AN INDEX THEOREM FOR TOEPLITZ OPERATORS WITH NONCOMMUTATIVE SYMBOL SPACE

by

JOHN PHILLIPS AND IAIN RAEBURN

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Abstract

We consider Toeplitz operators with symbol in a C^*-algebra A carrying an action α of $\mathbf{R}$. We prove that when A has an $\mathbf{R}$-invariant trace τ, the Toeplitz operators with invertible symbols are Fredholm elements of an appropriate von Neumann algebra, and give a formula for their Breuer index in terms of τ and the infinitesimal generator of the one-parameter automorphism group α. This theorem includes as special cases recent results of Curto-Muhly-Xia, Ji and Lesch, as well as classical results of Gohberg-Krein and Coburn-Douglas-Schaeffer-Singer, and provides a more direct approach to the corresponding index theorems for systems.

AN INDEX THEOREM FOR TOEPLITZ OPERATORS WITH NONCOMMUTATIVE SYMBOL SPACE

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The Toeplitz or Wiener-Hopf operator T_f with symbol $f \in L^\infty(\mathbf{R})$ is the compression of the multiplication operator M_f on $L^2(\mathbf{R})$ to the Hardy space $H^2(\mathbf{R})$. Gohberg and Krein proved that if f is continuous, $\lim_{|x|\to\infty} f(x)$ exists, and the resulting function $f : \mathbf{R} \cup \{\infty\} \to \mathbf{C}$ is nonvanishing, then T_f is a Fredholm operator with index equal to minus the winding number of the curve $f(\mathbf{R} \cup \{\infty\})$ about 0 [12]. This beautiful index theorem has been generalised in many directions: to wider classes of symbols, to operators defined on the boundaries of other regions of $\mathbf{C}$ and $\mathbf{C}^n$, and to operators which are only Fredholm in some weaker sense. In particular, there are far-reaching generalisations of the index theorem in which the operators are Fredholm relative to a II_∞-factor, and the index is the real-valued one of Breuer [2, 3]. Here we shall prove an index theorem of this kind, starting merely with an action of $\mathbf{R}$ on a C^*-algebra A and an invariant trace on A.

Let α be a strongly continuous action of $\mathbf{R}$ on a C^*-algebra A. If π is a representation of A on H, we define another representation $\tilde{\pi}$ of A on $L^2(\mathbf{R}, H)$ by

$$(\tilde{\pi}(a)\xi)(s) = \pi(\alpha_s^{-1}(a))(\xi(s)); \tag{0.1}$$

for motivation, note that if λ denotes the action of $\mathbf{R}$ by left translation on $L^2(\mathbf{R}, H)$, then $(\tilde{\pi}, \lambda)$ is a covariant representation of $(A, \mathbf{R}, \alpha)$ whose integrated form $\tilde{\pi} \times \lambda$ is the regular representation $\mathrm{Ind}\,\pi$ of the crossed product $A \times_\alpha \mathbf{R}$ induced from π [16, §7.7]. The **Toeplitz operator** T_a^π **with symbol** a is the compression of $\tilde{\pi}(a)$ to the Hardy space $H^2(\mathbf{R}, H) := H^2(\mathbf{R}) \otimes H$. Both $\tilde{\pi}(a)$ and the projection P onto $H^2(\mathbf{R}, H)$ lie in the von Neumann algebra $M = \mathrm{Ind}\,\pi(A \times_\alpha \mathbf{R})''$, so the Toeplitz operators lie in PMP. If π is the GNS-representation associated to an invariant trace τ, then there is an induced normal trace $\hat{\tau}$ on M, and hence also on PMP. Our index theorem asserts that, if a is invertible in A, then T_a^π is a Fredholm element of the semifinite von Neumann algebra PMP, and

$$\mathrm{ind}\, T_a^\pi = \frac{-1}{2\pi i}\, \tau\left(\frac{d}{dt}(\alpha_t(a)a^{-1})\bigg|_{t=0}\right). \tag{0.2}$$

When τ is a finite trace on A, this index theorem has been proved by Lesch [14], following earlier work in [5], [18], [11], [8] and [13]. However, his result does not strictly include that of Gohberg and Krein, because the natural invariant trace on $C_0(\mathbf{R})$ and $C(\mathbf{R} \cup \{\infty\})$ is given by Lebesgue measure, and is not finite. Our version works for densely defined, lower semicontinuous traces, and covers both the general theorem of Lesch and his predecessors, and the original Gohberg-Krein Theorem. Allowing infinite traces does pose some technical problems: both the trace and the infinitesimal generator δ of the one-parameter group α_t are only densely defined, and we have to use some old-fashioned Banach algebra methods to manipulate them simultaneously. In particular, we have to work harder to make sense of both sides of the index formula (0.2).

First we look at the right hand side of (0.2). We denote by δ the generator of α, so that this right-hand side is $(2\pi i)^{-1}\tau(\delta(a)a^{-1})$. In the classical case, δ is ordinary differentiation, and this reduces to the winding number of a about 0. In general, we can prove that it is a homotopy-invariant homomorphism of $\mathcal{D}(\delta)\cap GL(A)$ into $\mathbf{R}$, and hence extend it to all of $GL(A)$. (Here we have suppressed some problems involving the adjoining of identities, which we shall be careful about later.) We develop the necessary properties of the trace in an Appendix, and discuss our "winding number for C^*-algebras" in §1.

To get a real-valued Breuer-Fredholm index on the left of (0.2), we want to extend τ to a trace $\hat{\tau}$ on $M = \operatorname{Ind}\pi(A\times_\alpha \mathbf{R})''$. The idea is that $\hat{\tau}(\operatorname{Ind}\pi(x))$ should be $\tau(x(0))$ for nice functions x in $C_c(\mathbf{R},A)$; to fill in the details we need to use the theory of Hilbert algebras, as developed in [10]. (Lesch had already used similar arguments in the technically easier case of finite τ.) Then we have to modify Breuer's theory to use our trace $\hat{\tau}$ in place of his generalised dimension function, which is only real-valued on factors; we outline the necessary changes in Appendix B, and acknowledge that the same observations were implicitly used in [8] and [14].

We prove the index theorem in §3. Having overcome the technical difficulties caused by the infinite trace in §1 and Appendix A, we can follow in outline the second proof of [8, Theorem 25.2]. We finish with a section on examples, in which we show how to recover the Gohberg-Krein Theorem, and how to deal with systems. Both of these are quite straightforward, although some irritating multiplicities creep in because the GNS-representation π_τ is only a multiple of the naturally-occurring representation. Indeed, the ease with which we can handle systems is a feature of the noncommutative theory: a system $(T_{a_{ij}})$ of Toeplitz operators with symbol in A is really a single operator $T_{(a_{ij})}$ with symbol in $M_n(A)$, so that, even if A is commutative, we have to confront noncommutative symbol algebras. Previously, the extensions of index theorems to systems were nontrivial (cf. [5], where the case $A = AP(\mathbf{R})$ was left open, and [18], where it was settled).

We originally obtained our index theorem for the case of a finite trace, when the second author visited Victoria in 1990, and only learned of Lesch's work when the first author presented our results at the annual meeting of the Canadian Mathematical Society in May 1991. Obviously Lesch was there first, so we wanted to sort out the general case before writing up. However, even for finite traces, our approach has some different features:

our discussion of winding number in §1 provides a self-contained proof of its homotopy invariance (Lesch used it, but just referred to [6]), and the simplicity of our treatment of systems in §4 is a good advertisement for the noncommutative theory.

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1. Winding number in C^*-algebras.

In this section, A is a C^*-algebra with a densely defined, lower semicontinuous trace τ, and a continuous action $\alpha : \mathbf{R} \to \operatorname{Aut} A$ which leaves τ invariant: that is, $\tau(\alpha_t(a)) = \tau(a)$ for all $t \in \mathbf{R}$ and $a \in A_+$. While many of the details which follow are implicit in [6], our presentation is more elementary, and more completely describes the domain of the homomorphism $a \to (2\pi i)^{-1}\tau(\delta(a)a^{-1})$. Recall from Proposition A4 that $A^\tau = \{x \in A \mid \tau(|x|) < \infty\}$ is a Banach *-algebra in the norm $\|a\|_\tau = \|a\| + \|a\|_1$.

Lemma 1.1. *The automorphism group α restricts to a continuous action of $\mathbf{R}$ as isometric *-automorphisms of A^τ.*

Proof. It is easy to see that each α_t is $\|\cdot\|_\tau$-isometric, and hence we have to establish continuity. If $a \in A$ and $b \in A^\tau$, then $ab \in A^\tau$ and

$$\|ab\|_1 = \tau(|ab|) \le \tau(\|a\|_{\mathrm{op}}|b|) = \|a\|_{\mathrm{op}}\|b\|_1,$$

so that $\|ab\|_\tau \le \|a\|_{\mathrm{op}}\|b\|_\tau$. Similarly, $\|ba\|_\tau \le \|b\|_\tau\|a\|_{\mathrm{op}}$. Lemma A3 implies that products are dense in A^τ, and because α_t is isometric, it follows that $t \to \alpha_t(x)$ is $\|\cdot\|_\tau$-continuous for all $x \in A^\tau$.

Now, let δ (respectively, δ_τ) be the infinitesimal generator of α (respectively, $\alpha|_{A^\tau}$). As usual, $\mathcal{D}(\delta)$ and $\mathcal{D}(\delta_\tau)$ are dense *-subalgebras of A and A^τ. Further, if $x \in \mathcal{D}(\delta_\tau)$, then $x \in \mathcal{D}(\delta)$ and $\delta_\tau(x) = \delta(x)$.

Lemma 1.2. *$\cap_{n=1}^\infty \mathcal{D}((\delta_\tau)^n)$ is a dense *-subalgebra of the Banach *-algebra A^τ (and hence of A).*

Proof. For $x \in A^\tau$ and $f \in C_c^\infty(\mathbf{R})$, the A^τ-valued Bochner integral $x_f = \int f(t)\alpha_t(x)\,dt$ converges in the $\|\cdot\|_\tau$ norm, and $\delta_\tau(x_f) = x_{-f'}$ by the usual calculation. Thus x_f belongs to $\cap_{n=1}^\infty \mathcal{D}((\delta_\tau)^n)$. If we choose $f \ge 0$ to have integral 1 and support in a small neighbourhood of 0, then $\|x - x_f\|_\tau$ is small.

Lemma 1.3. *The trace τ restricts to a bounded functional on the Banach *-algebra A^τ, and for $x \in \mathcal{D}(\delta_\tau)$ we have $\tau(\delta(x)) = 0$.*

Proof. The first part is trivial:

$$|\tau(x)| \leq \tau(|x|) = \|x\|_1 \leq \|x\|_\tau,$$

and the second follows from the first:

$$\tau(\delta(x)) = \tau(\delta_\tau(x)) = \tau\left(\lim_{t\to 0} \frac{\alpha_t(x) - x}{t}\right) = \lim_{t\to 0}\left(\frac{\tau(\alpha_t(x)) - \tau(x)}{t}\right) = 0.$$

Notation 1.4. If A is a complex algebra without identity, then we let $\widetilde{A}$ denote the algebra obtained by adjoining an identity to A. If A already has an identity, then we take $\widetilde{A} = A$. We denote by $GL(A)$ the group of invertible elements in $\widetilde{A}$.

Definition 1.5. Let A be an algebra. If $x \in A$, we say x is **quasi-invertible** if and only if there exists $x' \in A$ such that $x + x' - xx' = 0 = x + x' - x'x$, in which case we call x' the **quasi-inverse** of x. Then

$$\mathrm{sp}_A(x) \setminus \{0\} = \{\lambda \in \mathbf{C} \setminus \{0\} \mid x/\lambda \text{ is not quasi-invertible}\},$$

and $0 \in \mathrm{sp}_A(x)$ if and only if x is not invertible. That this coincides with the usual definition of spectrum is proved in [17]. We shall denote by $GL'(A)$ the group of quasi-invertible elements of A.

Lemma 1.6. *If $x \in A^\tau$, then $\mathrm{sp}_A(x) = \mathrm{sp}_{A^\tau}(x)$, and $\nu_A(x) = \nu_{A^\tau}(x)$.*

Proof. If $1 \in A$, then since A^τ is a dense ideal in A, we have $A^\tau = A$ and we are done. If A is not unital, then neither is A^τ and therefore 0 is in the spectrum in any sense of the word. Thus it suffices to see that, if $x \in A^\tau$ and x' exists in A, then $x' \in A^\tau$. But in this case, $x' = x'x - x \in A^\tau$ because A^τ is an ideal.

Lemma 1.7. *If $x \in \mathcal{D}(\delta_\tau)$ satisfies $\|x\|_{\mathrm{op}} < 1$, then $\delta(x)x' \in A^\tau$ and $\tau(\delta(x)x') = 0$.*

Proof. We have $\tau(\delta(x)) = \tau(\delta(x^2)) = 0$ by Lemma 1.3. But $\tau(\delta(x^2)) = \tau(\delta(x)x + x\delta(x)) = 2\tau(\delta(x)x)$, and hence $\tau(\delta(x)x) = 0$. Similarly, by induction, we have $\tau(\delta(x)x^n) = 0$ for all $n \geq 1$. Now $\|x\|_{\mathrm{op}} < 1$ implies that $\nu_A(x) < 1$, and hence that $\nu_{A^\tau}(x) < 1$. Thus $-\sum_{n=1}^\infty x^n = x'$ converges in the $\|\cdot\|_\tau$ norm, $\delta(x)x' \in A^\tau$ and

$$\tau(\delta(x)x') = -\sum_{n=1}^{\infty} \tau(\delta(x)x^n) = 0.$$

Theorem 1.8. *Let A be a C^*-algebra with a densely defined, lower semicontinuous trace τ, and a continuous action $\alpha : \mathbf{R} \to \operatorname{Aut} A$ which leaves τ invariant. Let δ be the infinitesimal generator of α on A, and δ_τ the infinitesimal generator of α on A^τ. Then the map $x \mapsto \tau(\delta(x)x') : GL'(\mathcal{D}(\delta_\tau)) \to i\mathbf{R}$ is a group homomorphism which is constant on (operator norm) connected components.*

Proof. Let $B = \mathcal{D}(\delta_\tau)$, and recall that the group operation on $GL'(B)$ is $a \circ b := a + b - ab$. A messy computation, using the trace property of τ, the derivation property of δ, and the equation $\tau(\delta(x)) = 0$ for $x \in B$, gives

$$\tau(\delta(a \circ b) \cdot (a \circ b)') = \tau(\delta(a)a') + \tau(\delta(b)b').$$

To see that the map is constant on (operator norm) connected components, suppose $a, b \in B'$ and $\|a - b\|_{\text{op}} < 1/(1 + \|a'\|_{\text{op}})$. Then $\|a' \circ b\|_{\text{op}} \leq (\|a'\|_{\text{op}} + 1)\|a - b\|_{\text{op}} < 1$, the lemma gives $\tau(\delta(a' \circ b)(a' \circ b)') = 0$, and the homomorphism property implies that $\tau(\delta(a)a') = \tau(\delta(b)b')$.

To see that the range of this homomorphism lies in $i\mathbf{R}$, suppose x is self-adjoint, $x \in GL'(B)$ and $x < 1$. Then for all $t \in [0, 1]$, we have $tx < 1$, $1 - tx$ invertible in $\widetilde{B}$, and $tx \in GL'(B)$. Thus $t \mapsto tx$ is an (operator norm continuous) path from x to 0 in $GL'(B)$, and $\tau(\delta(x)x') = 0$. For general x in $GL'(B)$, x^* is also in $GL'(B)$, and $x^* \circ x \in GL'(B)$ is self-adjoint. Moreover,

$$1 - x^* \circ x = 1 - x^* - x + x^*x = (1 - x^*)(1 - x)$$

is bounded away from 0 in the operator norm because $x \in GL'(B)$ implies $(1 - x)$ invertible in $\widetilde{B}$. Thus $x^* \circ x < 1$, $\tau(\delta(x^* \circ x)(x^* \circ x)') = 0$, and $\tau(\delta(x^*)(x^*)') = -\tau(\delta(x)x')$. Since one easily calculates that $\tau(\delta(x^*)(x^*)') = \overline{\tau(\delta(x)x')}$, we deduce that $\tau(\delta(x)x') \in i\mathbf{R}$.

Remark 1.9. If $1 \in A$, then $A = A^\tau$ and τ is finite, so we may as well suppose $\tau(1) = 1$. In this case, $\mathcal{D}(\delta_\tau) = \mathcal{D}(\delta)$, and $x \in GL'(\mathcal{D}(\delta))$ exactly when $(1 - x) \in GL(\mathcal{D}(\delta))$, in which case $(1 - x)^{-1} = 1 - x'$. Then

$$\tau\big(\delta(1 - x)(1 - x)^{-1}\big) = \tau\big(\delta(-x)(1 - x')\big) = \tau(\delta(-x)) + \tau(\delta(x)x') = \tau(\delta(x)x').$$

Hence our homomorphism is a map of $GL(\mathcal{D}(\delta))$ into $i\mathbf{R}$ taking y to $\tau\big(\delta(y)y^{-1}\big)$.

If A does *not* have an identity, then we adjoin an identity: the action α extends uniquely to $\widetilde{A}$, and so does δ (with $\delta(1) = 0$). Then $y \in GL(\mathcal{D}(\delta_\tau))$ exactly when $y = \lambda(1 - x)$ for some $\lambda \neq 0$ and $x \in GL'(\mathcal{D}(\delta_\tau))$, in which case $y^{-1} = \lambda^{-1}(1 - x')$, and

$$\tau\big(\delta(y)y^{-1}\big) = \tau\big(\delta(\lambda(1 - x))\,(\lambda(1 - x))^{-1}\big) = \tau\big(\delta(1 - x)(1 - x)^{-1}\big) = \ldots = \tau(\delta(x)x').$$

Note that $\tau(\delta(y)y^{-1})$ does not change if we multiply y by a nonzero scalar γ, and we can again view our homomorphism as a map of $GL(\mathcal{D}(\delta_\tau))$ into $i\mathbf{R}$ sending y to $\tau(\delta(y)y^{-1})$.

In either case, we can extend our homomorphism to the full group $GL(A)$ of invertible elements of $\widetilde{A}$, since it is constant on operator norm connected components and $GL(\mathcal{D}(\delta_\tau))$ is dense in $GL(A)$. With this in mind, we restate our theorem.

Theorem 1.10. *Let A be a C^*-algebra with a densely defined, lower semicontinuous trace τ, and a continuous action $\alpha : \mathbf{R} \to \operatorname{Aut} A$ which leaves τ invariant. Let δ be the infinitesimal generator of α on A and δ_τ the infinitesimal generator of α on A^τ. Then the map $x \mapsto (2\pi i)^{-1}\tau(\delta(x)x^{-1}) : GL(\mathcal{D}(\delta_\tau)) \to \mathbf{R}$ extends uniquely to a group homomorphism $GL(A) \to \mathbf{R}$ which is constant on connected components.*

Remark 1.11. Let $\mathcal{K}$ denote the algebra of compact operators on a separable Hilbert space, and let $\{e_{ij}\}_{i,j=1}^\infty$ be matrix units for $\mathcal{K}$. Then for any C^*-algebra A and $x \in A \otimes \mathcal{K}$, there is a unique matrix $\{a_{ij}\}_{i,j=1}^\infty$ over A such that $x = \lim_{m\to\infty} \sum_{i,j=1}^m a_{ij} \otimes e_{ij}$. If τ is a lower semicontinuous trace on A, then the formula $\tau_\infty(x) = \sum_{i=1}^\infty \tau(a_{ii})$ defines a lower semicontinuous trace on $A \otimes \mathcal{K}$. If τ is densely defined or faithful, then so is τ_∞; if τ is invariant under an action α of $\mathbf{R}$ on A, then τ_∞ is invariant under the induced action $\alpha \otimes \mathrm{id}$ of $\mathbf{R}$ on $A \otimes \mathcal{K}$. Recalling that the K-group $K_1(A)$ can be defined as $GL(A \otimes \mathcal{K})/GL_0(A \otimes \mathcal{K})$, where $GL_0(A \otimes \mathcal{K})$ denotes the connected component of the identity, we obtain a K-theoretic corollary:

Corollary 1.12. *Let A be a C^*-algebra with a densely defined, lower semicontinuous trace τ, and a continuous action $\alpha : \mathbf{R} \to \operatorname{Aut} A$ which leaves τ invariant. Let δ_∞ be the generator of $\alpha \otimes \mathrm{id}$. Then the map $x \mapsto (2\pi i)^{-1}\tau_\infty(\delta_\infty(x)x^{-1})$ on $GL(\mathcal{D}(\delta_{\tau_\infty}))$ induces a homomorphism of $K_1(A)$ into $\mathbf{R}$.*

2. The trace on the crossed product von Neumann algebra.

Let A be a C^*-algebra with a faithful, densely defined, lower semicontinuous trace τ; we use the notation of Appendix B without comment. We let H_τ denote the Hilbert space completion of A_2^τ in the inner product $(a|b) = \tau(b^*a)$, and let $a \to \xi_a$ be the canonical embedding of A_2^τ in H_τ, so that we have by definition $(\xi_a|\xi_b) = \tau(b^*a)$. We set $\mathcal{A}_2^\tau = \{\xi_a | a \in A_2^\tau\}$ and $\mathcal{A}^\tau = \{\xi_a | a \in A^\tau\}$. Then $\mathcal{A}^\tau \subset \mathcal{A}_2^\tau$, and both are Hilbert algebras: that is, they are associative complex *-algebras, which are also pre-Hilbert spaces, and which satisfy

(1) $(\xi|\eta) = (\eta^*|\xi^*)$ for ξ, η in the algebra;

(2) $(\xi\eta|\zeta) = (\eta|\xi^*\zeta)$ for ξ, η, ζ in the algebra;

(3) the set of products $\xi\eta$ is dense in the algebra;

(4) for each ξ in the algebra, the operator $\pi(\xi) : \eta \to \xi\eta$ is bounded.

(See [10, §I.6].) In this case, the map $a \to \pi(\xi_a)$ is a *-representation of A_2^τ on H_τ.

Since τ is a lower semicontinuous weight on A, we can use the GNS-construction to obtain a representation π_τ of A, which also acts in the space H_τ [16, 5.1.3], and which satisfies $\pi(\xi_a) = \pi_\tau(a)$ for $a \in A_2^\tau$. Since A_2^τ is norm-dense in A, we have $\pi_\tau(A)'' = \pi(\mathcal{A}_2^\tau)''$; in other words, $R = \pi_\tau(A)''$ is precisely the left von Neumann algebra $\pi(\mathcal{A}_2^\tau)''$ of the Hilbert algebra $\mathcal{A}_2^\tau$ (or of $\mathcal{A}^\tau$). By [10, §I.6.2, Theorem 1], τ extends to a faithful, semifinite, ultraweakly lower semicontinuous trace $\overline{\tau}$ on R: by definition, $x \in R_+^{\overline{\tau}}$ if and only if $x^{1/2} = \pi(\eta)$ for some bounded element η of H_τ, and then $\overline{\tau}(x) = (\eta|\eta)$. We have $\pi_\tau(A^\tau) \subset R^{\overline{\tau}}$ and $\pi_\tau(A_2^\tau) \subset R_2^{\overline{\tau}}$.

If α is a continuous action of $\mathbf{R}$ on A which leaves τ invariant, then α restricts to continuous actions on A^τ and A_2^τ (see §1 for the case of A^τ; the other is similar). Thus for each t we have a unitary U_t on $\mathcal{A}_2^\tau$ such that $U_t(\xi_a) = \xi_{\alpha_t(a)}$ for $a \in A_2^\tau$, and the resulting homomorphism $U : \mathbf{R} \to U(H_\tau)$ is strong-operator continuous. The pair (π_τ, U) is a covariant representation of $(A, \mathbf{R}, \alpha)$, and hence $\operatorname{Ad} U$ extends α to an action $\overline{\alpha}$ of $\mathbf{R}$ on R which leaves the extended trace $\overline{\tau}$ invariant.

We now define a covariant representation $(\overline{\pi}_\tau, \lambda \otimes U)$ of $(A, \mathbf{R}, \alpha)$ on $L^2(\mathbf{R}) \otimes H_\tau = L^2(\mathbf{R}, H_\tau)$ by

$$\overline{\pi}_\tau(a) = 1 \otimes \pi_\tau(a) \qquad \text{and} \qquad (\lambda \otimes U)_t = \lambda_t \otimes U_t.$$

The corresponding representation $\pi = \overline{\pi}_\tau \times (\lambda \otimes U)$ of $A \times_\alpha \mathbf{R}$ on $L^2(\mathbf{R}, H_\tau)$ is given on the dense subalgebra $C_c(\mathbf{R}, A^\tau)$ by

$$(\pi(x)\xi)(s) = \int \pi_\tau(x(t)) U_t(\xi(s-t))\, dt.$$

If ξ belongs to $C_c(\mathbf{R}, A^\tau)$, which is also a dense subset of $L^2(\mathbf{R}, H_\tau)$, this formula becomes

$$(\pi(x)\xi)(s) = \int x(t)\alpha_t(\xi(s-t))\,dt = (x * \xi)(s),$$

where $x * \xi$ is the usual convolution product on the subalgebra $C_c(\mathbf{R}, A)$ of $A \times_\alpha \mathbf{R}$. Now $C_c(\mathbf{R}, A^\tau)$ is itself a Hilbert algebra in the inner product

$$(x|y) = \int \tau\big(y(t)^* x(t)\big)\,dt,$$

and this calculation shows that the representation π is precisely the left regular representation of the Hilbert algebra $C_c(\mathbf{R}, A^\tau)$ on its Hilbert space completion $L^2(\mathbf{R}, H_\tau)$. Thus $\pi(A \times_\alpha \mathbf{R})'' = \pi\big(C_c(\mathbf{R}, A^\tau)\big)'' = M_0$, say, and there is a faithful, semifinite, ultra-weakly lower semicontinuous trace $\overline{\sigma}$ on M_0 [10, §I.6.2, Theorem 1]. Moreover, two elements S, T of M_0 belong to $(M_0)_2^{\overline{\sigma}}$ if and only if there exist $\xi, \eta \in L^2(\mathbf{R}, H_\tau)$ such that $S = \pi(\xi)$, $T = \pi(\eta)$, and in this case we have $\overline{\sigma}(T^*S) = (\xi|\eta)$. In particular, if x, y belong to $C_0(\mathbf{R}, A^\tau) \cap L^2(\mathbf{R}, H_\tau)$, *and* if $\pi(x)$, $\pi(y)$ are bounded operators, then $\pi(x)$, $\pi(y)$ are in $(M_0)_2^{\overline{\sigma}}$, with

$$\overline{\sigma}\big(\pi(y)^*\pi(x)\big) = (x|y) = \int \tau\big(y(t)^* x(t)\big)\,dt.$$

It is well-known that the representation $\pi = \overline{\pi}_\tau \times (\lambda \otimes U)$ is one version of the regular representation of $A \times_\alpha \mathbf{R}$ induced from the representation π_τ of A, and we have just constructed a trace on the von Neumann algebra crossed product. However, our Toeplitz operators are defined using a different realisation of the regular representation, and in order to use the trace we need to carry it over to this realisation. To do this, we define a single unitary $V \in U\big(L^2(\mathbf{R}, H_\tau)\big)$ by $(V\xi)(t) = U_t\big(\xi(t)\big)$. We can easily check that

$$V\overline{\pi}_\tau(a)V^* = \widetilde{\pi}_\tau(a) \qquad \text{and} \qquad V(\lambda_t \otimes U_t)V^* = \lambda_t \otimes 1 = \lambda_t,$$

where

$$(2.1) \qquad \big(\widetilde{\pi}_\tau(a)\xi\big)(s) = \pi_\tau\big(\alpha_s^{-1}(a)\big)\big(\xi(s)\big) \qquad \text{and} \qquad (\lambda_t\xi)(s) = \xi(s-t).$$

We denote the integrated form $\widetilde{\pi}_\tau \times \lambda$ by $\operatorname{Ind}\pi_\tau$. By [16, 7.7.5], $\operatorname{Ind}\pi_\tau$ is a faithful representation of $A \times_\alpha \mathbf{R}$, and hence so is $\pi = \operatorname{Ad}V^* \circ \operatorname{Ind}\pi_\tau$. We let $M = VM_0V^* = \operatorname{Ind}\pi_\tau(A \times_\alpha \mathbf{R})''$, and define a faithful, semifinite, ultraweakly lower semicontinuous trace $\widehat{\tau}$ on M by $\widehat{\tau}(T) = \overline{\sigma}(V^*TV)$.

Now suppose that $x \in C_0(\mathbf{R}, A^\tau) \cap L^2(\mathbf{R}, H_\tau)$. If $\pi(x)$ is a bounded operator, then one can easily verify, at least on $C_c(\mathbf{R}, A^\tau) \subset L^2(\mathbf{R}, H_\tau)$, that $V\pi(x)V^*$ is given by

$$(2.2) \qquad (V\pi(x)V^*\xi)(s) = \int \pi_\tau\big(\alpha_s^{-1}(x(t))\big)\xi(s-t)\,dt,$$

and of course we denote this operator by $\mathrm{Ind}\,\pi_\tau(x)$. Conversely, if the right-hand side of (2.2) defines a bounded operator $\mathrm{Ind}\,\pi_\tau(x)$, then $\pi(x) = V^*\mathrm{Ind}\,\pi_\tau(x)V$ is bounded, hence belongs to $M_2^{\overline{\sigma}}$, and we can deduce that $\mathrm{Ind}\,\pi_\tau(x)$ belongs to $M_2^{\widehat{\tau}}$. Thus if y is another function in $C_0(\mathbf{R}, A^\tau) \cap L^2(\mathbf{R}, H_\tau)$ such that $\mathrm{Ind}\,\pi_\tau(y)$ is a bounded operator, then $\mathrm{Ind}\,\pi_\tau(x)^*\mathrm{Ind}\,\pi_\tau(y) \in M^{\widehat{\tau}}$, and

$$\widehat{\tau}(\mathrm{Ind}\,\pi_\tau(x)^*\mathrm{Ind}\,\pi_\tau(y)) = \int \tau\big(x(t)^*y(t)\big)\,dt. \tag{2.3}$$

3. The index theorem.

We consider a system $(A, \mathbf{R}, \alpha)$ and an invariant trace τ on A, exactly as in §2, and let $\pi_\tau : A \to B(H_\tau)$ be the GNS-representation associated to τ. The induced representation $\mathrm{Ind}\,\pi_\tau$ is given by the covariant pair $(\widetilde{\pi}_\tau, \lambda) : (A, \mathbf{R}, \alpha) \to B\big(L^2(\mathbf{R}, H_\tau)\big)$ of (2.1). We let $P_{\mathbf{R}}$ denote the orthogonal projection of $L^2(\mathbf{R})$ onto $H^2(\mathbf{R})$, and let $P = P_{\mathbf{R}} \otimes 1 : L^2(\mathbf{R}) \otimes H_\tau \to H^2(\mathbf{R}) \otimes H_\tau$. We write T_a for the Toeplitz operator $T_a^{\pi_\tau} = P\widetilde{\pi}_\tau(a)P$. Recall from §2 that we have a faithful, normal, semifinite trace $\widehat{\tau}$ on the von Neumann algebra $M = \mathrm{Ind}\,\pi_\tau(A \times_\alpha \mathbf{R})''$, whose restriction to $N = PMP$ has the same properties. As in §1, we let δ be the infinitesimal generator of α, and use Theorem 1.10 to extend the winding number $(2\pi i)^{-1}\tau\big(\delta(a)a^{-1}\big)$ to all of $GL(A)$. With these conventions, we can state our index theorem.

Theorem 3.1. *Let τ be a faithful, densely defined, lower semicontinuous trace on a C^*-algebra A, which is invariant for an action α of $\mathbf{R}$. Then for any $a \in GL(A)$, the Toeplitz operator T_a is Fredholm relative to the trace $\widehat{\tau}$ on $N = P\big(\mathrm{Ind}\,\pi_\tau(A \times_\alpha \mathbf{R})''\big)P$, and*

$$\widehat{\tau}\text{-}\mathrm{Ind}\,T_a = \frac{-1}{2\pi i}\,\tau\big(\delta(a)a^{-1}\big).$$

We intend to follow the second proof of [8, §25.2]. While we do face extra difficulties here, we have already addressed most of these in §1 and Appendix A using the Banach *-algebra $(A^\tau, \|\cdot\|_\tau = \|\cdot\|_{\mathrm{op}} + \|\cdot\|_1)$, and the derivation δ_τ which is the generator of α on A^τ. As in [8], a key observation is that $P_{\mathbf{R}} = (1 + H_{\mathbf{R}})/2$, where $H_{\mathbf{R}}$ is the Hilbert transform on $L^2(\mathbf{R})$, given by the principal-value integral

$$(H_{\mathbf{R}}\xi)(s) = \lim_{\epsilon \to 0} \int_{|t| \geq \epsilon} \frac{1}{\pi i t}\xi(s - t)\,dt,$$

which converges in $L^2(\mathbf{R})$. We let $H = H_{\mathbf{R}} \otimes 1$ be the corresponding operator on $L^2(\mathbf{R}, H_\tau)$, so that $P = (1 + H)/2$, and $[P, T] = [H, T]/2$ for any $T \in B\big(L^2(\mathbf{R}, H_\tau)\big)$.

Lemma 3.2. (cf. [8, 22.1]) *For any $a \in \mathcal{D}(\delta_\tau)$, $[H, \widetilde{\pi}_\tau(a)]$ belongs to $M_2^{\widehat{\mathcal{T}}}$. Indeed, we have $[H, \widetilde{\pi}_\tau(a)] = \operatorname{Ind} \pi_\tau(x)$, where $x \in C_0(\mathbf{R}, A^\tau) \cap L^2(\mathbf{R}, H_\tau)$ is given by*

$$x(t) = \frac{\alpha_t(a) - a}{\pi i t}.$$

Proof. To complete the following formal calculation, one just has to recall that the principal-value integrals converge in L^2:

$$\begin{aligned}
([H, \widetilde{\pi}_\tau(a)]\xi)(s) &= \int \frac{1}{\pi i t} (\widetilde{\pi}_\tau(a)\xi)(s-t)\, dt - \pi_\tau(\alpha_s^{-1}(a))(H\xi)(s) \\
&= \frac{1}{\pi i} \int \left(\frac{1}{t} \pi_\tau(\alpha_{s-t}^{-1}(a))\,(\xi(s-t)) - \frac{1}{t}\pi_\tau(\alpha_s^{-1}(a))\,(\xi(s-t)) \right) dt \\
&= \frac{1}{\pi i} \int \pi_\tau\left(\alpha_s^{-1}\left(\frac{\alpha_t(a) - a}{t} \right)\right) \xi(s-t)\, dt.
\end{aligned}$$

The function x belongs to $L^2(\mathbf{R}, H_\tau)$ because $x(t) \to (\pi i)^{-1}\delta(a)$ as $t \to 0$, and

$$\|x(t)\|_2^2 = \tau\big(x(t)^* x(t)\big) \leq \frac{4}{\pi^2 t^2} \|a\|_{\mathrm{op}} \|a\|_1.$$

Since we know that $[H, \widetilde{\pi}_\tau(a)]$ is a bounded operator, the result now follows from the discussion at the end of §2.

Corollary 3.3. *For $a, b \in \mathcal{D}(\delta_\tau)$ we have $T_a T_b - T_{ab} \in M^{\widehat{\mathcal{T}}} \cap N = N^{\widehat{\mathcal{T}}}$. If $ab = ba$, then $[T_a, T_b] \in N^{\widehat{\mathcal{T}}}$.*

Proof. We calculate:

$$\begin{aligned}
T_a T_b - T_{ab} &= \big(P\widetilde{\pi}_\tau(a)P\big)\big(P\widetilde{\pi}_\tau(b)P\big) - \big(P\widetilde{\pi}_\tau(ab)P\big) \\
&= P\widetilde{\pi}_\tau(a)(P-1)\widetilde{\pi}_\tau(b)P \\
&= -\big(P\widetilde{\pi}_\tau(a)(1-P)\big)\big((1-P)\widetilde{\pi}_\tau(b)P\big) \\
&= -P\big[P, \widetilde{\pi}_\tau(a)\big]\big[\widetilde{\pi}_\tau(b), P\big]P \\
&= \tfrac{1}{4} P\big[H, \widetilde{\pi}_\tau(a)\big]\big[H, \widetilde{\pi}_\tau(b)\big]P,
\end{aligned}$$

which is in $M^{\widehat{\mathcal{T}}} \cap PMP = N^{\widehat{\mathcal{T}}}$. If $ab = ba$, then

$$[T_a, T_b] = (T_a T_b - T_{ab}) + (T_{ba} - T_b T_a) \in N^{\widehat{\mathcal{T}}}.$$

Lemma 3.4. *Suppose $a, b \in \mathcal{D}(\delta_\tau)$ and $ab = ba$. Then*

$$T = \widetilde{\pi}_\tau(a) H \widetilde{\pi}_\tau(b) - \widetilde{\pi}_\tau(b) H \widetilde{\pi}_\tau(a)$$

belongs to $M_2^{\widehat{\tau}}$*; indeed, it has the form* $\operatorname{Ind}\pi_\tau(y)$*, where* y *is the function in* $C_0(\mathbf{R}, A^\tau) \cap L^2(\mathbf{R}, H_\tau)$ *given by* $y(t) = (\pi i t)^{-1}\big(\alpha_t(b)a - \alpha_t(a)b\big)$*. We have*

$$\widehat{\tau}\big([T_a, T_b]\big) = \tfrac{1}{4}\widehat{\tau}\big(PTP + (1-P)T(1-P)\big).$$

Proof. We know from Corollary 3.3 that $[T_a, T_b]$ is in $M^{\widehat{\tau}}$, and

$$\begin{aligned}\widehat{\tau}\big([T_a, T_b]\big) &= \widehat{\tau}\big((T_aT_b - T_{ab}) + (T_{ba} - T_bT_a)\big)\\ &= -\widehat{\tau}\big(P\widetilde{\pi}_\tau(a)(1-P)\widetilde{\pi}_\tau(b)P - P\widetilde{\pi}_\tau(b)(1-P)\widetilde{\pi}_\tau(a)P\big)\\ &= \widehat{\tau}\big(P\widetilde{\pi}_\tau(b)(1-P)\widetilde{\pi}_\tau(a)P - P\widetilde{\pi}_\tau(a)(1-P)\widetilde{\pi}_\tau(b)P\big).\end{aligned}$$

It follows from Lemma 3.2 that $P\widetilde{\pi}_\tau(a)(1-P) = \frac{1}{2}P[H, \widetilde{\pi}_\tau(a)]$ belongs to $M_2^{\widehat{\tau}}$, so by [10, I.6.12] we also have

$$\widehat{\tau}\big([T_a, T_b]\big) = \widehat{\tau}\big((1-P)\widetilde{\pi}_\tau(a)P\widetilde{\pi}_\tau(b)(1-P) - (1-P)\widetilde{\pi}_\tau(b)P\widetilde{\pi}_\tau(a)(1-P)\big).$$

Averaging the two formulas for $\widehat{\tau}\big([T_a, T_b]\big)$ gives

$$\begin{aligned}\widehat{\tau}\big([T_a, T_b]\big) = \tfrac{1}{2}\widehat{\tau}\Big(&P\big(\widetilde{\pi}_\tau(b)(1-P)\widetilde{\pi}_\tau(a) - \widetilde{\pi}_\tau(a)(1-P)\widetilde{\pi}_\tau(b)\big)P\\ &+ (1-P)\big(\widetilde{\pi}_\tau(a)P\widetilde{\pi}_\tau(b) - \widetilde{\pi}_\tau(b)P\widetilde{\pi}_\tau(a)\big)(1-P)\Big).\end{aligned}$$

Since $P = \frac{1}{2}(1+H)$, $(1-P) = \frac{1}{2}(1-H)$, and $\widetilde{\pi}_\tau(a)$ commutes with $\widetilde{\pi}_\tau(b)$, we deduce that

$$\begin{aligned}\widehat{\tau}\big([T_a, T_b]\big) &= \tfrac{1}{4}\widehat{\tau}\Big(P\big(\widetilde{\pi}_\tau(a)H\widetilde{\pi}_\tau(b) - \widetilde{\pi}_\tau(b)H\widetilde{\pi}_\tau(a)\big)P\\ &\qquad + (1-P)\big(\widetilde{\pi}_\tau(a)H\widetilde{\pi}_\tau(b) - \widetilde{\pi}_\tau(b)H\widetilde{\pi}_\tau(a)\big)(1-P)\Big)\\ &= \tfrac{1}{4}\widehat{\tau}\big(PTP + (1-P)T(1-P)\big),\end{aligned}$$

as claimed.

Next, notice that we can also write

$$T = \big[H, \widetilde{\pi}_\tau(b)\big]\widetilde{\pi}_\tau(a) - \big[H, \widetilde{\pi}_\tau(a)\big]\widetilde{\pi}_\tau(b). \tag{3.1}$$

From Lemma 3.2, $\big[H, \widetilde{\pi}_\tau(b)\big]\widetilde{\pi}_\tau(a) = \operatorname{Ind}\pi_\tau(z)$, where $z(t) = (\pi i t)^{-1}\big(\alpha_t(b)a - ba\big)$, and hence $T = \operatorname{Ind}\pi_\tau(y)$, where

$$y(t) = \frac{\alpha_t(b)a - ba}{2\pi i t} - \frac{\alpha_t(a)b - ab}{2\pi i t} = \frac{\alpha_t(b)a - \alpha_t(a)b}{2\pi i t}.$$

Because $y(t) \to (\pi i)^{-1}(\delta(b)a - \delta(a)b)$ in the $\|\cdot\|_\tau$-norm as $t \to 0$, y is a continuous A^τ-valued function, and because $\|y(t)\|_2^2 \le 4\|a\|_1\|b\|_{\mathrm{op}}/\pi^2t^2$ for $t \neq 0$, we know also that $y \in L^2(\mathbf{R}, H_\tau)$.

If the operator T were in $M^{\widehat{\tau}}$, we could immediately deduce from Lemma 3.4 that $\widehat{\tau}([T_a, T_b]) = \widehat{\tau}(T)$; however, as in [8], this is not necessarily the case, and we have to use an approximate identity argument to compute $\widehat{\tau}(PTP + (1-P)T(1-P))$. This is a little harder than in [8], since operators of the form $\lambda(f) \otimes 1$ for $f \in C_c(\mathbf{R})$ will not be in $M_2^{\widehat{\tau}}$ when 1 has infinite trace. Notice that in our notation, $\lambda(f)$, acting in $L^2(\mathbf{R}, H_\tau)$, is really $\lambda(f) \otimes 1$, acting in $L^2(\mathbf{R}) \otimes H_\tau$.

Lemma 3.5. *Suppose $a, b \in \mathcal{D}(\delta_\tau)$ satisfy $ab = ba$, and T is as in the previous lemma. Then for $f \in C_c(\mathbf{R})$, we have $T\lambda(f) \in M^{\widehat{\tau}}$, and*

$$\widehat{\tau}(T\lambda(f)) = \frac{1}{\pi i}\int_{\operatorname{supp} f} \tau\left(\frac{\alpha_t(b) - b}{t}a - \frac{\alpha_t(a) - a}{t}b\right) f(t)\,dt.$$

Proof. From Equation (3.1) we have

$$T\lambda(f) = [H, \widetilde{\pi}_\tau(b)]\widetilde{\pi}_\tau(a)\lambda(f) - [H, \widetilde{\pi}_\tau(a)]\widetilde{\pi}_\tau(b)\lambda(f),$$

and, since the commutators are in $M_2^{\widehat{\tau}}$, it is enough to show that $\widetilde{\pi}_\tau(a)\lambda(f)$ is in $M_2^{\widehat{\tau}}$. But for $\xi \in L^2(\mathbf{R}, H_\tau)$, we have

$$(\widetilde{\pi}_\tau(a)\lambda(f)\xi)(s) = \int \pi_\tau(\alpha_s^{-1}(af(t)))\xi(s-t)dt = ((\widetilde{\pi}_\tau \times \lambda)(af)\xi)(s);$$

in other words, $\widetilde{\pi}_\tau(a)\lambda(f) = (\widetilde{\pi}_\tau \times \lambda)(af)$. But af is in the Hilbert algebra $C_c(\mathbf{R}, A^\tau)$, and hence $(\widetilde{\pi}_\tau \times \lambda)(af)$ is in $M_2^{\widehat{\tau}}$, as required.

Proposition 3.6. *If $a, b \in \mathcal{D}(\delta_\tau)$ and $ab = ba$, then*

$$\widehat{\tau}([T_a, T_b]) = \frac{1}{2\pi i}\tau(\delta(b)a).$$

Proof. Since $S = PTP + (1-P)T(1-P)$ is in $M^{\widehat{\tau}}$, both $PS = PTP$ and $(1-P)S = (1-P)T(1-P)$ are in $M^{\widehat{\tau}}$, and each is a finite linear combination of products of elements of $L^2(\mathbf{R}, H_\tau)$. Thus, if $\{f_n\} \subset C_c(\mathbf{R})$ is sequence of nonnegative functions with integral 1 and supports shrinking to $\{0\}$, so that $\lambda(f_n) \to 1$ strongly, then we have $\widehat{\tau}(\lambda(f_n)PTP) \to \widehat{\tau}(PTP)$ and $\widehat{\tau}(\lambda(f_n)(1-P)T(1-P)) \to \widehat{\tau}((1-P)T(1-P))$. Since both $\lambda(f_n)$ and $P = \frac{1}{2}(1+H)$ lie in the abelian von Neumann algebra $\lambda(\mathbf{R})''$, we have

$$\widehat{\tau}(\lambda(f_n)PTP) = \widehat{\tau}(PTP\lambda(f_n)) = \widehat{\tau}(PT\lambda(f_n)P);$$

since $T\lambda(f_n)$ is in $M^{\widehat{\tau}}$ by Lemma 3.5, this equals $\widehat{\tau}(PT\lambda(f_n))$. Similarly,

$$\widehat{\tau}(\lambda(f_n)(1-P)T(1-P)) = \widehat{\tau}((1-P)T\lambda(f_n)).$$

Therefore

$$\begin{aligned}\widehat{\tau}(\lambda(f_n)PTP) &+ \widehat{\tau}(\lambda(f_n)(1-P)T(1-P)) \\ &= \widehat{\tau}(T\lambda(f_n)) = \frac{1}{\pi i}\int_{\operatorname{supp} f} \tau\left(\frac{\alpha_t(b)a - \alpha_t(a)b}{t}\right) f_n(t)\,dt,\end{aligned}$$

and so by Lemma 3.4,

$$\widehat{\tau}([T_a, T_b]) = \lim_{n\to\infty} \tfrac{1}{4}\widehat{\tau}(T\lambda(f_n)) = \lim_{n\to\infty} \frac{1}{4\pi i}\int_{\operatorname{supp} f} \tau\left(\frac{\alpha_t(b)a - \alpha_t(a)b}{t}\right) f_n(t)\,dt.$$

Since

$$\left\|\frac{\alpha_t(b)a - \alpha_t(a)b}{t} - (\delta(b)a - \delta(a)b)\right\|_1 \to 0 \qquad \text{as } t \to 0,$$

the integral converges to $\tau(\delta(b)a - \delta(a)b)$ as $n \to \infty$. Finally, we use $\delta(a)b = \delta(ab) - a\delta(b)$ and $\tau(\delta(ab)) = 0$ (which holds because $ab \in \mathcal{D}(\delta_\tau)$), to simplify

$$\widehat{\tau}([T_a, T_b]) = \frac{1}{4\pi i}\tau(\delta(b)a - \delta(a)b) = \frac{1}{4\pi i}\tau(\delta(b)a + a\delta(b)) = \frac{1}{2\pi i}\tau(\delta(b)a).$$

Proof of Theorem 3.1. Recall from the discussion in §1 that we may as well suppose $a \in GL(\mathcal{D}(\delta_\tau))$ has the form $a = 1 - x$ for some $x \in GL'(\mathcal{D}(\delta_\tau))$, so that $a^{-1} = 1 - x'$, and prove that

(3.2) $$\widehat{\tau}\text{-ind}\, T_a = \frac{-1}{2\pi i}\tau(\delta(a)a^{-1}) = \frac{-1}{2\pi i}\tau(\delta(x)x').$$

We immediately have $\widetilde{\pi}_\tau(a) = 1 - \widetilde{\pi}_\tau(x)$, $T_a = P - T_x$, and $[T_a, P] = -[T_x, P] \in N_2^{\widehat{\tau}}$, by Lemma 3.2. Since $\widetilde{\pi}_\tau(a)$ is invertible, this implies that $T_a = P\widetilde{\pi}_\tau(a)P$ is invertible modulo $\mathcal{K}_N$ (the norm closure of $N_2^{\widehat{\tau}}$), and hence, by Theorem B1, that T_a is $\widehat{\tau}$-Fredholm.

We want to reduce to the case where a is unitary, and, because both sides of Equation (3.2) are homotopy invariant, it is enough to check that the homotopy $t \to a|a|^{-t}$ lies in $GL(\mathcal{D}(\delta_\tau))$. Since a and a^*a are in $GL(A^\tau)$, and $\sigma_{A^\tau}(a^*a) = \sigma_A(a^*a) \subset (0,\infty)$, we can use the holomorphic functional calculus for $a^*a \in GL(A^\tau)$ to see that $|a|^t = (a^*a)^{t/2}$ lies in $GL(A^\tau)$. More explicitly, let γ be a simple closed contour in the right half-plane enclosing

$\sigma_{A^\tau}(a^*a)$, and choose a branch $f(z)$ of $z^{t/2}$ which is analytic in an open set containing γ and its interior. Then

$$|a|^t = (a^*a)^{t/2} = \frac{1}{2\pi i}\int_\gamma f(z)(z1 - a^*a)^{-1}\,dz,$$

and the integral converges in $((A^\tau)\tilde{}, \|\cdot\|_\tau)$. Now we can use the compactness of γ and the usual resolvent formula to see that $\delta_\tau(|a|^t)$ exists, and equals

$$\frac{1}{2\pi i}\int_\gamma f(z)(z1 - a^*a)^{-1}\delta_t(a^*a)(z1 - a^*a)^{-1}\,dz.$$

Thus we have $|a|^t$ in $GL(A^\tau)\cap\mathcal{D}(\delta_\tau) = GL(\mathcal{D}(\delta_\tau))$, as required.

It now suffices to prove the index formula for the unitary $u = a|a|^{-1}$ in $GL(\mathcal{D}(\delta_\tau))$. Relative to the decomposition $1 = P + (1-P)$, we can write

$$\widetilde{\pi}_\tau(u) = \begin{pmatrix} T_u & b \\ c & d \end{pmatrix},$$

where b and c are in $M_2^{\hat{\tau}}$ by Lemma 3.2. By [4, Proposition 2.5] and the previous Proposition, we therefore have

$$\begin{aligned}\hat{\tau}\text{-ind}(T_u) &= \hat{\tau}(c^*c - bb^*) = \hat{\tau}\big((1 - T_{u^*}T_u) - (1 - T_uT_{u^*})\big)\\ &= \hat{\tau}\big([T_u, T_{u^*}]\big) = \frac{-1}{2\pi i}\tau\big(\delta(u)u^*\big).\end{aligned}$$

This completes the proof of Theorem 3.1.

4. Examples and applications.

(a) The classical case.

Here $A = C_0(\mathbf{R})$, $\widetilde{A} = C(\mathbf{R} \cup \{\infty\})$, $\alpha_s(a)(t) = a(t-s)$, $\tau(a) = \int_{\mathbf{R}} a\,dm$, and π_τ is the representation M of A as multiplication operators on $L^2(\mathbf{R})$. The covariant representation $\widetilde{M} \times \lambda$ of $A \times_\alpha \mathbf{R}$ on $L^2(\mathbf{R}, L^2(\mathbf{R}))$ is spatially equivalent to the representation $(M \times \lambda) \otimes 1$ on $L^2(\mathbf{R}) \otimes L^2(\mathbf{R}) \cong L^2(\mathbf{R} \times \mathbf{R})$: indeed, if we identify both Hilbert spaces with $L^2(\mathbf{R} \times \mathbf{R})$, the unitary transformation W defined by $W\xi(s,t) = \xi(s-t,t)$ satisfies

$$W\widetilde{M}(a)W^* = M(a) \otimes 1, \qquad W(\lambda_r \otimes 1)W^* = \lambda_r \otimes 1.$$

Since $M \times \lambda(C_0(\mathbf{R}) \times \mathbf{R}) = \mathcal{K}(L^2(\mathbf{R}))$, the von Neumann algebra $\widetilde{M} \times \lambda(A \times_\alpha \mathbf{R})''$ is isomorphic to $\mathcal{K}(L^2(\mathbf{R}))'' = B(L^2(\mathbf{R}))$ (though not spatially). Now for z in $C_c(\mathbf{R} \times \mathbf{R}) \subset C_c(\mathbf{R}, C_0(\mathbf{R}))$, we have

$$M \times \lambda(z)\xi(s) = \int z(r,s)\xi(s-r)\,dr = \int z(s-p,s)\xi(p)\,dp,$$

and since the trace of the integral operator with kernel $k \in C_c(\mathbf{R} \times \mathbf{R})$ is $\int k(s,s)\,ds$, we have

$$\operatorname{tr}\big(M \times \lambda(z)\big) = \int z(s-s,s)\,ds = \tau\big(z(0)\big) \qquad \text{for } z \in C_c\big(\mathbf{R}, C_c(\mathbf{R})\big).$$

When z is the square in $C_0(\mathbf{R}) \times_\alpha \mathbf{R}$ of some element of $C_c\big(\mathbf{R}, C_c(\mathbf{R})\big)$, $\widetilde{M} \times \lambda(z)$ belongs to M^τ, and

$$\widehat{\tau}\big(\widetilde{M} \times \lambda(z)\big) = \tau\big(z(0)\big);$$

since such $\widetilde{M} \times \lambda(z)$ are dense in M^τ, it follows that the isomorphism of $\widetilde{M} \times \lambda\big(C_0(\mathbf{R}) \times \mathbf{R}\big)''$ onto $M \times \lambda\big(C_0(\mathbf{R}) \times \mathbf{R}\big)''$ carries $\widehat{\tau}$ into the usual trace tr on $B\big(L^2(\mathbf{R})\big) = M \times \lambda\big(C_0(\mathbf{R}) \times \mathbf{R}\big)''$. Since W commutes with $P \otimes 1$, and $W\widetilde{M}(a)W^* = M(a) \otimes 1$, the isomorphism also takes $T_a^{\pi_\tau}$ into the usual Toeplitz operator T_a on $H^2(\mathbf{R})$, and the index theorem says that, for a non-vanishing function $a(x)$ on $\mathbf{R}$ with limits as $|x| \to \infty$, T_a is Fredholm in the usual sense (i.e. tr-Fredholm) and

$$\operatorname{ind} T_a = \operatorname{tr-ind} T_a = \widehat{\tau}\text{-ind}\, T_a^{\pi_\tau} = \frac{-1}{2\pi i}\tau\big(\delta(a)a^{-1}\big).$$

Here the infinitesimal generator δ is ordinary differentiation, the right-hand side is

$$\frac{-1}{2\pi i}\int_{\mathbf{R}} \frac{a'}{a}\,dm = -\text{the winding number of } a \text{ about } 0,$$

and we recover the classical Gohberg-Krein Theorem.

(b) Operators with symbols in group algebras.

As we mentioned earlier, many group algebras and twisted group algebras have natural actions and invariant traces. For example, if Γ is a subgroup of $\mathbf{R}$, viewed as a discrete group, then $\mathbf{R} = \widehat{\mathbf{R}}$ sits naturally inside $\widehat{\Gamma}$ and hence acts on $C(\widehat{\Gamma}) \cong C^*(\Gamma)$ with Haar measure giving an invariant trace τ. On $l^1(\Gamma) \subset C^*(\Gamma)$ we have $\tau(f) = f(0)$, and the same formula characterises a finite trace on each twisted group algebra $C^*(\Gamma, \omega)$. The dual action of $\widehat{\Gamma}$ on $C^*(\Gamma, \omega)$ restricts to an action α of $\mathbf{R}$ on $C^*(\Gamma, \omega)$ satisfying $\alpha_s(f)(t) = e^{ist} f(t)$ for $s, t \in \mathbf{R}$, $f \in l^1(\Gamma)$, which leaves the trace invariant. The index theorem in this case reduces to that of [13] (at least for individual operators; we deal with systems below).

When G is locally compact and unimodular, the formula $\tau(f) = f(e)$ extends from $A(G) = L^2 * L^2$ to a densely defined lower semicontinuous trace on $C^*(G)$, and also on any twisted group algebra $C^*(G, \omega)$. (To see this last point, suppose $\omega \in Z^2(G, \mathbf{T})$ is a (Borel) multiplier, $G_\omega = \mathbf{T} \times_\omega G$ is the locally compact central extension in which $(w, s)(z, t) = (wz\omega(s, t), st)$, and τ_ω is the trace on $C^*(G_\omega)$. Then

$$\Phi(\{f_n\})(z, s) = \sum_n f_n(s) z^n$$

extends to an isomorphism of $\oplus_{n \in \mathbf{Z}} C^*(G, \omega^n)$ onto $C^*(G_\omega)$, such that $\tau_\omega(\Phi(\{f_n\})) = \sum f_n(e)$. Thus $C^*(G, \omega)$ is naturally embedded as an ideal in $C^*(G_\omega)$, and restricting τ_ω to this ideal gives a trace τ with the required property.) Many such groups carry actions of $\mathbf{R}$ leaving τ invariant: for example, the Heisenberg group

$$\left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbf{R} \right\}$$

has $G/Z(G) \cong \mathbf{R}^2$, lots of multipliers ω (cf. [15, §1.4(4)]), and lots of embeddings of $\mathbf{R}$ in $\widehat{\mathbf{R}^2} = \widehat{G/Z}$ giving actions of $\mathbf{R}$ on $C^*(G, \omega)$ which leave τ invariant. Other examples include the affine actions of $\mathbf{R}$ on $\mathbf{R}^m \times \mathbf{R}^n$ given by $t \cdot (u, v) = (u, v + tAu)$ for some $n \times m$-matrix A, for which the corresponding actions on $C_0(\mathbf{R}^m \times \mathbf{R}^n) \cong C^*(\mathbf{R}^m \times \mathbf{R}^n)$ leave Lebesgue measure invariant; however, adding in a multiplier does not seem to give anything particularly interesting, since $C^*(\mathbf{R}^m \times \mathbf{R}^n, \omega)$ is isomorphic to $\mathcal{K}(L^2(\mathbf{R}^n))$ if $m = n$ and ω is totally skew, and $C_0(\mathbf{R}^p, \mathcal{K})$ in general [1].

(c) The index theorem for systems.

Again τ will be a faithful, densely defined, lower semicontinuous trace on A which is invariant for the action $\alpha : \mathbf{R} \to \operatorname{Aut} A$. Suppose now that $(T_{a_{ij}})_{i,j=1}^n$ is an $n \times n$ system of Toeplitz operators associated to the GNS-representation π_τ of A. The trace $\widehat{\tau}$ on $\operatorname{Ind} \pi_\tau(A \times_\alpha \mathbf{R})''$ extends naturally to a trace $\widehat{\tau}_n = \widehat{\tau} \otimes \operatorname{tr}$ on $\operatorname{Ind} \pi_\tau(A \times_\alpha \mathbf{R})'' \otimes M_n = M_n(\operatorname{Ind} \pi_\tau(A \times_\alpha \mathbf{R})'')$ such that $\widehat{\tau}_n((T_{ij})) = \sum \widehat{\tau}(T_{ii})$.

Theorem 4.1. *Suppose $(a_{ij})_{i,j=1}^n$ is invertible in the algebra $M_n(\widetilde{A})$. Then the operator $(T_{a_{ij}})_{i,j=1}^n \in M_n(\operatorname{Ind}\pi_\tau(A \times_\alpha \mathbf{R})'')$ is Fredholm relative to $\widehat{\tau}_n$, and*

$$\widehat{\tau}_n\text{-ind}\left((T_{a_{ij}})\right) = \frac{1}{2\pi i}\tau_n\left((\delta(a_{ij}))(a_{ij})^{-1}\right).$$

Proof. The matrix $(T_{a_{ij}})$, viewed as a single operator on the space $H^2(\mathbf{R}, H_\tau) \otimes \mathbf{C}^n = H^2(\mathbf{R}, H_\tau \otimes \mathbf{C}^n)$, is the Toeplitz operator with symbol $(a_{ij}) \in M_n(A) = A \otimes M_n$ associated to the system $(A \otimes M_n, \mathbf{R}, \alpha \otimes \mathrm{id})$ and the representation $\pi_\tau \otimes \mathrm{id}$ of $A \otimes M_n$ on $H_\tau \otimes \mathbf{C}^n$. Since $\tau \otimes \mathrm{tr}$ is an $\mathbf{R}$-invariant trace on $A \otimes M_n$, we can apply our index theorem to the triple $(A \otimes M_n, \alpha \otimes \mathrm{id}, \tau \otimes \mathrm{tr})$. Unfortunately, it is not true that $\pi_\tau \otimes \mathrm{id} = \pi_{\tau \otimes \mathrm{tr}}$, so Theorem 3.1 does not immediately give information about $(T_{a_{ij}}) = T_{(a_{ij})}$.

The problem is that the GNS-representation π_{tr} is not the identity representation id of $M_n(\mathbf{C})$ on $\mathbf{C}^n$, but rather $\mathrm{id} \otimes 1$ acting in $\mathbf{C}^n \otimes \mathbf{C}^n$: for if $S = (s_{ij})$, $T = (t_{ij}) \in M_n(\mathbf{C})$, we have

$$\mathrm{tr}(S^*T) = \mathrm{tr}\left(\left(\sum_k \overline{s_{ki}} t_{kj}\right)_{i,j}\right) = \sum_{j,k} \overline{s_{kj}} t_{kj},$$

and $U((t_{ij})) = \sum t_{ij}(e_i \otimes e_j)$ defines a unitary transformation of $(M_n, (\cdot|\cdot)_{\mathrm{tr}})$ onto $\mathbf{C}^n \otimes \mathbf{C}^n$ which, by straightforward computation, satisfies $U\pi_{\mathrm{tr}}(S)U^* = S \otimes 1$. Since $\pi_\tau \otimes \pi_{\mathrm{tr}}$ is equivalent to $\pi_{\tau \otimes \mathrm{tr}}$, it follows that the map

$$\xi_{a \otimes (t_{ij})} \to \xi_a \otimes \left(\sum_{i,j} t_{ij}(e_i \otimes e_j)\right)$$

induces a unitary intertwining the representations $\pi_{\tau \otimes \mathrm{tr}}$ and $\pi \otimes \mathrm{id} \otimes 1$ of $A \otimes M_n$. Next, we observe that $\operatorname{Ind}(\pi \otimes \mathrm{id} \otimes 1)$ is naturally equivalent to $(\operatorname{Ind}\pi) \otimes \mathrm{id} \otimes 1$. Putting these equivalences together gives a unitary W which satisfies

$$W(\pi_{\tau \otimes \mathrm{tr}})^\frown(a \otimes T)W^* = \widetilde{\pi}(a) \otimes T \otimes 1 \qquad \text{for } a \in A,\ T \in M_n, \tag{4.1}$$

and which commutes with the various projections P onto $H^2(\mathbf{R}, \cdot)$. Thus $T_{(a_{ij})}^{\pi_{\tau \otimes \mathrm{tr}}}$ is unitarily equivalent to $(T_{a_{ij}}) \otimes 1$ acting in $(H^2(\mathbf{R}, H_\tau)^n) \otimes \mathbf{C}^n$. Formulas (4.1) and (2.3) imply that $\operatorname{Ad} W$ carries $(\tau \otimes \mathrm{tr})^\frown$ into $\widehat{\tau} \otimes \mathrm{tr} \otimes 1 = \widehat{\tau}_n \otimes 1$. We can therefore deduce from Theorem 3.1 that if (a_{ij}) is invertible in $M_n(A)$, then $(T_{a_{ij}}) \in M_n(B(H^2(\mathbf{R}, H_\tau)))$ is $\widehat{\tau}_n$-Fredholm, with

$$\begin{aligned}\widehat{\tau}_n\text{-ind}\left(T_{a_{ij}}\right) &= (\widehat{\tau} \otimes \mathrm{tr} \otimes 1)\text{-ind}\left((T_{a_{ij}}) \otimes 1\right) \\ &= (\tau \otimes \mathrm{tr})^\frown\text{-ind}\, T_{(a_{ij})}^{\pi_{\tau \otimes \mathrm{tr}}} \\ &= \frac{-1}{2\pi i}\tau \otimes \mathrm{tr}\left((\delta(a_{ij}))(a_{ij})^{-1}\right).\end{aligned}$$

Remark 4.2. In case A is a non-commutative torus, this reduces to [13, Theorem 7]. To recover [8, §32.5] for $A = C(X)$, we need to show also that

$$\tau \otimes \mathrm{tr}\left((\delta(a_{ij}))\,(a_{ij})^{-1}\right) = \tau\left(\delta\left(\det(a_{ij})\right)\left(\det(a_{ij})\right)^{-1}\right). \tag{4.2}$$

To see this, we note that, since $A \otimes M_n = C(X, M_n)$, we can use the formula

$$(a_{ij})^{-1} = \left(\frac{1}{\det(a_{ij})}A_{ji}\right)_{i,j=1}^{n},$$

where A_{ij} is the cofactor of the (i,j)th entry in (a_{ij}). Thus the left-hand side of (4.2) is

$$\sum_i \tau\left(\sum_j \delta(a_{ij})(A_{ij})\right)\det(a_{ij})^{-1} = \sum_{i,j} \tau\left(\delta(a_{ij})A_{ij}\det(a_{ij})^{-1}\right). \tag{4.3}$$

On the other hand, the derivation property of δ gives

$$\delta\big(\det(a_{ij})\big) = \sum_i \det\big(a_{1j}\; a_{2j}\; \cdots\; \delta(a_{ij})\; \cdots\; a_{nj}\big),$$

where the ith summand is the determinant of the matrix (a_{ij}) with just the ith column differentiated. But now we can expand this determinant by the ith column to see that

$$\delta\big(\det(a_{ij})\big) = \sum_i \delta(a_{ij})A_{ij},$$

and the right-hand side of (4.2) is also equal to (4.3).

APPENDICES

A. Unbounded traces on C^*-algebras.

While these elementary results about traces on C^*-algebras do not appear in the standard references [16, 9], they are probably known to experts. In particular, we realise that some can be proved using the GNS-representation associated to the trace τ, and the extension of this trace to an ultraweakly lower semicontinuous trace on the enveloping von Neumann algebra, but we prefer to give self-contained proofs avoiding this machinery. We do use the following observation: if A is a C^*-algebra on a Hilbert space H, $x \in A$ and $x = v|x|$ is the polar decomposition of x on H, then $v|x|^n = x|x|^{n-1} \in A$ for all $n \geq 1$, so that $vp(|x|) \in A$ for any polynomial with $p(0) = 0$, and $vf(|x|) \in A$ for any continuous function f on $[0,\infty)$ with $f(0) = 0$. Of course, this conclusion is independent of the choice of representation of A on Hilbert space. In general, we follow [16] for notation and definitions.

Definition. A **weight** on a C^*-algebra A is a function $\phi : A_+ \to [0,\infty]$ such that

(a) $\phi(\lambda x) = \lambda\phi(x)$ for $x \in A_+$ and $\lambda \in [0,\infty)$, with the convention $0 \cdot \infty = 0$, and

(b) $\phi(x+y) = \phi(x) + \phi(y)$ for $x, y \in A_+$.

A weight τ is called **a trace** if $\tau(u^*xu) = \tau(x)$ for all $x \in A_+$ and all unitaries $u \in \widetilde{A}$. This differs from Dixmier's definition [9, 6.1.1], but by [16, 5.2.2] agrees if the weight is **lower semicontinuous** on A_+, in the sense that the set $\{x \in A_+ | \phi(x) \leq \lambda\}$ is norm closed in A_+ for each $\lambda \in \mathbf{R}_+$. (It is easy to see that a weight ϕ is lower semicontinuous if and only if $\phi(x) \leq \liminf \phi(x_n)$ whenever $x_n \to x$ in A_+.)

Notation. If ϕ is a weight on A, then we let $A_+^\phi = \{x \in A_+ | \phi(x) < \infty\}$, and say that ϕ is **densely defined** if A_+^ϕ is dense in A_+. We let $A^\phi = \operatorname{sp} A_+^\phi$, and note that A^ϕ is a hereditary *-subalgebra of A with $(A^\phi)_+ = A_+^\phi$ [16, 5.1.2]. We let $A_2^\phi = \{x \in A \,|\, x^*x \in A_+^\phi\}$, and observe that by [16, 5.1.2], A_2^ϕ is a left ideal of A containing A^ϕ, and $(A_2^\phi)^* A_2^\phi = A^\phi$. Moreover, if ϕ is a trace on A, then both A^ϕ and A_2^ϕ are ideals in A (see the discussion preceding 5.2.2 in [16]), and $A^\phi = (A_2^\phi)^2$.

Proposition A1. *Let τ be a trace on a C^*-algebra A, and suppose either that τ is lower semicontinuous, or that A is closed under polar decomposition. Then*

(a) $A^\tau = \{x \in A \mid \tau(|x|) < \infty\}$;

(b) $|\tau(x)| \leq \tau(|x|)$ for $x \in A^\tau$;

(c) $\|x\|_1 := \tau(|x|)$ defines a seminorm on A^τ.

Proof. (We suppose τ is lower semicontinuous, since the other case is easier.) If $\tau(|x|) < \infty$, let $x = v|x|$ be the polar decomposition of x. Then $x = (v|x|^{1/2})|x|^{1/2}$ with both

$v|x|^{1/2}$ and $|x|^{1/2}$ in A — indeed, both $v|x|^{1/2}$ and $|x|^{1/2}$ are in A_2^τ. Since τ is a lower semicontinuous trace, A_2^τ is a *-ideal [16, 5.2.2], and

$$x = (v|x|^{1/2})|x|^{1/2} \in (A_2^\tau)(A_2^\tau) = (A_2^\tau)^*(A_2^\tau) \subset A^\tau$$

by [16, 5.1.2]. Thus $\{x \in A \mid \tau(|x|) < \infty\} \subset A^\tau$.

Before proving the other containment, we show that $\tau(|x|) < \infty$ implies $|\tau(x)| \le \tau(|x|)$. By the first paragraph, $x \in A^\tau$ and $v|x|^{1/2}$, $|x|^{1/2}$ are both in the *-ideal A_2^τ. Thus from [16, 5.1.2] we get:

$$|\tau(x)|^2 = |\tau(v|x|^{1/2}|x|^{1/2})|^2 \le \tau(|x|^{1/2}v^*v|x|^{1/2})\tau(|x|^{1/2}|x|^{1/2}) = \tau(|x|)^2,$$

as required. (This is half of (b)).

We now show that if $x, y \in A$ then $\tau(|x+y|) \le \tau(|x|) + \tau(|y|)$; this and a trivial induction argument will show that $A^\tau :=$ span $A_+^\tau \subset \{x \in A \mid \tau(|x|) < \infty\}$, which will complete the proof of (a), and then also finish off parts (b) and (c). To this end, let $x, y \in A$ and suppose without loss of generality that $\tau(|x|) < \infty$, $\tau(|y|) < \infty$ and $\|x+y\|_{\text{op}} \le 1$. Let $x + y = w|x+y|$ be the polar decomposition of $x+y$. Then for each $\epsilon > 0$ we have $|x+y|^{1+\epsilon} = |x+y|^\epsilon w^*(x+y) = z(x+y)$, where $z^* = w|x+y|^\epsilon \in A$ and $\|z\|_{\text{op}} \le 1$, and

$$\begin{aligned}
\tau(|x+y|^{1+\epsilon}) &= |\tau(zx) + \tau(zy)| \quad \text{(because } zx \text{ and } zy \text{ are in } A^\tau) \\
&\le |\tau(zx)| + |\tau(zy)| \\
&\le \tau(|zx|) + \tau(|zy|) \quad \text{(by the second paragraph)} \\
&\le \tau(\|z\|_{\text{op}}|x|) + \tau(\|z\|_{\text{op}}|y|) \quad \text{(since } \ |ab| \le \|a\|_{\text{op}}|b|\ ) \\
&\le \tau(|x|) + \tau(|y|).
\end{aligned}$$

We now let $\epsilon \to 0$, and the semicontinuity of τ gives $\tau(|x+y|) \le \tau(|x|) + \tau(|y|)$, as required.

Notation. For $y \in A_2^\tau$ we define $\|y\|_2 = \tau(y^*y)^{1/2}$. By [16, 5.1.2], this is a pre-Hilbert space semi-norm.

Lemma A2. *If τ is a lower semicontinuous trace on A, then $\|xy\|_1 \le \|x\|_2\|y\|_2$ for every $x, y \in A_2^\tau$.*

Proof. Since $xy \in A^\tau$, we have $\tau(|xy|) < \infty$. Without loss of generality we may suppose that $\|xy\|_{\text{op}} = 1$. If $xy = w|xy|$ is the polar decomposition of xy, then for every $\epsilon > 0$ we have $|xy|^{1+\epsilon} = |xy|^\epsilon w^*(xy) = z(xy)$, where $z^* = w|xy|^\epsilon \in A$ and $\|z\|_{\text{op}} \le 1$. Thus

$$\begin{aligned}\tau(|xy|^{1+\epsilon}) &= |\tau(zxy)| \\ &\leq \tau(zxx^*z^*)^{1/2}\tau(y^*y)^{1/2} \quad ([16, 5.1.2]) \\ &= \tau(x^*z^*zx)^{1/2}\tau(y^*y)^{1/2} \quad ([16, 5.2.2]) \\ &\leq \|z^*z\|_{\mathrm{op}}^{1/2}\tau(x^*x)^{1/2}\tau(y^*y)^{1/2} \\ &\leq \|x\|_2\|y\|_2.\end{aligned}$$

Finally, we have

$$\|xy\|_1 = \tau(|xy|) \leq \liminf \tau(|xy|^{1+\epsilon}) \leq \|x\|_2\|y\|_2.$$

Lemma A3. *Let τ be a lower semicontinuous trace on A. If $\{x_\alpha\}$ is a bounded approximate identity for A lying in A^τ, then $\{x_\alpha\}$ is an approximate identity for $(A^\tau, \|\cdot\|_1)$.*

Proof. By [16, 5.1.3], the GNS-representation (π_τ, H_τ) is nondegenerate, so $\{\pi_\tau(x_\alpha)\}$ converges strongly to 1, and $\|x_\alpha c - c\|_2 \to 0$ for any $c \in A_2^\tau$. Thus for $y = cd \in (A_2^\tau)^2 = A^\tau$, we have

$$\|x_\alpha y - y\|_1 = \|(x_\alpha c - c)d\|_1 \leq \|x_\alpha c - c\|_2\|d\|_2 \to 0.$$

Proposition A4. *If τ is a lower semicontinuous trace on the C^*-algebra A, then A^τ is a Banach *-algebra in the norm $\|\cdot\|_\tau = \|\cdot\|_{\mathrm{op}} + \|\cdot\|_1$.*

Proof. Without loss of generality we can assume that τ is densely defined. It is trivial to verify that $(A^\tau, \|\cdot\|_\tau)$ is a normed algebra. To see that the involution is isometric, let $x \in A^\tau$, and $x = v|x|$ be the polar decomposition of x. Then $xx^* = v|x|^2v^*$, so that

$$|x^*| = (xx^*)^{1/2} = v|x|v^* = v|x|^{1/2}|x|^{1/2}v^*$$

and $v|x|^{1/2}, |x|^{1/2}v^*$ are in A_2^τ. By [16, 5.2.2], we have

$$\|x^*\|_1 = \tau(|x^*|) = \tau(v|x|^{1/2}|x|^{1/2}v^*) = \tau(|x|^{1/2}v^*v|x|^{1/2}) = \tau(|x|) = \|x\|_1,$$

and $\|x\|_\tau = \|x^*\|_\tau$. To see that $(A^\tau, \|\cdot\|_\tau)$ is complete, suppose $\{x_n\}$ is a Cauchy sequence. Then it is also Cauchy in the operator norm, and hence there exists $x \in A$ such that $\|x_n - x\|_{\mathrm{op}} \to 0$. Moreover, $\|x_n\|_1 = \tau(|x_n|)$ is bounded, and the lower semicontinuity of τ and the continuity of the square root imply that $\|x\|_1 = \tau(|x|) \leq \liminf \tau(|x_n|) < \infty$. Thus $x \in A^\tau$. To complete the proof we need the following lemma:

Lemma A5. *If τ is a lower semicontinuous trace on A, then for $x \in A^\tau$, $\widehat{x}(y) := \tau(yx)$ defines a bounded linear functional $\widehat{x}$ on A with $\|\widehat{x}\| = \|x\|_1$.*

Proof. As A^τ is an ideal in A, $\widehat{x}$ is certainly a linear functional on A. For $y \in A$, we have

$$|\widehat{x}(y)| = |\tau(yx)| \leq \tau(|yx|) \leq \|y\|\tau(|x|) = \|y\| \, \|x\|_1,$$

so that $\widehat{x} \in A^*$ and $\|\widehat{x}\| \leq \|x\|_1$. To see the other inequality, we can assume that $\|x\|_{\mathrm{op}} = 1$. As usual, we let $x = v|x|$, so that for every $\epsilon > 0$, $|x|^\epsilon v^* \in A$ and $\| \, |x|^\epsilon v^*\|_{\mathrm{op}} = 1$. Then $\widehat{x}(|x|^\epsilon v^*) = \tau(|x|^\epsilon v^* v|x|) = \tau(|x|^{1+\epsilon})$, so that $\|\widehat{x}\| \geq \tau(|x|^{1+\epsilon})$ for every $\epsilon > 0$, and the semicontinuity gives $\|x\|_1 = \tau(|x|) \leq \liminf \tau(|x|^{1+\epsilon}) \leq \|\widehat{x}\|$.

We now return to the proof that $(A^\tau, \|\cdot\|_\tau)$ is complete. The given sequence $\{x_n\}$ is also a Cauchy sequence in the $\|\cdot\|_1$ norm, and hence by the lemma $\{\widehat{x}_n\}$ converges to some $\omega \in A^*$. For each $y \in A^\tau$ we have

$$\begin{aligned} |\widehat{x}(y) - \widehat{x}_n(y)| &= |\tau(y(x - x_n))| \\ &= |\tau((x - x_n)y)| \\ &\leq \tau(|(x - x_n)y|) \\ &\leq \tau(\|(x - x_n)\|_{\mathrm{op}}|y|) \\ &= \|x - x_n\|_{\mathrm{op}}\tau(|y|) \\ &\to 0. \end{aligned}$$

Hence $\widehat{x}(y) = \omega(y)$ for all y in the dense subspace A^τ of A. Thus $\widehat{x} = \omega$, $\|x_n - x\|_1 = \|\widehat{x}_n - \widehat{x}\| = \|\widehat{x}_n - \omega\| \to 0$, and $\|x_n - x\|_\tau \to 0$. This completes the proof of Proposition A4.

B. Fredholm theory relative to a trace on a von Neumann algebra.

Let N be a von Neumann algebra, and $\tau : N_+ \longrightarrow [0, \infty]$ a faithful, normal, semifinite trace. We aim to show that, if we use τ as a dimension function, minor modifications of Breuer's arguments in [2, 3] give a Fredholm theory involving a real-valued index, which has the usual algebraic and topological stability properties, and in which the role of the compact operators is played by the ideal $\mathcal{K}_N$ generated by the projections of finite trace. When $N = B(H)$ and τ is the usual trace, we recover the classical Fredholm theory; when N is a II_∞-factor, our theory coincides with Breuer's. When N is not a factor Breuer's index is not real-valued; however, his is canonically associated with the von Neumann algebra N, whereas ours depends on the choice of τ.

A projection E in N will be called **τ-finite** if $\tau(E) < \infty$. Since the projections which are Murray-von Neumann equivalent to E all have the same trace, every τ-finite projection is finite in the usual algebraic sense. An operator $T \in N$ is **Fredholm relative to τ**, or **τ-Fredholm**, if the projection N_T on $\ker T$ is τ-finite and there is a τ-finite projection $E \in N$ with the range of $1 - E$ contained in the range of T. Since τ-finite projections are finite, every τ-Fredholm operator is Fredholm in Breuer's sense. If T is τ-Fredholm, the **τ-index** of T is by definition $\tau(N_T) - \tau(N_{T^*})$; we shall see below that T^* must also be τ-Fredholm, so this is always a well-defined real number.

We observe that the ideal $\mathcal{K}_N$ can also be described as the *closure* of any of: the span of the τ-finite projections in N; the span of the τ-finite positive elements in N; the algebra of elements $T \in N$ whose range projection R_T is τ-finite. Of course, the ideal $\mathcal{K}_N$ is contained in Breuer's ideal $\mathcal{K}$, which is generated by all the finite projections in N.

We now look in detail at how Breuer's arguments carry over. Lemmas 1 through 8 of [2] remain true with "finite" replaced by "τ-finite": the only tricky one is Lemma 5, where we must use the equivalence of $\sup(E, F) - E$ and $F - \inf(E, F)$ to see that $\sup(E, F)$ is τ-finite if both E and F are τ-finite. Lemma 9 is merely a tool in the proof of Lemma 10, which is trivially true with "finite" replaced by "τ-finite". Lemmas 11 and 12 are not germane to our discussion; Lemma 13 involves no finite projections and so escapes unaltered. The Corollary to Lemma 13 remains true with "Fredholm" and "finite" replaced by "τ-Fredholm" and "τ-finite", respectively, in both the statement and its proof. Lemma 1 of [3] trivially remains true if we assume S and T are τ-Fredholm and not merely Fredholm. We now formally state our analogue of Theorem 1 of [2] and Theorems 1 and 2 of [3].

Theorem B1. *Let τ be a faithful, normal, semifinite trace on a von Neumann algebra N, and let $\mathcal{K}_N$ be the ideal in N generated by the τ-finite projections.*

(a) (The Fredholm alternative.) *If $T \in \mathcal{K}_N$, then $1-T$ is τ-Fredholm, and*

$$\tau\text{-Ind}\,(1-T) = 0.$$

(b) (Atkinson's Theorem.) *An operator $T \in N$ is τ-Fredholm if and only if $T + \mathcal{K}_N$ is invertible in $N/\mathcal{K}_N$.*

(c) *If S and T are τ-Fredholm, then so are S^* and ST, and*

$$\tau\text{-Ind}\,S^* = -(\tau\text{-Ind}\,S), \qquad \tau\text{-Ind}\,ST = \tau\text{-Ind}\,S + \tau\text{-Ind}\,T.$$

Proof. (a) We begin as in the proof of [2, Theorem 1], by first supposing that $\tau(R_T) < \infty$, so that $E = \sup(R_T, R_{T^*})$ is also τ-finite. We follow Breuer down to (4.7), where we find that $\tau(N_{1-T}) = \tau(N_{1-T^*})$. As in [2], we then drop the assumption $\tau(R_T) < \infty$, and instead choose T_0 with R_{T_0} τ-finite and $\|T - T_0\|$ small, so that $S = 1 - (T - T_0)$ is invertible. We then follow Breuer exactly (noting that equation (4.14) is a misprint, and should read: $N_{S^* - T_0^*} \sim N_{1-(S^*)^{-1}T_0^*}$), until equation (4.15), where we conclude that $\tau(N_{1-T}) = \tau(N_{1-T^*})$ in the general case $T \in \mathcal{K}_N$. Part (iii) of Breuer's proof gives a τ-finite projection F such that the range of $1-T$ contains the range of $1-F$. Thus $1-T$ is τ-Fredholm and $\tau\text{-Ind}\,(1-T) = \tau(N_{1-T}) - \tau(N_{1-T^*}) = 0$.

(b) If T is invertible modulo $\mathcal{K}_N$, then we proceed as in the proof of [3, Theorem 1] through equations (1.9) to (1.13), replacing Breuer's reference to [2, Theorem 1] with part (a) of this theorem, and making the usual change of nomenclature, to conclude that T is τ-Fredholm. The converse also follows Breuer's argument closely, with his implicit use of [2, Theorem 1] in equation (1.14) again replaced by part (a) of this theorem. (As usual, this characterization implies that T is τ-Fredholm if and only if T^* is τ-Fredholm, and hence that $\tau\text{-Ind}\,T = \tau(N_T) - \tau(N_{T^*})$ is a well-defined real number.)

(c) The first equality is immediate. Since τ-Fredholm operators are also Fredholm operators, we can use the proof of [3, Theorem 2] without change down to equation (2.7). At this point we observe that each of the finite projections is, in fact, τ-finite, so that an application of the trace τ yields

$$\tau(N_S) - \tau(N_{ST} - N_T) = \tau(N_{T^*}) - \tau(N_{(ST)^*} - N_{S^*}),$$

which implies the second equality of part (c).

Corollary B2. *The set $\mathcal{F}_\tau(N)$ of τ-Fredholm operators is open in the norm topology of N, and the index map $T \mapsto \tau\text{-Ind}\,T$ is locally constant on $\mathcal{F}_\tau(N)$.*

Proof. Any $T \in GL(N)$ is trivially τ-Fredholm with $\tau\text{-Ind}\,T = 0$. Thus if $S \in \mathcal{F}_\tau(N)$ and $T \in GL(N) \subset \mathcal{F}_\tau(N)$, then $ST \in \mathcal{F}_\tau(N)$ and

$$\tau\text{-Ind}\,ST = \tau\text{-Ind}\,S + \tau\text{-Ind}\,T = \tau\text{-Ind}\,S.$$

Thus τ-Ind is constant on the open neighbourhood $S \cdot GL(N)$ of S.

Remark. If τ is a normal trace on N which is not necessarily faithful or semifinite, then N decomposes as a direct sum $N = N_0 \oplus N_1 \oplus N_\infty$, where τ is identically 0 on $(N_0)_+$, τ is identically $+\infty$ on $(N_\infty)_+$, and τ is faithful and semifinite on $(N_1)_+$. Thus, with some minor modifications to the definitions, one can get a τ-Fredholm theory for any normal trace τ.

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J. Phillips
Department of Mathematics
University of Victoria
British Columbia V8W 3P4
Canada

I. Raeburn
Department of Mathematics
University of Newcastle
New South Wales 2308
Australia