The Heisenberg Spectral Triple and Associated Zeta Functions

by

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ABSTRACT

The construction of Butler, Emerson, and Schultz [2] produced a certain spectral triple, which they called the *Heisenberg cycle*, by way of the quantum mechanical annihilation and creation operators, $\frac{d}{dx} \pm x$, along with their relationships to the harmonic oscillator, $-\frac{d^2}{dx^2} + x^2$; Where all of these operators are defined (initially) to act on smooth functions over $\mathbb{R}$. In particular, their Heisenberg cycle was over a crossed-product generated by the natural translation action on the (commutative) C^*-algebra of uniformly continuous, bounded, functions on $\mathbb{R}$.

In this thesis, we generalize the Heisenberg cycle of Butler, Emerson, and Schultz to allow for the construction of a spectral triple over a crossed-product generated by the natural translation action on the C^*-algebra of uniformly continuous, bounded, functions on a Euclidean space, V, of arbitrary finite dimension n. For such a generalization, the annihilation and creation operators are replaced using the exterior derivative and codifferential, exterior and interior multiplication by a certain differential 1-form, and the relationship these four operators have to the n-dimensional harmonic oscillator acting on differential forms. Similarly to [2], we will show that our generalized Heisenberg cycle provides a new way of producing spectral triples over crossed-products of the form $C(M) \rtimes_\alpha \Gamma$, where Γ is a discrete subgroup of V and $\alpha : V \times M \to M$ is a smooth V-action on a compact manifold M.

In Chapter 1, we introduce the problem and briefly discuss some historical background behind Alain Connes program of noncommutative geometry, as well as touch on some elementary constructions in multi-linear algebra. Chapter 2 is where we define the classes of differential forms which appear most frequently in this thesis. Therein, we also rigorously define the operators mentioned in the paragraph above, and use them to produce the so-called Dirac-Heisenberg which will be associated to our generalization of the Heisenberg cycle. For the first half of Chapter 3, we discuss some basic C^*-algebra theory and introduce the crossed-product native to the Heisenberg cycle. In the latter half of that chapter, we verify that our Heisenberg cycle satisfies the conditions of a spectral triple, compute an integral formula for the resulting ζ-functions, and show how one uses the Heisenberg cycle to produce spectral triples over crossed-products generated by smooth actions of V on compact manifolds.

Table of Contents

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Chapter 1
Introduction and Preliminaries

1.1 Introduction

The connections and correspondences between geometric spaces and commutative algebras is a well-studied phenomena in mathematics, and a focal point for branches such as algebraic and differential geometry. For instance, the Gelfand-Naimark theorem implies that the category of locally compact Hausdorff spaces is dual to the category of commutative C^*-algebras. It is therefore reasonable to say that, in some sense, the topology of locally compact Hausdorff spaces is *commutative*. Such a philosophical stance allows one to view noncommutative C^*-algebras as a representative for more generalized notion of space, and has birthed the mathematical field of noncommutative topology.

Field Medalist Alain Connes program for *noncommutative geometry*, initiated in the 1980's, rests on the aforementioned idea and seeks to extend classical tools such as measure theory, differential calculus, and Riemannian geometry to the noncommutative situation. Such extensions require algebraic reformulations of the aforementioned tools but, in general, this can be notoriously difficult. On one hand, new interesting phenomena arise in the noncommutative case, such as the existence of canonical time evolutions for noncommutative measure spaces. On the other, constraints arising from development of these theories to the noncommutative case lead to new points of view and tools, even in the commutative case, such as cyclic cohomology and quantized differential calculus, which, unlike the theory of distributions, realizes products and gives meaning to expressions like $\int f(Z)|dZ|^p$ where Z is not differential and p is not necessarily an integer (Connes [3, Chapter I, V]).

Of particular importance to this thesis is Connes noncommutative extension of Riemannian geometry, encoded through what are called *spectral triples*, which pertain to exclusively operator theoretic information. The following is Connes original definition, and were called *K-cycles* as opposed to spectral triples.

Definition 1.1 (Connes [3]). A *K-cycle* over a unital $*$-algebra $\mathcal{A}$ is a triple,

$$(\mathfrak{H}, \pi, D),$$

where $\mathfrak{H}$ is a Hilbert space, π is a $*$-algebra representation of $\mathcal{A}$ on $\mathfrak{H}$ by bounded operators, and D is an unbounded self-adjoint operator with compact resolvent, such that the commutator $[D, \pi(a)]$ is bounded for any a in $\mathcal{A}$. A K-cycle $(\mathfrak{H}, \pi, D)$ is called *(n, ∞)-summable* if the eigenvalues μ_k of $|D|$ are of the order of $k^{\frac{1}{n}}$ as $k \to \infty$.

The motivating (commutative) example behind Connes K-cycles is that associated to an n-dimensional compact Riemannian spin manifold M: One constructs an n-dimensional spectral triple over the von Neumann algebra $\mathcal{A}$ of bounded measurable functions on M by taking $\mathfrak{H} := L^2(M; S)$ to be the Hilbert space of L^2-spinors over M, D to be the associated Dirac operator (cf. Gilkey [7]), and $\pi : \mathcal{A} \to \mathbb{B}(\mathfrak{H})$ the representation of $\mathcal{A}$ by multiplication operators. While independently one is not able to glean very much information from the data $(\mathfrak{H}, \pi)$ or D, the cumulative triple $(\mathfrak{H}, \pi, D)$ just described encodes enough information to reconstruct the metric space (M, d), with d the geodesic distance, the volume measure dv on M, the space of gauge potentials, and the Yang-Mills action functional [3, Section 6.1]. Most importantly, the metric is determined by the formula,

$$d(p, q) = \sup \left\{ |a(p) - a(q)| \;:\; a \in \mathcal{A}, \|[D, a]\|_{\mathbb{B}(\mathfrak{H})} \leq 1 \right\} \quad \text{for all} \quad p, q \in M,$$

where $\|[D, \pi(a)]\|_{\mathbb{B}(\mathfrak{H})}$ is the operator norm of the bounded commutator $[D, \pi(a)]$, while a formula for integration is given by,

$$\int_M f \; dv = c(n) \, \mathrm{Tr}_\omega(f |D|^{-n}) \quad \text{for all } f \in \mathcal{A}, \tag{1.1}$$

where $c(n)$ is a constant depending only on the dimension $n := \dim(M)$, and Tr_ω is the Dixmier trace [3, Chapter 4.2].

A more modern definition of spectral triples, and the one which we take, is found in Emerson [5]; Such a definition requires suitable hypotheses on the operator D and the $*$-algebra $\mathcal{A}$ which then give rise to a noncommutative analogue of integration on manifolds using the *residue trace*. In what follows, we fix the standard branch of λ^{-s} defined for a complex number s with $\mathrm{Re}(s) > 0$. If D is an unbounded self-adjoint invertible operator with discrete spectrum and finite spectral multiplicities, then we may apply λ^{-s} to $|D|$ using functional calculus. If the unbounded self-adjoint operator D is not invertible, then we let $|D|^{-s}$ denote the functional calculus application of λ^{-s} to the operator $|D| + p_{\ker(D)}$, where $p_{\ker(D)}$ is the orthogonal projection onto the kernel of D. For the remainder of this thesis, when we say *spectral triple*, we will mean of the kind given below.

Definition 1.2 ([5])**.** Let $\mathcal{A}$ be a unital C^*-algebra, and $\mathcal{A}^\infty \subseteq A$ a dense $*$-subalgebra. For $n \geq 1$, an *n-dimensional even spectral triple over $\mathcal{A}^\infty \subseteq A$* is a triple

$$(\mathfrak{H}, \pi, D)$$

consisting of a $\mathbb{Z}/2$-graded Hilbert space $\mathfrak{H}$, a representation $\pi : \mathcal{A} \to \mathbb{B}(\mathfrak{H})$ by even operators, and an unbounded self-adjoint operator D, which is odd with respect to the grading and has domain invariant under $\pi(\mathcal{A}^\infty)$, such that $[\pi(a), D]$ is bounded for all a in $\mathcal{A}^\infty$ and furthermore,

a) $(1 + D^2)^{-\frac{p}{2}}$ is trace-class for all $p > n$.

b) The analytic ζ-function,

$$s \longmapsto \operatorname{Tr}\left(\pi(a)|D|^{-s}\right), \quad \operatorname{Re}(s) > n,$$

extends to a meromorphic function on $\mathbb{C}$ for each a in $\mathcal{A}^\infty$.

An *odd* spectral triple is defined the same way, except one drops the assumptions of a $\mathbb{Z}/2$-grading. Further, *pre*-spectral triple is one for which condition *b*) above is dropped.

Remark 1.3. Definition 1.2 is similar to the original definition of Connes', given in Definition 1.1, with the differences being specification of a C^*-algebra containing a dense $*$-subalgebra, as well as conditions *a*) and *b*). However, the notion of (n, ∞)*-summability* used in Definition 1.1 is slightly stronger than condition *a*), with (n, ∞)-summability often holding in practice when constructing spectral triples; In the literature, condition *a*) is called *finite summability*, while condition *b*) is *the meromorphic continuation property.*

The meromorphic continuation property above implies that spectral triples endow the C^*-algebra which they are over with a certain trace, called the *residue trace.* This approach to constructing a trace is alternative to the Dixmier trace method originally used by Connes. For a proof of the following theorem, see [5, Theorem 9.6.11].

Theorem 1.4. *If $(\mathfrak{H}, \pi, D)$ is an n-dimensional spectral triple over $\mathcal{A}^\infty \subseteq \mathcal{A}$, then*

$$\operatorname{Res}\operatorname{Tr}(a) := \operatorname{Res}_{s=n} \operatorname{Tr}\left(\pi(a)|D|^{-s}\right)$$

is a positive trace on $\mathcal{A}^\infty$, called the residue trace. *In particular,* Res Tr *extends to a (positive) trace on $\mathcal{A}$.*

In the classical situation, with M an n-dimensional Riemannian spin^c-manifold, Equation (1.1) shows that the Dixmier trace method, originally used by Connes in conjunction with his K-cycles, produces a functional on the commutative $*$-algebra of bounded measurable functions over M which encodes important geometric information; Namely the volume measure associated to M. One should anticipate then that not only does this classical situation fit into the framework of spectral triples, as defined above, but also that the alternative residue-trace method of Theorem 1.4 produces a linear functional which again encodes the volume measure of M.

Indeed, the residue trace does, in the classical situation, encode the volume measure: If M is an n-dimensional compact Riemannian spin^c-manifold, and D_E a twisted Dirac operator on M *i.e.* a twist of the Dirac operator by a vector bundle $E \to M$ acting as an odd unbounded self-adjoint operator on the $\mathbb{Z}/2$-graded Hilbert space $L^2(M; S \otimes E)$ (cf. [5, Definition 8.6.1]), then an n-dimensional even spectral triple is given by,

$$\left(L^2(M; S \otimes E), \pi, D_E\right),$$

where $\pi : C(M) \to \mathbb{B}(L^2(M; S \otimes E))$ is the $*$-representation by multiplication operators. Moreover, the following result, originally based on a theorem of Weyl, shows that this triple admits meromorphic extension and the resulting residue-trace of Theorem 1.4 encodes the volume measure. For a proof, see Roe [15].

Theorem 1.5 (Weyl)**.** *Let M be an n-dimensional compact Riemannian spin^c-manifold, and D_E a twisted Dirac operator. If f is in $C^\infty(M)$, then the analytic function,*

$$\mathrm{Tr}\left(f|D|^{-s}\right), \quad \mathrm{Re}(s) > n,$$

extends meromorphically to $\mathbb{C}$, has a simple pole at $s = n$, and

$$\mathrm{Res}\,\mathrm{Tr}(f) := \mathrm{Res}_{s=n}\,\mathrm{Tr}(f|D|^{-s}) = c'(n) \int_M f dv,$$

where dv is the volume measure on M, and $c'(n)$ is a constant depending only on the dimension n.

Justification that the meromorphic extension property holds in this classical context of a Riemannian spin^c-manifold M relies on asymptotic expansions of the heat kernel, a kernel for the integral operator $e^{-t\Delta}$, where $\Delta = {D_E}^2$ and D_E is a twisted Dirac operator. Weyl's result points to the philosophical idea that a spectral triple over $\mathcal{A}^\infty \subseteq \mathcal{A}$, may endow, in a sense, the corresponding 'noncommutative space' with an analogue of Riemannian geometric structure.

From a dynamical systems perspective, a rich family of noncommutative C^*-algebras, and therefore a relevant family of 'noncommutative spaces' one may wish to endow with the structure of a spectral triple, is that of crossed-products. For instance, if we are interested in the set of all orbits of points in $\mathbb{T}$ under the $\mathbb{Z}$-action of rotation by an irrational angle $\hbar$, then the natural parametrization of this set, given by the quotient $\mathbb{T}/\mathbb{Z}$ has trivial topology, and the commutative C^*-algebra $C(\mathbb{T}/\hbar\mathbb{Z})$ is isomorphic to

$\mathbb{C}$. On the other hand, the well-known C^*-algebra crossed-product $A_\hbar := C(\mathbb{T}) \rtimes_\hbar \mathbb{Z}$, called the *irrational rotation algebra*, encodes a vast amount of information regarding arithmetic properties of the irrational number $\hbar$ and dynamics of the associated rotation action; Therefore serving, in a certain sense, as a noncommutative substitute for the C^*-algebra $C(\mathbb{T}/\hbar\mathbb{Z}) \cong \mathbb{C}$. For general reference on crossed-product C^*-algebras, see Williams [21] or [5].

In the preprint of Butler, Emerson, and Schulz ([2]), a 2-dimensional even pre-spectral triple, called the *Heisenberg cycle*, was constructed and analysed. Specifically, their Heisenberg cycle is over a certain dense $*$-subalgebra of the C^*-algebra crossed-product $C_u(\mathbb{R}) \rtimes \mathbb{R}_d$, with $C_u(\mathbb{R})$ the C^*-algebra of uniformly continuous and bounded functions on $\mathbb{R}$, and $\mathbb{R}_d$ the set of real numbers equipped discrete topology and acting by the natural translation action on $C_u(\mathbb{R})$.

Proposition 1.6 ([2]). *Let $C_u^\infty(\mathbb{R})[\mathbb{R}_d]$ denote the twisted group $*$-algebra generated by the translation action of $\mathbb{R}_d$ on the $*$-algebra $C_u^\infty(\mathbb{R})$ of smooth functions in $C_u(\mathbb{R})$ having bounded derivatives of all orders. Then the triple,*

$$\left(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), \pi \oplus \pi, D = \begin{pmatrix} 0 & x - \frac{d}{dx} \\ x + \frac{d}{dx} & \end{pmatrix}\right)$$

is a 2-dimensional even pre-spectral triple over $C_u^\infty(\mathbb{R})[\mathbb{R}_d] \subseteq C_u(\mathbb{R}) \rtimes \mathbb{R}_d$, where $\pi : C_u(\mathbb{R}) \rtimes \mathbb{R}_d \to \mathbb{B}(L^2(\mathbb{R}))$ is the $*$-representation on $L^2(\mathbb{R})$ induced from covariant pair given by representing $C_u(\mathbb{R})$ by multiplication operators, and $\mathbb{R}_d$ by (unitary) translations.

The reader familiar with the spectral triples and noncommutative geometry may immediately ask whether or not the associated even Fredholm module $[D]$ determined by the Heisenberg cycle, obtained by applying the normalizing function $\chi(x) = x(1 + x^2)^{-\frac{1}{2}}$ to the self-adjoint operator D via functional calculus, is nontrivial in the (even) K-homology group $\mathrm{KK}_0(C_u(V) \rtimes V_d, \mathbb{C})$. The answer is affirmative, following from by the fact that, as an element of $\mathrm{KK}_0(C_u(V) \rtimes V_d, \mathbb{C})$, the class $[D]$ has index $+1$. For a reference on analytic K-homology, we refer the reader to [5].

Note that the self-adjoint operator associated to the Heisenberg cycle above makes use of the annihilation and creation operators of quantum mechanics $x \pm \frac{d}{dx}$, which are closely related to the harmonic oscillator,

$$H := -\frac{d^2}{dx^2} + x^2,$$

acting as an unbounded operator on $L^2(\mathbb{R})$. The spectrum of H is well-studied; It is a diagonalizable, positive, unbounded operator, with each eigenvalue having multiplicity 1. Standard arguments show that H therefore admits a unique extension to an unbounded self-adjoint operator on $L^2(V)$. A heat kernel for the harmonic, *i.e.* an integral kernel for the compact operator $e^{-tH} : L^2(V) \to L^2(V)$, was constructed by Mehler [13], and is key in analysing the ζ-functions coming from Proposition 1.6.

Notoriety of the Heisenberg cycle lies in the numerous interesting C^*-subalgebras contained within $C_u(\mathbb{R}) \times \mathbb{R}_d$ and the natural pull-back operation for spectral triples: Given a C^*-algebra $\mathcal{A}$ containing a dense $*$-subalgebra $\mathcal{A}^\infty \subseteq \mathcal{A}$, any $*$-homomorphism $\mathcal{A} \to C_u(\mathbb{R}) \rtimes \mathbb{R}_d$ which takes $\mathcal{A}^\infty$ into $C_u^\infty(\mathbb{R})[\mathbb{R}_d]$ induces a 2-dimensional even pre-spectral triple over $\mathcal{A}^\infty \subseteq \mathcal{A}$.

Stating the obvious, examples of such $*$-homomorphism are given by the natural maps $C_u(\mathbb{R}) \rtimes \Gamma \to C_u(\mathbb{R}) \rtimes \mathbb{R}_d$ induced by the identity $C_u(\mathbb{R}) \to C_u(\mathbb{R})$ and the inclusion $\Gamma \hookrightarrow \mathbb{R}_d$, where Γ is any discrete subgroup of $\mathbb{R}$. More interesting are pull-backs determined by the data of a smooth flow on compact a manifold, a distinguished point of the manifold, and a discrete subgroup of $\mathbb{R}_d$.

Proposition 1.7 (Lemma 2.3 [2])**.** *Let M be a compact manifold, $\{\alpha_t\}_{t\in\mathbb{R}}$ a smooth flow on M, and Γ a discrete subgroup of $\mathbb{R}$. For fixed p in M and any continuous function f, the map,*

$$f_p : \mathbb{R} \longrightarrow \mathbb{C} : t \longmapsto f(\alpha_t(v)),$$

is is in $C_u(\mathbb{R})$. The induced $*$-homomorphism,

$$C(M) \longrightarrow C_u(\mathbb{R}) : f \longmapsto f_p,$$

together with the inclusion $\Gamma \hookrightarrow \mathbb{R}_d$, form a covariant pair and corresponding $*$-homomorphism

$$\sigma : C(M) \rtimes_\alpha \Gamma \longrightarrow C_u(V) \rtimes V_d,$$

such that $\sigma\left(C^\infty(M)[\Gamma]\right) \subseteq C_u^\infty(\mathbb{R})[\mathbb{R}_d]$, and which is injective if the flow $\{\alpha_t\}_{t\in\mathbb{R}}$ is minimal.

In particular, if $(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), \pi \oplus \pi, D)$ is the Heisenberg cycle of Proposition 1.6 then,

$$\left(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), (\pi \oplus \pi) \circ \sigma, D\right),$$

is a 2-dimensional even pre-spectral triple over $C^\infty(M)[\Gamma] \subseteq C(M) \rtimes \Gamma$.

If one takes an arbitrary point p in a compact manifold M equipped with a flow $\{\alpha_t\}_{t\in\mathbb{R}}$, and lets Γ be trivial subgroup of $\mathbb{R}_d$, then Proposition 1.7 yields a spectral triple over $C^\infty(M) \subseteq C(M)$, whose corresponding K-homology class in $\mathrm{KK}_0(C(M), \mathbb{C})$ will be equal to the K-homology class determined by the $*$-homomorphism,

$$\mathrm{ev}_p : C(M) \longrightarrow \mathbb{C} : f \longmapsto f(p),$$

of point evaluation at p in M (cf. Emerson, Duwenig [4] and [2, Proposition 2.13]). Thus, the only situations in which interesting *topological* information could be encoded in pull-back triples of the type in Proposition 1.7 are when one chooses suitable (nontrivial) discrete subgroups of $\mathbb{R}$.

On the other hand, the *geometry* of spectral triples arising from the Heisenberg cycle and Proposition 1.7 may still be interesting when choosing not to take into account a discrete subgroup of $\mathbb{R}_d$ (cf. [2, Proposition 3.4, Corollary 3.6]). One particular instance of this interesting geometry, is obtained when α is a smooth, ergodic, Riemannian, flow (in the sense that $\alpha_t : M \to M$ is an isometry for all t in $\mathbb{R}$) on a Riemannian manifold M equipped with an α-invariant probability measure μ: If α satisfies a certain *Diophantine condition* (see [2, Definition 3.7]), then for each point p in M, and function f in $C^\infty(M)$,

$$\mathrm{Res}_{s=1}\,\mathrm{Tr}(f_p H^{-s}) = \int_M f d\mu,$$

where $H = -\frac{d}{dx}^2 + x^2$ is the harmonic oscillator on $L^2(\mathbb{R})$, and f_p is the operator on $L^2(\mathbb{R})$ of multiplication by $f_p(t) = f(\alpha_t(p))$.

An extremely interesting scenario to apply Proposition 1.7 arises from the Kronecker flow on $\mathbb{T}^2$,

$$\alpha_t : \mathbb{T}^2 \longrightarrow \mathbb{T}^2 : (x,y) \longmapsto \left(x+t, y+\frac{t}{\hbar}\right), \qquad \text{for each} \qquad t \in \mathbb{R},$$

along the lines of slope $\frac{1}{\hbar}$, with $\hbar$ an irrational number in $\mathbb{R}$. Taking the point $(0,0)$ in $\mathbb{T}^2$, and the discrete subgroup,

$$\Lambda := \{m\hbar + k \mid m, k \in \mathbb{Z}\} \subseteq \mathbb{R},$$

known as the *homoclinic subgroup*, we obtain a $*$-homomorphism,

$$C(\mathbb{T}^2) \rtimes_\alpha \Lambda \longrightarrow C_u(\mathbb{R}) \rtimes \mathbb{R}_d, \tag{1.2}$$

which restricts to a map of $C^\infty(\mathbb{T}^2)[\Lambda]$ into $C_u^\infty(\mathbb{R})[\mathbb{R}_d]$. With $\mathcal{A}_\hbar$ and $\mathcal{A}_{\frac{1}{\hbar}}$ the irrational rotation algebras generated from rotation by $\hbar$, and $\frac{1}{\hbar}$, respectively, on $\mathbb{C}(\mathbb{T})$, one notices [2, Lemma 2.4],

$$\mathcal{A}_\hbar \otimes \mathcal{A}_{\frac{1}{\hbar}} \cong \mathbb{C}(\mathbb{T}^2) \rtimes_\alpha \Lambda. \tag{1.3}$$

In light of the $*$-isomorphism of Equation (1.3), we may view $C^\infty(\mathbb{T}^2)[\Lambda]$ as a dense $*$-subalgebra of $\mathcal{A}_\hbar \otimes \mathcal{A}_{\frac{1}{\hbar}}$, and obtain a pull-back of the Heisenberg cycle by the $*$-homomorphism of Equation (1.2).

Definition 1.8 ([2] Definition 2.6)**.** The *Heisenberg bi-cycle* is the 2-dimensional even pre-spectral triple over $C(\mathbb{T}^2)[\Lambda] \subseteq \mathcal{A}_\hbar \otimes \mathcal{A}_{\frac{1}{\hbar}}$ obtained by pulling back the Heisenberg spectral triple of Proposition 1.6 by the $*$-homomorphism,

$$\mathcal{A}_\hbar \otimes \mathcal{A}_{\frac{1}{\hbar}} \cong C(\mathbb{T}^2) \rtimes_\alpha \Lambda \longrightarrow C_u(\mathbb{R}) \rtimes \mathbb{R}_d.$$

It is the aim of this thesis is to generalize the methods used to construct the Heisenberg cycle of Proposition 1.6, so as to replace the use of a one-dimensional Euclidean space $\mathbb{R}$ with a general n-dimensional Euclidean space V. In particular, our generalization provides a $2n$-dimensional even pre-spectral triple over a smooth $*$-subalgebra of the C^*-algebra crossed-product $C_u(V) \rtimes V_d$ (see Example 3.15), where V_d denotes the underlying abelian group of V equipped with the discrete topology, and the action of V_d on the C^*-algebra $C_u(V)$ of uniformly continuous and bounded functions over V (see Example 3.5) is induced by translation.

This author's original motivation behind wishing to extend the Heisenberg cycle in such a manner was not only to provide a possibly interesting class of pre-spectral triples, but also arose from KK-theoretic results associated to (the K-homology class of) the Heisenberg bi-cycle of Definition 1.8; In the context of Kasparov's KK-theory, a joint generalization of analytic K-homology and K-theory (cf. Kasparov [10], [11], [12]), the K-homology class $[\Delta_\hbar]$ in $\mathrm{KK}_0\left(\mathcal{A}_\hbar \otimes \mathcal{A}_{\frac{1}{\hbar}}, \mathbb{C}\right)$ obtained from the Heisenberg bi-cycle determines a KK-duality between the C^*-algebra $\mathcal{A}_\hbar$ and $\mathcal{A}_{\frac{1}{\hbar}}$ ([2, Corollary 2.11]). Such a duality is an adjunction of functors and, in the aforementioned case, reduces to the construction of a pair consisting of a class in analytic K-homology and K-theory class for the C^*-algebra $\mathcal{A}_\hbar \otimes \mathcal{A}_{\frac{1}{\hbar}}$.

Taking $\mathbb{R}^n$ as opposed to $\mathbb{R}$, and a linear automorphism of $\mathbb{R}^n$ as opposed to a real number $\hbar$, a natural extension of the construction for the rotation algebras $\mathcal{A}_\hbar := C(\mathbb{T}) \rtimes_\hbar \mathbb{Z}$ and $\mathcal{A}_{\frac{1}{\hbar}} := C(\mathbb{T})\rtimes_{\frac{1}{\hbar}}$ is given by,

$$\mathcal{B}_g := C(\mathbb{T}^n) \rtimes_g \mathbb{Z}^n \quad \text{and} \quad \mathcal{B}_{g^{-1}} := C(\mathbb{T}^n) \rtimes_{g^{-1}} \mathbb{Z}^n,$$

where the actions of $\mathbb{Z}^n$ on $C(\mathbb{T}^n)$ are given by,

$$(\vec{z} \cdot_g f)(t) = f(t - g\vec{z}) \quad \text{and } (\vec{z} \cdot_{g^{-1}} f)(t) = f(t - g^{-1}\vec{z}),$$

respectively, for each continuous function f on $\mathbb{T}^n$, n-tuple of integers $\vec{z}$ in $\mathbb{Z}^n$, and point t in $\mathbb{T}^n$. We hope that this thesis will lay a good framework for answering the question of when, if at all, are the C^*-algebras $\mathcal{B}_g$ and $\mathcal{B}_{g^{-1}}$ dual in the KK-theoretic sense.

In the remainder of Chapter 1, we provide some rudimentary constructions, and outline for our basic notation; Section 1.2 is dedicated to defining the tensor and exterior algebras associated to a Euclidean space, exhibiting a natural inner product on such exterior algebras, as well as outlining our conventions for (complex) Hilbert space inner products, and introducing a number of function algebras which will appear throughout this thesis.

The goal of Chapter 2 is to rigorously construct the Dirac-Heisenberg operator, whose completion will serve as the operator of our generalized Heisenberg spectral triple. This process is broken into stages, with Section 2.1 and Section 2.2 focusing on algebras of differential forms over Euclidean space, and important linear operators associated to them such as exterior differentiation and codifferentiation; One should note that in neither of these sections do we make reference to any kind of topology on the algebra of forms. In Section 2.3 and 2.4, we introduce the Hilbert space structure associated to differential forms, and use the operators given in the prior sections to defined, and diagonalize, the Dirac-Heisenberg operator, which will initially only taken to be an essentially self-adjoint unbounded operator acting on the Hilbert space of forms.

Chapter 3 we be where the fruits of our labour are found. In Section 3.1 we first properly define, for V an n-dimensional Euclidean space, the C^*-algebra $C_u(V) \rtimes V_d$, along with its smooth $*$-subalgebra mentioned above which will appear in our $2n$-dimensional Heisenberg cycle. Section 3.2 will be the culmination of these efforts, and where we finally are able to argue that the triple we define is, indeed, a $2n$-dimensional pre-spectral triple, before finishing the section by analysing its ζ-functions and discussing a pull-back result similar to Proposition 1.7.

1.2 Basic Preliminaries

Throughout this section, assume V is a vector space over $\mathbb{R}$. We first briefly introduce the *tensor*, and subsequent *exterior*, algebras generated by V. Once these concepts have been treated, we discuss the notions of inner products, Euclidean and Hilbert spaces, and basic function algebras. In particular, we will discuss the Euclidean structure associated to the exterior algebra generated a Euclidean space.

1.2.1 The Tensor and Exterior Algebras

Here we introduce the *tensor algebra* associated to a finite dimensional vector space over $\mathbb{R}$. For a more detailed discussion of tensor algebras, see Warner [20, Chapter 2]. The use of tensor algebras will be in defining the *exterior algebra* of the next section.

Fix a finite dimensional vector space V over $\mathbb{R}$. Unless otherwise specified, we let the dimension of V be denoted by the natural number $n := \dim(V)$. For each natural number k, the *k-th tensor power of* V is the tensor product vector space of V with itself k times; That is, the k-th tensor power of V is the vector space

$$\mathcal{T}^k(V) := V^{\otimes k}$$

consisting of all k-th order tensor powers of V. We also define $\mathcal{T}^0(V) := \mathbb{R}$ and $\mathcal{T}^n(V) := \{0\}$ for all *negative* integers n

Each k-th tensor power of V has the structure of a real vector space over $\mathbb{R}$ and, under the convention that $\mathcal{T}^k(V) = \{0\}$ whenever $k < 0$, for each pair of integers m and n we obtain a natural (bilinear) isomorphism,

$$\otimes : \mathcal{T}^m(V) \times \mathcal{T}^n(V) \to \mathcal{T}^{m+n}(V) \tag{1.4}$$

defined for elementary tensors $v_1 \otimes \ldots \otimes v_m$ in $\mathcal{T}^m(V)$ and $v_1' \otimes \ldots \otimes v_n'$ in $\mathcal{T}^n(V)$ by

$$(v_1 \otimes \ldots \otimes v_m) \otimes (v_1' \otimes \ldots \otimes v_n') := v_1 \otimes \ldots \otimes v_m \otimes v_1' \otimes \ldots \otimes v_n'.$$

This serves as a product for the *tensor algebra* generated by V, defined below.

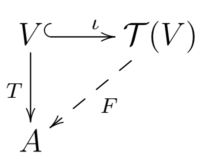
Definition 1.9. For each integer k, let $\mathcal{T}^k(V)$ denote the k-th tensor power of V, with the convention that $\mathcal{T}^0(V) := \mathbb{R}$ and $\mathcal{T}^k(V) := \{0\}$ when $k < 0$. The *tensor algebra of V* is the (associative) $\mathbb{R}$-algebra

$$\mathcal{T}(V) := \bigoplus_{k \in \mathbb{Z}} \mathcal{T}^k(V),$$

with multiplication induced by the canonical isomorphism of Equation (1.4). We denote the natural (linear) inclusion of V into $\mathcal{T}(V)$ by

$$\iota : V = \mathcal{T}^1(V) \hookrightarrow \mathcal{T}(V). \tag{1.5}$$

The tensor algebra $\mathcal{T}(V)$ is universal in the sense that any linear map from V into an associative $\mathbb{R}$-algebra A can be extended uniquely to an algebra homomorphism from $\mathcal{T}(V)$ into A; That is, for each linear map $T : V \to A$, there exists a unique algebra homomorphism $F : \mathcal{T}(V) \to A$ such that the following diagram commutes



Another way of defining the tensor algebra $\mathcal{T}(V)$ is as the universal, associative, noncommutative algebra generated by the elements of V.

Lemma 1.10. *The tensor algebra $\mathcal{T}(V)$ is a $\mathbb{Z}$-graded algebra, with component subspaces given by $\mathcal{T}^k(V)$ for each $k \in \mathbb{Z}$.*

Proof. By Definition 1.9, the tensor algebra $\mathcal{T}(V)$ admits a decomposition into the direct sum, over $k \in \mathbb{Z}$, of the subspaces $\mathcal{T}^k(V)$. It is clear from the definition of the product in $\mathcal{T}(V)$, as given in Equation (1.4), that

$$\otimes : \mathcal{T}^{k_1}(V) \times \mathcal{T}^{k_2}(V) \to \mathcal{T}^{k_1+k_2}(V),$$

for all integers k_1 and k_2, so that $\mathcal{T}(V)$ is, indeed, $\mathbb{Z}$-graded. □

The inclusion $\iota : V \hookrightarrow \mathcal{T}(V)$, together with the universal property of $\mathcal{T}(V)$, allows one to derive the following simple lemma showing that any linear operator on V extends uniquely to an algebra homomorphism from $\mathcal{T}(V)$ to itself.

Lemma 1.11. *Let $\mathcal{T}(V)$ be the tensor algebra of Definition 1.9. Any linear operator $T : V \to V$ induces a unique algebra homomorphism $\tilde{T} : \mathcal{T}(V) \to \mathcal{T}(V)$ such that the following diagram commutes*

$$\begin{array}{ccc} V & \xrightarrow{T} & V \\ \iota \downarrow & & \downarrow \iota \\ \mathcal{T}(V) & \xrightarrow[\tilde{T}]{} & \mathcal{T}(V) \end{array}$$

In more generality, if W is another finite dimensional real vector space, with exterior algebra $\mathcal{T}(W)$ and canonical inclusion $\iota_W : W \to \mathcal{T}(W)$, then it is an easy consequence of the universal property of tensor algebras that any linear map $T : V \to W$ extends uniquely to an algebra homomorphism $\tilde{T} : \mathcal{T}(V) \to \mathcal{T}(W)$ such that $\iota_W \circ T = \tilde{T} \circ \iota_V$.

It follows from Lemma 1.11 that we are able to produce algebra homomorphisms $\mathcal{T}(V) \to \mathcal{T}(V)$ whenever we have a linear operator on V. An important example of this principal is the following.

Corollary 1.12. *The tensor algebra $\mathcal{T}(V)$ is a $\mathbb{Z}/2$-graded algebra, with even and odd subspaces $\mathcal{T}^+(V)$ and $\mathcal{T}^-(V)$, respectively, defined by*

$$\mathcal{T}^+(V) := \bigoplus_{k \text{ even}} \mathcal{T}^k(V) \text{ and } \mathcal{T}^-(V) := \bigoplus_{k \text{ odd}} \mathcal{T}^k(V). \tag{1.6}$$

Proof. By Lemma 1.11, the linear map

$$V \to V : v \mapsto -v$$

induces a unique algebra homomorphism $\mathrm{gr}_2 : \mathcal{T}(V) \to \mathcal{T}(V)$. As is easy to verify, gr_2 is an automorphism of $\mathcal{T}(V)$ such that, for each integer k,

$$\mathrm{gr}_2(\omega) = (-1)^k \omega, \text{ for all } \omega \in \mathcal{T}^k(V).$$

In particular, $\mathrm{gr}_2 \circ \mathrm{gr}_2 = \mathrm{id} : \mathcal{T}(V) \to \mathcal{T}(V)$ so that gr_2 is an automorphism.

The even and odd subspaces induced by $\mathrm{gr}_2 : \mathcal{T}(V) \to \mathcal{T}(V)$ are exactly those defined in Equation (1.6). □

With the tensor algebra defined, and some of its more important properties discussed, we are ready to discuss the *exterior algebra* generated by the finite dimensional $\mathbb{R}$-vector space V. For a more in-depth reference on exterior algebras, see Sternberg [18, Chapter 1] and Warner [20, Chapter 2]. To this end, let V be a finite dimensional vector space. We let $\mathcal{T}^k(V)$ denote the k-th tensor power of V, defined above, for each integer k. Recall that, by convention, we define $\mathcal{T}^k(V)$ to be the trivial vector space whenever k is negative. Lastly, let $\mathcal{T}(V) = \bigoplus_{k\in\mathbb{Z}} \mathcal{T}^k(V)$ denote the tensor algebra generated by V, of Definition 1.9.

Definition 1.13. The *exterior algebra* generated by a finite dimensional vector space V, denoted ΛV, is the quotient algebra

$$\Lambda V := \mathcal{T}(V)/\mathscr{I},$$

where $\mathcal{T}(V)$ is the tensor algebra generated by V of Definition 1.9, and $\mathscr{I}$ is the two-sided ideal of $\mathcal{T}(V)$ generated by elements of the form $v \otimes v$ for v in V. We denote the induced algebra product on ΛV as

$$\cdot \wedge \cdot : \Lambda V \times \Lambda V \to \Lambda V.$$

For each integer k, the *k-th exterior power of* V, denoted $\Lambda^k V$, is the vector subspace of ΛV obtained as the image of $\mathcal{T}^k(V)$ under the canonical quotient map $\mathcal{T}(V) \to \Lambda V$.

Remark 1.14. Notice that, by convention, $\mathcal{T}^k(V)$ is trivial for negative integers k; In particular, for such k, the k-th exterior power $\Lambda^k V$ is the trivial vector subspace of the exterior algebra ΛV.

We may informally regard ΛV as the associative algebra generated by the vectors in V, subject to the relation

$$v \wedge v = 0, \text{ for all } v \in V. \tag{1.7}$$

We make this statement more formal in Lemma 1.17 by showing that a copy of the linear space V, and a copy of the algebra $\mathbb{R}$, sit inside ΛV.

By [20, 2.4], with the $\mathbb{Z}$-grading on $\mathcal{T}(V)$ given in 1.10, the ideal $\mathscr{I}$ of $\mathcal{T}(V)$, defined in Definition 1.13, is a $\mathbb{Z}$-graded ideal in the sense that $\mathscr{I} = \bigoplus_{k\in\mathbb{Z}} \mathscr{I} \cap \mathcal{T}^k(V)$. It follows that the k-th exterior power $\Lambda^k V$ may be equivalently defined as a vector space quotient of $\mathcal{T}^k(V)$ by its vector subspace $\mathscr{I} \cap \mathcal{T}^k(V)$ *i.e.*

$$\Lambda^k V := \mathcal{T}^k(V)/\left(\mathscr{I} \cap \mathcal{T}^k(V)\right).$$

Moreover, the graded ideal structure of $\mathscr{I}$ induces a $\mathbb{Z}$-grading on the exterior algebra ΛV, highlighted in the following lemma.

Lemma 1.15. *Under the convention that $\mathcal{T}^k(V)$, and hence $\Lambda^k V$, be trivial for all negative integers k, the exterior algebra ΛV is a $\mathbb{Z}$-graded algebra with component subspaces $\Lambda^k V$ for k ranging over $\mathbb{Z}$; That is, ΛV decomposes into a direct sum of linearly independent subspaces*

$$\Lambda V = \bigoplus_{k \in \mathbb{Z}} \Lambda^k V$$

and for any integers two integers k_1 and k_2, the algebra product in ΛV restricts to a bilinear map

$$\cdot \wedge \cdot : \Lambda^{k_1} V \times \Lambda^{k_2} V \to \Lambda^{k_1+k_2} V.$$

As we will be dealing with a number of graded algebras throughout this thesis, make an important definition regarding linear operators.

Definition 1.16. Let $A = \bigoplus_{k \in \mathbb{Z}} A_k$ be a $\mathbb{Z}$-graded algebra, in the sense that, for any integers k_1 and k_2, the algebra product of an element in A_{k_1} with an element in A_{k_2} yields an element in $A_{k_1+k_2}$. If j is an integer and $T : A \to A$ is a linear operator, we say that T has *grading degree* j, or that T is *degree* j, if for any integer k,

$$T|_{A_k} : A_k \longrightarrow A_{k+j}.$$

It is clear by definition that the ideal $\mathscr{I}$ of $\mathcal{T}(V)$, given in Definition 1.13, has trivial intersection with the tensor powers $\mathcal{T}^0(V)$ and $\mathcal{T}^1(V)$. It follows that the canonical quotient homomorphism $\mathcal{T}(V) \to \Lambda V$ restricts to an algebra isomorphism $\mathcal{T}^0(V) \to \Lambda^0 V$, and a linear isomorphism $\mathcal{T}^1(V) \to \Lambda^1 V$. By definition, $\mathbb{R} = \mathcal{T}^0(V)$ and $V = \mathcal{T}^1(V)$, and so we obtain a canonical algebra isomorphism identifying $\mathbb{R}$ with the subalgebra $\Lambda^0 V$ in ΛV, and a linear isomorphism identifying V with the subspace $\Lambda^1 V$ of ΛV. We regard both of these canonical mappings as inclusions, so that we may view $\mathbb{R}$ as a subalgebra of ΛV, and V as linear subspace of ΛV. This result restated in the following lemma for future reference.

Lemma 1.17. *Let ΛV be the exterior algebra of Definition 1.13. Then the algebra $\mathbb{R}$ is isomorphic to the subalgebra $\Lambda^0 V$ of ΛV, and the vector space V is linearly isomorphic to the subspace $\Lambda^1 V$ of ΛV.*

The condition that $v \wedge v$ is 0 in ΛV for any vector v in V implies a graded-commutative structure to the exterior algebra ΛV in the following sense.

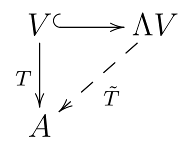
Lemma 1.18. *The exterior algebra $\Lambda V = \bigoplus_{k \in \mathbb{Z}} \Lambda^k V$ satisfies the following graded commutativity law: If ω_1 is in $\Lambda^{k_1} V$, and ω_2 is in $\Lambda^{k_2} V$, then*

$$\omega_1 \wedge \omega_2 + (-1)^{k_1 k_2 + 1} \omega_2 \wedge \omega_1 = 0. \tag{1.8}$$

For a reference, see [20, p. 57]

With the canonical identification of V with $\Lambda^1 V$ given in Lemma 1.17, we are able to describe the universal property satisfied by the exterior algebra ΛV.

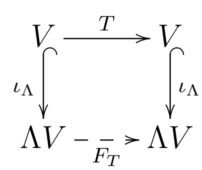
Proposition 1.19. *The exterior algebra ΛV is universal in the sense that if A is any associative $\mathbb{R}$-algebra, and $T : V \to A$ is a linear map from V into A such that $(Tv)^2 = 0$ as elements of A, then there exists a unique algebra homomorphism $\tilde{T} : \Lambda V \to A$ making the following diagram commute*



where the horizontal map is given by the canonical linear inclusion of V into ΛV.

Analogously to the tensor algebra and Lemma 1.11, one can apply the universal property of ΛV to extend linear operators on V to algebra endomorphisms $\Lambda V \to \Lambda V$. In particular, using the same linear map on V appearing in the proof of Corollary 1.21, it is possible to define a $\mathbb{Z}/2$-grading on ΛV.

Lemma 1.20. *Let ΛV be the exterior algebra generated by V, and let $T : V \to V$ be a linear map. Then there exists a unique algebra homomorphism $F_T : \Lambda V \to \Lambda V$ making the following diagram commute*



Proof. Let T be as in the lemma, and consider the linear map $\iota_\Lambda \circ T : V \to \Lambda V$. It follows from Lemma 1.18 that $[(\iota_\Lambda \circ T)(v)]^2 = 0$ in ΛV. Hence, by the universal property given in Proposition 1.19, there exists a unique algebra homomorphism $F_T : \Lambda V \to \Lambda V$ such that $F_T \circ \iota_\Lambda = \iota_\Lambda \circ T : V \to \Lambda V$. □

Corollary 1.21. *The exterior algebra ΛV of Definition 1.13 is a $\mathbb{Z}/2$-graded algebra, with even and odd subspaces $\Lambda^+ V$ and $\Lambda^- V$, respectively, defined by*

$$\Lambda^+ V := \bigoplus_{k \text{ even}} \Lambda^k V \text{ and } \Lambda^- V := \bigoplus_{k \text{ odd}} \Lambda^k V. \tag{1.9}$$

Proof. By Lemma 1.20, the linear map $V \to V : v \mapsto -v$ induces a unique algebra homomorphism $\varepsilon_\Lambda : \Lambda V \to \Lambda V$. As is easy to verify, ε_Λ is an automorphism of $\mathcal{T}(V)$ such that, for each integer k,

$$\varepsilon_\Lambda \omega = (-1)^k \omega, \text{ for all } \omega \in \Lambda^k V.$$

In particular, $\varepsilon_\Lambda \circ \varepsilon_\Lambda = \mathrm{id} : \Lambda V \to \Lambda V$.

The even and odd subspaces induced by the automorphism ε_Λ are exactly those defined in Equation (1.9); That is,

$$\Lambda^+ V = \ker(\varepsilon_\Lambda - \mathbb{1}) \quad \text{and} \quad \Lambda^- V = \ker(\varepsilon_\Lambda + \mathbb{1}),$$

where $\mathbb{1}$ is the identity homomorphism on ΛV, and $\ker(\varepsilon_\Lambda \pm \mathbb{1})$ denotes the kernel the homomorphism $\varepsilon_\Lambda \pm \mathbb{1}$. □

Thus far, we have made no real use of the finite dimensionality of V in our considerations of its exterior algebra ΛV. With the goal of defining a Euclidean structure on the exterior algebra, we now focus on how this finite dimensionality will come into play. The first key observation is that if V is finite dimensional, then the k-th exterior power Λ^k is not only trivial for negative integers k, but also when k is an integer greater than the dimension of V.

Lemma 1.22. *If V is a real vector space of finite dimension n, and $\Lambda^k V$ is the vector space of k-th exterior powers (see Definition 1.13), then $\Lambda^k V$ is trivial whenever k is an integer with $0 > k$ or $k > n$.*

Proof. The case where $k < n$ is observed in Remark 1.14.

Now assume k is an integer with $k > n$. If $w_1 \wedge w_2 \wedge \ldots \wedge w_k$ is an elementary vector in the k-th exterior power $\Lambda^k V$, then there is a linear dependence among the vectors $\{w_i\}_{i=1}^k \subseteq V$, say $w_s = \sum_{j \neq i} c_i w_i$ with $1 \leq s \leq k$, for a nontrivial combination of constants $c_i \neq 0 \in \mathbb{R}$. Then

$$w_1 \wedge \ldots \wedge w_s \wedge \ldots \wedge w_k = \sum_{i \neq s} c_i (w_1 \wedge \ldots \wedge w_i \wedge \ldots w_k).$$

Using Lemma 1.18, we can see that each term appearing in the sum on the right-hand side of the equality is contained in the ideal $\mathscr{I}$ of Definition 1.13. Hence, every elementary exterior k-tensor in $\Lambda^k V$ is zero, so it holds that $\Lambda^k V$ is trivial. □

Remark 1.23. From Lemma 1.22 and Lemma 1.22, one observes

$$\Lambda V = \bigoplus_{k=0}^{\dim(V)} \Lambda^k V.$$

To simplify our discussion of a basis for ΛV (and of tensor in general) we take a moment to define some *multi-index notation*.

Definition 1.24. Let β be a natural number. If k is a natural number with $k \leq \beta$, the *well-ordered multi-indices of length k, pulling from β*, is defined to be the set,

$$\mathcal{I}_k^{(\beta)} := \{(i_1, i_2, \ldots, i_k) \subset \{1, \ldots, \beta\}^k : \ i_1 < i_2 < \ldots < i_k\}.$$

By convention, we set $\mathcal{I}_0^{(\beta)} := \{\emptyset\}$.

The *complement map* is defined to be the invertible function

$$\mathcal{I}_k^{(\beta)} \to \mathcal{I}_{(\beta-k)}^{(\beta)} : I \mapsto I^c,$$

where, for a multi-index $I = (i_1, i_2, \dots, i_k)$ in $\mathcal{I}_k^{(\beta)}$, the multi-index $I^c = (i_1^c, i_2^c, \dots, i_{\beta-k}^c)$ in $\mathcal{I}_k^{\beta}$ is defined to be the unique well-ordered multi-index of length $(\beta - k)$ such that $i_{j_1} \neq i_{j_2}^c$ for all combinations of $i_1 = 1, \dots, k$ and $j_2 = 1, \dots, \beta - k$. In other words, if I is a multi-index in $\mathcal{I}_k^{(\beta)}$, then I^c is defined to be the unique well-ordered multi-index in $\mathcal{I}_{\beta-k}^{(\eta)}$ obtained by removing the indices of I from the set $\{1, 2, \dots, \beta\}$ and forming a well-ordered multi-index with the remaining $(\beta - k)$ numbers.

The set of *all well-ordered multi-indices, pulling from* β, is defined to be

$$\mathcal{I}^{(\beta)} := \bigcup_{k=0}^{\beta} \mathcal{I}_k^{(\beta)}.$$

Definition 1.24 may be cumbersome, but we will refer back to this notation several times throughout this thesis. The usefulness of well-ordered multi-index notation is made clear in the following lemma, which shows that any basis for a finite dimensional vector space V induces a finite basis for each k-th exterior power $\Lambda^k V$ which is indexed by the set of well-ordered multi-indices.

Proposition 1.25. *Let V be an n-dimensional vector space over $\mathbb{R}$, and let its exterior algebra be denoted ΛV. Further, take $\mathcal{I}_k^{(n)}$ to denote the set of well-ordered multi-indices of length k, pulling from n, of Definition 1.24.*

For any basis $\{e_i\}_{i=1}^n$ of V, and any integer k, define

$$\mathscr{E}_k := \left\{ e_{I_1} \wedge e_{I_2} \wedge \cdots \wedge e_{I_k} \in \Lambda^k V \;\middle|\; (I_1, I_2, \dots, I_k) \in \mathcal{I}_k^{(n)} \right\}, \tag{1.10}$$

where it is understood that $\mathscr{E}_k := \emptyset$ whenever $k < 0$ or $k > n$, and $\mathscr{E}_0 := \{1\} \subseteq \Lambda^0 V$. Then for each integer k, the set $\mathscr{E}_k$ is a basis for $\Lambda^k V$; In particular, the vector space dimension of $\Lambda^k V$ is given by

$$\dim\left(\Lambda^k V\right) = \begin{cases} \binom{n}{k}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise,} \end{cases}.$$

For proof of Proposition 1.25, see [20, 2.6] or [18, Theorem 4.1]. An immediate corollary to 1.25 is the following.

Corollary 1.26. *Let V have dimension n, and let $\{e_i\}_{i=1}^n$ be a basis for V. If $\mathscr{E}_k$ is* *the induced basis for $\Lambda^k V$ given in Proposition 1.25, then the set*

$$\mathscr{E} := \bigcup_{i=0}^{n} \mathscr{E}_k$$

is a basis for the vector space structure underlying the exterior algebra ΛV. In particular, ΛV has dimension 2^n as a real vector space.

Proof. This follows immediately from Lemma 1.15, Lemma 1.22, and Proposition 1.25. □

1.2.2 Hilbert Spaces and Basic Function Algebras

We recall some basic definitions and constructions in real and complex Hilbert space theory.

For a vector space X over the field $\mathbb{F}$, with either $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$, an *inner product on X* is a map of $X \times X$ into $\mathbb{F}$ which is conjugate symmetric, linear in the second coordinate, and positive definite in the sense that restricting the inner product to the diagonal is positive away from the zero vector in X. Note, in particular, that our inner products are therefore conjugate linear in the first coordinate, *i.e.* we are using the physicist notation. Often, we will denote our inner products by $\langle \cdot, \cdot \rangle_X : X \times X \to \mathbb{F}$, or simply $\langle \cdot, \cdot \rangle$ when the vector space on which $\langle \cdot, \cdot \rangle$ acts is clear. The norm on a vector space X induced by an inner product $\langle \cdot, \cdot \rangle_X : X \times X \to \mathbb{F}$ is defined as,

$$\| \cdot \|_X : X \longrightarrow [0, \infty) : x \longmapsto \sqrt{\langle x, x \rangle_X}.$$

By an *inner product space* $(X, \langle \cdot, \cdot \rangle_X)$, we will mean the data of a vector space X over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ such that X is equipped with both an inner product $\langle \cdot, \cdot \rangle_X : X \times X \to \mathbb{F}$ and the topology induced by the metric on X arising from $\langle \cdot, \cdot \rangle_X$; Explicitly, this metric is given by

$$d(x_1, x_2) = \|x_1 - x_2\|_X = \sqrt{\langle x_1 - x_2, x_1 - x_2 \rangle_X}, \text{ for all } x_1, x_2 \in X.$$

A *Hilbert space* is an inner product space $(X, \langle \cdot, \cdot \rangle_X)$ such that X is complete in the metric induced by $\langle \cdot, \cdot \rangle_X$. Of course, any inner product space $(X, \langle \cdot, \cdot \rangle_X)$ for which X has finite $\mathbb{F}$-vector space dimension is, in fact, a Hilbert space. Given two Hilbert spaces $(X, \langle \cdot, \cdot \rangle_X)$ and $(Y, \langle \cdot, \cdot \rangle_Y)$, with X and Y sharing the same ground field $\mathbb{F}$, we denote the bounded linear operators from X to Y by

$$\mathbb{B}(X; Y) := \left\{ T : X \to Y \;\middle|\; T \text{ is linear and } \sup_{\|v\|_X \le 1} \|Tv\|_Y < \infty \right\}.$$

Similarly, the bounded linear operators from a Hilbert space $(X, \langle\cdot,\cdot\rangle_X)$ to itself will be denoted

$$\mathbb{B}(X) := \left\{ T : X \to X \;\middle|\; T \text{ is linear and } \sup_{\|v\|_X \leq 1} \|Tv\|_X \right\}.$$

Finally, we reserve the name *Euclidean space* to mean a Hilbert space $(X, \langle\cdot,\cdot\rangle_X)$ in which X is an $\mathbb{R}$-vector space of finite dimension $\dim(X) < \infty$.

Example 1.27. Although trivial, we take a moment to consider that any Euclidean space $(V, \langle\cdot,\cdot\rangle_V)$ induces an isometrically isomorphic Euclidean space $(V^*, \langle\cdot,\cdot\rangle_{V^*})$, where V^* denotes the dual of V. To be precise, V^* consists of all linear functionals from V into $\mathbb{R}$, with vector space structure defined by pointwise addition and scalar multiplication. It is an elementary result that V^* is a finite dimensional vector space of the same dimension as V.

To obtain an inner product $\langle\cdot,\cdot\rangle_{V^*}$ on the Euclidean space $(V^*, \langle\cdot,\cdot\rangle_{V^*})$, one uses the linear isomorphism $\flat : V \to V^*$ where, for $v \in V$, the linear functional $\flat v : V \to \mathbb{R}$ is defined by

$$(\flat v)(x) := \langle v, x\rangle_V, \text{ for all } x \in V. \tag{1.11}$$

The fact that $\flat : V \to V^*$ defined by Equation (1.11) does, indeed, define a linear isomorphism is a basic linear algebra fact following from the Riesz Representation Theorem. In the situation where $(V, \langle\cdot,\cdot\rangle_V)$ is Hilbert, as opposed to Euclidean, Equation 1.11 defined a *conjugate-linear* isomorphism from V to its dual V^*, following again from Riesz Representation.

Explicitly, the inner product $\langle\cdot,\cdot\rangle_{V^*} : V^* \times V^* \to \mathbb{R}$ is defined by the equality

$$\langle v, w\rangle_V = \langle \flat v, \flat w\rangle_{V^*}, \text{ for all } v, w \in V. \tag{1.12}$$

The resulting Euclidean space $(V^*, \langle\cdot,\cdot\rangle_{V^*})$ is therefore isometrically isomorphic to the Euclidean space $(V, \langle\cdot,\cdot\rangle_V)$ via the map $\flat : V \to V^*$ determined by Equation (1.11). In particular, the dimension of Euclidean spaces $(V, \langle\cdot,\cdot\rangle_V)$ and $(V^*, \langle\cdot,\cdot\rangle_{V^*})$ are the same.

The second example of a Euclidean space we provide is that associated to the exterior algebra generated by a Euclidean vector space.

Example 1.28. Let $(V, \langle\cdot,\cdot\rangle_V)$ be a Euclidean space with $n := \dim(V)$. Recall, that for each non-negative integer k, the *k-th tensor power of* V is the vector space tensor product of V with itself k times, *i.e.* it is the real vector space $\mathcal{T}^k(V) := \bigoplus_{i=1}^k V$ under the convention $\mathcal{T}^0(V) := \mathbb{R}$. The *tensor algebra*, of Definition 1.9, is then defined by bilinearly extending the natural $\mathbb{R}$-bilinear isomorphism $\otimes : \mathcal{T}^{k_1}(V) \otimes \mathcal{T}^{k_2}(V) \to \mathcal{T}^{k_1+k_2}(V)$ to serve as an algebra product on the direct sum $\mathcal{T}(V) := \bigoplus_{k=0}^\infty \mathcal{T}^k(V)$. The *exterior algebra generated by* V, of Definition 1.13, which we denoted ΛV, is obtained as a quotient of the tensor algebra $\mathcal{T}(V)$ by its two-sided ideal, denoted $\mathscr{I}$, generated by the 2-tensors $v \otimes v$ for v ranging over V. We let $\cdot \wedge \cdot : \Lambda V \times \Lambda V \to \Lambda V$ denote the algebra product in ΛV. Moreover, recall that *k-th exterior power of* V, denoted $\Lambda^k V$, is the vector subspace of ΛV defined as the vector space quotient $\Lambda^k V := \mathcal{T}^k(V) / \left(\mathscr{I} \cap \mathcal{T}^k(V)\right)$.

From Lemma 1.15 and Lemma 1.22, the algebra ΛV decomposes as,

$$\Lambda V = \bigoplus_{k=0}^{\dim(V)} \Lambda^k V,$$

while Lemma 1.18 shows that a graded-commutativity condition on ΛV holds in the sense that

$$\omega_1 \wedge \omega_2 + (-1)^{k_1 k_2 + 1} \omega_2 \wedge \omega_1 = 0, \text{ for all } k_1, k_2 \in \mathbb{Z}.$$

As $\Lambda^k V$ is the trivial vector space whenever $k > \dim(V)$ or $k < 0$, we verified that ΛV is $\mathbb{Z}$-graded in the sense that

$$\Lambda V = \bigoplus_{k \in \mathbb{Z}} \Lambda^k V,$$

and the algebra product in ΛV restricts to a surjective, bilinear map

$$\cdot \wedge \cdot : \Lambda^{k_1} V \times \Lambda^{k_2} V \to \Lambda^{k_1 + k_2} V \text{ for all } k_1, k_2 \in \mathbb{Z},$$

with $\Lambda^0 V \cong \mathbb{R}$ sitting inside ΛV as an algebra, and $\Lambda^1 V \cong V$ sitting inside ΛV as a linear space. Lastly, we note the result of Proposition 1.25, which shows any basis for $\{e_i\}_{i=1}^n$ for V induces a basis for $\Lambda^k V$ given by

$$\mathscr{E}_k := \left\{ e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \in \Lambda^k V \ : (i_1, i_2, \ldots, i_k) \in \mathcal{I}_k^{(n)} \right\}, \tag{1.13}$$

where $\mathcal{I}_k^{(n)}$ denotes the set of well-ordered multi-indices of length k of Definition 1.24,

$$\mathcal{I}_k^{(n)} := \left\{ (i_1, i_2, \ldots, i_k) \in \{1, \ldots, n\}^k \ : i_1 < i_2 < \cdots < i_k \right\},$$

and subject to the convention $\mathscr{E}_0 = \{1\} \subseteq \Lambda^0 V$ and $\mathscr{E}_k = \emptyset$ when $k > n$ or $k < 0$. In particular, ΛV^k has dimension $\binom{n}{k}$ for all integers $0 \leq k \leq n$, and dimension zero otherwise, implying ΛV has dimension 2^n.

The inner product which we introduce on ΛV is defined via the following lemma.

Lemma 1.29. *Let $(V, \langle \cdot, \cdot \rangle_V)$ be a Euclidean space of dimension n, and let $\Lambda V = \bigoplus_{k=0}^n \Lambda^k V$ be its exterior algebra, with $\Lambda^k V$ the k-th exterior power, as seen in Definition 1.13. Then, for each integer k, there exists an inner product on the vector space $\Lambda^k V$, denoted $\langle \cdot, \cdot \rangle_{\Lambda^k} : \Lambda^k V \times \Lambda^k V \to \mathbb{R}$, such that for vecotrs $v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_k$*

in V, the elementary exterior k-tensors $v_1 \wedge \ldots \wedge v_k$ and $w_1 \wedge \ldots \wedge w_k$ in $\Lambda^k V$ satisfy,

$$\langle v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_k \rangle_{\Lambda^k} := \det\left(\left[\langle v_i, w_j \rangle_V \right]_{i,j} \right), \tag{1.14}$$

where $\left[\langle v_i, w_j \rangle_V\right]_{i,j}$ denotes the $k \times k$ matrix over $\mathbb{R}$ whose (i,j)-entry is $\langle v_i, w_j \rangle_V$.

Moreover, there exists an inner product on the underlying vector space of the exterior algebra ΛV, which we denote $\langle \cdot, \cdot \rangle_\Lambda : \Lambda V \times \Lambda V \to \mathbb{R}$, such that the restriction of $\langle \cdot, \cdot \rangle_\Lambda$ to $\Lambda^k V \times \Lambda^k V$ agrees with the inner product $\langle \cdot, \cdot \rangle_{\Lambda^k}$ on $\Lambda^k V$ defined above, and for which the subspaces $\Lambda^{k_1} V$ and $\Lambda^{k_2} V$ of ΛV are orthogonal with respect $\langle \cdot, \cdot \rangle_\Lambda$ whenever $k_1 \neq k_2$.

Proof. Let $\mathcal{T}^k(V)$ denote the k-th tensor power of V. For elementary k-tensors $v_1 \otimes v_2 \otimes \ldots \otimes v_k$ and $w_1 \otimes w_2 \otimes \ldots \otimes w_k$ in $\mathcal{T}^k(V)$, define

$$(v_1 \otimes \ldots \otimes v_k, w_1 \otimes \ldots \otimes w_k)_{\mathcal{T}^k} := \det\left(\left[\langle v_i, w_j \rangle \right]_{i,j} \right).$$

Extending by bilinearly, we obtain a symmetric, bilinear map

$$(\cdot, \cdot)_{\mathcal{T}^k} : \mathcal{T}^k(V) \times \mathcal{T}^k(V) \to \mathbb{R},$$

with kernel contained in the set

$$\left[\left(\mathscr{I} \cap \mathcal{T}^k(V) \right) \times \mathcal{T}^k(V) \right] \cup \left[\mathcal{T}^k(V) \times \left(\mathscr{I} \cap \mathcal{T}^k(V) \right) \right],$$

where $\mathscr{I}$ is the two sided ideal in the algebra $\mathcal{T}(V)$ given in Definition 1.13. It follows that $(\cdot, \cdot)_{\mathcal{T}^k} : \mathcal{T}^k(V) \times \mathcal{T}^k(V) \to \mathbb{R}$ descends to a map symmetric, bilinear map

$$\langle \cdot, \cdot \rangle_{\Lambda^k} : \Lambda^k V \times \Lambda^k V \to \mathbb{R}$$

satisfying

$$(a, b)_{\mathcal{T}^k} = \langle q(a), q(b) \rangle_{\Lambda^k}, \text{ for all } a, b \in \mathcal{T}^k(V),$$

with $q : \mathcal{T}^k(V) \to \Lambda^k V$ the canonical quotient map.

For positive definiteness of $\langle \cdot, \cdot \rangle_{\Lambda^k} : \Lambda^k V \times \Lambda^k V \to \mathbb{R}$, we use produce a basis for $\Lambda^k V$ which is orthonormal with respect to $\langle \cdot, \cdot \rangle_{\Lambda^k}$. In particular, let $\{e_i\}_{i=1}^n$ is be any orthonormal basis for V. Then taking $\mathcal{I}_k^{(n)}$ to be the set of well-ordered multi-indices of length k, of Definition 1.24, we have from 1.26 that set of vectors

$$\mathscr{E}_k := \{ e_{i_1} \wedge \ldots \wedge e_{i_k} \in \Lambda^k V \ : (i_1, \ldots, i_k) \in \mathcal{I}_k^{(n)} \},$$

is a basis for $\Lambda^k V$. It is easily checked that $\mathscr{E}_k$ is, in fact, an orthonormal set of vectors with respect to $\langle \cdot, \cdot \rangle_{\Lambda^k}$, from which it is easy to check positive definiteness. Hence, $\langle \cdot, \cdot \rangle_{\Lambda^k} : \Lambda^k V \times \Lambda^k V \to \mathbb{R}$ is an inner product on $\Lambda^k V$.

We now define the inner product on ΛV. First, recall that $\Lambda V = \bigoplus_{k=0}^{n} \Lambda^k V$. We define

$$\langle \cdot, \cdot \rangle_\Lambda : \Lambda V \times \Lambda V \to \mathbb{R}$$

by the formula, for ω in $\Lambda^{k_1} V$ and η in $\Lambda^{k_2} V$,

$$\langle \omega, \eta \rangle_\Lambda := \begin{cases} \langle \omega, \eta \rangle_{\Lambda^k}, & k = k_1 = k_2, \\ 0, & k_1 \neq k_2. \end{cases}.$$

As $\Lambda V = \bigoplus_{k=0}^{n} \Lambda^k V$, it is clear that

$$\langle \cdot, \cdot \rangle_\Lambda : \Lambda V \times \Lambda V$$

is a well-defined symmetric, bilinear map. Moreover, as $\langle \cdot, \cdot \rangle_{\Lambda^k}$ is an inner product on $\Lambda^k V$ for each k, it follows that $\langle \cdot, \cdot \rangle_\Lambda$ is also positive-definite on ΛV, and therefore determines an inner product on ΛV. By definition of this inner product on ΛV, whenever $k_1 \neq k_2$, $\Lambda^{k_1} V$ and $\Lambda^{k_2} V$ are orthogonal subspaces of ΛV. □

Remark 1.30. In the proof of Lemma 1.29, we showed that any orthonormal basis $\{e_i\}_{i=1}^n$ for an n-dimensional Euclidean space $(V, \langle \cdot, \cdot \rangle_V)$ induces an orthonormal basis for the $\binom{n}{k}$-dimensional Euclidean space $\left(\Lambda^k V, \langle \cdot, \cdot \rangle_{\Lambda^k}\right)$, with $0 \leq k \leq n$. In particular, the induced orthonormal basis of $\Lambda^k V$ is given by

$$\mathscr{E}_k := \{e_{i_1} \wedge \ldots \wedge e_{i_k} \in \Lambda^k V \ : (i_1, \ldots, i_k) \in \mathcal{I}_k^{(n)}\},$$

where by convention $\mathscr{E}_0 = \{1\} \subseteq \Lambda^0 V$, $\mathscr{E}_k = \emptyset$ when $k > n$ or $k < 0$, and $I_k^{(n)}$ denotes the set of well-ordered multi-indices of length k pulling from n as defined in Definition 1.24.

If we further set

$$\mathscr{E} := \bigcup_{k=0}^{n} \mathscr{E}_k,$$

then it is clear, by definition of the inner product $\langle \cdot, \cdot \rangle_\Lambda : \Lambda V \times \Lambda V \to \mathbb{R}$ on the exterior algebra ΛV, and by the fact that ΛV is the direct sum of orthogonal subspaces $\Lambda^k V$, that $\mathscr{E}$ is an orthonormal basis for ΛV.

With the Euclidean structure $(\Lambda V, \langle \cdot, \cdot \rangle_\Lambda)$ of Lemma 1.29 in hand, note that the canonical algebra isomorphism identifying $\mathbb{R}$ and $\Lambda^0 V$, and the canonical linear isomorphism identifying V and $\Lambda^1 V$, both given in Lemma 1.17, are isometric. Hence, there is no issues in continuing to identify $\Lambda^0 V$ with the algebra $\mathbb{R}$ and $\Lambda^1 V$ with the linear space V.

The following lemma shows that algebra product in ΛV is a continuous bilinear map with respect to the Euclidean topology on ΛV, and that one obtains families of bounded linear operators on $(\Lambda V, \langle\cdot,\cdot\rangle_V)$ which are given by left-multiplication, and the adjoints of such operators, respectively.

Lemma 1.31. *Let $(V, \langle\cdot,\cdot\rangle_V)$ be a Euclidean space of dimension n, and equip the exterior algebra with the Euclidean structure $(\Lambda V, \langle\cdot,\cdot\rangle_\Lambda)$ of Lemma 1.29. Take the norm induced by $\langle\cdot,\cdot\rangle_\Lambda$ to be denoted*

$$\|\omega\|_\Lambda = \sqrt{\langle\omega,\omega\rangle_\Lambda}, \text{ for all } \omega \in \Lambda V.$$

Then there exists a non-negative real constant $C > 0$ such that,

$$\|v \wedge \omega\|_\Lambda \leq C \|v\|_V \ \|\omega\|_\Lambda, \text{ for all } v \in \Lambda^1 V \text{ and } \omega \in \Lambda V.$$

Proof. We proceed using coordinates. Let $\{e_i\}_{i=1}^n$ be any orthonormal basis for $(V, \langle\cdot,\cdot\rangle_V)$. Letting $\mathcal{I}^{(n)} = \bigcup_{k=0}^n \mathcal{I}_k^{(n)}$, where

$$\mathcal{I}_k^{(n)} := \left\{(i_1, i_2, \ldots, i_k) \in \{1, 2, \ldots, n\}^k \ : i_1 < i_2 < \cdots < i_k\right\},$$

with the convention $\mathcal{I}_0^{(n)} = \{\emptyset\}$, we have from Remark 1.30 that the set

$$\mathscr{E} := \left\{e_I \in \Lambda V \ : I \in \mathcal{I}^{(n)}\right\}, \tag{1.15}$$

is an orthonormal basis for ΛV; With $e_I := e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$ for each $I = (i_1, i_2, \ldots, i_k) \in \mathcal{I}^{(n)}$, subject to the convention $e_\emptyset = 1 \in \Lambda^0 V$.

Now, taking any $\omega \in \Lambda V$, write $\omega = \sum_{I \in \mathcal{I}^{(n)}} \omega_I e_I$, with $\omega_I \in \mathbb{R}$ for each $I \in \mathcal{I}^{(n)}$. If $I = (i_1, \ldots, i_k) \in \mathcal{I}_k^{(n)}$ for some non-negative integer $k \leq n$, and if $j \in \{1, \ldots, n\}$, we will write $j \notin I$ to mean that j is not one of the indices i_t for any integer $1 \leq t \leq k$. Under this notation, we see that for $j \in \{1, \ldots, n\}$ and $I \in \mathcal{I}^{(n)}$,

$$e_j \wedge e_I = 0 \iff j \in I.$$

One sees that that $e_j \wedge e_I$ is a unit vector in ΛV whenever $j \notin I$, and that the vectors $e_i \wedge e_I$ and $e_j \wedge e_J$ and orthogonal whenever $I \neq J \in \mathcal{I}^{(n)}$.

Hence, taking any $\omega = \sum_{I \in \mathcal{I}^{(n)}} \omega_I e_I$ in ΛV with $\omega_I \in \mathbb{R}$ for each $I \in \mathcal{I}^{(n)}$, and any $v = \sum_{j=1}^n v_j e_j \in V$ with $v_i \in \mathbb{R}$ for each $j \in \{1, \ldots, n\}$, we have

$$\max_{1 \leq j \leq n} |v_i| \leq \sqrt{\sum_{j=1}^n |v_j|^2} = \|v\|_V$$

and

$$
\begin{aligned}
\|v \wedge \omega\|_\Lambda &\leq \sum_{j=1}^{n} |v_i| \, \|e_j \wedge \omega\|_\Lambda \\
&\leq \left(\max_{1 \leq j \leq n} |v_j| \right) \sum_{j=1}^{n} \|e_j \wedge \omega\|_\Lambda \\
&\leq \|v\|_V \sum_{j=1}^{n} \|e_j \wedge \omega\|_\Lambda \\
&= \|v\|_V \sum_{j=1}^{n} \left\| \sum_{\substack{I \in \mathcal{I}^{(n)} \\ j \notin I}} \omega_I e_j \wedge e_I \right\|_\Lambda \\
&= \|v\|_V \sum_{j=1}^{n} \sqrt{\sum_{\substack{I \in \mathcal{I}^{(n)} \\ j \notin I}} \omega_I^2} \\
&\leq n \|v\|_V \|\omega\|_\Lambda
\end{aligned}
$$

Since $v \in V$ and $\omega \in \Lambda V$ were arbitrary, the result holds. □

Lemma 1.32. *Let $(V, \langle \cdot, \cdot \rangle_V)$ be a Euclidean space with dim$(V) =: n$, and let $(\Lambda V, \langle \cdot, \cdot \rangle_\Lambda)$ be the Euclidean structure of the exterior algebra given in Lemma 1.29. We let $\| \cdot \|_\Lambda : \Lambda V \to [0, \infty)$ denote the norm induced by the inner product on ΛV, and $\mathbb{B}(\Lambda V)$ the bounded linear operators on $(\Lambda V, \langle \cdot, \cdot \rangle_\Lambda)$.*

For each vector v in V, define a function $\lambda_v : \Lambda V \to \Lambda V$ by

$$\lambda_v(\omega) := v \wedge \omega, \text{ for all } \omega \in \Lambda V. \tag{1.16}$$

Then $\lambda_v : \Lambda V \to \Lambda V$ satisfies the following conditions:

1) *$\lambda_v \in \mathbb{B}(\Lambda V)$, with $\|v\|_V \leq \|\lambda_v\|_{\mathbb{B}} \leq n\|v\|_V$ where $\| \cdot \|_{\mathbb{B}} : \mathbb{B}(\Lambda V) \to [0, \infty)$ is the operator norm,*

2) *λ_v is a grading degree 1 operator, in the sense of Definition 1.16, with respect to the $\mathbb{Z}$-grading on ΛV of Lemma 1.15; That is, for each integer k, $\lambda_v : \Lambda^k V \to \Lambda^{k+1} V$.*

3) *$\lambda_v^2 = 0 \in \mathbb{B}(\Lambda V)$.*

Moreover, the assignment $\lambda_{(\cdot)} : V \to \mathbb{B}(\Lambda V) : v \mapsto \lambda_v$ is linear.

Proof. It follows by linearity in the second coordinate of the algebra product in ΛV that the function $\lambda_v : \Lambda V \to \Lambda V$ defined by Equation (1.16) is linear, and that conditions 2) and 3) of the lemma are satisfied. Boundedness, and in particular that $\|\lambda_v\|_{\mathbb{B}} \leq n\|v\|_V$ follows is the result of Lemma 1.31. To see $\|\lambda_v\|_{\mathbb{B}} \geq \|v\|_V$, apply λ_v to the identity element in ΛV.

Linearity of the assignment $\lambda_{(\cdot)} : V \mapsto \mathbb{B}(\Lambda V) : v \mapsto \lambda_v$ follows by linearity in the first coordinate of the algebra product in ΛV. □

The following corollary is a result of Lemma 1.32 and the universal property of the exterior algebra given in Proposition 1.19.

Corollary 1.33. *The operator $\lambda_{(\cdot)} : V \to \mathbb{B}(\Lambda V)$ given in Lemma 1.32 extends uniquely to an algebra homomorphism, denoted $\lambda_{(\cdot)} : \Lambda V \to \mathbb{B}(\Lambda V)$, having the following properties:*

1) *For any η in ΛV, the operator $\lambda_\eta \in \mathbb{B}(\Lambda V)$ is given by the equation*

$$\lambda_\eta(\omega) := \eta \wedge \omega, \text{ for all } \omega \in \Lambda V. \tag{1.17}$$

2) *For any k-th exterior tensor η in $\Lambda^k V$, we have $\|\eta\|_\Lambda \leq \|\lambda_\eta\|_{\mathbb{B}} \leq \binom{n}{k}\|\eta\|_\Lambda$, where $\|\cdot\|_{\mathbb{B}} : \mathbb{B}(\Lambda V) \to [0,\infty)$ is the operator norm. In particular, for η in ΛV of mixed degree, $\|\eta\|_\Lambda \leq \|\lambda_\eta\|_{\mathbb{B}} \leq 2^n\|\eta\|_\Lambda$.*

3) *If k are integers and η is an exterior k-tensor in $\Lambda^k V$, then the linear operator λ_η is degree k with respect to the $\mathbb{Z}$-grading on ΛV; That is, for each integer k', $\lambda_\eta : \Lambda^{k'} V \to \Lambda^{k'+k} V$.*

Proof. The fact that the linear map $\lambda_{(\cdot)} : V \to \mathbb{B}(\Lambda V)$ extends to an algebra homomorphism $\lambda_{(\cdot)} : \Lambda V \to \mathbb{B}(\Lambda V)$ follows by point 3) of Lemma 1.32 and the universal property of ΛV (see Proposition 1.19).

For each $\eta \in \Lambda V$, we define a linear map $\tilde{\lambda}_\eta : \Lambda V \to \Lambda V$ given by the formula of Equation (1.17). Assuming $\tilde{\lambda}_\eta \in \mathbb{B}(\Lambda V)$, which we prove momentarily, it will follows that the induced map $\tilde{\lambda}_{(\cdot)} : \Lambda V \to \mathbb{B}(\Lambda V) : \eta \mapsto \tilde{\lambda}_\eta$ is an algebra homomorphism whose restriction to $V = \Lambda^1 V$ agrees with the linear map $\lambda_{(\cdot)} : V \to \mathbb{B}(\Lambda V)$ of Lemma 1.32. By uniqueness in the universal property of ΛV, it will follows that the universal extension $\lambda_{(\cdot)} : \Lambda V \to \mathbb{B}(\Lambda V)$ agrees with our defined map, $\tilde{\lambda}_{(\cdot)} : \Lambda V \to \mathbb{B}(\Lambda V)$. The remaining claims in Corollary 1.33 follow from this.

Let $\{e_i\}_{i=1}^n$ be an orthonormal basis for V, and let

$$\mathscr{E} := \left\{ e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \ : \ I \in \mathcal{I}^{(n)} \right\},$$

be the induced orthonormal basis for ΛV, where

$$\mathcal{I}^{(n)} := \bigcup_{k=0}^{n} \mathcal{I}_k^{(n)}$$

and

$$\mathcal{I}_k^{(n)} := \left\{(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \ : i_1 < i_2 < \ldots < i_k\right\},$$

subject to the convention $I_0^{(n)} := \{\emptyset\}$ and $e_\emptyset = 1 \in \Lambda^0 V$ (see Remark 1.30).

Assuming η is in $\Lambda^k V$, we show $\|\eta\|_\Lambda \leq \|\tilde{\lambda}_\eta\|_{\mathbb{B}} \leq 2^n \|\eta\|_\Lambda$ and, in particular, that $\tilde{\lambda}_\eta \in \mathbb{B}(\Lambda V)$. Assume first that $\eta = \eta_J e_J$ for some $J \in \mathcal{I}_k^{(n)}$ and $\eta_J \in \mathbb{R}$, *i.e.* that η has only one non-zero coordinate with respect to the basis $\mathscr{E}$ of ΛV. Then for any $\omega = \sum_{I \in \mathcal{I}^{(n)}} \omega_I e_I \in \Lambda V$, where $\omega_I \in \mathbb{R}$ for each $I \in \mathcal{I}_k^{(n)}$, we have,

$$\tilde{\lambda}_\eta(\omega) = \eta_I \sum_{I \in \mathcal{I}} \omega_I e_J \wedge e_I = \eta_I \sum_{\substack{I \in \mathcal{I} \\ I \cap J = \emptyset}} \omega_I e_J \wedge e_I,$$

with the notation $I \cap J = \emptyset$ meaning that the well-ordered multi-index $I \in \mathcal{I}$ and the well-ordered multi-index $J \in \mathcal{I}$ share no common indices. Hence,

$$\begin{aligned}
\|\tilde{\lambda}_{\eta_J e_J}(\omega)\|_\Lambda^2 :=& \langle \tilde{\lambda}_{\eta_J e_J}(\omega), \tilde{\lambda}_{\eta_J e_J}(\omega)\rangle_\Lambda \\
=& \eta_J^2 \left\langle \sum_{\substack{I \in \mathcal{I}^{(n)} \\ I \cap J = \emptyset}} \omega_I e_J \wedge e_I, \sum_{\substack{I \in \mathcal{I}^{(n)} \\ I \cap J = \emptyset}} \omega_I e_J \wedge e_I \right\rangle_\Lambda \\
=& \|\eta_J e_J\|_\Lambda^2 \left\| \sum_{\substack{I \in \mathcal{I}^{(n)} \\ I \cap J = \emptyset}} \omega_I e_J \wedge e_I \right\|_\Lambda^2 \\
=& \|\eta_J e_J\|_\Lambda^2 \left(\sum_{\substack{I \in \mathcal{I}^{(n)} \\ I \cap J = \emptyset}} \omega_I^2 \right) \\
\leq& \|\eta_J e_J\|_\Lambda^2 \left(\sum_{I \in \mathcal{I}^{(n)}} \omega_I^2 \right) = \|\eta_J e_J\|_\Lambda^2 \|\omega\|_\Lambda^2
\end{aligned}$$

where the fourth equality follows by the fact that if $I, K \in \mathcal{I}^{(n)}$ with $I \neq K$, then the vectors $e_J \wedge e_I$ and $e_J \wedge e_K$ are normal and orthogonal in ΛV. Thus, $\|\tilde{\lambda} \eta_J e_J(\omega)\|_\Lambda \leq |\eta_J| \|\omega\|_\Lambda$. Now let $\eta \in \Lambda^k V$ be a general k-th exterior tensor, say $\eta = \sum_{J \in \mathcal{I}_k^{(n)}} \eta_J e_J \in \Lambda V$ with $\eta_J \in \mathbb{R}$ for each $J \in \mathcal{I}^{(n)}$. Observe that $\tilde{\lambda}_\eta = \sum_{J \in \mathcal{I}_k^{(n)}} \tilde{\lambda}_{\eta_J e_J}$ and so, by the previous computation and the triangle inequality, we see

$$\begin{aligned}
\|\tilde{\lambda}_\eta(\omega)\|_\Lambda &= \left\| \sum_{J\in\mathcal{I}_k^{(n)}} \tilde{\lambda}_{\eta_J e_J}(\omega) \right\|_\Lambda \\
&\le \sum_{J\in\mathcal{I}_k^{(n)}} \|\tilde{\lambda}_{\eta_J e_J}(\omega)\|_\Lambda \\
&\le \left(\sum_{J\in\mathcal{I}_k^{(n)}} |\eta_J| \right) \|\omega\|_\Lambda \\
&\le \binom{n}{k} \left(\max_{J\in\mathcal{I}_k^{(n)}} |\eta_J| \right) \|\omega\|_\Lambda \\
&\le \binom{n}{k} \|\eta\|_\Lambda \ \|\omega\|_\Lambda
\end{aligned}$$

Hence, $\|\tilde{\lambda}_\eta\|_{\mathbb{B}} \le \|\eta\|_\Lambda$. To see $\|\tilde{\lambda}_\eta\|_{\mathbb{B}} \ge \|\eta\|_\Lambda$, apply λ_η to the identity $1 \in \Lambda^0 V$.

Finally, if η is a mixed exterior tensor in ΛV, then as $(\Lambda V, \langle\cdot,\cdot\rangle_\Lambda)$ decomposes into orthogonal subspaces as $\Lambda V = \bigoplus_{k=0}^n \Lambda^k V$ we have that $\eta = \sum_{k=0}^n \eta_j$ for unique $\eta_j \in \Lambda^j V$. Moreover, $\lambda_\eta = \sum_{k=0}^n \lambda_{\eta_j}$, so that

$$\begin{aligned}
\|\tilde{\lambda}_\eta(\omega)\|_\Lambda &= \left\| \sum_{k=0}^n \tilde{\lambda}_{\eta_J e_J}(\omega) \right\|_\Lambda \\
&\le \sum_{k=0}^m \|\tilde{\lambda}_{\eta_k}(\omega)\|_\Lambda \\
&\le \sum_{k=0}^m \binom{n}{k} \|\omega\|_\Lambda \\
&= 2^n \|\omega\|_\Lambda.
\end{aligned}$$

For the inequality $\|\eta\|_\Lambda \le \|\tilde{\lambda}_\eta\|_{\mathbb{B}}$, apply $\tilde{\lambda}_\eta$ to the identity in ΛV.

For the third claim, take k and k' to be integers and η to be an exterior k-tensor in $\Lambda^k V$. Then clearly our map satisfies $\tilde{\lambda}_\eta : \Lambda^{k'} V \to \Lambda^{k'+k} V$. Since the restriction of $\tilde{\lambda}_{(\cdot)}$ to the subspace $V = \Lambda^1 V$ agrees with the linear map $\lambda_{(\cdot)} : V \to \mathbb{B}(\Lambda V)$ of Lemma 1.32, the unique extension of $\lambda_{(\cdot)}$ agree with $\tilde{\lambda}_{(\cdot)}$, and so the claims of Corollary 1.33 hold. □

Remark 1.34. It follows form Corollary 1.33 that the algebra multiplication in ΛV is continuous with respect to the metric space topology induced by the Euclidean structure $(\Lambda V, \langle\cdot,\cdot\rangle_\Lambda)$ given in Lemma 1.29.

Finally, we consider the operation known as *interior multiplication* on the exterior algebra ΛV, which is defined, here, to be the adjoint of the left-wedge operation of Corollary 1.33. We first define interior multiplication by vectors in V through the following lemma.

Lemma 1.35. *Let $(V, \langle\cdot,\cdot\rangle_V)$ be an n-dimensional Euclidean space, and $(\Lambda V, \langle\cdot,\cdot\rangle_V)$ be the Euclidean structure on the exterior algebra of V described in Example 1.28. For a vector v in V, let λ_v denote the bounded linear operator defined in Lemma 1.32, and $\iota_v := \lambda_v^*$ its operator adjoint. Then, on elementary k-th exterior tensors $w_1 \wedge w_2 \wedge \ldots \wedge w_k$, a formula for the bounded linear operator ι_v is given by*

$$\iota_v(w_1 \wedge \cdots \wedge w_k) = \sum_{i=1}^{n} (-1)^{i+1} \langle v, w_i\rangle_V w_1 \wedge \cdots \wedge \widehat{w}_i \wedge \cdots \wedge w_k, \tag{1.18}$$

where $w_1 \wedge \cdots \wedge \widehat{w}_i \wedge \cdots \wedge w_k \in \Lambda^{k-1}V$ denotes the elementary $(k-1)$-th exterior tensor defined by removing w_i from the exterior k-tensor $w_1 \wedge w_2 \wedge \cdots \wedge w_k$ in $\Lambda^k V$.

In particular, for each $v \in V$,

1) *$\iota_v \in \mathbb{B}(\Lambda V)$ with $\|\iota_v\|_{\mathbb{B}} = \|\lambda_v\|_{\mathbb{B}}$ and $\|v\|_V \leq \|\iota_v\|_{\mathbb{B}} \leq n\|v\|_V$, where $\|\cdot\|_{\mathbb{B}} : \mathbb{B}(\Lambda V) \to [0, \infty)$ is the operator norm,*

2) *ι_v is a degree -1 linear operator with respect to the grading on ΛV of Lemma 1.15; That is, for each integer k, $\iota_v : \Lambda^k V \to \Lambda^{k-1}V$, and*

3) *$\iota_v^2 = 0 \in \mathbb{B}(\Lambda V)$.*

Proof. Let $v \in V$ and $\lambda_v \in \mathbb{B}(V)$ denote the left-wedge operation of Lemma 1.16. For any $k \in \mathbb{Z}$, take an elementary k-th exterior tensor $y_1 \wedge \cdots \wedge y_k \in \Lambda^k V$ and an elementary $(k+1)$-th exterior tensor $w_1 \wedge \cdots \wedge w_k \in \Lambda^{k+1}V$. Using cofactor expansion, it is easy to see

$$\begin{aligned}
\langle \lambda_v(y_1 \wedge \cdots \wedge y_k), w_1 \wedge \ldots \wedge w_{k+1}\rangle_\Lambda &= \langle v \wedge y_1 \wedge \cdots \wedge y_k, w_1 \wedge \cdots w_{k+1}\rangle_\Lambda \\
&= \det \begin{pmatrix} \langle v, w_1\rangle_V & \cdots & \langle v, w_{k+1}\rangle_V \\ \langle y_1, w_1\rangle_V & \cdots & \langle y_1, w_{k+1}\rangle_V \\ \vdots & \vdots & \vdots \\ \langle y_k, w_1\rangle_V & \cdots & \langle y_k, w_{k+1}\rangle_V \end{pmatrix} \\
&= \sum_{i=1}^{k} (-1)^{i+1} \langle v, w_i\rangle_V \langle y_1 \wedge \cdots \wedge y_k, w_1 \wedge \cdots \wedge \widehat{w}_i \wedge \cdots \wedge w_{k+1}\rangle_\Lambda \\
&= \left\langle y_1 \wedge \cdots \wedge y_k, \sum_{i=1}^{k} (-1)^{i+1} \langle v, w_i\rangle_V w_1 \wedge \cdots \wedge \widehat{w}_i \wedge \cdots w_{k+1} \right\rangle_\Lambda \\
&= \langle y_1 \wedge \cdots \wedge y_k, \iota_v(w_1 \wedge \cdots \wedge w_{k+1})\rangle_\Lambda .
\end{aligned}$$

Points 1), 2) and 3) are then immediate from the formula in Equation (1.18) above, and Lemma 1.32. □

Remark 1.36. From point 3) of Lemma 1.35, and the observation that the induced mapping $\iota_{(\cdot)} : V \to \mathbb{B}(\Lambda V)$ is linear, it follows that $\iota_{(\cdot)} : V \to \mathbb{B}(\Lambda V)$ extends to an algebra homomorphism $\iota_{(\cdot)} : \Lambda V \to \mathbb{B}(\Lambda V)$. Comparing this extension to the algebra homomorphism $\lambda_{(\cdot)} : \Lambda V \to \mathbb{B}(\Lambda V)$ of Corollary 1.33, we obtain the identity

$$\lambda_\omega^* = (-1)^{1+\deg(\omega)}\iota_\omega \in \mathbb{B}(\Lambda V), \text{ for all } \omega \in \Lambda V$$

where $\deg(\omega) = 1$ if $\omega \in \Lambda^+ V$ and $\deg(\omega) = -1$ if $\omega \in \Lambda^- V$, and $\Lambda^+ V$ and $\Lambda^- V$ are the even and odd subspaces, respectively, of the $\mathbb{Z}/2$-grading on ΛV given in Corollary 1.21. Moreover, if η is an exterior k-tensor in $\Lambda^k V$, then the induced linear operator ι_η is a degree k linear operator on ΛV in the sense that $\iota_\eta : \Lambda^{k'} V \to \Lambda^{k-k'} V$ for all integers k, and also $\|\eta\|_\Lambda \leq \|\iota_\eta\|_{\mathbb{B}} \leq \binom{n}{k}\|\eta\|_\Lambda$.

The last examples of this subsection are some basic function algebras, and the Hilbert space of *square-integrable* $\mathbb{C}$*-valued functions* on a Euclidean space of dimension $n := \dim(V)$. We begin with the case of standard Euclidean space $\mathbb{R}^n$.

Example 1.37. Let $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_{\mathbb{R}^n})$ denote the standard inner product structure of $\mathbb{R}^n$. Following the introduction of Warner [20, Chapter 1], for each integer j with $1 \leq j \leq n$, define the functional $r_j : \mathbb{R}^n \to \mathbb{R}$ to be the j-th (canonical) coordinate function on $\mathbb{R}^n$; That is, $r_j(t) = t_j$ when $t = (t_1, t_2, \ldots, t_n)$ is a point in $\mathbb{R}^n$. We call

$$r = (r_1, \ldots, r_n) : \mathbb{R}^n \longrightarrow \mathbb{R}^n \ : t \mapsto (r_1(t), r_2(t), \ldots, r_n(t)),$$

the *canonical system of coordinates on* $\mathbb{R}^n$.

Given an integer j with $1 \leq j \leq n$, and a map $h : \mathbb{R}^n \to \mathbb{C}$ for which the limit

$$\frac{\partial h}{\partial r_j}(t) := \lim_{s \xrightarrow{\mathbb{R}} 0} \frac{h\left((t_1, \ldots t_{j-1}, t_j + s, t_{j+1}, \ldots, t_n)\right) - h\left(t\right)}{s},$$

exists in $\mathbb{C}$ for each point $t = (t_1, \ldots, t_n)$ in $\mathbb{R}^n$, the induced function

$$\frac{\partial h}{\partial r_j} : \mathbb{R}^n \to \mathbb{C} : t \mapsto \frac{\partial h}{\partial r_j}(t),$$

is called the *partial derivative of* $h : \mathbb{R}^n \to \mathbb{C}$ *with respect to* r_j.

For functions $h_1, h_2 : \mathbb{R}^n \to \mathbb{C}$, we let their pointwise sum be denoted $h_1 + h_2 : \mathbb{R}^n \to \mathbb{C}$, while letting $h_1 \cdot h_2 : \mathbb{R}^n \to \mathbb{C}$ denote their pointwise product; We reserve the notation $h_1^k : \mathbb{R}^n \to \mathbb{R}$ to mean the pointwise product of h_1 with itself k-times, for some non-negative integer k. As well, for a complex number z, the pointwise product of z with h_1 will be written $zh_1 : \mathbb{R}^n \to \mathbb{C}$. To simplify the coming definitions, given an n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ of non-negative integers, also define

$$r^\alpha = r_1^{\alpha_1} \cdot r_2^{\alpha_2} \cdot \ldots \cdot r_n^{\alpha_n} : \mathbb{R}^n \longrightarrow \mathbb{R} \tag{1.19}$$

$$\|\alpha\|_1 := \sum_{j=1}^{n} \alpha_j, \quad \text{and} \quad \alpha! := \alpha_1!\alpha_2! \cdots \alpha_n!, \tag{1.20}$$

$$\frac{\partial^\alpha}{\partial r^\alpha} := \left(\frac{\partial}{\partial r_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial r_n}\right)^{\alpha_n}, \tag{1.21}$$

subject to the convention,

$$\frac{\partial^\alpha h}{\partial r^\alpha} = h, \text{ when } \alpha = (0, 0, \dots, 0) \text{ and } h : \mathbb{R}^n \to \mathbb{C}.$$

Definition 1.38. Let $h : \mathbb{R}^n \to \mathbb{C}$ be a function. We say that h is *differentiable of class* C^k, for k a non-negative integer, if the partial derivatives

$$\frac{\partial^\alpha h}{\partial r^\alpha} : \mathbb{R}^n \longrightarrow \mathbb{C}$$

exist, and are continuous, for all n-tuples of non-negative integers α satisfying $\|\alpha\|_1 \leq k$. In particular, h is C^0 if it is continuous.

Furthermore, h is said to be *smooth*, or C^∞, if it is C^k for all non-negative integers k. Lastly, h is *Schwartz-class* if it is both smooth and such that, for all n-tuples α and β of non-negative integers, the pointwise product

$$r^\beta \cdot \frac{\partial^\alpha h}{\partial r^\alpha} : \mathbb{R}^n \longrightarrow \mathbb{C} : t \longmapsto r^\beta(t)\, \frac{\partial^\alpha h}{\partial r^\alpha}(t),$$

is a bounded with respect to the norm on $\mathbb{C}$, where $r^\beta : \mathbb{R}^n \to \mathbb{R} \subseteq \mathbb{C}$ is given in Equation 1.19. We let $C^\infty(V)$ denote the set of all C^∞-functions, and $\mathcal{S}(V)$ its subset Schwartz-class functions.

The first observation one should make is that both $C^\infty(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ carry $\mathbb{C}$-algebras structures under the natural pointwise operations. In fact, $C^\infty(\mathbb{R}^n)$ is a unital, commutative $\mathbb{C}$-algebra, containing $\mathcal{S}(\mathbb{R}^n)$ as a non-unital subalgebra. For the smooth case, let $h_1, h_2 : \mathbb{R}^n \to \mathbb{R}$ be C^∞, and let be z complex number. Then, for any n-tuple of non-negative integers, it is easy to see that

$$\frac{\partial^\alpha (h_1 + zh_2)}{\partial r^\alpha} = \frac{\partial^\alpha h_1}{\partial r^\alpha} + z\frac{\partial^\alpha h_2}{\partial r^\alpha},$$

while, from Folland [6, pp. 236],

$$\frac{\partial^\alpha (h_1 \cdot h_2)}{\partial r^\alpha} = \sum_{\substack{\alpha=\beta+\gamma \\ \beta,\gamma \in \mathbb{Z}^n_{\geq 0}}} \frac{\alpha!}{\beta!\gamma!} \, \frac{\partial^\beta h_1}{\partial r^\beta} \cdot \frac{\partial^\gamma h_2}{\partial r}^\gamma, \tag{1.22}$$

where the notation $\alpha = \beta + \gamma$, for $\beta = (\beta_1, \ldots, \beta_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ in $\mathbb{Z}^n_{\geq 0}$, means $\alpha_j = \beta_j + \gamma_j$ for each integer j with $1 \leq j \leq n$. As the set of continuous functions from $\mathbb{R}^n$ into $\mathbb{C}$ is a commutative $\mathbb{C}$-algebra under pointwise addition, $\mathbb{C}$-scaling, and product, it follows from the identities above that $C^\infty(\mathbb{R}^n)$ is a commutative $\mathbb{C}$-algebra under such pointwise defined operations. The unit in $C^\infty(\mathbb{R}^n)$ is given by the function which is constantly 1 on all of $\mathbb{R}^n$. For the case of $\mathcal{S}(\mathbb{R}^n)$, note that each Schwartz-class function is smooth by definition. Moreover, trivial computations following from the triangle inequality in $\mathbb{C}$ and the previous paragraph's identities, verify that $\mathcal{S}(\mathbb{R}^n)$ is closed under the pointwise algebra operations in $C^\infty(\mathbb{R}^n)$. Hence, $\mathcal{S}(\mathbb{R}^n)$ is a commutative $\mathbb{C}$-algebra, and a subalgebra of $C^\infty(\mathbb{R}^n)$. When referring to either of $\mathcal{S}(\mathbb{R}^n)$ or $C^\infty(\mathbb{R}^n)$, as an $\mathbb{C}$-algebra, or even as a $\mathbb{C}$-linear space, we will mean with operations defined pointwise.

If we set,

$$\overline{h} : \mathbb{R}^n \longrightarrow \mathbb{C} \ : t \mapsto \overline{h(t)}, \tag{1.23}$$

to be the pointwise complex-conjugate of an arbitrary function $h : \mathbb{R}^n \to \mathbb{C}$, then equipping the algebra $C^\infty(\mathbb{R}^n)$ with the assignment $\bar{\cdot} : h \mapsto \overline{h}$ determines a $*$-algebra structure on $C^\infty(\mathbb{R}^n)$, under which $\mathcal{S}(\mathbb{R}^n)$ is a $*$-subalgebra. Indeed, if $h : \mathbb{R}^n \to \mathbb{C}$ is smooth, then for any n-tuple α of non-negative integers,

$$\frac{\partial^\alpha \left(\overline{h}\right)}{\partial r^\alpha} = \overline{\left(\frac{\partial^\alpha h}{\partial r^\alpha}\right)}, \tag{1.24}$$

the continuity of which follows by continuity of complex conjugation on $\mathbb{C}$, together with the smoothness of h. The identities required for the well-defined map $\bar{\cdot} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ to be an algebra involution are seen immediately from the pointwise structure of the algebra $C^\infty(\mathbb{R}^n)$. Moreover, since complex conjugation is an isometry on $\mathbb{C}$, it follows from Equation (1.24) that $\mathcal{S}(\mathbb{R}^n)$ is closed under the map $\bar{\cdot} : f \mapsto \overline{f}$, so that $\mathcal{S}(\mathbb{R}^n)$ is a $*$-subalgebra of the $*$-algebra $C^\infty(\mathbb{R}^n)$.

As shown in [6, Proposition 8.3], there are several equivalent definitions of Schwartz-class functions. Before providing them, we introduce

$$\| \cdot \|_{\mathbb{R}^n} : \mathbb{R}^n \to [0, \infty) : t \mapsto \sqrt{\langle t, t \rangle_{\mathbb{R}^n}}$$

to be the standard Euclidean norm on $\mathbb{R}^n$. If $r = (r_1, \ldots, r_n) : \mathbb{R}^n \to \mathbb{R}^n$ is the canonical system of coordinates on $\mathbb{R}^n$, observe that $r : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry in the sense that

$$\langle r(t_a), r(t_b)\rangle_{\mathbb{R}^n} = \langle t_a, t_b\rangle_{\mathbb{R}^n}, \text{ for all } t_a, t_b \in \mathbb{R}^n.$$

In particular, $\|r(t)\|_{\mathbb{R}^n} = \|t\|_{\mathbb{R}^n}$ for each vector $t \in \mathbb{R}^n$, and if we let

$$\|r\|_{\mathbb{R}^n} : \mathbb{R}^n \to [0,\infty) : t \mapsto \|r(t)\|_{\mathbb{R}^n} \tag{1.25}$$

denote the composition of the norm $\|\cdot\|_{\mathbb{R}^n} : \mathbb{R}^n \to [0,\infty)$ with $r : \mathbb{R}^n \to \mathbb{R}^n$, then $\|r\|_{\mathbb{R}^n} = \|\cdot\|_{\mathbb{R}^n}$ as functions on $\mathbb{R}^n$.

Proposition 1.39. *Let $h : \mathbb{R}^n \to \mathbb{C}$ be a smooth function. Then the following are equivalent:*

1. *h is a Schwartz-class function in the sense of Definition 1.38.*
2. *For all n-tuples α and β of non-negative integers, the function*

$$\frac{\partial^\alpha (r^\alpha \cdot h)}{\partial r^\alpha}\left(r^\beta \cdot h\right) : \mathbb{R}^n \longrightarrow \mathbb{C}$$

   is bounded; Where $r^\beta \cdot h : \mathbb{R}^n \to \mathbb{C}$ is the pointwise product $r^\beta : \mathbb{R}^n \to \mathbb{R} \subseteq \mathbb{C}$, as defined in Equation (1.19), and the function h.

3. *For all non-negative integers k, and all n-tuples α of non-negative integers, the pointwise product,*

$$(1 + \|r\|_{\mathbb{R}^n})^k \cdot \frac{\partial^\alpha h}{\partial r^\alpha} : \mathbb{R}^n \longrightarrow \mathbb{C}$$

   is a bounded function on $\mathbb{R}^n$; Where $1 : \mathbb{R}^n \to \mathbb{C}$ is the identity element in the algebra $C^\infty(\mathbb{R}^n)$, and $\|r\|_{\mathbb{R}^n} : \mathbb{R}^n \to [0,\infty)$ is defined in Equation (1.25)

Recall that a *derivation* on an algebra A is a linear map $T : A \to A$ such that $T(ab) = T(a)b + aT(b)$ for each a and b in A; This identity is called the Leibniz condition. Our next lemma introduces two families of linear operators on the algebra $C^\infty(\mathbb{R}^n)$, which also restrict to linear operators on its subalgebra $\mathcal{S}(\mathbb{R}^n)$, such that these families are indexed by the set of n-tuples with non-negative integer coordinates. In particular cases, wherein the sum of the coordinates of the n-tuple is 1, the operators arising from one of these families are not only linear on $C^\infty(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$, but are in fact linear derivations of these algebras.

Lemma 1.40. *Let α be an n-tuple of non-negative integers, and let $r^\alpha : \mathbb{R}^n \to \mathbb{R} \subseteq \mathbb{C}$ be defined as in Equation (1.19).*

- *The function r^α is smooth in the sense of Definition 1.38. Moreover, the assignment $r^\alpha\cdot : h \mapsto r^\alpha \cdot h$, taking a function $h : \mathbb{R}^n \to \mathbb{C}$ to its pointwise product with r^α, determines a well-defined linear operator on the algebra $C^\infty(\mathbb{R}^n)$, and on its subalgebra $\mathcal{S}(\mathbb{R}^n)$.*

- *The assignment* $\frac{\partial^\alpha}{\partial r^\alpha} : f \mapsto \frac{\partial^\alpha f}{\partial r^\alpha}$ *determines a well-defined linear operator on* $C^\infty(\mathbb{R}^n)$*, and on its subalgebra* $\mathcal{S}(\mathbb{R}^n)$*, such that*

$$\frac{\partial^\alpha(h_1 \cdot h_2)}{\partial r^\alpha} = \sum_{\substack{\alpha=\beta+\gamma \\ \beta,\gamma\in\mathbb{Z}^n_{\geq 0}}} \frac{\alpha!}{\beta!\gamma!} \frac{\partial^\beta h_1}{\partial r^\beta} \cdot \frac{\partial^\gamma h_2}{\partial r^\gamma},$$

where $h_1 \cdot h_2 : \mathbb{R}^n \to \mathbb{C}$ *is the pointwise product of smooth, or Schwartz-class, functions* h_1 *and* h_2*. In particular, for an integer* j *with* $1 \leq j \leq n$*, the assignment* $\frac{\partial}{\partial r_j} : h \mapsto \frac{\partial h}{\partial r_j}$*, determines a linear derivation on the algebra* $C^\infty(\mathbb{R}^n)$*, and on its subalgebra* $\mathcal{S}(\mathbb{R}^n)$*.*

Proof. For each integer j with $1 \leq j \leq n$, observe that the coordinate function $r_j : \mathbb{R}^n \to \mathbb{R} \subseteq \mathbb{C}$ is linear, and hence smooth. By definition, $r^\alpha : \mathbb{R}^n \to \mathbb{R} \subseteq \mathbb{C}$ is a pointwise product of the coordinate functions r_j, and since $C^\infty(\mathbb{R}^n)$ is an algebra under pointwise operations, we conclude r^α is a smooth function. Again applying the pointwise algebra structure of $C^\infty(\mathbb{R}^n)$, the smoothness of $r^\alpha : \mathbb{R}^n \to \mathbb{C}$ then implies

$$r^\alpha \cdot : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n) : h \longmapsto r^\alpha \cdot h,$$

is a well-defined linear operator on $C^\infty(\mathbb{R}^n)$. Observe that if β is any other n-tuple of non-negative integers, then

$$r^\beta \cdot r^\alpha \cdot = r^{\beta+\alpha} \cdot : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n).$$

Hence, if $h : \mathbb{R}^n \to \mathbb{C}$ is Schwartz-class, then for any n-tuples β and γ of non-negative integers,

$$\frac{\partial^\gamma(r^\beta \cdot r^\alpha \cdot h)}{\partial r^\gamma} = \frac{\partial^\gamma(r^{\beta+\gamma} \cdot h)}{\partial r^\gamma},$$

which is bounded by the assumption that h is Schwartz-class and Proposition 1.39. We therefore conclude

$$r^\alpha \cdot : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) : h \longmapsto r^\alpha \cdot h$$

is a well-defined linear operator on $\mathcal{S}(\mathbb{R}^n)$.

On the other hand, for a smooth function $h : \mathbb{R}^n \to \mathbb{C}$, Clairault's Theorem [19, Section 14.3] states that,

$$\frac{\partial}{\partial r_j}\left(\frac{\partial h}{\partial r_k}\right) = \frac{\partial}{\partial r_k}\left(\frac{\partial h}{\partial r_j}\right) : \mathbb{R}^n \to \mathbb{C},$$

for any integers j and k with $1 \leq j \leq k$. It follows that, for any n-tuple β,

$$\frac{\partial^\beta}{\partial r^\beta}\left(\frac{\partial^\alpha h}{\partial r^\alpha}\right) = \frac{\partial^{\alpha+\beta} h}{\partial r^{\alpha+\beta}},$$

which is continuous by the smoothness of $h : \mathbb{R}^n \to \mathbb{C}$. Thus,

$$\frac{\partial^\alpha}{\partial r^\alpha} : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n) : h \longmapsto \frac{\partial^\alpha h}{\partial r^\alpha} \tag{1.26}$$

is a well-defined map. To see that map given in Equation (1.26) is linear observe, for each integer $1 \leq j \leq n$, the partial derivative with respect to r_j,

$$\frac{\partial}{\partial r_j} : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n) : h \longmapsto \frac{\partial h}{\partial r_j},$$

is well-defined by the argument above, and is linear by linearity of the defining limit. We may therefore regard the map in Equation (1.26) as a composition of linear operators on $C^\infty(\mathbb{R}^n)$, and so is itself a linear operator. Moreover, if $h : \mathbb{R}^n \to \mathbb{C}$ is Schwartz-class, then for any n-tuples β and γ of non-negative integers,

$$r^\beta \cdot \frac{\partial^\gamma}{\partial r^\gamma}\left(\frac{\partial^\alpha h}{\partial r^\alpha}\right) = r^\beta \cdot \frac{\partial^{\gamma+\alpha} h}{\partial r^{\gamma+\alpha}} : \mathbb{R}^n \longrightarrow \mathbb{C}$$

is bounded by Proposition 1.39, so that $\frac{\partial^\alpha h}{\partial r^\alpha} : \mathbb{R}^n \to \mathbb{C}$ is Schwartz-class. It follows that the map

$$\frac{\partial^\alpha}{\partial r^\alpha} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) : h \longmapsto \frac{\partial^\alpha h}{\partial r^\alpha}$$

is well-defined and, in fact, a linear operator since $\mathcal{S}(\mathbb{R}^n)$ is a subalgebra of $C^\infty(V)$.

Finally, the fact that $\frac{\partial}{\partial r_j} : h \mapsto \frac{\partial h}{\partial r_j}$ is a derivation when acting on $C^\infty(\mathbb{R}^n)$, or on its subalgebra $\mathcal{S}(\mathbb{R}^n)$, is merely an application of the identity in Equation (1.22). □

With this modest review in hand, we equip the $\mathbb{C}$-linear space $\mathcal{S}(\mathbb{R}^n)$ with an inner product. To do so, let dr denote the Lebesgue measure on $\mathbb{R}^n$ and recall, for any Schwartz-class function $f : \mathbb{R}^n \to \mathbb{C}$, that $\overline{f} : \mathbb{R}^n \to \mathbb{C}$ is the Schwartz-class function defined by taking the pointwise complex-conjugate of f. Often referred to as the L^2-*inner product*, we set

$$\langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^n)} : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C} : \quad \langle f_1, f_2\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \overline{f_1} \cdot f_2 \; dr. \tag{1.27}$$

To verify this map is well-defined, note Hölder's Inequality [6, Theorem 6.2] implies we need only show that $\int_{\mathbb{R}^n} |f|^2 \; dr$ is finite for each Schwartz-class function $f : \mathbb{R}^n \to \mathbb{C}$. To see this, use Proposition 1.39 to deduce that for $f : \mathbb{R}^n \to \mathbb{C}$ Schwartz-class, and any integer k, there is a constant $C_k > 0$ for which the function $|f|^2 : t \mapsto |f(t)|^2$ is bounded above by

$$C_k \left(1 + \|r\|_{\mathbb{R}^n}\right)^{-2k} : \mathbb{R}^n \longrightarrow \mathbb{R} \; : t \longmapsto C_k \left(1 + \|t\|_{\mathbb{R}^n}\right)^{-2k},$$

which has finite Lebesgue integral whenever $k > n/2$ [6, Corollary 2.52]. Verifying that the map defined by Equation (1.27) is symmetric, linear in the second coordinate, and positive definite are clear from basic Lebesgue integral theory; For which we refer the reader to [6, Chapter 2, 6].

The resulting inner product space $\left(\mathcal{S}(\mathbb{R}^n), \langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^n)}\right)$ is not complete with respect to the induced inner product norm

$$\|\cdot\|_{L^2(\mathbb{R}^n)} : \mathcal{S}(\mathbb{R}^n) \longrightarrow [0,\infty) : h \mapsto \sqrt{\langle h, h\rangle_{L^2(\mathbb{R}^n)}}.$$

As such, we use the notation of a *Hilbert space completion.* The following theorem is found in Roman [16, Theorem 13.6].

Theorem 1.41. *Let $(X, \langle\cdot,\cdot\rangle_X)$ be a complex inner product space. Then there exists a complex Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$, and a $\mathbb{C}$-linear map $\tau : X \to H$, which is an isometry in the sense that,*

$$\langle a, b\rangle_X = \langle \tau(a), \tau(b)\rangle_H, \text{ for all } a, b \in X,$$

and such that range of $\tau : X \to H$ is a dense subspace of H. Moreover, $(H, \langle\cdot,\cdot\rangle_H)$ is unique up to isometric isomorphism.

Applying Theorem 1.41 to the inner product space $\left(\mathcal{S}(\mathbb{R}^n), \langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^n)}\right)$, we obtain the Hilbert space of *square-integrable functions*

$$\left(L^2(\mathbb{R}^n), \langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^n)}\right).$$

The name square-integrable "functions" for $L^2(\mathbb{R}^n)$ is somewhat misleading since, technically, $L^2(\mathbb{R}^n)$ no longer consists of Schwartz-class functions, or even functions at all. Rather, a constructive version of Theorem 1.41 shows tat the linear space $L^2(\mathbb{R}^n)$ identifies with equivalence classes of Cauchy sequences in $\mathcal{S}(\mathbb{R}^n)$, where by Cauchy we mean with respect to the inner product norm $\|\cdot\|_{L^2(\mathbb{R}^n)} : \mathcal{S}(\mathbb{R}^n) \to [0,\infty)$. In particular, two sequences of Schwartz-class functions $\{h_j\}_{j\in\mathbb{N}}$ and $\{f_j\}_{j\in\mathbb{N}}$, both Cauchy with respect to the norm $\|\cdot\|_{L^2(\mathbb{R}^n)} : \mathcal{S}(\mathbb{R}^n) \to [0,\infty)$, are of the same equivalence class if and only if

$$\|h_j - f_j\|_{L^2(\mathbb{R}^n)} \to 0 \text{ as } j \to \infty.$$

However, the existence of an isometric linear map $\tau : \mathcal{S}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with dense range ensures that the inner product space $\mathcal{S}(\mathbb{R}^n)$ identifies isometrically with a dense subspace of $L^2(\mathbb{R}^n)$. In particular, there is nor harm in viewing $\mathcal{S}(\mathbb{R}^n)$ as a dense subspace of $L^2(\mathbb{R}^n)$. We will make heavy use of this direct identification while noting that, in the context of $L^2(\mathbb{R}^n)$ consisting of equivalence classes of Cauchy sequences in $\mathcal{S}(\mathbb{R}^n)$, the map τ sends a Schwartz-class function $h : \mathbb{R}^n \to \mathbb{C}$ to the equivalence class of the constant Cauchy sequence $\{h\}_{j\in\mathbb{N}} \subseteq \mathcal{S}(\mathbb{R}^n)$.

The following extends Example 1.37 to the case of a general Euclidean space. Note that an arbitrary Euclidean space does not carry a canonical system of coordinates $r = (r_1, \ldots, r_n) : \mathbb{R}^n \to \mathbb{R}^n$. However, using the Euclidean structure of Example 1.27, we will see that any orthonormal basis for the dual space induces an orthonormal coordinate system which we may use to discuss notations of regularity for functions.

Example 1.42. Fix an arbitrary Euclidean space $(V, \langle \cdot, \cdot \rangle_V)$ of dimension $n := \dim(V)$, and let $(V^*, \langle \cdot, \cdot \rangle_{V^*})$ denote the induced n-dimensional Euclidean structure on the dual space V^* given in Example 1.27. For an orthonormal basis $\{x_j\}_{j=1}^n$ of V^*, we call the induced isometric linear isomorphism

$$x = (x_1, \ldots, x_n) : V \longrightarrow \mathbb{R}^n \ : v \mapsto (x_1(v), \ldots, x_n(v)),$$

an *orthonormal system of coordinates* on V.

The definition of smooth functions on V given below is that of [20, Definition 1.6], while Schwartz-class functions on V are defined in the obvious way; Both of these definitions make heavy use of orthonormal coordinate systems.

Definition 1.43. Let $f : V \to \mathbb{C}$ be a function. We say that f is *smooth*, if for each orthonormal system of coordinates $x : V \to \mathbb{R}^n$, the function

$$f \circ x^{-1} : \mathbb{R}^n \longrightarrow \mathbb{C},$$

is smooth on $\mathbb{R}^n$. Similarly, f is called *Schwartz-class* if the function

$$f \circ x^{-1} : \mathbb{R}^n \longrightarrow \mathbb{C},$$

is Schwartz-class on $\mathbb{R}^n$, for each orthonormal system of coordinates. We let $C^\infty(V)$ denote the set of smooth functions on V, and $\mathcal{S}(\mathbb{R}^n)$ its subset of Schwartz-class functions.

Lemma 1.44. *Equipped with pointwise addition, $\mathbb{C}$-scaling, and product, along with adjoint given by pointwise complex conjugation, the set $C^\infty(V)$ is a unital, commutative $*$-algebra containing $\mathcal{S}(V)$ as a non-unital $*$-subalgebra. Moreover, any orthonormal system of coordinates $x : V \to \mathbb{R}^n$ induces a commutative diagram,*

$$\begin{array}{ccc} \mathcal{S}(V) & \hookrightarrow & C^\infty(V) \\ \downarrow{\scriptstyle x_*} & & \downarrow{\scriptstyle x_*} \\ \mathcal{S}(\mathbb{R}^n) & \hookrightarrow & C^\infty(\mathbb{R}^n) \end{array} ,$$

where the horizontal maps are $*$-algebra inclusions, and the vertical maps are $*$-isomorphisms, both of which are given by the formula $x_*(f) = f \circ x^{-1} : \mathbb{R}^n \to \mathbb{C}$ for a smooth, or respectively Schwartz-class, function $f : V \to \mathbb{C}$.

Proof. If $f_1, f_2 : V \to \mathbb{C}$ are smooth, or Schwartz-class, and z is a complex number, then for any orthonormal coordinate system $x : V \to \mathbb{R}^n$,

$$\begin{aligned} (f_1 + zf_2) \circ x^{-1} &= f_1 \circ x^{-1} + z\left(f_2 \circ x^{-1}\right) : \mathbb{R}^n \longrightarrow \mathbb{C} \\ (f_1 \cdot f_2) \circ x^{-1} &= \left(f_1 \circ x^{-1}\right) \cdot \left(f_2 \circ x^{-1}\right) : \mathbb{R}^n \longrightarrow \mathbb{C} \\ \overline{f} \circ x^{-1} &= \overline{f \circ x^{-1}} : \mathbb{R}^n \to \mathbb{C}. \end{aligned}$$

As $C^\infty(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ are $*$-algebras under the natural pointwise operations, it follows from the identities above that $C^\infty(V)$ and $\mathcal{S}(V)$ are $*$-algebras under pointwise-defined operations operations. Clearly, the algebra product in $C^\infty(V)$ is commutative, and the unit in $C^\infty(V)$ is the function which takes every vector in V to the identity in $\mathbb{C}$; Note that this unit is not Schwartz-class.

Moreover, the identities above show that, given an orthonormal system of coordinates $x : V \to \mathbb{R}^n$, the assignment $x_* : f \mapsto f \circ x^{-1}$ determines a $*$-algebra homomorphism from $C^\infty(V)$ into $C^\infty(\mathbb{R}^n)$, and from $\mathcal{S}(V)$ into $\mathcal{S}(\mathbb{R}^n)$. To see that these $*$-homomorphisms are invertible, we show $h \circ x : V \to \mathbb{C}$ is smooth whenever $h : \mathbb{R}^n \to \mathbb{C}$ is smooth, for if this is the case then $x_*^{-1} : h \mapsto h \circ x$ is an inverse for both. Indeed, if $y : V \to \mathbb{R}^n$ is any other orthonormal system of coordinates, then $x \circ y^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isometry, and pre-composing any smooth function by a linear endomorphism of $\mathbb{R}^n$ produces another smooth function on $\mathbb{R}^n$. □

In Example 1.37, wherein smooth and Schwartz-class functions on $\mathbb{R}^n$ were considered, we took $r = (r_1, \ldots, r_n) : \mathbb{R}^n \to \mathbb{R}^n$ to denote canonical coordinate system on $\mathbb{R}^n$, and $\frac{\partial \mathrm{h}}{\partial r_j} : \mathbb{R}^n \to \mathbb{C}$ denote the partial derivative, with respect to r_j, of a function $h : \mathbb{R}^n \to \mathbb{C}$. In Lemma 1.40, we observed that the assignment $\frac{\partial}{\partial r_j} : h \mapsto \frac{\partial h}{\partial r_j}$ determines a linear derivation on the algebra $C^\infty(\mathbb{R}^n)$, and on its subalgebra $\mathcal{S}(\mathbb{R}^n)$. In the content of functions on V, one must start with an arbitrary orthonormal coordinate system $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ and, using Lemma 1.44, define the *partial derivative with respect to* x_j, denoted $\frac{\partial}{\partial x_k}$, to be the linear derivation which acts on the algebra $C^\infty(V)$, and its subalgebra $\mathcal{S}(V)$, defined by the following commutative diagram:

$$\begin{array}{ccccccc} C^\infty(V) & \xrightarrow{x_*} & C^\infty(\mathbb{R}^n) & \xrightarrow{\frac{\partial}{\partial r_j}} & C^\infty(\mathbb{R}^n) & \xrightarrow{x_*^{-1}} & C^\infty(V) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathcal{S}(V) & \xrightarrow{x_*} & \mathcal{S}(\mathbb{R}^n) & \xrightarrow{\frac{\partial}{\partial r_j}} & \mathcal{S}(\mathbb{R}^n) & \xrightarrow{x_*^{-1}} & \mathcal{S}(V) \end{array} \qquad \text{with } \frac{\partial}{\partial x_j} : C^\infty(V) \to C^\infty(V),\ \frac{\partial}{\partial x_j} : \mathcal{S}(V) \to \mathcal{S}(V) \tag{1.28}$$

From the definition of such partial derivatives, it is immediate that an analogue of

Clairault's Theorem holds in the sense that, for integers $1 \leq j \leq k \leq n$,

$$\frac{\partial}{\partial x_j}\,\frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_j}\,\frac{\partial}{\partial x_k},$$

as linear operators on $C^\infty(V)$, or $\mathcal{S}(V)$. Moreover, if for each n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers we set,

$$\frac{\partial^\alpha}{\partial x^\alpha} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n},$$

to be the composition of linear operators on $C^\infty(V)$ or $\mathcal{S}(V)$, then Lemma 1.40 implies

$$\frac{\partial^\alpha(f_1 \cdot f_2)}{\partial x^\alpha} = \sum_{\substack{\alpha=\beta+\gamma \\ \beta,\gamma\in\mathbb{Z}^n_{\geq 0}}} \frac{\alpha!}{\beta!\gamma!}\,\frac{\partial^\beta f_1}{\partial r^\beta}\cdot\frac{\partial^\gamma f_2}{\partial r^\gamma}. \tag{1.29}$$

In particular, $\frac{\partial}{\partial x_j}$ is a linear derivation when acting on either the algebra $C^\infty(V)$ or $\mathcal{S}(V)$.

Another operator on $C^\infty(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ which we considered in Example 1.37 was pointwise multiplication by a canonical coordinate function $r_j : \mathbb{R}^n \to \mathbb{R}$; We denoted this multiplication operator by

$$r_j\cdot : h \mapsto r_j \cdot h, \qquad \text{for } h \in C^\infty(V)$$

There is an analogous linear operator, acting on $C^\infty(V)$ and $\mathcal{S}(V)$, when one starts with an orthonormal system of coordinates for V. Taking $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ to be such an orthonormal system of coordinates, the *j-th coordinate function* is the function $x_j = r_j \circ x : V \to \mathbb{R}$. Hence, if $f : V \to \mathbb{C}$ then follows that the pointwise product of f with x_j, denoted,

$$x_j \cdot f : V \longrightarrow \mathbb{C} : v \longmapsto x_j(v)\ f(v),$$

satisfies

$$(x_j \cdot f) \circ x^{-1} = r_j \cdot (f \circ x^{-1}) : \mathbb{R}^n \longrightarrow \mathbb{C}.$$

Assuming $f : V \to \mathbb{C}$ is smooth, or Schwartz-class, then this equality of functions, coupled with Lemma 1.40, implies $x_j \cdot f : V \to \mathbb{C}$ is smooth, or Schwartz-class, respectively, and that the induced map $x_j\cdot : f \mapsto x_j \cdot f$ determines a linear operator on

$C^\infty(V)$ and $\mathcal{S}(V)$, satisfying the following commutative diagram:

$$\begin{array}{ccccccc} & & & \overset{x_j\cdot}{\longrightarrow} & & & \\ C^\infty(V) & \xrightarrow[x_*]{} & C^\infty(\mathbb{R}^n) & \xrightarrow[r_j\cdot]{} & C^\infty(\mathbb{R}^n) & \xrightarrow[x_*^{-1}]{} & C^\infty(V) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathcal{S}(V) & \xrightarrow{x_*} & \mathcal{S}(\mathbb{R}^n) & \xrightarrow{r_j\cdot} & \mathcal{S}(\mathbb{R}^n) & \xrightarrow{x_*^{-1}} & \mathcal{S}(V) \\ & & & \underset{x_j\cdot}{\longrightarrow} & & & \end{array} \tag{1.30}$$

For each n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers, and any orthonormal coordinate system $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$, we define the function

$$x^\alpha : V \longrightarrow \mathbb{C} : v \longmapsto (x_1(v))^{\alpha_1} (x_2(v))^{\alpha_2} \cdots (x_n(v))^{\alpha_n}, \tag{1.31}$$

as well as the compositions of linear operators

$$\frac{\partial^\alpha}{\partial x^\alpha} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}^{\alpha_n}\right).$$

Immediately from Lemma 1.40 and the definition of $\frac{\partial^\alpha}{\partial x^\alpha}$, note that if f_1 and f_2 are smooth functions, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of non-negative integers, then

$$\frac{\partial^\alpha (f_1 \cdot f_2)}{\partial x^\alpha} = \sum_{\substack{\alpha = \beta + \gamma \\ \beta, \gamma \in \mathbb{Z}^n_{\geq 0}}} \frac{\alpha!}{\beta! \gamma!} \frac{\partial^\beta h_1}{\partial x^\beta} \cdot \frac{\partial^\gamma h_2}{\partial x^\gamma}, \tag{1.32}$$

where $\alpha! := \prod_{j=1}^n \alpha_j!$ and the notation $\alpha = \beta + \gamma$ means $\alpha_j = \beta_j + \gamma_j$ for each integer $1 \leq j \leq n$. Moreover, it is also trivial to see that

$$\frac{\partial^\alpha \overline{h_1}}{\partial x^\alpha} = \overline{\frac{\partial^\alpha h_1}{\partial x^\alpha}}, \tag{1.33}$$

so that $\frac{\partial^\alpha}{\partial x^\alpha}$ commutes with the adjoint operation given by pointwise complex-conjugation.

As stated previously, we have many different orthonormal coordinate systems on V. However, given another such orthonormal coordinate system $y = (y_1, \ldots, y_n) : V \to \mathbb{R}^n$, then for each integer $1 \leq k \leq n$, we use the Euclidean structure $(V^*, \langle \cdot, \cdot \rangle_{V^*})$ to write

$$x_k = \sum_{j=1}^n \langle y_j, x_k \rangle_{V^*} \; y_j : V \longrightarrow \mathbb{R}.$$

From [20, Remark 1.20], this implies

$$\frac{\partial}{\partial y_k} = \sum_{j=1}^{n} \langle y_k, x_j \rangle_{V^*} \ \frac{\partial}{\partial x_j}, \tag{1.34}$$

as linear derivations on the algebras $C^\infty(V)$ and $\mathcal{S}(V)$. Similarly, with $y_k\cdot$ and $x_k\cdot$ denoting the operations of pointwise multiplication by the coordinate function $y_k : V \to \mathbb{R}$ and $x_k : V \to \mathbb{R}$, respectively, then

$$y_k \cdot = \sum_{j=1}^{n} \langle y_k, x_k \rangle_{V^*} \ x_j \cdot$$

as linear operators on $C^\infty(V)$ and $\mathcal{S}(V)$.

Given its pointwise $\mathbb{C}$-linear structure, we equip $\mathcal{S}(V)$ with an inner product using Example 1.37. Recall that we take dr to denote the Lebesgue measure on $\mathbb{R}^n$, and that we defined an inner product on $\mathcal{S}(\mathbb{R}^n)$ by

$$\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^n)} : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C} \ : (f_1, f_2) \longmapsto \int_{\mathbb{R}^n} \overline{f_1} \cdot f_2 \ dr.$$

Lemma 1.45. *Let $x : V \to \mathbb{R}^n$ be an orthonormal coordinate system, and $x_* : \mathcal{S}(V) \to \mathcal{S}(\mathbb{R}^n)$ the induced $*$-isomorphism of Lemma 1.44. Define*

$$\langle \cdot, \cdot \rangle_{L^2(V)} : \mathcal{S}(V) \times \mathcal{S}(V) \longrightarrow \mathbb{C} : (f_1, f_2) \longmapsto \langle x_*(f_1), x_*(f_2) \rangle_{L^2(\mathbb{R}^n)} . \tag{1.35}$$

Then $\langle \cdot, \cdot \rangle_{L^2(V)}$ is a well-defined inner product on $\mathcal{S}(V)$, which is independent of the choice of orthonormal coordinate system $x : V \to \mathbb{R}^n$. In particular, if $y : V \to \mathbb{R}^n$ is an orthonormal coordinate system, then

$$y_* : \mathcal{S}(V) \longrightarrow \mathcal{S}(\mathbb{R}^n) \ : f \mapsto f \circ y^{-1}$$

is an isometric linear isomorphism of complex inner product spaces.

Proof. For a fixed orthonormal coordinate system $x : V \to \mathbb{R}^n$, it is clear from the inner product structure of $\mathcal{S}(\mathbb{R}^n)$, and properties of the map $x_* : \mathcal{S}(V) \to \mathcal{S}(\mathbb{R}^n)$, that Equation (1.35) determines a well-defined inner product on $\mathcal{S}(V)$. Letting $y : V \to \mathbb{R}^n$ be another arbitrary orthonormal coordinate system, note $x \circ y^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is an isometric linear isomorphism. Hence, applying the change of variables formula in [6, Theorem 2.47] and the fact that x_* and y_* are $*$-isomorphisms,

$$\begin{aligned}\langle x_*(f_1), x_*(f_2)\rangle_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (\overline{f_1} \cdot f_2) \circ x^{-1}\ dr \\ &= \int_{\mathbb{R}^n} \left((\overline{f_1} \cdot f_2) \circ y^{-1}\right) \cdot \ \left|\det(x \circ y^{-1})\right|\ dr \\ &= \langle y_*(f_1), y_*(f_2)\rangle_{L^2(\mathbb{R}^n)},\end{aligned}$$

showing that the inner product of Equation (1.35) is independent of the chosen orthonormal coordinate system. On the other hand, notice that if $y_* : \mathcal{S}(V) \to \mathcal{S}(\mathbb{R}^n)$ is the $*$-isomorphism associated to the orthonormal coordinate system $y : V \to \mathbb{R}^n$, then it is immediate from the aforementioned independence that $y_* : \mathcal{S}(V) \to \mathcal{S}(\mathbb{R}^n)$ is an isometric linear isomorphism of inner product spaces. □

Remark 1.46. An equivalent formulation of this inner product on $\mathcal{S}(V)$ is given by defining an appropriately scaled Haar measure on V. In particular, the Euclidean space V, equipped with the Borel σ-algebra, is a measurable space. As any orthonormal coordinate system $x : V \to \mathbb{R}^n$ is a linear isomorphism, there is a well-defined Borel measure on V defined by

$$d\mu(B) := dr(x(B)),\ \text{for all Borel sets } B \subseteq V,$$

where dr is the Lebesgue measure on $\mathbb{R}^n$; The Borel measure $d\mu$ is a *pushfoward* of the Lebesgue measure by the map $x^{-1} : \mathbb{R}^n \to V$.

One verifies that the Borel measure $d\mu$ on V is a Haar measure, in the sense that $d\mu$ is invariant under the translation action of V on Borel sets. The inner product on $\mathcal{S}(V)$ may then be defined by

$$\langle f_1, f_2\rangle_{L^2(V)} := \int_V \overline{f_1} \cdot f_2\ d\mu,\ \text{for each } f_1, f_2 \in \mathcal{S}(V).$$

Well-definedness, and the agreement of this definition with that given in Equation (1.35), follows from Bogačev, [1, Section 3.6-3.7] and, in particular, by the change of variables formula for pushforward measures

$$\int_V f\ d\mu = \int_{\mathbb{R}^n} f \circ x^{-1}\ dr.$$

Though we have introduced an inner product on $\mathcal{S}(V)$ using Lemma 1.45, the induced inner product space $\left(\mathcal{S}(V), \langle \cdot, \cdot \rangle_{L^2(V)}\right)$ is not complete. However, Theorem 1.41 guarantees the existence of a complex Hilbert space $\left(L^2(V), \langle \cdot, \cdot \rangle_{L^2(V)}\right)$ and an isometry which takes the inner product space $\mathcal{S}(V)$ to a dense Hilbert subspace of $L^2(V)$. Just as in the case of Example 1.37 and $L^2(\mathbb{R}^n)$, the vectors in $L^2(V)$ are no longer functions. However, the isometry from $\mathcal{S}(V)$ into $L^2(V)$ allows us to directly identify $\mathcal{S}(V)$ with a dense subspace of $L^2(V)$.

If $x_* : \mathcal{S}(V) \to \mathcal{S}(\mathbb{R}^n)$ is the isometric linear isomorphism induced by an orthonormal coordinate system $x : V \to \mathbb{R}^n$ and Lemma 1.45, then uniqueness of Hilbert space completions up to isometric isomorphism implies the existence of a unitary isomorphism $x_* : L^2(V) \to L^2(\mathbb{R}^n)$ making the following diagram commute:

$$\begin{array}{ccc} L^2(V) & \xrightarrow{x_*} & L^2(\mathbb{R}^n) \\ \uparrow & & \uparrow \\ \mathcal{S}(V) & \xrightarrow[x_*]{} & \mathcal{S}(\mathbb{R}^n) \end{array} \quad (1.36)$$

That is to say, the isometric linear isomorphism $x_* : \mathcal{S}(V) \to \mathcal{S}(\mathbb{R}^n)$ extends to an isometric linear isomorphism of $L^2(V)$ into $L^2(\mathbb{R}^n)$, and we denote the extension by $x_* : L^2(V) \to L^2(\mathbb{R}^n)$.

We finish with a note relating the inner product of Lemma 1.45 with the operators given in the diagrams (1.28) and (1.30).

Proposition 1.47. *Let $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ be an orthonormal coordinate system and, for each integer $1 \leq j \leq n$, take $\frac{\partial}{\partial x_j} : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ and $x_j \cdot : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ be defined by the diagrams (1.28) and (1.30), respectively. Then for any Schwartz-class functions $f_1 : V \to \mathbb{C}$ and $f_2 : V \to \mathbb{C}$,*

$$\left\langle \frac{\partial f_1}{\partial x_j}, f_2 \right\rangle_{L^2(V)} = \left\langle f_1, -\frac{\partial f_2}{\partial x_j} \right\rangle_{L^2(V)} \qquad \textit{and,}$$

$$\langle x_j \cdot f_1, f_2 \rangle_{L^2(V)} = \langle f_1, x_j \cdot f_2 \rangle_{L^2(V)} .$$

Proof. Note that, by definition of the inner product $\langle \cdot, \cdot \rangle_{L^2(V)}$ and the $*$-algebra isomorphism $x_* : \mathcal{S}(V) \to \mathcal{S}(\mathbb{R}^n) : f \mapsto f \circ x^{-1}$,

$$\begin{aligned} \left\langle \frac{\partial f_1}{\partial x_j}, f_2 \right\rangle_{L^2(V)} &:= \int_{\mathbb{R}^n} \overline{\left(\frac{\partial f_1}{\partial x_j} \circ x^{-1} \right)} \cdot \left(f_2 \circ x^{-1} \right) dr \\ &= \int_{\mathbb{R}^n} \frac{\partial \left(\overline{f_1} \circ x^{-1} \right)}{\partial r_j} \cdot \left(f_2 \circ x^{-1} \right) dr, \end{aligned}$$

where

$$\frac{\partial (\overline{f_1} \circ x^{-1})}{\partial r_j} : \mathbb{R}^n \longrightarrow \mathbb{C}$$

is the partial derivative with respect to r_j of the function $\overline{f} \circ x^{-1}$ in $\mathcal{S}(\mathbb{R}^n)$. Hence, using integration by parts together with the fact that any Schwartz-class functions on

$\mathbb{R}^n$ vanish at infinity,

$$\begin{aligned}
\left\langle \frac{\partial f_1}{\partial x_j}, f_2 \right\rangle_{L^2(V)} &= \int_{\mathbb{R}^n} \frac{\partial \left(\overline{f_1} \circ x^{-1} \right)}{\partial r_j} \cdot \left(f_2 \circ x^{-1} \right) \, dr \\
&= - \int_{\mathbb{R}^n} \left(\overline{f_1} \circ x^{-1} \right) \cdot \frac{\partial (f_2 \circ x^{-1})}{\partial r_j} \, dr \\
&= - \int_{\mathbb{R}^n} \overline{(f \circ x^{-1})} \cdot \left(\frac{\partial f_2}{\partial x_j} \circ x^{-1} \right) \, dr \\
&= \left\langle f_1, -\frac{\partial f_2}{\partial x_j} \right\rangle_{L^2(V)}.
\end{aligned}$$

On the other hand, since $x_j \circ x^{-1} = r_j : \mathbb{R}^n \to \mathbb{R}$, we have

$$\begin{aligned}
\langle x_j \cdot f_1, f_2 \rangle_{L^2(V)} &:= \int_{\mathbb{R}^n} \overline{((x_j \cdot f_1) \circ x^{-1})} \cdot \left(f_2 \circ x^{-1} \right) \, dr \\
&= \int_{\mathbb{R}^n} r_j \cdot \overline{(f_1 \circ x^{-1})} \cdot \left(f_2 \circ x^{-1} \right) \, dr \\
&= \int_{\mathbb{R}^n} \overline{(f_1 \circ x^{-1})} \cdot \left((x_j \cdot f_2) \circ x^{-1} \right) \, dr \\
&= \langle f_1, x_j \cdot f_2 \rangle_{L^2(V)}.
\end{aligned}$$

□

Chapter 2
The Dirac-Heisenberg Operator

The *differential forms* over a manifold are a core construction in differential geometry. We give here an introduction to forms in the case for which the underlying manifold is a Euclidean space $(V, \langle\cdot,\cdot\rangle_V)$ with dimension $n := \dim(V)$. From Example 1.27, there is a canonical Euclidean structure on the dual space V^*, which we denote $(V^*, \langle\cdot,\cdot\rangle_{V^*})$. Explicitly, $(V^*, \langle\cdot,\cdot\rangle_{V^*})$ is isometrically isomorphic to $(V, \langle\cdot,\cdot\rangle_V)$ via the linear map $\flat : V \to V^*$, defined on vector v in V by,

$$(\flat v)(w) := \langle v, w\rangle_V, \text{ for all } w \in V.$$

Note $(V^*, \langle\cdot,\cdot\rangle_{V^*})$ has the same dimension as $(V, \langle\cdot,\cdot\rangle_V)$, and the inner product $\langle\cdot,\cdot\rangle_{V^*} : V^* \times V^* \to \mathbb{R}$ is determined by,

$$\langle v, w\rangle_V = \langle \flat v, \flat w\rangle_{V^*}, \text{ for all } v, w \in V.$$

We recall the construction of the exterior algebra generated by V^* given in Definition 1.13, and its inner product given in Example 1.28. For each integer k, the *k-th tensor power* of V^* is the real vector space,

$$\mathcal{T}^k(V^*) := \begin{cases} \bigoplus_{i=1}^k V^*, & k > 0 \\ \mathbb{R}, & k = 0 \\ \{0\}, & k < 0, \end{cases},$$

which is to say that $\mathcal{T}^k(V^*)$ is the direct sum of k-copies of V^* when $k > 0$, $\mathcal{T}^k(V^*)$ is $\mathbb{R}$ when $k = 0$, and $\mathcal{T}^k(V^*)$ is trivial when $k < 0$. The tensor algebra generated by V^*, given in Definition 1.9, is the unital $\mathbb{R}$-algebra with underlying vector space $\mathcal{T}(V^*) := \bigoplus_{k\in\mathbb{Z}} \mathcal{T}^k(V^*)$, and equipped algebra product induced by the canonical isomorphism $\otimes : \mathcal{T}^{k_1}(V^*) \times \mathcal{T}^{k_2}(V^*) \to \mathcal{T}^{k_1+k_2}(V^*)$. Using Definition 1.13, the exterior algebra of V^*, denoted ΛV^*, is then defined by taking an algebra quotient of $\mathcal{T}(V^*)$ by its two-sided ideal $\mathscr{I}$ generated from the 2-tensors $\omega \otimes \omega \in \mathcal{T}^2(V^*)$ for ω ranging over

V^*. We let the induced algebra product on ΛV^* be denoted $\cdot \wedge \cdot : \Lambda V^* \times \Lambda V^* \to \Lambda V^*$. Informally, ΛV^* should be thought of as the unital $\mathbb{R}$-algebra generated by the vectors in V^*, subject to the condition $\omega \wedge \omega$ be zero in ΛV^* for each ω in V^*.

From Lemma 1.15 and Lemma 1.18, the exterior algebra of V^* carries a $\mathbb{Z}$-grading, under which the product is graded-commutative; That is, ΛV^* decomposes as a direct sum, over $\mathbb{Z}$, of component subspaces $\Lambda V^* = \bigoplus_{k \in \mathbb{Z}} \Lambda^k V^*$, and the algebra product in ΛV^* restricts to a surjective bilinear map,

$$\cdot \wedge \cdot : \Lambda^{k_1} V^* \times \Lambda^{k_2} V^* \longrightarrow \Lambda^{k_1 + k_2} V^*,$$

satisfying the graded-commutativity condition,

$$\omega_1 \wedge \omega_2 + (-1)^{k_1 k_2 + 1} \omega_2 \wedge \omega_1 = 0, \text{ for all } \omega_1 \in \Lambda^{k_1} V^*, \omega_2 \in \Lambda^{k_2} V^*. \tag{2.1}$$

The k-th component subspace $\Lambda^k V^*$, called the *k-th exterior power of* V^*, is the vector space obtained as the image of $\mathcal{T}^k(V^*)$ under the canonical quotient homomorphism of $\mathcal{T}(V^*)$ into $\Lambda V^* := \mathcal{T}(V^*)/\mathscr{I}$. However, Lemma 1.22 showed that the only non-trivial components in this $\mathbb{Z}$-grading on ΛV^* are given by $\Lambda^k V$ for $0 \leq k \leq n := \dim(V)$, which is to say that $\Lambda^k V^*$ is the trivial subspace of ΛV^* whenever $k < 0$ or $k > n$. Thus,

$$\Lambda V^* = \bigoplus_{k \in \mathbb{Z}} \Lambda^k V^* = \bigoplus_{k=0}^{n} \Lambda^k V^*.$$

From Lemma 1.17, the exterior power $\Lambda^0 V^*$ is and algebra isomorphic to $\mathbb{R}$, and the exterior power $\Lambda^1 V^*$ is linearly isomorphic to V^*. We therefore regard $\mathbb{R}$ and V^* as a subalgebra and subspace, respectively, of ΛV^*. The linear dimension, and construction of bases, for the non-trivial exterior powers of V^* is discussed in Proposition 1.25 and Corollary 1.26; A basis for $\Lambda^0 V^*$ is given by the identity element, while if $\{x_i\}_{i=1}^n$ is a basis for V^*, k is an integer with $1 \leq k \leq n$, and

$$\mathcal{I}_k := \left\{ (i_1, i_2, \ldots, i_k) \in \{1, 2, \ldots, n\}^k \middle| \, i_1 < i_2 < \cdots < i_k \right\}$$

is the set of well-ordered multi-indices of Definition 1.24, then a basis for $\Lambda^k V^*$ is given by

$$\mathcal{E}_k := \left\{ dx_I := dx_{i_1} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Lambda^k V^* \middle| \, I = (i_1, \ldots, i_k) \in \mathcal{I}_k \right\}. \tag{2.2}$$

In particular, $\Lambda^k V^*$ has dimension $\binom{n}{k}$ for each integer $0 \leq k \leq n$, and dimension 0 otherwise, while the algebra $\Lambda V^* = \bigoplus_{k=0}^n \Lambda^k V^*$ has linear dimension 2^n, with a basis given by

$$\mathcal{E} := \bigcup_{k=0}^{n} \mathcal{E}_k = \{\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Lambda V^* |\, 0 \leq k \leq n, \text{ and } i_1 < i_2 < \ldots < i_k \}\,. \tag{2.3}$$

In Example 1.28, and particularly Lemma 1.29, we equipped ΛV^* with an inner product $\langle \cdot, \cdot \rangle_{\Lambda V^*} : \Lambda V^* \times \Lambda V^* \to \mathbb{R}$. Explicitly, if k_1 and k_2 are distinct integers, then the exterior powers $\Lambda^{k_1} V^*$ and $\Lambda^{k_2} V^*$ are orthogonal subspaces of ΛV^* with respect to $\langle \cdot, \cdot \rangle_{\Lambda V^*}$, while the inner product of two elementary exterior k-tensors $\omega_1 \wedge \cdots \wedge \omega_k$ and $\omega_1' \wedge \cdots \wedge \omega_k'$ in $\Lambda^k V^*$ is given by

$$\langle \omega_1 \wedge \cdots \wedge \omega_k, \omega_1' \wedge \cdots \wedge \omega_k' \rangle_{\Lambda V^*} = \det \left(\left[\left\langle \omega_i, \omega_j' \right\rangle_{V^*} \right]_{i,j=1}^{n} \right), \tag{2.4}$$

where $\left[\left\langle \omega_i, \omega_j' \right\rangle_{V^*} \right]_{i,j=1}^{k}$ is the $k \times k$ matrix with (i,j)-entry equal to $\left\langle \omega_i, \omega_j' \right\rangle_{V^*}$. The induced inner product norm on ΛV^* is then given by

$$\| \cdot \|_{\Lambda V^*} : \Lambda V^* \longrightarrow [0, \infty) \;:\; \|\omega\|_{\Lambda V^*} = \sqrt{\langle \omega, \omega \rangle_{\Lambda V^*}} \tag{2.5}$$

For each integer k, we let

$$\langle \cdot, \cdot \rangle_{\Lambda^k V^*} : \Lambda^k V^* \times \Lambda^k V^* \longrightarrow \mathbb{R}$$

denote the inner product obtained by restricting $\langle \cdot, \cdot \rangle_{\Lambda V^*}$ to an inner product on the subspace $\Lambda^k V^*$ of ΛV^*, with the induced inner product norm on $\Lambda^k V^*$ denoted $\|\cdot\|_{\Lambda^k V^*} : \Lambda^k V^* \to [0, \infty)$. In terms of this notation, the norm $\| \cdot \|_{\Lambda^k V^*}$ on $\Lambda^k V^*$ agrees with that obtained by restricting the norm $\| \cdot \|_{\Lambda V^*} : \Lambda V^* \to [0, \infty)$ to $\Lambda^k V^*$.

From Remark 1.30, if $\{x_i\}_{i=1}^n$ is an orthonormal basis for V^*, then the induced basis $\mathcal{E}_k$ for $\Lambda^k V^*$ given in Equation (2.2) is, in fact, an orthonormal basis for $\Lambda^k V^*$, with $\mathcal{E} := \bigcup_{k=0}^n \mathcal{E}_k$ an orthonormal basis for $\Lambda V^* = \bigoplus_{k=0}^n \Lambda^k V^*$. This also allows us to see that $\Lambda^0 V^*$, and $\Lambda^1 V^*$, isometrically identify with $\mathbb{R}$, and V^*, respectively under the canonical identifications of $\mathbb{R}$ with $\Lambda^0 V^*$, and of V^* with $\Lambda^1 V^*$, discussed above. As such, there is no harm continuing to identify $\mathbb{R}$ as a subalgebra, and V^* as a linear subspace, of the exterior algebra.

Using the decomposition of ΛV^* into a finite direct sum of orthogonal subspaces, $\Lambda V^* = \bigoplus_{k=0}^n \Lambda^k V^*$, one defines $\mathbb{Z}/2$-grading on the Euclidean space $(\Lambda V^*, \langle \cdot, \cdot \rangle_{\Lambda V^*})$, with even and odd subspaces defined, respectively, by

$$\Lambda^+ V^* := \bigoplus_{k \text{ even}} \Lambda^k V^*, \quad \text{and,} \quad \Lambda^- V^* := \bigoplus_{k \text{ odd}} \Lambda^k V^*.$$

Indeed, it is clear that $\Lambda V^* = \Lambda^+ V^* \oplus \Lambda^- V^*$ and, with respect to the inner product defined on ΛV^*, the subspaces $\Lambda^+ V^*$ and $\Lambda^- V^*$ are orthogonal.

Lastly, recall that, in the proof of Corollary 1.33, it was shown that algebra product $\cdot \wedge \cdot : \Lambda V^* \times \Lambda V^* \to \Lambda V^*$ is continuous with respect to the topology on ΛV^* induced by the inner product $\langle \cdot, \cdot \rangle_{\Lambda V^*}$ of Lemma 1.29. In particular, for any ω_1 and ω_2 in ΛV^*, there is a constant $C > 0$ such that,

$$\|\omega_1 \wedge \omega_2\|_\Lambda \leq C \|\omega_1\|_\Lambda \|\omega_2\|_\Lambda$$

2.1 Differential Forms

We define the algebra of smooth differential forms, and its subalgebra of Schwartz-class forms, as a tensor product of ΛV^* with the algebra of smooth, respectively Schwartz-class, $\mathbb{C}$-valued functions on V.

In the notation of Definition 1.43, let $C^\infty(V)$ to be the set of smooth $\mathbb{C}$-valued functions on V, and $\mathcal{S}(V)$ its subset of Schwartz-class $\mathbb{C}$-valued functions. Equipped with pointwise addition, $\mathbb{C}$-scaling, and product, and with adjoint given by pointwise complex-conjugation, Lemma 1.44 verifies that $C^\infty(V)$ is a unital, commutative $\mathbb{C}$-algebra which contains $\mathcal{S}(V)$ as a non-unital subalgebra. Since $C^\infty(V)$ and $\mathcal{S}(V)$ are algebras over $\mathbb{C}$, while ΛV^* is an algebra over $\mathbb{R}$, care must be taken when defining the tensor products $C^\infty(V) \otimes \Lambda V^*$ and $\mathcal{S}(V) \otimes \Lambda V^*$.

Definition 2.1. Let A and B be vector spaces over $\mathbb{C}$, and over $\mathbb{R}$, respectively. The tensor product of A and B is a $\mathbb{C}$-vector space, denoted $A \otimes_{\text{vect}} B$, equipped with a map

$$\otimes : A \times B \longrightarrow A \otimes_{\text{vect}} B : (a, b) \longmapsto a \otimes b,$$

which is $\mathbb{C}$-linear in the A-coordinate, $\mathbb{R}$-linear in the B-coordinate, and such that the following universal property is satisfied:

If W is a $\mathbb{C}$-vector space, and $h : A \times B \to W$ is $\mathbb{C}$-linear in the A-coordinate and $\mathbb{R}$-linear in the B-coordinate, then there exists a unique $\mathbb{C}$-linear map $\tilde{h} : A \otimes_{\text{vect}} B \to W$ such that $h(a, b) = \tilde{h}(a \otimes b)$.

Assuming the existence of such a universal object, and also that A and B are algebras over $\mathbb{C}$ and $\mathbb{R}$, respectively, the algebraic tensor product of A and B, denoted $A \otimes_{\mathbb{R}} B$, is the $\mathbb{C}$-algebra with underlying vector space $A \otimes_{\text{vect}} B$ and product given by the formula

$$\left(\sum_{j=1} a_j \otimes b_j \right) \cdot \left(\sum_k a'_k \otimes b'_k \right) = \sum_{j,k} a_j a'_k \otimes b_j b'_k, \tag{2.6}$$

for all $\sum_j a_j \otimes b_j$ and $\sum_k a'_k \otimes b'_k$ in $A \otimes_{\text{vect}} B$.

For the existence of such a universal vector space, take L to be the free $\mathbb{C}$-vector space with basis $A \times B$, and define S to be the set of vector in L of either the form

$$(a_1 + ca_2, b) - (a_1, b) - c(a_1, b) \qquad \text{for } c \in \mathbb{C}, a_1, a_2 \in A, b \in B,$$

or

$$(a, b_1 + kb_2) - (a, b_1) - k(a, b_2), \qquad \text{for } k \in \mathbb{R}, a \in A, b_1, b_2 \in B.$$

It is easily shown that the quotient $\mathbb{C}$-vector space $L/\mathrm{Span}_{\mathbb{C}}(S)$, together with the map

$$\otimes : A \times B \longrightarrow L/\mathrm{Span}_{\mathbb{C}}(S)$$

sending a pair (a, b) in $A \times B$ to its equivalence class in $L/\mathrm{Span}(S)$, is a tensor product in the sense given above.

To see that the algebra product in $A \otimes_{\mathbb{R}} B$ is well-defined, first fix a tuple (a, b) in $A \times B$. Applying the universal property to

$$A \times B \longrightarrow A \otimes B : (a', b') \longmapsto aa' \otimes bb',$$

we obtain, for each element (a, b) of $A \times B$, a $\mathbb{C}$-linear operator on $A \otimes_{\text{vect}} B$, which we denote $M_{(a,b)}$. With $\mathrm{End}(A \otimes_{\text{vect}} B)$ the complex vector space of $\mathbb{C}$-linear operators on $A \otimes_{\text{vect}} B$, we apply the universal property to

$$A \times B \longrightarrow \mathrm{End}_{\mathbb{C}}(A \otimes_{\text{vect}} B) : (a, b) \longmapsto M_{(a,b)},$$

yielding a $\mathbb{C}$-linear map $\tilde{M} : A \otimes_{\text{vect}} B \to \mathrm{End}(A \otimes_{\text{vect}} B)$. In particular, the formula,

$$\omega_1 \cdot \omega_2 := \left[\tilde{M}(\omega_1)\right](\omega_2), \qquad \text{for } \omega_1, \omega_2 \in A \otimes_{\text{vect}} B$$

determines an algebra product on $A \otimes_{\text{vect}} B$ which satisfies the identity in Equation (2.6).

Another important property of such tensor products is the following.

Lemma 2.2. *Let A, A' be complex vector spaces, and B, B' real vector spaces. If $T : A \to A'$ is a $\mathbb{C}$-linear map, and $F : B \to B'$ an $\mathbb{R}$-linear map, then there is a unique linear map*

$$T \otimes F : A \otimes_{vect} B \longrightarrow A' \otimes_{vect} B'$$

such that $(T \otimes F)(a \otimes b) = T(a) \otimes F(b)$ for all elementary tensors $a \otimes b$ in $A \otimes_{vect} B$.

Moreover, if $(B, \langle \cdot, \cdot \rangle_B)$ is a Euclidean space with orthonormal basis $\{e_i\}_{j=1}^{\dim(B)}$, then *any vector ω in $A \otimes B$ may be written*

$$\omega = \sum_{j=1}^{\dim(B)} f_j \otimes e_j \in A \otimes B,$$

for vectors $f_1, f_2, \ldots, f_{\dim(B)}$ in A. Such decompositions are unique in the sense that if $\{f_j\}_{j=1}^{\dim(B)}$ and $\{h_j\}_{j=1}^{\dim(B)}$ are sets of vectors in A with

$$\sum_{j=1}^{\dim(B)} f_j \otimes e_j = \sum_{j=1}^{\dim(B)} h_j \otimes e_j,$$

then $f_j = h_j$ for each $1 \leq j \leq \dim(B)$.

Proof. If $T : A \to A'$ is $\mathbb{C}$-linear, and $F : B \to B'$ is $\mathbb{R}$-linear, then the universal property of $A \otimes B$ ensures that, since

$$T \otimes F : A \times_{\text{vect}} B \longrightarrow A' \otimes_{\text{vect}} B' : (a, b) \mapsto T(a) \otimes T(b)$$

is $\mathbb{C}$-linear in A and $\mathbb{R}$-linear in B, there is a unique linear map

$$T \otimes F : A \otimes_{\text{vect}} B \longrightarrow A' \otimes_{\text{vect}} B',$$

which satisfied the identity claimed in the lemma.

For the second claim, assume $(B, \langle \cdot, \cdot \rangle_B)$ is a Euclidean space with basis $\{e_j\}_{j=1}^{\dim(B)}$. If $\omega = a \otimes b$ is an elementary tensor in $A \otimes_{\text{vect}} B$, then $b = \sum_{j=1}^{n} \langle e_j, b_j \rangle_B e_j$. Hence,

$$\omega = a \otimes b = \sum_{j=1}^{n} \langle e_j, b_j \rangle_B \; a \otimes e_j.$$

For arbitrary ω in $A \otimes_{\text{vect}} B$, we have by definition that there is a finite natural number t, and elementary tensors $a_1 \otimes b_1, a_2 \otimes b_2, \ldots, a_t \otimes b_t$ in $A \otimes B$ for which,

$$\omega = \sum_{k=1}^{t} a_k \otimes b_k.$$

It follows from the elementary tensor case, together with the identity $a \otimes b + a' \otimes b = (a + a') \otimes b$ for all a, a' in A and b in B, that

$$\omega = \sum_{k=1}^{t} \left(\sum_{j=1}^{\dim(B)} \langle e_j, b_k \rangle_B \; a_k \otimes e_j \right) = \sum_{j=1}^{\dim(B)} \left(\sum_{k=1}^{t} \langle e_j, b_k \rangle_B \; a_k \right) \otimes e_j.$$

Hence, any ω in $A \otimes_{\text{vect}} B$ may be written $\omega = \sum_{j=1}^{n} f_j \otimes e_j$ for some vectors $f_1, f_2, \ldots, f_n$

in A.

For the uniqueness portion of this decomposition, it suffices to show that

$$\sum_{j=1}^{\dim(B)} f_j \otimes e_j = 0 \in A \otimes_{\text{vect}} B \iff f_j = 0 \in A, \text{ for each } 1 \le j \le \dim(B).$$

The "if" implication is clear by the definition of the tensor product $A \otimes_{\text{vect}} B$. For the "only if" implication, let j be an integer with $1 \le j \le \dim(B)$, and define the map

$$A \times B \longrightarrow A \ : (a, b) \longmapsto \langle e_j, b \rangle_B \ a,$$

which is $\mathbb{C}$-linear in the A-coordinate, and $\mathbb{R}$-linear in the B-coordinate. By the universal property of $A \otimes_{\text{vect}} B$, there is a unique linear map

$$T_j : A \otimes_{\text{vect}} B \longrightarrow A$$

such that $T_j(a \otimes b) = \langle e_j, b \rangle_B \ a$ for each elementary tensor $a \otimes b$ in $A \otimes_{\text{vect}} B$. Assuming

$$\omega = \sum_{j=1}^{\dim(B)} f_j \otimes e_j = 0,$$

then by linearity of $T_j : A \otimes B \to A$ we have $a_j = T_j(\omega) = 0$. As this holds for each integer j with $1 \le j \le \dim(B)$, the "only if" implication is proven. $\square$

With this notation for tensor products, we may define the algebras of smooth, and Schwartz-class, differential forms.

Definition 2.3. The $\mathbb{C}$-algebras of *smooth differential forms over* V and *Schwartz-class differential forms over* V, respectively denoted $\Omega(V)$ and $\Omega_{\mathcal{S}}(V)$, are given in the notation of Definition 2.1, by

$$\Omega(V) := C^\infty(V) \otimes_{\mathbb{R}} \Lambda V^* \quad \text{and} \quad \Omega_{\mathcal{S}}(V) := \mathcal{S}(V) \otimes_{\mathbb{R}} \Lambda V^*.$$

Moreover, for each integer k, the $\mathbb{C}$-vector spaces of *smooth differential k-forms over* V and *Schwartz-class differential k-forms*, respectively denoted $\Omega^k(V)$ and $\Omega^k_{\mathcal{S}}(V)$, are given by

$$\Omega^k(V) := C^\infty(V) \otimes_{\text{vect}} \Lambda^k V^* \quad \text{and} \quad \Omega^k_{\mathcal{S}}(V) := \mathcal{S}(V) \otimes_{\text{vect}} \Lambda^k V^*.$$

Applying Lemma 2.2 to some combination of an identity operator, the canonical inclusion of $\mathbb{C}$-algebras $\mathcal{S}(V) \hookrightarrow C^\infty(V)$, and the canonical inclusion of $\mathbb{R}$-linear spaces $\Lambda^k V^* \hookrightarrow \Lambda V^*$ for each integer k, one sees that there is a commutative diagram of injective mappings, which we may consider to be inclusions, given by

$$\begin{array}{ccc} \Omega_{\mathcal{S}}(V) & \hookrightarrow & \Omega(V) \\ \uparrow & & \uparrow \\ \Omega^k_{\mathcal{S}}(V) & \hookrightarrow & \Omega^k(V) \end{array}$$

Note that the top-most horizontal map is a homomorphism of $\mathbb{C}$-algebras, while the other three are merely $\mathbb{C}$-linear.

The natural $\mathbb{Z}$-grading on the algebra $\Omega(V)$, which arises from the $\mathbb{Z}$-grading of ΛV^* given in Lemma 1.15, has component subspaces $\{\Omega^k(V)\}_{k\in\mathbb{Z}}$. Similarly, $\{\Omega^k_{\mathcal{S}}(V)\}_{k\in\mathbb{Z}}$ are the components of a natural $\mathbb{Z}$-grading on the algebra $\Omega_{\mathcal{S}}(V)$ given in Lemma 1.15 gives rise to a natural $\mathbb{Z}$-grading on the algebra $\Omega(V)$, and its subalgebra $\Omega^k_{\mathcal{S}}(V)$. Before showing this rigorously, note $\Omega^k(V)$ and $\Omega^k_{\mathcal{S}}(V)$ are both trivial when $k > n$ or $k < 0$, since the $\mathbb{R}$-linear space $\Lambda^k V^*$ is trivial for such k.

Lemma 2.4. *With component decompositions,*

$$\Omega(V) = \bigoplus_{k\in\mathbb{Z}} \Omega^k(V) = \bigoplus_{k=0}^{n} \Omega^k(V) \quad \textit{and} \quad \Omega^k_{\mathcal{S}}(V) = \bigoplus_{k\in\mathbb{Z}} \Omega^k_{\mathcal{S}}(V) = \bigoplus_{k=0}^{n} \Omega^k_{\mathcal{S}}(V),$$

both $\Omega(V)$ and $\Omega_{\mathcal{S}}(V)$ are $\mathbb{Z}$-graded algebras in the sense that, for any integers k_1 and k_2,

$$\Omega^{k_1}(V)\cdot\Omega^{k_2}(V) \subseteq \Omega^{k_1+k_2}(V) \quad \textit{and} \quad \Omega^{k_1}_{\mathcal{S}}(V)\cdot\Omega^{k_2}_{\mathcal{S}}(V) \subseteq \Omega^{k_1+k_2}_{\mathcal{S}}(V).$$

In particular, the algebra inclusion of $\Omega_{\mathcal{S}}(V) \hookrightarrow \Omega(V)$ is graded *in the sense that $\Omega^k_{\mathcal{S}}(V) \subseteq \Omega^k(V)$ for each integer k.*

Proof. We prove the result only for $\Omega(V)$, as the argument for $\Omega_{\mathcal{S}}(V)$ is the same. Take $\{x_j\}_{j=1}^n$ to be an orthonormal basis for the dual Euclidean space $(V^*, \langle\cdot,\cdot\rangle_{V^*})$, and let

$$\mathcal{E}_k := \{dx_I := dx_{i_1}\wedge dx_{i_2}\wedge\cdots\wedge dx_{i_j} \ : I = (i_1, i_2, \ldots, i_k) \in \mathcal{I}_k\}$$

be the orthonormal basis for $\left(\Lambda^k V^*, \langle\cdot,\cdot\rangle_{\Lambda^k V^*}\right)$ given in Equation (2.2). From Lemma 1.15,

$$\Lambda V^* = \bigoplus_{k=0}^{n} \Lambda^k V^* = \bigoplus_{k\in\mathbb{Z}} \Lambda^k V^*,$$

where the second equality follows by the triviality of $\Lambda^k V^*$ for integers $k > n$ or $k < 0$. Clearly then, $\mathcal{E} := \bigcup_{j=0}^n \mathcal{E}_k$, is an orthonormal basis for the Euclidean space $(\Lambda V^*, \langle\cdot,\cdot\rangle_{\Lambda V^*})$.

We now show $\Omega(V) = \bigoplus_{k\in\mathbb{Z}} \Omega^k(V)$. Since $\Omega^k(V)$ is trivial for $k > n$ or $k < 0$, it suffices to prove $\Omega(V) = \bigoplus_{k=0}^n \Omega^k(V)$. By Lemma 2.2 applied to the basis $\mathcal{E}$ given above, any smooth differential form ω may be written

$$\omega = \sum_{k=0}^{n} \sum_{dx_I \in \mathcal{E}_k} f_I \otimes dx_I,$$

for smooth functions $f_I : V \to \mathbb{C}$, in exactly one way. For each integer $0 \leq k \leq n$, the summand $\omega_k := \sum_{dx_I \in \mathcal{E}_k} f_I \otimes dx_I$ is a smooth k-form in $\Omega^k(V)$. The uniqueness result of Lemma 2.2 then implies that if, for each integer $0 \leq k \leq n$, there exists a smooth differential k-form η_k such that

$$\omega = \sum_{k=0}^{n} \omega_k = \sum_{k=0}^{n} \eta_k,$$

then it must be

$$\omega_k := \sum_{dx_I \in \mathcal{E}_k} f_I \otimes dx_I = \eta_k.$$

Hence, $\Omega(V)$ decomposes as $\Omega(V) = \bigoplus_{k \in \mathbb{Z}} \Omega^k(V)$.

To see that that, for any integers k_1 and k_2, the algebra product in $\Omega(V)$ restricts to a $\mathbb{C}$-bilinear map

$$\Omega^{k_1}(V) \times \Omega^{k_2}(V) \longrightarrow \Omega^{k_1+k_2}(V),$$

simply observe that if $f_1 \otimes \omega_{k_1}$ is an elementary tensor in $\Omega^{k_1}(V)$, and $f_2 \otimes \omega_{k_2}$ an elementary tensor in $\Omega^{k_2}(V)$, then their algebra product in $\Omega(V)$ is defined to be the elementary tensor $(f_1 \cdot f_2) \otimes (\omega_{k_1} \wedge \omega_{k_2})$ in $\Omega(V)$. Such an elementary tensor is, in fact, a vector in the subspace $\Omega^{k_1+k_2}(V)$. Indeed, $\omega_{k_1} \wedge \omega_{k_2}$ is contained in $\Lambda^{k_1+k_2} V^*$ by the $\mathbb{Z}$-grading structure of ΛV^*, while $f_1 \cdot f_2$ is in $C^\infty(V)$ by the algebra structure of $C^\infty(V)$. We have already seen above that, for each integer k, $\Omega^k_{\mathcal{S}}(V)$ is a linear subspace of $\Omega^k(V)$, and thus the algebra inclusion $\Omega_{\mathcal{S}}(V) \hookrightarrow \Omega(V)$ respect the $\mathbb{Z}$-gradings exhibited here. □

2.2 Operators on Differential Forms

In the notation of Definition 2.3, the $\mathbb{C}$-algebra of smooth differential forms over an n-dimensional Euclidean space $(V, \langle \cdot, \cdot \rangle_V)$ is denoted by

$$\Omega(V) := C^\infty(V) \otimes_{\mathbb{R}} \Lambda V^*,$$

and its subalgebra of Schwartz-class differential forms by,

$$\Omega_{\mathcal{S}}(V) := \mathcal{S}(V) \otimes_{\mathbb{R}} \Lambda V^*.$$

From Lemma 2.4, both $\Omega(V)$ and $\Omega_{\mathcal{S}}(V)$ are $\mathbb{Z}$-graded algebras, with components decompositions,

$$\bigoplus_{k\in\mathbb{Z}}\Omega^k(V)=\bigoplus_{k=0}^{n}\Omega^k(V)\quad\text{and}\quad\Omega_{\mathcal{S}}(V)=\bigoplus_{k\in\mathbb{Z}}\Omega^k_{\mathcal{S}}(V)=\bigoplus_{k=0}^{n}\Omega^k_{\mathcal{S}}(V),$$

respectively; Where $\Omega^k(V):=C^\infty(V)\otimes_{\text{vect}}\Lambda^kV^*$ is the subspace of $\Omega(V)$ consisting of smooth differential k-forms, and $\Omega^k_{\mathcal{S}}(V):=\mathcal{S}(V)\otimes_{\text{vect}}\Lambda^kV^*$ the linear subspace of $\Omega_{\mathcal{S}}(V)$ consisting Schwartz-class differential k-forms. The goal of this subsection is to define a number of linear operators acting on the algebras of smooth and Schwartz-class forms over V.

In a majority of the cases, we will first define a linear operator of degree k on the $\mathbb{Z}$-graded algebra $\Omega(V)$ by way of Lemma 2.2, and then wish to verify such an operator restricts to one of degree k on the $\mathbb{Z}$-graded $\Omega_{\mathcal{S}}(V)$; The following proposition shows that this is possible in the obvious case.

Proposition 2.5. *Let $T_1:C^\infty(V)\to C^\infty(V)$ to be a $\mathbb{C}$-linear operator which restricts to a well-defined operator $T_1|_{\mathcal{S}(V)}:\mathcal{S}(V)\to\mathcal{S}(V)$. Further, take $T_2:\Lambda V^*\to\Lambda V^*$ be $\mathbb{R}$-linear, and let*

$$T_1\otimes T_2:\Omega(V)\longrightarrow\Omega(V),\qquad\textit{and}\qquad T_1|_{\mathcal{S}(V)}\otimes T_2:\Omega_{\mathcal{S}}(V)\longrightarrow\Omega_{\mathcal{S}}(V),$$

be defined as in Lemma 2.2. Then $T_1\otimes T_2$ restricts to a well-defined linear operator on $\Omega_{\mathcal{S}}(V)$, and this restriction agrees with $T_1|_{\mathcal{S}(V)}\otimes T_2$.

Moreover, if the linear operator $T_2:\Lambda V^*\to\Lambda V^*$ is degree j with respect to the $\mathbb{Z}$-grading on ΛV^*, then the linear operators

$$T_1\otimes T_2:\Omega(V)\longrightarrow\Omega(V)\qquad\textit{and}\qquad T_1|_{\mathcal{S}(V)}\otimes T_2:\Omega_{\mathcal{S}}(V)\longrightarrow\Omega_{\mathcal{S}}(V),$$

are degree j with respect to the $\mathbb{Z}$-grading on $\Omega(V)$, and on $\Omega_{\mathcal{S}}(V)$, respectively.

Proof. Let $x=(x_1,\dots,x_n):V\to\mathbb{R}^n$ be an orthonormal coordinate system for V. In particular, $\{x_j\}_{j=1}^n$ is an orthonormal basis for V^*, and we let $\mathcal{E}_k$ be the orthonormal basis for $\left(\Lambda^kV^*,\langle\cdot,\cdot\rangle_{\Lambda^kV^*}\right)$ given in Equation (2.2), with $\mathcal{E}:=\bigcup_{k=0}^n\mathcal{E}_k$ the corresponding orthonormal basis for $(\Lambda V^*,\langle\cdot,\cdot\rangle_{\Lambda V^*})$.

From Lemma 2.2, we have that any Schwartz-class differential form ω in $\Omega(V)$ may be written uniquely as

$$\omega=\sum_{k=0}^{n}\sum_{dx_I\in\mathcal{E}_k}f_I\otimes dx_I,$$

for Schwartz-class functions $f_I : V \to \mathbb{C}$. Hence,

$$(T_1 \otimes T_2)|_{\Omega_{\mathcal{S}(V)}}(\omega) = \sum_{k=0}^{n} \sum_{dx_I \in \mathcal{E}_k} T_1(V)(f_I) \otimes T_2(dx_I),$$

and since restricting T_1 to $\mathcal{S}(V)$ yields a linear operator on $\mathcal{S}(V)$, it follows that each elementary tensor $T_1|_{\mathcal{S}(V)}(f_I) \otimes T_2(dx_I)$ is in $\Omega_{\mathcal{S}}(V)$, and

$$(T_1 \otimes T_2)|_{\Omega_{\mathcal{S}(V)}} = T_1|_{\Omega_{\mathcal{S}(V)}} \otimes T_2 : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V).$$

On the other hand, for each integer k, any ω in $\Omega^k(V)$ may be written uniquely as,

$$\omega = \sum_{dx_I \in \mathcal{E}_k} f_I \otimes dx_I,$$

for smooth functions $f_I : V \to \mathbb{C}$. Assuming T_2 has grading degree j as a linear operator on the algebra ΛV^*, then

$$(T_1 \otimes T_2)|_{\Omega^k(V)}(\omega) = \sum_{dx_I \in \mathcal{E}_k} T_1(f_I) \otimes T_2(dx_I),$$

where each elementary tensor $T_1(f_I) \otimes T_2(dx_I)$ is in $\Omega^{k+j}(V)$. It follows that,

$$(T_1 \otimes T_2)|_{\Omega^k(V)} = T_1 \otimes T_2|_{\Lambda^k V^*} : \Omega^k(V) \longrightarrow \Omega^{k+j}(V),$$

which is to say that $T_1 \otimes T_2$ is a degree j linear operator on the $\mathbb{Z}$-graded algebra $\Omega(V)$. A similar argument shows, for each integer k, that

$$\big(T_1|_{\mathcal{S}(V)} \otimes T_2\big)|_{\Omega^k_{\mathcal{S}}(V)} = T_1|_{\mathcal{S}(V)} \otimes T_2|_{\Lambda^k V^*} : \Omega^k_{\mathcal{S}}(V) \longrightarrow \Omega^{k+j}_{\mathcal{S}}(V).$$

Thus $T_1|_{\mathcal{S}(V)} \otimes T_2$ is a degree j linear operator on the $\mathbb{Z}$-graded algebra $\Omega_{\mathcal{S}}(V)$. □

We recall from Lemma 1.32, that there is a linear assignment

$$\lambda_{(\cdot)} : \Lambda^1 V^* \longrightarrow \mathbb{B}(\Lambda V^*) : d\tau \longmapsto \lambda_{d\tau},$$

where, for each $d\tau$ in $\Lambda^1 V^*$,

$$\lambda_{d\tau} : \Lambda V^* \longrightarrow \Lambda V^* : \eta \longmapsto d\tau \wedge \eta. \tag{2.7}$$

In particular, we showed that for each $d\tau$ in $\Lambda^1 V^*$, the bounded linear operator $\lambda_{d\tau}$ is degree 1 with respect to the $\mathbb{Z}$-grading of $\Lambda V^* = \bigoplus_{k=0}^{n} \Lambda^k V^*$. From the graded-commutativity of ΛV^* observed in Lemma 1.18, it follows that,

$$\lambda_{d\tau_1}\lambda_{d\tau_2} = -\lambda_{d\tau_2}\lambda_{d\tau_1}, \qquad \text{for all } d\tau_1, d\tau_2 \in \Lambda^1 V^*. \tag{2.8}$$

Moreover, for each $d\tau$ in $\Lambda^1 V^*$, Lemma 1.35 shows that the formal adjoint of $\lambda_{d\tau}$, which we denoted,

$$\iota_{d\tau} = \lambda_{d\tau}^* : \Lambda V^* \longrightarrow \Lambda V^*, \tag{2.9}$$

is a degree -1 bounded linear operator on the $\mathbb{Z}$-graded algebra ΛV^* called *interior multiplication by $d\tau$*. Using Equation (2.8) and properties of the adjoint operation in $\mathbb{B}(\Lambda V^*)$, we see that for any exterior 1-tensors $d\tau_1$ and $d\tau_2$,

$$\iota_{d\tau_1}\iota_{d\tau_2} = -\iota_{d\tau_2}\iota_{d\tau_1}, \qquad \text{for all } d\tau_1, d\tau_2 \in \Lambda^1 V^*, \tag{2.10}$$

while Equation (1.18) can be used to show that as linear operators on ΛV^*,

$$\iota_{d\tau_1}\lambda_{d\tau_2} + \lambda_{d\tau_2}\iota_{d\tau_1} = \langle d\tau_1, d\tau_2\rangle_{\Lambda^1 V^*}\ \mathbb{1} : \Lambda V^* \longrightarrow \Lambda V^*, \tag{2.11}$$

where $\mathbb{1} : \Lambda V^* \to \Lambda V^*$ is the identity operator.

On the other hand, if if $x = (x_1, \ldots, x_n)$ is an orthonormal coordinate system and $1 \le j \le n$, recall that the partial derivative with respect to x_j is the linear operator $\frac{\partial}{\partial x_j} : C^\infty(V) \to C^\infty(V)$ given by the diagram (1.28), while pointwise multiplication by a coordinate function x_j is the linear operator $x_j\cdot : C^\infty(V) \to C^\infty(V)$ given by the diagram (1.30). In particular, both $\frac{\partial}{\partial x_j} : C^\infty(V) \to C^\infty(V)$ and $x_j\cdot : C^\infty(V) \to C^\infty(V)$ restrict to linear operators on the algebra of Schwartz-class functions $\mathcal{S}(V)$.

2.2.1 The Hodge-de Rham Operator and Hodge-Laplacian

The goal of this subsection is to first introduce the exterior derivative and the codifferential, two introductory linear operators acting on the algebras of smooth and Schwartz-class differential forms over V. Once this is completed, we will briefly discuss the Hodge-de Rham operator, a sum of the exterior derivative and codifferential, along with its square, the Hodge-Laplacian. For a more classical discussion of the Hodge-de Rham operator and the Hodge-Laplacian, specifically its relation to the Hodge Decomposition Theorem, see Helgason [8, pp. 385] and [20].

Let $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ be an orthonormal system of coordinates and, for each integer $1 \le j \le n$, let dx_j denote the element of $\Lambda^1 V^*$ determined by the functional $x_j : V \to \mathbb{R}$ and the canonical identification $V^* \cong \Lambda^1 V^*$ of Lemma 1.17. An immediate corollary to Proposition 2.5 is that the linear operator on $\Omega(V)$ defined by an application of Lemma 2.2,

$$\frac{\partial}{\partial x_j} \otimes \lambda_{dx_j} : \Omega(V) \longrightarrow \Omega(V),$$

is degree 1 with respect to the $\mathbb{Z}$-grading on $\Omega(V)$, and restricts to a degree 1 linear operator on the subalgebra $\Omega_{\mathcal{S}}(V)$. Similarly,

$$\frac{\partial}{\partial x_j} \otimes \iota_{dx_j} : \Omega(V) \longrightarrow \Omega(V),$$

is a degree -1 linear operator on $\Omega(V)$ which restricts to a degree -1 operator on $\Omega_{\mathcal{S}}V)$. Hence,

$$d = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \otimes \lambda_{dx_j} : \Omega(V) \longrightarrow \Omega(V) \tag{2.12}$$

is a degree 1 linear operator on the $\mathbb{Z}$-graded algebra $\Omega(V)$ which restricts to a degree 1 linear operator on the $\mathbb{Z}$-graded algebra $\Omega_{\mathcal{S}}(V)$, and

$$d^* = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \otimes \iota_{dx_j} : \Omega(V) \longrightarrow \Omega(V), \tag{2.13}$$

is degree -1 operator on $\Omega(V)$ which restricts to a degree -1 linear operator on $\Omega_{\mathcal{S}}(V)$.

If $y = (y_1, \ldots, y_n) : V \to \mathbb{R}^n$ is another system of orthonormal coordinates, then using the Euclidean structure $(V^*, \langle \cdot, \cdot \rangle_{V^*})$, write

$$y_k = \sum_{j=1}^{n} \langle x_j, y_k \rangle_{V^*} \; x_j \quad \text{ and } \quad x_k = \sum_{j=1}^{n} \langle x_k, y_j \rangle_{V^*} \; y_j.$$

Linearity of the maps $\lambda_{(\cdot)} : \Lambda^1 V^* \to \mathbb{B}(\Lambda V^*)$ and $\iota_{(\cdot)} : \Lambda^1 V^* \to \mathbb{B}(\Lambda V^*)$, respectively, yield that,

$$\lambda_{dy_k} = \sum_{j=1}^{n} \langle x_j, y_k \rangle_{V^*} \; \lambda_{dx_j} : \Lambda V^* \longrightarrow \Lambda V^* \quad \text{and,} \tag{2.14}$$

$$\iota_{dy_k} = \sum_{j=1}^{n} \langle x_j, y_k \rangle_{y_k} \; \iota_{dx_j} : \Lambda V^* \longrightarrow \Lambda V^*, \tag{2.15}$$

while, from Equation (1.34),

$$\frac{\partial}{\partial y_k} = \sum_{j=1}^{n} \langle y_k, x_j \rangle_{V^*} \; \frac{\partial}{\partial x_j} : C^\infty(V) \longrightarrow C^\infty(V).$$

Hence,

$$\sum_{k=1}^{n} \frac{\partial}{\partial y_k} \otimes \lambda_{dy_k} = \sum_{k,j,i=1}^{n} \langle y_k, x_j \rangle_{V^*} \ \langle x_i, y_k \rangle_{V^*} \ \frac{\partial}{\partial x_j} \otimes \lambda_{dx_i}$$
$$= \sum_{i,j=1}^{n} \langle x_i, x_j \rangle_{V^*} \ \frac{\partial}{\partial x_j} \otimes \lambda_{dx_i}$$
$$= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \otimes \lambda_{dx_i}.$$

and

$$-\sum_{k=1}^{n} \frac{\partial}{\partial y_k} \otimes \iota_{dy_k} = -\sum_{k,j,i=1}^{n} \langle y_k, x_j \rangle_{V^*} \ \langle x_i, y_k \rangle_{V^*} \ \frac{\partial}{\partial x_j} \otimes \iota_{dx_i}$$
$$= -\sum_{i,j=1}^{n} \langle x_i, x_j \rangle_{V^*} \ \frac{\partial}{\partial x_j} \otimes \iota_{dx_i}$$
$$= -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \otimes \iota_{dx_i},$$

What we have just shown in the paragraph above is that the operators coming from Equation (2.12) and Equation (2.13) are independent of the initial orthonormal coordinate system $x = (x_1, \dots, x_n) : V \to \mathbb{R}^n$. This leads us to our definition of the exterior derivative and codifferential.

Definition 2.6. The *exterior derivative* is the unique linear operator $d : \Omega(V) \to \Omega(V)$ such that, if $x = (x_1, \dots, x_n) : V \to \mathbb{R}^n$ is any orthonormal system of coordinates, then

$$d = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \otimes \lambda_{dx_j} : \Omega(V) \longrightarrow \Omega(V).$$

The *codifferential* is the unique linear operator $d^* : \Omega(V) \to \Omega(V)$ such that, if $x = (x_1, \dots, x_n) : V \to \mathbb{R}^n$ is any orthonormal coordinate system, then

$$d^* = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \otimes \lambda_{dx_j} : \Omega(V) \longrightarrow \Omega(V).$$

We following this definition with basic first properties of the exterior derivative and codifferential. Note the first two conditions were a corollary to Proposition 2.5.

Proposition 2.7. *Let $d : \Omega(V) \to \Omega(V)$ denote the exterior derivative, and $d^* : \Omega(V) \to \Omega(V)$ the codifferential. Then the following hold:*

1. *Both the d and d^* both restrict to linear operators on $\Omega_{\mathcal{S}}(V)$.*

2. *With respect to the $\mathbb{Z}$-grading on the algebra $\Omega(V)$, the linear operators d and d^* have grading degree 1, and -1, respectively. Moreover, with respect to the $\mathbb{Z}$-grading on $\Omega_{\mathcal{S}}(V)$, the linear operators obtained by restricting d, and d^*, to $\Omega_{\mathcal{S}}(V)$ have grading degree 1, and -1, respectively.*

3. *Squaring either d, or d^*, yields the zero operator on $\Omega(V)$.*

4. *If ω is a smooth k-form for some integer k, and η is a smooth differential form of any degree, then*

$$d(\omega \cdot \eta) = d(\omega) \cdot \eta + (-1)^k \ \omega \cdot d(\eta), \tag{2.16}$$

where $\cdot$ denotes the algebra product in $\Omega(V)$. In particular, this holds when ω is a Schwartz-class differential k-form and η a Schwartz-class form of arbitrary degree.

Proof. The third condition is merely an application of Clairault's Theorem in conjunction with (2.8), for the case of d, and Equation (2.10), for the case of d^*.

For the fourth claim of the proposition, take any orthonormal coordinate system $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ and let $\mathcal{E}_k$ be the induced orthonormal basis for $(\Lambda V^*, \langle \cdot, \cdot \rangle_{\Lambda V^*})$. From Lemma 2.2, any smooth differential k-form ω may be written uniquely as

$$\omega = \sum_{dx_I \in \mathcal{E}_k} f_I \otimes dx_I,$$

for smooth functions $f_I : V \to \mathbb{C}$, and any smooth differential form η of arbitrary degree may be written uniquely as

$$\eta = \sum_{k=0}^{n} \sum_{dx_I \in \mathcal{E}_k} g_I \otimes dx_I,$$

for smooth functions $g_I : V \to \mathbb{C}$. Using this decomposition, together Lemma 1.18 and the observation that each partial derivative operator $\frac{\partial}{\partial x_j} : C^\infty(V) \to C^\infty(V)$ is a derivation on the algebra $C^\infty(V)$, one then easily computes that the identity in Equation (2.16) holds true. □

There are, in fact, four defining properties of the exterior derivative acting on smooth differential forms, each of which was shown above. In particular, the exterior derivative $d : \Omega(V) \to \Omega(V)$ defined by Equation (2.12) is the unique linear operator on $\Omega(V)$ such that

1. $d : \Omega(V) \to \Omega(V)$ is a degree 1 linear operator on the $\mathbb{Z}$-graded algebra $\Omega(V)$.

2. For any orthonormal coordinate system $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$,

$$d(f \otimes 1) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \otimes dx_j \in \Omega^1(V), \quad \text{for all } f \otimes 1 \in \Omega^0(V),$$

3. $d^2(f \otimes 1) = 0$ for all 0-forms $f \otimes 1$ in $\Omega^0(V)$.

4. If ω is in $\Omega^k(V)$, and η is in $\Omega(V)$, then

$$d(\omega \cdot \eta) = d(\omega) \cdot \eta + (-1)^k \, \omega \cdot d(\eta),$$

   where $\cdot$ denote the algebra product in $\Omega(V)$.

For a more classical definition of the exterior derivative using these conditions, see [20, Theorem 2.20]. An argument showing that such an operator agrees with that of Definition 2.6 is found in Sternberg [18, pp. 99].

With the exterior derivative and codifferential defined, we are able to define the *Hodge-de Rham operator* and the *Hodge-Laplacian*, both of which are operators acting on smooth, and Schwartz-class, differential forms over V.

Definition 2.8. Let $d : \Omega(V) \to \Omega(V)$ and $d^* : \Omega(V) \to \Omega(V)$ denote the exterior derivative, and codifferential, respectively, of Definition 2.6. The linear operator

$$d + d^* : \Omega(V) \longrightarrow \Omega(V)$$

is called the *Hodge-de Rham operator*, while the *Hodge-Laplacian*, or simply the *Laplacian*, on smooth forms is the linear operator,

$$(d + d^*)^2 = dd^* + d^*d : \Omega(V) \longrightarrow \Omega(V). \tag{2.17}$$

Observe that the Laplacian is obtained by simply squaring the Hodge-de Rham operator, while the equality of linear operators in Equation (2.17) follows from Proposition 2.7; Particularly since d and d^* square to the zero operator on $\Omega(V)$.

Proposition 2.9. *In terms of an arbitrary system of orthonormal coordinates $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$, the Laplacian satisfies,*

$$dd^* + d^*d = \left(- \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \right) \otimes \mathbb{1} : \Omega(V) \longrightarrow \Omega(V), \tag{2.18}$$

where $\mathbb{1} : \Lambda V^* \to \Lambda V^*$ is the identity operator, $\frac{\partial^2}{\partial x_j^2} = \left(\frac{\partial}{\partial x_j}\right)^2 : C^\infty(V) \to C^\infty(V)$. In particular, Lemma 2.5 implies that the Laplacian is a degree 0 linear operator on the

$\mathbb{Z}$-graded algebra $\Omega(V)$ which restricts to a degree 0 linear operator on the $\mathbb{Z}$-graded algebra $\Omega_{\mathcal{S}}(V)$.

Proof. The fact that $\Omega_{\mathcal{S}}(V)$ is closed under the Hodge-de Rham operator follows from the algebra $\Omega_{\mathcal{S}}(V)$ being closed under exterior differentiation and the codifferential, as argued in Proposition 2.7.

A description of Laplacian in terms of a local coordinate system through Equation (2.18) is found in [20, pp. 252]. For the sake of completeness, let $d\tau_1$ and $d\tau_2$ be arbitrary exterior 1-tensors. Then using Equation (2.11) and Clairault's Theorem we see,

$$\begin{aligned} d^*d &= -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} \otimes \iota_{dx_i}\lambda_{dx_j} \\ &= -\sum_{i,j=1}^{n} \langle x_i, x_j\rangle_{V^*}\ \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} \otimes \mathbb{1} + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} \otimes \lambda_{dx_j}\iota_{dx_i} \\ &= -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \otimes \mathbb{1} - dd^* \end{aligned}$$

Since $\mathbb{1} : \Lambda V^* \to \Lambda V^*$ is a degree 0 linear operator on the $\mathbb{Z}$-graded algebra ΛV^*, and since $\frac{\partial^2}{\partial x_j^2} : C^\infty(V) \to C^\infty(V)$ restricts to a linear operator on $\mathcal{S}(V)$, we use Proposition 2.5 and the local coordinate description above to deduce that the Laplacian, and its restriction to $\Omega_{\mathcal{S}}(V)$, are both degree 0 linear operators on the $\mathbb{Z}$-graded algebras $\Omega(V)$ and $\Omega_{\mathcal{S}}(V)$, respectively. □

2.2.2 The Clifford Operator and Harmonic Oscillator

We will now introduce the *Clifford operator* acting on $\Omega(V)$ and $\Omega_{\mathcal{S}}(V)$. Once this operator is introduced, we discuss the *Harmonic Oscillator*, which is defined as a sum of the Laplacian and the square of the Clifford operator.

Letting $x = (x_n, \dots, x_n) : V \to \mathbb{R}^n$ be an orthonormal coordinate system and $1 \le j \le n$, take $x_j\cdot : C^\infty(V) \to C^\infty(V)$ to be the linear operator acting on smooth functions given by the diagram (1.30). In particular, $x_k\cdot : C^\infty(V) \to C^\infty(V)$ restricts to a linear operator on the algebra $\mathcal{S}(V)$, and if $f : V \to \mathbb{C}$ is smooth or Schwartz-class, then

$$x_j \cdot f : V \longrightarrow \mathbb{C} : v \longmapsto x_j(v)f(v).$$

Using the operators $x_j\cdot : C^\infty(V) \to C^\infty(V)$, we apply Lemma 2.2 and Proposition 2.5 to obtain,

$$\rho := \sum_{j=1}^{n} x_j \otimes \lambda_{dx_j} : \Omega(V) \longrightarrow \Omega(V), \tag{2.19}$$

a degree 1 linear operator on the $\mathbb{Z}$-graded algebra $\Omega(V)$ which restricts to a linear operator of degree 1 on the $\mathbb{Z}$-graded algebra $\Omega_{\mathcal{S}}(V)$. A second application of these results yields,

$$\rho^* := \sum_{j=1}^{n} x_j \otimes \iota_{dx_j} : \Omega(V) \longrightarrow \Omega(V), \tag{2.20}$$

a degree -1 linear operator on $\Omega(V)$ which restricts to a degree -1 linear operator on $\Omega_{\mathcal{S}}(V)$.

If $y = (y_1, \ldots, y_n) : V \to \mathbb{R}^n$ is another orthonormal coordinate system, observe that linearity of the assignments,

$$\lambda_{(\cdot)} : \Lambda^1 V^* \longrightarrow \mathbb{B}(\Lambda V^*) : d\tau \longmapsto \lambda_{d\tau} \quad \text{and} \quad \iota_{(\cdot)} : \Lambda^1 V^* \longrightarrow \mathbb{B}(\Lambda V^*) : d\tau \longmapsto \iota_{d\tau},$$

together with,

$$x_k = \sum_{j=1}^{n} \langle y_j, x_k \rangle_{V^*} \; y_j \quad \text{and} \quad y_k = \sum_{j=1}^{n} \langle x_j, y_k \rangle_{V^*} \; x_j, \quad \text{for each integer } 1 \leq k \leq n,$$

implies by direct computation that

$$\sum_{j=1}^{n} y_j \otimes \lambda_{dy_j} = \sum_{i,k=1}^{n} \langle x_i, x_k \rangle_{V^*} \; x_i \otimes \lambda_{dx_k} = \sum_{i=1}^{n} x_i \otimes \lambda_{dx_i},$$

and

$$\sum_{j=1}^{n} y_j \otimes \iota_{dy_j} = \sum_{i,k=1}^{n} \langle x_i, x_k \rangle_{V^*} \; x_i \otimes \iota_{dx_k} = \sum_{i=1}^{n} x_i \otimes \iota_{dx_i}.$$

Hence, the definitions of ρ and ρ^* as operators on $\Omega(V)$ given in Equation (2.19), and Equation (2.20), respectively, do not depend on the initial choice of orthonormal coordinate system $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$.

We may have equivalently defined $\rho : \Omega(V) \to \Omega(V)$ as the operator whose action on $\Omega(V)$ is left-multiplication by a particular smooth, exact, 1-form in $\Omega^1(V)$. Indeed, let

$$\| \cdot \|_V : V \longrightarrow \mathbb{R} : v \longmapsto \sqrt{\langle v, v \rangle_V}$$

denote the inner product norm of the Euclidean space $(V, \langle \cdot, \cdot \rangle_V)$, and consider the function

$$\chi : V \longrightarrow \mathbb{C} : v \longmapsto \frac{\|v\|_V^2}{2}.$$

Since any orthonormal coordinate system $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ is an isometry of Euclidean spaces, it is plain to see that $\chi : V \to \mathbb{C}$ is a smooth function such that $\frac{\partial \chi}{\partial x_j} = x_j$ for each $1 \leq j \leq n$. Hence, for $\chi \otimes 1$ the corresponding element of $\Omega^0(V)$, it follows by definition of exterior differentiation that,

$$d(\chi \otimes 1) = \sum_{j=1}^{n} \frac{\partial \chi}{\partial x_j} \otimes dx_j = \sum_{j=1}^{n} x_j \otimes dx_j \tag{2.21}$$

As $\Omega(V)$ is an algebra, one obtains a well-defined linear operator on $\Omega(V)$ of left-multiplication by the smooth 1-form $d(\chi \otimes 1)$; Namely,

$$M_{d(\chi\otimes 1)} : \Omega(V) \longrightarrow \Omega(V) : \omega \longmapsto d(\chi \otimes 1) \cdot \omega,$$

where $\cdot$ denotes the algebra product in $\Omega(V)$. From Equation (2.21), it follows easily that

$$M_{d(\chi\otimes 1)} = \sum_{j=1}^{n} x_j \otimes \lambda_{dx_j} = \rho.$$

As a corollary to the paragraph above, observe that $d\rho + \rho d : \Omega(V) \to \Omega(V)$ identifies with the zero operator. Indeed, from the third and fourth properties in Proposition 2.7, together with the identification of $\rho : \Omega(V) \to \Omega(V)$ and left-multiplication by the smooth, exact, differential 1-form $d(\chi \otimes 1)$, we see that for each smooth differential form η,

$$d\rho(\eta) = d(d(\chi \otimes 1) \cdot \eta) = d^2(\chi \otimes 1) \cdot \eta - \rho d(\eta) = -\rho d(\eta).$$

Hence $d\rho = -\rho d$ as operators on $\Omega(V)$ or equivalently $d\rho + \rho d = 0$.

The following lemma highlights the important properties of ρ and ρ^* given by Equation (2.19) and Equation (2.20), respectively.

Lemma 2.10. *There are unique linear operators $\rho : \Omega(V) \to \Omega(V)$ and $\rho^* : \Omega(V) \to \Omega(V)$ such that, for any orthonormal system of coordinates $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$,*

$$\rho := \sum_{j=1}^{n} x_j \otimes \lambda_{dx_j} : \Omega(V) \longrightarrow \Omega(V) \quad \textit{and,}$$

$$\rho^* := \sum_{j=1}^{n} x_j \otimes \iota_{dx_j} : \Omega(V) \longrightarrow \Omega(V).$$

Squaring either of the operators ρ or ρ^* yields the zero operator on $\Omega(V)$. Moreover, with respect to the $\mathbb{Z}$-grading on $\Omega(V)$, the linear operator ρ is degree 1, while the linear operator ρ^* is a degree -1. As well, ρ and ρ^* restrict to linear operators, of degree 1 and -1, respectively, on the $\mathbb{Z}$-graded algebra $\Omega_{\mathcal{S}}(V)$.

If $d : \Omega(V) \to \Omega(V)$ is the exterior derivative of Definition 2.6, then

$$d\rho + \rho d : \Omega(V) \longrightarrow \Omega(V)$$

is the zero operator.

Proof. The only claim which we have yet to show is that ρ and ρ^* both square to the zero operator on $\Omega(V)$. Indeed, from a simple computation involving commutation of the operators $x_i \cdot : \Omega(V) \to \Omega(V)$ and $x_j \cdot : C^\infty(V) \to C^\infty(V)$ for $1 \leq i, j \leq n$, along with Equation (2.8) for the case of ρ, and Equation (2.10) for the case of ρ^*. $\square$

The Clifford operator, defined below, is a sum of the unique operators ρ and ρ^* given in Lemma 2.10.

Definition 2.11. Let $\rho : \Omega(V) \to \Omega(V)$, and $\rho^* : \Omega(V) \to \Omega(V)$, be the unique linear operators of Lemma 2.10. The *Clifford operator* on $\Omega(V)$ is the linear operator

$$\rho + \rho^* : \Omega(V) \longrightarrow \Omega(V).$$

We end this discussion by highlighting the important properties of the Clifford operator, with most being simple corollaries of our previous results.

Proposition 2.12. *Let $\rho + \rho^* : \Omega(V) \to \Omega(V)$ be the Clifford operator of Definition 2.11. Then*

$$(\rho + \rho^*)^2 = \rho\rho^* + \rho^*\rho : \Omega(V) \longrightarrow \Omega(V)$$

is a degree 0 linear operator on the $\mathbb{Z}$-graded algebra $\Omega(V)$, and restricts to a degree 0 linear operator on the $\mathbb{Z}$-graded algebra $\Omega_{\mathcal{S}}(V)$. If $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ is an orthonormal system of coordinates, then

$$\rho\rho^* + \rho^*\rho = \sum_{j=1}^{n} x_j^2 \otimes \mathbb{1} : \Omega(V) \longrightarrow \Omega(V), \tag{2.22}$$

where $\mathbb{1} : \Lambda V^* \to \Lambda V^*$ is the identity operator, and $x_j^2 : C^\infty(V) \to C^\infty(V)$ is the operator obtained by squaring $x_j \cdot : C^\infty(V) \to C^\infty(V)$ defined in the diagram (1.30).

Proof. The equality of linear operators $(\rho + \rho^*)^2 = \rho\rho^* + \rho^*\rho$ is immediate from Lemma 2.10. For the coordinate description, let $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ be an orthonormal system of coordinates. Then from Equation (2.11), commutativity of the operators $x_i : C^\infty(V) \to C^\infty(V)$ and $x_j : C^\infty(V) \to C^\infty(V)$ for each $1 \leq i, j \leq n$, and orthonormality of the coordinate system, we can directly compute

$$\begin{aligned}\rho^*\rho &= \sum_{i,j=1}^{n} x_i x_j \otimes \iota_{dx_i}\lambda_{dx_j}\\ &= \sum_{i,j=1}^{n} \langle x_i, x_j\rangle_{V^*}\ x_i x_j \otimes \mathbb{1} - \sum_{i,j=1}^{n} x_j x_i \otimes \lambda_{dx_j}\iota_{dx_i}\\ &= \sum_{j=1}^{n} x_j^2 \otimes \mathbb{1} - \rho\rho^*.\end{aligned}$$

Hence, Equation (2.22) is satisfied. As a corollary to Proposition 2.5 and the coordinate description of $\rho\rho^* + \rho^*\rho$ obtained above, the Clifford operator is a degree 0 linear operator on $\Omega(V)$, and restricts to a degree 0 linear operator on $\Omega_{\mathcal{S}}(V)$. □

With the Clifford operator and its square analysed, we define the *harmonic oscillator* on $\Omega(V)$ as follows.

Definition 2.13. Let $d + d^* : \Omega(V) \to \Omega(V)$ be the Hodge-de Rham operator of Definition 2.8, and $\rho + \rho^* : \Omega(V) \to \Omega(V)$, and the Clifford operator of Definition 2.11. The *harmonic oscillator* on $\Omega(V)$ is the linear operator $\mathcal{H} : \Omega(V) \to \Omega(V)$ defined as the sum of squares of $d + d^*$ and $\rho + \rho^*$; That is,

$$\mathcal{H} := (d + d^*)^2 + (\rho + \rho^*)^2 : \Omega(V) \longrightarrow \Omega(V).$$

Proposition 2.14. *If $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ is an orthonormal coordinate system, then the harmonic oscillator $\mathcal{H} : \Omega(V) \to \Omega(V)$ of Definition 2.13 satisfies*

$$\mathcal{H} = \sum_{j=1}^{n} -\frac{\partial^2}{\partial x_j^2} \otimes \mathbb{1} : \Omega(V) \longrightarrow \Omega(V), \tag{2.23}$$

where $-\frac{\partial^2}{\partial x_j^2} = \left(\frac{\partial}{\partial x_j}\right)^2 : \Omega(V) \to \Omega(V)$ and $\mathbb{1} : \Lambda V^* \to \Lambda V^*$ is the identity. In particular, Lemma 2.5 implies that the harmonic oscillator is a degree 0 linear operator on the $\mathbb{Z}$-graded algebra $\Omega(V)$ which restricts to a degree 0 liner operator on the $\mathbb{Z}$-graded algebra $\Omega_{\mathcal{S}}(V)$.

Proof. If $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ is an orthonormal coordinate system, then Equation (2.23) follows from the local coordinate descriptions in Proposition 2.18 and Proposition 2.12. Moreover, using such a local coordinate description of the harmonic oscillator, together with Proposition 2.5, it is plain to see that $\mathcal{H} : \Omega(V) \to \Omega(V)$ is a degree 0 linear operator which restricts to a degree 0 linear operator on $\Omega_{\mathcal{S}}(V)$. □

2.3 Square-integrable Differential Forms

For a Euclidean space $(V, \langle\cdot,\cdot\rangle_V)$ of dimension n, we introduce the Hilbert space of *square-integrable differential forms over* V. Take $\mathcal{S}(V)$ denote the $*$-algebra, with operations defined pointwise, of Schwartz-class functions from V into $\mathbb{C}$, as given in Definition 1.43. Recall from Lemma 1.45 that Equation (1.35) determined an inner product on $\mathcal{S}(V)$, which we denoted

$$\langle\cdot,\cdot\rangle_{L^2(V)} : \mathcal{S}(V) \times \mathcal{S}(V) \longrightarrow \mathbb{C}. \tag{2.24}$$

The resulting inner product space $\Big(\mathcal{S}(V), \langle\cdot,\cdot\rangle_{L^2(V)}\Big)$ was, however, not complete with respect to the indued norm,

$$\|\cdot\|_{L^2(V)} : \mathcal{S}(V) \longrightarrow [0,\infty) : f \longmapsto \sqrt{\langle f, f\rangle_{L^2(V)}},$$

so a completion was taken, allowing us to define the Hilbert space of *square-integrable functions* $\Big(L^2(V), \langle\cdot,\cdot\rangle_{L^2(V)}\Big)$; That is, $\Big(L^2(V), \langle\cdot,\cdot\rangle_{L^2(V)}\Big)$ is defined up to isometric isomorphism as a Hilbert space together with a linear isometry, with dense range, from $\Big(\mathcal{S}, \langle\cdot,\cdot\rangle_{L^2(V)}\Big)$ into $\Big(L^2(V), \langle\cdot,\cdot\rangle_{L^2(V)}\Big)$. We use such an isometry to isometrically identify $\mathcal{S}(V)$ as a dense subspace of $L^2(V)$.

The underlying vector space of square-integrable differential forms is given in terms of Definition 2.1, with complex vector space $L^2(V)$, and real vector space ΛV^*.

Definition 2.15. The $\mathbb{C}$-vector space of *square-integrable differential forms over* V is given, in the notation of Definition 2.1, by

$$\Omega_{L^2}(V) := L^2(V) \otimes_{\text{vect}} \Lambda V^*. \tag{2.25}$$

Similarly, for each integer k, the $\mathbb{C}$-vector space of *square-integrable differential k-forms over* V is given by,

$$\Omega^k_{L^2}(V) := L^2(V) \otimes_{\text{vect}} \Lambda^k V^*.$$

As with the case of smooth and Schwartz-class differential forms, $\Omega^k_{L^2}(V)$ identifies naturally with a linear subspace of $\Omega_{L^2}(V)$, and such an identification gives way to the internal direct sum decomposition

$$\Omega_{L^2}(V) = \bigoplus_{k=0}^{n} \Omega^k_{L^2}(V).$$

Indeed, if $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ is a system of orthonormal coordinates and k any integer, let $\mathcal{E}_k$ denote the orthonormal basis for $\Lambda^k V^*$ of Equation (2.2), and $\mathcal{E} := \bigcup_{k=0}^n \mathcal{E}_k$ the corresponding orthonormal basis for ΛV^*. Now, from Lemma 2.2, each ω in $\Omega_{L^2}(V)$ may be written uniquely as,

$$\omega = \sum_{k=0}^{n} \sum_{dx_I \in \mathcal{E}_k} \xi_I \otimes dx_I, \tag{2.26}$$

with each ξ_I a vector in the Hilbert space $L^2(V)$. From this decomposition, it is becomes clear that $\Omega_{L^2}(V)$ is an internal direct sum of its linearly independent subspaces $\Omega^k_{\mathcal{S}}(V)$, for $0 \leq k \leq n$.

To equip $\Omega_{L^2}(V)$ with an inner product, let $\mathcal{E} = \{dx_I \ : \ I \in \mathcal{I}\}$ be an orthonormal basis for ΛV^*, indexed by the finite set $\mathcal{I}$. Using Proposition 2.2, we may write any ω and η in $\Omega_{L^2}(V)$ uniquely as,

$$\omega = \sum_{I \in \in \mathcal{I}} f_I \otimes dx_I, \quad \text{and} \quad \eta = \sum_{I \in \mathcal{I}} g_I \otimes dx_I,$$

such that, for every I in the index set $\mathcal{I}$, both f_I and g_I are vectors in $L^2(V)$. We then define,

$$\langle \omega, \eta \rangle_\Omega := \sum_{I \in \mathcal{I}} \langle f_I, g_I \rangle_{L^2(V)},$$

where $\langle \cdot, \cdot \rangle_{L^2(V)}$ is the inner product given in Equation (2.24). If $\mathcal{E}' := \{dy_I \ : I \in \mathcal{I}\}$ is another orthonormal basis for ΛV^*, then we may similarly decompose ω and η as,

$$\omega = \sum_{I \in \mathcal{I}} f'_I \otimes dy_I, \quad \text{and} \quad \eta = \sum_{I \in \mathcal{I}} g'_I \otimes dy_I.$$

By basic properties of the tensor product $\Omega_{L^2}(V) := L^2(V) \otimes_{\text{vect}} \Lambda V^*$, it follows that

$$\sum_{J \in \mathcal{I}} f'_J \otimes dy_J = \omega = \sum_{I \in \mathcal{I}} f_I \otimes dx_I = \sum_{J \in \mathcal{I}} \sum_{I \in \mathcal{I}} \langle dy_J, dx_I \rangle_{\Lambda V^*} f_I \otimes dy_J.$$

Hence, by uniqueness of the decomposition given in Lemma 2.2,

$$f'_J = \sum_{I \in \mathcal{I}} \langle dy_J, dx_I \rangle_{\Lambda V^*} \ f_I, \quad \text{for each } J \in \mathcal{I}$$

Similarly, we may conclude

$$g'_J = \sum_{I \in \mathcal{I}} \langle dy_J, dx_I \rangle_{\Lambda V^*} \ g_I, \quad \text{for each } J \in \mathcal{I},$$

and from the previous two equations one sees that,

$$\sum_{J \in \mathcal{I}} \langle f'_J, g'_J \rangle_{L^2(V)} = \sum_{I,K \in \mathcal{I}} \langle dx_I, dx_K \rangle_{\Lambda V^*} \ \langle f_I, g_K \rangle_{L^2(V)} = \sum_{J \in \mathcal{I}} \langle f_J, g_J \rangle_{L^2(V)}.$$

The preceding argument shows that we have a well-defined map,

$$\langle \cdot, \cdot \rangle_\Omega : \Omega_{L^2}(V) \times \Omega_{L^2}(V) \longrightarrow \mathbb{C}, \tag{2.27}$$

such that if $\mathcal{E} = \{dx_I \ : I \in \mathcal{I}\}$ is any orthonormal basis for ΛV^*, then

$$\langle \omega, \eta \rangle_\Omega = \sum_{I \in \mathcal{I}} \langle f_I, g_I \rangle_{L^2(V)},$$

whenever ω and η are the unique elements of $\Omega_{L^2}(V)$ satisfying

$$\omega = \sum_{I \in \mathcal{I}} f_I \otimes dx_I, \quad \text{and} \quad \eta = \sum_{I \in \mathcal{I}} f_I \otimes dx_I,$$

where f_I and g_I are in $L^2(V)$ for each index I in $\mathcal{I}$. The fact that $\langle \cdot, \cdot \rangle_\Omega$ of Equation (2.27) is conjugate symmetric and linear in the second coordinate are immediate from basic tensor product properties and $\langle \cdot, \cdot \rangle_{L^2(V)}$ being an inner product on $L^2(V)$. Clearly, $\langle \cdot, \cdot \rangle_\Omega$ is also positive semidefinite, following from $\langle \cdot, \cdot \rangle_{L^2(V)}$ being an inner product on $L^2(V)$; To see it is strictly positive definite, note that if $\mathcal{E} = \{dx_I \ : I \in \mathcal{I}\}$ is an orthonormal basis for ΛV^*, then $\sum_I f_I \otimes dx_I$ is zero in $\Omega_{L^2}(V)$ if and only if each f_I is zero as an element of $L^2(V)$.

Proposition 2.16. *With respect to the inner product,*

$$\langle \cdot, \cdot \rangle_\Omega : \Omega_{L^2(V)}(V) \times \Omega_{L^2(V)}(V) \longrightarrow \mathbb{C},$$

defined above, $\Omega_{\mathcal{S}}(V)$ is a Hilbert space; That is, $\Omega_{L^2}(V)$ is complete with respect to the metric induced by the inner product norm

$$\| \cdot \|_\Omega : \Omega_{L^2}(V) \longrightarrow [0, \infty) : \omega \longmapsto \sqrt{\langle \omega, \omega \rangle_\Omega}. \tag{2.28}$$

Moreover, if $f_1 \otimes \tau_1$ and $f_2 \otimes \tau_2$ are elementary tensors in $\Omega_{L^2}(V)$, then

$$\langle f_1 \otimes \tau_1, f_2 \otimes \tau_2 \rangle_\Omega = \langle f_1, f_2 \rangle_{L^2(V)} \ \langle \tau_1, \tau_2 \rangle_{\Lambda V^*}. \tag{2.29}$$

In particular, for integers $0 \leq k_1, k_2 \leq n$ such that $k_1 \neq k_2$, then $\Omega^{k_1}_{L^2}(V)$ and $\Omega^{k_2}_{L^2}(V)$ are orthogonal subspaces of $\Omega_{L^2}(V) = \bigoplus_{k=0}^n \Omega^k(V)$ with respect to $\langle \cdot, \cdot \rangle_\Omega$.

Proof. Let $\mathcal{E} := \{dx_I \ : I \in \mathcal{I}\}$ be an orthonormal basis for ΛV^*. If $\{\omega^{(j)}\}_{j \in \mathbb{N}}$ is a sequence in $\Omega_{L^2}(V)$, then for each natural number j, write

$$\omega_j = \sum_{I \in \mathcal{I}} f_I^{(j)} \otimes dx_I,$$

where $f_I^{(j)}$ is in $L^2(V)$ for all indices I in $\mathcal{I}$. Assuming $\{\omega_j\}_{j \in \mathbb{N}}$ is Cauchy in $\Omega_{L^2}(V)$, it follows by definition of the inner product norm $\|\cdot\|_\Omega$ in Equation (2.28) that, for every index I in $\mathcal{I}$, the sequence $\{f_I^{(j)}\}_{j\in\mathbb{N}}$ is Cauchy with respect to the Hilbert space norm on $L^2(V)$. In particular, as a Cauchy sequence in a Hilbert space, $\{f_I^{(j)}\}_{j\in\mathbb{N}}$ converges to some element, say f_I in $L^2(V)$, for each index I. By direct computation, it is then easy to see that the Cauchy sequence $\{\omega_j\}_{j\in\mathbb{N}}$ in $\Omega_{L^2}(V)$ converges, in the norm defined in Equation (2.28), to the square-integrable differential form,

$$\omega = \sum_{I\in\mathcal{I}} f_I \otimes dx_I.$$

To see that Equation (2.29) holds, take $f_1\otimes\tau_1$ and $f_2\otimes\tau_2$ to be elementary tensors in $\Omega_{L^2}(V)$, and notice that since $\mathcal{E}$ is an orthonormal basis for the Euclidean space $(\Lambda V^*, \langle\cdot,\cdot\rangle_{\Lambda V^*})$,

$$f_1\otimes\tau_1 = \sum_{I\in\mathcal{I}} \langle dx_I, \tau_1\rangle_{\Lambda V^*}\ f_1\otimes dx_I, \qquad \text{and,}$$

$$f_2\otimes\tau_2 = \sum_{I\in\mathcal{I}} \langle dx_I, \tau_2\rangle_{\Lambda V^*}\ f_2\otimes dx_I.$$

As the inner product $\langle\cdot,\cdot\rangle_{L^2(V)}$ on $L^2(V)$ is certainly $\mathbb{R}$-bilinear, we conclude

$$\begin{aligned}\langle f_1\otimes\tau_1, f_2\otimes\tau_2\rangle_\Omega &= \sum_{I\in\mathcal{I}} \langle\langle dx_I,\tau_1\rangle_{\Lambda V^*} f_1, \langle dx_I,\tau_2\rangle_{\Lambda V^*} f_2\rangle_{L^2(V)}\\ &= \langle f_1, f_2\rangle_{L^2(V)}\ \sum_{I\in\mathcal{I}} \langle dx_I,\tau_1\rangle_{\Lambda V^*}\ \langle dx_I,\tau_2\rangle_{\Lambda V^*}\\ &= \langle f_1, f_2\rangle_{L^2(V)}\ \langle\tau_1,\tau_2\rangle_{\Lambda V^*}.\end{aligned}$$

If $k_1\neq k_2$ are integers, $f_1\otimes\tau_1$ is an elementary tensor in $\Omega^{k_1}_{L^2}(V)$, and $f_2\otimes\tau_2$ is an elementary tensor in $\Omega^{k_2}_{L^2}(V)$, then $f_1\otimes\tau_1$ and $f_2\otimes\tau_2$ are orthogonal by combining Equation (2.29) with the orthogonality of the subspaces $\Lambda^{k_1}V^*$ and $\Lambda^{k_2}V^*$ of ΛV^* with respect to the inner product $\langle\cdot,\cdot\rangle_{\Lambda V^*}$. Since any vector in $\Omega^{k_1}_{L^2}(V)$ is a finite sum of elementary tensors in $\Omega^{k_1}_{L^2}(V)$, and any in $\Omega^{k_2}_{L^2}(V)$ is a finite sum of elementary tensors in $\Omega^{k_2}_{L^2}(V)$, additivity in each coordinate of the inner product $\langle\cdot,\cdot\rangle_\Omega$ implies that, indeed, $\Omega^{k_1}_{L^2}(V)$ and $\Omega^{k_2}_{L^2}(V)$ are orthogonal subspaces of $\Omega_{L^2}(V)$ with respect to the inner product $\langle\cdot,\cdot\rangle_\Omega$. □

Since $\Big(L^2(V), \langle\cdot,\cdot\rangle_{L^2(V)}\Big)$ was defined by taking a Hilbert space completion of the inner product space $\Big(\mathcal{S}(V), \langle\cdot,\cdot\rangle_{L^2(V)}\Big)$, we have from Theorem 1.41 that there is a canonical linear injection,

$$[\cdot] : \mathcal{S}(V) \longrightarrow L^2(V) : f \longmapsto [f],$$

which has dense range in $L^2(V)$. Applying Lemma 2.2 to this linear injection, along with the identity on ΛV^*, one obtains the linear operator,

$$[\cdot] \otimes \mathbb{1} : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{L^2}(V), \tag{2.30}$$

which takes any elementary tensor $f \otimes \tau$ in $\Omega_{\mathcal{S}}(V)$ to the elementary tensor $[f] \otimes \tau$ in $\Omega_{L^2}(V)$. It is not hard to see that the linear operator defined in Equation (2.30) is injective. Indeed, if $\mathcal{E} = \{dx_I \ : \ I \in \mathcal{I}\}$ is an orthonormal basis for ΛV^*, then use Lemma 2.2 to write and arbitrary Schwartz-class differential form ω uniquely as,

$$\omega = \sum_{I \in \mathcal{I}} f_I \otimes dx_I,$$

with $f_I : V \to \mathbb{C}$ a Schwartz-class function for every index I in $\mathcal{I}$. Applying the operator in question, we see

$$([\cdot] \otimes \mathbb{1})(\omega) = \sum_{I \in \mathcal{I}} [f_I] \otimes dx_I,$$

which, again by Lemma 2.2, is zero in $\Omega_{L^2}(V)$ if and only if, for each index I in $\mathcal{I}$, the vector $[f_I]$ is zero in $L^2(V)$. However, since $[\cdot] : \mathcal{S}(V) \longrightarrow L^2(V)$ is injective, it follows $f_I : V \to \mathbb{C}$ is the zero vector in $\mathcal{S}(V)$ for each index I in $\mathcal{I}$, so that ω is the zero vector in $\Omega_{\mathcal{S}}(V)$. Moreover, one uses a similar argument to see that, for each integer k, restricting the injective linear operator of Equation (2.30) to the subspace of Schwartz-class differential k-forms yields an injective linear map,

$$[\cdot] \otimes \mathbb{1} : \Omega^k_{\mathcal{S}}(V) \longrightarrow \Omega^k_{L^2}(V). \tag{2.31}$$

In light of the linear injections given by Equation (2.30) and Equation (2.31), we shall view $\Omega_{\mathcal{S}}(V)$ as a linear subspace of $\Omega_{L^2(V)}(V)$, and $\Omega^k_{\mathcal{S}}(V)$ as a linear subspace of $\Omega^k_{L^2}(V)$. In particular, the following diagram, with morphisms given by linear inclusions, commutes:

$$\begin{array}{ccc} \Omega_{\mathcal{S}}(V) & \xrightarrow[{[\cdot]\otimes\mathbb{1}}]{} & \Omega_{L^2}(V) \\ \uparrow & & \uparrow \\ \Omega^k_{\mathcal{S}}(V) & \xrightarrow[{[\cdot]\otimes\mathbb{1}}]{} & \Omega^k_{L^2}(V) \end{array}$$

Moreover, if $\left(\Omega^k_{L^2}(V), \langle \cdot, \cdot \rangle_{\Omega^k}\right)$ is the Hilbert space structure induced by inclusion of the subspace $\Omega^k_{L^2}(V)$ into $(\Omega_{L^2}(V), \langle \cdot, \cdot \rangle_{\Omega})$, it is not hard to see that $\Omega^k_{\mathcal{S}}(V)$ is a dense subspace of $\left(\Omega^k_{L^2}(V), \langle \cdot, \cdot \rangle_{\Omega^k}\right)$. Indeed, the result is clear when $k > n$ or $k < 0$, for in that situation both $\Omega^k_{L^2}(V)$ and $\Omega^k_{\mathcal{S}}(V)$ are trivial. On the other hand, for $0 \leq k \leq n$, let ω in $\Omega^k_{L^2}(V)$ be arbitrary. Taking $\mathcal{E}_k = \{dx_I \ : \ I \in \mathcal{I}_k\}$ to be an orthonormal basis for $\Lambda^k V^*$, we have that the number of vectors in $\mathcal{E}_k$, use Lemma 2.2 to write ω uniquely as,

$$\omega = \sum_{I \in \mathcal{I}_k} \xi_I \otimes dx_I.$$

where ξ_I is a vector in $L^2(V)$ for each index I in $\mathcal{I}_k$. With $\mathcal{S}(V) \to L^2(V) : f \mapsto [f]$ the canonical linear mapping with dense range obtained by defining $L^2(V)$ as a completion of $\Big(\mathcal{S}(V), \langle \cdot, \cdot \rangle_{L^2(V)}\Big)$, we may find, for each index I in $\mathcal{I}_k$, a Schwartz-class function $f_I : V \to \mathbb{C}$ such that $\|\xi_I - [f_I]\|_{L^2} < \varepsilon$. By definition of the inner product norm in Equation (2.28), and since the number of vectors in the orthonormal basis $\mathcal{E}_k$ must be $\binom{n}{k}$, it follows that,

$$\left\| \omega - \sum_{I \in \mathcal{I}_k} [f_I] \otimes dx_I \right\|_\Omega = \left\| \sum_{I \in \mathcal{I}_k} (\xi_I - \omega_I) \right\|_\Omega < \sqrt{\binom{n}{k}}\, \varepsilon,$$

from which we conclude that the Schwartz-class differential k-forms identify with a dense subspace of the Hilbert space of differential k-forms. By applying the observations,

$$\Omega_{L^2}(V) = \bigoplus_{k=0}^{n} \Omega^k_{L^2}(V) \quad \text{and} \quad \Omega_{\mathcal{S}}(V) = \bigoplus_{k=0}^{n} \Omega^k_{\mathcal{S}}(V),$$

it is immediate that the Schwartz-class differential forms are also a dense subspace of the Hilbert space $(\Omega_{L^2}(V), \langle \cdot, \cdot \rangle_\Omega)$, of square-integrable differential forms. For future reference, we summarize density results of this paragraph in the lemma below.

Proposition 2.17. *Let $(\Omega_{L^2}(V), \langle \cdot, \cdot \rangle_\Omega)$ denote the Hilbert space of square-integrable differential forms, and $\big(\Omega^k_{L^2}(V), \langle \cdot, \cdot \rangle_{\Omega^k}\big)$ the Hilbert space of square-integrable differential k-forms induced by the inclusion of $\Omega^k_{L^2}(V)$ into $\Omega_{L^2}(V)$.*

Then the Schwartz-class differential forms linearly identifies with a dense subspace of $\Omega_{L^2}(V)$ via the injection given in Equation (2.30), while the Schwartz-class differential k-forms linearly identify with a dense subspace $\Omega^k_{L^2}(V)$ via the map given in Equation (2.31).

The final observation we make of the Hilbert space of square-integrable differential forms is that there is a natural $\mathbb{Z}/2$-grading.

Proposition 2.18. *With even and odd subspaces defined, respectively, by*

$$\Omega^+_{L^2}(V) := \bigoplus_{k \text{ even}} \Omega^k_{L^2}(V) \quad \text{and} \quad \Omega^-_{L^2}(V) := \bigoplus_{k \text{ odd}} \Omega^k_{L^2}(V), \tag{2.32}$$

the Hilbert space $(\Omega_{L^2}(V), \langle \cdot, \cdot \rangle_\Omega)$ is $\mathbb{Z}/2$-graded; That is, $\Omega^+_{L^2}(V)$ and $\Omega^-_{L^2}(V)$ are orthogonal subspaces of $\Omega_{L^2}(V)$ with respect to $\langle \cdot, \cdot \rangle_\Omega$, and $\Omega_{L^2}(V) = \Omega^+_{L^2}(V) \oplus \Omega^-_{L^2}(V)$.

2.4 The Dirac-Heisenberg Operator

Our goal in this section is to define the *Dirac-Heisenberg operator*, a densely defined, unbounded, self-adjoint, odd linear operator acting on the Hilbert space of square-integrable differential forms over Euclidean space. Recall that if $(H, \langle\cdot,\cdot\rangle_H)$ is a Hilbert space, then an *unbounded* operator on H is a linear operator $D : \mathrm{Dom}(D) \to H$, where $\mathrm{Dom}(D)$ is a subspace of H, called the domain of D. One says that an unbounded operator $D : \Omega(V) \to H$ is *densely defined* if the domain, $\mathrm{Dom}(D)$, is a dense subspace of H. All of the unbounded linear operators which appear in this thesis are densely defined, and so we will often say *unbounded operator* to mean one which is densely defined. If $D_1 : \mathrm{Dom}(D_1) \to H$ and $D_2 : \mathrm{Dom}(D_2) \to H$ are unbounded linear operators, one says D_2 *extends* D_1, written $D_1 \subset D_2$, if $\mathrm{Dom}(D_1) \subset \mathrm{Dom}(D_2)$ and $D_1\varphi = D_2\varphi$ for all φ in $\mathrm{Dom}(D_1)$; Further, one says that D_1 *equals* D_2, written $D_1 = D_2$, if D_2 extends D_1 and D_1 extends D_2.

The *adjoint* of a densely defined unbounded operator $D : \mathrm{Dom}(D) \to H$ as follows: Set $\mathrm{Dom}(D^*)$ to be the subspace of H containing all vectors φ in H such that there exists an η in H satisfying,

$$\langle D\psi, \varphi\rangle_H = \langle \psi, \eta\rangle_H, \quad \text{for all} \quad \psi \in H.$$

For each such φ in $\mathrm{Dom}(D^*)$, we defined $D^*\varphi = \eta$. The resulting unbounded operator $D^* : \mathrm{Dom}(D^*) \to H$ is the *adjoint* of $D : \mathrm{Dom}(D) \to H$. Note that well-definedness of $D^* : \mathrm{Dom}(D^*) \to H$ is implied by the fact that $\mathrm{Dom}(D)$ is dense in H, while $\mathrm{Dom}(D^*)$ need not be a dense subspace of H, and could even possible be non-empty. A *self-adjoint* unbounded operator $D : \mathrm{Dom}(D) \to H$ is one for which $D = D^*$, as unbounded linear operators on H.

A condition on unbounded operators which is strictly weaker than self-adjoint is symmetric: One says that a densely defined unbounded operator $D : \mathrm{Dom}(D) \to H$ is *symmetric* if

$$\langle D\varphi, \psi\rangle_H = \langle \varphi, D\psi\rangle_H \quad \text{for all} \quad \varphi, \psi \in \mathrm{Dom}(D).$$

Equivalently, $D : \mathrm{Dom}(D) \to H$ is symmetric if and only if its adjoint $D^* : \mathrm{Dom}(D^*) \to H$ is an extension of D. Lying between self-adjoint and symmetric is the notion of *essentially self-adjoint*; Though certainly not the formal definition, we will call a densely defined, symmetric, unbounded operator $D : \mathrm{Dom}(D) \to H$ *essentially self-adjoint*, if both $D + i : \mathrm{Dom}(D) \to H$ and $D - i : \mathrm{Dom}(D) \to H$ have dense range. We refer the reader to [14, Chapter 8] for the classical definition of essentially self-adjoint unbounded operators. It should be noted that self-adjoint operators are essentially self-adjoint, essentially self-adjoint operators are symmetric, and the converse of both those implications is false.

The importance of essentially self-adjoint operators is that they admit one and only one self-adjoint extension [14, pp. 256]. While the reason one cares about self-adjoint unbounded operators is that they admit a functional calculus, as seen in [14, Theorem 8.5].

2.4.1 Basic Operators on Square-integrable Forms

Recall that $(\Omega_{L^2}(V), \langle\cdot,\cdot\rangle_\Omega)$ denotes the complex Hilbert space of square-integrable differential forms, wherein the vector space $\Omega_{L^2}(V)$, given in Definition 2.15, is equipped with the inner product $\langle\cdot,\cdot\rangle_\Omega$ of Proposition 2.16. In particular, Proposition 2.18 shows that the square-integrable differential forms exhibits a $\mathbb{Z}/2$-grading, with even and odd subspaces,

$$\Omega^+_{L^2}(V) := \bigoplus_{k \text{ even}} \Omega^k_{L^2}(V) \quad \text{and} \quad \Omega^-_{L^2}(V) := \bigoplus_{k \text{ odd}} \Omega^k_{L^2}(V),$$

respectively.

From Proposition 2.17, we may identify the $\mathbb{C}$-vector space of Schwartz-class differential forms $\Omega_{\mathcal{S}}(V)$, of Definition 2.3, as a dense subspace of $\Omega_{L^2}(V)$. It follows by the identification that any linear operator on $\Omega_{\mathcal{S}}(V)$ may be regarded as a densely defined unbounded linear operator on the Hilbert space $\Omega_{L^2}(V)$. As we have previously discussed a number of linear operators acting on the smooth differential forms over V which restrict to $\Omega_{\mathcal{S}}(V)$, we are left with a number of densely defined unbounded linear operators on $\Omega_{L^2}(V)$, which we now analyse.

Take $d : \Omega(V) \to \Omega(V)$ and $d^* : \Omega(V) \to \Omega(V)$ to denote the exterior derivative, and codifferential, respectively, of Definition 2.6. As shown in Proposition 2.7, both d and d^* restrict to linear operators on $\Omega_{\mathcal{S}}(V)$, and in an abuse of notation we denote these restrictions by $d : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ and $d^* : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$. Similarly, we take $\rho : \Omega(V) \to \Omega(V)$ and $\rho^* : \Omega(V) \to \Omega(V)$ to be the linear operators of Lemma 2.10, both of which restrict to linear operators on $\Omega_{\mathcal{S}}(V)$ which we also denote, in an abuse of notation, by $\rho : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ and $\rho^* : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$, respectively. It follows that d, d^*, ρ, and ρ^* can be viewed as densely defined linear operators on $\Omega_{L^2}(V)$, each having domain $\Omega_{\mathcal{S}}(V)$.

Lemma 2.19. *For any Schwartz-class differential forms ω and η,*

$$\langle d(\omega), \eta\rangle_\Omega = \langle \omega, d^*(\eta)\rangle_\Omega \quad \text{and} \quad \langle \rho(\omega), \eta\rangle_\Omega = \langle \omega, \rho^*(\eta)\rangle_\Omega.$$

In particular, the Hodge-de Rham operator $d + d^* : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ of Definition 2.8 and the Clifford operator $\rho+\rho^* : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ of Definition 2.11 determine densely defined, symmetric, unbounded linear operators on $\Omega_{L^2}(V)$, both having domain $\Omega_{\mathcal{S}}(V)$.

Proof. Let $x = (x_1, \dots, x_n) : V \to \mathbb{R}^n$ be an orthonormal system of coordinates and, for each integer $1 \le j \le n$, let dx_j be the exterior 1-tensor induced by the functional

$x_j : V \to \mathbb{R}$ and the canonical isometric identification $V^* \cong \Lambda^1 V^*$. Further, take $\mathcal{E}_k := \{dx_I \ : \ I \in \mathcal{I}_k\}$ be the orthonormal basis for $\Lambda^k V^*$ induced by $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ as in Equation (2.2).

Assume first that $\omega = f_1 \otimes \tau_1$ and $f_2 \otimes \tau_2$ are elementary tensors in $\Omega_{\mathcal{S}}(V)$. Then by definition of the exterior derivative and codifferential, we have

$$d(\omega) = \sum_{j=1}^{n} \frac{\partial f_1}{\partial dx_j} \otimes \lambda_{dx_j}(\tau_1) \quad \text{and} \quad d(\eta) = \sum_{j=1}^{n} -\frac{\partial f_2}{\partial x_j} \otimes \iota_{dx_j}(\tau_2).$$

On the other hand, by definition of ρ and ρ^*,

$$\rho(\omega) = \sum_{j=1}^{n} (x_j \cdot f_1) \otimes \lambda_{dx_j}(\eta) \quad \text{and} \quad \rho^*(\eta) = \sum_{j=1}^{n} (x_j \cdot f_2) \otimes \iota_{dx_j}(\tau_2).$$

From the identity in Equation (2.29), it follows that

$$\langle d(\omega), \eta \rangle_\Omega = \sum_{j=1}^{n} \left\langle \frac{\partial f_1}{\partial x_j}, f_2 \right\rangle_{L^2(V)} \left\langle \lambda_{x_j}(\tau_1), \tau_2 \right\rangle_{\Lambda V^*} \quad \text{and,} \tag{2.33}$$

$$\langle \rho(\omega), \eta \rangle_\Omega = \sum_{j=1}^{n} \langle x_j \cdot f_1, f_2 \rangle_{L^2(V)} \ \langle \lambda_{dx_j}(\tau_1), \tau_2 \rangle_{\Lambda V^*}. \tag{2.34}$$

Using the fact that $\iota_{dx_j} : \Lambda V^* \to \Lambda V^*$ is adjoint to the bounded linear operator $\lambda_{dx_j} : \Lambda V^* \to \Lambda V^*$, together with Proposition 1.47, it is easy to see that Equation (2.33) and Equation (2.34) imply, respectively,

$$\langle d(\omega), \eta \rangle_\Omega = \sum_{j=1}^{n} \left\langle f_1, -\frac{\partial f_2}{\partial x_j} \right\rangle_{L^2(V)} \left\langle \tau_1, \iota_{dx_j}(\tau_2) \right\rangle_{\Lambda V^*} = \langle \omega, d^*(\eta) \rangle_\Omega$$
$$\langle \rho(\omega), \eta \rangle_\Omega = \sum_{j=1}^{n} \langle f_1, x_j \cdot f_2 \rangle_{L^2(V)} \ \left\langle \tau_1, \iota_{dx_j}(\tau_2) \right\rangle_{\Lambda V^*} = \langle \omega, \rho^*(\eta) \rangle_\Omega$$

If ω and η are general Schwartz-class differential forms over V, then one may use Lemma 2.2 to write ω and η as finite sums of elementary tensors in $\Omega_{\mathcal{S}}(V)$. The fact that $\langle d(\omega), \eta \rangle_\Omega = \langle \omega, d^*(\eta) \rangle_\Omega$ and $\langle \rho(\omega), \eta \rangle_\Omega$ then follow by the previous paragraph's results, the linearity of d, d^*, ρ, and ρ^* as operators on $\Omega_{\mathcal{S}}(V)$, and additivity of the inner product $\langle \cdot, \cdot \rangle_\Omega$ in both coordinates.

Finally, the Hodge-de Rham operator $d + d^* : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ and the Clifford operator $\rho + \rho^* : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ clearly determine densely defined unbounded linear operators on $\Omega_{L^2}(V)$. The fact that both are symmetric is then a simple application of linearity and $\langle d(\omega), \eta \rangle_\Omega = \langle \omega, d^*(\eta) \rangle_\Omega$, and $\langle \rho(\omega), \eta \rangle_\Omega = \langle \omega, \rho^*(\eta) \rangle_\Omega$. □

Obtained as a corollary to Lemma 2.10 and Lemma 2.19, one may see quite easily that $d^*\rho^* + \rho^* d^*$ acts as the zero operator on $\Omega_{\mathcal{S}}(V)$.

Corollary 2.20. *As densely defined unbounded linear operators on $\Omega_{L^2}(V)$ with domain $\Omega_{\mathcal{S}}(V)$,*

$$d^*\rho^* + \rho^* d^* = 0 : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V) \subset \Omega_{L^2}(V).$$

Proof. As shown in Lemma 2.10,

$$d\rho + \rho d = 0 : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V)$$

as linear operators on $\Omega_{\mathcal{S}}(V)$. From Lemma 2.19 we have, for any ω and η in $\Omega_{\mathcal{S}}(V)$,

$$0 = \langle (d\rho + \rho d)(\omega), \eta \rangle_{\Omega} = \langle \omega, (\rho^* d^* + d^* \rho^*)(\eta) \rangle_{\Omega}.$$

Hence, as d^* and ρ^* are linear operators on Schwartz-class differential forms, we may take ω to be the Schwartz-class differential form $\omega = (\rho^* d^* + d^* \rho^*)(\eta)$, from which one sees $0 = \|(d^*\rho^* + \rho^* d^*)(\eta)\|_{\Omega}$, where

$$\|\cdot\|_{\Omega} : \Omega_{L^2(V)} \longrightarrow [0, \infty) : \omega \longmapsto \sqrt{\langle \omega, \omega \rangle_{\Omega}},$$

is the inner product norm on $\Omega_{L^2}(V)$. Since $\Omega_{\mathcal{S}}(V)$ identifies with a subspace of $\Omega_{L^2}(V)$, and we have shown that $\|(d^*\rho^* + \rho^* d^*)(\eta)\|_{\Omega} = 0$ for each η in $\Omega_{\mathcal{S}}(V)$, it follows that $d^*\rho^* + \rho^* d^* : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ is the zero operator. □

2.4.2 Definition and First Properties

We are now in a position to define the Dirac-Heisenberg operator on $\Omega_{L^2}(V)$, a densely defined unbounded linear operator which we shall show to be essentially self-adjoint and *odd* with respect to the $\mathbb{Z}/2$-grading on $\Omega_{L^2}(V)$ of Proposition 2.18.

Definition 2.21. The *Dirac-Heisenberg operator* is the unbounded linear operator on $\Omega_{L^2}(V)$ obtained by taking the sum of the Hodge-de Rham operator $d + d^* : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ and the Clifford operator $\rho + \rho^* : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$; That is, the Dirac-Heisenberg operator is the unbounded linear operator

$$D := d + d^* + \rho + \rho^* : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V) \subset \Omega_{L^2}(V).$$

Immediately from Lemma 2.19, one observes that since the Hodge-de Rham and Clifford operators are densely defined, symmetric, unbounded linear operators on $\Omega_{L^2}(V)$ with domain $\Omega_{\mathcal{S}}(V)$, so too is the Dirac-Heisenberg operator. Also easily observed is that the Dirac-Heisenberg operator is odd with respect to the $\mathbb{Z}/2$-grading on $\Omega_{L^2}(V)$ given in Proposition 2.18, in the sense that if

$$\mathrm{Dom}(D)^+ := \mathrm{Dom}(D) \cap \Omega^+_{L^2}(V) \quad \text{and} \quad \mathrm{Dom}(D)^- := \mathrm{Dom}(D) \cap \Omega^-_{L^2}(V),$$

then,

$$D : \mathrm{Dom}(D)^\pm \longrightarrow \mathrm{Dom}(D)^\mp,$$

Indeed, from Proposition 2.17 we have, for each integer k, that $\Omega^k_{\mathcal{S}}(V)$ identifies with a dense subspace of $\Omega^k_{L^2}(V)$; In particular,

$$\mathrm{Dom}(D)^+ = \bigoplus_{k \text{ even}} \Omega^k_{\mathcal{S}}(V) \quad \text{and} \quad \mathrm{Dom}(D)^- = \bigoplus_{k \text{ odd}} \Omega^k_{\mathcal{S}}(V).$$

From Proposition 2.7 and Lemma 2.10, it holds that d, d^*, ρ, and ρ^* restrict to linear maps from $\mathrm{Dom}(D)^\pm$ into $\mathrm{Dom}(D)^\mp$. Hence, as a sum of operators with such a property, it is plain to see that the Dirac-Heisenberg operator also takes the subspaces $\mathrm{Dom}(D)^\pm$ into $\mathrm{Dom}(D)^\mp \subseteq \Omega^\mp_{L^2}(V)$. We record this property in the following proposition.

Proposition 2.22. *With respect to the $\mathbb{Z}/2$-grading on the Hilbert space $\Omega_{L^2}(V) = \Omega^+_{L^2}(V) \oplus \Omega^-_{L^2}(V)$ given in Proposition 2.18, the Dirac-Heisenberg operator is a densely-defined, odd, unbounded linear operator on $\Omega_{L^2}(V)$.*

What makes the Dirac-Heisenberg operator so important is that it squares to a sum of the harmonic oscillator on differential forms, (Definition 2.13), and the *number operator*, defined below, which acts as multiplication by a certain scalar when restricted to differential forms of a fixed degree.

Definition 2.23. The *number operator*, denoted $N : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$, is the linear operator acting on $\Omega_{\mathcal{S}}(V) = \bigoplus_{k=0}^n \Omega^k_{\mathcal{S}}(V)$ such that, for each integer $0 \leq k \leq n := \dim(V)$,

$$N(\omega) = (2k - n)\omega, \quad \text{for all} \quad \omega \in \Omega^k_{\mathcal{S}}(V).$$

Theorem 2.24. *Let*

$$\mathcal{H} := (d + d^*)^2 + (\rho + \rho^*)^2 : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V)$$

denote the harmonic oscillator on Schwartz-class differential forms, as given in Definition 2.13, and let $N : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ be the number operator of Definition 2.23. If $D : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ is the Dirac-Heisenberg operator, then

$$D^2 = \mathcal{H} + N : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V) \subset \Omega_{L^2}(V),$$

as unbounded linear operators on $\Omega_{L^2}(V)$.

Proof. Note that, by definition of the harmonic oscillator and the Dirac-Heisenberg operator,

$$D^2 = \mathcal{H} + (d+d^*)(\rho+\rho^*) + (\rho+\rho^*)(d+d^*) : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V),$$

so it suffices to prove,

$$N = (d+d^*)(\rho+\rho^*) + (\rho+\rho^*)(d+d^*) : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V).$$

However, from Lemma 2.10 and Corollary 2.20,

$$0 = d\rho + \rho d = d^*\rho^* + \rho^* d^* : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V),$$

so we need only show,

$$N = (d\rho^* + \rho^* d) + (d^*\rho + \rho d^*) : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V).$$

Let $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ be an orthonormal system of coordinates and, for each integer $1 \leq j \leq n$, let dx_j denote the exterior 1-tensor induced by the functional $x_j : V \to \mathbb{R}$ and the canonical isometric identification $V^* \cong \Lambda^1 V^*$. Using the local coordinate descriptions of d and ρ^*, together with the identity in Equation (2.11), we see

$$\rho^* d = \sum_{i,j=1}^{n} x_i \frac{\partial}{\partial x_j} \otimes \iota_{dx_i}\lambda_{dx_j} = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \otimes \mathbb{1} - \sum_{i,j=1}^{n} x_i \frac{\partial}{\partial x_j} \otimes \lambda_{dx_j}\iota_{dx_i}. \tag{2.35}$$

On the other hand, for integers $1 \leq i, j \leq n$,

$$\frac{\partial}{\partial x_j} x_i = \langle x_i, x_j \rangle_{V^*} \mathbb{1} + x_i \frac{\partial}{\partial x_j} : \mathcal{S}(V) \longrightarrow \mathcal{S}(V), \tag{2.36}$$

and so

$$d\rho^* = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} x_i \otimes \lambda_{dx_j}\iota_{dx_i} = \sum_{i=1}^{n} \mathbb{1} \otimes \lambda_{dx_i}\iota_{dx_i} + \sum_{i,j=1}^{n} x_i \frac{\partial}{\partial x_j} \otimes \lambda_{dx_j}\iota_{dx_i}. \tag{2.37}$$

Combining Equation (2.35) and Equation (2.37), we have

$$\rho^* d + d\rho^* = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \otimes \mathbb{1} + \sum_{i=1}^{n} \mathbb{1} \otimes \lambda_{dx_i}\iota_{dx_i}. \tag{2.38}$$

To obtain a local coordinate formula for $d^*\rho + \rho d^* : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$, note that Proposition 1.47 imply $d^*\rho + \rho d^*$ is the unique linear operator on $\Omega_{\mathcal{S}}(V)$ such that, for all Schwartz-class differential forms ω and η,

$$\langle (d\rho^* + \rho^* d)(\omega), \eta \rangle_\Omega = \langle \omega, (\rho d^* + d^* \rho)(\eta) \rangle_\Omega.$$

Using Equation (2.38), Proposition 1.47, and the fact that λ_{dx_i} is adjoint to ι_{dx_i} as operators on ΛV^*, observe,

$$d^*\rho + \rho d^* = \sum_{i=1}^{n} -\frac{\partial}{\partial x_i}\ x_i \otimes \mathbb{1} + \sum_{i=1}^{n} \mathbb{1} \otimes \lambda_{dx_i} \iota_{dx_j} : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V) \tag{2.39}$$

An application of Equation (2.36) to Equation (2.39) then yields,

$$d^*\rho + \rho d^* = -n\ \mathbb{1} \otimes \mathbb{1} - \sum_{i=1}^{n} x_i\ \frac{\partial}{\partial x_i} \otimes \mathbb{1} + \sum_{i=1}^{n} \mathbb{1} \otimes \lambda_{dx_i} \iota_{dx_i}. \tag{2.40}$$

Hence, using Equation (2.38) and Equation (2.40), we obtain

$$(d\rho^* + \rho d^*) + (\rho d^* + d^* \rho) = 2 \left(\sum_{i=1}^{n} \mathbb{1} \otimes \lambda_{dx_i} \iota_{dx_i} \right) - n\ \mathbb{1} \otimes \mathbb{1} : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V). \tag{2.41}$$

The result then follows by the observation that,

$$\sum_{i=1}^{n} \lambda_{dx_i} \iota_{dx_i} (dx_{j_1} \wedge \cdots \wedge dx_{j_k}) = k\ dx_{j_1} \wedge \cdots \wedge dx_{j_k},$$

for each exterior k-tensor $dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_k}$ in $\Lambda^k V^*$. For if this is the case, then by taking the basis for $\Lambda^k V^*$ induced by the coordinate system $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$, and applying the decomposition in Lemma 2.2, one concludes

$$\left(-n\ \mathbb{1} \otimes \mathbb{1} + 2 \sum_{i=1}^{n} \mathbb{1} \otimes \lambda_{dx_i} \iota_{dx_i} \right) (\omega) = (-n + 2k)\ \omega,$$

for each Schwartz-class differential k-form ω. □

2.4.3 Spectral Considerations

In this subsection, we investigate the spectral theory and, in particular, the principal values of the densely defined unbounded operator, acting on square-integrable differential forms, determined by the Dirac-Heisenberg operator. This will be accomplished by using the identity in Theorem 2.24 to construct a *diagonalization* for the square of the Dirac-Heisenberg operator; That is, if $D : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ is the Dirac-Heisenberg operator of Definition 2.21, then we will produce an orthonormal basis for the Hilbert space $\Omega_{L^2}(V)$ such that each basis vector is a Schwartz-class differential form and an eigenvector for the unbounded operator D^2. As a corollary to the existence, and regularity, of such a diagonalization, it will follow easily that the Dirac-Heisenberg determines an essentially self-adjoint unbounded operator on $\Omega_{L^2}(V)$.

In order to construct our diagonalization, fix, for the remainder of this subsection, an orthonormal system of coordinates $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ and, for each integer $1 \leq j \leq n$, let dx_j be the exterior 1-tensor induced by the functional $x_j : V \to \mathbb{R}$ and the canonical isometric identification $V^* \cong \Lambda^1 V^*$. Further, for each integer $0 \leq k \leq n$, take

$$\mathcal{E}_k := \left\{ dx_I \in \Lambda^k V^* \ : \ I \in \mathcal{I}_k \right\},$$

be the orthonormal basis for $\Lambda^k V^*$ constructed as in Equation (2.2); Recalling that $\mathcal{E}_0 \subseteq \Lambda^0 V^*$ contains only the identity element of ΛV^*, and $\mathcal{E} := \bigcup_{k=0}^n \mathcal{E}_k$ is an orthonormal basis for the exterior algebra. To simplify notation, we continue to denote the Schwartz-class functions on V by $\mathcal{S}(V)$, and we define the *scalar harmonic oscillator* to be,

$$\mathcal{H}_s := \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} + x_j^2 : \mathcal{S}(V) \longrightarrow \mathcal{S}(V) : f \longmapsto \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2} + x_j^2 \cdot f. \tag{2.42}$$

In terms of this notation, if $\mathcal{H} : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ is the harmonic oscillator acting on Schwartz-class differential forms, as given in Definition 2.13, then Proposition 2.14 showed,

$$\mathcal{H} = \mathcal{H}_s \otimes \mathbb{1} : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V).$$

In particular, if $N : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ is the number operator of Definition 2.23, then the result of Theorem 2.24 may be rewritten as,

$$D^2 = \mathcal{H}_s \otimes \mathbb{1} + N : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V).$$

Now for an integer $0 \leq k \leq n$, if we take a basis element dx_I in $\mathcal{E}_k \subseteq \Lambda^k V^*$ and a Schwartz-class function $f : V \to \mathbb{C}$, one easily deduces that,

$$D^2(f \otimes dx_I) = \mathcal{H}_s(f) \otimes dx_I + (2k - n)\ f \otimes dx_I. \tag{2.43}$$

Hence, the Schwartz-class differential form $f \otimes dx_I$, with $f : V \to \mathbb{C}$ a Schwartz-class function and dx_I in $\mathcal{E}_k$, is an eigenvector of $D^2 : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ whenever f is an eigenvector of the scalar harmonic oscillator $\mathcal{H}_s : \mathcal{S}(V) \to \mathcal{S}(V)$. In order to construct a diagonalization for D^2, we will therefore turn our attention to eigenvectors of the scalar harmonic oscillator, acting on the inner product product space $\Big(\mathcal{S}(V), \langle \cdot, \cdot \rangle_{L^2(V)}\Big)$ of Example 1.42.

To accomplish this, we restrict, for the moment, to the case wherein $(V, \langle \cdot, \cdot \rangle_V)$ is given by the 1-dimensional Euclidean space $\mathbb{R}$. Recall that $*$-algebra of Schwartz-class functions on $\mathbb{R}$, given in Example 1.42, is denoted $\mathcal{S}(\mathbb{R})$ and, when equipped with the inner product,

$$\langle\cdot,\cdot\rangle_{L^2(\mathbb{R})} : \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C} \ : (h_1, h_2) \longmapsto \int_{\mathbb{R}} \overline{h_1(t)}\ h_2(t)\ dt, \tag{2.44}$$

the resulting inner product space $\left(\mathcal{S}(\mathbb{R}), \langle\cdot,\cdot\rangle_{L^2(\mathbb{R})}\right)$ has Hilbert space completion $L^2(\mathbb{R})$. For each non-negative integer k, we inductively define $\psi_k : \mathbb{R} \to \mathbb{R}$ by,

$$\psi_0 : t \mapsto (2\pi)^{-\frac{1}{2}} e^{-t^2/2}, \text{ and } \psi_k : t \mapsto (2k)^{\frac{1}{2}} \left(-\frac{d\psi_{k-1}}{dt}(t) + t\psi_{k-1}(t) \right) \text{ for } k \in \mathbb{N}. \tag{2.45}$$

From Roe [15, Chapter 9], each $\psi_k : \mathbb{R} \to \mathbb{R} \subseteq \mathbb{C}$ is a Schwartz-class function and an eigenvector, with eigenvalue $2k+1$, for the linear operator,

$$-\frac{\partial^2}{\partial t^2} + t^2 : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}) : h \longmapsto -\frac{\partial^2 h}{\partial t^2} + t^2 \cdot h.$$

Moreover, the same reference shows that the set of functions,

$$\{\psi_k : V \to \mathbb{R} \subseteq \mathbb{C} \ : \ k \in \mathbb{Z}_{\geq 0}\} \subseteq \mathcal{S}(\mathbb{R})$$

determines an orthonormal basis for $\left(L^2(\mathbb{R}), \langle\cdot,\cdot\rangle_{L^2(\mathbb{R})}\right)$.

Returning to the case where $(V, \langle\cdot,\cdot\rangle_V)$ is a general Euclidean space with orthonormal coordinate system $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$, for each n-tuple $\vec{k} = (k_1, k_2, \ldots, k_n)$ of non-negative integers, we use the functions given in Equation (2.45) to define,

$$\psi_{\vec{k}} = \prod_{j=1}^{n} (\psi_{k_j} \circ x_j) : V \longrightarrow \mathbb{R} \subseteq \mathbb{C} : v \longmapsto \prod_{j=1}^{n} \psi_{k_j}(x_j(v)) \tag{2.46}$$

Proposition 2.25. *For each n-tuple $\vec{k} := (k_1, k_2, \ldots, k_n)$ of non-negative integers, define $\psi_{\vec{k}} : V \to \mathbb{R} \subseteq \mathbb{C}$ as in Equation (2.46). Then each $\psi_{\vec{k}}$ is a Schwartz-class function on V, in the sense of Definition 1.43, and under the identification of $\mathcal{S}(V)$ with a dense linear subspace of $L^2(V)$, the set of functions,*

$$\mathscr{B} := \left\{ \psi_{\vec{k}} : V \to \mathbb{R} \,\middle|\, \vec{k} \in \mathbb{Z}^n_{k \geq 0} \right\} \subseteq \mathcal{S}(V),$$

determine an orthonormal basis for the Hilbert space $\left(L^2(V), \langle\cdot,\cdot\rangle_{L^2(V)}\right)$.

Moreover, for each n-tuple $\vec{k} := (k_1, k_2, \ldots, k_n)$ of non-negative integers, the Schwartz-class function $\psi_{\vec{k}} : V \to \mathbb{R}$ is an eigenvector, with eigenvalue $\sum_{j=1}^{n} 2k_j+1$, for the scalar *harmonic oscillator*

$$\mathcal{H}_s : \mathcal{S}(V) \longrightarrow \mathcal{S}(V) : f \longmapsto \sum_{j=1}^{n} -\frac{\partial^2 f}{\partial x_j} + x_j^2 \cdot f$$

given in Equation (2.42).

Proof. By orthonormality of the coordinate system $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$, the map $v \mapsto x(v)$ is a linear isometry and thus,

$$\psi_{\vec{0}} : V \longmapsto \mathbb{R}^n : v \longmapsto \pi^{\frac{n}{4}} \; e^{-\|v\|_V^2/2},$$

where $\|\cdot\|_V : V \to [0,\infty) : v \mapsto \sqrt{\langle v, v\rangle_V}$ is the Euclidean norm on V and $\vec{0}$ is the n-tuple in $\mathbb{Z}^n_{\geq 0}$ consisting of all zeros. It is therefore plain to see that $\psi_{\vec{0}} : V \to \mathbb{R} \subseteq \mathbb{C}$ is a Schwartz-class function, *i.e.* $\psi_{\vec{0}}$ determines an element of $\mathcal{S}(V)$. To see that, for any n-tuple $\vec{k} := (k_1, \ldots, k_n)$ of non-negative integers, the functions $\psi_{\vec{k}} : V \to \mathbb{R} \subseteq \mathbb{C}$ are also in $\mathcal{S}(V)$, we first define, for each integer $1 \leq j \leq n$, the linear operator,

$$A_j^* := -\frac{\partial}{\partial x_j} + x_j : \mathcal{S}(V) \longrightarrow \mathcal{S}(V) : f \longmapsto -\frac{\partial f}{\partial x_j} + x_j \cdot f.$$

By construction of the functions $\psi_k : \mathbb{R} \to \mathbb{R} \subseteq \mathbb{C}$ given Equation (2.45), for each natural number k we have,

$$\psi_k \circ x_j = (2k)^{-\frac{1}{2}} \; A_j^* \left(\psi_{k-1} \circ x_j\right) = \left(2^k k!\right)^{-\frac{1}{2}} \left(A_j^*\right)^k \left(\psi_0 \circ x_j\right)$$

as functions from V into $\mathbb{R} \subseteq \mathbb{C}$. Now, for any n-tuple $\vec{k} := (k_1, k_2, \ldots, k_n)$ of non-negative integers, and any integer $1 \leq j \leq n$, note

$$A_j^* \psi_{\vec{k}} = \left[A_j^* \left(\psi_{k_j} \circ x_j\right)\right] \cdot \prod_{\substack{i=1 \\ i \neq j}}^{n} \left(\psi_{k_i} \circ x_j\right).$$

This identity, together with the observations that A_i^* and A_j^* commute whenever $i \neq j$, it follows that for any n-tuple $\vec{k} := (k_1, k_2, \ldots, k_n)$ of non-negative integers,

$$\begin{aligned} \psi_{\vec{k}} &= \left(\prod_{\substack{1 \leq j \leq n \\ k_j \neq 0}} \left(2^{k_j} k_j!\right)^{-\frac{1}{2}} \right) \prod_{m=1}^{n} \left[\left(A_m^*\right)^{k_s} \left(\psi_0 \circ x_j\right) \right] \\ &= \left(\prod_{\substack{1 \leq i \leq n \\ k_i \neq 0}} \left(2^{k_j} k_j!\right)^{-\frac{1}{2}} \right) \left(A_1^*\right)^{k_1} \left(A_2^*\right)^{k_2} \cdots \left(A_n^*\right)^{k_n} \psi_{\vec{0}}. \end{aligned}$$

Since $A_j^* : \mathcal{S}(V) \to \mathcal{S}(V)$ is a well-defined linear operator for each integer j, and since $\psi_{\vec{0}} : V \to \mathbb{R} \subseteq \mathbb{C}$ is in $\mathcal{S}(V)$, we conclude that the set of functions,

$$\mathscr{B} := \left\{ \psi_{\vec{k}} : V \to \mathbb{R} \subseteq \mathbb{C} \;\middle|\; \vec{k} \in \mathbb{Z}^n_{\geq 0} \right\}, \tag{2.47}$$

is contained in the inner product space $\Big(\mathcal{S}(V), \langle \cdot, \cdot \rangle_{L^2(V)} \Big)$.

We now argue that the set of Schwartz-class functions determined by $\mathscr{O}$ is orthonormal with respect to the inner product,

$$\langle \cdot, \cdot \rangle_{L^2(V)} : \mathcal{S}(V) \times \mathcal{S}(V) \longrightarrow \mathbb{C} : (f_1, f_2) \longmapsto \int_{\mathbb{R}^n} \overline{(f_1 \circ x^{-1})(r)} \;\left(f_2 \circ x^{-1} \right)(r) \; dr,$$

defined in Lemma 1.45. First, note that if $\vec{k} = (k_1, k_2, \ldots, k_n)$ and $\vec{l} := (l_1, l_2, \ldots, l_n)$ are two n-tuple of non-negative integers, and $\psi_{\vec{k}}, \psi_{\vec{l}} : V \to \mathbb{R} \subseteq \mathbb{C}$ are the corresponding Schwartz-class functions in $\mathscr{B}$, then by Fubini's Theorem (see Hewitt and Ross [9, Theorem 13.8]) and the fact that each Schwartz-class function in $\mathscr{B}$ is real-valued, one observes,

$$\begin{aligned} \left\langle \psi_{\vec{k}}, \psi_{\vec{l}} \right\rangle_{L^2(V)} &:= \int_{\mathbb{R}^n} \overline{\left(\psi_{\vec{k}} \circ x^{-1}\right)(r)} \;\left(\psi_{\vec{l}} \circ x^{-1}\right)(r) \; dt \\ &= \prod_{j=1}^{n} \left(\int_{\mathbb{R}} \psi_{k_j}(t) \; \psi_{l_j}(t) \; dt \right), \end{aligned} \tag{2.48}$$

where, for each integer $1 \leq j \leq n$, both $\psi_{k_j}, \psi_{l_j} : \mathbb{R} \to \mathbb{R} \subseteq \mathbb{C}$ are defined by Equation (2.45). However, for each integer $1 \leq j \leq n$, it was shown in Lemma 9.6 and Lemma 9.7 of [15] that,

$$\int_{\mathbb{R}} \psi_{k_j}(t) \psi_{l_j}(t) \; dt = \begin{cases} 1, & k_j = l_j \\ 0, & k \neq r. \end{cases} \tag{2.49}$$

From Equation (2.48), we conclude,

$$\left\langle \psi_{\vec{k}}, \psi_{\vec{l}} \right\rangle_{L^2(V)} = \begin{cases} 1, & \text{if } \vec{k} = \vec{l}, \\ 0, & \text{if } \vec{k} \neq \vec{l}, \end{cases}$$

which is to say that the set of Schwartz-class functions $\mathscr{B}$ defined in Equation (2.47) is orthonormal with respect to the inner product $\langle \cdot, \cdot \rangle_{L^2(V)}$ on $\mathcal{S}(V)$.

Now take an n-tuple $\vec{k} := (k_1, \ldots, k_n)$ of non-negative integers, and let $\mathcal{H}_s : \mathcal{S}(V) \to \mathcal{S}(V)$ denote the scalar harmonic oscillator. If we define, for each integer $1 \leq j \leq n$, the linear operator,

$$\mathcal{H}_s^{(j)} := -\frac{\partial^2}{\partial x_j} + x_j^2 : \mathcal{S}(V) \longrightarrow \mathcal{S}(V) : f \longmapsto -\frac{\partial^2 f}{\partial x_j^2} + x_j^2 \cdot f,$$

then clearly,

$$\mathcal{H}_s = \sum_{j=1}^{n} \mathcal{H}_s^{(j)} : \mathcal{S}(V) \longrightarrow \mathcal{S}(V). \tag{2.50}$$

Moreover, for any n-tuple $\vec{k} := (k_1, k_2, \ldots, k_n)$ of non-negative integers,

$$\begin{aligned} \mathcal{H}_s^{(j)} \left(\psi_{\vec{k}}\right) &= \left[\mathcal{H}_s^{(j)}(\psi_{k_j} \circ x^{-1})\right] \cdot \prod_{\substack{i=1 \\ i\neq j}}^{n} \left(\psi_{k_i} \circ x^{-1}\right) && (2.51) \\ &= (2k_j + 1)\ \psi_{\vec{k}}, && (2.52) \end{aligned}$$

where the last equality follows from [15, Lemma 9.6]. Hence, using Equation (2.50), one concludes that for any n-tuple $\vec{k} := (k_1, k_2, \ldots, k_n)$ of non-negative integers, and with $\psi_{\vec{k}} : V \to \mathbb{R} \subseteq \mathbb{C}$ the corresponding element of $\mathscr{O}$,

$$H_{\mathcal{S}}(\psi_{\vec{k}}) = \sum_{j=1}^{n} H_{\mathcal{S}}^{(j)}(\psi_{\vec{k}}) = \left(\sum_{j=1}^{n} 2k_j + 1\right)\ \psi_{\vec{k}} : V \longrightarrow \mathbb{R} \subseteq \mathbb{C}.$$

Therefore, for an n-tuple of non-negative integers $\vec{k} := (k_1, \ldots, k_n)$, we have that the corresponding element of $\mathscr{B}$ determined by the Schwartz-class functions $\psi_{\vec{k}} : V \to \mathbb{R} \subseteq \mathbb{C}$ is, indeed, an eigenvector for the scalar harmonic oscillator $\mathcal{H}_s : \mathcal{S}(V) \to \mathcal{S}(V)$ with eigenvalue $\sum_{j=1}^{n} 2k_j + 1$.

It remains to show that, viewing $\mathcal{S}(V)$ as a dense subspace of the Hilbert space $\left(L^2(V), \langle\cdot,\cdot\rangle_{L^2(V)}\right)$, the set of Schwartz-class functions determined by $\mathscr{B}$ is a basis for the Hilbert space $L^2(V)$. Indeed, from [15, Proposition 9.8], the set of vectors,

$$\mathscr{O} := \{\psi_k : \mathbb{R} \to \mathbb{R} \subseteq \mathbb{C} \ : \ k \in \mathbb{Z}_{\geq 0}\} \subseteq \mathcal{S}(\mathbb{R}) \tag{2.53}$$

is an orthonormal basis for the Hilbert space $L^2(\mathbb{R})$ consisting of Schwartz-class functions; Where $L^2(\mathbb{R})$ is the Hilbert space completion of $\mathcal{S}(\mathbb{R})$ with respect to the inner product $\langle\cdot,\cdot\rangle_{L^2(\mathbb{R})}$ given in Equation (2.44). If we let $\otimes_{k=1}^{n} L^2(\mathbb{R})$ denote the Hilbert space tensor product of $L^2(\mathbb{R})$ with itself n times, as defined in [14, Section II.4], then it is clear that we may identify the tensor product vector space $\otimes_{j=1}^{n}\mathcal{S}(\mathbb{R})$ with a dense subspace of $\otimes_{j=1}^{n} L^2(\mathbb{R})$. Moreover, from [14, pp. 50] the set of vectors

$$\mathscr{O}_{\otimes} \left\{\otimes_{j=1}^{n} \psi_{k_j} \ : \ \psi_{k_j} \in \mathscr{O} \text{ for all } 1 \leq j \leq n\right\},$$

is both contained in $\otimes_{j=1}^n \mathcal{S}(\mathbb{R})$ and is an orthonormal basis for the Hilbert space $\otimes_{j=1}^n L^2(\mathbb{R})$. Moreover, [14, pp, 51] shows if $L^2(\mathbb{R}^n)$ is the Hilbert space constructed in Example 1.37, then there is a unitary isomorphism of Hilbert spaces,

$$U : \otimes_{j=1}^n L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}^n)$$

which maps the orthonormal basis $\mathscr{B}_\otimes$ to an orthonormal basis for $L^2(\mathbb{R}^n)$. Recalling the diagram (1.36), we also have the existence of a unitary isomorphism $x_*^{-1} : L^2(\mathbb{R}^n) \to L^2(V)$. Precomposing $x_*^{-1} : L^2(\mathbb{R}^n) \to L^2(V)$ by the unitary $U : \otimes_{j=1}^n L^2(\mathbb{R}) \to L^2(\mathbb{R}^n)$, we obtain a unitary mapping of Hilbert spaces,

$$x_*^{-1} U : \otimes_{k=1}^n L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}) \longrightarrow L^2(V)..$$

By construction of U and x_*^{-1}, it is clear the basis $\mathscr{O}_\otimes$ is mapped under $x_*^{-1}U$ to the set of vectors $\mathscr{B}$ defined in Equation (2.47); Indeed, if $\otimes_{j=1}^n \psi_{k_j}$ is in $\mathscr{O}_\otimes$, then,

$$x_*^{-1} U \left(\otimes_{j=1}^n \psi_{k_j} \right) = \psi_{\vec{k}} \in \mathscr{B} \subseteq L^2(V)$$

where $\vec{k} = (k_1, k_2, \ldots, k_n)$. Since $\mathscr{O}_\otimes$ and $\mathscr{B}$ are in bijective correspondence under the unitary map $x_*^{-1}U : \otimes_{j=1}^n L^2(\mathbb{R}) \to L^2(V)$, and since $\mathscr{O}_\otimes$ is an orthonormal basis for $\otimes_{j=1}^n L^2(\mathbb{R})$, it follows that $\mathscr{B}$ must then be an orthonormal basis for $L^2(V)$. □

We use the diagonalization of the scalar harmonic oscillator given in Proposition 2.25 to construct a diagonalization of the real Dirac-Heisenberg operator. To state this result, recall that the well-ordered multi-indices of length k, with $1 \leq k \leq n$, are defined by

$$\mathcal{I}_k := \{(i_1, i_2, \ldots, i_k) \in \{1, 2, \ldots, n\}^k \ : i_1 < i_2 < \cdots < i_k\},$$

and we set $\mathcal{I}_\emptyset := \{\emptyset\}$ (see Definition 1.24). For each multi-index I in $\mathcal{I} := \bigcup_{j=0}^n \mathcal{I}_k$, let $|I| = k$ mean that I is in $\mathcal{I}_k$. Given an orthonormal coordinate system $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$, we may construct an orthonormal basis for the exterior k-tensors $\Lambda^k V^*$ by setting,

$$\mathcal{E}_k := \left\{ dx_I \in \Lambda^k V^* \ : I \in \mathcal{I}_k \right\},$$

where,

$$dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} \in \Lambda^k V^*, \quad \text{for } I \in \mathcal{I}_k, 1 \leq k \leq n \tag{2.54}$$

$$dx_\emptyset = 1 \in \Lambda^0 V^*.$$

In particular,

$$\mathcal{E} = \bigcup_{k=0}^n \mathcal{E}_k,$$

is an orthonormal basis for the exterior algebra ΛV^* with respect to the inner product given in Example 1.28.

Corollary 2.26. *Let $D : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ be the Dirac-Heisenberg operator of Definition 2.21, and $x = (x_1, \dots, x_n) : V \to \mathbb{R}^n$ be an orthonormal coordinate system. Further, use $x = (x_1, \dots, x_n) : V \to \mathbb{R}^n$ to define the orthonormal basis $\mathcal{E} = \bigcup_{k=0}^{n} \mathcal{E}_k$ for ΛV^* discussed above, and the orthonormal basis,*

$$\mathscr{B} := \left\{ \psi_{\vec{k}} \in \mathcal{S}(V) \;\middle|\; \vec{k} \in \mathbb{Z}^n_{\geq 0} \right\},$$

constructed in Proposition 2.25, for the Hilbert space $L^2(V)$.

Then the set,

$$\mathscr{B}_\Omega := \left\{ \psi_{\vec{k}} \otimes dx_I \in \Omega_{\mathcal{S}}(V) \;\middle|\; \psi_{\vec{k}} \in \mathscr{B} \text{ and } dx_I \in \mathcal{E} \right\}$$

is an orthonormal basis for the Hilbert space of square-integrable differential forms $\Omega_{L^2(V)}$. Moreover, if $\vec{k} = (k_1, \dots, k_n)$ is an n-tuple of non-negative integers and dx_I is in $\mathcal{E}_k$, then the corresponding element $\psi_{\vec{k}} \otimes dx_I$ in $\mathscr{B}_\Omega$ is an eigenvector of $D^2 : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ with,

$$D^2 \left(\psi_{\vec{k}} \otimes dx_I \right) = 2 \left(k + \sum_{j=1}^{n} k_i \right) \psi_{\vec{k}} \otimes dx_I$$

Proof. Since $\mathscr{B}$ contains only Schwartz-class functions, it is clear by Lemma 2.2 that $\mathscr{B}_\Omega$ contains only Schwartz-class differential forms. Orthonormality of the set $\mathscr{B}_\Omega \subseteq \Omega_{\mathcal{S}}(V) \subseteq \Omega_{L^2}(V)$ with respect to the inner product $\langle \cdot, \cdot \rangle_\Omega$ on $\Omega_{L^2}(V)$ follows from the identity in Proposition 2.16 given for $\langle \cdot, \cdot \rangle_\Omega$ acting on elementary tensor. Indeed, if $\psi_{\vec{k}_1} \otimes dx_I$ and $\psi_{\vec{k}_2} \otimes dx_J$ are in $\mathscr{B}_\Omega$, then since $\mathscr{B}$ is an orthonormal basis for $L^2(V)$ and $\mathcal{E}$ is an orthonormal basis for ΛV^*, we have

$$\begin{aligned} \langle \psi_{\vec{k}_1} \otimes dx_I, \psi_{\vec{k}_2} \otimes dx_J \rangle_\Omega &= \langle \psi_{\vec{k}_1}, \psi_{\vec{k}_2} \rangle_{L^2(V)} \; \langle dx_I, dx_J \rangle_{\Lambda V^*} \\ &= \begin{cases} 1, & \vec{k}_1 = \vec{k}_2 \text{ and } dx_I = dx_J \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

To see that $\mathscr{B}_\Omega$ is a basis for $\Omega_{L^2}(V)$, take ω in $\Omega_{L^2}(V)$ to be arbitrary and use the basis $\mathcal{E}$ of ΛV^*, together with the decomposition given in Lemma 2.2, to write ω uniquely as

$$\omega = \sum_{k=0}^{n} \sum_{I \in \mathcal{I}_k} f_I \otimes dx_I,$$

where f_I is in $L^2(V)$ for each I in $\mathcal{I} := \bigcup_{k=0}^{n} \mathcal{I}_k$. Since $\mathscr{B}$ is a basis for the Hilbert space $L^2(V)$, for any $\varepsilon > 0$ and any multi-index I, one may find a φ_I lying in the $\mathbb{C}$-linear span of $\mathscr{B}$ such that

$$\|f_I - \varphi_I\|_{L^2(V)} := \sqrt{\langle f_I - \varphi_I, f_I - \varphi_I \rangle_{L^2(V)}} < \varepsilon.$$

By basic properties of tensor products, it is clear that $\varphi_I \otimes dx_I$ is in the $\mathbb{C}$-linear span of $\mathscr{B}_\Omega$, and from Equation 2.29 we conclude,

$$\left\| \omega - \sum_{k=0}^{n} \sum_{I \in \mathcal{I}_k} \varphi_I \otimes dx_I \right\|_\Omega = \sum_{k=0}^{n} \sum_{I \in \mathcal{I}_k} \|f_I - \varphi_I\|_{L^2(V)} \leq 2^n \varepsilon,$$

where,

$$\| \cdot \|_\Omega : \Omega_{L^2}(V) \longrightarrow [0, \infty) : \omega \longmapsto \sqrt{\langle \omega, \omega \rangle_\Omega},$$

is the Hilbert space norm on $\Omega_{L^2}(V)$. Since ω was arbitrary, it follows that the $\mathbb{C}$-linear span of $\mathscr{B}_\Omega$ is dense in $\Omega_{L^2}(V)$, so that $\mathscr{B}_\Omega$ is an orthonormal basis for the Hilbert space $\Omega_{L^2}(V)$.

Lastly, we show that each vector in $\mathscr{B}_\Omega$ is also an eigenvector of $D^2 : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}$. Indeed, from Theorem 2.24,

$$D^2 = \mathcal{H}_s \otimes \mathbb{1} + N : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V),$$

where $\mathcal{H}_s : \mathcal{S}(V) \to \mathcal{S}(V)$ is the scalar harmonic oscillator of Equation (2.42) and $N : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ is the number operator of Definition 2.23. Therefore, taking any well-ordered multi-index I of length $0 \leq m \leq n$, and any $\psi_{\vec{k}}$ in $\mathscr{B}$, with $\vec{k} = (k_1, k_2, \ldots, k_n)$ an n-tuple of non-negative integers, we have

$$\begin{aligned} D^2(\psi_{\vec{k}} \otimes dx_I) &= \mathcal{H}_s(\psi_{\vec{k}}) \otimes dx_I + N\left(\psi_{\vec{k}} \otimes dx_I\right) \\ &= \left(\sum_{j=1}^{n} 2k_j + 1\right) \psi_{\vec{k}} \otimes dx_I + (2m - n)\ \psi_{\vec{k}} \otimes dx_I \\ &= 2\left(m + \sum_{j=1}^{n} k_j\right) \psi_{\vec{k}} \otimes dx_I. \end{aligned}$$

□

Corollary 2.27. *The Dirac-Heisenberg operator $D : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ of Definition 2.21, viewed as a densely defined unbounded linear operator on $\Omega_{L^2}(V)$, is essentially self-adjoint. Moreover, the kernel of D and D^2 are one dimensional, both being spanned by the vector $\psi_{\vec{0}} \otimes 1$ in $\Omega_{\mathcal{S}}(V)$.*

Proof. Let $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ be an orthonormal coordinate system, and $\mathscr{B}_\Omega$ the orthonormal basis for $\Omega_{L^2}(V)$ constructed in Corollary 2.26. By [14, pp. 257], it suffices to show that the operators $D \pm i : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V) \subseteq \Omega_{L^2}(V)$ have range which is dense in $\Omega_{L^2}(V)$. However, since $\mathscr{B}_\Omega$ is an orthonormal basis, it suffices to show that $\mathscr{B}_\Omega$ is in the range of $D \pm i$. Indeed this is the case, for if $\psi_{\vec{k}} \otimes dx_I$ is in $\mathscr{B}_\Omega$, with $\vec{k} := (k_1, \ldots, k_n)$ and n-tuple of non-negative integers, and dx_I in an orthonormal basis element in $\mathcal{E}_k \subseteq \Lambda^k V^*$, then

$$D^2(\psi_{\vec{k}} \otimes dx_I) = 2\left(k + \sum_{j=1}^{n} k_j\right)\ \psi_{\vec{k}} \otimes dx_I,$$

In particular, $c := 2\left(k + \sum_{j=1}^{n} k_j\right) + 1 > 1$ and

$$(D \pm i)(D \mp i)(c^{-1}\psi_{\vec{k}} \otimes dx_I) = c^{-1}\left(D^2 + 1\right)(\psi_{\vec{k}} \otimes dx_I) = \psi_{\vec{k}} \otimes dx_I.$$

Thus, $\mathscr{B}_\Omega$ is in the range of $D \pm i : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$.

To see that the kernel of $D : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ is 1-dimensional note that, from Corollary 2.26, the kernel of $D^2 : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ is spanned by the Schwartz-class differential form $\psi_{\vec{0}} \otimes 1$ in $\mathscr{B}_\Omega$. Clearly the kernel of D is contained in the kernel of D^2, so it suffices to show that $\psi_{\vec{0}} \otimes 1$ is contained in the kernel of D. Indeed, let d and d^* be the exterior derivative and codifferential of Definition 2.6, and let ρ and ρ^* be the linear operators of Lemma 2.10; Recall that d^* and ρ^* both act as the zero operators on $\Omega^0_{\mathcal{S}}(V)$. Hence, by definition of the Dirac-Heisenberg operator as,

$$D := d + d^* + \rho + \rho^* : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V),$$

and since $\psi_{\vec{0}} \otimes 1$ is in $\Omega^0_{\mathcal{S}}(V)$, it follows that,

$$D(\psi_{\vec{0}} \otimes 1) = (d + \rho)(\psi_{\vec{0}} \otimes 1) = \sum_{j=1}^{n}\left(\frac{\partial \psi_{\vec{0}}}{\partial x_j} + x_j \cdot \psi_{\vec{0}}\right) \otimes 1. \tag{2.55}$$

By definition,

$$\psi_{\vec{0}} : V \longrightarrow \mathbb{C} : v \longmapsto (2\pi)^{-\frac{n}{2}} e^{-\|v\|_V^2/2},$$

from which it is easy to compute that, for each integer $1 \leq j \leq n$,

$$\frac{\partial \psi_{\vec{0}}}{\partial x_j} + x_j \cdot \psi_{\vec{0}} = 0 : V \longrightarrow \mathbb{C}.$$

Hence the Dirac-Heisenberg operator has kernel spanned by the Schwartz-class differential 0-form $\psi_{\vec{0}} \otimes 1$. □

Using Corollary 2.26, one also observes that the sequence of eigenvalues of D^2, and hence the principal values of the Dirac-Heisenberg operator, are unbounded. However, for our purposes relating to spectral triples, this is an insufficient characterization of the spectral information carried by D. The following lemma provides a growth rate for the eigenvalues of D^2 acting on $\Omega_{\mathcal{S}}(V)$, and hence a growth rate for the principal values of the Dirac-Heisenberg operator D, acting as an unbounded operator on $\Omega_{L^2}(V)$.

Proposition 2.28. *Let $\{\mu_j\}_{j\in\mathbb{N}}$ be a non-decreasing enumeration of the eigenvalues (repeated with multiplicity) of the linear operator $D^2 : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$, where D is the Dirac-Heisenberg operator Definition 2.21. If $n := \dim(V)$, then for any natural number k with $k \geq 8e$,*

$$\left(\frac{n}{4e}\right) k \leq \mu_{k^n} \leq (2n)\, k.$$

Proof. Define the function

$$\|\cdot\|_1 : \mathbb{Z}^n_{\geq 0} \to \mathbb{Z}_{\geq 0} : (z_1, z_2, \ldots, z_n) \mapsto \sum_{j=1}^{n} z_j,$$

and, for each non-negative integer j, define the set,

$$B^{\mathbb{Z}}_j := \left\{ \vec{z} \in \mathbb{Z}^n_{\geq 0} \middle|\ \|z\|_1 \leq j \right\}. \tag{2.56}$$

Denote the set of well-ordered multi-indices of length k, with $1 \leq j \leq n$, by

$$\mathcal{I}_k := \left\{ (i_1, i_2, \ldots, i_j) \in \{1, 2, \ldots, n\}^k\ : i_1 < i_2 < \cdots < i_j \right\},$$

where, by convention, $\mathcal{I}_0 := \{\emptyset\}$. Letting

$$S := \left(\bigcup_{j=0}^{n} \mathcal{I}_j \right) \times \mathbb{Z}^n_{\geq 0},$$

we analogously define the function $\|\cdot\|_S : S \to \mathbb{Z}_{\geq 0}$ by

$$\|(I, \vec{z})\|_S := k + \|\vec{z}\|_1, \text{ for each } I \in \mathcal{I}_k \text{ and } \vec{z} \in \mathbb{Z}^n_{\geq 0}$$

and, for each j in $\mathbb{Z}_{\geq 0}$, the set

$$B^S_j := \{ (I, \vec{z}) \in S \mid\ \|(I, \vec{z})\| \leq j \}. \tag{2.57}$$

By definition, the set $\mathscr{B}_\Omega \subseteq \Omega_{\mathcal{S}}(V)$ given in Corollary 2.26, consisting of orthogonal eigenvectors for the operator D^2 acting on $\Omega_{\mathcal{S}}(V)$, is in bijection with the set S; In particular, each element $(I, \vec{z})$ in S corresponds to an eigenvector $\psi_{\vec{z}} \otimes dx_I$ in $\mathscr{B}_\otimes$, which in turn corresponds to a unique term $\mu_\tau = \mu_{\tau(I,\vec{k})}$ of the enumeration $\{\mu_j\}_{j\in\mathbb{N}}$ such that $\mu_\tau = 2\,\|(I, \vec{z})\|_S$. Using this bijection between the set S and the enumeration

$\{\mu_j\}_{j\in\mathbb{N}}$, it is easy to see that, for any r in $\mathbb{Z}_{\geq 0}$

$$\#B_r^S = \#\{\mu_j \ : \mu_j \leq 2r\},$$

where B_r^S is defined as in Equation (2.57), and $\#A$ denotes the number of elements in an arbitrary set A. It follows immediately that,

$$2r = \mu_{\#B_r^S}. \tag{2.58}$$

Now, let k be a natural number and define the set,

$$A_k := \{\emptyset\} \times \{0, 1, \dots, k-1\}^n \subseteq S.$$

Since $A_k \subseteq B_{nk}^S$, it follows that A_k satisfies $k^n = \#A_k \leq \#B_{nk}^S$. Hence,

$$\mu_{k^n} \leq \mu_{\#B_{nk}^S} = 2nk.$$

For the other inequality, we must be more careful. First, observe that, for arbitrary j in $\mathbb{Z}_{\geq 0}$, the set B_j^S may be written as a disjoint union,

$$B_j^S = \bigsqcup_{j=0}^{n} \left\{ (I, \vec{z}) \in S \mid \ I \in \mathcal{I}_j, \vec{z} \in B_{r-j}^{\mathbb{Z}} \right\} = \bigsqcup_{j=0}^{n} \mathcal{I}_j \times B_{r-j}^{\mathbb{Z}},$$

where $B_{r-j}^{\mathbb{Z}} \subseteq \mathbb{Z}_{\geq 0}^n$ is defined as in Equation (2.56). It follows that,

$$\#B_r^S = \sum_{j=0}^{n} \# \left(\mathcal{I}_j \times B_{r-j}^{\mathbb{Z}}\right) = \sum_{j=0}^{n} \binom{n}{j} \left(\#B_{r-j}^{\mathbb{Z}}\right) \leq 2^n \left(\#B_r^{\mathbb{Z}}\right), \tag{2.59}$$

where the last inequality holds since $B_{r-j}^{\mathbb{Z}} \subseteq B_r^{\mathbb{Z}}$ for any $j = 0, 1, \dots, n$. Counting the number of elements in $B_r^{\mathbb{Z}}$ is an application of the "Stars and Bars" theorem, which shows,

$$\#B_r^{\mathbb{Z}} = \sum_{j=0}^{r} \binom{n+j-1}{j} = \binom{n+r}{n}.$$

Well-known bounds on the binomial coefficient, together with Equation (2.59), then imply,

$$\#B_r^S \leq 2^n \left(\#B_r^{\mathbb{Z}}\right) \leq 2^n \left(\frac{e(n+r)}{n}\right)^n = \left(\frac{2e}{n}\right)^n (n+r)^n. \tag{2.60}$$

If we assume that $r \geq n$, then Equation (2.60) implies the bound,

$$\#B_r^S \leq \left(\frac{4er}{n}\right)^n, \tag{2.61}$$

Letting $c = \frac{n}{8e}$ and k be an integer with $k \geq 8e$, we have $ck \geq n \geq 1$, and hence $\lceil ck \rceil \geq n \geq 1$. Therefore, by our choice of c and Equation (2.61),

$$\#B^S_{\lceil ck \rceil} \leq \left(\frac{4e\lceil ck \rceil}{n} \right)^n \leq \left(\frac{4e\,(ck+1)}{n} \right)^n \leq \left(\frac{8eck}{n} \right)^n = k^n. \tag{2.62}$$

Finally, by Equation (2.58) and Equation (2.62), we see

$$\frac{nk}{4e} = 2ck \leq 2\lceil ck \rceil = \mu_{\#B^S_{\lceil ck \rceil}} \leq \mu_{k^n}.$$

□

Chapter 3
C*-algebras and Heisenberg Cycles

3.1 C*-Algebras: Definition and Examples

We begin with some basic definitions in C^*-algebra theory, taken mainly from Emerson [5].

Definition 3.1. An *associative algebra over* $\mathbb{C}$ is a $\mathbb{C}$-vector space A equipped with an associative, bilinear multiplication operation $A \times A \to A; (a, b) \mapsto ab$. An algebra is *unital* if it contains an element $1 \in A$ such that $1a = a1 = a$ for all $a \in A$. An associative algebra over $\mathbb{R}$ is defined similarly.

All of the algebras which will, and have, appeared in this thesis are associative, and so we generally drop the word associative and refer to them simply as *algebras*.

Definition 3.2. A *complex C^*-algebra* A is an (associative) algebra over the complex numbers equipped with a map $* : A \to A$ (which we call the *adjoint*) and a norm $\| \cdot \| : A \to [0, \infty)$ satisfying

a) The map $*$ is a conjugate-linear, involutive anti-homomorphism, *i.e.* satisfies

- $(\lambda a + b)^* = \overline{\lambda} a^* + b^*$ for all $\lambda \in \mathbb{C}$, $a, b \in A$,
- $(ab)^* = b^* a^*$ for all $a, b \in A$, and
- $(a^*)^* = a$ for all $a \in A$.

b) With the metric induced by the norm $\| \cdot \| : A \to [0, \infty)$, A is complete, *i.e.* $(A, \| \cdot \|)$ is a Banach space.

c) $\|ab\| \leq \|a\| \|b\|$ for all $a, b \in A$.

d) $\|a^* a\| = \|a\|^2$ for all $a \in A$.

One says that a C^*-algebra A is *unital* if there exists an element $1 \in A$ acting as identity under multiplication, and that A is *commutative* if its algebra product is commutative.

Condition d) is often called the C^*-*condition*, and it is from this C^*-condition that much of the rich structure of C^*-algebras arises.

Briefly, we remind the reader that a $*$-*homomorphism* $\varphi : A \to B$ between two C^*-algebras is an algebra homomorphism such that $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$. The notion of isomorphisms, and automorphisms, of C^*-algebras are then the obvious one: the existence of a pair of $*$-homomorphisms which compose to the identity. We do not prove it here (see [5]), but any $*$-homomorphism is automatically continuous.

Example 3.3. [The Real and Complex Numbers]

The prototypical example of a commutative complex C^*-algebra is that of $\mathbb{C}$. Indeed, equipping $\mathbb{C}$ with its natural commutative Banach algebra structure, one may take the adjoint $* : \mathbb{C} \to \mathbb{C}$ to be defined by complex conjugation. In this setting, conditions *a*) through *d*) are easily seen to hold.

On the other hand, the prototypical example of a commutative real C^*-algebra is that of the real numbers $\mathbb{R}$. Indeed, equipping the algebra $\mathbb{R}$ with the absolute value norm, and with adjoint $* : \mathbb{R} \to \mathbb{R}$ given by the identity map, one not only sees that $\mathbb{R}$ forms a C^*-algebra, but also a real C^*-subalgebra of $\mathbb{C}$.

The next example of a C^*-algebra is that of bounded $\mathbb{C}$-valued functions on a Euclidean space. While not of vital importance to this thesis, a number of the interesting C^*-algebras we discuss will be contained in the C^*-algebra of bounded functions.

Example 3.4. [Bounded Functions on Euclidean Space] Let $(V, \langle\cdot,\rangle_V)$ be a finite dimensional Euclidean space, and equip $\mathbb{C}$ with its natural norm $|\cdot| : \mathbb{C} \to [0, \infty)$. The *supremum norm* of a function $f : V \to \mathbb{C}$ is defined by,

$$\|f\| := \sup_{x \in V} |f(x)| \tag{3.1}$$

and we say that the function $f : V \to \mathbb{C}$ is *bounded* if $\|f\| < \infty$. Denoting the set of bounded functions from V to $\mathbb{C}$ by,

$$\mathcal{B}(V; \mathbb{C}) := \{f : V \to \mathbb{C} \ : \ \|f\| < \infty\}, \tag{3.2}$$

we aim to prescribe $\mathcal{B}(V; \mathbb{C})$ with the structure of a complex C^*-algebra.

To this end, we define an (associative, unital) $\mathbb{C}$-algebra structure on $\mathcal{B}(V; \mathbb{C})$, by taking the pointwise operations of addition, scalar multiplication, and product. The adjoint $* : \mathcal{B}(V; \mathbb{C}) \to \mathcal{B}(V; \mathbb{C})$ is defined by taking the pointwise complex conjugate; That is, for any $f \in \mathcal{B}(V; \mathbb{C})$, we define $f^* \in \mathcal{B}(V; \mathbb{C})$,

$$(f^*)(v) := \overline{f(v)}, \quad \text{for all} \quad v \in V, \tag{3.3}$$

One easily verifies that condition *a*) of Definition 3.2 holds, so that $\mathcal{B}(V; \mathbb{C})$ is a $*$-algebra. Endowing $\mathcal{B}(V; \mathbb{C})$ with the supremum norm $\|\cdot\| : \mathcal{B}(V; \mathbb{C}) \to [0, \infty)$ of

Equation (3.1), a routine verification shows conditions *c*) and *d*) of Definition 3.2 hold under this choice of norm and $*$-algebra structure for $\mathcal{B}(V;\mathbb{C})$. For condition *b*), *i.e.* the completeness of $\mathcal{B}(V;\mathbb{C})$ with respect to the supremum norm, we refer to Folland [6, Proposition 4.13].

In conclusion, the set of bounded, complex-valued, functions on a Euclidean space, denoted $\mathcal{B}(V;\mathbb{C})$, forms a unital commutative C^*-algebra over the complex numbers under the natural pointwise operations and supremum norm.

The next example of a C^*-algebra we provide is one of particular importance to this thesis: that of the *bounded, uniformly continuous functions on a finite dimensional Euclidean space*. Such a C^*-algebra will be used in the definition of our Heisenberg spectral triple.

Example 3.5 (Bounded, Uniformly Continuous Functions on Euclidean Space)**.**

Let $(V, \langle\cdot,\cdot\rangle)$ be a Euclidean space, with inner product norm

$$\|\cdot\|_V : V \longrightarrow [0,\infty) : v \longmapsto \sqrt{\langle v, v\rangle_V}$$

and let $\mathcal{B}(V;\mathbb{C})$ denote the C^*-algebra of bounded functions on V, described in Example 3.4. We denote the norm on $\mathcal{B}(V;\mathbb{C})$ by

$$\|f\| := \sup_{v\in V}|f(v)|, \quad \text{for all} \quad f\in\mathcal{B}(V;\mathbb{C}).$$

Recall that, in the context complex valued functions on Euclidean spaces, a function $f : V \to \mathbb{C}$ is said to be *uniformly continuous* if,

$$\lim_{\|v\|_V\to 0}\left(\sup_{x\in V}|f(x+v) - f(x)|\right) = 0. \tag{3.4}$$

If we define, for each vector v in V and function $f : V \to \mathbb{C}$, a new function $f_v : V \to \mathbb{C}$ by

$$f_v(x) := f(x - v), \text{ for all } x \in V,$$

then by definition, uniform continuity of $f : V \to Y$ is equivalent to

$$\lim_{\|v\|_V\to 0}\|f_v - f\| = 0.$$

We will now provide a C^*-subalgebra of $\mathcal{B}(V;\mathbb{C})$ consisting of bounded functions on V which are also uniformly continuous. First, we let the set of all *bounded, uniformly* *continuous functions from* V *to* $\mathbb{F}$ be denoted

$$C_u(V) := \left\{ f \in \mathcal{B}(V;\mathbb{C}) \;\middle|\; \lim_{\|v\|_V \to 0} (\|f_v - f\|) = 0 \right\} \tag{3.5}$$
$$= \left\{ f : V \to \mathbb{C} \;\middle|\; \|f\| < \infty \text{ and } \lim_{\|v\|_V \to 0} \|f_v - f\| = 0 \right\}$$

By definition, each function in $C_u(V)$ is bounded, and hence $C_u(V)$ is a subset of the C^*-algebra $\mathcal{B}(V)$. In fact, $C_u(V)$ is also a $*$-subalgebra of $\mathcal{B}(V;\mathbb{C})$: For all f and h in $C_u(V)$, and all complex numbers z, we have that for all vectors v in V that

$$\|(f+h)_v - (f+h)\| = \|f_v + h_v - f + h\| \leq \|f_v - f\| + \|h_v - h\|, \tag{3.6}$$

$$\|(cf)_v - (cf)\| = |c| \cdot \|f_v - f\|, \tag{3.7}$$

$$\|(f \cdot h)_v - (f \cdot h)\| \leq \|f_v - f\| \; \|h_v\| + \|f\| \; \|h_v - h\| \tag{3.8}$$

$$\|(f^*)_v - (f^*)\| = \|(f_v - f)^*\| = \|f_v - f\|_{\mathbb{F}}. \tag{3.9}$$

Taking a limit as $\|v\|_V \to 0$ in the above four inequalities yields the desired closure of $C_u(V)$ under pointwise addition, multiplication, $\mathbb{C}$-scaling, and adjoint operations in $\mathcal{B}(V;\mathbb{C})$. Thus, $C_u(V)$ is a $*$-subalgebra of the C^*-algebra $\mathcal{B}(V;\mathbb{C})$.

Restricting the norm on $\mathcal{B}(V;\mathbb{C})$ to the $*$-subalgebra $C_u(V)$, conditions c) and d) of Definition 3.2 follow immediately from the previous paragraph. Thus, all we must do to verify $C_u(V)$ is a C^*-algebra is show that it is complete in the metric determined by restricting the norm on $\mathcal{B}(V;\mathbb{C})$ to the $*$-subalgebra $C_u(V)$. This simply amounts to checking that $C_u(V)$ is a closed subset of the complete metric space $\mathcal{B}(V;\mathbb{C})$. Taking f and h in $C_u(V)$, and any vector v in V, we compute,

$$\|f_v - f\| \leq \|f_v - h_v\| + \|h_v - h\| + \|h - f\|$$
$$= 2\|f - h\| + \|h_v - h\|,$$

from which it follows that if a sequence of elements in $C_u(V)$ converges, in supremum norm, to a bounded function in $\mathcal{B}(V;\mathbb{C})$, then that bounded function must also be uniformly continuous.

Hence, $C_u(V)$ is a C^*-subalgebra of $\mathcal{B}(V;\mathbb{C})$ and, in particular, a C^*-algebra in its own right.

Before moving on from Example 3.5, we consider a dense $*$-subalgebra of the C^*-algebra $C_u(V)$, consisting of smooth functions which have bounded partial derivatives of all orders; This will serve as the smooth subalgebra for our Heisenberg spectral triple. For the definition of smooth functions from V into $\mathbb{C}$, see Definition 1.43; Recall that the set of all such smooth functions is denoted $C^\infty(V)$. As was shown in Lemma 1.44, $C^\infty(V)$ is a $*$-algebra with operations given by pointwise addition, $\mathbb{C}$-scaling, product, and complex conjugation. Moreover, given an orthonormal coordinate system $x = (x_1, \dots, x_n) : V \to \mathbb{R}^n$, which is simply a unitary isomorphism of Euclidean spaces, we defined the linear operator of partial differentiation by x_j,

$$\frac{\partial}{\partial x_j} : C^\infty(V) \longrightarrow C^\infty(V) : f \longmapsto \frac{\partial f}{\partial x_j},$$

using the diagram (1.28). Further, for each n-tuple $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of non-negative integers, we introduced the notation for a composition of such operators,

$$\frac{\partial^\alpha}{\partial x^\alpha} := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \tag{3.10}$$

In terms of the notation in Equation (3.10), we introduce the class of smooth functions with *bounded partial derivatives of all orders.*

Definition 3.6. Let $f : V \to \mathbb{C}$ be a smooth function. We say that f has *bounded partial derivatives of all orders*, or simply that f is *smooth with bounded derivatives*, if for every orthonormal coordinate system $x = (x_1, \dots, x_n) : V \to \mathbb{R}^n$ and any n-tuple α of non-negative integers, there exists a non-negative real constant C_α such that,

$$\left\| \frac{\partial^\alpha f}{\partial x^\alpha} \right\| = \sup_{v \in V} \left| \frac{\partial^\alpha f}{\partial x^\alpha}(v) \right| \le C_\alpha.$$

The set of all smooth $\mathbb{C}$-valued functions with bounded partial derivatives of all orders is denoted $C_u^\infty(V)$.

The following lemma shows that not only is every smooth function with bounded partial derivatives of all orders automatically bounded and uniformly continuous, but also that $C_u^\infty(V)$ is closed under the $*$-algebra operations in $C_u(V)$; In particular, $C_u^\infty(V)$ is a $*$-subalgebra of the C^*-algebra $C_u(V)$.

Lemma 3.7. *Let $C_u(V)$ denote the C^*-algebra uniformly continuous functions, as defined in Example 3.5, and let $C_u^\infty(V)$ denote the set of smooth functions with bounded derivatives given in Definition 3.6. Then $C_u^\infty(V; \mathbb{F})$ is a subset of $C_u(V)$ which closed under the C^*-algebra operations in $C_u(V)$. In particular, $C_u^\infty(V)$ is a $*$-subalgebra of $C_u(V)$.*

Proof. It follows immediately from Definition 3.6 that each function in $C_u^\infty(V)$ is bounded. To see that functions in $C_u^\infty(V)$ are also uniformly continuous, and therefore that $C_u^\infty(V)$ is a subset of $C_u(V)$, let $x = (x_1, \dots, x_n) : V \to \mathbb{R}^n$ be an orthonormal system of coordinates. Note that, since this coordinate system if orthonormal, if $\{e_j\}_{j=1}^n$

the standard orthonormal basis for $\mathbb{R}^n$, then $\{x^{-1}(e_j)\}_{j=1}^n$ is an orthonormal basis for V. If f is a function in $C_u^\infty(V)$, and v, w are two vectors in V, then define,

$$\delta_0 := 0 \in V, \quad \text{and} \quad \delta_j = \delta_{j-1} + \langle x^{-1}(e_j), v\rangle_V \; x^{-1}(e_j) \quad \text{for integers } 1 \leq j \leq n.$$

Using a telescoping series, we may write,

$$f(w+v) - f(w) = \sum_{j=1}^{n} f(w+\delta_j) - f(w+\delta_{j-1}),$$

which implies

$$|f(w+v) - f(w)| \leq \sum_{j=1}^{n} |f(w+\delta_j) - f(w+\delta_{j-1})|.$$

For each integer $1 \leq j \leq n$, define

$$\chi_j : \mathbb{R} \longrightarrow \mathbb{C} : r \longmapsto f(w + \delta_{j-1} + tx^{-1}(e_j)),$$

and observe that ϕ_j is continuously differentiable on all of $\mathbb{R}$, with

$$\chi_j'(t) = \frac{\partial f}{\partial x_j}(w + \delta_{j-1} + tx^{-1}(e_j)) \quad \text{for all} \quad t \in \mathbb{R},$$

and satisfies

$$\chi_j\left(\langle x^{-1}(e_i), v\rangle_V\right) - \chi_j(0) = f(w+\delta_j) - f(w+\delta_{j-1}).$$

Identifying $\mathbb{C}$ isometrically with the Euclidean space $\mathbb{R}^2$ in the usual way, it follows from Rudin [17, Theorem 5.19] that,

$$|f(w+v) - f(w)| \leq \sum_{j=1}^{n} \left|\langle x^{-1}(e_j), v\rangle_V\right| \; \left\|\frac{\partial f}{\partial x_j}\right\|.$$

Since $|\langle x^{-1}(e_j), v\rangle_V| \leq \|v\|_V$ and f has bounded partial derivatives of all orders, it follows that there is a constant $C > 0$, which is independent of v and w, such that $|f(w+v) - f(w)| \leq C\|v\|_V$. Taking a supremum in w, it follows that $\|f_v - f\| \leq C\|v\|_V$, allowing us to conclude $\lim_{\|v\|_V \to 0} \|f_v - f\| = 0$; That is, f is uniformly continuous.

The above paragraph shows that $C_u^\infty(V)$ is a subset of $C_u(V)$, and so it remains to show that $C_u^\infty(V)$ is closed under the pointwise $*$-algebra operations in $C_u(V)$. To see this, take f and h to be two functions in $C_u^\infty(V)$, z to be a complex number, and α

to be an n-tuple of non-negative integers. By definition, there are constants $C_\alpha^{(f)}$ and $C_\alpha^{(h)}$ such that, for any orthonormal coordinate system $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$,

$$\left\| \frac{\partial^\alpha f}{\partial x^\alpha} \right\| \le C_\alpha^{(f)} \quad \text{and} \quad \left\| \frac{\partial^\alpha h}{\partial x^\alpha} \right\| \le C_\alpha^{(h)}.$$

We have already shown in Lemma 1.44 that since f and h are smooth, so too are the four pointwise defined functions,

$$f + h : V \to \mathbb{C} : v \mapsto f(v) + h(v), \qquad zf : V \to \mathbb{C} : v \mapsto zf(v),$$

$$f \cdot h : V \to \mathbb{C} : v \mapsto f(v)\ h(v), \qquad f^* : V \to \mathbb{C} : v \mapsto \overline{f(v)}.$$

Hence, if $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ is any orthonormal coordinate system, then by linearity of the operator $\frac{\partial^\alpha}{\partial x^\alpha}$ acting on smooth functions,

$$\frac{\partial^\alpha (f + h)}{\partial x^\alpha} = \frac{\partial^\alpha f}{\partial x^\alpha} + \frac{\partial \alpha}{\partial h} x^\alpha \quad \text{and} \quad \frac{\partial^\alpha z f}{\partial x^\alpha} = z\ \frac{\partial^\alpha f}{\partial x^\alpha}$$

as functions from V to $\mathbb{C}$; In particular,

$$\left\| \frac{\partial^\alpha (f + h)}{\partial x^\alpha} \right\| \le \left\| \frac{\partial^\alpha f}{\partial x^\alpha} \right\| + \left\| \frac{\partial^\alpha h}{\partial x^\alpha} \right\| \le C_\alpha^{(f)} + C_\alpha^{(h)} \quad \text{and,}$$

$$\left\| \frac{\partial^\alpha (zf)}{\partial x^\alpha} \right\| = |z|\ \left\| \frac{\partial^\alpha f}{\partial x^\alpha} \right\| = |z|\ C_\alpha^{(f)},$$

from which we conclude that $C_u^\infty(V)$ is closed pointwise addition and $\mathbb{C}$-scalar multiplication. On the other hand, Equation (1.32) and Equation (1.33), respectively, show that if $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ is an arbitrary system of orthonormal coordinates, then

$$\frac{\partial^\alpha (f \cdot h)}{\partial x^\alpha} = \sum_{\substack{\alpha = \beta + \gamma \\ \gamma, \beta \in \mathbb{Z}^n_{\ge 0}}} \frac{\alpha!}{\beta! \gamma!} \frac{\partial^\beta f}{\partial x^\beta} \cdot \frac{\partial^\gamma h}{\partial x^\gamma} : V \longrightarrow \mathbb{C} \qquad \text{and,}$$

$$\frac{\partial^\alpha (f^*)}{\partial x^\alpha} = \left(\frac{\partial^\alpha f}{\partial x^\alpha} \right)^* : V \longrightarrow \mathbb{C}.$$

Hence, for an arbitrary orthonormal coordinate system $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$,

$$\left\| \frac{\partial^\alpha (f \cdot h)}{\partial x^\alpha} \right\| \le \sum_{\substack{\alpha = \beta + \gamma \\ \gamma, \beta \in \mathbb{Z}^n_{\ge 0}}} \frac{\alpha!}{\beta! \gamma!} \left\| \frac{\partial^\beta f}{\partial x^\beta} \right\|\ \left\| \frac{\partial^\gamma h}{\partial x^\gamma} \right\| \le \sum_{\substack{\alpha = \beta + \gamma \\ \gamma, \beta \in \mathbb{Z}^n_{\ge 0}}} \frac{\alpha!}{\beta! \gamma!} C_\beta^{(f)}\ C_\gamma^{(h)} \qquad \text{and ,}$$

$$\left\| \frac{\partial^\alpha f^*}{\partial x^\alpha} \right\| = \left\| \frac{\partial^\alpha f}{\partial x^\alpha} \right\| \le C_\alpha^{(f)}$$

from which it follows that $C_u^\infty(V)$ is closed under the $*$-algebra product and adjoint in $C_u(V)$. ☐

Verifying that $C_u^\infty(V)$ is, in fact, a dense $*$-subalgebra of $C_u(V)$ will require the use of *convolution*, which we briefly define here in the case of continuous functions. For a more thorough analysis of convolutions, see [6, §8.2]

Definition 3.8. Let μ denote Haar measure on the Euclidean space $(V, \langle\cdot,\cdot\rangle_V)$, and let $C_u(V)$ be the C^*-algebra of bounded, uniformly continuous functions from V to $\mathbb{C}$, as given in Example 3.5. If f is a function in $C_u(V)$ and $\phi : V \to \mathbb{C}$ is a smooth function on V with compact support, then the *convolution of f and ϕ* is the function

$$f * \phi : V \longrightarrow \mathbb{C} : v \longmapsto \int_V f(w)\phi(v-w)\ d\mu(w).$$

Of course, one should verify that the convolution of a function in $C_u(V)$ with a compactly supported smooth function determines a well-defined $\mathbb{C}$-valued function on V. The next lemma ensure this is the case, and moreover that such convolutions are uniformly continuous and bounded. For the proof, see [6, Proposition 8.8].

Lemma 3.9. *Let μ denote Haar measure on $(V, \langle\cdot,\cdot\rangle_V)$. If f is in $C_u(V)$ and $\phi : V \to \mathbb{C}$ is a smooth function with compact support, then the convolution $f * \phi : V \to \mathbb{C}$ is in $C_u(V)$ with,*

$$\|f * \phi\| \leq \|f\|\ \|\phi\|\ \mu\left(\mathrm{supp}(\phi)\right),$$

where $\mathrm{supp}(\phi)$ *is the support of $\phi : V \to \mathbb{C}$.*

Convoluting sufficiently regular functions produces ones which are not only bounded and uniformly continuous, as Lemma 3.9 shows, but also, under appropriate conditions, the resulting convolution is a smooth function with bounded partial derivatives of all orders in the sense of Definition 3.6.

Lemma 3.10. *If f is a function in $C_u(V)$, and $\phi : V \to \mathbb{C}$ is smooth, real valued, non-negative, and compactly supported, then the convolution $f * \phi : V \to \mathbb{C}$ is in $C_u^\infty(V)$. In particular, if α is an n-tuple of non-negative integers and $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ an orthonormal coordinate system of coordinates, then*

$$\frac{\partial^\alpha (f * \phi)}{\partial x^\alpha} = f * \left(\frac{\partial^\alpha \phi}{\partial x^\alpha}\right) : V \longrightarrow \mathbb{C} : v \longmapsto \int_V f(w)\ \frac{\partial^\alpha \phi}{\partial x^\alpha}(v-w)\ d\mu(w).$$

Proof. Let $\mathrm{supp}(\phi)$ denote the (compact) support of ϕ, and take $\varepsilon > 0$ such that,

$$\mathrm{supp}(\phi) \subseteq B_\varepsilon(0) := \{v \in V\ :\ \|v\|_V < \varepsilon\}.$$

Fix an arbitrary orthonormal system of coordinates $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$. If α is any n-tuple of non-negative integers, then

$$\frac{\partial^\alpha \phi}{\partial x^\alpha} : V \longrightarrow \mathbb{R} \subseteq \mathbb{C},$$

is a smooth function with compact support contained in $B_\varepsilon(0)$. In particular, from Lemma 3.9, it holds that the convolution,

$$f * \frac{\partial^\alpha \phi}{\partial x^\alpha} : V \longrightarrow \mathbb{C},$$

is continuous and bounded.

To see $f * \phi : V \to \mathbb{C}$ is smooth, it suffices to show that, for each integer $1 \leq j \leq n$,

$$\frac{\partial (f * \phi)}{\partial x_j} = f * \frac{\partial \phi}{\partial x_j} : V \longrightarrow \mathbb{C}.$$

Indeed, if this identity holds then, by induction, it will follow that for any n-tuple α of non-negative integers,

$$\frac{\partial^\alpha (f * \phi)}{\partial x^\alpha} = f * \frac{\partial^\alpha \phi}{\partial x^\alpha} : V \longrightarrow \mathbb{C},$$

which is continuous and bounded by the observations made in the previous paragraph.

To this end, let $\{e_j\}_{j=1}^n$ be the canonical orthonormal basis for $\mathbb{R}^n$. From the definition of $\frac{\partial}{\partial x_j}$ and linearity of Lebesgue integration,

$$\frac{\partial (f * \phi)}{\partial x_j}(v) = \lim_{t \to 0} \int_V f(w) \frac{\phi\left(v - w + t x^{-1}(e_j)\right) - \phi(v - w)}{t} \, d\mu(w). \tag{3.11}$$

With the aim of applying the Dominated Convergence Theorem, denote the difference quotient in the integrand on the right-hand side by

$$\Delta_t : V \longrightarrow \mathbb{R} \subseteq \mathbb{C} : y \longmapsto \frac{\phi\left(v - y + t x^{-1}(e_j)\right) - \phi(v - y)}{t},$$

so that the integrand is,

$$h_t : V \longrightarrow \mathbb{C} : w \longmapsto f(w)\Delta_t(w).$$

For each real number t, the support of $\Delta_t : V \to \mathbb{R} \subseteq \mathbb{C}$ lies in

$$B_\varepsilon(v) \cup B_\varepsilon\left(v + t x^{-1}(e_j)\right).$$

However, since the orthonormal coordinate system $x : V \to \mathbb{R}^n$ is an isometric linear isomorphism, we have $\|x^{-1}(e_j)\|_V$ and thus, whenever $0 < |t| < 1$, the support of

$\Delta_t : V \to \mathbb{R} \subseteq \mathbb{C}$, and therefore also the support of $\chi_t : V \to \mathbb{C}$, lies in the bounded set $B_{2\varepsilon}(v)$. Therefore, for each $0 < |t| < 1$, the integrand $h_t : V \to \mathbb{C}$ is continuous with compact support lying in $B_{2\varepsilon}(v)$ which, in particular, implies that it is a Lebesgue integrable function over V. Also clear is that $h_t : V \to \mathbb{C}$ converges pointwise to the function $V \to \mathbb{C} : w \mapsto f(w)\frac{\partial \phi}{\partial x_j}(v-w)$ as t tends to 0. Lastly, let $\chi_{B_{2\varepsilon}(v)} : V \to \{0,1\}$ denote the indicator function for the set $B_{2\varepsilon}(v)$, and note that if $0 < |t| < 1$ and w is in V, then the fact that $h_t : V \to \mathbb{C}$ is supported in $B_{2\varepsilon}(v)$, together with the Mean Value Theorem, implies,

$$|h_t(w)| = |h_t(w)|\, \chi_{B_{2\varepsilon}(v)}(w) \leq \|f\| \, \left\| \frac{\partial \phi}{\partial x_j} \right\| \, \chi_{B_{2\varepsilon}(v)}(w).$$

This shows that for $0 < |t| < 1$, the integrand $h_t : V \to \mathbb{C}$ is dominated by the Lebesgue integrable function

$$V \longrightarrow \mathbb{C} : w \longmapsto \|f\| \, \left\| \frac{\partial \phi}{\partial x_j} \right\| \, \chi_{B_{2\varepsilon}}(w).$$

Applying the Dominated Convergence Theorem [6, Theorem 2.25], we conclude that

$$\frac{\partial f * \phi}{\partial x_j} = f * \frac{\partial \phi}{\partial x_j} : V \longrightarrow \mathbb{C}.$$

Continuity of the function $\frac{\partial (f*\phi)}{\partial x_j} : V \to \mathbb{C}$ is then a result of Lemma 3.9 and the above identity.

As noted, induction then implies that the convolution $f * \phi : V \to \mathbb{C}$ is smooth, and

$$\frac{\partial^\alpha (f * \phi)}{\partial x^\alpha} = f * \frac{\partial^\alpha \phi}{\partial x^\alpha},$$

for each n-tuple of non-negative integers. To see that the smooth function $f * \phi : V \to \mathbb{C}$ has bounded derivatives of all orders in the sense of Definition 3.6, simply note $\frac{\partial^\alpha \phi}{\partial x^\alpha} : V \to \mathbb{C}$ is smooth with compact support for each n-tuple α of non-negative integers. Hence, by Lemma 3.9, there exists existence of a constant C_α such that,

$$\left\| \frac{\partial^\alpha (f * \phi)}{\partial x^\alpha} \right\| = \left\| f * \frac{\partial^\alpha \phi}{\partial x^\alpha} \right\| \leq C_\alpha.$$

□

Finally, we are in a position to verify that not only is C_u^∞ a $*$-subalgebra of the C^*-algebra $C_u(V)$, but that this $*$-subalgebra is also dense in $C_u(V)$.

Theorem 3.11. *Let $C_u(V)$ denote the C^*-algebra of bounded, uniformly continuous, $\mathbb{C}$-valued functions on the Euclidean space $(V, \langle \cdot, \cdot \rangle_V)$, as defined in Example 3.5, and let $C_u^\infty(V)$ denote the smooth functions with bounded derivatives of all orders from Definition 3.6. Then $C_u^\infty(V; \mathbb{F})$ is a dense $*$-subalgebra of $C_u(V)$.*

Proof. We have shown in Lemma 3.7 that $C_u^\infty(V)$ is a $*$-subalgebra of $C_u(V)$, so we need only show $C_u^\infty(V)$ is dense in $C_u(V)$ with respect to the supremum norm,

$$\|\cdot\| : C_u(V) \longrightarrow [0,\infty) : f \longmapsto \sup_{v\in V} |f(v)|.$$

To this end, let μ denote Haar measure on $(V, \langle\cdot,\cdot\rangle_V)$, and let $\phi : V \to \mathbb{F}$ be a smooth function with compact support such that the range of ϕ lies in the non-negative real numbers and $\int_V \phi(x)\ d\mu(x) = 1$. Take a positive real number r such that the support of $\phi : V \to \mathbb{R} \subseteq \mathbb{C}$ lies in the open ball $B_r(0) := \{v \in V\ : \|v\|_V < r\}$, and denote $n := \dim(V)$. For each natural number k, define,

$$\phi_k : V \longrightarrow \mathbb{R} \subseteq \mathbb{C} : v \longmapsto k^n\phi(kx).$$

It is clear that, for each natural number k, the function $\phi_k : V \to \mathbb{C}$ is smooth and compactly supported, with support lying in the open ball $B_{\frac{r}{k}}(0) \subseteq V$, and hence that the function

$$V \longmapsto \mathbb{R} \subseteq \mathbb{C} : w \longmapsto \phi_k(v - w)$$

is supported in $B_{\frac{r}{k}}(v)$ for each v in V. Moreover, by a routine substitution,

$$\int_V \phi_k(w)\ d\mu(w) = \int_V k^n\phi(kw)\ d\mu(w) = \int_V \phi(w)\ d\mu(w) = 1.$$

For arbitrary $f \in C_u(V)$, we have from Lemma 3.10 that, for each natural number k, the convolution $f * \phi_k : V \to \mathbb{C}$ is in $C_u^\infty(V)$.

We claim that the limit, as k tends to infinity, of $\|f * \phi_k - f\|$ is 0, from which it will follow that $C_u^\infty(V)$ is dense in $C_u(V)$. Indeed, let $\varepsilon > 0$ and, by uniform continuity of $f : V \to \mathbb{C}$, find $\delta > 0$ such that $|f(v_1) - f(v_2)| < \varepsilon$ whenever $\|v_1 - v_2\|_V < \delta$. For any natural number k with $k > \frac{r}{\delta}$, we see that for each v in V,

$$\begin{aligned}
|(f * \phi_k)(v) - f(v)| &= \left| \left(\int_V f(w)\phi_k(v-w)\ d\mu(w) \right) - f(v) \int_V \phi_k(v-w)\ d\mu(w) \right| \\
&= \left| \int_V (f(w) - f(v))\, \phi_k(v-w)\ d\mu(w) \right| \\
&\leq \int_V |f(w) - f(v)|\ \phi_k(v-w)\ d\mu(w) \\
&= \int_{B_{\frac{r}{k}}(v)} |f(w) - f(v)|\ \phi_k(v-w)\ d\mu(w) \\
&\leq \left(\sup_{\|v_1 - v_2\|_V < \delta} |f(v_1) - f(v_2)| \right) \int_{B_{\frac{r}{k}}(v)} \phi_k(v-w)\ d\mu(w) \\
&= \left(\sup_{\|v_1 - v_2\|_V < \delta} |f(v_1) - f(v_2)| \right) \\
&< \varepsilon.
\end{aligned}$$

Taking a supremum, we conclude $\|f * \phi_k - f\|$ tends to 0 as k tends to infinity, proving $C_u^\infty(V)$ is dense in $C_u(V)$. □

Our next example is that of a *crossed-product C^*-algebra*. In general, crossed-products arise whenever one a C^*-algebra equipped with (sufficiently continuous) group action by $*$-automorphisms. For our purposes however, we will only concern ourselves with the action discrete groups on C^*-algebras by automorphisms, and the *reduced* crossed-product generated by such actions. For a general reference on crossed-products, see [5, Chapter 1].

Prior to giving this example, we recall the definition of a group action on a C^*-algebra

Definition 3.12. Let G be a discrete topological group. A *G-$*$-algebra* is a $*$-algebra A together with a group homomorphism

$$\theta : G \longrightarrow \mathrm{Aut}(A) : g \longmapsto \theta_g,$$

where $\mathrm{Aut}(A)$ is the group of $*$-automorphisms on A. The notion of a *G-C^*-algebra* is defined in the obvious fashion. To simplify notation, we will sometimes write $g(a)$, as opposed to $\theta_g(a)$, to denote the image of a in A under the automorphism determined by g in G.

Definition 3.13 (Definition 1.12.2 [5])**.** Let G be a discrete group, and A a G-$*$-algebra, with the action denoted $G \to \mathrm{Aut}(A) : g \mapsto \theta_g$. The *twisted group algebra* $A[G]$ is the $*$-algebra defined as follows.

Take the underlying set of $A[G]$ to be $C_c(G, A)$, the compactly supported continuous functions from G into A. As G is discrete, we may write any element of $A[G]$ in the form $\sum_{g\in G} a_g[g]$, with a_g in A being the value of the function at g in G, and where it is understood that $a_g = 0$ for all but finitely many g in G. In terms of this notation, we equip $A[G]$ with vector space operations of pointwise addition and scalar multiplication are given, respectively, by the formulae,

$$\left(\sum_{g\in G} a_g[g]\right) + \left(\sum_{g\in G} b_g[g]\right) := \sum_{g\in G}(a_g + b_g)[g], \tag{3.12}$$

$$z\left(\sum_{g\in G} a_g[g]\right) := \sum_{g\in G}(za_g)[g], \tag{3.13}$$

for any z in $\mathbb{C}$ and $\sum_{g\in G} a_g[g]$, $\sum_{g\in G} b_g[g]$ in $A[G]$. The (associative) algebra product and involution on $A[G]$ are given, respectively, by,

$$\left(\sum_{g\in V_d} a_g[g]\right) * \left(\sum_{g\in V_d} b_g[g]\right) := \sum_{g_1,g_2\in G} a_{g_1} \cdot \theta_{g_1}(b_{g_2})[g_1g_2], \tag{3.14}$$

$$\left(\sum_{g\in V_d} a_g[g]\right)^* := \sum_{g\in V_d} \theta_g^{-1}(a_g^*)[g^{-1}], \tag{3.15}$$

for $\sum_{g\in G} a_g[g]$ and $\sum_{g\in G} b_g[g]$ in $A[G]$.

While we will not prove the following properties, as most are well-known and easily verifiable facts, the reader should be aware of the following.

Lemma 3.14. *Let G be a discrete topological group, A a G-$*$-algebra, and $A[G]$ the induced twisted group algebra.*

1. *If e is the identity of the group G, then the map $A \to A[G] : a \mapsto a[e]$, is a $*$-homomorphism. Hence, A can be viewed as a $*$-subalgebra of $A[G]$, consisting of functions supported at the identity in G.*

2. *If A is unital, say with identity 1_A in A, then for each g in G the elements $[g] := 1_A[g]$ are unitary in the sense that $[g]^* = [g^{-1}] = [g]^{-1}$. Moreover, the resulting copy of the group G as unitaries in $A[G]$ satisfies $[g]a[g^*] = \theta_g(a) = \theta_g(a)[e]$, for each a in A and g in G.*

3. *Let G' be a discrete group, A' a G'-$*$-algebra with action $\theta' : G' \to \mathrm{Aut}(B)$, and $A'[G']$ the resulting twisted group algebra. If $\upsilon : A \to B$ is a $*$-homomorphism, $\varphi : G \to G'$ is a group homomorphism and,*

$$\varphi(g)(\upsilon(a)) = \upsilon(\theta'_g(a)), \qquad \textit{for all } g \in G, a \in A,$$

is satisfied, then the map

$$A[G] \longrightarrow B[G'] : \sum_{g\in G} a_g[g] \longmapsto \sum_{g\in G} \upsilon(a_g)[\varphi(g)],$$

is a $*$-homomorphism.

4. *Let B be a unital $*$-algebra, and denote the group of unitary elements in B by $\mathbb{U}$. If $\upsilon : A \to B$ a $*$-homomorphism, $\varphi : G \to \mathbb{U}(B)$ a group homomorphism, and the* covariance condition*,*

$$\varphi(g)\upsilon(a)\varphi(g)^* = \upsilon(g(a)), \quad \textit{for all} \quad g \in G, a \in A, \tag{3.16}$$

is satisfied, then the map,

$$A[G] \longrightarrow B : \sum_{g\in G} a_g[g] \longmapsto \sum_{g\in G} \upsilon(a)\varphi(g),$$

is a $*$-algebra homomorphism. The pair (υ, φ) is called a covariant pair*.*

Example 3.15. Let $(V, \langle\cdot,\cdot\rangle_V)$ be a Euclidean space, and let V_d denote the discrete, abelian, topological group generated by the underlying group of the vector space V. As constructed in Example 3.4 and Definition 3.6, respectively, let $C_u(V)$ denote the C^*-algebra of bounded, uniformly continuous, $\mathbb{C}$-valued functions on V, and let $C_u^\infty(V)$ denote its dense $*$-subalgebra of smooth functions with bounded partial derivatives of all orders, cf Theorem 3.11. The goal of this section is to define the reduced crossed-product C^*-algebra generated by the natural translation action of V_d on $C_u(V)$, and to consider a dense $*$-subalgebra of this crossed product arising from the inclusion of $C_u^\infty(V)$ into $C_u(V)$. As is often the case when constructing C^*-algebras, this crossed-product will be defined by completing a $*$-algebra under an appropriate choice of norm.

We begin by observing that if g is a group element in V_d, and $f : V \to \mathbb{C}$ is bounded and uniformly continuous, then the function

$$g(f) : V \longrightarrow \mathbb{C} : v \longmapsto f(v - g),$$

is also in $C_u(V)$. A routine verification shows that $g : C_u(V) \to C_u(V) : f \mapsto g(f)$ is a well-defined $*$-automorphism, and the induced map $G \to \mathrm{Aut}(C_u(V))$ is a group homomorphism. We may therefore consider $C_u(V)$ to be a V_d-C^*-algebra in the sense of Definition 3.12, and form the twisted group algebra $C_u(V)[V_d]$ of Definition 3.13.

Equipping $C_u(V)[V_d]$ with an appropriate norm will require us to define an injective $*$-homomorphism from $C_u(V)$ into a Hilbert space, which we choose to be the square-integrable functions $\left(L^2(V), \langle\cdot,\cdot\rangle_{L^2(V)}\right)$ of Example 1.42. Recall that the inner product norm on $L^2(V)$ is given by,

$$\|\cdot\|_{L^2(V)} : L^2(V) \longrightarrow [0,\infty) : \xi \longmapsto \sqrt{\langle \xi, \xi, \rangle_{L^2(V)}},$$

and that $\mathcal{S}(V)$, the Schwartz-class functions of Definition 1.43, are to be viewed as a dense linear subspace of $L^2(V)$; In fact, we defined $L^2(V)$ as the Hilbert space completion of $\mathcal{S}(V)$ with respect to the inner product

$$\langle \cdot, \cdot \rangle_{L^2(V)} : \mathcal{S}(V) \times \mathcal{S}(V) \longrightarrow \mathbb{C},$$

of Lemma 1.45.

If f is in $C_u^\infty(V)$ and $h : V \to \mathbb{C}$ is in $\mathcal{S}(V)$, then it follows by the definition of $C_u^\infty(V)$ that,

$$\pi(f)\xi : V \longrightarrow \mathbb{C} : v \longmapsto f(v)\xi(v),$$

is a Schwartz-class function in V. Routine computations then verify that,

$$\pi(f) : \mathcal{S}(V) \longrightarrow \mathcal{S}(V) : \eta \longmapsto \tilde{\pi}(f)\eta,$$

is a well-defined linear map which, by Hölder's Inequality [6, Theorem 6.2], satisfies $\|\pi(f)\eta\|_{L^2(V)} \leq \|f\|\|\eta\|_{L^2(V)}$ for each η in $\mathcal{S}(V)$. If fact, using a standard smooth bump function argument, one can show,

$$\sup_{\substack{\|\eta\|_{L^2(V)}=1 \\ \eta \in \mathcal{S}(V)}} \|\pi(f)\eta\|_{L^2(V)} = \|f\|. \tag{3.17}$$

From basic properties of Hilbert space completions, it follows that $\pi(f) : \mathcal{S}(V) \to \mathcal{S}(V)$ extends uniquely to a bounded linear operator on $L^2(V)$, also denoted $\pi(f) : L^2(V) \to L^2(V)$. From Equation (3.17) and the density of $\mathcal{S}(V)$ in $L^2(V)$, this induced bounded linear operator on $L^2(V)$ has norm $\|f\|$ in the sense that,

$$\|\pi(f)\|_{\mathbb{B}(L^2(V))} := \sup_{\substack{\|\xi\|_{L^2(V)}=1 \\ \xi \in L^2(V)}} \|\pi(f)\xi\|_{L^2(V)} = \|f\|.$$

As the algebra operations in $C_u(V)$ and $C_u^\infty(V)$ are defined pointwise, it is plain to see that,

$$\pi : C_u^\infty(V) \longrightarrow \mathbb{B}(L^2(V)) : f \longmapsto \pi(f),$$

is a unital $*$-homomorphism which is isometric with respect to the supremum norm on $C_u^\infty(V)$ and the operator norm on $\mathbb{B}(L^2(V))$. By the density of $C_u^\infty(V)$ in $C_u(V)$, it follows that this $*$-homomorphism $\pi : C_u^\infty(V) \to \mathbb{B}(L^2(V))$ canonically extends to an isometric, unital, $*$-homomorphism,

$$\pi : C_u(V) \longrightarrow \mathbb{B}(L^2(V)) : f \longmapsto \pi(f).$$

We record the observations made in the previous paragraph through the following proposition.

Proposition 3.16. *For each f in $C_u^\infty(V)$ and η in $\mathcal{S}(V)$, the function,*

$$f \cdot \eta : V \longrightarrow \mathbb{C} : v \longmapsto f(v)\eta(v),$$

is Schwartz-class, and the map

$$\pi(f) : \mathcal{S}(V) \longrightarrow \mathcal{S}(V) : \eta \longmapsto f \cdot \eta,$$

is well-defined, linear, and extends canonically to a bounded linear operator on $L^2(V)$ whose operator norm agrees with $\|f\|$, the supremum norm of f. Moreover, the induced mapping,

$$\pi : C_u^\infty(V) \longrightarrow \mathbb{B}(L^2(V)) : f \longmapsto \pi(f),$$

is a unital, isometric, $*$-homomorphism which canonically extends to a unital, isometric, $*$-homomorphism

$$\pi : C_u(V) \longrightarrow \mathbb{B}(L^2(V))$$

representing the C^*-algebra $C_u(V)$ as bounded operators on $L^2(V)$.

We use the representation $\pi : C_u(V) \to \mathbb{B}(L^2(V))$, as constructed in Proposition 3.16, to define the *left-regular representation* of the twisted group algebra $C_u(V)[V_d]$. Let $\ell^2(V_d; L^2(V))$ denote the Hilbert space of square-integrable functions from the discrete group V_d into the Hilbert space $L^2(V)$; That is, $\ell^2(V_d; L^2(V))$ is the Hilbert space completion of $C_c(V_d; L^2(V))$, the compactly supported functions from V_d into $L^2(V)$, with respect to the inner product,

$$\langle \cdot, \cdot \rangle : C_c(V_d; L^2(V)) \times C_c(V_d; L^2(V)) \longrightarrow \mathbb{C} : (\tau_1, \tau_2) \longmapsto \sum_{g \in V_d} \langle \tau_1(g), \tau_2(g) \rangle_{L^2(V)}.$$

We define a covariant pair, in the sense of Lemma 3.14, and induced $*$-homomorphism which we denote by

$$\mathrm{Ind}(\pi) : C_u(V)[V_d] \longrightarrow \mathbb{B}(\ell^2(V_d; L^2(V))), \tag{3.18}$$

by defining a $*$-representation of $C_u(V)$ on $\ell^2(V_d; L^2(V))$, and a unitary representation of V_d on $\ell^2(V_d; L^2(V))$, respectively, via the formulae,

$$(f \cdot \Upsilon)(g_0) := [\pi(g_0^{-1}(f)) \cdot \Upsilon](g_0), \text{ for all } g_0 \in V_d, f \in C_u(V),$$

$$(g \cdot \Upsilon)(g_0) = \Upsilon(g_0 - g), \quad \text{for all} \quad g, g_0 \in V_d,\ \Upsilon \in \ell^2(V_d; L^2(V)).$$

Definition 3.17. Let $C_u(V)[V_d]$ denote the twisted group algebra generated by the translation action of V_d on $C_u(V)$, and let $\mathrm{Ind}(\pi) : C_u(V)[V_d] \to \mathbb{B}(\ell^2(V_d; L^2(V)))$ denote the $*$-homomorphism of Equation (3.18), where $\mathbb{B}(\ell^2(V_d; L^2(V)))$ is the C^*-algebra of bounded linear operators on the Hiblert space $\ell^2(V_d; L^2(V))$.

The (reduced) *crossed-product* C^*-algebra $C_u(V) \rtimes V_d$ is the completion of the twisted group algebra $C_u(V)[V_d]$ in the norm,

$$\|\cdot\| : C_u(V)[V_d] \longrightarrow [0, \infty) : \sum_{g \in V_d} f_g[g] \longmapsto \left\| \mathrm{Ind}(\pi) \left(\sum_{g \in V_d} f_g[g] \right) \right\|_{\mathbb{B}}, \tag{3.19}$$

where $\|\cdot\|_{\mathbb{B}}$ denotes operator norm on $\mathbb{B}(\ell^2(V_d; L^2(V)))$.

It is easy to see that the $*$-algebra $C_u^\infty(V)$ of Definition 3.6, which is a $*$-subalgebra of $C_u(V)$ by Lemma 3.9, is closed under the group action of V_d on $C_u(V)$. We therefore also regard $C_u^\infty(V)$ as a V_d-$*$-algebra, allowing us to construct the twisted group algebra $C_u^\infty(V)[V_d]$. Applying the third point of Lemma 3.14 to the canonical inclusion $C_u^\infty(V) \hookrightarrow C_u(V)$ and the identity homomorphism $V_d \to V_d$, we obtain a $*$-homomorphism of twisted group algebras,

$$C_u^\infty(V)[V_d] \longrightarrow C_u(V)[V_d] : \sum_{g \in V_d} f_g[g] \longmapsto \sum_{g \in V_d} f_g[g],$$

which is easily verified to be injective. Hence, we may view $C_u^\infty(V)[V_d]$ as a $*$-subalgebra of $C_u(V)[V_d]$, and therefore also as a $*$-subalgebra of the crossed-product $C_u(V) \rtimes V_d$. This $*$-subalgebra $C_u^\infty(V)[V_d]$ is, in fact, dense in the C^*-algebra crossed-product $C_u(V) \rtimes V_d$. Indeed, by the density of $C_u^\infty(V)$ in $C_u(V)$, shown in Theorem 3.11, it follows that with respect to the norm of Equation (3.19), $C_u^\infty(V)[V_d]$ is a dense $*$-subalgebra of $C_u(V)[V_d]$, and therefore must also be a dense $*$-subalgebra of the completion $C_u(V) \rtimes V_d$.

3.2 The Heisenberg Cycle

Fixing an n-dimensional Euclidean space $(V, \langle \cdot, \cdot \rangle_V)$, and letting V_d denote the discrete, abelian, topological group generated by the underlying group structure of V, we constructed the crossed-product C^*-algebra $C_u(V) \rtimes V_d$ in Example 3.15; In particular, this crossed product is generated by the group action of V_d on $C_u(V)$ in which a group element g in V_d acts by the $*$-automorphism $C_u(V) \to C_u(V) : \xi \mapsto g(\xi)$, and

$$(g\xi)(v) = \xi(x - v), \quad \text{for all} \quad \xi \in C_u(V), v \in V.$$

In that same example, we argued that the twisted group algebra $C_u^\infty[V_d]$ determines a dense $*$-subalgebra of the crossed product $C_u(V) \rtimes V_d$.

The aim of this section is to define the *Heisenberg spectral triple*, a $2n$-dimensional even spectral triple over $C_u^\infty(V)[V_d] \subset C_u(V) \rtimes V_d$ in the sense of Definition 1.2. The Hilbert space associated to this spectral triple will be that of square-integrable differential forms $\Omega_{L^2}(V) := L^2(V) \otimes \Lambda V^*$ of Definition 2.15, equipped with the inner product,

$$\langle \cdot, \cdot \rangle_\Omega : \Omega_{L^2}(V) \times \Omega_{L^2}(V) \longrightarrow \mathbb{C},$$

of Proposition 2.16 and the $\mathbb{Z}/2$-grading $\Omega_{L^2}(V) = \Omega^+_{L^2}(V) \oplus \Omega^-_{L^2}(V)$ of Proposition 2.18. We denote the corresponding inner product norm on $\Omega_{L^2}(V)$ by,

$$\| \cdot \|_\Omega : \Omega_{L^2}(V) \longrightarrow [0, \infty) : \omega \longmapsto \sqrt{\langle \omega, \omega \rangle_\Omega}.$$

In the following subsection, we construct a $*$-homomorphism $\sigma : C_u(V) \rtimes V_d \to \mathbb{B}(\Omega_{L^2}(V))$ representing $C_u(V) \rtimes V_d$ as even bounded linear operators on the $\mathbb{Z}/2$-graded Hilbert space $\Omega_{L^2}(V)$. Once such a representation is defined, the subsection after will concern itself with defining a densely defined, self-adjoint, unbounded linear operator $\overline{D} : \mathrm{Dom}(\overline{D}) \to \Omega_{L^2}(V)$ which is odd (grading reversing) with respect to the $\mathbb{Z}/2$-grading, *i.e.*

$$\overline{D} : \mathrm{Dom}(\overline{D}) \cap \Omega^\pm_{L^2}(V) \longrightarrow \Omega^\mp_{L^2}(V).$$

One such an operator is obtained, we will argue that $(1 + \overline{D}^2)^{-\frac{1}{2}}$, a bounded linear operator on $\Omega_{L^2}(V)$ defined via functional calculus, is Schatten p-class for each $p > 2n$. From that point, all one will need to verify in order to conclude that the data $(\Omega_{L^2}(V), \sigma, \overline{D})$ determines a $2n$-dimensional even spectral triple over $C_u^\infty[V_d] \subset C_u(V) \rtimes V_d$ is check that, for each θ in $C_u^\infty(V)[V_d]$, the domain $\mathrm{Dom}(\overline{D})$ is invariant under the linear operator $\sigma(\theta)$, and the resulting commutator,

$$[\overline{D}, \sigma(\theta)] : \mathrm{Dom}(\overline{D}) \longrightarrow \Omega_{L^2}(V),$$

extends uniquely to a bounded linear operator on all of $\Omega_{L^2}(V)$.

3.2.1 The Hilbert Space and Representation

As stated above, the $\mathbb{Z}/2$-graded Hilbert space of our Heisenberg spectral triple is that of the square-integrable differential forms over V, denoted $(\Omega_{L^2}(V), \langle \cdot, \cdot \rangle_\Omega)$, where the even and odd subspaces of $\Omega_{L^2}(V)$ are defined by

$$\Omega^+_{L^2}(V) := \bigoplus_{k \text{ even}} \Omega^k_{L^2}(V) \quad \text{and} \quad \Omega^-_{L^2}(V) := \bigoplus_{k \text{ odd}} \Omega^k_{L^2}(V),$$

where $\Omega^k_{L^2}(V)$ is the closed subspace of $\Omega_{L^2}(V)$ consisting of square-integrable differential k-forms of Definition 2.15.

As is usually often the case with crossed-product C^*-algebras, we construct a representation of $C_u(V) \rtimes V_d$ by Lemma 3.14 and, in particular, *covariant pairs*; That is, we will first construct a representation of $C_u(V)$ by bounded linear operators on $\Omega_{L^2}(V)$, and a representation of V_d by unitary operators on $\Omega_{L^2}(V)$, such that these representations combine to satisfy the *covariance condition* of Equation (3.16). Any such covariant pair will induced a $*$-homomorphism of $C_u(V)[V_d]$ into $\mathbb{B}(\Omega_{L^2}(V))$, which extends uniquely to a $*$-homomorphism of $C_u(V) \rtimes V_d$ into $\mathbb{B}(\Omega_{L^2}(V))$.

To construct this covariant pair, we will need to use some basic notation, which we now recall. First let,

$$(\Lambda V^*, \langle \cdot, \cdot \rangle_{\Lambda V^*}) \quad \text{and} \quad \Big(L^2(V), \langle \cdot, \cdot \rangle_{L^2(V)}\Big),$$

denote, respectively, the Hilbert space of square-integrable functions constructed in Example 1.42, and the Euclidean space determined by the exterior algebra constructed in Example 1.28; Their respective inner products are denoted,

$$\langle \cdot, \cdot \rangle_{L^2(V)} : L^2(V) \times L^2(V) \longrightarrow \mathbb{C} \quad \text{and} \quad \langle \cdot, \cdot \rangle_{\Lambda V^*} : \Lambda V^* \times \Lambda V^* \longrightarrow \mathbb{R},$$

while the respective inner product norms are,

$$\| \cdot \|_{L^2(V)} : L^2(V) \longrightarrow [0, \infty) : \xi \longmapsto \sqrt{\langle \xi, \xi \rangle_{L^2(V)}}$$

$$\| \cdot \|_{\Lambda V^*} : \Lambda V^* \longrightarrow [0, \infty) : \tau \longmapsto \sqrt{\langle \tau, \tau \rangle_{\Lambda V^*}}.$$

Moreover, by definition of $L^2(V)$ as the Hilbert space completion of the Schwartz-class functions on V, denoted $\mathcal{S}(V)$, with respect to the inner product defined in Lemma 1.45, we may view $\mathcal{S}(V)$ as a dense subspace of the Hilbert space $L^2(V)$. In terms of these inner product spaces, the square-integrable differential forms are given by $\Omega_{L^2}(V) := L^2(V) \otimes \Lambda V^*$, while the Schwartz-class differential forms are given by $\Omega_{\mathcal{S}}(V) := \mathcal{S}(V) \otimes \Lambda V^*$; Both of these tensor product are of the type seen in Definition 2.1. In particular, we argued in Proposition 2.17 that $\Omega_{\mathcal{S}}(V)$ may be viewed as a dense subspace of the Hilbert space $\Omega_{L^2}(V)$.

For the representation of $C_u(V)$ as bounded linear operators on $\Omega_{L^2}(V)$, recall from 3.16 that there exists a representation $\pi : C_u(V) \to \mathbb{B}(L^2(V))$. Hence, for each function f in $C_u(V)$, we may apply Lemma 2.2 to the bounded operator $\pi(f) : L^2(V) \to L^2(V)$, and the identity on ΛV^*, in order to obtain a linear map,

$$\pi_\Omega(f) := \pi(f) \otimes \mathbb{1} : \Omega_{L^2}(V) \longrightarrow \Omega_{L^2}(V). \tag{3.20}$$

Such a linear map is determined by the formula,

$$\pi_\Omega(f)(\xi \otimes \tau) = (\pi(f)\xi) \otimes \tau,$$

for each elementary tensor $\xi \otimes \tau$ in $\Omega_{L^2}(V) := L^2(V) \otimes \Lambda V^*$.

Proposition 3.18. *For each function f in $C_u(V)$, the linear operator $\pi_\Omega(f) : \Omega_{L^2}(V) \to \Omega_{L^2}(V)$ defined in Equation (3.20) is bounded. The induced map,*

$$C_u(V) \longrightarrow \mathbb{B}(\Omega_{L^2}(V)) : f \longmapsto \pi_\Omega(f),$$

is a unital, isometric, $*$-homomorphism. Moreover, if f is in $C_u^\infty(V)$, then the dense subspace $\Omega_{\mathcal{S}}(V)$ in $\Omega_{L^2}(V)$ is closed under the operator $\pi_\Omega(f)$, i.e.

$$\pi_\Omega(f) : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V).$$

Proof. Let $\mathcal{E} := \{dx_I \ : \ I \in \mathcal{I}\}$ be an orthonormal basis for the Euclidean space $(\Lambda V^*, \langle \cdot, \cdot \rangle_{\Lambda V^*})$. By Lemma 2.2, any ω in $\Omega_{L^2}(V)$ may be written uniquely as,

$$\omega = \sum_{I \in \mathcal{I}} \xi_I \otimes dx_I,$$

with ξ_I in $L^2(V)$ for each index I in $\mathcal{I}$. If f in $C_u(V)$, then we have already shown that the operator norm of $\pi(f) : L^2(V) \to L^2(V)$ is $\|\pi(f)\|_{\mathbb{B}(L^2(V))} = \|f\|$, with $\|\cdot\| : C_u(V) \to [0, \infty)$ the supremum norm. By definition of the linear operator $\pi_\Omega(f) : \Omega_{L^2}(V) \to \Omega_{L^2}(V)$, orthonormality of the basis $\mathcal{E} \subseteq \Lambda V^*$, and Equation (2.29), it follows that,

$$\begin{aligned} \|\pi_\Omega(f)\omega\|_\Omega^2 = \left\| \sum_{I \in \mathcal{I}} (\pi(f)\xi) \otimes dx_I \right\|_\Omega^2 &= \sum_{I \in \mathcal{I}} \|\pi(f)\xi_I\|_{L^2(V)}^2 \\ &\leq \|f\|^2 \sum_{I \in \mathcal{I}} \|\xi_I\|_{L^2(V)}^2 \\ &= \|f\|^2 \ \|\omega\|_\Omega^2. \end{aligned}$$

We therefore conclude, $\|\pi_\Omega(f)\omega\|_\Omega \leq \|f\| \ \|\omega\|_\Omega$ for arbitrary ω in $\Omega_{L^2}(V)$, and hence $\pi_\Omega(f) : \Omega_{L^2}(V) \to \Omega_{L^2}(V)$ is a bounded linear operator.

Seeing that the induced map

$$\pi_\Omega : C_u(V) \to \mathbb{B}(\Omega_{L^2}(V)) : f \mapsto \pi_\Omega(f)$$

determines a unital, isometric, $*$-homomorphism is a simple consequence of the map $\pi : C_u(V) \to \mathbb{B}(L^2(V))$ being a unital, isometric, $*$-homomorphism.

For the final claim, take f in $C_u^\infty(V)$, and recall from Proposition 3.16 that, if $\xi : V \to \mathbb{C}$ is in $\mathcal{S}(V)$, then

$$\pi(f)\xi : V \to \mathbb{C} : v \longmapsto f(v)\ \xi(v)$$

is also a Schwartz-class function. Hence, if $\xi \otimes \tau$ is an elementary tensor in $\Omega_{\mathcal{S}}(V) := \mathcal{S}(V) \otimes \Lambda V^*$, then it follows by definition that ξ is in $\mathcal{S}$ and τ is in ΛV^*. Hence, by the definition of $\pi_\Omega(f)$,

$$\pi_\Omega(f)(\xi \otimes \tau) = (\pi(f)\xi) \otimes \tau,$$

which is again an elementary tensor in $\Omega_{\mathcal{S}}(V)$ as $\pi(f)\xi$ is in $\mathcal{S}(V)$. Therefore, the map $\pi_\Omega(f)$ takes elementary tensors in $\Omega_{\mathcal{S}}(V)$ to elementary tensors in $\Omega_{\mathcal{S}}(V)$. As a general element of $\Omega_{\mathcal{S}}(V)$ is a finite sum of elementary tensors, it follows that $\Omega_{\mathcal{S}}(V)$ is closed under $\pi_\Omega(f)$ when f is a function in $C_u^\infty(V)$. □

We will similarly turn to a unitary group representation of V_d on $L^2(V)$ to obtain a unitary representation of V_d on $\Omega_L^2(V)$ by way of the identity operator on ΛV^* and Lemma 2.2. For a group element g in V_d and a Schwartz-class function $\xi : V \to \mathbb{C}$, it is easily shown that,

$$u(g)\xi : V \longrightarrow \mathbb{C} : v \longmapsto \xi(v - g),$$

determines a Schwartz-class function on V, and the induced map $u(g) : \mathcal{S}(V) \to \mathcal{S}(V)$ is linear, invertible, and satisfies,

$$u(g_1)u(g_2) = u(g_1 + g_2) : \mathcal{S}(V) \longrightarrow \mathcal{S}(V), \quad \text{for all} \quad g_1, g_2 \in V_d. \tag{3.21}$$

Moreover, by a simple change-of-variables argument,

$$\langle u(g)\xi_1, \xi_2\rangle_{L^2(V)} = \langle \xi_1, u(g^{-1})\xi_2\rangle_{L^2(V)}, \quad \text{for all} \quad \xi_1, \xi_2 \in \mathcal{S}(V), \tag{3.22}$$

where $\langle\cdot,\cdot\rangle_{L^2(V)}$ denotes the inner product on $\mathcal{S}(V)$ of Lemma 1.45. In particular, $u(g) : \mathcal{S}(V) \to \mathcal{S}(V)$ satisfies,

$$\|u(g)\|_{\mathbb{B}} := \sup_{\substack{\|\xi\|_{L^2(V)} \le 1 \\ \xi \in \mathcal{S}(V)}} \|u(g)\xi\|_{L^2(V)} = 1. \tag{3.23}$$

As $L^2(V)$ is defined as the Hilbert space completion of $\mathcal{S}(V)$ under the inner product $\langle\cdot,\cdot\rangle_{L^2(V)}$, we are able to conclude that $u(g) : \mathcal{S}(V) \to \mathcal{S}(V)$ extends uniquely to a bounded linear operator on $L^2(V)$. In fact, if $\mathbb{U}(L^2(V))$ denotes the group of unitary operators on $L^2(V)$, then from Equation (3.22) this extension $u(g) : L^2(V) \to L^2(V)$ is an element of $\mathbb{U}(L^2(V))$. Seeing that the induced map,

$$u : V_d \longrightarrow \mathbb{U}(L^2(V)) : g \longmapsto u(g), \tag{3.24}$$

is a group homomorphism is a simple consequence of Equation (3.21). We state the result of this paragraph in a lemma for future reference.

Lemma 3.19. *Let $\mathbb{U}(L^2(V))$ denote the group of unitary linear operators on the Hilbert space $L^2(V)$. Then there is a group homomorphism,*

$$u : V_d \longrightarrow \mathbb{U}(L^2(V)) : g \longmapsto u(g),$$

which is uniquely determined by the formula,

$$(u(g)\xi)(v) := \xi(v - g), \quad \text{for all} \quad g \in V_d, \xi \in \mathcal{S}(V), v \in V.$$

With the group representation of V_d on $L^2(V)$ given in Lemma 3.19, one may then apply Lemma 2.2 to obtain, for each group element g in V_d, a linear operator on $\Omega_{L^2}(V) := L^2(V) \otimes \Lambda V^*$,

$$u_\Omega(g) := u(g) \otimes \mathbb{1} : \Omega_{L^2}(V) \longrightarrow \Omega_{L^2}(V). \tag{3.25}$$

In particular, for each elementary tensor $\xi \otimes \tau$ in $\Omega_{L^2}(V)$,

$$u_\Omega(g)(\xi \otimes \tau) = (u(g)\xi) \otimes \tau.$$

The proof of the following proposition is extremely similar to that of Proposition 3.18, and so we omit the argument.

Proposition 3.20. *For each g in V_d, the linear map $u_\Omega(g) : \Omega_{L^2}(V) \to \Omega_{L^2}(V)$ given in Equation (3.25) is a unitary Hilbert space operator on $\Omega_{L^2}(V)$. If $\mathbb{U}(\Omega_{L^2}(V))$ denotes the group of unitary operators on $\Omega_{L^2}(V)$, then the induced map,*

$$u_\Omega : V_d \longrightarrow \mathbb{U}(\Omega_{L^2}(V)) : g \longmapsto u_\Omega(g),$$

is an injective group homomorphism. Moreover, for each group element g in V_d, the dense subspace $\Omega_{\mathcal{S}}(V)$ of $\Omega_{L^2}(V)$ is closed under the unitary operator $u_\Omega(g)$, i.e.

$$u_\Omega(g) : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V).$$

Finally, we show that the representation $\pi_\Omega : C_u(V) \to \mathbb{B}(\Omega_{L^2}(V))$ of Proposition 3.18 and the representation $u_\Omega : V_d \to \mathbb{U}(\Omega_{L^2}(V))$ of Proposition 3.20 form a covariant pair.

Theorem 3.21. *Equip $C_u(V)$ with the V_d-action of Example 3.15, where the $*$-automorphism determined by a group element g in V_d is defined by the formula,*

$$[g(f)](v) = f(v - g), \quad \text{for all} \quad f \in C_u(V),\ g \in V_d, v \in V.$$

The C^*-algebra representation $\pi_\Omega : C_u(V) \to \mathbb{B}(\Omega_{L^2}(V))$ of Proposition 3.18, and the unitary group representation $u_\Omega : V_d \to \mathbb{U}(\Omega_{L^2}(V))$ of Proposition 3.20, are a covariant pair in the sense that, for all f in $C_u(V)$ and group elements g in V_d,

$$u_\Omega(g)\ \pi_\Omega(f)\ u_\Omega(g^{-1}) = \pi_\Omega(g(f)) \in \mathbb{B}(\Omega_{L^2}(V)). \tag{3.26}$$

It follows that there is a unique $*$-algebra homomorphism,

$$\sigma : C_u(V) \rtimes V_d \longrightarrow \mathbb{B}(\Omega_{L^2}(V)),$$

such that, for each $\theta = \sum_{g\in V_d} f_g[g]$ in the twisted group algebra $C_u(V)[V_d] \subseteq C_u(V)\rtimes V_d$,

$$\sigma(\theta) = \sum_{g\in V_d} \pi_\Omega(f_g)\ u_\Omega(g) \in \mathbb{B}(\Omega_{L^2}(V)). \tag{3.27}$$

Proof. Take any g in V_d and any function f in $C_u(V)$. By definition of the representations π_Ω and u_Ω, it follows that,

$$u_\Omega(g)\ \pi_\Omega(f)\ u_\Omega(g^{-1}) = (u(g)\ \pi(f)\ u(g^{-1})) \otimes \mathbb{1} : \Omega_{L^2}(V) \longrightarrow \Omega_{L^2}(V) \quad \text{and,}$$

$$\pi_\Omega(g(f)) = \pi(g(f)) \otimes \mathbb{1} : \Omega_{L^2}(V) \longrightarrow \Omega_{L^2}(V),$$

where $\pi : C_u(V) \to \mathbb{B}(L^2(V))$ is the representation of Proposition 3.16, and $u : V_d \to \mathbb{U}(L^2(V))$ is the unitary representation of Proposition 3.19. It therefore suffices to prove that the representations,

$$\pi : C_u(V) \longrightarrow \mathbb{B}(L^2(V)) \quad \text{and} \quad u : V_d \longrightarrow \mathbb{U}(L^2(V))$$

satisfy the covariance condition,

$$u(g)\ \pi(f)\ u(g^{-1}) = \pi(g(f)) : L^2(V) \longrightarrow L^2(V), \tag{3.28}$$

for any g in V_d and f in $C_u(V)$.

To this end, fix g in V_d, f in $C_u(V)$, and ξ in $\mathcal{S}(V)$. Then for each v in V,

$$\begin{aligned}\left[u(g)\pi(f)u(g^{-1})\ \xi\right](v) &= \left[\pi(f)u(g^{-1})\ \xi\right](v-g)\\ &= f(v-g)\left[u(g^{-1})\ \xi\right](v-g)\\ &= f(v-g)\xi(v)\\ &= \left[\pi(g(f))\cdot\xi\right](v).\end{aligned}$$

Since $\mathcal{S}(V)$ is dense in the Hilbert space $L^2(V)$, we can conclude that Equation (3.28) is satisfied for each g in V_d and f in $C_u(V)$. As noted above, this implies that the pair

$\pi_\Omega : C_u(V) \to \mathbb{B}(\Omega_{L^2}(V))$ and $u_\Omega : V_d \to \mathbb{U}(\Omega_{L^2}(V))$ is covariant.

From Lemma 3.14, there exists an induced $*$-homomorphism $\sigma : C_u(V)[V_d] \to \mathbb{B}(\Omega_{L^2}(V))$ such that,

$$\sigma\left(\sum_{g\in V_d} f_g[g]\right) = \sum_{g\in V_d} \pi_\Omega(f)u_\Omega(g) \quad \text{for all} \quad \sum_{g\in V_d} f_g[g] \in C_u(V)[V_d].$$

By the density of $C_u(V)[V_d]$ in $C_u(V) \rtimes V_d$, it follows that $\sigma : C_u(V)[V_d] \to \mathbb{B}(\Omega_{L^2}(V))$ extends uniquely to a unital $*$-homomorphism $\sigma : C_u(V) \rtimes V_d \to \mathbb{B}(\Omega_{L^2}(V))$ which satisfies Equation (3.27) by definition. □

We discuss two final important properties of the C^*-algebra representation $\sigma : C_u(V) \rtimes V_d \to \mathbb{B}(\Omega_{L^2}(V))$ defined in Theorem 3.21. First, recall from Proposition 2.18 that $\Omega_{L^2}(V)$ is a $\mathbb{Z}/2$-graded algebra, with orthogonal even and odd subspaces denoted, respectively, by $\Omega^+_{L^2}(V)$ and $\Omega^-_{L^2}(V)$. By definition of the representation $\sigma : C_u(V) \rtimes V_d \to \mathbb{B}(\Omega_{L^2}(V))$, for each element $\sum_{g\in V_d} f_g[g]$ in the twisted group algebra $C_u(V)[V_d]$,

$$\sigma(\sum_{g\in V_d} f_g[f]) = \sum_{g\in V_d} \pi(f)u(g) \otimes \mathbb{1} : \Omega_{L^2}(V) \longrightarrow \Omega_{L^2}(V),$$

where $\pi : C_u(V) \to \mathbb{B}(L^2(V))$ and $u : V_d \to \mathbb{U}(L^2(V))$ are the representations given in Proposition 3.18 and Equation (3.24). Using similar arguments to those of Proposition 2.5, it is easy to see that for any integer $0 \leq k \leq n$ and any θ in $C_u(V)[V_d]$,

$$\sigma(\theta) : \Omega^k_{L^2}(V) \longrightarrow \Omega^k_{L^2}(V).$$

Since $C_u(V) \rtimes V_d$ is the completion of $C_u(V)[V_d]$ with respect to the norm of Example 3.15, one then easily verifies, using the fact that $\Omega^k_{L^2}(V)$ is a closed subspace of $\Omega_{L^2}(V)$, that for arbitrary θ in $C_u(V) \rtimes V_d$,

$$\sigma(\theta) : \Omega^k_{L^2}(V) \longrightarrow \Omega^k_{L^2}(V).$$

In particular, this implies the following proposition.

Proposition 3.22. *The representation $\sigma : C_u(V) \rtimes V_d \to \mathbb{B}(\Omega_{L^2}(V))$ given in Theorem 3.21 is even with respect to the $\mathbb{Z}/2$-grading on $\Omega_{L^2}(V) = \Omega^+_{L^2}(V) \oplus \Omega^-_{L^2}(V)$ discussed in Proposition 2.18; That is, for any θ in $C_u(V) \rtimes V_d$,*

$$\sigma(\theta) : \Omega^+_{L^2}(V) \longrightarrow \Omega^+_{L^2}(V) \quad \textit{and} \quad \sigma(\theta) : \Omega^-_{L^2}(V) \longrightarrow \Omega^-_{L^2}(V).$$

For the second important property of the representation $\sigma : C_u(V) \rtimes V_d \to \mathbb{B}(\Omega_{L^2}(V))$ we wish to discuss, recall that we ended Example 3.15 by arguing that $C^\infty_u(V)[V_d]$ is a dense $*$-subalgebra of $C_u^\infty(V) \rtimes V_d$, where $C_u^\infty(V)[V_d]$ is the twisted group algebra generated by the translation action of V_d on $C_u^\infty(V)$.

Lemma 3.23. *Let $\sigma : C_u(V) \rtimes V_d \to \mathbb{B}(\Omega_{L^2}(V))$ be the representation of Theorem 3.21, and $C_u^\infty(V)[V_d]$ the dense $*$-subalgebra of $C_u(V) \rtimes V_d$ discussed above. If θ is in $C_u^\infty(V)[V_d]$, then the dense subspace $\Omega_{\mathcal{S}}(V)$ in $\Omega_{L^2}(V)$ is closed under $\pi(\omega) : \Omega_{L^2}(V) \to \Omega_{L^2}(V)$,* i.e.

$$\sigma(\theta) : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V).$$

Proof. Every element of $C_u^\infty[V_d]$ is of the form $\sum_{g\in V_d} f_g[g]$, where this sum is understood to be finite, and by definition of σ,

$$\sigma\left(\sum_{g\in V_d} f_g[g]\right) = \sum_{g\in V_d} \pi(f)u(g) \otimes \mathbb{1} : \Omega_{L^2}(V) \longrightarrow \Omega_{L^2}(V),$$

where $\pi : C_u(V) \to \mathbb{B}(L^2(V))$ and $u : V_d \to \mathbb{U}(L^2(V))$ are the representations given in Proposition 3.18 and Lemma 3.19, respectively. It therefore suffices to show that $\Omega_{\mathcal{S}}(V)$ is closed under the linear operator

$$\sigma(f[g]) := \pi(f)u(g) \otimes \mathbb{1} : \Omega_{L^2}(V) \longrightarrow \Omega_{L^2}(V)$$

, for any function f in $C_u^\infty(V)$ and any group element g in V_d.

With the above in mind, fix f in $C_u^\infty(V)$ and g in V_d. Recall that any vector in $\Omega_{\mathcal{S}}(V) := \mathcal{S}(V) \otimes \Lambda V^*$ is, by definition, a finite sum of elementary tensors in $\Omega_{\mathcal{S}}(V)$. Hence, we need only show that $\pi(f)u(g) \otimes \mathbb{1} : \Omega_{L^2}(V) \to \Omega_{L^2}(V)$ takes elementary tensors in $\Omega_{\mathcal{S}}(V)$ to elementary tensors in $\Omega_{\mathcal{S}}(V)$. Indeed, this is the case, for if $\xi : V \to \mathbb{C}$ is a Schwartz-class function and τ is in the exterior algebra ΛV^*, then we see

$$(\pi(f)u(g) \otimes \mathbb{1})(\xi \otimes \tau) = (\pi(f)u(g)\xi) \otimes \tau.$$

However, by definition of $u : V_d \to \mathbb{U}(L^2(V))$ and $\pi : C_u(V) \to \mathbb{B}(L^2(V))$, we have that $u(g) : \mathcal{S}(V) \to \mathcal{S}(V)$ and, since f is in $C_u^\infty(V)$, also that $\pi(f) : \mathcal{S}(V) \to \mathcal{S}(V)$. Hence, $\pi(f)u(g)\xi$ is in $\mathcal{S}(V)$, and we are bale to conclude that $(\pi(f)u(g)\xi) \otimes \tau$ is an elementary tensor in $\Omega_{\mathcal{S}}(V) := \mathcal{S}(V) \otimes \Lambda V^*$. □

3.2.2 The Operator

Thus far, we have discussed the $\mathbb{Z}/2$-graded Hilbert space $\Omega_{L^2}(V)$, and the even representation $\sigma : C_u(V) \rtimes V_d \to \mathbb{B}(\Omega_{L^2}(V))$ given in Theorem 3.21, which will determine two thirds of our Heisenberg (pre-)spectral triple. It remains to introduce the densely defined, self-adjoint, odd, unbounded linear operator on $\Omega_{L^2}(V)$. This subsection defines such an operator, and verifies that it satisfies the required properties given in

Definition 1.2 to form a pre-spectral triple when combined with the aforementioned Hilbert space and representation of $C_u^\infty(V)[V_d] \subseteq C_u(V) \rtimes V_d$

Take $D : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ to denote the Dirac-Heisenberg operator of Definition 2.21, viewed initially as an unbounded operator on the Hilbert space $\Omega_{L^2}(V)$ with dense domain $\mathrm{Dom}(D) := \Omega_{\mathcal{S}}(V)$. We showed in Corollary 2.27 that D is essentially self-adjoint, and therefore admits a unique maximal extension to a densely defined self-adjoint unbounded operator on $\Omega_{L^2}(V)$. The operator of our Heisenberg spectral triple will be the unique maximal self-adjoint extension of D, which we denote by $\overline{D} : \mathrm{Dom}(\overline{D}) \to \Omega_{L^2}(V)$.

From basic unbounded operator theory, the domain $\mathrm{Dom}(\overline{D})$ of the extension $\overline{D}$ is a dense subspace of $\Omega_{L^2}(V)$ which identifies with the Hilbert space completion of $\mathrm{Dom}(D) := \Omega_{\mathcal{S}}(V)$ under the inner product,

$$\langle \cdot, \cdot \rangle_D : \Omega_{\mathcal{S}}(V) \times \Omega_{\mathcal{S}}(V) \longrightarrow \mathbb{C} : (\omega_1, \omega_2) \longmapsto \langle \omega_1, \omega_2 \rangle_\Omega + \langle D\omega_1, D\omega_2 \rangle_\Omega, \tag{3.29}$$

where $\langle \cdot, \cdot \rangle_\Omega$ is the inner product on $\Omega_{L^2}(V)$. Since,

$$\|\omega\|_\Omega \leq \|\omega\|_D \quad \text{for all} \quad \omega \in \Omega_{\mathcal{S}}(V),$$

it follows that the canonical inclusion $\Omega_{\mathcal{S}}(V) \hookrightarrow \Omega_{L^2}(V)$ extends to a continuous linear map from $\mathrm{Dom}(\overline{D})$ into $\Omega_{L^2}(V)$ which is easily seen to be injective. With this consideration in mind, we view $\mathrm{Dom}(\overline{D})$ as a subspace of $\Omega_{L^2}(V)$. Note the induced hierarchy of vector space inclusions,

$$\Omega_{\mathcal{S}}(V) \hookrightarrow \mathrm{Dom}(\overline{D}) \hookrightarrow \Omega_{\mathcal{S}}(V).$$

As $\Omega_{\mathcal{S}}(V)$ is dense in $\Omega_{L^2(V)}(V)$, so too is the subspace $\mathrm{Dom}(\overline{D})$.

The following lemma gives a more concrete description of $\mathrm{Dom}(\overline{D})$ and the corresponding unbounded linear operator $\overline{D} : \mathrm{Dom}(\overline{D}) \to \Omega_{L^2}(V)$.

Lemma 3.24. *Take any ω in $\Omega_{L^2}(V)$, and let $\|\cdot\|_\Omega$ denote the inner product norm associated to $\Omega_{L^2}(V)$. Then ω is in $\mathrm{Dom}(\overline{D})$ if and only if there a sequence $\{\omega_j\}_{j\in\mathbb{N}} \subseteq \Omega_{\mathcal{S}}(V)$ such that $\{D\omega_j\}_{j\in\mathbb{N}} \subseteq \Omega_{\mathcal{S}}(V)$ converges to an element of $\Omega_{L^2}(V)$ with respect to the inner product norm $\|\cdot\|_\Omega$, and*

$$0 = \lim_{j\to 0} \|\omega_j - \omega\|_\Omega.$$

Moreover, $\overline{D} : \mathrm{Dom}(\overline{D}) \to \Omega_{L^2}(V)$ is defined, for each η in $\mathrm{Dom}(\overline{D})$, by the formula,

$$\overline{D}\omega := \lim_{j\to 0} D\omega_j \in \Omega_{L^2}(V),$$

where $\{\omega_j\}_{j\in\mathbb{N}} \subseteq \Omega_{\mathcal{S}}(V)$ is any sequence of Schwartz-class differential forms such that

$\lim_{j\to\infty}\|\omega_j - \omega\|_\Omega = 0$ *and* $\{D\omega_j\}_{j\in\mathbb{N}}$ *converges to an element of* $\Omega_{L^2}(V)$ *with respect to* $\|\cdot\|_\Omega$.

Remark 3.25. To see that the map $\overline{D} : \mathrm{Dom}(\overline{D}) \to \Omega_{L^2}(V)$ given above is well-defined, we use the fact that D is essentially self-adjoint and, in particular, symmetric as an unbounded operator on $\mathrm{Dom}(D) := \Omega_{\mathcal{S}}(V)$. Indeed, let ω be in $\mathrm{Dom}(\overline{D})$, and assume $\{\omega_j\}_{j\in\mathbb{N}}, \{\eta_j\}_{j\in\mathbb{N}}$ are any two sequences of Schwartz-class differential forms with

$$0 = \lim_{j\to 0}\|\omega_j - \omega\|_\Omega = \|\eta_j - \omega\|_\Omega$$

and such that there are χ_1 and χ_2 in $\Omega_{L^2}(V)$ for which,

$$0 = \lim_{j\to\infty}\|D\omega_j - \chi_1\|_\Omega = \lim_{j\to 0}\|D\eta_j - \chi_2\|_\Omega.$$

Fixing β in $\Omega_{\mathcal{S}}(V)$, it follows by continuity of the inner product and from the Dirac-Heisenberg operator being symmetric on $\Omega_{\mathcal{S}}(V)$ that,

$$\begin{aligned} |\langle \chi_1 - \chi_2, \beta\rangle_\Omega| &= \lim_{j\to\infty} |\langle D(\omega_j - \eta_j), \beta\rangle_\Omega| \\ &= \lim_{j\to\infty} |\langle \omega_j - \eta_j, D\beta\rangle_\Omega| \\ &\leq \lim_{j\to\infty} \|\omega_j - \eta_j\|_\Omega \; \|D\beta\|_\Omega \\ &= 0. \end{aligned}$$

As $\Omega_{\mathcal{S}}(V)$ is dense in $\Omega_{L^2}(V)$, we are able to conclude $\|\chi_1 - \chi_2\|_\Omega = 0$, which is to say,

$$\lim_{j\to\infty} D\omega_j = \chi_1 = \chi_2 = \lim_{j\to\infty} D\eta_j.$$

Hence, $\overline{D} : \mathrm{Dom}(\overline{D}) \to \Omega_{L^2}(V)$ is well-defined.

The densely defined, self-adjoint, unbounded linear operator $\overline{D} : \mathrm{Dom}(\overline{D}) \to \Omega_{L^2}(V)$ described in Lemma 3.24 will be the operator associated to the Heisenberg spectral triple. As such, the remainder of this subsection is dedicated to exhibiting that this operator, in conjunction with C^*-algebra $C_u(V) \rtimes V_d$, the dense $*$-subalgebra $C_u^\infty(V)[V_d]$, and the representation $\sigma : C_u(V) \rtimes V_d \to \mathbb{B}(\Omega_{L^2}(V))$, satisfy the properties of a $2n$-dimensional even spectral triple without the meromorphic extension property.

We first tackle verifying that $\overline{D}$ determines an odd linear operator, and that the finite summability condition in Definition 1.2 holds true.

Proposition 3.26. *The densely defined self-adjoint unbounded linear operator* $\overline{D} : \mathrm{Dom}(\overline{D}) \to \Omega_{L^2}(V)$ *of Lemma 3.24 is odd with respect to the* $\mathbb{Z}/2$*-grading on* $\Omega_{L^2}(V)$

given in Proposition 2.18; That is,

$$\overline{D} : \mathrm{Dom}(\overline{D}) \cap \Omega^{\pm}_{L^2}(V) \longrightarrow \Omega^{\mp}_{L^2}(V),$$

where $\Omega^{+}_{L^2}(V)$ *and* $\Omega^{-}_{L^2}(V)$ *are the even and odd subspaces of* $\Omega_{L^2}(V)$*, respectively.*

Moreover, with $n := \dim(V)$ *and* $\{\mu_j\}_{j\in\mathbb{N}}$ *a non-decreasing enumeration of the principal values (repeated with multiplicity) of* $\overline{D}$*, there are non-negative real constants* C_1 *and* C_2 *such that, for all but finitely many natural numbers* j*,*

$$C_1 k^{\frac{1}{2n}} \leq \mu_j \leq C_2 \frac{1}{2n}.$$

In particular, the bounded linear operator $(1+\overline{D}^2)^{-\frac{1}{2}} : \Omega_{L^2}(V) \to \Omega_{L^2}(V)$*, defined via functional calculus, is Schatten* p*-class for all* $p > 2n$*.*

Proof. For notational convenience, define the two orthogonal subspaces of $\Omega_{\mathcal{S}}(V)$,

$$\Omega^{+}_{\mathcal{S}}(V) := \bigoplus_{k \text{ even}} \Omega^{k}_{\mathcal{S}}(V) \quad \text{and } \Omega^{-}_{\mathcal{S}}(V) := \bigoplus_{k \text{ odd}} \Omega^{k}_{\mathcal{S}}(V).$$

Note that $\Omega_{\mathcal{S}}(V) = \Omega^{+}_{\mathcal{S}}(V) \oplus \Omega^{-}_{\mathcal{S}}(V)$, while $\Omega^{+}_{\mathcal{S}}(V)$ is a dense subspace of $\Omega^{+}_{L^2}(V)$ and $\Omega^{-}_{\mathcal{S}}(V)$ a dense subspace of $\Omega^{-}_{L^2}(V)$.

Let ω be in $\mathrm{Dom}(\overline{D}) \cap \Omega^{+}_{L^2}(V)$. Then, by definition of $\mathrm{Dom}(\overline{D})$, there is a sequence $\{\omega_j\}_{j\in\mathbb{N}} \subseteq \Omega_{\mathcal{S}}(V)$ such that,

$$\lim_{j\to\infty} \|\omega - \omega_j\|_{\Omega} = 0 = \lim_{j\to\infty} \|\overline{D}\omega - D\omega_j\|_{\Omega}.$$

Fix an arbitrary β in $\Omega^{+}_{\mathcal{S}}(V)$ and observe that Proposition 2.22 implies $D\beta$ is in $\Omega^{-}_{\mathcal{S}}(V) \subseteq \Omega^{-}_{L^2}(V)$. Since ω is, in particular, a vector in $\Omega^{+}_{L^2}(V)$, we have from the orthogonality of $\Omega^{+}_{L^2}(V)$ and $\Omega^{-}_{L^2}(V)$ that,

$$\langle \overline{D}\omega, \beta\rangle_{\Omega} = \lim_{j\to\infty} \langle \omega_j, D\beta\rangle_{\Omega} = \langle \omega, D\beta\rangle_{\Omega} = 0,$$

As this holds for every β in $\Omega^{+}_{\mathcal{S}}(V)$, and since $\Omega^{+}_{\mathcal{S}}(V)$ is dense in the closed subspace $\Omega^{+}_{L^2}(V)$ of $\Omega_{L^2}(V)$, we conclude that $\overline{D}\omega$ lies in the orthogonal complement of $\Omega^{+}_{L^2}(V)$. As this orthogonal complement is exactly $\Omega^{-}_{L^2}(V)$, it must be that $\overline{D}\omega$ is in $\Omega^{-}_{L^2}(V)$ whenever ω is in $\mathrm{Dom}(\overline{D}) \cap \Omega^{+}_{L^2}(V)$. By a similar reasoning, one shows that if ω is in $\mathrm{Dom}(\overline{D}) \cap \Omega^{-}_{L^2}(V)$, then $\overline{D}\omega$ is in $\Omega^{+}_{L^2}(V)$. Thus, $\overline{D} : \mathrm{Dom}(\overline{D}) \to \Omega_{L^2}(V)$ is an odd operator with respect to the $\mathbb{Z}/2$-grading on $\Omega_{L^2}(V)$.

For the second claim, observe that Proposition 2.28 implies the existence of non-negative real constants C_1 and C_2 such that, if $\{\mu_j\}_{j\in\mathbb{N}}$ is a non-decreasing enumeration (repeated with multiplicity) of the principal values for $D : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$, then,

$$C_1 \, j^{\frac{1}{2n}} \leq \mu_j \leq C_2 \, j^{\frac{1}{2n}}, \quad \text{for all but finitely many } j.$$

Since the principal values of $D : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V) \subseteq \Omega_{L^2}(V)$ are exactly the principal values of its extension $\overline{D} : \mathrm{Dom}(\overline{D}) \to \Omega_{L^2}(V)$, it is clear that principal values of $\overline{D}$ are also on the order of $j^{\frac{1}{2n}}$ for sufficiently large j. As a consequence of the observations made in Remark 1.3, we may therefore conclude that bounded linear operator $(1 + D^2)^{-\frac{1}{2}}$ on $\Omega_{L^2}(V)$ is Schatten p-class for all $p > 2n$. □

We recall that $\sigma : C_u(V) \rtimes V_d \to \mathbb{B}(\Omega_{L^2}(V))$ denotes C^*-algebra representation of Theorem 3.21, and $C_u^\infty(V)[V_d]$ the dense $*$-subalgebra in $C_u(V) \rtimes V_d$ which is generated by the action V_d acting by translation operators. With the goal of showing that the triple, $(\Omega_{L^2}(V), \sigma, D)$ is spectral, it remains to show that, for each θ in $C_u^\infty[V_d]$, the subspace $\mathrm{Dom}(\overline{D}) \subseteq \Omega_{L^2}(V)$ is invariant under linear operator $\sigma(\theta) : \Omega_{L^2}(V) \to \Omega_{L^2}(V)$, and the commutator,

$$[\overline{D}, \sigma(\theta)] = \overline{D}\sigma(\theta) - \sigma(\theta)\overline{D} : \mathrm{Dom}(\overline{D}) \longrightarrow \Omega_{L^2}(V),$$

extends to a bounded linear operator on all of $\Omega_{L^2}(V)$.

We first show that such results hold when one replaces $\overline{D}$ with the the Dirac-Heisenberg operator D.

Lemma 3.27. *Let $D : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ be the Dirac-Heisenberg operator of Definition 2.21; Where $\Omega_{\mathcal{S}}(V)$ the subspace of $\Omega_{L^2}(V)$ determined by Schwartz-class differential forms. If $\sigma : C_u(V) \rtimes V_d \to \mathbb{B}(\Omega_{L^2}(V))$ is the representation of Theorem 3.21, then for any θ in $C_u^\infty(V)[V_d]$, the commutator*

$$[D, \sigma(\theta)] : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V),$$

is well-defined and extends uniquely to a bounded linear operator on all of $\Omega_{L^2}(V)$.

Proof. The Dirac-Heisenberg operator, with domain $\Omega_{\mathcal{S}}(V)$, is defined by,

$$D = d + d^* + \rho + \rho^* : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V),$$

where d and d^* are the linear operators on $\Omega_{\mathcal{S}}(V)$ of Proposition 2.7, while ρ and ρ^* are the linear operators on $\Omega_{\mathcal{S}}(V)$ of Lemma 2.10. If θ is an arbitrary element in $C_u^\infty(V)[V_d]$, then we have already shown in Lemma 3.23 that $\Omega_{\mathcal{S}}(V)$ is invariant under the operator $\sigma(\theta) : \Omega_{L^2}(V) \to \Omega_{L^2}(V)$. Hence, the commutators $[D, \sigma(\theta)]$, $[d+d^*, \sigma(\theta)]$, and $[\rho + \rho^*, \sigma(\theta)]$ are well-defined linear operators on $\Omega_{\mathcal{S}}(V)$ satisfying,

$$[D, \sigma(\theta)] = [d + d^*, \sigma(\theta)] + [\rho + \rho^*, \sigma(\theta)] : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V). \tag{3.30}$$

In order to provide a formula for $[d+d^*, \sigma(\theta)]$ and $[\rho+\rho^*, \sigma(\theta)]$, recall that, if $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ is an orthonormal system of coordinates, then the operators $\lambda_{dx_j} : \Lambda V^* \to \Lambda V^*$ and $\iota_{dx_j} : \Lambda V^* \to \Lambda V^*$ are defined by Lemma 1.32 and Lemma 1.35, respectively, both being bounded linear operators such that,

$$\|\lambda_{dx_j}\|_{\mathbb{B}(\Lambda V^*)} \leq n \quad \text{and} \quad \lambda^*_{dx_j} = \iota_{dx_j} : \Lambda V^* \to \Lambda V^*.$$

Moreover, recall the linear operators $\frac{\partial}{\partial x_j} : \mathcal{S}(V) \to \mathcal{S}(V)$ and $x_j \cdot : \mathcal{S}(V) \to \mathcal{S}(V)$ given by the diagrams (1.28) and (1.30), respectively. In terms of this notation,

$$d + d^* = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \otimes \left(\lambda_{dx_j} - \iota_{dx_j}\right) : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V), \text{ and}$$

$$\rho + \rho^* = \sum_{j=1}^{n} x_j \otimes \left(\lambda_{dx_j} + \iota_{dx_j}\right) : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V).$$

We check that, for an element $f[g]$ in $C_u^\infty(V)[V_d]$, the operator $[D, \sigma(f[g])] : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ extends uniquely to a bounded linear operator on all of $\Omega_{L^2}(V)$. By the definition of $\sigma : C_u(V) \rtimes V_d \to \mathbb{B}(\Omega_{L^2}(V))$, observe

$$\sigma(f[g]) = \pi(f)u(g) \otimes \mathbb{1} : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V),$$

where $\pi(f) : \mathcal{S}(V) \to \mathcal{S}(V)$ and $u(g) : \mathcal{S}(V) \to \mathcal{S}(V)$ are well-defined and given in Proposition 3.16 and Lemma 3.19, respectively. We therefore have,

$$[d+d^*, \sigma(f[g])] = \sum_{j=1}^{n} \left[\frac{\partial}{\partial x_j}, \pi(f)u(g)\right] \otimes (\lambda_{dx_j} - \iota_{dx_j}) : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V), \quad (3.31)$$

$$[\rho+\rho^*, \sigma(f[g])] = \sum_{j=1}^{n} [x_j, \pi(f)u(g)] \otimes (\lambda_{dx_j} + \iota_{dx_j}) : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V). \quad (3.32)$$

However, for each integer $1 \leq j \leq n$ it is easy to compute that,

$$\left[\frac{\partial}{\partial x_j}, u(g)\right] = 0 = [x_j, \pi(f)] : \mathcal{S}(V) \longrightarrow \mathcal{S}(V),$$

while,

$$\left[\frac{\partial}{\partial x_j}, \pi(f)\right] = \pi\left(\frac{\partial f}{\partial x_j}\right) : \mathcal{S}(V) \longrightarrow \mathcal{S}(V),$$

and

$$[x_j, u(g)] = x_j(g)\mathbb{1} : \mathcal{S}(V) \longrightarrow \mathcal{S}(V),$$

where $x_j(g)$ is the evaluation of $x_j : V \to \mathbb{R}$ at g in V_d, and $\mathbb{1} : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ is the identity. Therefore, from Equation (3.31) we obtain,

$$[d + d^*, \sigma(f[g])] = \sum_{j=1}^{n} \pi\left(\frac{\partial f}{\partial x_j}\right) u(g) \otimes \left(\lambda_{dx_j} - \iota_{dx_j}\right), \tag{3.33}$$

while Equation (3.32) yields,

$$[\rho + \rho^*, \sigma(f[g])] = \sum_{j=1}^{n} x_j(g)\ \pi(f) \otimes (\lambda_{dx_j} + \iota_{dx_j}). \tag{3.34}$$

As constructed in Equation (2.2), let

$$\mathcal{E}_k := \{dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_k}\ \ : I = (i_1, \ldots, i_j) \in \mathcal{I}_k\}$$

be the orthonormal basis for $\Lambda^k V^*$ induced by the orthonormal coordinate system $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$, and $\mathcal{E} := \bigcup_{k=0}^{n}$ the corresponding orthonormal basis for ΛV^*; Where,

$$\mathcal{I}_k := \bigcup_{k=0}^{n} \{(i_1, i_2, \ldots, i_k) \in \{1, 2, \ldots, n\}^k\ \ :\ i_1 < i_2 < \cdots < i_k\},$$

subject to the conventions $\mathcal{I}_0 = \emptyset$ and $dx_\emptyset = 1$ in $\Lambda^0 V^*$. In particular, $\mathcal{E}_0$ is the set containing only the identity in ΛV^* and, if $\mathcal{I} := \bigcup_{j=0}^{n} \mathcal{I}_k$, then $\mathcal{E} = \{dx_I\ \ :\ \ I \in \mathcal{I}\}$. With respect to this notation, given two indices I, J in $\mathcal{I}$, along with any integer $1 \leq j \leq n$, one observes,

$$\left\langle \iota_{dx_j}(dx_I), \iota_{dx_j}(dx_J) \right\rangle_{\Lambda V^*} = \left\langle \lambda_{dx_j}(dx_I), \lambda_{dx_j}(dx_J) \right\rangle_{\Lambda V^*} = \begin{cases} 1, & I = J \\ 0, & I \neq J \end{cases}. \tag{3.35}$$

where $\langle \cdot, \cdot \rangle_{\Lambda V^*}$ is the inner product on the Euclidean space ΛV^* of Example 1.28.

Now, from Lemma 2.2, any Schwartz-class differential form η may be written uniquely as,

$$\eta = \sum_{I \in \mathcal{I}} h_I \otimes dx_I \in \Omega_{\mathcal{S}}(V),$$

with $h_I : V \to \mathbb{C}$ a Schwartz-class function for each index I in $\mathcal{I}$. Hence, for a fixed integer $1 \leq j \leq n$, we have from Equation (2.29) and Equation (3.35) that,

$$\begin{aligned}\left\| \left(x_j(g)\ \pi(f) \otimes \lambda_{dx_j} \right) \eta \right\|_\Omega^2 &= \left\| \sum_{I \in \mathcal{I}} x_j(g) \pi(f) h_I \otimes \lambda_{dx_j} dx_I \right\|_\Omega^2 \\ &= \sum_{I \in \mathcal{I}} \| x_j(g) \pi(f) h_I \|_{L^2(V)}^2 \\ &\leq |x_j(g)|^2\ \|\pi(f)\|_{\mathbb{B}(L^2(V))}^2 \sum_{I \in \mathcal{I}} \|h_I\|_{L^2(V)} \\ &= |x_j(g)|^2\ \|\pi(f)\|_{\mathbb{B}(L^2(V))}^2\ \|\eta\|_\Omega^2,\end{aligned}$$

and similarly,

$$\left\| \left(x_j(g)\ \pi(f) \otimes \iota_{dx_j} \right) \eta \right\|_\Omega^2 \leq |x_j(g)|^2\ \|\pi(f)\|_{\mathbb{B}(L^2(V))}^2\ \|\eta\|_\Omega^2.$$

Since $\|\pi(f)\|_{\mathbb{B}(\Omega_{L^2}(V))} = \|f\|$, these two equations allow us to conclude,

$$\left\| x_j(g) \pi(f) \otimes (\lambda_{dx_j} + \iota_{dx_j}) \eta \right\|_\Omega \leq 2 |x_j(g)|\ \|f\|\ \|\eta\|_\Omega,$$

for each η in $\Omega_{L^2}(V)$. Hence, by Equation (3.34),

$$\| [\rho + \rho^*, \sigma(f[g])]\ \xi \|_\Omega \leq 2 \|f\| \left(\sum_{j=1}^n |x_j(g)| \right) \|\eta\|_\Omega. \tag{3.36}$$

It follows by the density of $\Omega_{\mathcal{S}}(V)$ in the Hilbert space $\Omega_{L^2}(V)$ that the linear operator $[\rho + \rho^*, \sigma(f[g])] : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ extends uniquely to a bounded linear operator on all of $\Omega_{L^2}(V)$.

By analogous reasoning to that used in the previous paragraph one shows easily that, for each integer $1 \leq j \leq n$ and any Schwartz-class differential form η,

$$\left\| \left(\pi\left(\frac{\partial f}{\partial x_j} \right) u(g) \otimes (\lambda_{dx_j} - \iota_{dx_j}) \right) \eta \right\|_\Omega \leq 2 \left\| \frac{\partial f}{\partial x_j} \right\|\ \|\eta\|_\Omega.$$

Thus, from Equation (3.33),

$$\| [d + d^*, \sigma(f[g])] \eta \|_\Omega \leq 2 \left(\sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\| \right) \|\eta\|_\Omega,$$

for each η in $\Omega_{\mathcal{S}}(V)$, and we may conclude that $[d + d^*, \sigma(f[g])] : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ extends uniquely to a bounded linear operator on $\Omega_{L^2}(V)$.

From the two paragraphs above, together with Equation (3.30), we conclude that the commutator,

$$[D, \sigma(f[g])] : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V),$$

extends uniquely to a bounded linear operator on $\Omega_{L^2}(V)$ for each $f[g]$ in $C_u^\infty(V)[V_d]$. A simple linearity argument shows that, since every element of $C_u^\infty(V)[V_d]$ is a finite sum of elements of the type $f[g]$, it must be that, for each θ in $C_u^\infty(V)[V_d]$,

$$[D, \sigma(\theta)] : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V)$$

extends uniquely to a bounded linear operator on all of $\Omega_{L^2}(V)$. □

As a corollary to Lemma 3.27, we arrive at our desired result regarding the domain of $\mathrm{Dom}(\overline{D})$ and the boundedness of the commutator $[\overline{D}, \sigma(\theta)]$ for each θ in $C_u^\infty(V)[V_d]$.

Theorem 3.28. *Let $\overline{D} : \mathrm{Dom}(\overline{D}) \to \Omega_{L^2}(V)$ be the unique self-adjoint extension of the Dirac-Heisenberg operator, as discussed in Lemma 3.24, and let $\sigma : C_u(V) \rtimes V_d \to \mathbb{B}(\Omega_{L^2}(V))$ be the representation of Theorem 3.21. Then for any θ in $C_u^\infty(V)[V_d]$, the subspace $\mathrm{Dom}(\overline{D})$ of $\Omega_{L^2}(V)$ is closed under the bounded linear operator $\sigma(\theta)$, and the commutator,*

$$[\overline{D}, \sigma(\theta)] : \mathrm{Dom}(\overline{D}) \longrightarrow \Omega_{L^2}(V),$$

extends uniquely to a bounded linear operator on all of $\Omega_{L^2}(V)$.

Proof. Let η be in $\mathrm{Dom}(\overline{D})$. From Proposition 3.26, there is a sequence of Schwartz-class differential forms $\{\eta_j\}_{j\in\mathbb{N}} \subseteq \Omega_{\mathcal{S}}(V)$ such that $\{D\eta_j\}_{j\in\mathbb{B}} \subseteq \Omega_{\mathcal{S}}(V)$ is a convergent sequence with respect to the inner product norm $\|\cdot\|_\Omega$ on $\Omega_{L^2}(V)$, and

$$0 = \lim_{j\to 0} \|\eta_j - \eta\|_\Omega.$$

Take any θ in $C_u^\infty(V)[V_d]$, and note that

$$\lim_{j\to\infty} \|\sigma(\theta)\eta_j - \sigma(\theta)\eta\|_\Omega \leq \lim_{j\to\infty} \|\sigma(\theta)\|_{\mathbb{B}(\Omega_{L^2}(V))} \, \|\eta_j - \eta\|_\Omega = 0,$$

so that $\sigma(\theta)\eta_j$ converges to $\sigma(\theta)\eta$ with respect to the norm $\|\cdot\|_\Omega$. To prove $\sigma(\theta)\eta$ is in $\mathrm{Dom}(\overline{D})$, it suffices to show that $\{D\sigma(\theta)\eta_j\}_{j\in\mathbb{N}} \subseteq \Omega_{\mathcal{S}}(V)$ is convergent with respect to the norm $\|\cdot\|_\Omega$. As we showed in Lemma 3.27 that $\|[D, \pi(\theta)]\|_{B(\Omega_{L^2}(V))} < \infty$, it follows that,

$$\begin{aligned}
\|D\sigma(\theta)\eta_j - D\pi(\theta)\eta_m\|_\Omega &= \|D\pi(\theta)(\eta_j - \eta_m)\|_\Omega \\
&\leq \|[D, \sigma(\theta)](\eta_j - \eta_m)\| + \|\sigma(\theta)D(\eta_j - \eta_m)\|_\Omega \\
&\leq \Big(\|[D, \sigma(\theta)]\|_{\mathbb{B}(\Omega_{L^2}(V))} + \|\sigma(\theta)\|_{\mathbb{B}(\Omega_{L^2}(V))}\Big) \|\eta_j - \eta_m\|_\Omega.
\end{aligned}$$

Since the sequence $\{\eta_j\}_{j\in\mathbb{N}} \subseteq \Omega_{\mathcal{S}}(V)$ is Cauchy with respect to the norm $\|\cdot\|_\Omega$, the above inequality implies that the sequence $\{D\pi(\theta)\eta_j\}_{j\in\mathbb{N}}$ is also Cauchy with respect to $\|\cdot\|_\Omega$, and therefore convergent to an element of $\Omega_{L^2}(V)$. Hence, if η is in $\mathrm{Dom}(\overline{D})$, then so too is $\sigma(\theta)\eta$ for any θ in $C_u^\infty(V)[V_d]$.

The fact that, for each θ in $C_u^\infty(V)[V_d]$, the commutator $[\overline{D}, \sigma(\theta)] : \mathrm{Dom}(\overline{D}) \to \Omega_{L^2}(V)$ extends uniquely to a bounded linear operator on $\Omega_{L^2}(V)$ is an immediate corollary to Lemma 3.27. Indeed, observe that,

$$[\overline{D}, \sigma(\theta)]|_{\Omega_{\mathcal{S}}(V)} = [D, \sigma(\theta)] : \Omega_{\mathcal{S}}(V) \longrightarrow \Omega_{\mathcal{S}}(V),$$

where $[\overline{D}, \sigma(\theta)]|_{\Omega_{\mathcal{S}}(V)}$ is the restriction of $[\overline{D}, \sigma(\theta)]$ to $\Omega_{\mathcal{S}}(V) \subseteq \mathrm{Dom}(\overline{D})$. As we showed $[D, \sigma(\theta)] : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ extends uniquely to a bounded linear operator on all of $\Omega_{L^2}(V)$, it follows that so too does $[\overline{D}, \sigma(\theta)]$, and that these extensions to $\Omega_{L^2}(V)$ must agree. □

We conclude this subsection with a culminating theorem.

Theorem 3.29. *With $\Omega_{L^2}(V) = \Omega^+_{L^2}(V) \oplus \Omega^-_{L^2}(V)$ the $\mathbb{Z}/2$-graded Hilbert space of square-integrable differential forms, $\sigma : C_u(V) \rtimes V_d \to \mathbb{B}(\Omega_{L^2}(V))$ the representation of Theorem 3.21, and $\overline{D} : \mathrm{Dom}(\overline{D}) \to \Omega_{L^2}(V)$ the densely defined, self-adjoint, unbounded linear operator of Lemma 3.24, the triple*

$$\left(\ \Omega^+_{L^2}(V) \oplus \Omega^-_{L^2}(V),\ \sigma : C_u(V) \rtimes V_d \to \mathbb{B}(\Omega_{L^2}(V)),\ \overline{D} : \mathrm{Dom}(\overline{D}) \to \Omega_{L^2}(V)\ \right)$$

is a $2n$-dimensional even pre-spectral triple over $C_u^\infty(V)[V_d] \subset C_u(V) \rtimes V_d$, where $n := \dim(V)$; We call this the Heisenberg spectral triple*, or simply the* Heisenberg cycle.

3.2.3 Analysis of the Heisenberg Cycle

We analyse the ζ-functions associated to the Heisenberg spectral triple $(\Omega_{L^2}(V), \sigma, \overline{D})$ outlined in Theorem 3.29: A $2n$-dimensional even pre-spectral triple over $C_u^\infty(V)[V_d] \subseteq C_u(V) \rtimes V_d$. Our method of analysis is via the theory of *heat kernels*, and as such we begin with a lemma concerning the heat kernel of the *scalar harmonic oscillator*. For the remainder of this subsection, fix an orthonormal coordinate system $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$.

Recall that the scalar harmonic oscillator, denoted $\mathscr{H}_s$, is a linear operator acting on the vector space of Schwartz-class functions $\mathcal{S}(V)$ defined by,

$$\mathcal{H}_s = \sum_{j=1}^{n} -\frac{\partial}{\partial x_j^2} + x_j^2 : \mathcal{S}(V) \longrightarrow \mathcal{S}(V).$$

Identifying $\mathcal{S}(V)$ with a dense subspace of the Hilbert space $L^2(V)$ of square-integrable functions, we view $\mathcal{H}_s$ as a densely defined unbounded linear operator on $L^2(V)$. Recall, from Proposition 2.25, that for each n-tuple $\vec{k} = (k_1, \ldots, k_n)$ of non-negative integers, we constructed a function $\psi_{\vec{k}}$ in $\mathcal{S}(V)$ which was an eigenvector of $\mathcal{H}_s$; Where,

$$\mathcal{H}_s \psi_{\vec{k}} = \nu_{\vec{k}} \psi_{\vec{k}},$$

and $\nu_{\vec{k}}$ the (positive) eigenvalue $\nu_k = n + \sum_{j=1}^{n} 2k_j$ of $\mathcal{H}_s$. Further, we argued that the set of Schwartz-class functions,

$$\mathscr{B} := \left\{ \psi_{\vec{k}} \in \mathcal{S}(V) \ : \ \vec{k} \in \mathbb{Z}^n_{\geq 0} \right\} \subseteq \mathcal{S}(V) \subseteq L^2(V),$$

was an orthonormal basis for $L^2(V)$, thus exhibiting a diagonalization for the densely defined unbounded operator $\mathcal{H}_s$.

What we did not state explicitly in Subsection 2.4.3 was that, using this diagonalization $\mathscr{B}$, one may verify $\mathcal{H}_s : \mathcal{S}(V) \to \mathcal{S}(V) \subseteq L^2(V)$ is an essentially self-adjoint unbounded operator on $L^2(V)$: It is clear from our diagonalization and positivity of the eigenvalues $\nu_{\vec{k}}$ that $\mathcal{H}_s : \mathcal{S}(V) \to L^2(V)$ is symmetric with range containing the basis $\mathscr{B}$. Hence, the scalar harmonic oscillaotr $\mathcal{H}_s$ admits a unique, maximal, self-adjoint extension,

$$\overline{\mathcal{H}_s} : \mathrm{Dom}(\overline{\mathcal{H}_s}) \longrightarrow L^2(V).$$

Applying functional calculus for self-adjoint unbounded operators (see [14, Chapter 8]), we define, for each positive real number t, the compact operator,

$$e^{-t\overline{\mathcal{H}_s}} : L^2(V) \longrightarrow L^2(V).$$

Since $\mathscr{B}$ is an orthonormal basis for $L^2(V)$ and,

$$\mathscr{B} \subseteq \mathcal{S}(V) \subseteq \mathrm{Dom}(\overline{\mathcal{H}_s}) \subseteq L^2(V),$$

it follows that this self-adjoint extension $\overline{\mathcal{H}_s}$ is also diagonalized by $\mathscr{B}$, satisfying,

$$\overline{\mathcal{H}_s} \psi_{\vec{k}} = \nu_k \psi_{\vec{k}} = \mathcal{H}_s \psi_{\vec{k}} \quad \text{for each } \psi_{\vec{k}} \in \mathscr{B}.$$

If we take a non-decreasing enumeration $\{\chi_j\}_{j \in \mathbb{N}}$ of the (positive) eigenvalues $\{\nu_{\vec{k}}\}_{\vec{k} \in \mathbb{Z}^n_{\geq 0}}$ for $\overline{\mathcal{H}_s}$, then by an argument analogous to the proof of Proposition 2.28, there are positive real constants c_1 and c_2 such that, for all sufficiently large natural numbers j,

$$c_1 j^{\frac{1}{n}} \leq \chi_j \leq c_2 j^{\frac{1}{n}}.$$

In turn, for each positive real number t, we conclude may that the compact operator

$e^{-t\overline{\mathcal{H}_s}} : L^2(V) \to L^2(V)$ is in the (ideal) of trace class operators. In particular, $\pi : C_u(V) \rtimes \mathbb{B}(L^2(V))$ to be the C^*-algebra representation of Proposition 3.18, and $u : V_d \to \mathbb{U}(L^2(V))$ the unitary group representation of Lemma 3.19, then for any f in $C_u(V)$ and g in V_d, the operator,

$$\pi(f)u(g)e^{-t\overline{\mathcal{H}_s}} : L^2(V) \longrightarrow L^2(V),$$

is also trace class. As the next lemma shows, the trace of this operator has a relatively simple integral formula.

Lemma 3.30. *Let $\pi : C_u(V) \to \mathbb{B}(L^2(V))$ be the C^*-algebra representation of Proposition 3.18, and $u : V_d \to \mathbb{U}(L^2(V))$ be the unitary group representation of Lemma 3.19. For any positive real number t, any f in $C_u(V)$, and any g in V_d,*

$$\begin{aligned} &\operatorname{Tr}\left(\pi(f)u(g)e^{-t\overline{\mathcal{H}_s}}\right) \\ &\quad = (2\pi)^{-\frac{n}{2}} \operatorname{csch}^n(t) e^{-\coth(t)\frac{\|g\|_V^2}{4}} \int_V e^{-\|v\|_V^2} f\left(\coth^{\frac{1}{2}}(t)v + g/2\right)\, d\mu(v) \end{aligned} \tag{3.37}$$

where $d\mu$ is a Haar measure on V, normalized so that, with respect to Lebesgue measure on $\mathbb{R}^n$, any orthonormal coordinate system is a measure preserving map from V to $\mathbb{R}^n$.

Proof. We define the function, $k : \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ for each (t, y_1, y_2) in $\mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}$ by

$$k_t^{(\mathbb{R})}(y_1, y_2) = (2\pi \sinh(2t))^{-\frac{1}{2}} \exp\left(-\tanh(t)\frac{(y_1+y_2)^2}{4} - \coth(t)\frac{(y_1-y_2)^2}{4}\right).$$

The formula for $k : \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ given above is the *(one dimensional) Mehler's formula* [13], and is the heat kernel for scalar harmonic oscillator on $L^2(\mathbb{R})$. In particular, if k_j is a non-negative integer, and $\psi_{k_j} : \mathbb{R} \to \mathbb{R}$ is the eigenfunction of the one dimensional harmonic oscillator on $\mathbb{R}$ defined in Equation (2.45), then for each positive real number t,

$$\int_{\mathbb{R}} k_t^{(\mathbb{R})}(y_1, y_2)\psi_{k_j}(y_2)\, dy_2 = e^{-t(2k_j+1)}\psi_{k_j}(y_1) \quad \text{for all } y_1 \in \mathbb{R}, \tag{3.38}$$

where the integration is with respect Lebesgue integral over $\mathbb{R}$. In particular, since

$$\{\psi_j \in \mathcal{S}(\mathbb{R}) \ : \ j \in \mathbb{Z}_{\geq 0}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$, this heat kernel satisfies,

$$\lim_{t\to 0} \int_{\mathbb{R}} k_t^{(\mathbb{R})}(y_1, y_2)\xi(y_2)\, dy_2 = \xi(y_2), \text{ for all } \xi \in \mathcal{S}(\mathbb{R}).$$

We use this 1-dimensional Mehler's formula to construct the heat kernel for $\overline{\mathcal{H}_s}$, the self-adjoint extension of the n-dimensional scalar harmonic oscillator on V.

Define the function $k : \mathbb{R}_{>0} \times V \times V \to \mathbb{C}$ by,

$$k_t(v_1, v_2) := \prod_{j=1}^{n} k_t^{(\mathbb{R})}(t, x_j(v_1), x_j(v_2)), \text{ for all } (t, v_1, v_2) \in \mathbb{R}_{>0} \times V \times V,$$

where $x_j : V \to \mathbb{R}$ is the j-th coordinate function of the orthonormal coordinate system $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ and $k_t^{(\mathbb{R})}$ is the one-dimensional heat kernel discussed above. In particular,

$$k_t(v_1, v_2) = (2\pi \sinh(2t))^{-\frac{n}{2}} \exp\left(- \tanh(t)\frac{\|v_1 + v_2\|_V^2}{4} - \coth(t)\frac{\|v_1 - v_2\|_V^2}{4} \right),$$

where $\|\cdot\|_V : V \to [0, \infty)$ denotes the Euclidean norm in V. If one takes an n-tuple $\vec{k} = (k_1, \ldots, k_n)$ of non-negative integers, then recall that the corresponding basis element $\psi_{\vec{k}}$ in $\mathscr{B} \subseteq \mathcal{S}(V) \subseteq L^2(V)$ satisfies,

$$\psi_{\vec{k}}(v) = \prod_{j=1}^{n} \psi_{k_j}(x_j(v)), \quad \text{for all} \quad v \in V,$$

where $\psi_{k_j} : \mathbb{R} \to \mathbb{R}$ is defined by Equation (2.45). If we let dr denote the Lebesgue measure on $\mathbb{R}$, then since $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ is measure preserving, it follows that for each w in V,

$$\begin{aligned}
\int_V k_t(w, v)\psi_{\vec{k}}(v)\ d\mu(v) &= \int_V \prod_{j=1}^{n} k_t^{(\mathbb{R})}(x_j(w), x_j(v))\ \psi_{k_j}(x_j(v))\ d\mu(v) \\
&= \prod_{j=1}^{n} \int_{\mathbb{R}} k_t^{(\mathbb{R})}(x_j(w), r)\ \psi_{k_j}(r)\ dr \\
&= \prod_{j=1}^{n} e^{-t(2k_j+1)}\psi_{k_j}(x_j(w)) \\
&= e^{-t\left(n + \sum_{j=1}^{n} 2k_j\right)} \prod_{j=1}^{n} \psi_{k_j}(x_j(w)) \\
&= \left(e^{-t\overline{\mathcal{H}_s}}\psi_{\vec{k}}\right)(w).
\end{aligned}$$

As this holds for each $\psi_{\vec{k}}$ in the orthonormal basis $\mathscr{B} \subseteq \mathcal{S}(V) \subseteq L^2(V)$, we conclude $e^{-t\overline{\mathcal{H}_s}} : L^2(V) \to L^2(V)$ satisfies,

$$\left(e^{-t\overline{\mathcal{H}_s}}\xi\right)(w) = \int_V k_t(w, v)\xi(v)\ d\mu(v),$$

for every ξ in $\mathcal{S}(V)$ and w in V, and so $e^{-t\overline{\mathcal{H}_s}}$ is a compact integral operator on $L^2(V)$ with kernel $k_t : V \times V \to \mathbb{C}$ defined above.

Taking f to be in the C^*-algebra $C_u(V)$, and g a group element in V_d, it follows that $\pi(f)u(g)e^{-t\overline{\mathcal{H}_s}}$ is also a compact integral operator with kernel,

$$V \times V \longrightarrow \mathbb{C} : (v_1, v_2) \longmapsto f(v_1)k_t(v_1 - g, v_2).$$

As the trace of an integral operator is obtained by integrating its kernel along the diagonal we obtain,

$$\begin{aligned}\operatorname{Tr}\left(\pi(f)u(g)e^{-t\overline{\mathcal{H}_s}}\right) &= \int_V f(v)k_t(v-g, v)\ d\mu(v) \\ &= (2\pi \sinh(2t))^{-\frac{n}{2}} \int_V f(v) \exp\left(\tanh(t)\frac{\|2v-g\|_V^2}{4} - \coth(t)\frac{\|g\|_V^2}{4}\right)\ d\mu(v).\end{aligned}$$

Applying the change of variables $v \mapsto \sqrt{\coth(t)}v + g/2$, it follows that,

$$\begin{aligned}&\operatorname{Tr}\left(\pi(f)u(g)e^{-t\overline{\mathcal{H}_s}}\right) \\ &= (2\pi \sinh(2t))^{-\frac{n}{2}} (\coth(t))^{\frac{n}{2}} e^{-\coth(t)\frac{\|g\|_V^2}{4}} \int_V e^{-\|v\|_V^2} f\left(\sqrt{\coth(t)}v + g/2\right)\ d\mu(v),\end{aligned}$$

from which the identity $\frac{\coth(t)}{\sinh(2t)} = \operatorname{csch}^2(t)$ yields Equation (3.37). □

Having discussed the heat kernel for the scalar harmonic oscillator, and the resulting trace formula, we advance to considering the ζ-functions associated to the Heisenberg spectral triple. As the unbounded operator $\overline{D}$ acts on $\Omega_{L^2}(V) := L^2(V) \otimes \Lambda V^*$, we will first recall Corollary 2.26 and the construction of an orthonormal basis of eigenfunctions for $D^2 : \Omega_{\mathcal{S}}(V) \to \Omega_{L^2}(V)$ induced by the coordinate system $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^m$.

For each integer $1 \leq j \leq n$, let dx_j denote the exterior 1-tensor in $\Lambda^1 V^* \subseteq \Lambda V^*$ induced by the coordinate functional $x_j : V \to \mathbb{R}$ and the isometric identification of Euclidean spaces $V^* \cong \Lambda^1 V^*$. Moreover, for each integer $0 \leq m \leq n$, let,

$$\mathcal{I}_m := \begin{cases} \{(i_1, \ldots, i_m) \in \{1, \ldots, n\}^j \ : \ i_1 < i_2 < \cdots < i_m\}, & \text{if } 1 \leq m \leq n, \\ \{\emptyset\}, & \text{if } m = 0 \end{cases},$$

be the set of well-ordered multi-indices of length j, as seen in Definition 1.24, while $\mathcal{I} := \bigsqcup_{m=0}^n \mathcal{I}_m$ is the set of all well-ordered multi-indices of length less than or equal to n. As constructed in Equation (2.2), the set,

$$\mathcal{E}_m := \{dx_I := dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m} \ : \ I = (i_1, i_2, \ldots, i_m) \in \mathcal{I}_j\} \subseteq \Lambda^m V^*,$$

subject to the convention that $dx_\emptyset$ is the identity in ΛV^*, is an orthonormal basis for $\Lambda^m V^*$. Since ΛV^* admits an internal orthogonal direct sum decomposition $\Lambda V^* = \bigoplus_{m=0}^{n} \Lambda^m V^*$, the set,

$$\mathcal{E} := \bigsqcup_{m=0}^{n} \mathcal{E}_k = \{dx_I \ : \ I \in \mathcal{I}\},$$

is an orthonormal basis for the exterior algebra ΛV^*. In Corollary 2.26, we used $\mathcal{E}$, together with the orthonormal basis $\mathscr{B} \subseteq \mathcal{S}(V) \subseteq L^2(V)$, to construct an orthonormal basis for $\Omega_{L^2}(V) := L^2(V) \otimes \Lambda V^*$: For each $\psi_{\vec{k}}$ in $\mathscr{B}$ and dx_I in $\mathcal{E}_j$ with $0 \leq j \leq n$, we defined the elementary tensor,

$$\psi_{(\vec{k},I)} := \psi_{\vec{k}} \otimes dx_I \in \Omega_{\mathcal{S}}(V) := \mathcal{S}(V) \otimes \Lambda V^*,$$

and argued that the set,

$$\mathscr{B}_\Omega^j := \left\{\psi_{(\vec{k},I)} \ : \ (\vec{k}, I) \in \mathbb{Z}_{\geq 0} \times \mathcal{I}_j\right\} \subseteq \Omega_{\mathcal{S}}^j(V) \subseteq \Omega_{L^2}^j(V) \tag{3.39}$$

is an orthonormal basis for the Hilbert space $\Omega_{L^2}^j(V) := L^2(V) \otimes \Lambda^j V^*$ of square-integrable differential j-forms. As we may decompose $\Omega_{L^2}(V)$ into an internal direct sum $\Omega_{L^2}(V) = \bigoplus_{j=0}^{n} \Omega_{L^2}^j(V)$, it is obvious that,

$$\mathscr{B}_\Omega := \bigsqcup_{j=0}^{m} \mathscr{B}_\Omega^j = \left\{\psi_{(\vec{k},I)} \ : \ (\vec{k}, I) \in \mathbb{Z}_{\geq 0}^n \times \mathcal{I}\right\} \subseteq \Omega_{\mathcal{S}}(V) \subseteq \Omega_{L^2}(V)$$

is an orthonormal basis for the Hilbert space $\Omega_{L^2}(V)$. Moreover, we verified that if $D : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ is the Dirac-Heisenberg operator of Definition 2.21, then for each n-tuple $\vec{k} = (k_1, \ldots, k_n)$ of non-negative integers and each $I \in \mathcal{I}_j$, with $0 \leq j \leq n$, the orthonormal basis element $\psi_{(\vec{k},I)} \in \mathscr{B}_\Omega^j \subseteq \mathscr{B}_\Omega$ satisfies,

$$D^2 \psi_{(\vec{k},I)} = \mu_{(\vec{k},I)} \ \psi_{(\vec{k},I)},$$

where,

$$\mu_{(\vec{k},I)} := 2\left(j + \sum_{j=1}^{n} k_j\right) \in \mathbb{R}_{\geq 0}. \tag{3.40}$$

As noticed in Corollary 2.27, the kernel of $D : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ agrees with the kernel of $D^2 : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$; The kernel of both is spanned by the basis vector $\psi_{(\vec{0},\emptyset)}$ in $\mathscr{B}_\Omega^0 \subseteq \mathscr{B}_\Omega$. Of course, it follows that the kernel of $\overline{D} : \mathrm{Dom}(\overline{D}) \to \Omega_{L^2}(V)$, which is defined in Lemma 3.24 as the maximal self-adjoint extension of the Dirac-Heisenberg operator D, certainly contains the span of $\psi_{(\vec{0},\emptyset)}$. However, as the following lemma shows, the kernel of $\overline{D}$ is one dimensional, and so agrees with the kernel of D and D^2.

Lemma 3.31. *The kernel of* $\overline{D} : \mathrm{Dom}(\overline{D}) \to \Omega_{L^2}(V)$ *is one dimensional, being spanned by the basis vector* $\psi_{(\vec{0},\emptyset)}$ *in* $\mathscr{B}^0_\Omega$. *In particular, the unbounded operators* $D, D^2 : \Omega_{\mathcal{S}}(V) \to \Omega_{L^2}(V)$ *and* $\overline{D} : \mathrm{Dom}(\overline{D}) \to \Omega_{L^2}(V)$ *share the same kernel.*

Proof. Let η be in $\mathrm{Dom}(\overline{D})$, and assume $\overline{D}\eta = 0$. Denote the Hilbert space inner product on $\Omega_{L^2}(V)$ by $\langle \cdot, \cdot \rangle_\Omega$, and the corresponding Hilbert space norm by $\|\cdot\|_\Omega : \Omega_{L^2}(V) \to [0, \infty)$. From Lemma 3.24, there exists a sequence of Schwartz-class differential forms η_k such that,

$$\lim_{j\to 0} \|\eta - \eta_j\|_\Omega = 0,$$

and the sequence $\{D\eta_j\}_{j\in\mathbb{N}} \subseteq \Omega_{\mathcal{S}}(V)$ converges to an element in $\Omega_{L^2}(V)$ with respect to the norm $\|\cdot\|_\Omega$. By continuity of the inner product, and since the Dirac-Heisenberg operator is essentially self-adjoint, it follows that for any Schwartz-class differential form ω,

$$0 = \langle \overline{D}\eta, \omega \rangle_\Omega = \lim_{j\to\infty} \langle \eta_j, D\omega \rangle_\Omega = \langle \eta, D\omega \rangle_\Omega.$$

In particular, for any basis element $\psi_{(\vec{k},I)}$ in $\mathscr{B}_\Omega$, it holds that $D\psi_{(\vec{k},I)}$ is a Schwartz-class differential form, and we may set $\omega = D\psi_{(\vec{k},I)}$ to observe,

$$0 = \langle \eta, D\omega \rangle_\Omega = \mu_{(\vec{k},I)} \left\langle \eta, \psi_{(\vec{k})} \right\rangle.$$

Since $\mu_{(\vec{k},I)} = 0$ if and only if $(\vec{k}, I) = (\vec{0}, \emptyset)$, it follows that η is orthogonal to all but one basis element in $\mathscr{B}_\Omega$, namely $\psi_{(\vec{0},\emptyset)}$ the single 0-eigenvector for $D^2 : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$. Hence, the kernel of $\overline{D}$ is spanned by the basis vector $\psi_{(\vec{0},\emptyset)}$ in $\mathscr{B}^0_\Omega$.

In Corollary 2.27, we showed that kernel of $D : \Omega_{\mathcal{S}}(V) \to \Omega_{\mathcal{S}}(V)$ is also spanned by $\psi_{(\vec{0},\emptyset)}$. Hence, D and its extension $\overline{D}$ share the same kernel. □

Since the unbounded operator $\overline{D} : \mathrm{Dom}(\overline{D}) \to \Omega_{L^2}(V)$ is densely defined and self-adjoint, it has a unique polar decomposition. In particular, there is a unique densely defined, self-adjoint, unbounded linear operator,

$$|\overline{D}| : \mathrm{Dom}(\overline{D}) \longrightarrow \Omega_{L^2}(V), \tag{3.41}$$

known as the *modulus of* $\overline{D}$, which satisfies,

$$|D|\psi_{(\vec{k},I)} = \sqrt{\mu_{(\vec{k},I)}}\ \psi_{(\vec{k},I)} \quad \text{for all} \quad \psi_{(\vec{k},I)} \in \mathscr{B}_\Omega, \tag{3.42}$$

where $\mu_{(\vec{k},I)} \geq 0$ is given in Equation (3.40). In particular, $|\overline{D}|$ has the same kernel as $\overline{D}$.

As per the conventions made prior to Definition 1.2, fix the standard branch of λ^{-s}, defined for complex numbers s with $\mathrm{Re}(s) > 0$ and λ a real number. Denoting the rank one orthogonal projection onto the kernel of $\overline{D} : \mathrm{Dom}(\overline{D}) \to \Omega_{L^2}(V)$ by,

$$p : \Omega_{L^2}(V) \longrightarrow \Omega_{L^2}(V) : \omega \longmapsto \langle \psi_{(\vec{0},\emptyset)}, \omega \rangle_\Omega \ \psi_{(\psi_{\vec{0},\emptyset})},$$

it is plain to see that,

$$|\overline{D}| + p : \mathrm{Dom}(\overline{D}) \longrightarrow \Omega_{L^2}(V) : \eta \longmapsto |\overline{D}|\eta + p\eta, \tag{3.43}$$

has no kernel, is self-adjoint, and has spectrum lying in $(1, \infty)$. Hence, for a fixed complex number s with $\mathrm{Re}(s) > 0$, we may use the Borel functional calculus of [14, Chapter 8] to apply the function $\mathbb{R} \to \mathbb{C} : t \mapsto t^{-s}$ to the operator $|D| + p$. The result is a compact linear operator,

$$(|\overline{D}| + p)^{-s} : \Omega_{L^2}(V) \longrightarrow \Omega_{L^2}(V),$$

such that, if $\psi_{\vec{k},I}$ is a basis element in $\mathscr{B}_\Omega \subseteq \Omega_{\mathcal{S}}(V) \subseteq \Omega_{L^2}(V)$, then

$$\big(|\overline{D}| + p\big)^{-s} \psi_{(\vec{k},I)} = \begin{cases} \Big(\mu_{(\vec{k},I)}\Big)^{-\frac{s}{2}} \ \psi_{(\vec{k},I)}, & \text{if } (\vec{k}, I) \neq (\vec{0}, \emptyset) \in \mathbb{Z}^n_{\geq 0} \times \mathcal{I}, \\ \psi_{(\vec{0},\emptyset)}, & \text{otherwise.} \end{cases} \tag{3.44}$$

We similarly use functional calculus to define, for each non-negative real number t, the compact operator,

$$e^{-t\overline{D}^2} : \Omega_{L^2}(V) \longrightarrow \Omega_{L^2}(V),$$

by applying the bounded Borel function $\mathbb{R} \to \mathbb{C} : m \mapsto e^{-tm^2}$ to $|\overline{D}|$, as well as,

$$e^{-tp} : \Omega_{L^2}(V) \longrightarrow \Omega_{L^2}(V),$$

by applying $\mathbb{R} \to \mathbb{C} : m \mapsto e^{-tm^2}$ to the orthogonal projection $p = p^2 : \Omega_{L^2}(V) \to \Omega_{L^2}(V)$. It is clear that $e^{-t\overline{D}^2}$ and e^{-tp} commute as linear operators, as is easily verified on all basis vectors in $\mathscr{B}_\Omega$, and so we introduce a special notation for their product:

$$e^{-t\left(\overline{D}^2 + p\right)} := e^{-tp} e^{-t\overline{D}^2} = e^{-t\overline{D}^2} e^{-tp} : \Omega_{L^2}(V) \longrightarrow \Omega_{L^2}(V). \tag{3.45}$$

Note that,

$$e^{-t(\overline{D}^2 + p)} = e^{-t\overline{D}^2} + (e^{-t} - 1)p : \Omega_{L^2}(V) \longrightarrow \Omega_{L^2}(V), \tag{3.46}$$

which is also seen by verification on basis vectors in $\mathscr{B}$.

Lemma 3.32. *Let V have dimension $n := \dim(V)$, and assume s is a complex number with $\mathrm{Re}(s) > 2n$. If $a : \Omega_{L^2}(V) \to \Omega_{L^2}(V)$ is any bounded linear operator on $\Omega_{L^2}(V)$,* *then the compact operator,*

$$a(|\overline{D}| + p)^{-s} : \Omega_{L^2}(V) \longrightarrow \Omega_{L^2}(V),$$

is trace class, and

$$\Gamma\left(\frac{s}{2}\right) \operatorname{Tr}\left(a\left(|\overline{D}| + p\right)^{-s}\right) = \int_0^\infty t^{\frac{s}{2}-1} \operatorname{Tr}\left(ae^{-t\left(\overline{D}^2+p\right)}\right) \, dt$$

Proof. It is a simple consequence of Proposition 3.26 that the operator $\left(|\overline{D}| + p\right)^{-s}$ is trace-class for each complex number s with $\mathrm{Re}(s) > 2n$. As the trace class operators form an ideal in the C^*-algebra of all bounded operators on $\Omega_{L^2}(V)$, we conclude $a\left(|\overline{D}| + p\right)^{-s}$ is trace-class.

For the identity satisfied by $\operatorname{Tr}\left(a\left(|\overline{D}| + p\right)^{-s}\right)$, recall that the complex gamma function satisfies, by definition,

$$\Gamma\left(\frac{s}{2}\right) := \int_0^\infty t^{\frac{s}{2}-1} e^{-t} \, dt,$$

for each s in $\mathbb{C}$ such that $\mathrm{Re}(s) > 2n$. Performing the substitution $t \mapsto ct$, for c a positive real number, it follows that,

$$\Gamma\left(\frac{s}{2}\right) \, c^{-\frac{s}{2}} = \int_0^\infty t^{\frac{s}{2}-1} e^{-tc} \, dt.$$

In particular, this applies when c is any eigenvalue of $|\overline{D}| + p$, from which we have by properties of functional calculus that,

$$\Gamma\left(\frac{s}{2}\right) \left(|\overline{D}| + p\right)^{-s} = \int_0^\infty t^{\frac{s}{2}-1} e^{-t\left(\overline{D}^2+p\right)} dt,$$

where the right-hand side is a limit of operator valued (definite) integrals which converges, in operator norm, to $\Gamma\left(\frac{s}{2}\right)\left(|\overline{D}| + p\right)^{-s}$. As such, if a is a bounded linear operator acting on $\Omega_{L^2}(V)$,

$$\Gamma\left(\frac{s}{2}\right) \, a\left(|\overline{D}| + p\right)^{-s} = \int_0^\infty t^{\frac{s}{2}-1} a e^{-t\left(\overline{D}^2+p\right)} \, dt,$$

and taking traces yields the desired result.

□

With Lemma 3.32 in hand, we argue that,

$$\Gamma\left(\frac{s}{2}\right) \operatorname{Tr}\left(a\left(|\overline{D}| + p\right)^{-s}\right) - \int_0^1 t^{\frac{s}{2}-1} \operatorname{Tr}\left(ae^{-t\left(\overline{D}^2+p\right)}\right) \, dt, \quad \mathrm{Re}(s) > 2n, \tag{3.47}$$

is analytic on $\mathbb{C}$. The following lemma provides an asymptotic estimate for,

$$\mathbb{R}_{\geq 0} \to \mathbb{C} : t \longmapsto \operatorname{Tr}\left(ae^{-t\left(\overline{D}^2+p\right)}\right),$$

which ensures that the difference in Equation (3.47) is well-defined and analytic for all s in $\mathbb{C}$

Lemma 3.33. *For any natural number d and any bounded operator a on $\Omega_{L^2}(V)$, there exist positive real constants t_0 and C such that,*

$$\operatorname{Tr}\left(ae^{-t\left(\overline{D}^2+p\right)}\right) \leq Ct^{-d} \quad \textit{for all} \quad t > t_0.$$

Proof. Let $\{\mu_k\}_{k=0}^{\infty}$ be a non-decreasing enumeration of the eigenvalues for D^2, and let $\{\omega_k\}_{k\in\mathbb{Z}_{\geq 0}} \subseteq \mathscr{B}$ denote the corresponding orthonormal basis of eigenvectors $\mathscr{B}$; So that, $D^2\omega_k = \mu_k\omega_k$ for each non-negative integer k. Recall, $\mu_0 = 0$ and $\mu_k \geq 1$ for each $k \geq 1$. Moreover, by Proposition 2.28, there exists a natural number N, along with a constant c, such that,

$$\mu_k \geq ck^{\frac{1}{n}}, \quad \text{for all} \quad k \geq n.$$

With $\|a\|_{\mathbb{B}}$ denote the operator norm of the bounded operator a, and $\langle\cdot,\cdot\rangle_\Omega$ the inner product on $\Omega_{L^2}(V)$, it follows by the orthonormality of the vectors $\{\omega_k\}_{k\in\mathbb{Z}_{\geq 0}}$ that, for any positive number t,

$$\begin{aligned}
\operatorname{Tr}\left(ae^{-t\left(\overline{D}^2+p\right)}\right) &= \sum_{k=0}^{\infty}\left\langle \omega_k, ae^{-t\left(\overline{D}^2+p\right)}\omega_k\right\rangle_\Omega \\
&\leq \|a\|_{\mathbb{B}}\left(e^{-t} + \sum_{k=1}^{\infty} e^{-t\mu_k}\right) \\
&\leq \|a\|_{\mathbb{B}}\left(Ne^{-t} + \sum_{j=N}^{\infty} e^{-tck^{\frac{1}{n}}}\right) \\
&\leq \|a\|_{\mathbb{B}}\left(Ne^{-t} + \int_N^{\infty} e^{-tck^{\frac{1}{n}}}\, dk\right),
\end{aligned}$$

where the integration in the last line is with respect to the Lebesgue measure on $\mathbb{R}$.

From the above string of inequalities, and since $\lim_{t\to 0} t^d e^{-t} = 0$ for each natural number d, it suffices to exhibit positive constants t_0 and C such that,

$$\int_N^{\infty} e^{-tck^{\frac{1}{n}}}\, dk < Ct^{-d} \quad \text{for all} \quad t > t_0.$$

To this end, note that a simple substitution yields,

$$\int_N^\infty e^{-tck^{\frac{1}{n}}}\ dk = n(ct)^{-n}\Gamma\left(n; ctN^{\frac{1}{n}}\right), \tag{3.48}$$

for all $t > 0$, where,

$$\Gamma\left(n; ctN^{\frac{1}{n}}\right) := \int_{ctN^{\frac{1}{n}}}^\infty k^{n-1}e^{-k}\ dk,$$

is the upper incomplete Γ-function.

For d a natural number with $d \leq n$, we have from Equation (3.48) and the non-increasing nature of the map $t \mapsto \Gamma\left(v, ctN^{\frac{1}{n}}\right)$ that,

$$\int_N^\infty e^{-tck^{\frac{1}{n}}}\ dk \leq \left[nc^{-n}\Gamma\left(n; cN^{\frac{1}{n}}\right)\right]\ t^{-d} \quad \text{for all} \quad t \geq 1,$$

where

$$0 < nc^{-n}\Gamma\left(n; cN^{\frac{1}{n}}\right) < \infty.$$

Hence the result holds when $d \leq n$.

In the case where $d > n$, take $t_0 \geq 1$ and $C > 0$ such that $e^{-t} \leq Ct^{-d}$ for all $t \geq t_0$. Then, for all

$$t > t_0' := \max\left\{1, \frac{t_0}{cN^{\frac{1}{n}}}\right\},$$

we have,

$$\Gamma\left(n; ctN^{\frac{1}{n}}\right) \leq C\frac{\left(ctN^{\frac{1}{n}}\right)^{n-d}}{d-n} = C\frac{c^nN}{d-n}\ t^{n-d},$$

and hence Equation (3.48) yields,

$$\int_N^\infty e^{-tck^{\frac{1}{n}}}\ dk = n(ct)^{-n}\Gamma\left(n; ctN^{\frac{1}{n}}\right) \leq C\frac{nN}{d-n}\ t^{-d},$$

whenever $t > t_0'$.

From the sufficiency observations made earlier in the proof, this proves that the claim made in the lemma is true. □

Corollary 3.34. *For any bounded operator a on $\Omega_{L^2}(V)$, the analytic function*

$$s \longmapsto \int_1^\infty t^{\frac{s}{2}-1}\,\mathrm{Tr}\left(ae^{-t\left(D^2+p\right)}\right)\ dt, \quad \textit{for} \quad \mathrm{Re}(s) > 2n,$$

extends to an entire function on $\mathbb{C}$, with the extension given allowing s to be any complex number in the formula above; In particular, the derivative at s in $\mathbb{C}$ of this *entire extension is given by,*

$$s \longmapsto \int_1^\infty \frac{\ln(t)}{2} t^{\frac{s}{2}-1} \operatorname{Tr}\left(a e^{-t(\overline{D}^2+p)}\right)\, dt.$$

From Corollary 3.34, it follows that for each bounded operator a on $\Omega_{L^2}(V)$ and each complex number s with $\operatorname{Re}(s) > 2n$,

$$\Gamma\left(\frac{s}{2}\right) \operatorname{Tr}\left(a(\overline{D}^2+p)^{-s}\right) \;\sim\; \int_0^1 t^{\frac{s}{2}-1} \operatorname{Tr}\left(a e^{-t\left(\overline{D}^2+p\right)}\right)\, dt \tag{3.49}$$

where $\sim$ means up to addition by an entire function in the complex variable s. We further dissect the right-hand side of this up-to-entire equivalence by way of Equation (3.46), and requiring the linear operator a to be induced by an element $f[g]$ in $C_u^\infty(V)[V_d] \subseteq C_u(V) \rtimes V_d$ acting via the $*$-representation of Theorem 3.21. Recall that,

$$\sigma(f[g]) = \pi(f)u(g) \otimes \mathbb{1} : \Omega_{L^2}(V) \longrightarrow \Omega_{L^2}(V),$$

where $\pi : C_u(V) \to \mathbb{B}(L^2(V))$ is the $*$-representation by multiplication operators of Proposition 3.16 and $u : V_d \to \mathbb{U}(L^2(V))$ is the unitary group representation of Lemma 3.19.

Proposition 3.35. *Let $\sigma : C_u(V) \rtimes V_d \to \mathbb{B}(\Omega_{L^2}(V))$ be the $*$-representation of Theorem 3.21, and $d\mu$ a scaled Haar measure on V such that any orthonormal coordinate system $x = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ is measure preserving with respect to the Lebesgue measure on $\mathbb{R}^n$. For each function f in $C_u^\infty(V)$, group element g in V_d, and complex number s with* $\operatorname{Re}(s) > 2n$*,*

$$\int_0^1 t^{\frac{s}{2}-1} \operatorname{Tr}\left(a e^{-t\left(\overline{D}^2+p\right)}\right)\, dt = \eta(s) + \int_0^1 t^{\frac{s}{2}-1} \operatorname{Tr}\left(a e^{-t\overline{D}^2}\right)\, dt,$$

where $\eta : \mathbb{C} \to \mathbb{C}$ is a meromorphic function with simple pole at $s = 0$ and residue there,

$$\operatorname*{Res}_{s=0}(\eta) = -c(n) e^{-\frac{\|g\|_V^2}{4}} \int_V e^{-\|v-g/2\|_V^2} f(v)\, d\mu(v),$$

where $c(n) := 2^{1-n}\pi^{-n}$ and $\|\cdot\|_V : V \to [0,\infty)$ is the Euclidean norm on V.

Proof. For s a complex number with $\operatorname{Re}(s) > 2n$ and a an arbitrary bounded linear operator, Equation (3.46) and linearity of the trace imply,

$$\begin{aligned}
&\int_0^1 t^{\frac{s}{2}-1} \operatorname{Tr}\left(a e^{-t\left(\overline{D}^2+p\right)}\right)\, dt \\
&\quad = \int_0^1 t^{\frac{s}{2}-1} \operatorname{Tr}\left(a e^{-t\overline{D}^2}\right) + t^{-\frac{s}{2}-1}(e^{-t}-1) \operatorname{Tr}(ap)\, dt.
\end{aligned} \tag{3.50}$$

Moreover, since $\mathrm{Re}(s) > 2n$, it is easily verified that the integral,

$$\int_0^1 t^{\frac{s}{2}-1}\,\mathrm{Tr}\left(ae^{-t\overline{D}^2}\right)\,dt$$

is convergent.

Denote the lower incomplete Γ-function by,

$$\gamma(\cdot;1) : \mathbb{C} \longrightarrow \mathbb{C} : s \longmapsto \int_0^1 t^s e^{-t}\,dt,$$

and recall that this function admits an analytic extension to all of $\mathbb{C}$. Assuming $\mathrm{Re}(s) > 2n$, a direct computation shows,

$$\int_0^1 t^{\frac{s}{2}-1}\,dt = 2\frac{\overline{s}}{|s|^2} = \frac{2}{s}.$$

Thus, for s in $\mathbb{C}$ with $\mathrm{Re}(s) > 2n$, the function,

$$\begin{aligned}\eta(s) &:= \int_0^1 t^{\frac{s}{2}-1}(e^{-t}-1)\,\mathrm{Tr}\,(ap)\,\,dt \\ &= \mathrm{Tr}(ap)\int_0^1 t^{\frac{s}{2}-1}e^{-t} - \mathrm{Tr}\,(ap)\int_0^1 t^{\frac{s}{2}-1}\,dt \\ &= \mathrm{Tr}(ap)\gamma\left(\frac{s}{2};1\right) - \frac{2\,\mathrm{Tr}(ap)}{s},\end{aligned}$$

extends meromorphically to $\mathbb{C}$ with a simple pole at $s = 0$, and the residue there is given by,

$$\operatorname*{Res}_{s=0}(\eta) = -2\,\mathrm{Tr}(ap). \tag{3.51}$$

It remains to compute $\mathrm{Tr}(ap)$, with $p : \Omega_{L^2}(V) \to \Omega_{L^2}(V)$ the orthogonal projection onto the kernel of $\overline{D}$ and $a = \sigma(f[g])$ for $f[g]$ in $C_u^\infty[V_d]$. From Lemma 3.31, the range of p is spanned by the normalized Schwartz-class differential 0-form $\psi_{(\vec{0},\emptyset)}$ which is contained in the set $\mathscr{B}_\Omega^0$ given in Equation (3.39). In particular, $\sigma(f[g])$ acts on $\psi_{(\vec{0},\emptyset)} := \psi_{\vec{0}} \otimes 1$ by,

$$\sigma(f[g])(\psi_{(\vec{0},\emptyset)}) = (\pi(f)u(g)\psi_{\vec{k}}) \otimes 1 \in \Omega^0_{\mathcal{S}}(V),$$

where the Schwartz-class functions $\psi_{\vec{0}}$ and $\pi(f)u(g)\psi_{\vec{0}}$ are given by,

$$\begin{aligned}\psi_{\vec{0}} : V \longrightarrow \mathbb{R} : v \longmapsto (2\pi)^{-\frac{n}{2}}e^{-\frac{\|v\|_V^2}{2}} \quad &\text{and,} \\ \pi(f)u(g)\psi_{\vec{0}} : V \longrightarrow \mathbb{C} : v \longmapsto (2\pi)^{-\frac{n}{2}}f(v)e^{-\frac{\|v-g\|_V^2}{2}}&,\end{aligned}$$

with $\|v\|_V := \sqrt{\langle v, v\rangle_V}$ is the Euclidean norm on V. Hence, if $\langle \cdot, \cdot \rangle_\Omega$ denotes the Hilbert space inner product on $\Omega_{L^2}(V)$ then,

$$\begin{aligned}\mathrm{Tr}\left(\sigma(f[g])p\right) :=& \langle \psi_{\vec{0}} \otimes 1, (\pi(f)u(g)\psi_{\vec{0}}) \otimes 1\rangle_\Omega \\ =& \langle \psi_{\vec{0}}, \pi(f)u(g)\psi_{\vec{0}}\rangle_{L^2(V)} \\ =&(2\pi)^{-n} \int_V e^{-\frac{\|v-g\|_V^2}{2} - \frac{\|v\|_V^2}{2}} f(v)\ d\mu(v) \\ =&(2\pi)^{-n} e^{-\frac{\|g\|_V^2}{4}} \int_V e^{-\|v-g/2\|_V^2} f(v)\ d\mu(v),\end{aligned}$$

with the second equality following from Equation (2.29) and the last from the parallelogram law. Our desired residue result is then obtained by combining,

$$\mathrm{Tr}\left(\sigma(f[g])p\right) = (2\pi)^{-n} e^{-\frac{\|g\|_V^2}{4}} \int_V e^{-\|v-g/2\|_V^2} f(v)\ d\mu(v),$$

with Equation (3.51). □

Proposition 3.35 and the equality (up to an entire function) given in Equation (3.49) reduces the problem of extending the function,

$$\Gamma\left(\frac{s}{2}\right) \mathrm{Tr}\left(\sigma(f[g])(\overline{D}^2 + p)\right) \qquad \mathrm{Re}(s) > 2n,$$

meromorphically to finding a meromorphic extension of,

$$\int_0^1 t^{\frac{s}{2}-1} \left(\sigma(f[g])e^{-t\overline{D}^2}\right) \qquad \mathrm{Re}(s) > 2n.$$

This is where Lemma 3.30 comes into play, as we are able to write $e^{-t\overline{D}^2}$ as a direct sum of a number of copies of the scalar harmonic oscillator up to addition by a scalar multiple of the identity.

Lemma 3.36. *Let $\sigma : C_u(V) \rtimes V_d \to \Omega_{L^2}(V)$ be the representation of Theorem 3.21. Fix a function f in $C_u(V)$, and a group element g in V_d, and recall,*

$$\sigma(f[g]) = \pi(f)u(g) \otimes \mathbb{1} : \Omega_{L^2}(V) \longrightarrow \Omega_{L^2}(V),$$

where $\pi : C_u(V) \to \mathbb{B}(L^2(V))$ is the C^*-algebra representation of Proposition 3.16 and $u : V_d \to \mathbb{U}(L^2(V))$ is the unitary group representation of Lemma 3.19. Then, for any *complex number s with* $\mathrm{Re}(s) > 2n$*,*

$$\begin{aligned}&\mathrm{Tr}\left(\sigma(f[g])e^{-t\overline{D}^2}\right)\\&=\left(\frac{2}{\pi}\right)^{\frac{n}{2}}\coth^n(t)e^{-\coth(t)\frac{\|g\|_V^2}{4}}\int_V e^{-\|v\|_V^2}f\left(\coth^{\frac{1}{2}}(t)v+g/2\right)\,d\mu(v).\end{aligned}$$

Proof. For each integer $0 \leq m \leq n$ we have,

$$e^{-t\overline{D}^2}:\Omega^m_{L^2}(V)\longrightarrow\Omega^m_{L^2}(V)\quad\text{and}\quad\sigma(f[g]):\Omega^m_{L^2}(V)\longrightarrow\Omega^m_{L^2}(V),$$

and thus $\sigma(f[g])e^{-t\overline{D}^2}$ is also an endomorphism of the square-integrable m-forms. Since $\Omega_{L^2}(V)$ decomposes as a direct sum of orthogonal subspaces $\Omega_{L^2}(V) = \bigoplus_{m=0}^n \Omega^m_{L^2}(V)$, it follows that,

$$\mathrm{Tr}\left(\sigma(f[g])e^{-t\overline{D}^2}\right)=\sum_{m=0}^n\mathrm{Tr}\left(\sigma(f[g])e^{-t\overline{D}^2}\Big|_{\Omega^m_{L^2}(V)}\right). \tag{3.52}$$

Now, for each integer $0 \leq m \leq n$, each n-tuple $\vec{k} = (k_1, \ldots, k_n)$ of non-negative integers, and any basis element $\psi_{(\vec{k},I)}$ in $\mathscr{B}^m_\otimes \subseteq \Omega_{\mathcal{S}}(V)^m \subseteq \Omega^m_{L^2}(V)$, we have by basic functional calculus properties that,

$$e^{-t\overline{D}^2}\Big|_{\Omega^m_{L^2}(V)}\psi_{(\vec{k},I)}=e^{-t\mu_{(\vec{k},I)}}\psi_{(\vec{k},I)}=e^{-t(2m-n)}e^{-t\left(n+\sum_{j=1}^n k_j\right)}\psi_{(\vec{k},I)},$$

where,

$$\mu_{(\vec{k},I)}:=2\left(m+\sum_{j=1}^n k_j\right)=(2m-n)+\left(n+\sum_{j=1}^n 2k_j+1\right).$$

Moreover, as

$$\sigma(f[g])=\pi(f)u(g)\otimes\mathbb{1}:\Omega^m_{L^2}(V)\to\Omega^m_{L^2}(V),$$

it follows from Equation (2.29) that, for each n-tuple $\vec{k} = (k_1, \ldots, k_n)$ and each $\psi_{(\vec{k},I)}$ in the orthonormal basis $\mathscr{B}^m_\otimes$ of $\Omega^m_{L^2}(V)$,

$$\begin{aligned}&\left\langle\psi_{(\vec{k},I)},\sigma(f[h])e^{-t\overline{D}^2}\Big|_{\Omega_{L^2}(V)}\psi_{(\vec{k},I)}\right\rangle_\Omega\\&=e^{-t(2m-n)}e^{-t\left(n+\sum_{j=1}^n k_j\right)}\left\langle\psi_{\vec{k}}\otimes dx_I,\pi(f)u(g)\psi_{\vec{k}}\otimes dx_I\right\rangle_\Omega\\&=e^{-t(2m-n)}e^{-t\left(n+\sum_{j=1}^n k_j\right)}\left\langle\psi_{\vec{k}},\pi(f)u(g)\psi_{\vec{k}}\right\rangle_{L^2(V)}\\&=e^{-t(2m-n)}\left\langle\psi_{\vec{k}},\pi(f)u(g)e^{-t\overline{\mathcal{H}_s}}\psi_{\vec{k}}\right\rangle_{L^2(V)},\end{aligned}$$

As there are exactly $\binom{n}{m}$ multi-indices in the set $\mathcal{I}_m$, and since,

$$\mathscr{B}^m_\Omega = \left\{\psi_{(\vec{k},I)} = \psi_{\vec{k}} \otimes dx_I \;:\; (\vec{k}, I) \in \mathbb{Z}^n_{\geq 0} \times \mathcal{I}_m\right\} \subseteq \Omega^m_{\mathcal{S}}(V) \subseteq \Omega^m_{L^2}(V),$$

is an orthonormal basis for $\Omega^m_{L^2}(V)$ while $\mathscr{B} := \{\psi_{\vec{k}} \;:\; \vec{k} \in \mathbb{Z}^n_{\geq 0}\}$ is an orthonormal basis for $L^2(V)$, one easily computes,

$$\begin{aligned}
\operatorname{Tr}\left(\sigma(f[g])e^{-t\overline{D}^2}\Big|_{\Omega^m_{L^2}(V)}\right) &= \sum_{(\vec{k},I)\in\mathbb{Z}^n_{\geq 0}\times\mathcal{I}_m} \left\langle \psi_{(\vec{k},I)}, \sigma(f[g])e^{-t\overline{D}^2}\right\rangle_\Omega \\
&= e^{-t(2m-n)} \sum_{(\vec{k},I)\in\mathbb{Z}^n_{\geq 0}\times\mathcal{I}_m} \left\langle \psi_{\vec{k}}, \pi(f)u(g)e^{-t\overline{\mathcal{H}_s}}\psi_{\vec{k}}\right\rangle_{L^2(V)} \\
&= e^{-t(2m-n)}\binom{n}{m} \sum_{\vec{k}\in\mathbb{Z}^n_{\geq 0}} \left\langle \psi_{\vec{k}}, \pi(f)u(g)e^{-t\overline{\mathcal{H}_s}}\psi_{\vec{k}}\right\rangle_{L^2(V)} \\
&= e^{-(2m-n)}\binom{n}{m} \operatorname{Tr}\left(\pi(f)u(g)e^{-t\overline{\mathcal{H}_s}}\right),
\end{aligned}$$

where $\overline{\mathcal{H}_s} : \operatorname{Dom}(\overline{\mathcal{H}_s}) \to L^2(V)$ is the self-adjoint extension of the scalar harmonic oscillator $\mathcal{H}_s : \mathcal{S}(V) \to \mathcal{S}(V)$ discussed previously in this subsection, and a formula for $\operatorname{Tr}\left(\pi(f)u(g)e^{-t\overline{\mathcal{H}_s}}\right)$ is given in Lemma 3.30.

Combining the above with Equation (3.52), we obtain,

$$\begin{aligned}
\operatorname{Tr}\left(\sigma(f[g])e^{-t\overline{D}^2}\right) &= \sum_{m=0}^{n}\binom{n}{m} e^{-t(2m-n)} \operatorname{Tr}\left(\pi(f)u(g)e^{-t\overline{\mathcal{H}_s}}\right) \\
&= (e^t + e^{-t})^n \operatorname{Tr}\left(\pi(f)u(g)e^{-t\overline{\mathcal{H}_s}}\right), \\
&= 2^n \cosh^n(t) \operatorname{Tr}\left(\pi(f)u(g)e^{-t\overline{\mathcal{H}_s}}\right).
\end{aligned}$$

Lemma 3.30 then yields the desired result. □

Finally, we obtain a criteria for when one is able to meromorphically extend

$$s \longmapsto \Gamma\left(\frac{s}{2}\right) \operatorname{Tr}\left(\sigma(f[g])(\overline{D}+p)^{-s}\right) \qquad \operatorname{Re}(s) > 2n$$

to all of $\mathbb{C}$.

Theorem 3.37. *If $f[g]$ is in $C^\infty_u(V)[V_d]$ and $\sigma : C_u(V) \rtimes V_d \to \mathbb{B}(\Omega_{L^2}(V))$ is the representation of Theorem 3.21 then the analytic function,*

$$s \longmapsto \Gamma\left(\frac{s}{2}\right) \operatorname{Tr}\left(\sigma(f[g])(\overline{D}+p)^{-s}\right), \qquad \operatorname{Re}(s) > 2n,$$

extends meromorphically to $\mathbb{C}$ *if and only if the analytic function,*

$$s \longmapsto \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \int_0^1 t^{\frac{s}{2}-1} \coth^n(t)\phi_{f,g}(t)\ d\mu(t), \quad \operatorname{Re}(s) > 2n,$$

extends meromorphically to $\mathbb{C}$ *where,*

$$\phi_{f,g}(t) := e^{-\coth(t)\frac{\|g\|_V^2}{4}} \int_V e^{-\|v\|_V^2} f\left(\coth^{\frac{1}{2}}(t)v + g/2\right)\ d\mu(v).$$

3.2.4 Concluding Remarks on the Heisenberg Cycle

As pointed out in the introduction, our original goal in extending the Heisenberg cycle of Proposition 1.6 was with a focus on KK-duality. In particular, we are motived by the following extension of 1.7 when applied to 'higher dimensional' rotation algebras of the form $C(\mathbb{T}^n) \rtimes_g \mathbb{Z}^n$ and $C(\mathbb{T}^n) \rtimes_{g^{-1}} \mathbb{Z}^n$ with g an automorphism of V and where the actions on $C(\mathbb{T}^n)$ are those induced by the translation actions of the lattices $g\mathbb{Z}^n$ and $g^{-1}\mathbb{Z}^n$ on $\mathbb{T}^n$. We continue to let $(V, \langle\cdot,\cdot\rangle_V)$ be a Euclidean space of dimension n.

Theorem 3.38. *Let* M *be a compact manifold,* $\alpha : V \times M \to M$ *a smooth action of* V *on* M*, and* Γ *a discretely topologized subgroup of* V*. For a fixed point* m *in* M *and any continuous function* f *on* M*, the map,*

$$f_m : V \longrightarrow \mathbb{C} : v \longmapsto f(\alpha_v(m))$$

is contained in the C^**-algebra* $C_u(V)$ *of Example 3.5. The induced* $*$*-homomorphism,*

$$C(M) \longrightarrow C_u(V) : f \longmapsto f_m,$$

together with the inclusion $\Gamma \hookrightarrow V$*, form a covariant pair and corresponding* $*$*-homomorphism,*

$$\operatorname{Ind}_m : C(M) \rtimes_\alpha \Gamma \longrightarrow C_u(V) \rtimes V_d,$$

such that $\operatorname{Ind}_m(C^\infty(M)[\Gamma]) \subseteq C^\infty(V)[V_d]$.

In particular, if $(\Omega_{L^2}(V), \sigma, D)$ *is the Heisenberg cycle of Theorem 3.29 then,*

$$(\Omega_{L^2}(V), \sigma \circ \operatorname{Ind}_m, D)$$

is a $2n$*-dimensional even pre-spectral triple over* $C^\infty(M)[\Gamma] \subseteq C(M) \rtimes_\alpha \Gamma$.

Proof. Boundedness and uniform continuity of $f_m : V \to \mathbb{C}$, for each continuous function f in $C(M)$, follows from the compactness of the manifold M and regularity of the action α, while smoothness of $f_m : V \to \mathbb{C}$ for each f in $C^\infty(M)$ is plain to see by the smoothness of the action α.

The covariance condition follow from the following simple computation: Let f be in $C(M)$ and g be in $\Gamma \subseteq V_d$. Then, for each $v \in V$,

$$[g(f)]_m(v) = [g(f)](\alpha_v(m)) = f(\alpha_{-g} \circ \alpha_v(m)) = f_m(v-g) = [g(f_m)](v).$$

If $\mathrm{Ind}_m : C(M) \rtimes_\alpha \Lambda \to C_u(V) \rtimes V_d$ is the induced $*$-homomorphism, then clearly $\mathrm{Ind}_m\left(C^\infty(M)[\Gamma]\right) \subseteq C_u^\infty(V)[V_d]$ $\square$

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