). Thus if q has separable height 1, $(q_{F(q)})_{an} \simeq (q_{ts})_{F(q)}$ must be defined over F. Recall that we say q is a *close Pfister neighbour* if q is a Pfister neighbour and $r+s = 2^n$. This just means $dim(q) = 2^{n+1}-s$. Also, notice that a close Pfister neighbour has separable height 1. Then we have the following conjecture as a special case of Conjecture 6.3.2.

Conjecture 6.3.3. *If q has separable height 1 and $s < 2^{n+1}/3$, then q is a close Pfister neighbour.*

Proposition 6.3.4. *Suppose $(q_{F(q)})_{an}$ is defined over F, i.e., $(q_{F(q)})_{an} \simeq \tau_{F(q)}$ for some form τ over F. We let $q \simeq q_r \perp q_{ts}$ and $\tau \simeq \tau_r \perp \tau_{ts}$ where q_{ts} and τ_{ts} are the totally singular parts of q and τ respectively. Consider the form $\phi := (q_r \perp -\tau_r) \perp q_{ts}$, we have the following results.*

(1) ϕ is an anisotropic form of dimension $2^{n+1} - s$ and has separable height 1.

(2) ϕ is stably birational to q. In particular, $2^n < dim(\phi) \leq 2^{n+1}$.

(3) ϕ has maximal splitting.

Proof. Note that q is not totally singular, thus $F(q)$ is a separable field extension by Proposition 3.4.5. Recall that totally singular forms stay anisotropic over separable field extension (Lamma 3.4.6). Since $(\tau_{ts})_{F(q)} \simeq (q_{ts})_{F(q)}$ and τ_{ts}, q_{ts} are all totally singular form, we know that $\tau_{ts} \simeq q_{ts}$ by Theorem 3.4.11.

Now we assume that τ is of type (r', s) for some integer r'. Obviously, we have that $r - r' = \mathfrak{i}_1(q)$. So ϕ is a form of type $(r + r', s)$. Notice that $\phi_{F(q)} \sim (q_r \perp -\tau_r)_{F(q)} \perp (q_{ts})_{F(q)} \sim (\tau_r \perp -\tau_r)_{F(q)} \perp (q_{ts})_{F(q)} \sim (q_{ts})_{F(q)}$. This shows that $i_W(\phi_{F(q)}) = r + r'$.

Now we want to show ϕ is anisotropic. To see this, we assume that $i_W(\phi) = i$. Since $i_W(\phi_{F(q)}) = r+r'$, we have that $i_W((\phi_{an})_{F(q)}) = r+r'-i$. For any separable field extension E/F, if $i_W((\phi_{an})_E) > r + r' - i - 1$, we have that $(\phi_{an})_E \sim (q_r \perp -\tau_r)_E \perp q_{ts} \sim q_{ts}$. This means $q_E = (q_r \perp q_{ts})_E \sim (q_{ts} \perp \tau_r)_E = \tau_E$. Since $dim(\tau) < dim(q)$, we have that q_E is isotropic. Conversely, if $i_W(q_E)$ is isotropic, then there exists an F-place $F(q) \rightharpoonup E$ by Proposition 3.4.8. Thus we have $i_W((\phi_{an})_E) \geq i_W((\phi_{an})_{F(q)}) = r + r' - i$ by Proposition 3.3.9.

Finally, for any separable field extension E/F, $i_W((\phi_{an})_E) > r + r' - i - 1$ is equivalent to $i_W(q_E) > 0$. Now we let X be the associated quadric of ϕ_{an}. Note that $0 \in \Lambda(U(Q))$ and $(dim(q) - i_1(q) - 1) \in \Lambda(U(Q))$ by Corollary 5.3.9. Thus by Corollary 5.5.3, there exists a minimal cycle $\beta \in \overline{Ch}_{dim(X)}(X \times X)$ such that $r+r'-i-1 \in \Lambda(\beta)$ and $dim(q)-i_1(q)-1+r+r'-i-1 \in \Lambda(\beta)$. Since $i_1(q) = r-r'$, we have that $dim(q) - i_1(q) - 1 + r + r' - i - 1 = dim(q) + 2r' - i - 2$. However, we know that $dim(\phi_{an}) = dim(q) + 2r' - 2i$. Thus the maximal integer in $\Lambda(X)$ is $dim(X) = dim(q) + 2r' - 2i - 2$, which forces $i = 0$. We conclude that ϕ is an anisotropic form.

To show that $\phi \overset{st}{\sim} q$, we need to show that $i_1(\phi) = r + r'$ (Proposition 3.4.8). Consider the cycle β as before. We know that $a(\beta) = r+r'-1 \in \Lambda(\beta)$ and $dim(X) = dim(q) + 2r' - 2 \in \Lambda(\beta)$ ($dim(X) \in \Lambda(X)$ is just $0 \in \Lambda(X)^{up}$). Since the maximal integer in $\Lambda(X)_{lo}$ is $r+r'-1$ and the maximal integer in $\Lambda(X)$ is $dim(X)$, we conclude that β is a binary correspondence. However, since $0 \in \Lambda(X)^{up}$ is always connected to $i_1(\phi) - 1 \in \Lambda(X)_{lo}$ by Corollary 5.3.9, it forces $i_1(\phi) = r + r'$. Since ϕ is anisotropic of type $(r + r', s)$, we know that ϕ has separable height 1.

Also, note that $dim(\phi) = 2r + 2r' + s = 2^n + m + 2r'$. Since $\phi_{F(q)}$ is isotropic, by the Separation Theorem (Theorem 3.4.10), it forces $m + 2r' \leq 2^n$. Also, since $q_{F(\phi)}$ is isotropic ($\phi \overset{st}{\sim} q$), by the Separation Theorem (Theorem 3.4.10), it forces $m+2r' \geq 1$.

Thus we have $1 \leq m + 2r' \leq 2^n$ and $2^n < dim(\phi) \leq 2^{n+1}$.

Since $r + s \leq 2^n$ and q is not totally singular, we have that $s < 2^n$. Notice that $dim(q) = 2^n + m = 2r + s$, we then obtain that $2r > m$ and $r > m/2$. In the proof of part(1), we obtained that $\mathfrak{i}_1(\phi) = r + r'$. Hence we have $\mathfrak{i}_1(\phi) > (m + 2r')/2$. Consider all possible values of $\mathfrak{i}_1(\phi)$ allowed by Theorem 3.5.9. We conclude that $\mathfrak{i}_1(\phi) = r + r' = m + 2r'$ (Example 3.5.12). Thus we know that ϕ has maximal splitting and $dim(\phi) = 2^{n+1} - s$. □

Combining Proposition 6.3.4 and Proposition 5.8.8, we immediately have the following statement.

Theorem 6.3.5. *Under the same situation as in Proposition 6.3.4, we have the following results.*

(1) q has maximal splitting.

(2) $U(Q)$ is binary

(3) q is a virtual Pfister neighbour.

In addition, we see the relation between Conjecture 6.3.2 and Conjecture 6.3.3. If Conjecture 6.3.3 is true, then Conjecture 6.3.2 is also true. We summarize the observation as the corollary below.

Corollary 6.3.6. *In order to prove Conjecture 6.3.2, it suffices to prove Conjecture 6.3.3. Moreover, if q is as in Conjecture 6.3.3, then $dim(q) = 2^{n+1} - s$.*

Proof. Let p be an anisotropic quadratic form over F which is not totally singular. Let $dim(p) = 2^n + m$ for non-negative integers n and $1 \leq m \leq 2^n$. We suppose that p is of type (r, s) with $s < 2^{n+1}/3$. By Proposition 6.3.4, we have a form ϕ of dimension $2^{n+1} - s$ which is stably birational equivalent to p. Also, ϕ has separable height 1.

We suppose that Conjecture 6.3.3 is true. Then we know that ϕ is a Pfister neighbour. As a result, we have the associated Pfister form π of dimension 2^{n+1} such that $\phi \overset{st}{\sim} \pi$. Since $p \overset{st}{\sim} \phi$, we conclude that p is also a Pfister neighbour. □

At the same time, Proposition 6.3.4 also allows us, under a slightly stronger restriction on s, to relate Hoffmann and Laghribi's conjecture to two major open problems in the study of non-degenerate quadratic forms. We recall here their statements:

Conjecture 6.3.7. [21, A.Vishik, Conjecture 4.21] *For any smooth quadric P, P is given by a Pfister neighbour if and only if $U(P)$ is a binary correspondence.*

Conjecture 6.3.8. [5, Hoffmann, Page 475] *Let q be an anisotropic non-degenerate quadratic form over F. We suppose $2^n < dim(q) \leq 2^{n+1}$. If $dim(q) > 2^n + 2^{n-2}$ and q has maximal splitting, then q is a Pfister neighbour.*

Proposition 6.3.9. *Let q and ϕ be under the same setting as in Proposition 6.3.4. Moreover, we assume that $s \leq 2^{n-1}$.*

(1) If Conjecture 6.3.7 is true, then Conjecture 6.3.2 holds for q.

(2) If $s \leq 2^{n-2} + 2^{n-3}$ and Conjecture 6.3.8 is true, then Conjecture 6.3.2 holds for q.

Proof. Let $\phi = q_r \perp -\tau_r \perp q_{ts}$ be the same as in Proposition 6.3.4. We consider any subform ϕ' of ϕ which is of type $(r + r', 1)$. By Proposition 6.3.4, ϕ has separable height 1. Thus we have $2^n - m - 2r' = s$. This gives that $2r - m = 2^n - s$. Recall that in the proof of Proposition 6.3.4, we showed that $r + r' = m + 2r'$. Note that $r' = r - m$. Thus we obtain that $\mathfrak{i}_1(\phi) = r + r' = 2r - m = 2^n - s$. Now we claim that $\phi \overset{st}{\sim} \phi'$. Since $s \leq 2^{n-1}$, we have that $\mathfrak{i}_1(\phi) = 2^n - s > s - 1$. Thus any maximal totally isotropic subspace V' of $\phi_{F(\phi)}$ has dimension larger than or equal to s. Notice the codimension of ϕ' in ϕ, i.e., $dim(\phi) - dim(\phi') = s - 1$. Thus $dim(V_{\phi'}) + dim(V') > dim(V_\phi)$. Hence $V_{\phi'_{F(\phi)}} \cap V' \neq$ and $\phi'_{F(\phi)}$ is isotropic. Conversely, since $\phi' \prec \phi$, we know that $\phi_{F(\phi')}$ is also isotropic. Hence $\phi \overset{st}{\sim} \phi'$.

(1): We assume that Conjecture 6.3.7 is true. Recall that in the proof of Proposition 6.3.4, we showed that $U(X)$ (X is the associated quadric of ϕ) is binary. Thus $U(Y)$ is also binary by Corollary 5.5.3. Then ϕ' is a Pfister neighbour, which means that there exists an anisotropic Pfister form π such that $\phi' \overset{st}{\sim} \pi$. We have shown that $\phi' \overset{st}{\sim} \phi$. Thus we conclude that $\phi \overset{st}{\sim} \pi$ and ϕ is a Pfister neighbour.

(2): Notice that the condition $s < 2^{n+1}/3$ of Conjecture 6.3.2 is already satisfied under the condition $s \leq 2^{n-1}$. By Proposition 6.3.4, we have that $dim(\phi) = 2^{n+1} - s$. Hence $dim(\phi') = dim(\phi) - (s - 1) = 2^n + (2^n - 2s + 1)$. Since $s \leq 2^{n-2} + 2^{n-3}$, we have that $2^n - 2s + 1 \geq 2^{n-2} + 1$. Hence $dim(\phi') > 2^n + 2^{n-2}$. Since $\phi \overset{st}{\sim} \phi'$ and ϕ has maximal splitting, by Theorem 3.6.4, we know that ϕ' also has maximal splitting. As a result, if Conjecture 6.3.8 is true, we know that ϕ' is a Pfister neighbour. Let π be the associated Pfister form of ϕ'. Then we know that $\phi \overset{st}{\sim} \pi \overset{st}{\sim} \phi'$, which shows that ϕ is also a Pfister neighbour. □

6.4 Isotropy of quadratic forms over function fields of quadrics

Fix a field F. In this final section, we apply our methods to the study of isotropy of quadratic forms over function fields of quadrics.

We fix an anisotropic quadratic form p over F which is not totally singular and let s be the unique non-negative integer for which $2^s < dim(p) \leq 2^{s+1}$. We suppose that q is an anisotropic non-degenerate quadratic form over F. We set $k := dim((q_{F(p)})_{an}) = dim(q) - 2i_W(q_{F(p)})$. The following is conjectured by S.Scully [18].

Conjecture 6.4.1. [18, Conjecture 1.1] *Under the same setting as above, we have $dim(q) = a2^{s+1} + \epsilon$ for some non-negative integer a and some $-k \leq \epsilon \leq k$.*

Recall the Separation Theorem (Theorem 3.4.10) saying that if $dim(q) \leq 2^s$, then $q_{F(p)}$ stays anisotropic. When $dim(q) \leq 2^s$, the conjecture claims that $k = dim(q)$, which means q is anisotropic over $F(p)$. As a result, Conjecture 6.4.1 can be viewed as an extension of the Separation Theorem.

When p, q are non-degenerate quadratic forms, there is much evidence for Conjecture 6.4.1. However, if we only suppose that q is non-degenerate, then the proofs of many statements in [18] are not applicable any more. But we can still prove the following proposition. The proof below is basically the same as that of [18, Proposition 6.1].

Proposition 6.4.2. *Let p be the quadratic form as we fixed above and q an anisotropic non-degenerate form. If either of the following conditions holds:*

(1) $i_W(q_K) > i_W(q_{F(p)}) - 1$ is equivalent to $i_W(p_K) > 0$ for any separable field extension K/F;

(2) $dim(q) \leq 2^{s+2} + 2^{s+1} + k$.

Then Conjecture 6.4.1 is true.

Proof. We may assume that $k \leq 2^s - 2$ since otherwise Conjecture 6.4.1 holds trivially. Notice that $2^{s+2} + 2^{s+1} = (2+1)2^{s+1}$. Thus the conjecture also holds trivially when $2^{s+2} + 2^{s+1} - k \leq dim(q) \leq 2^{s+2} + 2^{s+1} + k$. We may replace condition (2) by

$$dim(q) < 2^{s+2} + 2^{s+1} - k \tag{6.1}$$

Now we let $F = F_0 \subset F_1 \subset \cdots \subset F_{h(q)}$ be the Knebusch splitting tower of q, and let $0 \le r \le h(q)$ be the unique integer such that $q_r = (q_{F_r})_{an}$ has dimension k. We now induct on r. If $r = 0$, then there is nothing to prove. So we assume that $r \ge 1$. Since q_{r-1} becomes isotropic over $F_{r-1}(p)$, and since $dim(p) > 2^s$, the Separation Theorem (Theorem 3.4.10) implies that $dim(q_{r-1}) > 2^s$. Since $dim(q_r) = k < 2^s$, Theorem 3.5.9 on the first Witt index implies that $dim(q_{r-1}) = 2^{n+1} - k$ for some integer $n \ge s$. Note that Conjecture 6.4.1 holds if $r = 1$. So we assume that $r \ge 2$. Applying the induction hypothesis to q_1, we get that $dim(q_1) = a_1 2^{s+1} + \epsilon_1$ for some positive integer a_1 and some $-k \le \epsilon_1 \le k$. If $\mathfrak{i}_1(q) \le (k - \epsilon)/2$, then $a_1 2^{s+1} - k \le dim(q) \le a_1 2^{s+1} + k$ and so the desire assertion holds with $a = a_1$. We can therefore assume that $\mathfrak{i}_1(q) > (k - \epsilon_1)/2$. Let u be the smallest non-negative integer satisfying the inequality $\mathfrak{i}_1(q) \le 2^u$. By another application of Theorem 3.5.9, we have that

$$a_1 2^{s+1} + \epsilon_1 + \mathfrak{i}_1(q) = dim(q) - \mathfrak{i}_1(q) \equiv 0 \pmod{2^u}.$$

We claim that $u \ge s$. If the claim is correct, we have that $\mathfrak{i}_1(q) \equiv -\epsilon_1 \pmod{2^s}$, and hence

$$dim(q) = a_1 2^{s+1} + \epsilon_1 + 2\mathfrak{i}_1(q) \equiv -\epsilon_1 \pmod{2^{s+1}}.$$

Then the desired assertion holds with $\epsilon = -\epsilon_1$.

To prove the claim, we suppose that $u < s$, so that $\mathfrak{i}_1(q) = \mu 2^u - \epsilon_1$ for some integer μ. Since $\mathfrak{i}_1(q) > (k - \epsilon_1)/2$, we have

$$\mu 2^u = \mathfrak{i}_1(q) + \epsilon_1 > (\frac{k - \epsilon_1}{2}) + \epsilon_1 = \frac{k + \epsilon_1}{2} \ge 0.$$

At the same time, we also have

$$\mu 2^u = \mathfrak{i}_1(q) + \epsilon_1 \le 2^u + k < 2^u + 2^s,$$

and so $0 < \mu 2^u \le 2^s$. If $\mu 2^u = 2^s$, then again we have that $\mathfrak{i}_1(q) \equiv -\epsilon_1 \pmod{2^s}$, and the result follows as above. So we only need to show that we cannot have $0 < \mu 2^u < 2^s$.

Suppose otherwise. We let P, Q be the associated quadrics of p, q respectively. Since $dim(q) - \mathfrak{i}_1(q) = a_1 2^{s+1} + \epsilon_1 + \mathfrak{i}_1(q) = a_1 2^{s+1} + \mu 2^u$. By Theorem 6.1.2(as a corollary of the excellent connections), we obtain that $a_1 2^s \in \Lambda(U(Q))$, where $U(Q)$

is the upper summand in $\overline{Ch}_{dim(Q)}(Q \times Q)$. Let $j = i_W(q_{F(p)}) - 1 - a_1 2^s$. Then

$$j = \frac{dim(q) - k}{2} - 1 - a_1 2^s = \mathfrak{i}_1(q) - 1 - (\frac{k - \epsilon_1}{2})$$

and so $0 \leq j < \mathfrak{i}_1(q)$ (here we use our assumption that $\mathfrak{i}_1(q) > (k - \epsilon_1)/2$). By Corollary 5.3.4, there exists a minimal cycle α starting with $h^j \times l_j$ such that $\Lambda(\alpha) = \Lambda(U(Q))[j]$. Thus $j + a_1 2^s = i_W(q_{F(p)}) - 1 \in \Lambda(\alpha)$. This means j is always connected to $i_W(q_{F(p)}) - 1$ in $\Lambda(Q)$. If we can find a minimal cycle $\beta \in \overline{Ch}_{dim(Q)}(Q \times Q)$ such that $i_W(q_{F(p)} - 1) \in \Lambda(\beta)$ while $j \notin \Lambda(\beta)$, then this will give us a contradiction.

Suppose we are under condition (1). By Corollary 5.5.3, there is a minimal cycle $\beta \in \overline{Ch}_{dim(Q)}(Q \times Q)$ such that $\Lambda(U(P))[i_W(q_{F(p)}) - 1] = \Lambda(\beta)$. Notice that $i_W(q_{F(p)}) - 1 \in \Lambda(U(P))[i_W(q_{F(p)}) - 1]$, hence $i_W(q_{F(p)}) - 1 \in \Lambda(\beta)$.

But since $j < i_W(q_{F(p)}) - 1 - a_1 2^s < i_W(q_{F(p)}) - 1$ (recall that $a_1 > 1$), we know that $j \notin \Lambda(\beta)$. So the cycle β will give us a contradiction.

Suppose we are under the condition (6.1), i.e., $dim(q) < 2^{s+2} + 2^{s+1} - k$. Recall that $dim(q_{r-1}) = 2^{n+1} - k$ for some integer $n \geq s$. At first, we claim that $p_{F_{r-1}}$ is anisotropic: if $p_{F_{r-1}}$ is isotropic, by Proposition 3.4.4, we know that $F(p_{F_{r-1}}) = F_{r-1}(p)$ is a purely transcendental extension over F. Thus q_{r-1} stays anisotropic under $F_{r-1}(p)/F_{r-1}$ by Lemma 3.4.3, which is incorrect. Thus we conclude that $p_{F_{r-1}}$ is anisotropic.

If $n = s$, we know that $p_{F_{r-1}}$ is stably birational equivalent to q_{r-1} by Theorem 3.6.5. This means, for any separable field extension K/F which contains F_{r-1}, we have that $i_W((p_{F_{r-1}})_K) > 0$ is equivalent to $i_W((q_{r-1})_K) > 0$. Since $dim(q_r) = k = dim((q_{F(p)})_{an})$, we can reduce our case to condition (1) by Proposition 3.4.8 and Corollary 3.3.9, which has already been shown.

Now we assume that $n > s$. We then have that

$$\mathfrak{i}_r(q) = \mathfrak{i}_1(q_{r-1}) = 2^n - k \geq 2^{s+1} - k > k > dim(q_t) > \mathfrak{i}_t(q)$$

for all $r < t \leq h(q)$. This means that any minimal cycle of $\overline{Ch}_{dim(Q)}(Q \times Q)$ starting from the rth shell triangle is binary. Also, our assumption on $dim(q)$ gives that

$$\mathfrak{j}_{r-1}(q) = \frac{dim(q) - (2^{n+1} - k)}{2} < \frac{(2^{s+2} + 2^{s+1} - k) - (2^{s+2} - k)}{2} = 2^s \leq min\{\mathfrak{i}_r(q), 2^{n-1}\}.$$

Then by [18, Proposition 5.1], we have a minimal cycle $\alpha \in \overline{Ch}_{dim(Q)}(Q \times Q)$ starting from the rth shell triangle. Hence α is binary. By Proposition 5.2.12, we may assume that $\Lambda(\alpha)$ consists of $\mathfrak{j}_{r-1}(q)$ and $\mathfrak{j}_{r-1}(q) - 2^n - 1$. Since $i_W(q_{F(p)}) = \mathfrak{j}_r(q)$, we have a minimal cycle $\beta \in \overline{Ch}_{dim(Q)}(Q \times Q)$ such that $\Lambda(\beta) = \Lambda(\alpha)[\mathfrak{i}_r(q) - 1]$. Since $j < i_W(q_{F(p)}) - 1$, the cycle β will give the contradiction. This finishes the induction and gives us the proof. □

Now we define the *Izhboldin dimension* of p to be $dim_{Izh}(p) := dim(p) - \mathfrak{i}_1(p)$. For each integer $r \geq 1$, we let y_r be such that $y_r 2^r < dim_{Izh}(p) \leq (y_r + 1)2^r$. For each $x \geq 0$, we set

$$I_r(x) := [(y_r + x)2^{r+1} - k, x2^{r+1} + k].$$

The following strengthens Conjecture 6.4.1 for non-Pfister neighbours. When p is a Pfister neighbour, Conjecture 6.4.3 is optimal.

Conjecture 6.4.3. *We assume that p is not a Pfister neighbour. If $dim(q) > k$, then $dim(q) = a2^{s+2} + \epsilon$ for some $a \geq 0$ and $\epsilon \in I_r(x)$ for some r, x.*

Remark 6.4.4. (1) Note that when $k < 2^r y_r, I_r(x) = \emptyset$.

(2) When $r = s + 1$ and $x = 0$, we just obtain that $dim(q) = a2^{s+2} + \epsilon$ with $\epsilon \in [-k, k]$.

Under an assumption similar to condition 1 of Proposition 6.4.2, we checked several examples of $dim_{Izh}(p) \leq 32$ for Conjecture 6.4.3, which give evidence for Conjecture 6.4.3 to be true. As a result, we have the following question.

Question 6.4.5. *Let p be the fixed form as above and q be an anisotropic non-degenerate quadratic form. If $i_W(q_K) > i_W(q_{F(p)}) - 1$ is equivalent to $i_W(p_K) > 0$ for any separable field extension K/F, then Conjecture 6.4.3 is true.*

Chapter 7

Conclusion and Future Work

In this thesis, we considered the algebraic cycles (mod-2) on products of generically smooth quadrics. Fix a field F and let $\overline{F}$ be an algebraic closure of F. For generically smooth quadrics $X_1, \cdots, X_n$ which are given by quadratic forms with anisotropic totally singular parts, we studied the image of the change of field homomorphism $Ch(X_1 \times \cdots \times X_n) \to Ch((X_1 \times \cdots \times X_n)_{\overline{F}})$, which is denoted by $\overline{Ch}(X_1 \times \cdots \times X_n)$. We showed that $\overline{Ch}(X_1 \times \cdots \times X_n)$ inherits a ring structure from the Chow ring (modulo-2) of the smooth locus of $X_1 \times \cdots \times X_n$. Using the ring structure, we then introduced and studied a composition of rational correspondences (modulo 2) for products of generically smooth projective quadrics.

With these results, we introduced the rational correspondence types for generically smooth quadrics, which extends Vishik's motivic decomposition types for smooth quadrics. We computed the rational correspondence types of certain family of quadrics, e.g., quadrics given by quasi-strongly excellent forms, generic forms of even dimensions and we described the rational correspondence type of a Pfister neighbour in terms of its complementary form. The future work will be based on some outstanding questions for generically smooth but singular quadrics.

(1) In Chapter 6, we discussed a fundamental result of Vishik on the motivic decomposition type of smooth quadrics, concerning the existence of "excellent connections". Using the methods described in the thesis, we can only show that the excellent connections exist for singular forms up to dimension 9. Even for 10-dimensional singular forms, we don't know whether the excellent connections still exist. This is a very interesting and important open problem that we will investigate in the future.

(2) The methods used in this thesis give means to relate questions about singular forms to question about non-degenerate forms, e.g., computation of splitting patterns. In Chapter 6, we discussed the possible value of the first index of 8-dimensional singular forms. The future work will include the computation of splitting patterns of lower dimensional singular quadratic forms. Also, it will include the refinements of Karpenko's theorem (Theorem 3.5.9) on the first Witt index of singular forms.

(3) In Chapter 6, we used the results in this thesis to get partial information on a conjecture of Hoffmann and Laghribi which concerns the classification of Pfister neighbours when the characteristic of the base field is 2. First, we reduced the conjecture to the study of forms which have separable height 1. Second, we related the conjecture to a major conjecture by Vishik in this area which concerns the upper summand of a smooth quadric in the motivic decomposition type. This gives two lines of approach to Hoffmann and Laghribi's conjecture that we plan to investigate further.

(4) The existing literature also includes a number of important results concerning algebraic cycles (mod-2) on the quadratic Grassmannians defined by non-degenerate forms (e.g., Vishik's theory of elementary discrete and J-invariants). We plan to investigate the extension of some of the results to singular forms.

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