NILPOTENCY AND GENERALIZED LIE IDEALS

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0. INTRODUCTION

The motivation for this paper arises from two desires. The first is to generalize in a natural way the notion of a Lie inner ideal of an associative ring originally studied by Benkart [1]. We recall that a Lie inner ideal of a ring R is an additive subgroup V such that $[V,[V,R]] \subseteq V$. The second is to find a Lie–type notion that will include not only Lie ideals but will also include in an appropriate sense nilpotent associative subrings (since these sometimes arise in a Lie setting, e.g., strictly triangular matrices.)

The appropriate concept we are looking for appears to be that of a generalized Lie ideal which we proceed to define after making a few remarks concerning notation. Let V and W be subsets of a ring R. We define $V^{(m)}(W)$ inductively as follows: $V^{(1)}(W) = [V,W]$, $V^{(m+1)}(W) = [V,V^{(m)}(W)]$. The notation $V^{(m)}$ will mean $V^{(m-1)}(V)$. V^m will have its usual associative meaning.

DEFINITION. *An additive subgroup V of a ring R is a generalized Lie ideal* (GLI) *of index $\leq n$ if $V^{(n)}(R) \subseteq V$.*

Our main goal is to characterize generalized Lie ideals in a prime ring. The proof of this requires us to pass from a prime ring to its central closure, and so at this point we will fix some notation. Let R be a prime ring. Then Z will denote the center of R, C will

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denote the extended centroid of R, and $A = RC + C$ will denote the central closure of R. We fulfill our main goal in §3 by proving

THEOREM *3. If R is a prime ring of char. 0 and V is a GLI of index $\leq n$, then either (a) $V \subseteq Z$, or (b) $[J,R] \subseteq V$ for some nonzero ideal J of R, or (c) $V \subseteq C + S$ where $S \subseteq A$ and* $S^{3n-3} = 0$.

The proof of Theorem 3 requires Theorem 1 and Theorem 2, both of which are of independent interest, and these are taken up in §§ 1 and 2, respectively.

THEOREM *1. If R is a prime ring of* char. 0 *and V is a nonempty subset satisfying $V^{(n)}(R) = 0$ then $V \subseteq C + S$, where $S \subseteq A$ and* $S^m = 0$, $m = [n+1/2]$.

Before stating Theorem 2 we need to establish some notation concerning derivations. Let D be a nonempty collection of derivations of R. Then $\Delta = \delta_1 \delta_2 \cdots \delta_k$, $\delta_i \in D$, will denote a typical product of k derivations from D and we let $|\Delta| = k$. These mappings will be written on the right in the form of superscripts.

THEOREM *2. Let R be a prime ring, let $n > 0$ be fixed, let D be a nonempty set of derivations of R, and suppose $[x^\Delta, y^\Omega] = 0$ for all $x,y \in R$ and all $\Delta. \Omega$ such that $|\Delta| \geq n$ and $|\Omega| \geq n$. Then either R is commutative or $x^\Gamma = 0$ for all $x \in R$ and all Γ for which* $|\Gamma| \geq 3n - 1$.

The paper is largely self–contained, but we do assume the reader is familiar with the notion of extended centroid. We do cite in §1 a result of ours [4] on derivations, and in §3 a result of Herstein [2] on Lie structure.

1. LIE NILPOTENCY

For a nonempty subset V of a ring R we define $V^{(k)}(R)$ inductively by $V^{(1)}(R) = [V,R]$, $V^{(k+1)}(R) = [V, V^{(k)}(R)]$. We assume throughout this section that R is prime with extended centroid C and central closure $A = RC + C$. Our aim is to prove

THEOREM 1. *Let* R *be a prime ring of* char 0 *and let* V *be a nonempty subset of* R *satisfying*

$$(*) \quad V^{(n)}(R) = 0$$

for a fixed positive integer n*. Then* $V \subseteq C + S$*, where* S *is a subset of* A *such that* $S^m = 0$, $m = [n+1/2]$.

Before beginning the proof proper we will make some preparatory remarks. Without loss of generality we may assume that $n = 2m$ is even. It is immediately clear that (*) is equivalent to

$$(*) \quad V^{(q)}(R) = 0 \quad \text{for all} \quad q \geq n.$$

In particular $(\text{ad } v)^n = 0$ for all $v \in V$, and so by ([4], Corollary 1(b), p. 182) $(v-\lambda)^m = 0$, $v \in V$, $\lambda = \lambda(v) \in C$. We remark that λ is uniquely determined by v since $v - \lambda$, $v - \mu$ both nilpotent, $\lambda,\mu \in C$, implies that $(v-\lambda) - (v-\mu) = \mu - \lambda$ is nilpotent, which forces $\mu = \lambda$. We set $S = \{a \in A \mid a = v - \lambda,\ a^m = 0,\ v \in V\}$. We note that S also satisfies

$$(*) \quad S^{(q)}(R) = 0 \qquad q \geq n$$

and in addition to this that $a^m = 0$ for all $a \in S$. Letting x_ℓ and x_r denote the left and right multiplications of R respectively determined by the element x of R we may spell

(*) out in more detail.

$$(*)\quad (s_{1\ell}-s_{1r})(s_{2\ell}-s_{2r})\cdots(s_{q\ell}-s_{qr})=0$$

for all $s_i \in S$. We expand (*) to achieve an even more detailed version.

$$(*)\quad \sum(-1)^h\, s_{i_1\ell}s_{i_2\ell}\cdots s_{i_g\ell}\; s_{j_1r}s_{j_2r}\cdots s_{j_hr}=0, \qquad s_i\in S$$

where

$$i_1<i_2<\cdots<i_g,\ j_1<j_2<\cdots<j_h,\ g+h=q$$

$$\{i_1,\cdots,i_g,\ j_1,\cdots,j_h\}=\{1,2,\cdots,q\}$$

$$\{i_1,\cdots,i_g\}\cap\{j_1,\cdots,j_h\}=\emptyset.$$

(Although superficially cumbersome this is easily shown by induction on q).

By use of the isomorphism $A_\ell A_r \cong A \underset{C}{\otimes} A^{\circ}$, (*) can be translated to

$$(*)\quad \sum(-1)^h\, s_{i_1}s_{i_2}\cdots s_{i_g}\otimes s_{j_1}\bullet s_{j_2}\bullet\cdots\bullet s_{j_h}=0$$

where $x\bullet y=yx$, $x,y\in A$, is the multiplication in the opposite algebra A°. It is understood that if, e.g., $h=0$ then the corresponding summand in (*) is $s_1s_2\cdots s_q\otimes 1$.

Now choose and fix $a_1,a_2,\cdots,a_m\in S$. We will find it useful to partition q into $2m-1$ sumands

$$q=\sigma_m+\sigma_{m-1}+\cdots+\sigma_2+\sigma_1+\tau_2+\tau_3+\cdots+\tau_m\,.$$

Of course many of these summands may well be 0. This will dictate the type of substitutions we will make in (*) for $s_1, s_2, \cdots, s_q$:

$$\underbrace{a_m, a_m, \cdots, a_m}_{\sigma_{\mathrm{m}}};\ \underbrace{a_{m-1}, \cdots, a_{m-1}}_{\sigma_{\mathrm{m-1}}};\ \cdots;\ \underbrace{a_2, \cdots, a_2}_{\sigma_2};\ \underbrace{a_1, \cdots, a_1}_{\sigma_1};\ \underbrace{a_2, \cdots, a_2}_{\tau_2};\ \cdots;\ \underbrace{a_m, \cdots, a_m}_{\tau_{\mathrm{m}}}$$

With the above substitution (*) becomes

$$(*)\ \sum (-1)^h \epsilon_k a_m^{\alpha_m} a_{m-1}^{\alpha_{m-1}} \cdots a_1^{\alpha_1} a_2^{\beta_2} \cdots a_m^{\beta_m} \otimes a_m^{\gamma_m} \bullet a_{m-1}^{\gamma_{m-1}} \bullet \cdots \bullet a_1^{\gamma_1} \bullet a_2^{\delta_2} \bullet \cdots \bullet a_m^{\delta_m} = 0$$

where

$$\alpha_i + \gamma_i = \sigma_i\,, \quad \beta_i + \delta_i = \tau_i$$

$$h = \gamma_m + \cdots + \gamma_1 + \delta_2 + \cdots + \delta_m$$

$$\epsilon_k = \begin{bmatrix} \sigma_{\mathrm{m}} \\ \alpha_{\mathrm{m}} \end{bmatrix} \begin{bmatrix} \sigma_{\mathrm{m-1}} \\ \alpha_{\mathrm{m-1}} \end{bmatrix} \cdots \begin{bmatrix} \sigma_1 \\ \alpha_1 \end{bmatrix} \begin{bmatrix} \tau_2 \\ \beta_2 \end{bmatrix} \cdots \begin{bmatrix} \tau_{\mathrm{m}} \\ \beta_{\mathrm{m}} \end{bmatrix} .$$

The important thing for us is that the ϵ_k's are always nonzero, since char $R = 0$.

It is natural to focus attention on $\mathbb{N}^{2m-1}$, the set of all $(2m-1)$–sequences with components in $\mathbb{N} = \{0,1,2,\cdots\}$. The connection with the summands of (*) is given as follows. For $x = (\alpha_m, \alpha_{m-1}, \cdots, \alpha_1, \beta_2, \cdots, \beta_m) \in \mathbb{N}^{2m-1}$ we define

$$\bar{x} = a_m^{\alpha_m} a_{m-1}^{\alpha_{m-1}} \cdots a_1^{\alpha_1} a_2^{\beta_2} \cdots a_m^{\beta_m}$$

$$\tilde{x} = a_m^{\alpha_m} \bullet a_{m-1}^{\alpha_{m-1}} \bullet \cdots \bullet a_1^{\alpha_1} \bullet a_2^{\beta_2} \bullet \cdots \bullet a_m^{\beta_m} .$$

For an element x of $\mathbb{N}^{2m-1}$ (as detailed above) we also make the following definitions

$$\ell(x) = \alpha_m + \cdots + \alpha_2 + \alpha_1 + \beta_2 + \cdots + \beta_m$$

$$\ell_1(x) = \alpha_1$$

$$\ell_i(x) = \alpha_i + \beta_i \qquad i = 2,3,\cdots,m$$

$$h_i(x) = \max\{\alpha_i,\beta_i\} \qquad i = 2,3,\cdots,m\,.$$

We then define a partial ordering on $\mathbb{N}^{2m-1}$ in the following way. For any two elements $x,y \in \mathbb{N}^{2m-1}$ we compute ℓ, ℓ_1, ℓ_2, $\cdots$, ℓ_m, h_2, $\cdots$, h_m (in that order). At the first place where they disagree (if at all) (e.g., $\ell_5(x) \neq \ell_5(y)$) we can then say accordingly whether $x > y$ or $x < y$ (e.g., if $\ell_5(x) > \ell_5(y)$ then $x > y$). If ℓ, ℓ_1, $\cdots$, h_m all agree on x and y then x and y are not comparable. In this case we just say $x \sim y$ (x is equivalent to y). For example, if $m = 3$, we have $(11,2,3,4,1) < (4,7,3,1,6)$ whereas $(1,4,2,5,9) \sim (9,5,2,4,1)$.

Proof of Theorem 1. Our aim is to show that $a_1 a_2 \cdots a_m = 0$ for $a_i \in S$. We suppose this is not always the case, i.e., there is some product $\bar{v} = a_1 a_2 \cdots a_m \neq 0$ where $v = (0,0,\cdots,0,1,1,\cdots,1) \in \mathbb{N}^{2m-1}$ (m-1 0's and m 1's). On the other hand there exists $t \in \mathbb{N}^{2m-1}$, e.g. $t = (m,m,\cdots,m)$, for which $\bar{x} = 0$ and $\tilde{x} = 0$ for all $x \geq t$. Therefore we choose $z = (\alpha_m, \cdots, \alpha_2, \alpha_1, \beta_2, \cdots, \beta_m)$ maximal with respect to the condition that $\bar{z} \neq 0$ or $\tilde{z} \neq 0$. Without loss of generality $\bar{z} \neq 0$, since if $\tilde{z} \neq 0$ by setting $z_0 = (\beta_m, \cdots, \beta_2, \alpha_1, \alpha_2, \cdots, \alpha_m)$ we have $\bar{z}_0 = \tilde{z} \neq 0$, with $z_0 \sim z$. We do know (from the element v above) that $\ell(z) \geq m$.

We now set

$$Y_1 = \{y \in \mathbb{N}^{2m-1} \mid \ell_i(y) = \ell_i(z),\ \ i = 1,2,\cdots,m\}.$$

Writing $y = (\epsilon_m, \cdots, \epsilon_2, \epsilon_1, \rho_2, \cdots, \rho_m) \in Y_1$ we note that $\epsilon_1 = \alpha_1$. We remark also that $\ell(y) = \ell(z)$ for all $y \in Y_1$. Our aim is to show that for all $y \in Y_1$ we have $\tilde{y} = 0$. We assume this is not the case and proceed to choose a particular $y \in Y_1$ for which $\tilde{y} \neq 0$ according to the following process:

1. Suppose $h_1(z) = \alpha_2$ (i.e., $\alpha_2 \geq \beta_2$). Among all $y \in Y_1$ such that $\tilde{y} \neq 0$ let Y_2 consist of all those for which ϵ_2 is maximal. Suppose $h_2(z) = \beta_2$ (i.e., $\beta_2 > \alpha_2$). Among all $y \in Y_1$ such that $\tilde{y} \neq 0$ let Y_2 be all those for which ρ_2 is maximal.

2. If $h_3(z) = \alpha_3$ (i.e. $\alpha_3 \geq \beta_3$) let Y_3 be all those y in Y_2 for which ϵ_3 is maximal. If $h_3(z) = \beta_3$ (i.e., $\beta_3 > \alpha_3$) let Y_3 be all those y in Y_2 for which ρ_3 is maximal.

3. It is now clear how this process works and we thereby choose nonempty sets $Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq \cdots \supseteq Y_m$. Thus in Y_m we choose $y = (\epsilon_m, \cdots, \epsilon_2, \epsilon_1, \rho_2, \cdots, \rho_m)$ with ϵ_i or ρ_i chosen maximally at each step. In view of $\epsilon_i + \rho_i = \ell_i(y) = \ell_i(z) = \alpha_i + \beta_i$ it is clear that the y we have chosen in Y_m is in fact unique.

In (*) we now make the substitution

$$w = (\alpha_m + \epsilon_m, \cdots, \alpha_2 + \epsilon_2, \alpha_1 + \epsilon_1, \beta_2 + \rho_2. \cdots, \beta_m + \rho_m).$$

This is an allowable substitution since $\ell(w) = 2\ell(z) \geq 2m = n$. We proceed to examine which summands are nonzero.

Clearly $\bar{z} \otimes \tilde{y} \neq 0$. Let $\bar{x} \otimes \tilde{u}$ be any other nonzero summand appearing in (*), where we write $x = (\gamma_m, \cdots, \gamma_1, \delta_2, \cdots, \delta_m)$ and $u = (\lambda_m, \cdots, \lambda_1, \mu_2, \cdots, \mu_m)$. Clearly $x + u = w = z + y$. Thus $\ell(x) + \ell(u) = \ell(w) = \ell(z) + \ell(y) = 2\ell(z)$. If either

$\ell(x) > \ell(z)$ or $\ell(u) > \ell(z)$ then $x > z$, whence $\bar{x} = 0$ or $\tilde{u} = 0$. It follows that $\ell(x) = \ell(u) = \ell(z)$.

Next we claim that for all i, $\ell_i(x) = \ell_i(u) = \ell_i(z)$. If not let i be the first subscript for which this fails. We note first that $\ell_i(x) \leq \ell_i(z)$. (Otherwise $\ell_i(x) > \ell_i(z)$ implies $x > z$ and so $\bar{x} = 0$.) Similarly $\ell_i(u) \leq \ell_i(z)$ (otherwise $u > z$ and so $\tilde{u} = 0$). On the other hand we have $\ell_i(x) + \ell_i(u) = (\gamma_i+\delta_i) + (\lambda_i+\mu_i) = (\gamma_i+\lambda_i) + (\delta_i+\mu_i) = (\alpha_i+\epsilon_i) + (\beta_i+\rho_i) = (\alpha_i+\beta_i) + (\epsilon_i+\rho_i) = 2(\alpha_i+\beta_i) = 2\ell_i(z)$. It follows that $\ell_i(x) = \ell_i(u) = \ell_i(z)$, a contradiction to our present assumption.

We are now ready to embark on showing that $x = z$ and $u = y$.

Let us first suppose $h_2(z) = \alpha_2 = \beta_2$. If $\gamma_2 > \alpha_2$ then $h_2(x) > h_2(z)$ and so $x > z$, whence $\bar{x} = 0$. If $\gamma_2 < \alpha_2$ then from $\gamma_2 + \lambda_2 = \alpha_2 + \epsilon_2$ we have $\lambda_2 > \epsilon_2$. Since $u \in Y_1$ we then have $\tilde{u} = 0$ by the maximality of ϵ_2. Therefore in this case $\gamma_2 = \alpha_2$ (and hence $\lambda_2 = \epsilon_2$, $\delta_2 = \beta_2$, $\mu_2 = \rho_2$).

Let us next suppose that $h_2(z) = \beta_2$. If $\delta_2 > \beta_2$ then $h_2(x) > h_2(z)$ and so $x > z$ whence $\bar{x} = 0$. If $\delta_2 < \beta_2$ then from $\delta_2 + \mu_2 = \beta_2 + \rho_2$ we have $\mu_2 > \rho_2$. Since $u \in Y_1$ we have $\tilde{u} = 0$ by the maximality of ρ_2. Thsu $\delta_2 = \beta_2$ (and hence $\mu_2 = \rho_2$, $\gamma_2 = \alpha_2$, $\lambda_2 = \epsilon_2$).

So far we have $x = (\cdots, \alpha_2, \alpha_1, \beta_2, \cdots)$ and $u = (\cdots, \epsilon_2, \epsilon_1, \rho_2, \cdots)$.

We spell out the details of the process for the next step. Suppose $h_3(z) = \alpha_3$. If $\gamma_3 > \alpha_3$ then $h_3(x) > h_3(z)$ and so $x > z$, whence $\bar{x} = 0$. If $\gamma_3 < \alpha_3$ then from $\gamma_3 + \lambda_3 = \alpha_3 + \epsilon_3$ we have $\lambda_3 > \epsilon_3$. Since $u \in Y_2$ we have the contradiction $\tilde{u} = 0$ by the maximality of ϵ_3. Therefore $\gamma_3 = \alpha_3$ (and hence $\lambda_3 = \epsilon_3$, $\delta_3 = \beta_3$, $\mu_3 = \rho_3$). Suppose $h_3(z) = \beta_3$. If $\delta_3 > \beta_3$ then $h_3(x) > h_3(z)$ and so $x > z$, whence $\bar{x} = 0$. If $\delta_3 < \beta_3$ then from $\delta_3 + \mu_3 = \beta_3 + \rho_3$ we have $\mu_3 > \rho_3$. Since $u \in Y_2$ we have the contradiction $\tilde{u} = 0$ by the maximality of ρ_3. Thus $\delta_3 = \beta_3$ (and hence $\mu_3 = \rho_3$, $\gamma_3 = \alpha_3$, $\lambda_3 = \epsilon_3$).

At this point we have

$$x = (\cdots, \alpha_3, \alpha_2, \alpha_1, \beta_2, \beta_3, \cdots)$$

and

$$u = (\cdots, \epsilon_3, \epsilon_2, \epsilon_1, \rho_2, \rho_3, \cdots).$$

By continuing the process it is clear that we are led to $x = z$ and $u = y$, and therefore the only nonzero summand appearing in (*) is just $\tilde{z} \otimes \tilde{y}$ itself. Thus (*) reduces to $\epsilon\, \tilde{z} \otimes \tilde{y} = 0$, where ϵ is an appropriate nonzero scalar in view of char $R = 0$, and we are left with an evident contradiction. Thus we have accomplished our aforementioned aim of showing that $\tilde{y} = 0$ for all $y \in Y_1$. In particular $z_0 = (\beta_m, \cdots, \beta_2, \alpha_1, \alpha_2, \cdots, \alpha_m)$, which lies in Y_1 since $\ell_i(z_0) = \beta_i + \alpha_i = \ell_i(z)$, is such that $\tilde{z}_0 = 0$. But $\tilde{z}_0 = \tilde{z} \neq 0$ which gives us our final contradiction, and we are forced to conclude that $a_1 a_2 \cdots a_m = 0$ for all $a_i \in S$. From the very definition of S we then have $V \subseteq C + S$ and the proof of the theorem is now complete. ■

A careful examination of the proof shows that $q \leq m(2m-1)$ and accordingly, in view of the expressions for the coefficients ϵ_k in (*), the theorem also holds for char $R > m(2m-1)$.

2. COMMUTING PRODUCTS OF DERIVATIONS

Let R be any ring and let Der R be the set of all derivations of R. We let $\Delta = \delta_1\delta_2\cdots\delta_n$ denote the product of $\delta_1,\delta_2,\cdots,\delta_n \in$ Der R and let $|\Delta| = n$. For $x,y \in R$ we have the Leibnitz Formula

$$(*) \quad (xy)^{\Delta} = \sum_{\Delta_i} x^{\Delta_i} y^{\Delta_{n-i}}$$

where

$$\Delta_i = \delta_{j_1} \delta_{j_2} \cdots \delta_{j_i}, \quad j_p < j_{p+1}$$

$$\Delta_{n-i} = \delta_{k_1} \delta_{k_2} \cdots \delta_{k_{n-i}}, \quad k_q < k_{q+1}$$

$$\{j_1,j_2,\cdots,j_i, k_1,k_2,\cdots, k_{n-i}\} = \{1,2,\cdots,n\}$$

$$\{j_1,j_2,\cdots j_i\} \cap \{k_1,k_2,\cdots, k_{n-i}\} = \emptyset.$$

It is understood that Δ_0 (product of no derivations) is just the identity map, i.e., $x^{\Delta_0} = x$. We omit the proof but suggest that if the reader verifies the details for $n = 1,2,3$ he will readily see why this formula holds.

We now let D be a given nonempty subset of Der R, we fix a positive integer n and we let W denote the associative subring of R generated by all elements of the form x^{Δ}, $\Delta = \delta_1\delta_2 \cdots \delta_m$, $|\Delta| \geq n$, where x varies over R and $\delta_1,\delta_2,\cdots,\delta_m$ vary in D. We first prove a lemma of a technical nature.

Lemma *A. For* $\ell = 1,2,\cdots,n$ $x^{\Omega_{n-\ell}} y^{\Lambda_{2n+\ell-1}} \in W$ *where* $x,y \in R$ *and* $\Omega_{n-\ell}$ *and* $\Lambda_{2n+\ell-1}$ *are products of elements of* D *of lengths* $n-\ell$ *and* $2n+\ell-1$ *respectively.*

Proof. The proof is by induction on ℓ. For $|\Delta| = n$ we write the Leibnitz formula (*) as

$$(*) \quad (xy)^{\Delta} = xy^{\Delta} + \sum_{|\Delta_i|>0} x^{\Delta_i} y^{\Delta_{n-i}}$$

Replacing x by x^{Φ} and y by y^{Γ} in (*), where $|\Phi| = n-1$ and $|\Gamma| = n$, we obtain

$$(*) \quad (x^{\Phi}y^{\Gamma})^{\Delta} = x^{\Phi}\, y^{\Gamma\Delta} + \sum_{|\Delta_i|>0} x^{\Phi\Delta_i} y^{\Gamma\Delta_{n-i}} .$$

It follows that $x^{\Phi}\, y^{\Gamma\Delta} \in W$, $|\Phi| = n-1$, $|\Gamma\Delta| = (2n+1)-1$ and so the lemma has been proved for $\ell = 1$.

We now replace x by x^{Φ} and y by y^{Γ} in (*), where $|\Phi| = n-\ell$ and $|\Gamma| = n+\ell-1$, and obtain

$$(*) \quad (x^{\Phi}y^{\Gamma})^{\Delta} = x^{\Phi}\, y^{\Gamma\Delta} +$$

$$\sum_{0<i<\ell} x^{\Phi\Delta_i} y^{\Gamma\Delta_{n-i}} + \tag{a}$$

$$\sum_{i\geq\ell} x^{\Phi\Delta_i} y^{\Gamma\Delta_{n-i}} . \tag{b}$$

Each summand in (a) lies in W by the induction hypothesis since $|\Phi\Delta_i| = n-\ell+i = n-(\ell-i)$ and $|\Gamma\Delta_{n-i}| = n+\ell-1+n-\mathrm{i} = 2\mathrm{n}+(\ell-\mathrm{i})-1$ and $0 < \ell-i < \ell$. Each summand of (b) lies in W since

$$|\Phi\Delta_i| = n - \ell + i \geq n \text{ and } |\Gamma\Delta_{n-i}| = n + \ell - 1 + n - i \geq n$$

It follows that $x^{\Phi}\, y^{\Gamma\Delta} \in W$ which, in view of $|\Phi| = n - \ell$ and $|\Gamma\Delta| = n + \ell - 1 + n = 2n + \ell - 1$, completes the proof of the lemma. ■

It is now an easy matter to prove the main result of this section (which we may remark is a generalization of [4], Theorem 3).

THEOREM 2. *Let R be a prime ring, let $n > 0$ be fixed, let D be a nonempty subset of* Der R, *and suppose $[x^{\Delta}, y^{\Omega}] = 0$ for all $x, y \in R$ and all Δ, Ω such that $|\Delta| \geq n$ and $|\Omega| \geq n$. Then either R is commutative or $x^{\Gamma} = 0$ for all $x \in R$ and all Γ for which $|\Gamma| \geq 3n - 1$.*

Proof. Letting W denote the subring generated by all x^{Δ}, $x \in R$, $|\Delta| = n$, we see from our hypothesis that W is a commutative subring of R. Setting $\ell = n$ in Lemma A we have $R\, y^{\Omega} \subseteq W$ for all $y \in R$ and $|\Omega| = 3n - 1$. If $y^{\Omega} \neq 0$ for some $y \in R$ and some $|\Omega| = 3n - 1$ then $R\, y^{\Omega}$ is a commutative left ideal of R. This forces R to be commutative, and the proof is now complete. ■

3. GENERALIZED LIE IDEALS

We let R be a ring and recall that a generalized Lie ideal of index $\leq n$ is an additive subgroup V of R such that $V^{(n)}(R) \subseteq V$. For $n = 1$ we simply have the notion of an ordinary Lie ideal and for $n = 2$ we see that V is a Lie inner ideal in the sense of Benkart [1].

Following Benkart, for any additive subgroup V of R we define

$$T(V) = \{t \in R \mid [V,[t,R]] \subseteq V\}.$$

Lemma B. *$T(V)$ is (a) an associative subring and (b) a Lie inner ideal of R.*

Proof. For $t,u \in T$, $r \in R$, $v \in V$ we have $[v,[tu,r]] = [v,[t,ur]] + [v,[u,rt]] \in V$ which establishes (a). To prove (b), for $t,u \in T$, $x,r \in R$, $v \in V$, we commute v with the equation

$$[[t,[u,r]],x] = [[t,x],[u,r]] + [t,[[u,r],x]]$$

to obtain

$$[[v,[t,x]],[u,r]] + [[t,x],[v,[u,r]]] + [v,[t,[[u,r],x]]] \in V. \blacksquare$$

If $r_1, r_2, \cdots, r_k \in R$ the notation $[r_1,r_2, \cdots, r_k]$ will designate any one of the various k-fold Lie products of $r_1, r_2, \cdots, r_k$. An obvious induction combined with the Jacobi identity yields the following result.

Lemma C. *For $v_1,v_2, \cdots, v_k, \ r \in R$*

$$[[v_1,v_2,\cdots, v_k],r] = \sum \pm [w_1,\cdots [w_{k-1},[w_k,r]]\cdots]$$

where $w_1, w_2, \cdots, w_k$ *is a suitable permutation of* $v_1, v_2, \cdots, v_k$.

In particular $[[V,V],R] \subseteq V$ *if* V *is a Lie inner ideal of* R *and* $V^{(n-1)} \subseteq T(V)$ *if* V *is a* GLI *of index* $\leq n$.

For completeness we now reprove a result of Benkart.

Lemma D. ([1], Theorem 3.7). *Let* R *be a prime ring of* char $\neq 2$ *and let* P *be both a subring and a Lie inner ideal of* R. *Then either* P *contains a nonzero ideal of* R *or* $[P,P] = 0$.

Proof. We let $x \in [P,P]$, $y \in P$, and $r \in R$. From $[x,yr] = [x,y]r + y[x,r]$ we see that $[x,y]R \subseteq P$. Using this fact we conclude next from expanding $[[r_0,[x,y]r_1]\ [x,y]r_2] \in P$, $r_0,r_1,r_2 \in R$, that $r_0[xy]r_1[x,y]r_2 \in P$. In other words for all $r_1 \in R$ we have $I = R[x,y]r_1[x,y]R \subseteq P$. If for some r_1, $I \neq 0$ we are finished, and so we may now assume $[x,y]R[x,y] = 0$ whence $[x,y] = 0$ for all $x \in [P,P]$ and $y \in P$. In particular for $y = [x,r] \in P$ we see that $[x[x,r]] = 0$ for all $x \in [P,P]$ and $r \in R$. It is then well known that $x \in Z$, the center of R, thus we now have $[P,P] \subseteq Z$. Now suppose $x = [u,v] \neq 0$ for some $u,v \in P$. Then $u[u,v] = [u,uv] \in Z$ (since $uv \in P$) which forces the contradiction that $u \in Z$. ■

We come now to the main result of this section. For a prime ring R we remind the reader that Z denotes the center, C denotes the extended centroid and $A = RC + C$ denotes the central closure.

Theorem 3. *Let* R *be a prime ring of* char. 0 *and let* V *be a generalized Lie ideal of index* $\leq n$. *Then one of the following must occur:*

(*a*) $V \subseteq Z$

(*b*) $[J,R] \subseteq V$ *for some nonzero ideal* J *of* R.

(*c*) $V \subseteq C + S$, *where* S *is a subset of* A *such that* $S^{3n-3} = 0$.

Proof. We form $T = T(V)$ which by Lemma B is a subring and a Lie inner ideal. By Lemma D we know that either (i) T contains a nonzero ideal I of R or (ii) $[T,T] = 0$.

In case (i) we have $[V,U] \subseteq V$ where $U = [I,R]$ is a Lie ideal of R. By ([2], Theorem 5), either $[V,U] = 0$ or $[J,R] \subseteq V$ for some nonzero ideal J of R. This latter possibility is just conclusion (b) of the theorem. If $[V,U] = 0$ then in particular $[v,[v,I]] = 0$ for all $v \in V$. In this case it is well known that $V \subseteq Z$ which is just conclusion (a).

In case (ii) we see from Lemma C that $V^{(n-1)} \subseteq T$ and hence $[V^{(n-1)},V^{n=1}] = 0$. We let D be the set of all derivations of the form ad v, $v \in V$, and we let Δ and Γ be arbitrary products of elements of D such that $|\Delta| = |\Gamma| = 2n-2$. Using the fact that $V^{(n)}(R) \subseteq V$ we see that $x^{\Delta} \in V^{(n-1)}$ and $y^{\Gamma} \in V^{(n-1)}$ for all $x,y \in R$. Therefore we have $[x^{\Delta},y^{\Gamma}] = 0$ for all $x,y \in R$ and so by Theorem 2 we conclude that $V^{(6n-7)}(R) = 0$. By Theorem 1 we then have $V \subseteq C + S$, where S is a subset of A such that $S^{3n-3} = 0$. This is just conclusion (c) and our proof is complete. ■

Our remark at the end of §1 indicating that Theorem 1 holds for suitably high characteristic carries the obvious implication that Theorem 3 also holds for char. R greater than a suitable function of n which can readily be determined.

Another remark we can make at this point is that without loss of generality the subset S appearing in conclusion (c) of Theorem 3 may be assumed to be a nilpotent subring of index $\leq 3n-3$.

We close this paper with several corollaries.

Corollary 3a. *If* R *is simple of* char. 0 *and* V *is a* GLI *of index* $\leq n$ *then one of the following holds*:

(*a*) $V \subseteq Z$

(*b*) $[R,R] \subseteq V$

(*c*) $V \subseteq C + S$, *where* C *is the centroid of* R *and* S *is a nilpotent subring of* $R + C$ *of index* $\leq 3n - 3$.

Corollary 3b. *If* $R = M_n(F)$, F *a field of* char. 0, *and if* V *is a* GLI *of* R, *then either*

(*i*) $[R,R] \subseteq V$, *or*

(*ii*) *there is an invertible matrix* τ *such that* $\tau^{-1} V \tau \subseteq F1 + N$, *where* N *is a strictly upper triangular subring.*

Proof. If (c) of Corollary 3a occurs then by a well known theorem of Levitzki [3] the subring S is strictly upper triangularizable. Conclusion (a) of Corollary 3a is also included under (ii).

Corollary 3c. *If* R *is a division ring of* char. 0 *and* V *is a GLI of* R, *then either* $[R,R] \subseteq V$ *or* $V \subseteq Z$.

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