

**BIPARTITE CHARACTERIZATION OF SIGN
NONSINGULARITY**

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Abstract

Sign nonsingularity of $A \in M_n(\mathbb{R})$ is characterized in terms of conditions on the cycles of the signed bipartite graph of A . For $n \leq 5$, all maximal sign nonsingular patterns (up to transposition, permutation and signature equivalence) are listed.

1 Introduction

We say that matrices $A = [a_{ij}]$ and $B = [b_{ij}] \in M_n(\mathbb{R})$ have the same sign pattern if $a_{ij}b_{ij} > 0$ or $a_{ij} = b_{ij} = 0$, for all i, j . The matrix $A \in M_n(\mathbb{R})$ is called *sign nonsingular* if every matrix with the same sign pattern as A is nonsingular. For an n -by- n matrix $A = [a_{ij}]$, each term, $\text{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}$, in its determinant is called a *signed elementary product* from A . It follows that the matrix $A \in M_n(\mathbb{R})$ is sign nonsingular if and only if all signed elementary products from A are weakly of the same sign and at least one is nonzero.

Sign nonsingularity is invariant under transposition, permutation equivalence and diagonal equivalence. Sign nonsingularity of A has heretofore been most succinctly characterized in terms of a signed directed graph (signed digraph) associated with A ; see, for example, [BMQ, KLM, LM, T]. This seems unnatural, as the digraph is not permutation *equivalence* invariant; moreover, the signed digraph that is used is not uniquely determined. Because nonsingularity is permutation equivalence invariant, it seems more natural to think of sign nonsingularity in terms of the bipartite graph (which is permutation equivalence invariant). For a discussion of bipartite graphs

versus digraphs via $(0,1)$ matrices, see [BHM].

Our purpose here is to present a simple characterization of sign nonsingularity in terms of the signed bipartite graph. We note that the classical signed digraph characterization [BMQ] follows from our characterization. We also list all irreducible and reducible maximal sign nonsingular patterns (up to equivalence) for $n \leq 5$.

2 Signed Bipartite Graphs

For index sets $\alpha, \beta \subseteq N \equiv \{1, \dots, n\}$, the submatrix of A lying in the rows indicated by α and the columns indicated by β is denoted by $A[\alpha, \beta]$. We denote the complement of α in N by α^c so that the submatrix complementary to $A[\alpha, \beta]$ is $A[\alpha^c, \beta^c]$. Matrix A is *combinatorially singular* if every matrix with the same zero pattern as A is singular. This is equivalent to A having a submatrix $A[\alpha, \beta]$ equal to zero with $|\alpha| + |\beta| > n$. For a discussion of this result attributed to Frobenius and to König, see [MP].

The Laplace expansion for $\det A$ [MM, 2.4.11] is based upon sequences of integers from $Q_{r,n}$, the totality of strictly increasing sequences of r integers ($1 \leq r \leq n-1$) taken from N . By letting α be a fixed sequence from $Q_{r,n}$ and defining $s(\beta) = \sum_{t=1}^r \beta_t$ for the sequences $\beta = (\beta_1, \dots, \beta_r) \in Q_{r,n}$ we have

$$\det A = (-1)^{s(\alpha)} \sum_{\beta \in Q_{r,n}} (-1)^{s(\beta)} \det A[\alpha, \beta] \det A[\alpha^c, \beta^c]. \quad (2.1)$$

Given a matrix $A = [a_{ij}]$, we construct its *signed bipartite graph*, $B(A)$, hereafter called the *signed bigraph* of A , as follows. Let $V(B(A))$, the *vertex set* of $B(A)$, be defined by $\{r_1, \dots, r_n, c_1, \dots, c_n\}$ where $\{r_i\}$ and $\{c_i\}$ correspond to the rows and columns of A , respectively. The edge set is $E(B(A)) \equiv \{\{r_i, c_j\} : a_{ij} \neq 0\}$. The edge $\{r_i, c_j\}$ is signed with $\text{sgn}(a_{ij})$. Note that $B(A) = B(F)$ for all F with the same sign pattern as A .

A *cycle* of length $2p$ in $B(A)$ is a sequence of $2p$ edges

$$\{r_{i_1}, c_{j_1}\}, \{r_{i_2}, c_{j_1}\}, \{r_{i_2}, c_{j_2}\}, \dots, \{r_{i_p}, c_{j_p}\}, \{r_{i_1}, c_{j_p}\},$$

with $p \geq 2$, in which all vertices r_{i_x} and c_{j_x} are distinct; it has *cycle sign* equal to $\text{sgn}(a_{i_1, j_1} a_{i_2, j_1} a_{i_2, j_2} \cdots a_{i_p, j_p} a_{i_1, j_p})$. The submatrix corresponding to this cycle is $A[\alpha, \beta]$ where $\alpha = \{r_{i_1}, \dots, r_{i_p}\}$, $\beta = \{c_{j_1}, \dots, c_{j_p}\}$, and the submatrix complementary to the cycle is $A[\alpha^c, \beta^c]$.

A subset M of $E(B(A))$ is a *matching* in $B(A)$ if no two edges in M are incident with the same vertex. A *perfect matching* is a matching with every vertex of $B(A)$ incident with an edge in M . There is an obvious one-to-one correspondence between perfect matchings in $B(A)$ and nonzero signed elementary products from A ; matrix A is combinatorially singular if and only if there is no perfect matching in $B(A)$.

If $G = A^T$, $G = PAQ$ or $G = D_1 A D_2$, where P and Q are permutation matrices and D_1 and D_2 are nonsingular diagonal matrices, then the *unsigned* bigraphs of A and G are isomorphic. Moreover, there is a one-to-one correspondence between the cycles in $B(A)$ and those in each of $B(A^T)$, $B(PAQ)$ and $B(D_1 A D_2)$. These three properties of $B(A)$ correspond to the three invariance properties for sign nonsingularity mentioned in the introduction. Thus the bigraph seems more natural for characterizing sign nonsingularity than does the digraph.

3 Characterization

Given index sets $\alpha, \beta \subseteq N$ with $|\alpha| = |\beta|$, we define the n -by- n matrix $J_{\alpha, \beta} = [j_{pq}]$ by

$$j_{pq} = \begin{cases} 1 & \text{if } p \in \alpha, q \in \beta \text{ or } p \in \alpha^c, q \in \beta^c, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $J_{\alpha, \beta}$ is permutation equivalent to $J_{|\alpha|} \oplus J_{|\alpha^c|}$, in which J_k is the k -by- k matrix of 1's. Also $J_{\alpha, \beta} = J_{\alpha^c, \beta^c}$.

Relating $J_{\alpha, \beta}$ to the Laplace expansion (2.1) of $\det A$ about any arbitrary but fixed row set α , we find one term in the expansion of $\det A$ is

$$\det(A \circ J_{\alpha, \beta}) = (-1)^{s(\alpha)+s(\beta)} \det A[\alpha, \beta] \det A[\alpha^c, \beta^c], \quad (3.1)$$

in which \circ denotes the Hadamard (entrywise) product. The right hand side of (3.1) is a sum of signed elementary products from A .

We first give a characterization of sign nonsingularity based upon complementary submatrices.

LEMMA 3.1 *Let $A \in M_n(\mathbb{R})$ be nonsingular. Then A is sign nonsingular if and only if for all $\alpha, \beta \subseteq N$ with $1 \leq |\alpha| = |\beta| \leq n - 1$,*

- (a) *submatrices $A[\alpha, \beta]$ and $A[\alpha^c, \beta^c]$ are both sign nonsingular or one is combinatorially singular, and*
- (b) $\det A \det(A \circ J_{\alpha, \beta}) \geq 0$.

Proof: Assume A is sign nonsingular and assume (a) is not true for some fixed α, β . Thus neither $A[\alpha, \beta]$ nor $A[\alpha^c, \beta^c]$ is combinatorially singular, and they are not both sign nonsingular. Suppose, for argument, $A[\alpha, \beta]$ is not sign nonsingular; then $A[\alpha, \beta]$ must have two signed elementary products of opposite sign. Using (3.1), we can produce two oppositely signed elementary products from A by multiplying any nonzero elementary product from $A[\alpha^c, \beta^c]$ by $(-1)^{s(\alpha)+s(\beta)}$ and by the two oppositely signed products from $A[\alpha, \beta]$. Thus (a) is true.

For (b), each signed elementary product from A is weakly of the same sign, as A is sign nonsingular. Thus the expression (3.1) must have weakly the same sign as $\det A$.

Conversely, we now assume (a) and (b) hold for all $\alpha, \beta \subseteq N$ with $1 \leq |\alpha| = |\beta| \leq n - 1$. Let $F \in M_n(\mathbb{R})$ have the same sign pattern as A . We must show that F is nonsingular. From (2.1), for any fixed $\alpha \in Q_{r,n}$

$$\det F = (-1)^{s(\alpha)} \sum_{\beta \in Q_{r,n}} (-1)^{s(\beta)} \det F[\alpha, \beta] \det F[\alpha^c, \beta^c]. \quad (3.2)$$

If $A[\alpha, \beta]$ and $A[\alpha^c, \beta^c]$ are both sign nonsingular, then $\det F[\alpha, \beta]$ has the same sign as $\det A[\alpha, \beta]$ and $\det F[\alpha^c, \beta^c]$ has the same sign as $\det A[\alpha^c, \beta^c]$. If the alternate condition in (a) is true, that is, if one of $A[\alpha, \beta]$ or $A[\alpha^c, \beta^c]$ is combinatorially singular, then the corresponding submatrix of F is also combinatorially singular, and $\det F[\alpha, \beta] \det F[\alpha^c, \beta^c] = 0$. So (b) and (3.1) imply that each term in the summation in (3.2) has weakly the same sign as $\det A$. Since A is nonsingular, at least one of the summands in (2.1), and hence in (3.2), must be nonzero. Thus $\det F$ is nonzero, proving that A is sign nonsingular. ■

Note that necessity can also be proved by using results of [DJvD] which develop properties of several classes of nonsingular matrices. The set of sign nonsingular matrices is exactly equal to the set $B_{n,2}$ defined in [DJvD] restricted by condition (a) of lemma 3.1.

We use the above lemma to prove our main result, a characterization of sign nonsingularity in terms of the signed bigraph. We define the matrix

$P_k = [p_{ij}] \in M_k(\mathbb{R})$ to be the permutation matrix with $p_{k1} = 1$ and $p_{i,i+1} = 1$, $i = 1, \dots, k-1$.

THEOREM 3.2 *Let $A \in M_n(\mathbb{R})$ be nonsingular. Then A is sign nonsingular if and only if for every cycle of length $2p$ in $B(A)$ either there is no perfect matching in the submatrix complementary to the cycle or the cycle sign is $(-1)^{p-1}$.*

Proof. First assume that $A = [a_{ij}]$ is sign nonsingular. If $B(A)$ contains no cycle, there is nothing to show. If there is a cycle, first consider the case in which its length is $2n$. Such a cycle corresponds to two disjoint perfect matchings in $B(A)$. Because of the invariance of the problem under permutation equivalence, we may assume without loss of generality that the matchings correspond respectively to the nonzero entries a_{ij} , $i = 1, \dots, n$, and to the nonzero entries $a_{i,i+1}$, $i = 1, \dots, n-1$, and a_{n1} . These matchings give respectively the signed elementary products $a_{11} \cdots a_{nn}$ and $(-1)^{n-1}(a_{12}a_{23} \cdots a_{n1})$ from A . These two products enter $\det A$ with the same signs when n is odd and opposite signs when n is even. Since A is sign nonsingular, these two signed elementary products must have the same sign. This means that the cycle must have positive cycle sign when n is odd and negative cycle sign when n is even, verifying the desired conclusion in case $p = n$.

Consider then a cycle with length strictly less than n and the square (not necessarily principal) submatrix corresponding to this cycle. By lemma 3.1, either its complement is combinatorially singular or both our submatrix and its complement are sign nonsingular. The complement is combinatorially singular exactly when it contains no perfect matching. Alternatively, our submatrix is sign nonsingular, which, by replacing A with our submatrix, reduces the problem to the case considered above. This completes the proof of necessity.

Since A is nonsingular, there is at least one nonzero signed elementary product from A . If there is only one, A is clearly sign nonsingular. If there are two or more, we need to show that any two (not necessarily disjoint) have the same sign, in order to verify the converse. Without loss of generality again, we may assume that one nonzero signed elementary product arises from the diagonal entries and the second from $A \circ (P_{k_1} \oplus P_{k_2} \oplus \dots \oplus P_{k_t} \oplus I_q)$, in which $q = n - \sum_{i=1}^t k_i$. Together these entries form t cycles, with lengths $2k_1, 2k_2, \dots, 2k_t$, in $B(A)$. For example, the nonzero entries from P_{k_1} and the first k_1 diagonal entries form a cycle of length $2k_1$. The contribution of this cycle to the first signed elementary product is $a_{11}a_{22} \dots a_{k_1 k_1}$ and to the second is $(-1)^{k_1-1}(a_{12}a_{23} \dots a_{k_1 1})$. Since the cycle sign is assumed to be $(-1)^{k_1-1}$, the two contributions have the same sign. The same argument applies to each of the t cycles, and any overlapping diagonal entries clearly contribute to both signed elementary products in the same way. Thus, all signed elementary products from A are of the same sign and A is sign nonsingular. ■

As stated in the introduction, there is a well known signed digraph characterization of sign nonsingularity due to [BMQ]. Given a nonsingular matrix A , a signature matrix S and a permutation matrix P are chosen so that SPA

has all diagonal entries negative. This is always possible and preserves nonsingularity, but is, in general, not unique. Matrix A is sign nonsingular if and only if the signed digraph of SPA has only negative cycles. Consider a simple cycle of length p in this digraph. Then the entries of SPA on this cycle, together with the corresponding p diagonal entries, form a cycle of length $2p$ in $B(\text{SPA})$. The signed digraph cycle is negative if and only if the signed bigraph cycle has sign $(-1)(-1)^p$, showing the equivalence of the signed digraph characterization and theorem 3.2.

4 Maximal Sign Nonsingular Patterns

The *sign pattern matrix* associated with a given $A \in M_n(\mathbb{R})$ is the n -by- n matrix with its (i,j) entry equal to $\text{sgn}(a_{ij})$. If A is sign nonsingular, then its sign pattern matrix is a *sign nonsingular pattern*. A sign nonsingular pattern is *maximal* if no zero entry can be changed to a plus or a minus while retaining sign nonsingularity. (Note that "maximal" refers to the set of positions of nonzeros.) All other sign nonsingular patterns can be obtained from the maximal ones by changing particular nonzero entries to zero. The *minimum* number of zero entries in a (maximal) sign nonsingular pattern of size n may be deduced from corollary 2 in [G], see also [T].

THEOREM 4.1 *If $A \in M_n(\mathbb{R})$ is sign nonsingular, then A has at least $(n-1)(n-2)/2$ zero entries. If A is sign nonsingular, then it has exactly $(n-1)(n-2)/2$ zero entries if and only if A is permutation equivalent to a Hessenberg matrix.*

We now list equivalence classes (with respect to transposition, permutation and signature equivalence) of maximal sign nonsingular patterns for $2 \leq n \leq 5$. The only maximal sign nonsingular pattern for $n = 2$ is

$$H_2 \equiv \begin{bmatrix} - & + \\ - & - \end{bmatrix}$$

and the only one for $n = 3$ is

$$H_3 \equiv \begin{bmatrix} - & + & 0 \\ - & - & + \\ - & - & - \end{bmatrix}.$$

For $n = 4$, there are three irreducible maximal sign nonsingular patterns, namely,

$$\mathbf{H}_4 \equiv \begin{bmatrix} - & + & 0 & 0 \\ - & - & + & 0 \\ - & - & - & + \\ - & - & - & - \end{bmatrix}, \quad \mathbf{G}_4 \equiv \begin{bmatrix} - & + & 0 & 0 \\ - & - & + & - \\ - & - & - & + \\ 0 & 0 & - & - \end{bmatrix}, \quad \mathbf{F}_4 \equiv \begin{bmatrix} - & + & 0 & - \\ - & - & + & 0 \\ - & 0 & - & + \\ 0 & - & - & - \end{bmatrix}.$$

There are also two reducible maximal sign nonsingular patterns for $n = 4$, namely,

$$\left[\begin{array}{c|c} \mathbf{H}_2 & \mathbf{0} \\ \hline + & - \\ - & - \\ \hline & \mathbf{H}_2 \end{array} \right], \quad \left[\begin{array}{c|c} - & \mathbf{0} \\ \hline - & \\ - & \\ \hline + & \mathbf{H}_3 \end{array} \right].$$

Note that the pattern \mathbf{H}_3 given for $n = 3$ and the pattern \mathbf{H}_4 for $n = 4$ above are Hessenberg and are examples of the case of equality in theorem 4.1.

For $n = 5$, we found by computer search that there are only six irreducible maximal sign nonsingular patterns, namely,

$$\begin{bmatrix} - & + & 0 & 0 & 0 \\ - & - & + & 0 & 0 \\ - & - & - & + & 0 \\ - & - & - & - & + \\ - & - & - & - & - \end{bmatrix}, \quad \begin{bmatrix} - & + & 0 & 0 & 0 \\ - & - & + & + & 0 \\ - & - & - & + & 0 \\ - & - & 0 & - & + \\ - & - & 0 & - & - \end{bmatrix}, \quad \begin{bmatrix} - & + & 0 & 0 & 0 \\ - & - & + & 0 & 0 \\ - & - & - & + & - \\ - & - & - & - & + \\ 0 & 0 & 0 & - & - \end{bmatrix},$$

$$\begin{bmatrix} - & - & + & 0 & 0 \\ + & - & 0 & + & 0 \\ - & 0 & - & + & 0 \\ 0 & - & - & - & + \\ 0 & - & - & - & - \end{bmatrix}, \quad \begin{bmatrix} - & + & 0 & 0 & 0 \\ - & - & + & - & 0 \\ - & - & - & + & 0 \\ 0 & 0 & - & - & + \\ 0 & 0 & - & - & - \end{bmatrix}, \quad \begin{bmatrix} - & + & 0 & 0 & - \\ - & - & + & 0 & 0 \\ 0 & - & - & + & 0 \\ - & 0 & - & - & + \\ 0 & - & 0 & - & - \end{bmatrix}.$$

Note that the first pattern is Hessenberg (with 6 zero entries) and each of the other patterns has 7, 8 or 9 zero entries.

For $n = 5$, a computer search found the following thirteen reducible maximal sign nonsingular patterns, each of which has either 7 or 8 zero entries:

$$\begin{array}{ccc}
\left[\begin{array}{c|c} \mathbf{H}_2 & \mathbf{0} \\ \hline + & + \\ + & + \\ - & - \end{array} \right], & \left[\begin{array}{c|c} \mathbf{H}_2 & \mathbf{0} \\ \hline - & - \\ - & + \\ - & - \end{array} \right], & \left[\begin{array}{c|c} \mathbf{H}_2 & \mathbf{0} \\ \hline - & + \\ - & - \\ - & - \end{array} \right], \\
\left[\begin{array}{c|c} \mathbf{H}_2 & \mathbf{0} \\ \hline + & - \\ + & + \\ - & - \end{array} \right], & \left[\begin{array}{c|c} - & \mathbf{0} \\ \hline - & \\ + & \\ - & \\ - & \end{array} \right], & \left[\begin{array}{c|c} - & \mathbf{0} \\ \hline + & \\ + & \\ + & \\ - & \end{array} \right], \\
\left[\begin{array}{c|c|c} - & \mathbf{0} & \mathbf{0} \\ \hline + & & \\ + & \mathbf{H}_3 & \mathbf{0} \\ - & & \\ \hline - & + & - & - \\ - & & \end{array} \right], & \left[\begin{array}{c|c|c} - & \mathbf{0} & \mathbf{0} \\ \hline + & & \\ + & \mathbf{H}_3 & \mathbf{0} \\ - & & \\ \hline - & - & + & + \\ - & & \end{array} \right], & \left[\begin{array}{c|c|c} - & \mathbf{0} & \mathbf{0} \\ \hline - & & \\ + & \mathbf{H}_3 & \mathbf{0} \\ - & & \\ \hline - & + & - & - \\ - & & \end{array} \right], \\
\left[\begin{array}{c|c} - & \mathbf{0} \\ \hline - & \\ - & \\ - & \\ - & \end{array} \right], & \left[\begin{array}{c|c} - & \mathbf{0} \\ \hline - & \\ - & \\ - & \\ - & \end{array} \right], & \left[\begin{array}{c|c} - & \mathbf{0} \\ \hline - & \\ + & \\ + & \\ + & \end{array} \right], \\
\left[\begin{array}{c|c} - & \mathbf{0} \\ \hline - & \\ - & \\ - & \\ - & \end{array} \right]. & &
\end{array}$$

The above computer searches were carried out by first showing that the set of all 5-by-5 matrices with each entry equal to ± 1 consists of 30 equivalence classes (with respect to transposition, permutation and signature equivalence). Then m ($7 \leq m \leq 9$) zero entries were introduced into a representative of each equivalence class in all possible ways, and the resultant matrices were checked for sign nonsingularity and maximality. By theorem 4.1, only values of $m \geq 7$ need be considered, and a partial search with $m = 10$ found no further examples.

It is an open problem to determine the *maximum* number of zeros in an n -by- n maximal sign nonsingular matrix. For $n = 2$, it is 0; for $n = 3$, it

is 1; and for $n = 4$, it is 4. For $n = 5$, we believe that it is 9. For the case $n = 7$, the interesting example $A = -(I_7 + P_7 + P_7^3)$ has 28 zero entries (see [T, fig. 3(c)]). We also note that for irreducible maximal sign nonsingular matrices, there is at most one example for a given zero pattern, up to equivalence. This is not the case for reducible patterns as the above examples show.

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