A characterization of faithful representations of the Toeplitz algebra of the $ax + b$-semigroup of a number ring

by

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ABSTRACT

In their paper [2] Cuntz, Deninger, and Laca introduced a C^*-algebra $\mathfrak{T}[R]$ associated to a number ring R and showed that it was functorial for injective ring homomorphisms and had an interesting KMS-state structure, which they computed directly. Although isomorphic to the Toeplitz algebra of the $ax+b$-semigroup $R \rtimes R^\times$ of R, their C^*-algebra $\mathfrak{T}[R]$ was defined in terms of relations on a generating set of isometries and projections. They showed that a homomorphism $\varphi : \mathfrak{T}[R] \to A$ is injective if and only if φ is injective on a certain commutative $*$-subalgebra of $\mathfrak{T}[R]$. In this thesis we give a direct proof of this result, and go on to show that there is a countable collection of projections which detects injectivity, which allows us to simplify their characterization of faithful representations of $\mathfrak{T}[R]$.

Contents

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Chapter 1

Introduction

For nearly 20 years mathematicians have been studying various C^*-algebras arising from algebraic number theory. The idea is that the KMS states of a conveniently chosen C^*-dynamical system (A, σ) (a pair consisting of a C^*-algebra A together with a one-parameter group of automorphisms σ) can give us information about a number field K. As an example, a C^*-dynamical system which preserves enough of the structure of a number field could shed light on Hilbert's 12th problem (for a precise formulation of the problem see [3]).

Obviously, in order for (A, σ) to have any hope of recovering the structure of K, the C^*-algebra itself should be, in some way, related to K. In general a C^*-algebra can be generated by the multiplicative structure of a group-like object via a representation as operators on a Hilbert space. There are many groups and semigroups associated with K which we could represent; its ring of integers R is a particularly good source. Basic examples are the additive group $(R, +)$, the multiplicative semigroup $(R^\times, \cdot)$ $(R^\times = R \setminus \{0\})$, the unit group $(R^*, \cdot)$, the semigroup consisting of all non-zero ideals of R, and the ideal class group. Perhaps the simplest non-abelian semigroup we can construct from R is the semi-direct product $R \rtimes R^\times$ of $(R, +)$ and $(R^\times, \cdot)$. The image of the left-regular representation $\mathfrak{T}_{R\rtimes R^\times}$ of $R \rtimes R^\times$ on $\ell^2(R \rtimes R^\times)$ contains the image of a unitary representation of $(R, +)$, the image of an isometric representation of $(R^\times, \cdot)$, and the image of a projective representation of the semigroup of non-zero ideals of R with the operation of intersection.

In their paper [2] Cuntz, Deninger, and Laca showed that $\mathfrak{T}_{R\rtimes R^\times}$ is functorial for injective ring homomorphisms and has an interesting KMS-structure, which they computed directly. Their preferred presentation of $\mathfrak{T}_{R\rtimes R^\times}$ was as a universal C^*-algebra $\mathfrak{T}[R]$ defined in terms of relations on a generating set of isometries. They showed that a homomorphism $\varphi : \mathfrak{T}[R] \to A$ is injective if and only if φ is injective on a certain $*$-subalgebra of $\mathfrak{T}[R]$. In this thesis we use elementary methods to prove this, and go on to give a simpler characterization of injectivity.

1.1 C^*-Algebras Generated by Isometries

A bounded operator V on a Hilbert space (or an element of a C^*-algebra) is called an *isometry* if $V^*V = 1$, and *unitary* if, in addition, $VV^* = 1$. The prototype example of a non-unitary isometry is the *unilateral shift* S. If $\xi_1, \xi_2, \ldots$ is the standard orthonormal basis for $\ell^2(\mathbb{N})$, then the map $\xi_n \mapsto \xi_{n+1}$ extends to all of $\ell^2(\mathbb{N})$ and gives a bounded linear operator S. The Wold decomposition states that every isometry is a direct sum of a unitary and copies of the unilateral shift.

The C^*-algebra $C^*(V)$ generated by a unitary V is isomorphic to the space of continuous complex valued functions on the spectrum of V. Thus we would rightly expect two different unitaries to generate two different C^*-algebras. Surprisingly, the situation becomes much simpler when V is a non-unitary isometry. In 1967 Coburn [1] proved that all C^*-algebras generated by a single non-unitary isometry are isomorphic. In fact any C^*-algebra $C^*(V)$ generated by a non-unitary isometry satisfies the following universal property. If V is an isometry in a C^*-algebra A, then the map $S \mapsto V$ extends to a homomorphism from $C^*(S)$ to A. Moreover this homomorphism is injective if and only if $1 - VV^* \neq 0$.

Isometries are the natural operators to use when we want to represent semigroups. Let Γ be a semigroup with identity e. An *isometric representation* of Γ on a Hilbert space $\mathcal{H}$ is a map $V : \Gamma \to \mathcal{B}(\mathcal{H})$ such that V_x is an isometry, $V_e = 1$, and $V_x V_y = V_{xy}$ for $x, y \in \Gamma$ (see [4]). As an example consider the collection of isometries $T_{(m,n)} : \ell^2(\mathbb{N}^2) \to \ell^2(\mathbb{N}^2)$, indexed by $\mathbb{N}^2$, where $T_{(m,n)}$ extends the map $\xi_{(x,y)} \mapsto \xi_{(m+x,n+y)}$ by linearity and continuity. One can easily see that this is the image of an isometric representation of $\mathbb{N}^2$. We call the map $(m, n) \mapsto T_{(m,n)}$ the *left-regular representation* of $\mathbb{N}^2$ on $\ell^2(\mathbb{N}^2)$, and the C^*-algebra $\mathfrak{T}_{\mathbb{N}^2}$ generated by these isometies the *left-regular C^*-algebra* of $\mathbb{N}^2$ on $\ell^2(\mathbb{N}^2)$, or its *Toeplitz algebra*.

The left-regular representation of a semigroup is often the prototype example of an isometric representation. Since a C^*-algebra is an object not tied to any particular Hilbert space we might try to find another Hilbert space $\mathcal{H}$ on which $\mathfrak{T}_{\mathbb{N}^2}$ could act, i.e. find a homomorphism from $\mathfrak{T}_{\mathbb{N}^2}$ to $\mathcal{B}(\mathcal{H})$. Doing this directly can be tricky; the norm gets in the way. One approach is to find a universal C^*-algebra that is isomorphic to $\mathfrak{T}_{\mathbb{N}^2}$. We will define it by imposing relations on a generating set. It is important to note that this does not always work. For example, there is a universal C^*-algebra generated by an isometry, but not one generated by a normal element. The reason that normal does not work as a universal property follows from the fact that all $*$-homomorphisms must be norm-decreasing, but there is no normal element of largest norm. A universal C^*-algebra will exist if the relations imply that the generators are bounded in norm. In particular when the generators are unitaries, isometries, or projections we can impose any other relations we wish. The trick is to find the essential relations on the generators that describe the structure of $\mathfrak{T}_{\mathbb{N}^2}$.

Since $\mathfrak{T}_{\mathbb{N}^2}$ is generated by $T_{(1,0)}$ and $T_{(0,1)}$ the universal C^*-algebra will be generated by two isometries. We will need three more relations to describe the structure of $\mathfrak{T}_{\mathbb{N}^2}$. The first is that $T_{(1,0)}$ and $T_{(0,1)}$ commute, this is obtained from the fact that $T_{(1,0)}T_{(0,1)} = T_{(1,1)} = T_{(0,1)}T_{(1,0)}$. To find the other two we need to examine the multiplicative structure of the range projections. It is not hard to see that

$$T_{(m,n)}T^*_{(m,n)}(\xi_{(x,y)}) = \begin{cases} \xi_{(x,y)} & \text{if } m \leq x \text{ and } n \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $T_{(m_1,n_1)}T^*_{(m_1,n_1)}T_{(m_2,n_2)}T^*_{(m_2,n_2)} = T_{(m,n)}T^*_{(m,n)}$ where $m = \max\{m_1, m_2\}$ and $n = \max\{n_1, n_2\}$. In particular this shows that

$$T_{(1,0)}T^*_{(1,0)}T_{(0,1)}T^*_{(0,1)} = T_{(1,0)}T_{(0,1)}T^*_{(1,0)}T^*_{(0,1)}$$

which implies that $T^*_{(1,0)}$ commutes with $T_{(0,1)}$. Similarly we can see that $T_{(1,0)}$ commutes with $T^*_{(0,1)}$, in this situation we say $T_{(1,0)}$ and $T_{(0,1)}$ $*$-commute. The following theorem tells us that we have found the right relations, and it is very similar to our main theorem (Theorem 1.2.5).

Theorem 1.1.1. *Let $C^*(S,T)$ be the universal C^*-algebra generated by two $*$-commuting isometries S and T. If V and W are $*$-commuting isometries in a C^*-algebra A, then the map*

$$S \mapsto V, \quad T \mapsto W$$

extends to a homomorphism $\varphi : C^*(S,T) \to A$. Moreover φ is injective if and only if $(1 - VV^*)(1 - WW^*) \neq 0$. [6, Theorem 4.4]

Notice the similarity of the injectivity condition $(1 - VV^*)(1 - WW^*) \neq 0$ to that of Coburn. Since $T_{(1,0)}$ and $T_{(0,1)}$ are $*$-commuting isometries that satisfy the injectivity condition, and since the homomorphism from $C^*(S,T)$ to $\mathfrak{T}_{\mathbb{N}^2}$ is surjective, the two C^*-algebras must be isomorphic. Now that we have $\mathfrak{T}_{\mathbb{N}^2} \cong C^*(S,T)$, this theorem gives us a convenient way of finding homomorphic images of $\mathfrak{T}_{\mathbb{N}^2}$: any two $*$-commuting isometries generate a C^*-algebra which is the homomorphic image of $\mathfrak{T}_{\mathbb{N}^2}$. Moreover checking that a homomorphism is faithful amounts to verifying that a single expression does not vanish.

Remark 1.1.2. In $\mathfrak{T}_{\mathbb{N}^2}$, $(1 - T_{(m+1,0)}T^*_{(m+1,0)})T_{(m,0)}T^*_{(m,0)}$ is the projection onto the subspace of $\ell^2(\mathbb{N}^2)$ spanned by basis vectors with m in the first coordinate, while $(1 - T_{(0,n+1)}T^*_{(0,n+1)})T_{(0,n)}T^*_{(0,n)}$ is the projection onto the subspace spanned by basis vectors with n in the second coordinate. The product of these two projections is the rank-one projection onto the subspace spanned by $\xi_{(m,n)}$ and can be written as

$$T_{(m,n)}(1 - T_{(1,0)}T^*_{(1,0)})(1 - T_{(0,1)}T^*_{(0,1)})T^*_{(m,n)}.$$

In an arbitrary C^*-algebra containing two $*$-commuting isometries V and W, the condition $(1 - VV^*)(1 - WW^*) \neq 0$ is algebraically equivalent to all products of the form

$$\Big((1 - V^{m+1}(V^{m+1})^*)V^m(V^m)^*\Big)\Big((1 - W^{n+1}(W^{n+1})^*)W^n(W^n)^*\Big)$$

being non-zero.

1.2 Summary of Main Results

Let R be a number ring (see chapter 2 for a brief introduction to number rings and an overview of the relevant results). The semidirect product $R \rtimes R^\times$ of $(R, +)$ and

$(R^\times, \cdot)$ where $R^\times = R \setminus \{0\}$ is the set $R \times R^\times$ with the binary operation

$$(x,a)(y,b) = (x + ay, ab).$$

This may seem like an odd way to multiply, but it makes sense when looking at other presentations of $R \rtimes R^\times$. Here are two more:

- The set of 2×2 matrices of the form $\left(\begin{smallmatrix} a & x \\ 0 & 1 \end{smallmatrix}\right)$, where $a \in R^\times$ and $x \in R$.
- The set of functions $f : R \to R$ of the form $f(x) = ax + b$ with $a \in R^\times$ and $b \in R$ with the operation of function composition. Because of this, $R \rtimes R^\times$ is sometimes called the *$ax+b$-semigroup of R*.

Remark 1.2.1. An element (x,a) is invertible in $R \rtimes R^\times$ if a has a multiplicative inverse, in which case $(x,a)^{-1} = (-a^{-1}x, a^{-1})$.

The characteristic properties of $R \rtimes R^\times$ are that it contains copies of the additive group and of the multiplicative semigroup of R:

$$(R, +) \cong \{(x,1) : x \in R\}, \text{and}$$
$$(R^\times, \cdot) \cong \{(0,a) : a \in R^\times\},$$

that every element of $R \rtimes R^\times$ can be decomposed as a product of a member of each of these sets, $(x,a) = (x,1)(0,a)$, and that the relation $(0,a)(x,1) = (ax, a)$ holds.

Let $\mathfrak{T}_{R \rtimes R^\times}$ be the left-regular C^*-algebra of $R \rtimes R^\times$, that is the C^*-algebra generated by the family of isometries $T_{(x,a)} : \ell^2(R \rtimes R^\times) \to \ell^2(R \rtimes R^\times)$ indexed by $R \rtimes R^\times$ defined to be the continuous linear extensions of the maps $\xi_{(y,b)} \mapsto \xi_{(x,a)(y,b)}$.

Just like $\mathfrak{T}_{\mathbb{N}^2}$, the Toeplitz algebra of $R \rtimes R^\times$ comes attached to a particular Hilbert space, but it can act on others. As in the example, to make it easier to identify other Hilbert spaces on which $\mathfrak{T}_{R\rtimes R^\times}$ can act, we will define a universal C^*-algebra $\mathfrak{T}[R]$ defined by relations on generators (which will be isomorphic to $\mathfrak{T}_{R\rtimes R^\times}$). Then we will show that a homomorphism $\varphi : \mathfrak{T}[R] \to A$ is injective if and only if A satisfies a condition similar to $(1 - VV^*)(1 - WW^*) \neq 0$ in the $\mathfrak{T}_{\mathbb{N}^2}$ example.

Since $R \rtimes R^\times$ is more complicated than $\mathbb{N}^2$, it is understandable that both $\mathfrak{T}[R]$ and the injectivity condition will also be more complicated. The relations on the generators used to define $\mathfrak{T}[R]$ will not make sense unless we take some time to get to know $\mathfrak{T}_{R\rtimes R^\times}$. Our first observation is that the map $(x, 1) \mapsto T_{(x,1)}$ is a unitary representation of $(R, +)$, and the map $(0, a) \mapsto T_{(0,a)}$ is an isometric representation of $(R^\times, \cdot)$. Moreover the elements $T_{(x,1)}$ and $T_{(0,a)}$ generate $\mathfrak{T}_{R\rtimes R^\times}$.

In the previous section we saw that in order to find all the relations describing $\mathfrak{T}_{\mathbb{N}^2}$ we needed to examine how the range projections multiplied. The relations defining $\mathfrak{T}[R]$ will recover the multiplicative structure of a family of projections which contains, but in general is not limited to, the range projections of the isometries generating $\mathfrak{T}_{R\rtimes R^\times}$. We motivate this with the following example.

Example 1.2.2. Let $R = \mathbb{Z}$. In this example we will compute what the projections $T_{(x,a)}T^*_{(x,a)}$ do to basis vectors and how to multiply two of these projections.

For all $\xi_{(y,b)}$ and $\xi_{(z,c)}$ in $\ell^2(\mathbb{Z} \rtimes \mathbb{Z}^\times)$, the adjoint $T^*_{(x,a)}$ of $T_{(x,a)}$ satisfies

$$\langle T^*_{(x,a)}(\xi_{(y,b)}), \xi_{(z,c)}\rangle = \langle \xi_{(y,b)}, T_{(x,a)}(\xi_{(z,c)})\rangle = \begin{cases} 1 & \text{if } (y, b) = (x, a)(z, c), \\ 0 & \text{otherwise.} \end{cases}$$

From this we can conclude that $T^*_{(x,a)}(\xi_{(y,b)})$ is non-zero if and only if we can write $(y, b) = (x, a)(z, c) = (x+az, ac)$ for some (z, c) in $\mathbb{Z}\rtimes\mathbb{Z}^\times$, in which case $T^*_{(x,a)}(\xi_{(y,b)}) = \xi_{(z,c)}$. Notice that the existence of such a (z, c) is equivalent to $a \mid y - x$ and $a \mid b$. We claim that it is also equivalent to $y + b\mathbb{Z} \subseteq x + a\mathbb{Z}$. To see this observe that if $a \mid y - x$

and $a \mid b$, then we can write $y = x + az$ and $b = ac$, and $y + b\mathbb{Z} = x + az + ac\mathbb{Z} \subseteq x + a\mathbb{Z}$. On the other hand, if the inclusion holds, we can find $(z, c) \in \mathbb{Z} \rtimes \mathbb{Z}^\times$ by solving $y + b \cdot 0 = x + az$ followed by $x + az + b \cdot 1 = x + a(c + z)$. Using the notation $e_{(x,a\mathbb{Z})} := T_{(x,a)} T^*_{(x,a)}$, we have shown that

$$e_{(x,a\mathbb{Z})}(\xi_{(y,b)}) = \begin{cases} \xi_{(y,b)} & \text{if } y + b\mathbb{Z} \subseteq x + a\mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

moreover the following are equivalent:

1. we can find $(z, c) \in \mathbb{Z} \rtimes \mathbb{Z}^\times$ such that $(y, b) = (x, a)(z, c)$,
2. $a \mid y - x$ and $a \mid b$, and
3. $y + b\mathbb{Z} \subseteq x + a\mathbb{Z}$.

We will now make sense of the product of two of these projections. Using (1.1), we can compute the action of $e_{(x,a\mathbb{Z})} e_{(y,b\mathbb{Z})}$ on the standard orthonormal basis:

$$e_{(x,a\mathbb{Z})} e_{(y,b\mathbb{Z})}(\xi_{(w,d)}) = \begin{cases} \xi_{(w,d)} & \text{if } w + d\mathbb{Z} \subseteq (x + a\mathbb{Z}) \cap (y + b\mathbb{Z}), \\ 0 & \text{otherwise.} \end{cases}$$

We will prove that when the intersection is non-empty, any $z \in (x + a\mathbb{Z}) \cap (y + b\mathbb{Z})$ satisfies $z + (a\mathbb{Z} \cap b\mathbb{Z}) = (x + a\mathbb{Z}) \cap (y + b\mathbb{Z})$. Since $a\mathbb{Z} \cap b\mathbb{Z} = c\mathbb{Z}$, where c is the least common multiple of a and b, we will have shown that set of range projections of the isometries generating $\mathfrak{T}_{\mathbb{Z} \rtimes \mathbb{Z}^\times}$ is closed under multiplication, and that

$$e_{(x,a\mathbb{Z})} e_{(y,b\mathbb{Z})} = \begin{cases} 0 & \text{if } (x + a\mathbb{Z}) \cap (y + b\mathbb{Z}) = \varnothing, \\ e_{(z,a\mathbb{Z} \cap b\mathbb{Z})} & \text{if } (x + a\mathbb{Z}) \cap (y + b\mathbb{Z}) \neq \varnothing, \text{ for any } z \in (x + a\mathbb{Z}) \cap (y + b\mathbb{Z}). \end{cases} \tag{1.2}$$

Assuming that the intersection is non-empty, let $z \in (x+a\mathbb{Z}) \cap (y+b\mathbb{Z})$ be arbitrary and write $z = x+ar = y+bs$. If k is in $(x+a\mathbb{Z}) \cap (y+b\mathbb{Z})$, then we can find integers m and n so that $k = x+am = z+a(m-r)$ and $k = y+bn = z+b(n-s)$. Since $a(m-r)$ and $b(n-s)$ are then equal, k is in $z+(aZ \cap b\mathbb{Z})$. On the other hand, if $k \in z+a\mathbb{Z} \cap b\mathbb{Z}$, then we can write $k = z + am = x + a(m + r)$ and $k = z + bn = x + b(m + s)$, where m and n are integers. Hence k is in both $x + a\mathbb{Z}$ and $y + b\mathbb{Z}$. □

The example suggests that when R is a general number ring, $\mathfrak{T}_{R \rtimes R^\times}$ contains a family of projections $e_{(x,I)}$, where $x \in R$ and I is a non-zero ideal of R. These projections are characterized by their action on the basis elements

$$e_{(x,I)}(\xi_{(y,b)}) = \begin{cases} \xi_{(y,b)} & \text{if } y + bR \subseteq x + I, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, as we will show in Lemma 4.1.3(b), they multiply according to the rule

$$e_{(x,I)}e_{(y,J)} = \begin{cases} 0 & \text{if } (x + I) \cap (y + J) = \varnothing, \\ e_{(z,I \cap J)} & \text{if } (x + I) \cap (y + J) \neq \varnothing, \text{ for any } z \in (x + I) \cap (y + J). \end{cases}$$

For non-zero principal ideals aR, $e_{(x,aR)} = T_{(x,a)}T^*_{(x,a)}$. But unlike $\mathbb{Z}$, a general number ring may have ideals which are not principal, in which cases $e_{(x,I)}$ is not of that form. We will see that given a non-zero ideal I we can find $a, b \in R$ such that $I = \frac{a}{b}R \cap R$ (Proposition 2.0.14), thus we can write

$$e_{(x,I)} = T_{(x,1)}T^*_{(0,b)}T_{(0,a)}T^*_{(0,a)}T_{(0,b)}T_{(-x,1)}$$

(Proposition 4.1.6(a)). We are now ready to define the universal C^*-algebra.

Definition 1.2.3. Following [2] we define the C^*-algebra $\mathfrak{T}[R]$ as the universal C^*-

algebra generated by elements u^x, $x \in R$, s_a, $a \in R^\times$, e_I, I a non-zero ideal in R, with the following relations

- Ta: The u^x are unitary and satisfy $u^x u^y = u^{x+y}$, the s_a are isometries and satisfy $s_a s_b = s_{ab}$. Moreover we require the relation $s_a u^x = u^{ax} s_a$ for all $x \in R$, $a \in R^\times$.
- Tb: The e_I are projections and satisfy $e_{I\cap J} = e_I e_J$, $e_R = 1$.
- Tc: We have $s_a e_I s_a^* = e_{aI}$.
- Td: For $x \in I$ one has $u^x e_I = e_I u^x$, for $x \notin I$ one has $e_I u^x e_I = 0$.

Before going further, we will explain the relations. Ta simply says that $\mathfrak{T}[R]$ is an image of an isometric representation of $R \rtimes R^\times$. The other three relations recover the structure of the projections described above. The relation Tc directly implies that $e_{aR} = s_a s_a^*$, and assuming that $I = \frac{a}{b} R \cap R$, it guarantees that $e_I = s_b^* s_a s_a^* s_b$ (proof: $bI = aR \cap bR$ implies that $s_b e_I s_b^* = e_{bI} = e_{aR} e_{bR} = s_a s_a^* s_b s_b^*$). In chapter 3 we will see that Tb together with Td imply that the projections $e_I^x := u^x e_I u^{-x}$ multiply in the same way as the $e_{(x,I)} \in \mathfrak{T}_{R\rtimes R^\times}$ (defined on page 9):

$$e_I^x e_J^y = \begin{cases} 0 & \text{if } (x+I)\cap(y+J) = \varnothing, \\ e_{I\cap J}^z & \text{if } (x+I)\cap(y+J) \neq \varnothing, \text{ for any } z \in (x+I)\cap(y+J). \end{cases}$$

The C^*-subalgebra generated by these projections will be called $\overline{\mathcal{D}}$.

Remark 1.2.4. When R is a principal ideal domain, Tb, Tc, and Td can be replaced with the single condition

$$\sum_{x \in R/aR} u^x s_a s_a^* u^{-x} \leq 1.$$

for all $a \in R$. This is Remark 2.2 in [2].

We now present our main theorem.

Theorem 1.2.5. *Let A be a C^*-algebra that contains elements U^x, $x \in R$, S_a, $a \in R^\times$, and E_I, I a non-zero ideal of R, that satisfy Ta-Td. Then the map*

$$u^x \mapsto U^x, \quad s_a \mapsto S_a, \quad e_I \mapsto E_I,$$

extends to a homomorphism $\varphi : \mathfrak{T}[R] \to A$. For each prime ideal P in R and non-negative integer t, we define the projection $\delta_{(t,P)} = (1 - \sum_{x \in R/P^{t+1}} U^x E_{P^{t+1}} U^{-x}) E_{P^t}$. Then the homomorphism φ is injective if and only if all projections of the form $\delta_{(t_1,P_1)}\delta_{(t_2,P_2)} \cdots \delta_{(t_n,P_n)}$, with $P_1, P_2 \ldots, P_n$ distinct, are non-zero.

Once we show that the elements $T_{(x,1)}$, $T_{(0,a)}$ and $e_{(0,I)}$ in $\mathfrak{T}_{R \rtimes R^\times}$ satisfy Ta-Td and the injectivity condition is satisfied, we will have that $\mathfrak{T}[R] \cong \mathfrak{T}_{R \rtimes R^\times}$. This makes finding other Hilbert spaces for $\mathfrak{T}_{R \rtimes R^\times}$ to act on a matter of checking that Ta-Td are satisfied. To show that a homomorphism from $\mathfrak{T}_{R \rtimes R^\times}$ to another C^*-algebra is faithful, we only need to check a simple condition.

To illustrate how to apply the theorem we present the following example. Note that

1. non-zero ideals in $\mathbb{Z}$ are of the form $a\mathbb{Z}$, where $a \in \mathbb{Z}^\times$,
2. prime ideals in $\mathbb{Z}$ are of the form $p\mathbb{Z}$, where p is a prime, and
3. in $\mathfrak{T}_{\mathbb{Z} \rtimes \mathbb{Z}^\times}$,

$$T_{(x,1)} e_{(0,a\mathbb{Z})} T_{(-x,1)} = T_{(x,1)} T_{(0,a)} T^*_{(0,a)} T_{(-x,1)} = T_{(x,a)} T^*_{(x,a)} = e_{(x,a\mathbb{Z})}.$$

Lemma 1.2.6. *The elements $T_{(x,1)}$, $T_{(0,a)}$, and $e_{(0,b\mathbb{Z})}$ in $\mathfrak{T}_{\mathbb{Z} \rtimes \mathbb{Z}^\times}$ satisfy Ta-Td. Moreover, if $p_1, \ldots, p_n \in \mathbb{Z}$ are distinct primes, $t_1, \ldots, t_n$ are non-negative integers, and*

$\delta_{(t_i,p_i\mathbb{Z})} = (1 - \sum_{x=1}^{p_i^{t_i+1}} e_{(x,p_i^{t_i+1}\mathbb{Z})})e_{(0,p_i^{t_i}\mathbb{Z})}$, *then*

$$\delta_{(t_1,p_1\mathbb{Z})} \cdots \delta_{(t_n,p_n\mathbb{Z})} \neq 0.$$

Proof. Ta: Elements of the form $T_{(x,1)}$ and $T_{(0,a)}$ are isometries because $\mathfrak{T}_{\mathbb{Z}\rtimes\mathbb{Z}^\times}$ is an image of an isometric representation of $\mathbb{Z}$, $T_{(x,1)}$ is unitary because, by Remark 1.2.1, $(x, 1) \in \mathbb{Z}\rtimes\mathbb{Z}^\times$ is invertible. The fact that $T_{(0,a)}T_{(x,1)} = T_{(ax,1)}T_{(0,a)}$ follows from the definition of multiplication in $\mathbb{Z} \rtimes \mathbb{Z}^\times$.

Tb: This follows easily from (1.1) and (1.2).

Tc: Use the definitions of $T_{(0,a)}$ and $e_{(0,b\mathbb{Z})}$ to verify that $T_{(0,a)}e_{(0,b\mathbb{Z})} = e_{(0,ab\mathbb{Z})}T_{(0,a)}$, then

$$T_{(0,a)}e_{(0,b\mathbb{Z})}T^*_{(0,a)} = e_{(0,ab\mathbb{Z})}T_{(0,a)}T^*_{(0,a)} = e_{(0,ab\mathbb{Z})}e_{(0,a\mathbb{Z})} = e_{(0,ab\mathbb{Z})},$$

where the last equality comes from Tb (or (1.2)) together with the fact that $ab\mathbb{Z} \cap a\mathbb{Z} = ab\mathbb{Z}$.

Td: Use (1.2) to show that $T_{(x,1)}e_{(0,b\mathbb{Z})}T_{(-x,1)} = e_{(x,b\mathbb{Z})} = e_{(0,b\mathbb{Z})}$ if $x \in b\mathbb{Z}$, and $e_{(0,b\mathbb{Z})}(T_{(x,1)}e_{(0,b\mathbb{Z})}T_{(-x,1)}) = e_{(0,b\mathbb{Z})}e_{(x,b\mathbb{Z})} = 0$ if $x \notin b\mathbb{Z}$. Together these are equivalent to Td.

To prove the second part, will first show that

$$\delta_{(t_i,p_i\mathbb{Z})}(\xi_{(y,b)}) = \begin{cases} \xi_{(y,b)} & \text{if } p_i^{t_i} \mid y \text{ and } b = p_i^{t_i}c \text{ where } \gcd(c, p_i) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

If both $e_{(0,p_i^{t_i}\mathbb{Z})}(\xi_{(y,b)}) = \xi_{(y,b)}$ and $(1 - \sum_{x=1}^{p_i^{t_i+1}} e_{(x,p_i^{t_i+1}\mathbb{Z})})(\xi_{(y,b)}) = \xi_{(y,b)}$ hold, then $\delta_{(t_i,p_i\mathbb{Z})}(\xi_{(y,b)}) = \xi_{(y,b)}$. Otherwise at least one of $e_{(0,p_i^{t_i}\mathbb{Z})}(\xi_{(y,b)})$ and $(1-\sum_{x=1}^{p_i^{t_i+1}} e_{(x,p_i^{t_i+1}\mathbb{Z})})(\xi_{(y,b)})$ will be zero, in which case $\delta_{(t_i,p_i\mathbb{Z})}(\xi_{(y,b)})$ will also be zero. As we saw in Example 1.2.2,

$e_{(0,p_i^{t_i}\mathbb{Z})}(\xi_{(y,b)}) = \xi_{(y,b)}$ if and only if $p_i^{t_i} \mid y$ and $p_i^{t_i} \mid b$. Thus we only need to show that $(1 - \sum_{x=1}^{p_i^{t_i+1}} e_{(x,p_i^{t_i+1}\mathbb{Z})})(\xi_{(y,b)}) = \xi_{(y,b)}$ if and only if $p_i^{t_i+1} \nmid b$. Since there is exactly one $1 \leq x \leq p_i^{t_i+1}$ such that $p_i^{t_i+1} \mid y - x$, there can be at most one $1 \leq x \leq p_i^{t_i+1}$ for which $e_{(x,p_i^{t_i+1}\mathbb{Z})}(\xi_{(y,b)}) = \xi_{(y,b)}$. It follows that

$$\Big(1 - \sum_{x=1}^{p_i^{t_i+1}} e_{(x,p_i^{t_i+1}\mathbb{Z})}\Big)(\xi_{(y,b)}) = \begin{cases} \xi_{(y,b)} & \text{if } e_{(x,p_i^{t_i+1}\mathbb{Z})}(\xi_{(y,b)}) = 0 \text{ for all } 1 \leq x \leq p_i^{t_i+1}, \\ 0 & \text{otherwise,} \end{cases}$$

and that y does not affect whether or not all the $e_{(x,p_i^{t_i+1}\mathbb{Z})}(\xi_{(y,b)})$ are zero. From this we can conclude that $(1 - \sum_{x=1}^{p_i^{t_i+1}} e_{(x,p_i^{t_i+1}\mathbb{Z})})(\xi_{(y,b)}) = \xi_{(y,b)}$ if and only if $p_i^{t_i+1} \nmid b$, which proves the claim.

Using (1.3) we can see that $\delta_{(t_i,p_i\mathbb{Z})}(\xi_{(0,p_1^{t_1}p_2^{t_2}\cdots p_n^{t_n})}) = \xi_{(0,p_1^{t_1}p_2^{t_2}\cdots p_n^{t_n})}$, which shows that $\delta_{(t_1,p_1\mathbb{Z})}\delta_{(t_2,p_2\mathbb{Z})}\cdots\delta_{(t_n,p_n\mathbb{Z})}$ is non-zero. □

Remark 1.2.7. The condition in the first case of (1.3) can be restated in terms of ideals: $\delta_{(t,p\mathbb{Z})}(\xi_{(y,b)})$ is non-zero if and only if $y \in p^t\mathbb{Z}$ and $b \in p^t\mathbb{Z} \setminus p^{t+1}\mathbb{Z}$. This characterization of $\delta_{(t,p\mathbb{Z})}$, generalizes. In Lemma 4.1.3 we will see that when P is a non-zero prime ideal in a general number ring R, $\delta_{(t,P)} \in \mathfrak{T}_{R \rtimes R^\times}$ satisfies

$$\delta_{(t,P)}(\xi_{(y,b)}) = \begin{cases} \xi_{(y,b)} & \text{if } y \in P^t \text{ and } b \in P^t \setminus P^{t+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Method of Proof of Theorem 1.2.5. The map φ extends to a homomorphism because of the universal property defining $\mathfrak{T}[R]$. The method that is used to prove that φ is injective has been used by many authors. To help visualize the method it will be useful to have the following commutative diagram.

$$
\begin{array}{ccc}
\mathfrak{T}[R] & \xrightarrow{\ \varphi\ } & A \\
{\scriptstyle E}\downarrow & & \downarrow \\
\overline{\mathcal{D}} & \xrightarrow[\varphi|_{\overline{\mathcal{D}}}]{} & \overline{\text{span}}\{U^x E_I U^{-x}\}
\end{array}
$$

The map E, defined in Lemma 3.0.23, is a norm-decreasing projection onto $\overline{\mathcal{D}}$ that is faithful as positive map (i.e. if $E(h^*h) = 0$ then $h = 0$). In Chapter 4 we will prove that the condition that all $\delta_{(t_1,P_1)}\delta_{(t_2,P_2)} \cdots \delta_{(t_n,P_n)} \neq 0$ is equivalent to the injectivity of the restriction of φ to $\overline{\mathcal{D}}$, and that the map from A to $\overline{\text{span}}\{U^x E_I U^{-x}\}$ is norm-decreasing. Assuming that $\varphi|_{\overline{\mathcal{D}}}$ is injective and that the map from A to $\overline{\text{span}}\{U^x E_I U^{-x}\}$ is norm decreasing, we prove that φ is injective because

$$\varphi(h) = 0 \Rightarrow \varphi(h^*h) = 0 \Rightarrow \varphi(E(h^*h)) = 0 \Rightarrow E(h^*h) = 0 \Rightarrow h = 0.$$

The details of this argument are given in Chapter 3.

Chapter 2

Summary of Algebraic Number Theory

In this chapter we give a few facts from algebraic number theory, followed by the results we will use. Most of this chapter is summarized from [5]. We will assume the reader is familiar with the basics of elementary number theory and ring theory.

A *number field* K is a finite extension field of the rationals. In other words K is a field that contains $\mathbb{Q}$ and has finite dimension as a vector space over $\mathbb{Q}$. If $\alpha \in K$ then there is some n for which $\alpha, \alpha^2, \ldots, \alpha^n$ are linearly dependant, this means that we can find rationals $a_0, a_1, \ldots, a_n$ such that

$$a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n = 0.$$

Thus α is the root of a polynomial with rational coefficients, and hence is an *algebraic number*. If α satisfies a non-zero monic polynomial with integer coefficients, we call α an *algebraic integer*. It turns out that the sum and product of two algebraic integers is again an algebraic integer [5, p. 16], the set of all algebraic integers in a number field is a ring R, which we call the *ring of integers* of K. Such a ring is called a *number ring.*

Example 2.0.8. 1. The ring of integers in the field $\mathbb{Q}[i]$ is $\mathbb{Z}[i]$, often called the *Gaussian Integers*

2. The ring of integers of $\mathbb{Q}[\sqrt{-5}]$ is $\mathbb{Z}[\sqrt{-5}]$.

3. Let ζ_n be a primitive n^{th} root of unity. The field $\mathbb{Q}[\zeta_n]$ is called the n^{th} cyclotomic field [5, p. 12], and its ring of integers is $\mathbb{Z}[\zeta_n]$. For example, $\frac{1+\sqrt{-3}}{2}$ is a primitive cube root of unity, and since $\mathbb{Q}[\sqrt{-3}] = \mathbb{Q}[\frac{1+\sqrt{-3}}{2}]$ its ring of integers is $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$. This shows that the ring of integers of $\mathbb{Q}[\sqrt{n}]$ need not be $\mathbb{Z}[\sqrt{n}]$. □

Number fields and number rings are generalizations of the rationals and the integers. A good question one might ask is: how does the Unique Factorization of Arithmetic generalize? That is, does every element of R factor as a product of primes uniquely up to reordering and multiplication by units? It turns out that this does not happen in general number rings. Before giving an example, we need a few definitions.

- An element $u \in R$ is called a *unit* if it has a multiplicative inverse in R. Examples: The units of $\mathbb{Z}[i]$ are $\{\pm 1, \pm i\}$ and the units of $\mathbb{Z}[\sqrt{2}]$ are $\{\pm(1+\sqrt{2})^n : n \in \mathbb{Z}\}$.

- An element $r \in R$ is *irreducible* if whenever $r = ab$, either a or b is a unit.

- An element $p \in R$ is *prime* if $p \mid ab$ implies that either $p \mid a$ or $p \mid b$. Here $p \mid a$ means that there is some $x \in R$ such that $a = px$.

In $\mathbb{Z}$ and some other special number rings R, the notions of prime and irreducible are equivalent. When this is the case, the elements of R factor uniquely as a product of primes. Although in a general number ring, all primes are irreducible, the converse is not necessarily true. This is where unique prime factorization can fail, as the following example illustrates.

Example 2.0.9. There are two distinct ways to factor 6 into irreducibles in $\mathbb{Z}[\sqrt{-5}]$:

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

We define the norm of $a \in \mathbb{Z}[\sqrt{-5}]$ to be $N(a) = a\overline{a}$, where $\overline{a}$ is the complex conjugate of a. The norm is multiplicative ($N(ab) = N(a)N(b)$) because multiplication is commutative, and an easy argument can be used to show that $u \in \mathbb{Z}[\sqrt{-5}]$ is a unit if and only if $N(u) = 1$. Since we can always write $a = x + y\sqrt{-5}$ with $x, y \in \mathbb{Z}$, and since neither $x^2 + 5y^2 = 2$ nor $x^2 + 5y^2 = 3$ have integer solutions, no element of $\mathbb{Z}[\sqrt{-5}]$ can have norm 2 or 3. Also since the only solutions to $x^2 + 5y^2 = 1$ are $x = \pm 1$ and $y = 0$, we can see that the only units are ± 1, which makes it is clear that the two given factorization of 6 are different. We claim that $2, 3, (1+\sqrt{-5})$, and $(1 - \sqrt{-5})$ are irreducible.

Suppose that $2 = ab$, where $a, b \in \mathbb{Z}[\sqrt{-5}]$ are not units. Since $4 = N(2) = N(ab) = N(a)N(b)$, and since neither a nor b has norm one, we must have $N(a) = N(b) = 2$; but this is impossible, thus we can conclude that 2 is irreducible. We can show $3, (1 + \sqrt{-5})$, and $(1 - \sqrt{-5})$ are irreducible using a similar method. $\square$

All is not lost; we will see shortly that we can recover some form of unique factorization using ideals. Recall that an *ideal* I of R is a subgroup of $(R, +)$ such that $xy \in I$ when $x \in I$ and $y \in R$, and that (for commutative rings) an ideal P is called *prime* if $ab \in P$ implies that either a or b is in P. In general, an ideal is prime if and only if the quotient of the ring by the ideal is an integral domain. In the case of number rings, R/I is always finite, hence an ideal P is prime if and only if R/P is a field. An important fact (for us) is that a number ring has countably many prime ideals.

Example 2.0.10.

- The ideals of $\mathbb{Z}$ are all of the form $n\mathbb{Z}$, the prime ideals are those for which n is prime.

- In $\mathbb{Z}[\sqrt{-5}]$ the following ideals are prime

$$P_1 = 2\mathbb{Z} + (1+\sqrt{-5})\mathbb{Z}$$
$$P_2 = 3\mathbb{Z} + (1+\sqrt{-5})\mathbb{Z}$$
$$P_3 = 3\mathbb{Z} + (1-\sqrt{-5})\mathbb{Z}.$$

The easiest way to prove this would be to use the first isomorphism theorem for rings to show that $\mathbb{Z}[\sqrt{-5}]/P_1 \cong \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}[\sqrt{-5}]/P_2 \cong \mathbb{Z}[\sqrt{-5}]/P_3 \cong \mathbb{Z}/3\mathbb{Z}$.

□

An ideal is *principal* if it is of the form aR for some $a \in R$. If all the ideals in R are principal, then we say that R is a *principal ideal domain*. Examples of PIDs include $\mathbb{Z}$ and $\mathbb{Z}[i]$.

The product of two ideals I and J is defined to be the smallest ideal containing all products of the form ab, where $a \in I$ and $b \in J$. Explicitly,

$$IJ = \{\sum_{i=1}^{n} a_i b_i : a_i \in I, b_i \in J\}.$$

Notice that unlike multiplication of integers, IJ (as a set) is smaller than both I and J. Unsurprisingly, along with multiplication comes a notion of divisibility. We say that I *divides* J, and write $I \mid J$ if $J \subseteq I$. Notice that if $J = aR$, the divisibility condition can be simplified to $a \in I$.

Example 2.0.11. We have seen that $6 \in R = \mathbb{Z}[\sqrt{-5}]$ has two distinct factorizations into irreducibles. Although we can write

$$6R = 2R \cdot 3R = (1+\sqrt{-5})R \cdot (1-\sqrt{-5})R,$$

none of those ideals are prime. Each of those ideals can be factored uniquely as a product of the prime ideals given in Example 2.0.10:

$$P_1^2 P_2 P_3 = (P_1^2)(P_2 P_3) = (P_1 P_2)(P_1 P_3). \quad \square$$

Theorem 2.0.12. *Every ideal in a number ring R, other than $\{0\}$ and R, factors uniquely as a product of finitely many prime ideals. This factorization is unique up to re-ordering. [5, p. 60]*

We observe that this theorem together with the fact that there are countably many prime ideals imply that a number ring has countably many ideals.

Concepts that play an important role in elementary number theory are the notions of the greatest common divisor, least common multiple, and the notion of two numbers being relatively prime. We will now see how these generalize to ideals.

We define two other binary operations on ideals:

$$I + J = \{a + b : a \in I, b \in J\}, \text{and}$$
$$I \cap J = \{b : a \in I \text{ and } b \in J\}.$$

In words $I + J$ is the smallest ideal containing both I and J. This operation plays the role of the *greatest common divisor*. Just like the integers, if $I = P_1^{e_1} \cdots P_n^{e_n}$ and $J = P_1^{f_1} \cdots P_n^{f_n}$, then $I + J = P_1^{t_1} \cdots P_n^{t_n}$ where $t_i = \min\{e_i, f_i\}$. If $I + J = R$ then we say that I and J are *relatively prime*.

The operation $I \cap J$ gives the largest ideal contained in both I and J, and the resulting ideal is the *least common multiple*. Again, just like the integers, $I \cap J = P_1^{t'_1} \cdots P_n^{t'_n}$ where $t'_i = \max\{e_i, f_i\}$. Observe that when I and J are relatively prime $I \cap J = IJ$.

There are two more results that we will need. The first is the Chinese Remainder

Theorem [5, p. 253] which tells us that for fixed $a_i \in R$ we can always solve the congruence

$$a \equiv a_i \mod I_i,\ i = 1, 2, \ldots, n$$

when $I_1, \ldots, I_n$ are pairwise relatively prime ideals of R. The other is that every ideal in R can be written in the form $I = \frac{a}{b}R \cap R$.

Theorem 2.0.13 (Chinese Remainder Theorem)**.** *Let $I_1, \ldots, I_n$ be pairwise relatively prime ideals in R. Then*

$$R/(I_1 \cdots I_n) \cong R/I_1 \times \cdots \times R/I_n.$$

Proposition 2.0.14. *Every non-zero ideal in R can be written in the form $I = \frac{a}{b}R \cap R$, where $a, b \in R^\times$.*

Proof. Suppose I is an ideal of R with prime decomposition $P_1^{t_1} \cdots P_n^{t_n}$. Fix $a_i \in P_i^{t_i} \setminus P_i^{t_i+1}$ and solve the congruence

$$a \equiv a_i \mod P_i^{t_i+1},\ i = 1, 2, \ldots, n.$$

Since the highest power of P_i that divides aR is t_i, we can write $aR = IJ$ with $I + J = R$ (J could be equal to R, in which case I would be principal). Let $Q_1^{r_1} \cdots Q_m^{r_m}$ be the prime decomposition of J. Fix $b_i \in Q_i^{r_i} \setminus Q_i^{r_i+1}$ and $c_i \notin P_i$, and solve the congruence

$$b \equiv b_i \mod Q_i^{r_i+1},\ i = 1, \ldots, m,$$
$$b \equiv c_i \mod P_i,\ i = 1, \ldots, n.$$

Then $bR = JM$ where M is relatively prime to I and J. Thus we have

$$bI = IJM = IJ \cap JM = aR \cap bR.$$

□

Chapter 3

The Universal C^*-Algebra $\mathfrak{T}[R]$

In this chapter and the next, we will prove Theorem 1.2.5. In this chapter, we go as far as we can without using a specific homomorphic image of $\mathfrak{T}[R]$ (see Definition 1.2.3). The relations defining $\mathfrak{T}[R]$ tell us a lot about its multiplicative structure. Thus we will begin by deriving some multiplicative identities. The highlights will be that the projections $e_I^x := u^x e_I u^{-x}$ recover the multiplicative structure of the $e_{(x,I)} \in \mathfrak{T}_{R\rtimes R^\times}$, and that the linear span of the collection of elements of the form $s_a^* e_I^x u^y s_b$ is dense in $\mathfrak{T}[R]$. Following this, we will prove a version of Theorem 1.2.5 which does not include the injectivity condition.

Lemma 3.0.15. *Let $x \in R$ and let I and J be non-zero ideals in R. Then $e_I u^x e_J = 0$ if $x \notin I + J$ and $e_I u^x e_J = u^{x_1} e_{I\cap J} u^{x_2}$ if $x = x_1 + x_2$ with $x_1 \in I$ and $x_2 \in J$.*

Proof. Since $I = I \cap (I+J)$ and $J = J \cap (I+J)$, we can write $e_I u^x e_J = e_I e_{I+J} u^x e_{I+J} e_J = 0$ which by Td vanishes if $x \notin I$. On the other hand, if x is in $I+J$, then we can write $x = x_1 + x_2$ with $x_1 \in I$ and $x_2 \in J$. By Td, u^{x_1} commutes with e_I and u^{x_2} commutes with e_J, which together with Tb implies that $e_I u^x e_J = e_I u^{x_1} u^{x_2} e_J = u^{x_1} e_I e_J u^{x_2} =$

$u^{x_1} e_{I\cap J} u^{x_2}$. □

Let $\overline{\mathcal{D}}$ be the $*$-subalgebra generated by the projections e_I^x. Note that the projection e_I^x is not uniquely determined by x, in fact by Td we see that e_I^x equals $e_I^{x'}$ if $x - x'$ is in I. Next we see that as promised, the generators of $\overline{\mathcal{D}}$ multiply according to (1.2).

Proposition 3.0.16. *For every x and y in R and non-zero ideals I and J of R,*

$$e_I^x e_J^y = \begin{cases} 0 & \text{if } (x+I)\cap(y+J) = \varnothing, \\ e_{I\cap J}^z & \text{if } (x+I)\cap(y+J) \neq \varnothing, \text{ for any } z \in (x+I)\cap(y+J). \end{cases}$$

Remark 3.0.17. This formula is well-defined since if $z, z' \in (x+I)\cap(y+J)$, then $z - z' \in I\cap J$ which by Td implies $e_{I\cap J}^z = e_{I\cap J}^{z'}$. (Reason: Since $z, z' \in (x+I)$, $z - z' \in I$. Similarly $z - z' \in J$, whence $z - z' \in I \cap J$.)

Proof. Suppose $(x+I)\cap(y+J) = \varnothing$. Then $x - y \notin I + J$. Since $x - y \in I + J$ would imply that the intersection was non-empty, we have by Lemma 3.0.15 $e_I u^{-x} u^y e_J = 0$. On the other hand, if $z \in (x+I)\cap(y+J)$, then $x - z \in I$ and $y - z \in J$, thus we can write $e_I^x = e_I^z$ and $e_J^y = e_J^z$ which gives us $e_I^x e_J^y = u^z e_I (u^{-z} u^z) e_J u^{-z} = u^z e_{I\cap J} u^{-z} = e_{I\cap J}^z$. □

Lemma 3.0.18. *For any $x \in R$, $a \in R^\times$, and non-zero ideal I, we have*

(a) $s_a e_I^x s_a^* = e_{aI}^{ax}$, *and*

(b) $$s_a^* e_I^x s_a = \begin{cases} 0 & \text{if } aR\cap(x+I) = \varnothing, \\ e^y_{\frac{1}{a}(aR\cap I)} & \text{if } aR\cap(x+I) \neq \varnothing, \text{ for } y \in \frac{1}{a}(aR\cap(x+I)). \end{cases}$$

Proof. (a) This follows easily from Ta and Tc.

(b) Observe that we can write

$$s_a^* e_I^x s_a = s_a^* s_a s_a^* e_I^x s_a = s_a^* e_{aR} e_I^x s_a.$$

If $aR \cap (x + I) = \varnothing$, then by Proposition 3.0.16 the product $e_{aR} e_I^x$ is zero and we are done. On the other hand, if the intersection is non-empty, then the proposition tells us that $e_{aR} e_I^x = e_{aR\cap I}^{ay}$ for any $ay \in aR \cap (x + I)$. Applying part (a) we have

$$s_a^* e_{aR\cap I}^{ay} s_a = s_a^* (s_a e_{\frac{1}{a}(aR\cap I)}^{y} s_a^*) s_a = e_{\frac{1}{a}(aR\cap I)}^{y}.$$

□

Lemma 3.0.19. *The linear span of set of elements of the form $s_a^* e_I^x u^y s_b$ is dense in $\mathfrak{T}[R]$.*

Proof. The set contains all the generators. All we need to show is that the collection of elements of that form is closed under multiplication by the generators, and under taking adjoints.

Multiplying $s_a^* e_I^x u^y s_b$ on the right by s_c or on the left by s_c^* is easy. The calculations that deal with multiplying by s_c on the right and s_c^* on the left are more difficult but quite similar. The former is as follows:

$$\begin{aligned} s_c s_a^* e_I^x u^y s_b &= (s_a^* s_a) s_c s_a^* (s_c^* s_c) e_I^x (s_c^* s_c) u^y s_b \\ &= s_a^* (s_c s_a s_a^* s_c^*)(s_c e_I^x s_c^*)(s_c u^y) s_b \\ &= s_a^* (e_{acR} e_{cI}^{cx}) u^{cy} s_{cb}, \end{aligned}$$

the other, $s_a^* e_I^x u^y s_b s_c^* = s_{ac}^* (e_{cI}^{cx} e_{bc}^{cy}) u^{cy} s_b$, is left as an exercise.

If we multiply by u^z on the right, we can apply Ta directly. When multiplying on the left by u^z we get

$$\begin{aligned} u^z s_a^* e_I^x u^y s_b &= (u^{-z})^* s_a^* e_I^x u^y s_b \\ &= (s_a u^{-z})^* e_I^x u^y s_b \\ &= (u^{-az} s_a)^* e_I^x (u^{-az} u^{az}) u^y s_b \\ &= s_a^* (u^{az} e_I^x u^{-az}) u^{y+az} s_b \\ &= s_a^* e_I^{x+az} u^{y+az} s_b. \end{aligned}$$

Using the above identities, we can deduce the ones for multiplication by e_J^z and taking adjoints. For example:

$$\begin{aligned} s_a^* e_I^x u^y (s_b e_J) &= s_a^* e_I^x (u^y e_{bJ} s_b) \\ &= s_a^* e_I^x e_{bJ}^z u^z s_b. \end{aligned}$$

□

We are interested in finding a necessary and sufficient condition for a homomorphism $\varphi : \mathfrak{T}[R] \to A$ to be injective. The theorem at the end of the chapter is the first step. It will tell us that, with an additional assumption on A (later proved to be unnecessary), φ is injective if its restriction to $\overline{\mathcal{D}}$ is faithful. The critical component in the proof of this theorem is the existence of a conditional expectation[1] $E : \mathfrak{T}[R] \to \overline{\mathcal{D}}$ that is faithful ($E(h^*h) = 0$ implies that $h = 0$) and satisfies

$$E(s_a e_J^y u^z s_b) = \begin{cases} e_I^x & \text{if } s_a^* e_J^y u^z s_b = e_I^x \text{ for some } e_I^x, \\ 0 & \text{otherwise.} \end{cases}$$

[1]See Definition A.2.1.

Remark 3.0.20. In the first part of this chapter we saw that the relations defining $\mathfrak{T}[R]$ can be used to prove many identities involving multiplication. Since Ta-Td are multiplicative in nature, useful statements about the additive structure of $\mathfrak{T}[R]$ are more difficult to find. Using E, we can give a necessary condition for a linear combination of the generating set in Lemma 3.0.19 to be zero:

$$\sum_{i=1}^{N} \lambda_i e_{I_i}^{x_i} + \sum_{i=1}^{M} \lambda_i' s_{a_i}^* e_{J_i}^{y_i} u^{z_i} s_{b_i} = 0,$$

where $a_i \neq b_i$ or $z_i \neq 0$, only if $\sum_{i=1}^{N} \lambda_i e_{I_i}^{x_i} = 0$. This will follow from the fact, stated in Lemma 3.0.23, that $E(s_a^* e_J^y u^z s_b) \neq 0$ if and only if $a = b$ and $z = 0$.

The typical faithful conditional expectation arises by averaging over the orbits of a compact group action (see Proposition A.2.2). Unfortunately E cannot be produced in this way. Instead E will be the composition of two faithful conditional expectations, the first will be obtained from an action of the dual $\widehat{K^\times}$ of the multiplicative group of the number field K associated with R, and the second from an action of the dual $\widehat{R}$ of the additive group of $\widehat{R}$ on the fixed point algebra of the $\widehat{K^\times}$ action. (see Appendix A.1).

Lemma 3.0.21. *Let $B = \overline{\mathrm{span}}\{e_I^x u^y\}$. There exists a faithful conditional expectation $\theta_\alpha : \mathfrak{T}[R] \to B$ that satisfies*

$$\theta_\alpha(s_a^* e_I^x u^y s_b) = \begin{cases} s_a^* e_I^x u^y s_a & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, when $s_a^* e_I^x u^y s_a$ is not zero it can be written in the from $e^{z}_{\frac{1}{a}(aR \cap I)} u^{y/a}$.

Proof. Suppose R is the ring of integers of the number field K. The multiplicative group of K is a discrete abelian group ($K^\times := K \setminus \{0\}, \cdot$) therefore, by Appendix

A.1, its dual $\widehat{K^\times}$ is a compact group which we can assume to have total Haar measure one.

In order to apply Proposition A.2.2, we will need show that $\widehat{K^\times}$ acts continuously on $\mathfrak{T}[R]$ by automorphisms. That is: (i) if $\chi \in \widehat{K^\times}$, the map

$$u^x \mapsto u^x, \quad s^a \mapsto \chi(a)s^a, \quad e_I \mapsto e_I$$

extends by linearity and continuity to an automorphism $\alpha_\chi : \mathfrak{T}[R] \to \mathfrak{T}[R]$, and (ii) the function $\chi \mapsto \alpha_\chi$ is continuous.

(i) The elements u^x, $\chi(a)s^a$, and e_I clearly satisfy Ta-Td, hence the universal property gives a homomorphism α_χ. It is an automorphism because $\alpha_\chi^{-1} = \alpha_{\overline{\chi}}$.

(ii) Suppose that the sequence $\chi_n \in \widehat{K^\times}$ converges to $\chi \in \widehat{K^\times}$. That is, given $\varepsilon > 0$ and $a_1, \ldots, a_k \in K^\times$, there exists an N such that $|\chi_n(a_i) - \chi(a)| < \varepsilon$ for all $n > N$ and $i = 1, \ldots, k$.

We need to show that α_{χ_n} converges point-wise to α_χ. Given $h \in \mathfrak{T}[R]$ and $\varepsilon > 0$ we can find by Lemma 3.0.19 $z = \sum_{i=1}^k \lambda_i s_{a_i}^* e_{I_i}^{x_i} u^{y_i} s_{b_i}$ such that $\|h - z\| < \varepsilon/3$. Observe that since $\alpha_\chi(s_a^*) = (\chi(a)s_a)^* = \overline{\chi(a)}s_a^* = \chi^{-1}(a)s_a = \chi(a^{-1})s_a^*$, we have that

$$\alpha_\chi(z) = \sum_{i=1}^k \alpha_\chi(\lambda_i s_{a_i}^* e_{I_i}^{x_i} u^{y_i} s_{b_i}) = \sum_{i=1}^k \lambda_i \chi(b_i/a_i) s_{a_i}^* e_{I_i}^{x_i} u^{y_i} s_{b_i}.$$

By assumption there is an N such that for all $n > N$ and $i = 1, \ldots, k$,

$|\chi_n(b_i/a_i) - \chi(b_i/a_i)| < \varepsilon/(3k\|z\|)$, so

$$\begin{aligned}\|\alpha_{\chi_n}(z) - \alpha_\chi(z)\| &= \Big\|\Big(\sum_{i=1}^{k} \chi_n(b_i/a_i) - \chi(b_i/a_i)\Big)z\Big\| \\ &= \Big|\sum_{i=1}^{k} \chi_n(b_i/a_i) - \chi(b_i/a_i)\Big|\|z\| \\ &\leq \Big(\sum_{i=1}^{k} |\chi_n(b_i/a_i) - \chi(b_i/a_i)|\Big)\|z\| \\ &< k\Big(\frac{\varepsilon}{3k\|z\|}\Big)\|z\| = \varepsilon/3,\end{aligned}$$

and we have

$$\begin{aligned}\|\alpha_{\chi_n}(h) - \alpha_\chi(h)\| &= \|\alpha_{\chi_n}(h - z + z) - \alpha_\chi(h - z + z)\| \\ &\leq \|\alpha_{\chi_n}(h - z)\| + \|\alpha_\chi(h - z)\| + \|\alpha_{\chi_n}(z) - \alpha_\chi(z)\| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

Thus α_{χ_n} converges point-wise to α_χ, and the map $\chi \mapsto \alpha_\chi$ is continuous.

We can now conclude, by Proposition A.2.2, that the formula

$$\theta_\alpha(h) = \int_{\widehat{K^\times}} \alpha_\chi(h)d\chi$$

defines a faithful conditional expectation on $\mathfrak{T}[R]$.

Next, by (A.1) we know that for $c \in K^\times$

$$\int_{\widehat{K^\times}} \chi(c)d\chi = \begin{cases} 1 & \text{if } c = 1, \\ 0 & \text{otherwise,} \end{cases}$$

thus we can conclude that

$$\begin{aligned}\theta_\alpha(s_a^* e_I^x u^y s_b) &= \int_{\widehat{K^\times}} \alpha_\chi(s_a^* e_I^x u^y s_b) d\chi \\ &= \int_{\widehat{K^\times}} \chi(b/a) s_a^* e_I^x u^y s_b d\chi \\ &= \Big(\int_{\widehat{K^\times}} \chi(b/a) d\chi \Big) s_a^* e_I^x u^y s_b \\ &= \begin{cases} s_a^* e_I^x u^y s_a & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Finally, in order to prove that the range of θ_α is contained in B, we will show that

$$s_a^* e_I^x u^y s_a = \begin{cases} e^z_{\frac{1}{a}(aR\cap I)} u^{y/a} & \text{if } y \in aR \text{ and } aR \cap (x+I) \neq \varnothing, \text{ for any } az \in aR \cap (x+I), \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $s_a^* e_I^x u^y s_a$ is not zero. Then

$$s_a^* e_I^x u^y s_a = s_a^* s_a s_a^* e_I^x u^y s_a s_a^* s_a = s_a^* e_I^x (e_{aR} u^y e_{aR}) s_a,$$

and we can conclude from Td that $y \in aR$, hence $y/a \in R$, and that $u^y s_a = s_a u^{y/a}$ by Ta. Now apply Lemma 3.0.18(b) to get $s_a^* e_I^x u^y s_a = e^z_{\frac{1}{a}(aR\cap I)} u^{y/a}$. It is clear from the above argument that $s_a^* e_I^x u^y s_a = 0$ unless $y \in aR$ and $aR \cap (x+I) \neq \varnothing$. This proves the claim. □

Lemma 3.0.22. *There is a faithful conditional expectation $\theta_\beta : B \to \overline{\mathcal{D}}$ that satisfies*

$$\theta_\beta(e_I^x u^y) = \begin{cases} e_I^x & \textit{if } y = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The proof is nearly identical to that of Lemma 3.0.21. This time the dual $\widehat{R}$ of the additive group of R acts continuously on B by automorphism β_χ, $\chi \in \widehat{R}$ defined by $\beta_\chi u^x = \chi(x)u^x$ and $\beta_\chi(e_I) = e_I$. The proof that β_χ is an automorphism of B follows the same line of reasoning as the proof of (i) in Lemma 3.0.21, and uses the fact that B is the universal C^*-algebra generated by elements u^x, $x \in R$, and e_I, I a non-zero ideal of R, satisfying the relevant relation of $\mathfrak{T}[R]$ (the first part of Ta, Tb, and Td); this can be proved by a method similar to the one used in the proof of Proposition 4.1.1. One can show that $\chi \mapsto \beta_\chi$ is continuous using a similar argument as the one in (ii) of the proof of Lemma 3.0.21. Thus by Proposition A.2.2, the formula

$$\theta_\beta(h) = \int_{\widehat{R}} \beta_\chi(h) d\chi$$

defines a faithful conditional expectation on B. Since, by (A.1),

$$\int_{\widehat{R}} \chi(y) d\chi = \begin{cases} 1 & \text{if } y = 0, \\ 0 & \text{otherwise,} \end{cases}$$

we have that

$$\begin{aligned} \theta_\beta(e_I^x u^y) &= \int_{\widehat{R}} \beta_\chi(e_I^x u^y) d\chi \\ &= \int_{\widehat{R}} \chi(y) e_I^x u^y d\chi \\ &= \Big(\int_{\widehat{R}} \chi(y) d\chi \Big) e_I^x u^y \\ &= \begin{cases} e_I^x u^y & \text{if } y = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

It follows that the range of θ_β is $\overline{\mathcal{D}}$. □

Lemma 3.0.23. *The composition $\theta_\beta \circ \theta_\alpha : \mathfrak{T}[R] \to \overline{\mathcal{D}}$ is a faithful conditional expectation E that satisfies*

$$E(s_a^* e_I^x u^y s_b) = \begin{cases} s_a^* e_I^x s_a & \text{if } a = b \text{ and } y = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover $s_a^* e_I^x s_a \in \overline{\mathcal{D}}$.

Proof. It is not hard to see that composition $\theta_\beta \circ \theta_\alpha$ is a conditional expectation that satisfies the formula. By Lemma 3.0.18(b), $s_a^* e_I^x s_a \in \overline{\mathcal{D}}$. □

Theorem 3.0.24. *Let A be a C^*-algebra that contains elements U^x, $x \in R$, S_a, $a \in R^\times$, and E_I, I a non-zero ideal of R, that satisfy Ta-Td, and define $E_I^x = U^x E_I U^{-x}$. Then the homomorphism $\varphi : \mathfrak{T}[R] \to A$ that extends the map*

$$u^x \mapsto U^x, \quad s_a \mapsto S_a, \quad e_I \mapsto E_I,$$

is injective if

(i) φ is injective on $\overline{\mathcal{D}}$, and

(ii) $\| \sum_{i=1}^N \lambda_i E_{I_i}^{x_i} \| \leq \| \sum_{i=1}^N \lambda_i E_{I_i}^{x_i} + \sum_{i=1}^M \lambda_i' S_{a_i}^* E_{J_i}^{y_i} U^{z_i} S_{b_i} \|$

where $a_i \neq b_i$ or $z_i \neq 0$.

Remark 3.0.25. This is our first major step toward proving Theorem 1.2.5. In the next chapter we will see that (i) is true if and only if A satisfies the injectivity condition of Theorem 1.2.5 (see Proposition 4.1.7), and that (ii) is always true (see Proposition 4.2.4). Together, these three result will prove Theorem 1.2.5.

Proof. Let h in $\mathfrak{T}[R]$ be positive such that $\varphi(h) = 0$, and let $\varepsilon > 0$. Since the span of elements of the form $s_a^* e_I^x u^y s_b$ is dense in $\mathfrak{T}[R]$, we may find $z = \sum_{i=1}^N \lambda_i e_{I_i}^{x_i} +$

$\sum_{i=1}^{M} \lambda_i' s_{a_i}^* e_{J_i}^{y_i} u^{z_i} s_{b_i}$ with $a_i \neq b_i$ or $y_i \neq 0$ such that $\|h - z\| < \varepsilon$. Then

$$\begin{aligned}
\|E(z)\| &= \Big\| \sum_{i=1}^{N} \lambda_i e_{I_i}^{x_i} \Big\| && \text{by Lemma 3.0.23} \\
&= \Big\| \varphi\Big(\sum_{i=1}^{N} \lambda_i e_{I_i}^{x_i} \Big) \Big\| && \text{since } \varphi \text{ is injective on } \overline{\mathcal{D}} \\
&\leq \|\varphi(z)\| && \text{by (ii)} \\
&\leq \|\varphi(z - h)\| + \|\varphi(h)\| \\
&\leq \|z - h\| + 0 < \varepsilon,
\end{aligned}$$

and

$$\|E(h)\| = \|E(h - z) + E(z)\| \leq \|E(h - z)\| + \|E(z)\| < 2\varepsilon.$$

We can conclude that $E(h) = 0$, and because E is faithful on positive elements, h must also be zero. $\square$

Chapter 4

The $*$-Subalgebra $\overline{\mathcal{D}}$

4.1 The Condition of Being Proper

In this chapter we will prove our main theorem, which we will now restate:

Theorem 1.2.5. *Let A be a C^*-algebra that contains elements U^x, $x \in R$, S_a, $a \in R^\times$, and E_I, I a non-zero ideal of R, that satisfy Ta-Td. Then the map*

$$u^x \mapsto U^x, \quad s_a \mapsto S_a, \quad e_I \mapsto E_I,$$

extends to a homomorphism $\varphi : \mathfrak{T}[R] \to A$. For each prime ideal P in R and non-negative integer t, we define the projection $\delta_{(t,P)} = (1 - \sum_{x \in R/P^{t+1}} U^x E_{P^{t+1}} U^{-x}) E_{P^t}$. Then the homomorphism φ is injective if and only if all projections of the form $\delta_{(t_1,P_1)} \delta_{(t_2,P_2)} \cdots \delta_{(t_n,P_n)}$, with $P_1, P_2 \ldots, P_n$ distinct, are non-zero.

In the last chapter we saw in Theorem 3.0.24 that a homomorphism $\varphi : \mathfrak{T}[R] \to A$ is injective if

(i) φ is injective on $\overline{\mathcal{D}}$, and

(ii) $\| \sum_{i=1}^{N} \lambda_i E_{I_i}^{x_i} \| \leq \| \sum_{i=1}^{N} \lambda_i E_{I_i}^{x_i} + \sum_{i=1}^{M} \lambda_i' S_{a_i}^* E_{J_i}^{y_i} U^{z_i} S_{b_i} \|$

where $a_i \neq b_i$ or $z_i \neq 0$. To prove Theorem 1.2.5 we must show that the injectivity condition implies (i) and (ii). The first implication will be the focus of this section, while the second will be proved in the next section.

To simplify matters, we will study homomorphisms of $\overline{\mathcal{D}}$ rather than restrictions of homomorphisms of $\mathfrak{T}[R]$. Our strategy is the same as our strategy for studying homomorphisms of $\mathfrak{T}_{R \rtimes R^\times}$; we will find a universal C^*-algebra that is isomorphic to $\overline{\mathcal{D}}$. The difference will be that rather than defining the universal C^*-algebra using relations on generators, we will prove that the concrete C^*-algebra generated by the projections $e_{(x,I)}$ on $\ell^2(R \rtimes R^\times)$ defined on the basis as

$$e_{(x,I)}(\xi_{(y,b)}) = \begin{cases} \xi_{(y,b)} & \text{if } y + bR \subseteq x + I, \\ 0 & \text{otherwise.} \end{cases}$$

satisfies the universal property in Proposition 4.1.1. This has two benefits. The first is that some calculations will be simplified when we have a Hilbert space to act on. The second is that once we show $e_{(x,I)} \in \mathfrak{T}_{R \rtimes R^\times}$, and that $\mathfrak{T}_{R \rtimes R^\times}$ is a realization of the relations Ta-Td, we will get an isomorphism from $\overline{\mathcal{D}}$ to $C^*\{e_{(x,I)}\}$ for free.

Proposition 4.1.1. *Suppose B is a C^*-algebra that contains projections E_I^x with $x \in R$ and I a non-zero ideal of R, satisfying the following two relations*

(Pa) $E_R^0 = 1$, and

(Pb) the projections multiply according to the rule

$$E_I^x E_J^y = \begin{cases} 0 & \text{if } (x+I) \cap (y+J) = \varnothing, \\ E_{I \cap J}^z & \text{if } (x+I) \cap (y+J) \neq \varnothing, \text{ for any } z \in (x+I) \cap (y+J). \end{cases} \tag{4.1}$$

Then the map $e_{(x,I)} \mapsto E_I^x$ extends to a homomorphism $\psi : C^*\{e_{(x,I)}\} \to B$. For *each prime ideal P of R, $y \in R$, and non-negative integer t, we define the projection $\delta^y_{(t,P)} = (1 - \sum_{z \in R/P^{t+1}} E^z_{P^t})E^y_{P^t}$. Then the homomorphism ψ is injective if and only if*

$$\delta^y_{(t_1,P_1)}\delta^y_{(t_2,P_2)} \cdots \delta^y_{(t_n,P_n)} \neq 0 \tag{4.2}$$

whenever with $P_1, \ldots, P_n$ are distinct prime ideals of R, $y \in R$, and $t_1, \ldots, t_n$ are non-negative integers.

Definition 4.1.2. Any collection $\{E^x_I\}$ of projections in a C^*-algebra that satisfies the two relations in the proposition will be called *proper* if it also satisfies the injectivity condition (4.2). We will use the notation $F_I = \sum_{x \in R/I} E^x_I$.

The first step toward proving Proposition 4.1.1 is to show that the collection $\{e_{(x,I)}\}$ satisfies (Pb) (it is clear that it satisfies relation (Pa)) and that it is proper.

Lemma 4.1.3. *(a) Let $x, y \in R$ and let I and J be non-zero ideals in R. Then*

$$e_{(x,I)}e_{(y,J)} = \begin{cases} 0 & \text{if } (x+I) \cap (y+J) = \varnothing, \\ e_{(z,I\cap J)} & \text{if } (x+I) \cap (y+J) \neq \varnothing, \text{ for any } z \in (x+I) \cap (y+J). \end{cases}$$

(b) The condition $y + bR \subseteq x + I$ is equivalent to $(y - x), b \in I$.

(c) Let $P_1, \ldots, P_n$ be distinct prime ideals in R, $x \in R$, and $t_1, \ldots, t_n$ be non-negative integers. Then

$$\delta^x_{(t_1,P_1)} \cdots \delta^x_{(t_n,P_n)}(\xi_{(y,b)}) = \begin{cases} \xi_{(y,b)} & \text{if } y - x \in P_i^{t_i} \text{ and } b \in P_i^{t_i} \setminus P_i^{t_i+1} \text{for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

(d) The collection $\{e_{(x,I)}\}$ is proper.

Proof. (a) It is clear that if $\xi_{(w,d)}$ is an element of the standard orthonormal basis of $\ell^2(R \rtimes R^\times)$, then

$$e_{(x,I)}e_{(y,J)}(\xi_{(w,d)}) = \begin{cases} \xi_{(w,d)} & \text{if } w + dR \subseteq (x+I) \cap (y+J), \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $(x+I) \cap (y+J) \neq \varnothing$ and let $z \in (x+I) \cap (y+J)$. To prove the multiplication relation 2. we simply need to show that $z + (I \cap J) = (x+I) \cap (y+J)$.

We can easily see that $z+(I \cap J) \subseteq (x+I)\cap(y+J)$ by observing that $z \in (x+I)$ and $I \cap J \subseteq I$ implies that $z + (I \cap J) \subseteq (x+I)$, and that $z \in (y+J)$ together with $I \cap J \subseteq J$ implies that $z + (I \cap J) \subseteq (y+J)$.

To prove the reverse inclusion let $z, z' \in (x+I) \cap (y+J)$. Since $z, z' \in (x+I)$ implies that $z - z' \in I$, and since $z, z' \in (y+J)$ is implies that $z - z' \in J$, we have $z - z' \in I \cap J$ which is equivalent to $z' \in z + (I \cap J)$.

(b) Assume $y + bR \subseteq (x+I)$. Clearly $y \in (x+I)$, and hence $(y-x) \in I$. Using the fact that $y + bR \subseteq (x+I) = (y+I)$, it is easy to see that $b \in I$.

Conversely, if $(y-x), b \in I$, then $y + bR \subseteq (y+I) = (x+I)$.

(c) Let $P_1, \ldots, P_n$ be distinct prime ideals in R, $x \in R$, and $t_1, \ldots, t_n$ be non-negative integers, and fix i. Assume that $y - x \in P_i^{t_i}$ and $b \in P_i^{t_i} \setminus P_i^{t_i+1}$. Then by part (b), $e_{(x,P_i^{t_i})}(\xi_{(y,b)}) = \xi_{(y,b)}$ and $e_{(z,P_i^{t_i+1})}(\xi_{(y,b)}) = 0$ for all $z \in R$. It follows that $\delta^x_{(t_i,P_i)}(\xi_{(y,b)}) = \xi_{(y,b)}$. On the other hand suppose that we do not have $y - x \in P_i^{t_i}$ and $b \in P_i^{t_i} \setminus P_i^{t_i+1}$. There are two cases:

Case 1: If $y - x$ or b is not in $P_i^{t_i}$, then by part (b) we have $e_{(x,P_i^{t_i})}(\xi_{(y,b)}) = 0$.

Case 2: If b is in $P_i^{t_i+1}$, then by part (b) there is exactly one $z \in R/P_i^{t_i+1}$ for which $e_{(z,P_i^{t_i+1})}(\xi_{(y,b)}) = \xi_{(y,b)}$, and it follows that

$$\Big(1 - \sum_{z\in R/P_i^{t_i+1}} e_{(z,P_i^{t_i+1})}\Big)(\xi_{(y,b)}) = 0.$$

Since both cases imply that $\delta^x_{(t_i,P_i)}(\xi_{(y,b)}) = 0$, we have shown that

$$\delta^x_{(t_i,P_i)}(\xi_{(y,b)}) = \begin{cases} \xi_{(y,b)} & \text{if } y - x \in P_i^{t_i} \text{ and } b \in P_i^{t_i} \setminus P_i^{t_i+1}, \\ 0 & \text{otherwise,} \end{cases}$$

from which the result follows.

(d) Using the notation of (c), fix $b_i \in P_i^{t_i} \setminus P_i^{t_i+1}$. The Chinese Remainder Theorem tells us we can solve the congruence

$$b \equiv b_i \mod P_i^{t_i+1}, \; i = 1, 2, \dots, n$$

which yields a non-zero vector $\xi_{(x,b)}$ that satisfies $\delta^x_{(t_1,P_1)} \cdots \delta^x_{(t_n,P_n)}(\xi_{(x,b)}) = \xi_{(x,b)}$. Thus the collection $\{e_{(x,I)}\}$ is proper.

□

We now turn our attention to linear combinations of the projections E_I^x. Given a finite subset $\{E_{I_i}^{x_i}\}_{i\in X}$ of $\{E_I^x\}$ we can write

$$1 = \prod_{i\in X}(E_{I_i}^{x_i\perp} + E_{I_i}^{x_i}) = \sum_{Y\subseteq X}\Big(\prod_{i\in Y^c} E_{I_i}^{x_i\perp} \prod_{i\in Y} E_{I_i}^{x_i}\Big).$$

Because the $E_{I_i}^{x_i}$'s commute, the product

$$Q_Y := \prod_{i \in Y^c} E_{I_i}^{x_i \perp} \prod_{i \in Y} E_{I_i}^{x_i}$$

is a projection. Observe that $E_{I_i}^{x_i} Q_Y$ equals Q_Y if i is in Y and zero otherwise, some of the Q_Y's may be zero, and the Q_Y's are mutually orthogonal.

Any linear combination of the $E_{I_i}^{x_i}$'s can be written as a linear combination of the Q_Y's in the following way

$$\sum_{i \in X} \lambda_i E_{I_i}^{x_i} = \Big(\sum_{i \in X} \lambda_i E_{I_i}^{x_i}\Big)\Big(\sum_{Y \subseteq X : Q_Y \neq 0} Q_Y\Big) = \sum_{Y \subseteq X : Q_Y \neq 0} \Big(\sum_{i \in Y} \lambda_i\Big) Q_Y.$$

Since the Q_Y are mutually orthogonal,

$$(\mu_Y)_{Y \subseteq X : Q_Y \neq 0} \mapsto \sum_{Y \subseteq X : Q_Y \neq 0} \mu_Y Q_Y$$

is an injective $*$-homomorphism $\mathbb{C}^N \to B$ (possibly non-unital) where $N = |\{Y \subseteq X : Q_Y \neq 0\}|$. Since the spectrum of $(\sum_{i \in Y} \lambda_i)_{Y \subseteq X : Q_Y \neq 0}$ is $\{\sum_{i \in Y} \lambda_i : Y \subseteq X : Q_Y \neq 0\}$ we can conclude that the the spectrum of $\sum_{Y \subseteq X : Q_Y \neq 0}(\sum_{i \in Y} \lambda_i) Q_Y$ is also $\{\sum_{i \in Y \subseteq X} \lambda_i : Q_Y \neq 0\}$ (and possibly 0). We can then use the C^*-identity together with the fact that for self-adjoint elements the spectral radius equals the norm to get

$$\Big\| \sum_{i \in X} \lambda_i E_{I_i}^{x_i} \Big\|^2 = \Big\| \sum_{Y \subseteq X : Q_Y \neq 0} \Big(\sum_{i \in Y} \lambda_i\Big) Q_Y \Big\|^2 = \max_{Y \subseteq X : Q_Y \neq 0} \Big| \sum_{i \in Y} \lambda_i \Big|^2. \tag{4.3}$$

Using only the multiplication condition (4.1) we can identify two sufficient conditions on $Y \subseteq X$ for Q_Y to be zero. First, if $\bigcap_{i \in Y}(x_i + I_i) = \varnothing$, then $\prod_{i \in Y} E_{I_i}^{x_i} = 0$, and second, if $\bigcap_{i \in Y}(x_i + I_i) \subseteq (x_j + I_j)$ for some $j \in Y^c$, then $E_{I_j}^{x_j \perp} \prod_{i \in Y} E_{I_i}^{x_i} = 0$. This motivates the following definition.

Definition 4.1.4. Using the above notation, we call $Y \subseteq X$ *good* if the following two conditions are true:

$$\bigcap_{i \in Y}(x_i + I_i) \neq \varnothing \text{ and} \tag{i}$$

$$\bigcap_{i \in Y}(x_i + I_i) \nsubseteq (x_j + I_j),\ \forall j \in Y^c. \tag{ii}$$

We have nearly proved the following lemma.

Lemma 4.1.5. *Let $\{E_I^x\}$ be a collection of projections in a C^*-algebra that satisfy the two relations in Proposition 4.1.1. If X is a finite index set, then*

$$\Big\| \sum_{i \in X} \lambda_i E_{I_i}^{x_i} \Big\| \leq \max_{Y \subseteq X \text{ good}} \Big| \sum_{i \in Y} \lambda_i \Big|. \tag{4.4}$$

Moreover we have equality if $\{E_I^x\}$ is proper.

Proof. The first part summarizes the discussion preceding the lemma. By (4.3), to prove equality in (4.4), it is enough to show that for a proper family $\{E_I^x\}$, $Q_Y \neq 0$ when $Y \subseteq X$ is good.

Let $P_1, \dots, P_n$ be a list of distinct prime ideals that includes the prime divisors of each I_i, and suppose that $Y \subseteq X$ is good. Part (i) of the goodness condition guarantees that $\bigcap_{i \in Y}(x_i + I_i) = x + P_1^{t_1} \cdots P_n^{t_n}$ for some non-negative integers $t_1, \dots, t_n$. Let $I = P_1^{t_1} \cdots P_n^{t_n}$. We claim that

$$Q_Y = \Big(\prod_{j \in Y^c} E_{I_j}^{x_j \perp} \Big) E_{P_1^{t_1} \cdots P_n^{t_n}}^x \geq \Big(\prod_{i=1}^n F_{P_i^{t_i+1}}^\perp \Big) E_{P_1^{t_1} \cdots P_n^{t_n}}^x = \delta_{(t_1, P_1)}^x \cdots \delta_{(t_n, P_n)}^x. \tag{4.5}$$

The projections on the right hand side are equal because the collection $\{E_I^x\}$ commutes and $E_{P_1^{t_1} \cdots P_n^{t_n}}^x = E_{P_1^{t_1} \cap \cdots \cap P_n^{t_n}}^x = E_{P_1^{t_1}}^x \cdots E_{P_n^{t_n}}^x$. To prove the inequality it is enough to show that

$$E_{I_j}^{x_j\perp} \geq \Big(\prod_{i=1}^{n} F_{P_i^{t_i+1}}^{\perp}\Big) E_{P_1^{t_1}\cdots P_n^{t_n}}^{x}$$

for each $j \in Y^c$.

The second half of being good means that for each $j \in Y^c$ either $(x+I)\cap(x_j+I_j) = \varnothing$ or $I \cap I_j \neq I$. In the first case $E_{I_j}^{x_j\perp} \geq E_I^x$ and we are done. The second case implies that there is some i for which $P_i^{t_i+1}$ divides I_j. Then since there can be only one $y \in R/P_i^{t_i+1}$ for which $(x_j+I_j)\cap(y+P_i^{t_i}) \neq \varnothing$, we can conclude that $E_{I_j}^{x_j\perp} \geq F_{P_i^{t_i+1}}^{\perp}$, which proves the result. □

We are now ready to prove Proposition 4.1.1.

Proof. Observe that for all finite linear combinations

$$\Big\|\sum_{i\in X} \lambda_i E_{I_i}^{x_i}\Big\| \leq \max_{Y\subseteq X \text{ good}} \Big|\sum_{i\in Y} \lambda_i\Big| = \Big\|\sum_{i\in X} \lambda_i e_{(x_i,I_i)}\Big\|$$

where the inequality follows from Lemma 4.1.5 and the equality from combining Lemmas 4.1.3(d) and 4.1.5. Notice also that the inequality becomes equality if $\{E_I^x\}$ is proper. It follows that

$$\psi\Big(\sum_{i\in X} \lambda_i e_{(x_i,I_i)}\Big) = \sum_{i\in X} \lambda_i E_{I_i}^{x_i}$$

gives a well-defined contractive linear map on span$\{e_{(x,I)}\}$, that ψ extends to $C^*\{e_{(x,I)}\}$, and that ψ is isometric if $\{E_I^x\}$ is proper. If $\{E_I^x\}$ is not proper then ψ is obviously not injective because the kernel contains a non-zero projection of the from $\delta_{(t_1,P_1)}^{y}\cdots\delta_{(t_n,P_n)}^{y}$. □

Recall that the Toeplitz algebra $\mathfrak{T}_{R\rtimes R^\times}$ of $R \rtimes R^\times$ (defined at the beginning of Section 1.2) is the C^*-algebra generated by the isometries $T_{(x,a)} : \ell^2(R \rtimes R^\times) \to$

$\ell^2(R \rtimes R^\times)$ defined to be the continuous linear extensions of the maps $\xi_{(y,b)} \mapsto \xi_{(x,a)(y,b)} = \xi_{(x+ay,ab)}$. In Example 1.2.2 we showed that $e_{(x,a\mathbb{Z})} = T_{(x,a)}T^*_{(x,a)} \in \mathfrak{T}_{\mathbb{Z}\rtimes\mathbb{Z}^\times}$. The argument can easily be generalized to show that $e_{(x,aR)} = T_{(x,a)}T^*_{(x,a)} \in \mathfrak{T}_{R\rtimes R^\times}$, where R is an arbitrary number ring. However if I is a non-zero ideal of R which is not principal, in order to show that $e_{(x,I)} \in \mathfrak{T}_{R\rtimes R^\times}$, we have to be more subtle.

Proposition 4.1.6. *(a) Let $x \in R$ and let I be a non-zero ideal of R. Then there exists $a, b \in R^\times$ such that*

$$e_{(x,I)} = T_{(x,1)}T^*_{(0,b)}T_{(0,a)}T^*_{(0,a)}T_{(0,b)}T_{(-x,1)}.$$

(b) The elements $U^x = T_{(x,1)}$, $S_a = T_{(0,a)}$, and $e_{(0,I)}$ in $\mathfrak{T}_{R\rtimes R^\times}$, where $x \in R$, $a \in R^\times$, and I is a non-zero ideal of R, satisfy Ta-Td.

Proof. (a) Let $x \in R$ and I be a non-zero ideal in R. By Proposition 2.0.14 I can be written as $\frac{a}{b}R \cap R$ with $a, b \in R^\times$. Lemma 4.1.3(b) tells us that $e_{(x,I)}(\xi_{(y,c)}) = \xi_{(y,c)}$ if and only if $y - x, c \in I$, equivalently if there exists $z \in R$ and $d \in R^\times$ such that

$$y - x = \frac{a}{b}z \text{ and } c = \frac{a}{b}d \iff by - bx = az \text{ and } bc = ad.$$

Using the same argument as the one in Example 1.2.2, we can show that $T^*_{(0,a)}T_{(0,b)}T_{(-x,1)}(\xi_{(y,c)}) = T^*_{(0,a)}(\xi_{(by-bx,bc)}) \neq 0$ if and only if there exists $(z,d) \in R \rtimes R^\times$ such that $(by - bx, bc) = (0,a)(z,d)$. Since the same argument shows that $T^*_{(0,b)}(\xi_{(by-bx,bc)}) = \xi_{(y-x,c)}$, we can conclude that

$$T_{(x,1)}T^*_{(0,b)}T_{(0,a)}T^*_{(0,a)}T_{(0,b)}T_{(-x,1)}(\xi_{(y,c)}) = \xi_{(y,c)}$$

if and only if there exists $(z,d) \in R \rtimes R^\times$ such that $(by - bx, bc) = (0,a)(z,d)$.

This shows that

$$e_{(x,I)} = T_{(x,1)}T^*_{(0,b)}T_{(0,a)}T^*_{(0,a)}T_{(0,b)}T_{(-x,1)}.$$

(b) Ta: These follow easily from the definitions of the left-regular representation.

Tb: This just Lemma 4.1.3(a).

Tc: Because $y, b \in I$ if and only if $ay, ab \in aI$, we can use Lemma 4.1.3(b) to conclude that $S_a e_{(0,I)} = e_{(0,aI)}S_a$. Since $aI \cap aR = aI$,

$$S_a e_{(0,I)} S_a^* = e_{(0,aI)} S_a S_a^* = e_{(0,aI)} e_{(0,aR)} = e_{(0,aI)},$$

which shows that Tc holds.

Td: The first part of Td requires that for $x \in I$ we have $U^x e_{(0,I)} = e_{(0,I)} U^x$. This follows from the fact that if $x \in I$, then $y \in I$ if and only if $y + x \in I$. The second part demands that for $x \notin I$ we have $e_{(0,I)} U^x e_{(0,I)} = 0$. This follows from the fact that $e_{(0,I)} e_{(x,I)} = 0$ (i.e. $I \cap (x + I) = \varnothing$) when $x \notin I$.

□

Proposition 4.1.7. *Let A be a C^*-algebra that contains elements U^x, $x \in R$, S_a, $a \in R^\times$, and E_I, I a non-zero ideal of R, that satisfy Ta-Td. Let $E_I^x = U^x E_I U^{-x}$, and for each prime ideal P in R and non-negative integer t define the projection $\delta_{(t,P)} = (1 - \sum_{x \in R/P^{t+1}} E^x_{P^{t+1}})E_{P^t}$. Then the homomorphism $\varphi : \mathfrak{T}[R] \to A$ that extends the map*

$$u^x \mapsto U^x, \quad s_a \mapsto S_a, \quad e_I \mapsto E_I,$$

is injective on $\overline{\mathcal{D}}$ if and only if all projections of the form $\delta_{(t_1,P_1)}\delta_{(t_2,P_2)}\cdots\delta_{(t_n,P_n)}$, with $P_1, P_2 \ldots, P_n$ distinct, are non-zero.

Proof. Proposition 4.1.6 gives us a homomorphism from $\overline{\mathcal{D}} \subset \mathfrak{T}[R]$ onto $C^*\{e_{(x,I)}\}$ which extends the map $e_I^x \mapsto e_{(x,I)}$. This shows that the homomorphism in Proposition 4.1.1 from $C^*\{e_{(x,I)}\}$ onto $\overline{\mathcal{D}}$ which extends the map $e_{(x,I)} \mapsto e_I^x$ has an inverse, from which we can conclude $\overline{\mathcal{D}}$ is isomorphic to $C^*\{e_{(x,I)}\}$, and that $\overline{\mathcal{D}}$ satisfies the universal property of Proposition 4.1.1. If we show that $\delta_{(t_1,P_1)} \cdots \delta_{(t_n,P_n)} \neq 0$ implies that $\delta^y_{(t_1,P_1)} \cdots \delta^y_{(t_n,P_n)} \neq 0$ for all y in R, we can apply Proposition 4.1.1 to prove the result.

If I is a non-zero ideal of R, then for all y in R the sets $\{E_I^x\}_{x\in R/I}$ and $\{U^y E_I^x U^{-y}\}_{x\in R/I}$ are equal, thus $F_I = U^y F_I U^{-y}$, and we have

$$\begin{aligned}
U^y \delta_{(t_i,P_i)} U^{-y} &= U^y (1 - F_{P_i^{t_i+1}}) E_{P_i^{t_i}} U^{-y} \\
&= (1 - U^y F_{P_i^{t_i+1}} U^{-y}) U^y E_{P_i^{t_i}} U^{-y} \\
&= (1 - F_{P_i^{t_i+1}}) E^y_{P_i^{t_i}} \\
&= \delta^y_{(t_i,P_i)}.
\end{aligned}$$

Then

$$\begin{aligned}
U^y \delta_{(t_1,P_1)} \cdots \delta_{(t_n,P_n)} U^{-y} &= (U^y \delta_{(t_1,P_1)} U^{-y}) U^y \cdots U^{-y} (U^y \delta_{(t_n,P_n)} U^{-y}) \\
&= \delta^y_{(t_1,P_1)} \cdots \delta^y_{(t_n,P_n)},
\end{aligned}$$

which proves the result. □

4.2 The Second Assumption

Our final task in proving Theorem 1.2.5 is to show that if A is a C^*-algebra that contains elements U^x, S_a, and E_I, with $x \in R$, $a \in R^\times$, and I is a non-zero ideal of

R, that satisfy Ta-Td, then the inequality

$$\Big\| \sum_{i=1}^{N} \lambda_i E_{I_i}^{x_i} \Big\| \leq \Big\| \sum_{i=1}^{N} \lambda_i E_{I_i}^{x_i} + \sum_{i=1}^{M} \lambda_i' S_{a_i}^* E_{J_i}^{y_i} U^{z_i} S_{b_i} \Big\|$$

where $a_i \neq b_i$ or $y_i \neq 0$ and $\lambda_i, \lambda_i' \in \mathbb{C}$, holds when $\{E_I^x\}$ is proper. Given such a linear combination, our strategy is to find a projection $\delta = \delta_{(t_1,P_1)}^x \cdots \delta_{(t_n,P_n)}^x$ that satisfies

$$\Big\| \delta \Big(\sum_{i=1}^{N} \lambda_i E_{I_i}^{x_i} + \sum_{i=1}^{M} \lambda_i' S_{a_i}^* E_{J_i}^{y_i} U^{z_i} S_{b_i} \Big) \delta \Big\| = \Big\| \sum_{i=1}^{N} \lambda_i E_{I_i}^{x_i} \Big\|.$$

Lemma 4.2.1. *Let A be a C^*-algebra that contains elements U^x, $x \in R$, S_a, $a \in R^\times$, and E_I, I a non-zero ideal of R, that satisfy Ta-Td, and define $E_I^x = U^x E_I U^{-x}$. Let $z \in R$ and suppose $a, b \in R^\times$ satisfy $aR \neq bR$. Let P be a prime ideal in R whose exponent in the prime power decomposition of aR differs from its exponent in the prime power decomposition of bR. Then for any non-negative integer t and $x \in R$*

$$\delta_{(t,P)}^x S_a^* U^z S_b \delta_{(t,P)}^x = 0.$$

Proof. By Lemma 3.0.18(a), $E_{P^t}^x S_a^* = S_a^* E_{aP^t}^{ax}$ and $S_b E_{P^t}^x = E_{bP^t}^{bx} S_b$, thus we can write

$$\begin{aligned} E_{P^t}^x S_a^* U^z S_b E_{P^t}^x &= S_a^* E_{aP^t}^{ax} U^z E_{bP^t}^{bx} S_b \\ &= S_a^* U^{ax} E_{aP^t} U^{z-ax+bx} E_{bP^t} U^{-bx} S_b. \end{aligned}$$

If $E_{aP^t} U^{z-ax+bx} E_{bP^t} \neq 0$, then by Lemma 3.0.15 we can find $ax_1 \in aP^t$ and $bx_2 \in bP^t$ such that $z - ax + bx = ax_1 - bx_2$ and

$$\begin{aligned} S_a^* U^{ax} E_{aP^t} U^{z-ax+bx} E_{bP^t} U^{-bx} S_b &= S_a^* U^{ax} U^{ax_1} E_{aP^t \cap bP^t} U^{-bx_2} U^{-bx} S_b \\ &= S_a^* U^{a(x+x_1)} E_{aP^t \cap bP^t} U^{-b(x_2+x)} S_b. \end{aligned}$$

Write $aR = P^m I$ and $bR = P^n J$ where I and J are relatively prime to P, and assume without loss of generality that $m > n$. Then $aP^t \cap bP^t = IP^{t+m} \cap JP^{t+n} = IP^{t+m} \cap JP^{t+m}$ and

$$\begin{aligned} S_a^* U^{a(x+x_1)} E_{aP^t \cap bP^t} U^{-b(x_2+x)} S_b &= S_a^* U^{a(x+x_1)} E_{IP^{t+m}} E_{JP^{t+m}} U^{-b(x_2+x)} S_b \\ &= S_a^* U^{a(x+x_1)} E_{IP^{t+m}} U^{-b(x+x_2)} U^{b(x+x_2)} E_{JP^{t+m}} U^{-b(x_2+x)} S_b \\ &= S_a^* U^{a(x+x_1)} E_{IP^{t+m}} U^{-b(x+x_2)} E^{b(x_2+x)}_{JP^{t+m}} S_b. \end{aligned}$$

Since $t + m - n \geq t + 1$, we can write $JP^{t+m} = bP^{t+m-n}$ and apply Lemma 3.0.18(a) to get

$$S_a^* U^{a(x+x_1)} E_{IP^{t+m}} U^{-b(x+x_2)} E^{b(x_2+x)}_{JP^{t+m}} S_b = S_a^* U^{a(x+x_1)} E_{IP^{t+m}} U^{-b(x+x_2)} S_b E^{x_2+x}_{P^{t+m-n}}.$$

We have shown

$$\delta^x_{(t,P)} S_a^* U^z S_b \delta^x_{(t,P)} = \Big(1 - \sum_{y \in R/P^{t+1}} E^y_{P^{t+1}}\Big) E^x_{P^t} S_a^* U^z S_b E^x_{P^t} \Big(1 - \sum_{y \in R/P^{t+1}} E^y_{P^{t+1}}\Big)$$

$$= \Big(1 - \sum_{y \in R/P^{t+1}} E^y_{P^{t+1}}\Big) S_a^* U^{a(x+x_1)} E_{IP^{t+m}} U^{-b(x+x_2)} S_b \left(E^{x_2+x}_{P^{t+m-n}} \Big(1 - \sum_{y \in R/P^{t+1}} E^y_{P^{t+1}}\Big)\right).$$

But $t + m - n \geq t + 1$, so

$$E^{x_2+x}_{P^{t+m-n}} \Big(1 - \sum_{y \in R/P^{t+1}} E^y_{P^{t+1}}\Big) = 0$$

and it follows that $\delta^x_{(t,P)} S_a^* U^z S_b \delta^x_{(t,P)} = 0$. □

Lemma 4.2.2. *Let A be a C^*-algebra that contains elements U^x, $x \in R$, S_a, $a \in R^\times$, and E_I, I a non-zero ideal of R, that satisfy Ta-Td, and define $E^x_I = U^x E_I U^{-x}$. Let $x, z \in R$, $a, b \in R^\times$, and P be a prime ideal of R. Suppose that $aR = bR$, P is* *relatively prime to aR, and $z + x(b - a) \notin P$. Then for all $t \geq 1$*

$$\delta^x_{(t,P)} S^*_a U^z S_b \delta^x_{(t,P)} = 0.$$

Proof. It is enough to show that $E^x_{P^t} S^*_a U^z S_b E^x_{P^t} = 0$. By Lemma 3.0.18(a) and the fact that $aR = bR$ is relatively prime to P we have $E^x_{P^t} S^*_a = S^*_a E^{ax}_{aR} E^{ax}_{P^t}$ and $S_b E^x_{P^t} = E^{bx}_{P^t} E^{bx}_{bR} S_b$, thus we can write

$$\begin{aligned} E^x_{P^t} S^*_a U^z S_b E^x_{P^t} &= S^*_a E^{ax}_{aR} E^{ax}_{P^t} U^z E^{bx}_{P^t} E^{bx}_{bR} S_b \\ &= S^*_a E^{ax}_{aR} U^{ax} E_{P^t} U^{-ax} U^z U^{bx} E_{P^t} U^{-bx} E^{bx}_{bR} S_b \\ &= S^*_a E^{ax}_{aR} U^{ax} \Big(E_{P^t} U^{z+x(b-a)} E_{P^t} \Big) U^{-bx} E^{bx}_{bR} S_b. \end{aligned}$$

Since, by assumption, $z - x(b-a) \notin P^t \subseteq P$, Lemma 3.0.15 tells us that $E_{P^t} U^{z+x(b-a)} E_{P^t}$, and hence $\delta^x_{(t,P)} S^*_a U^z S_b \delta^x_{(t,P)}$ equals zero. □

Lemma 4.2.3. *Let A be a C^*-algebra that contains elements U^x, $x \in R$, S_a, $a \in R^\times$, and E_I, I a non-zero ideal of R, that satisfy Ta-Td, and define $E^x_I = U^x E_I U^{-x}$. Given a linear combination of the form*

$$\sum_{i=1}^{N} \lambda_i E^{x_i}_{I_i} + \sum_{i=1}^{M} \lambda'_i S^*_{a_i} E^{y_i}_{J_i} U^{z_i} S_{b_i}$$

with $a_i \neq b_i$ or $z_i \neq 0$, we can find $\delta = \delta^x_{(t_1,P_1)} \cdots \delta^x_{(t_n,P_n)}$ such that

(i) $\|\delta(\sum_{i=1}^N \lambda_i E^{x_i}_{I_i})\delta\| = \|\sum_{i=1}^N \lambda_i E^{x_i}_{I_i}\|$, *and*

(ii) $\delta(S^*_{a_i} E^{y_i}_{J_i} U^{z_i} S_{b_i})\delta = 0$, *for $i = 1, 2, \ldots M$.*

Proof. Since $\{E^x_I\}$ is proper, we know from Lemma 4.1.5 that

$$\Big\| \sum_{i \in X} \lambda_i E^{x_i}_{I_i} \Big\| = \max_{Y \subseteq X \text{ good}} \Big| \sum_{j \in Y} \lambda_j \Big|,$$

Let Y_0 be a good subset of X that maximizes $|\sum_{j\in Y_0}\lambda_j|$, and recall that $Q_{Y_0} = \prod_{j\in Y_0^c} E_{I_j}^{x_j\perp}\prod_{j\in Y_o} E_{I_j}^{x_j}$ satisfies

$$\Big(\sum_{i=1}^{N}\lambda_i E_{I_i}^{x_i}\Big)Q_{Y_0} = \Big(\sum_{i=1}^{N}\lambda_i\Big)Q_{Y_0}.$$

In order to meet the first requirement, δ will be a non-zero projection such that $Q_{Y_0}\geq\delta$, then

$$\Big\|\delta\Big(\sum_{i=1}^{N}\lambda_i E_{I_i}^{x_i}\Big)\delta\Big\| = \Big\|\Big(\sum_{j\in Y_0}\lambda_j\Big)\delta\Big\| = \Big|\sum_{j\in Y_0}\lambda_j\Big| = \max_{Y\subseteq X\text{ good}}\Big|\sum_{j\in Y}\lambda_j\Big| = \Big\|\sum_{i=1}^{N}\lambda_i E_{I_i}^{x_i}\Big\|.$$

By (4.5) and since $\{E_I^x\}$ is proper, we can find a non-zero projection of the from $\delta_{(t_1,P_1)}^{x'}\cdots\delta_{(t_k,P_k)}^{x'}$ that bounds Q_{Y_0} from below. Although it satisfies (i), the projection $\delta_{(t_1,P_1)}^{x'}\cdots\delta_{(t_k,P_k)}^{x'}$ projection might not satisfy (ii), in which case we can refine it to obtain δ.

1. First, since either $b_i - a_i\neq 0$ or $z_i\neq 0$, we can find an $x\in x' + P_1^{t_1}\cdots P_k^{t_k}$ such that $z_i + x(b_i - a_i)\neq 0$ for $1\leq i\leq M$.

2. Next, let $P_{k+1},\dots,P_{n-1}$ be the distinct prime divisors of the a_iR's and b_iR's that do not appear among $P_1,\dots,P_k$, and let $t_{k+1}=\cdots=t_{n-1}=1$.

3. Finally choose any prime P_n, distinct from $P_1,\cdots,P_{n-1}$, that does not divide any $(z_i+x(b_i-a_i))R$; such a prime will not contain $z_i+x(b_i-a_i)$ for $1\leq i\leq M$. Let $t_n = 1$.

Let $\delta = \delta_{(t_1,P_1)}^{x}\cdots\delta_{(t_n,P_n)}^{x}$. Since $P_1,\dots,P_n$ are distinct, and since $x\in x'+P_1^{t_1}\cdots P_k^{t_k}$,

$$Q_{Y_0}\geq\delta_{(t_1,P_1)}^{x'}\cdots\delta_{(t_k,P_k)}^{x'}\geq\delta\neq 0,$$

thus δ meets the first requirement.

To prove that δ satisfies the second requirement it will be enough to show that for each $1 \leq i \leq M$ the projections $S_{a_i}\delta S_{a_i}^*$ and $U^{z_i}S_{b_i}\delta S_{b_i}^* U^{-z_i}$ are orthogonal. Fix i. There are two cases:

case 1: If $a_iR \neq b_iR$, then by 2., there is a prime P_j among $P_1, \dots P_{n-1}$ whose exponent in the prime factorization of a_iR is different from its exponent in the prime factorization of b_iR. By Lemma 4.2.1, the projection $S_{a_i}\delta^x_{(t_j,P_j)}S_{a_i}^*$ is orthogonal to $U^{z_i}S_{b_i}\delta^x_{(t_j,P_j)}S_{b_i}^*U^{-z_i}$, and it follows that the projections $S_{a_i}\delta S_{a_i}^*$ and $U^{z_i}S_{b_i}\delta S_{b_i}^*U^{-z_i}$ are orthogonal.

case 2: If $a_iR = b_iR$, then by Lemma 4.2.2, the projection $S_{a_i}\delta^x_{(t_n,P_n)}S_{a_i}^*$ is orthogonal to $U^{z_i}S_{b_i}\delta^x_{(t_n,P_n)}S_{b_i}^*U^{-z_i}$, and it follows that the projections $S_{a_i}\delta S_{a_i}^*$ and $U^{z_i}S_{b_i}\delta S_{b_i}^*U^{-z_i}$ are orthogonal.

□

Proposition 4.2.4. *Let A be a C^*-algebra that contains elements U^x, $x \in R$, S_a, $a \in R^\times$, and E_I, I a non-zero ideal of R, that satisfy Ta-Td, and define $E_I^x = U^xE_IU^{-x}$. If $\{E_I^x\}$ is proper, then the inequality $\|\sum_{i=1}^N \lambda_i E_{I_i}^{x_i}\| \leq \|\sum_{i=1}^N \lambda_i E_{I_i}^{x_i} + \sum_{i=1}^M \lambda_i' S_{a_i}^* E_{J_i}^{y_i} U^{z_i} S_{b_i}\|$ where $a_i \neq b_i$ or $z_i \neq 0$, holds.*

Proof. By Lemma 4.2.3 we can find a $\delta \in A$ such that

$$\Big\|\delta\Big(\sum_{i=1}^N \lambda_i E_{I_i}^{x_i} + \sum_{i=1}^M \lambda_i' S_{a_i}^* E_{J_i}^{y_i} U^{z_i} S_{b_i}\Big)\delta\Big\| = \Big\|\sum_{i=1}^N \lambda_i E_{I_i}^{x_i}\Big\|.$$

□

This completes the proof of Theorem 1.2.5, which we will now apply to $\mathfrak{T}_{R \rtimes R^\times}$.

Proposition 4.2.5. *The Toeplitz algebra $\mathfrak{T}_{R\rtimes R^\times}$ of $R \rtimes R^\times$ is isomorphic to $\mathfrak{T}[R]$.*

Proof. By Proposition 4.1.6, there is a homomorphism $\varphi : \mathfrak{T}[R] \to \mathfrak{T}_{R \rtimes R^\times}$ that extends the map

$$u^x \mapsto T_{(x,1)}, \quad s_a \mapsto T_{(0,a)}, \quad e_I \mapsto e_{(0,I)},$$

where $x \in R$, $a \in R^\times$, and I is non-zero ideal in R. This homomorphism is surjective because $T_{(x,1)}$ and $T_{(0,a)}$ generate $\mathfrak{T}_{R \rtimes R^\times}$. By Lemma 4.1.3(d) the collection $\{e_{(x,I)}\}$ is proper, and we can conclude from Theorem 1.2.5 that φ is injective. Thus $\mathfrak{T}_{R \rtimes R^\times}$ and $\mathfrak{T}[R]$ are isomorphic. □

Chapter 5

Conclusion

Our goal was to find homomorphisms from the Toeplitz algebra of $R \rtimes R^\times$ to other C^*-algebras, and characterize the faithful ones. To that end we defined the universal C^*-algebra $\mathfrak{T}[R]$ generated by isometries and projections satisfying the relations of Ta-Td. The majority of these relations were needed to ensure that the $e_I^x \in \mathfrak{T}[R]$ recovered the multiplicative structure and decomposition of the projections $e_{(x,I)} \in \mathfrak{T}_{R \rtimes R^\times}$. Theorem 1.2.5 stated that a homomorphism φ from $\mathfrak{T}[R]$ to a C^*-algebra A is faithful if and only if the image of the set $\{e_I^x\}$ was proper. To prove this we first did as much as we could using only $\mathfrak{T}[R]$, and were able to prove Theorem 3.0.24, which was a weaker version of Theorem 1.2.5 that told us that if we assumed that φ was faithful on $\overline{\mathcal{D}}$ and that A satisfied a condition on the norms of certain elements, then φ was faithful on all of $\mathfrak{T}[R]$. Next we used $\mathfrak{T}_{R \rtimes R^\times}$ to show that the first assumption was equivalent to the image of $\{e_I^x\}$ being proper. We then completing the proof of Theorem 1.2.5 by showing that the norm condition on A was implied if the image of $\{e_I^x\}$ was proper. After showing that $\mathfrak{T}[R] \cong \mathfrak{T}_{R \rtimes R^\times}$, our theorem gave us a convenient way of identifying representations of $\mathfrak{T}_{R \rtimes R^\times}$, and a straightforward way to check when they are faithful.

Appendix A

Faithful Conditional Expectations

A.1 The Dual of a Group

Let G be an abelian group (written multiplicatively) with the discrete topology. For our purposes, we will be concerned with two specific cases: the multiplicative group of a number field $(K^\times, \cdot)$ and the additive group of a number ring $(R, +)$. A character of G is a group homomorphism (automatically continuous) χ from G to the circle group $\mathbb{T}$. The set of all characters of G is a group $\widehat{G}$ under the operation of pointwise multiplication. The topology on $\widehat{G}$ is that of uniform convergence on compact sets. Since G was assumed to be discrete, $\widehat{G}$ is a compact Hausdorff space. We use Haar measure μ to make $\widehat{G}$ into a measure space, normalized to have $\mu(\widehat{G}) = 1$. The defining property of Haar measure is that it is translation invariant, that is if E is a measurable subset of G, then $\mu(gE) = \mu(E)$.

Lemma A.1.1. *Let A and B be subgroups of a discrete abelian group G. Suppose $\alpha \in \widehat{A}$ and $\beta \in \widehat{B}$ agree on $A \cap B$. Then $\chi(ab) := \alpha(a)\beta(b)$ defines a character on the subgroup $A \cdot B = \{ab : a \in A, b \in B\}$ of G.*

Remark A.1.2. Since G is abelian, $A \cdot B$ defines a subgroup.

Proof. First, to see that χ is well-defined suppose $a_1b_1 = a_2b_2$ where $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Then $a_1a_2^{-1} = b_2b_1^{-1} \in A \cap B$, and by assumption we have $\alpha(a_1a_2^{-1}) = \beta(b_2b_1^{-1})$. Thus

$$\chi(a_1b_1) = \alpha(a_1)\beta(b_1) = \alpha(a_2)\beta(b_2) = \chi(a_2b_2),$$

and χ is well-defined. Next, χ is a (continuous) homomorphism because

$$\begin{aligned}\chi(a_1b_1a_2b_2) = \chi(a_1a_2b_1b_2) = \alpha(a_1a_2)\beta(b_1b_2) &= \alpha(a_1)\alpha(a_2)\beta(b_1)\beta(b_2) \\ &= \alpha(a_1)\beta(b_1)\alpha(a_2)\beta(b_2) = \chi(a_1b_1)\chi(a_2b_2).\end{aligned}$$

□

Lemma A.1.3. *Let A and B be subgroups of a discrete abelian group G and let β be a character on B. If A is cyclic, then β can be extended to a character on $A \cdot B$.*

Proof. By the preceding lemma, we need to show there is a character α on A which agrees with β on $A \cap B$. We will extend $\beta|_{A\cap B}$ from a character on $A \cap B$ to a character α on A.

If $A \cap B = \{1\}$, then any character α on A extends $\beta|_{A\cap B}$, and we are done. Otherwise write $A = \langle a \rangle$, and suppose n is the least positive integer for which $a^n \in B$. Then $\alpha(a^k) = \beta|_{A\cap B}(a^n)^{\frac{k}{n}}$ is a character on A which extends $\beta|_{A\cap B}$ and we are done. □

Lemma A.1.4. *Let H be a subgroup of a discrete abelian group G. Then any character on H can be extended to a character on G.*

Proof. Let G be an abelian group and let χ be a character defined on H. By a simple Zorn's lemma argument, there is a maximal extension $\overline{\chi}$ of χ defined on a subgroup $\overline{H}$ of G. If $\overline{H} \neq G$, then there is an $a \in G \setminus \overline{H}$. But, by the preceding lemma, we can extend $\overline{\chi}$ to $\langle a \rangle \cdot \overline{H}$, contradicting maximality. □

Proposition A.1.5. *Let g be an element of a discrete abelian group G. Then there exists $\widehat{\chi} \in \widehat{G}$ such that $\widehat{\chi}(g) \neq 1$.*

Proof. If $|\langle g \rangle| = n$, let $z \in \mathbb{T}$ be a primitive nth rood of unity, otherwise if $\langle g \rangle$ is infinite, let $z \in \mathbb{T}$ be anything but 1. Then $\chi(g^k) = z^k$ defines a character on $\langle g \rangle$ for which $\chi(g) \neq 1$, which by the above lemma can be extended to a character $\widehat{\chi}$ on G. □

To each element g of G we can associate a continuous function $\widehat{g} : \widehat{G} \to \mathbb{T}$ which evaluates characters at g, $\widehat{g}(\chi) = \chi(g)$. Consider the integral

$$\int_{\widehat{G}} \chi(g) d\chi = \int_{\widehat{G}} \widehat{g}(\chi) d\chi.$$

If g is the identity, then since $\mu(\widehat{G}) = 1$,

$$\int_{\widehat{G}} \chi(g) d\chi = \int_{\widehat{G}} 1 d\chi = 1.$$

On the other hand, suppose that g is not the identity. Let χ_0 be some character for which $\chi_0(g) \neq 1$. Then using the fact that Haar measure is left-translation invariant we see that

$$\chi_0(g) \int_{\widehat{G}} \chi(g) d\chi = \int_{\widehat{G}} \chi_0(g)\chi(g) d\chi = \int_{\widehat{G}} \chi_0\chi(g) d\chi = \int_{\widehat{G}} \chi(g) d\chi,$$

which shows that $\int_{\widehat{G}} \chi(g) d\chi = 0$. Summarizing we get

$$\int_{\widehat{G}} \chi(g) d\chi = \begin{cases} 1 & \text{if } g \text{ is the identity,} \\ 0 & \text{otherwise.} \end{cases} \tag{A.1}$$

A.2 Averaging Over a Compact Group

Definition A.2.1. Let A be a C^*-algebra, and suppose B is a $*$-subalgebra of A. A *conditional expectation* $E : A \to B$ is a projection (a linear map such that $E(E(a)) = E(a)$) of norm one from A into B. When $E(a^*a) = 0$ only if $a = 0$, we call E *faithful.*

Proposition A.2.2. *Let G be a compact group with Haar measure normalized so that G has measure 1, and let A be a C^*-algebra. Suppose G acts continuously by automorphisms α_χ, $\chi \in G$. Then the formula $\theta_\alpha(h) = \int_G \alpha_\chi(h)d\chi$ defines a faithful conditional expectation.*

Before proving this, we will first make sense of the integral itself. Let $\phi : A \to \mathbb{C}$ be a continuous linear functional. The composition $\phi(\alpha_\chi(h)) : G \to \mathbb{C}$ is a continuous complex-valued function and thus can be integrated. It turns out that there is a unique element of A, we will call $\int_G \alpha_\chi(h)d\chi$, such that

$$\phi\Big(\int_G \alpha_\chi(h)d\chi\Big) = \int_G \phi(\alpha_\chi(h))d\chi,$$

holds for every continuous linear functional. (This integral is called the Pettis integral, or the weak integral.)

Proof. By using a standard fact about integration, and that α_χ is isometric, one can easily see that θ_α is norm decreasing, indeed

$$\Big\| \int_G \alpha_\chi(h)d\chi\Big\| \leq \int_G \|\alpha_\chi(h)\|d\chi = \|x\|.$$

The fact that θ maps elements of A to A^α follows from the left-translation invariance of Haar measure and linearity of integration,

$$\alpha_{\chi'}\Big(\int_G \alpha_\chi(h)d\chi\Big) = \int_G \alpha_{\chi'}(\alpha_\chi(h))d\chi = \int_G \alpha_{\chi'\chi}(h)d\chi = \theta_\alpha(h).$$

If h is already in A^α, then

$$\int_G \alpha_\chi(h)d\chi = \int_G h\, d\chi = h.$$

Finally we will prove that θ is faithful. Suppose that $\theta(h^*h) = 0$. Then every positive linear functional ϕ on A gives

$$0 = \phi\Big(\int_G \alpha_\chi(h^*h)d\chi\Big) = \int_G \phi\big(\alpha_\chi(h)^*\alpha_\chi(h)\big)d\chi.$$

Since $\phi\big(\alpha_\chi(h)^*\alpha_\chi(h)\big) : G \to \mathbb{C}$ is a continuous function taking positive values whose integral is zero, we can conclude that $\phi\big(\alpha_\chi(h^*h)\big)$ is zero for every positive linear functional on A, hence h^*h must also be zero. $\square$

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