Chronological Rectangle Digraphs

by

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B.Sc., Mount Allison University, 2006
M.Sc., Memorial University of Newfoundland, 2008

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ABSTRACT

Interval graphs admit elegant ordering and structural characterizations. A natural digraph analogue of interval graphs, called chronological interval digraphs, has recently been identified and studied.

We introduce the class of chronological rectangle digraphs, and show that they are a higher dimensional analogue of chronological interval digraphs. A main goal of this thesis is to establish a foundation of knowledge about this class, including basic properties and an ordering characterization. Our most significant result is a forbidden induced subdigraph characterization for the series-parallel digraphs which are chronological rectangle. We also discuss obtaining chronological rectangle digraphs from orientations of graphs.

In addition we introduce the related concept of the chronological interval dimension of a digraph, and determine the digraphs for which it is defined. Unit and proper chronological rectangle digraphs, defined analogously to unit and proper interval graphs, are also introduced and studied.

Contents

List of Figures

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Chapter 1

Introduction

1.1 Background

Interval graphs are famous for their rich structure, elegant characterizations, and practical applications. The ability to admit simple geometric representations makes interval graphs a very interesting class of graphs. While it is easy to give a superficial description of the interval graphs, they have many powerful properties which only emerge when examining them a little deeper.

A graph G is an *interval graph* if there is a one-to-one correspondence between the vertex set of G and a family of intervals on the real line such that two vertices are adjacent in G if and only if the corresponding two intervals intersect. The family of intervals is called an *interval model* for G.

Some of the early interest in interval graphs was motivated by their applications to genetics [42]. In particular, since genes are arranged in a linear way in chromosomes, Benzer [5] was interested in determining whether or not the subelements of genes also had linear arrangements. Benzer's problem then became how to determine whether or not the observed data was consistent with intersections of linear structures. Interval graphs also have applications in other fields including archaeology [39], operations research, and scheduling theory [4].

Suppose that $G = (V, E)$ is an interval graph and $\mathcal{F}$ is an interval model for G. Order the vertices of G according to a non-decreasing order of the left endpoints of the intervals in $\mathcal{F}$. Let $v_1, v_2, \ldots, v_n$ be such an ordering of vertices of G. Then for all $i < j < k$, $v_iv_k \in E$ implies $v_iv_j \in E$. Conversely, if a graph has a vertex ordering which satisfies the stated property then one can obtain an interval model for the graph. This ordering characterization of interval graphs, observed by several authors [47, 50, 51], thus provides a correspondence between vertex orderings and interval models of interval graphs.

A graph is *chordal* if it does not contain an induced cycle of length four or more. Since no cycle of length four or more can be an interval graph, every interval graph is chordal. An *asteroidal triple* in a graph is a triple of pairwise non-adjacent vertices such that between any two of the three vertices there is a path joining the two vertices which does not contain the third vertex or any of its neighbours. It is easy to see that no interval graph contains an asteroidal triple. The celebrated theorem of Lekkerkerker and Boland [42] states that a graph is an interval graph if and only if it is chordal and contains no asteroidal triple.

Another well-known characterization of interval graphs due to Gilmore and Hoffman [31] says that a graph is an interval graph if and only if the maximal cliques can be ordered in such a way that for each vertex the cliques containing that vertex are consecutive with respect to the ordering. An equivalent characterization of interval graphs in terms of the adjacency matrices was given by Fulkerson and Gross [30].

Interval graphs have been generalized in different ways. A graph G is called a *k-box graph* if there is a one-to-one correspondence between the vertex set of G and a family of axis-parallel boxes in the k-dimensional space $\mathbb{R}^k$ such that two vertices are adjacent in G if and only if the corresponding two boxes intersect. Since intervals are 1-boxes, interval graphs are the 1-box graphs. The *boxicity* of a graph G is the minimum number k for which G is a k-box graph. Interval graphs are precisely the graphs of boxicity one. The concept of boxicity was introduced by Roberts [54] who gave an upper bound on the boxicity of a general graph in terms of the number of vertices.

Boxicity has been studied extensively in the literature, frequently in conjunction with graph classes and parameters. Quest and Wegner [49] gave a matrix characterization of the graphs with boxicity at most 2 which generalizes the results of Fulkerson and Gross [30] for interval graphs. Outerplanar graphs were shown to have boxicity at most 2 by Scheinerman [55], while Thomassen [59] proved that planar graphs have boxicity at most 3. More recent work has shown that graphs which are embeddable on a torus have boxicity at most 7, while graphs embeddable on a surface of genus g have boxicity at most $5g + 3$ [28]. Boxicity has been bounded in terms of parameters such as treewidth [16] and maximum degree [1, 13, 27]. Boxicity of various other graph classes have also been studied [6, 7, 12, 21], where in particular it was shown in [12] that there exist chordal graphs with arbitrarily high boxicity.

Interval graphs can be recognized in linear time. Booth and Lueker [10] designed the first linear time algorithm, using a data structure called PQ-trees. Several linear time algorithms for recognizing interval graphs have since been developed [35, 40]. In particular, Corneil et. al. [19] devised a linear-time algorithm using lexicographic breadth-first search.

Computationally, Cozzens [20] showed that computing the boxicity of a given graph is an NP-hard problem. Yannakakis [63] proved that determining whether or not the boxicity of a given graph is at most k for $k \geq 3$ is NP-complete, while Kratochvíl [41] showed the NP-completeness for $k = 2$.

A digraph analogue of interval graphs, called interval digraphs, was pioneered in [56]. A digraph $G = (V, A)$ is an interval digraph if there exists a family of ordered pairs of closed intervals $(I_v, J_v), v \in V$, also called an *interval model* for G, such that $uv \in A$ if and only if $I_u \cap J_v \neq \emptyset$. Interval digraphs have also been extensively studied [11, 23, 24, 29, 36, 46, 56, 57, 58, 62]. In particular, there are characterizations of interval digraphs in terms of matrices [56, 57, 62], as well as a polynomial time recognition algorithm [46]. However, the most attractive aspects of interval graphs are absent, namely, an ordering characterization and a forbidden substructure characterization.

A natural alternative digraph analogue of interval graphs was proposed and studied in [22]. A digraph $G = (V, A)$ is a *chronological interval digraph* if there exists a family of closed intervals $I_v, v \in V$, called a *chronological interval model* for G, such that $uv \in A$ if and only if I_u contains the left endpoint of I_v. (Equivalently, $uv \in A$ if and only if $I_u \cap I_v \neq \emptyset$ and the left endpoint of I_u is not greater than the left endpoint of I_v.) Since every interval contains its own left endpoint, the digraph G is reflexive. For the same reason, every interval graph is reflexive. Chronological interval digraphs correspond to orientations of interval graphs according to the order of the left endpoints of an interval representation. As a consequence, an undirected graph is an interval graph if and only if it admits an orientation which is a chronological interval digraph.

A digraph $D = (V, A)$ is a *Ferrers digraph* if the out-neighbourhoods of its vertices can be linearly ordered by inclusion. Ferrers digraphs were introduced independently by Guttman [34] and Riguet [52]. Every digraph is the intersection of a finite number of Ferrers digraphs. The *Ferrers dimension* of a digraph D is the minimum number k of Ferrers digraphs whose intersection is D. Cogis [17, 18] characterized the digraphs of Ferrers dimension at most 2.

There is a fundamental connection between interval and Ferrers digraphs. In fact, every interval digraph is the intersection of two Ferrers digraphs whose union is complete [56]. In particular, every interval digraph has Ferrers dimension at most 2.

A *pointed set* is a set with a distinguished element referred to as its *base point*. A digraph $D = (V, A)$ is a *catch digraph* if every vertex is associated with a pointed set, and there is an arc from u to v if and only if the set corresponding to u contains the base point of the set corresponding to v. Of particular interest is the case when the set is an interval I and the base point is in I. A digraph $D = (V, A)$ has an *interval catch representation* if there exists a family of intervals $I_v, v \in V$ and a collection of points $p_v \in I_v, v \in V$ such that $uv \in E$ if and only if $p_v \in I_u$. Note that interval catch digraphs are interval digraphs such that J_v is a point in I_v.

A characterization of interval catch digraphs analogous to Lekkerkerker and Boland's asteroidal triple characterization for interval graphs [42] was given by Prisner [48]. Descriptions of interval catch digraphs analogous to the ordering and matrix characterizations for interval graphs are also known [44]. Maehara also showed that for every digraph D, there exists a value of k so that D can be represented as the catch digraph of a family of pointed boxes in $\mathbb{R}^k$ whose base points are at their respective centers [44].

1.2 Chronological Interval Digraphs

Recall that a *chronological interval digraph* is a digraph $D = (V, A)$ which has a *chronological interval model* $I_v, v \in V$, where each I_v is a closed interval on the real line such that $uv \in A$ if and only if the left endpoint of I_v is contained in I_u.

Let $D = (V, A)$ be a chronological interval digraph and $I_v, v \in V$ be a chronological interval model for D. Since each interval contains its own left endpoint, every chronological interval digraph is reflexive. If u, u' are both in-neighbours of a vertex v in D, then $I_u, I_{u'}$ intersect as they both contain the left endpoint of I_v, so there is at least one arc between u, u'. Hence, D is in-semicomplete. Suppose that u, v are in the same strong component of D. Then there exists a directed path from u to v and a directed path from v to u. It follows from the definition that the left endpoint of I_u is not greater than the left endpoint of I_v and vice versa. Consequently, the left endpoints of I_u and I_v are the same which means that uv is a symmetric arc in D. Hence every strong component of D is complete. It follows that the in-neighbourhood of u and v are the same and their out-neighbourhoods are comparable, in the sense that either $N^+(u) \subseteq N^+(v)$ or $N^+(v) \subseteq N^+(u)$. A digraph D is *clustered* if it satisfies all these properties (that is, it is in-semicomplete and for any two vertices u, v in the same strong component, the in-neighbourhoods of u and v are the same and their out-neighbourhoods are comparable).

Chronological interval digraphs admit an ordering characterization. Let $D = (V, A)$ be a reflexive digraph and let S be the set of symmetric arcs in D. A vertex ordering $\prec$ of D is a *chronological ordering* of D if it satisfies the following four properties, for any $u \prec v$ (for P_1) and any $u \prec v \prec w$ (for $P_2 - P_4$).

(P_1) $vu \notin A - S$

(P_2) $uw \in S$ implies $uv, vw \in S$

(P_3) $uw \in A - S$ implies either $uv \in A - S$ or both $uv \in S$ and $vw \in A - S$

(P_4) $uw \notin A$ implies $uv \notin A$ or $vw \notin S$.

Theorem 1.1. *[22] A digraph G is a chronological interval digraph if and only if it admits a chronological ordering.* ■

Chronological interval digraphs also admit a structural characterization similar to that of Lekkerkerker and Boland for interval graphs. An *asynchronous triple* in a digraph $D = (V, A)$ is a triple of vertices a, b, c, such that for every ordering $u < v < w$ of a, b, c, there exists a sequence $u = u_1, u_2, \ldots, u_p = w$ of vertices of G, such that for every $i = 1, 2, \ldots, p-1$ the triple u_i, v, u_{i+1} is *bad*, in the sense that it violates one of the properties $P_1 - P_4$ above (with u, w replaced by u_i, u_{i+1} respectively), that is, at least one of the following four properties holds.

(Q_1) one of $vu_i, u_{i+1}v, u_{i+1}u_i$ is in $A - S$

(Q_2) $u_iu_{i+1} \in S$ and at least one of $u_iv \notin S, vu_{i+1} \notin S$

(Q_3) $u_iu_{i+1} \in A - S$ and either $u_iv \notin A$ or both $u_iv \in S$ and $vu_{i+1} \notin A - S$

(Q_4) $u_iu_{i+1} \notin A$, $u_iv \in A$ and $vu_{i+1} \in S$.

Theorem 1.2. *[22] A digraph G is a chronological interval digraph if and only if it is clustered and contains no asynchronous triple.* ■

In addition to the above ordering and structural characterizations, Das, Francis, Hell, and Huang [22] also showed that chronological interval digraphs have another characterization in terms of so-called parallel vertices. All these characterizations together lead to a linear time recognition algorithm for the class of chronological interval digraphs.

A chronological interval digraph D is called *proper* if it has a chronological interval model in which no interval is contained another. Clearly, proper chronological interval digraphs do not contain symmetric arcs. Proper chronological interval digraphs have been studied previously under a different name by Deng, Hell, and Huang [25]. A digraph is *straight* if there exists an ordering $v_1, v_2, \ldots, v_n$ of its vertices such that for each i there exist nonnegative integers ℓ and k (which depend on i) such that the in-neighbours of v_i are $v_i, v_{i-1}, v_{i-2}, \ldots, v_{i-\ell}$ and the out-neighbours of v_i are $v_i, v_{i+1}, v_{i+2}, \ldots, v_{i+k}$. It is easy to see that proper chronological interval digraphs are precisely the straight digraphs.

1.3 Chronological Rectangle Digraphs

A *chronological rectangle digraph* is a digraph $D = (V, A)$ which has a *chronological rectangle model* $R_v, v \in V$, where R_v is an axis-parallel rectangle in the plane $\mathbb{R}^2$, such that $uv \in A$ if and only if the lower-left corner of R_v is contained in R_u [37]. An example of a chronological rectangle digraph D with a corresponding chronological rectangle model is given in Figure 1.1.

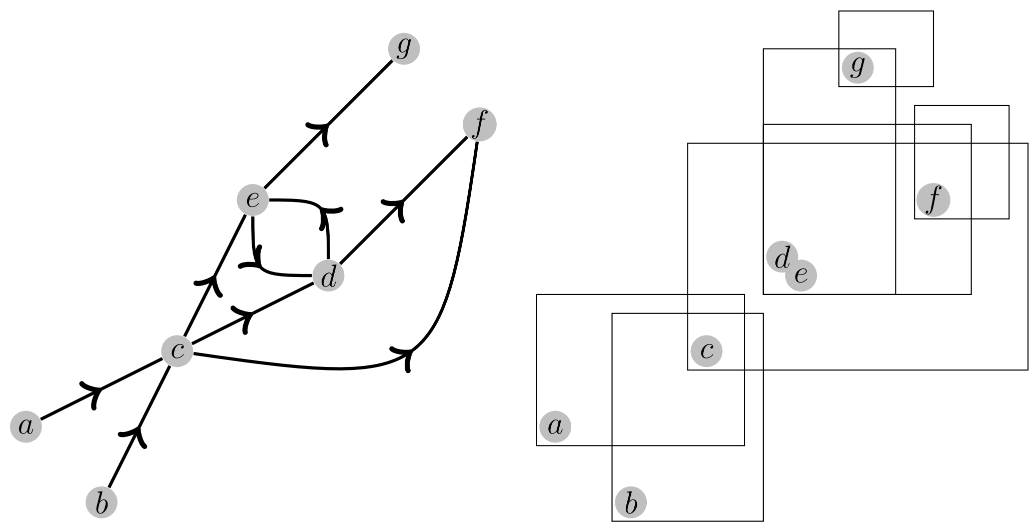

Figure 1.1: On the left is a chronological rectangle digraph D, while a chronological rectangle model for D is given on the right. Loops have been omitted from the digraph for convenience.

The rectangles are taken to be closed so that each rectangle contains its boundary and in particular its own lower-left corner. As a consequence, there are loops at every vertex so chronological rectangle digraphs are reflexive, just like chronological interval digraphs. We will, however, omit loops from all figures for convenience. Now suppose that two rectangles have the same lower-left corner. If u and v are the corresponding vertices, there is an arc from u to v as well as an arc from v to u. Conversely, vertices with symmetric arcs between them correspond to rectangles with the same lower-left corner. For this reason, vertices with symmetric arcs between them have the same in-neighbourhood.

Like chronological interval digraphs and various families of intersection digraphs, the class of chronological rectangle digraphs is closed under taking induced subdigraphs. This follows from the fact that if H is an induced subdigraph of a chronological rectangle digraph, then the set of rectangles corresponding to the vertices of H is a chronological rectangle model for the digraph induced by the vertices of H.

Chronological rectangle digraphs are a class of catch digraphs, where the catch representations $(R_v, p_v), v \in V$, are such that each R_v is a rectangle in the plane, and the distinguished point p_v is the lower-left corner of the rectangle R_v.

The k-box graphs generalize interval graphs in the sense that a k-box graph is the intersections of k interval graphs. We will show that a chronological rectangle digraph is the intersection of two chronological interval digraphs. As such, chronological rectangle digraphs are a natural higher dimensional analogue of chronological interval digraphs.

1.4 Outline

In Chapter 2, we study the basic structure of chronological rectangle digraphs. We show that they exhibit several interesting properties and have an ordering characterization which is both intuitive and rich. This ordering characterization is useful in proving that digraphs belonging to some classes are chronological rectangle. We also use the ordering characterization to derive some chronological rectangle obstructions.

In contrast with boxicity and Ferrers dimension, the chronological interval dimension is not well-defined for every digraph. However in Chapter 3 we are able to identify which digraphs can be represented as the intersection of some finite number of chronological interval digraphs, and prove that in this case the dimension is bounded by the number of vertices. Furthermore, we show that there exists a digraph whose chronological interval dimension is exactly k for all values of k. A contrast between the chronological interval dimension of a digraph and the boxicity of the underlying graph is also explored.

Series-parallel digraphs are studied in Chapter 4. Using the ordering characterization developed in Chapter 2, we are able to prove that a special digraph and an infinite family of series-parallel digraphs are not chronological rectangle. We are able to use these obstructions to prove a forbidden induced subdigraph characterization for the series-parallel digraphs which are chronological rectangle digraphs. The proof is quite involved, relying on developing a partition of the vertices and recursively constructing orderings in various cases.

We also consider how to obtain chronological rectangle digraphs from graphs by adding loops to all vertices and orienting the edges. In Chapter 5, it is shown that each orientation of a tree leads to a chronological rectangle digraph. We also show that for every k-tree there exists an orientation from which a chronological rectangle digraph can be constructed. At the same time, there exists a reflexive split graph such that no orientation is chronological rectangle.

Finally, in Chapter 6 we introduce the classes of unit and proper chronological rectangle digraphs. These classes have ordering characterizations which are more restricted. We study some of the basic properties of these two classes, as well as their relationships to chronological interval and chronological rectangle digraphs.

1.5 Definitions, Conventions, and Notation

A *graph* G is an unordered pair $G = (V(G), E(G))$, where $V(G)$ is a finite set and $E(G)$ is a set of unordered pairs of elements from $V(G)$. The set $V(G)$ is called the *vertex set* of G and its elements are called *vertices*, while $E(G)$ is the *edge set* of

G and its elements *edges*. An edge $\{u, v\}$ will be denoted by uv for simplicity. In contexts where the graph G is clear, the vertex set will be denoted by V and the edge set by E. A set of non-loop edges without any common vertices is a *matching*.

Two vertices u, v in a graph G are *adjacent* if $uv \in E$. The edge uv is *incident* with the two vertices u and v. For a vertex u, define its *neighbourhood* $N(u)$ to be the set of vertices which are adjacent to u. An *independent set* of G is a set of pairwise non-adjacent vertices. A *clique* of G is a set of pairwise adjacent vertices. A graph G is *complete* if $V(G)$ is a clique. A *split graph* has the property that its vertices can be partitioned into a clique and an independent set.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Given $X \subseteq V(G)$, we write $G - X$ for the subgraph of G obtained by deleting the set of vertices X as well as the edges incident with at least one vertex of X. An *induced subgraph* of G is a subgraph obtained by deleting a subset of the vertex set of G. In particular, if $Y = V(G)\backslash X$ then the *subgraph of G induced by Y* is $G - X$.

A *path* in a graph is a sequence of distinct vertices $v_1, v_2, \ldots, v_m$ such that $v_i v_{i+1} \in E$ for $i = 1, 2, \ldots, m-1$. If there is a path between any two vertices then the graph is *connected*. The *components* of a graph G are its maximal connected subgraphs.

A *cycle* on three or more vertices in a graph is a set of vertices which can be arranged in a cyclic sequence in such a way that two vertices are adjacent if they are consecutive in the sequence and nonadjacent otherwise. A graph with no cycles is a *forest*. The components of a forest are *trees*. The class of *k-trees* is defined recursively as follows. The complete graph on $k+1$ vertices is a k-tree, and so is any graph obtained from a k-tree by adding a vertex which is adjacent to exactly k vertices which form a clique.

A graph G is *bipartite* if there exists a partition (X, Y) of V such that there are no edges between vertices of X or between vertices of Y. Bipartite graphs will sometimes be referred to as *bigraphs*. If xy is an edge for every $x \in X$ and $y \in Y$, then the bipartite graph is called *complete bipartite*.

A *digraph* D is an unordered pair $D = (V(D), A(D))$, where $V(D)$ is a finite vertex set and $A(D)$ is a set of ordered pairs of elements from $V(G)$ called *arcs*. An arc from u to v will typically be written as uv for convenience. The vertices u and v are the *endpoints* of the arc uv. Let $S(D) = \{uv \in A \mid vu \in A\}$ denote the set of symmetric arcs of D. In contexts where the digraph D is clear, the vertices, arcs, and symmetric arcs of D will be denoted by V, A, and S respectively. A digraph is *reflexive* if $uu \in A$ for all $u \in V$.

For a vertex v, the *out-neighbourhood of* v is $N^+(v) = \{u \in V \mid vu \in A\}$ while the *in-neighbourhood of* v is $N^-(v) = \{u \in V \mid uv \in A\}$. Given $X \subseteq V$, define $N^+(X) = \{v \in V\backslash X | \exists u \in X \text{ with } uv \in A\}$ to be the set of vertices which are outneighbours of at least one vertex of X, but are not in X.

A digraph H is a *subdigraph* of a digraph D if $V(H) \subseteq V(D)$ and $A(H) \subseteq A(D)$. Given $X \subseteq V(D)$, we write $D - X$ for the subdigraph of D obtained by deleting the set of vertices X as well as the arcs with at least one endpoint in X. An *induced subdigraph* of D is a subdigraph obtained by deleting a subset of the vertex set of D. If $Y = V(D)\backslash X$ then the *subdigraph of* D *induced by* Y is $D - X$. If $R \subseteq A(D)$ then the *subdigraph of* D *induced by* R consists of the arcs of R together with all vertices which are endpoints of some vertex in R. Given subdigraphs $D_1, D_2, \ldots, D_n$ of a digraph D, the *intersection* of digraphs D_1, D_2, …, D_n is the subdigraph H of D with $V(H) = V(D_1) \cap V(D_2) \cap \cdots \cap V(D_n)$ and $A(H) = A(D_1) \cap A(D_2) \cap \cdots \cap A(D_n)$.

A *directed path* P in a digraph D is a sequence of vertices $v_1, v_2, \ldots, v_m$ in D such that $v_1v_2, v_2v_3, \ldots, v_{m-1}v_m \in A$. The *directed distance from* u *to* v, denoted $d(u, v)$, is the number of arcs in a shortest directed path from u to v (if any such directed path exists). A *strong component* S in a digraph is a set of vertices such that for all $u, v \in S$ there is a directed path from u to v as well as a directed path from v to u.

Given a graph $G = (V, E)$, an *orientation* of G is a directed graph $D = (V, A)$ where $V(G) = V(D)$ and every edge uv of G is replaced by an arc between u and v. A *directed cycle* on three or more vertices is an orientation of a cycle in which each vertex is an in-neighbour of its successor in the sequence; a vertex with a loop is a directed cycle on one vertex and a pair of vertices with symmetric arcs between them is a directed cycle on two vertices. A *transitive tournament* is an orientation of a complete graph which has no directed cycles.

Given a digraph $D = (V, A)$, the *underlying graph of* D is the undirected graph $G = (V, E)$ where $E = \{uv | u \neq v$ and at least one of $uv, vu \in A\}$. A digraph D is *connected* if the underlying graph of D is connected. The *components* of a digraph D are the maximal connected subdigraphs of D.

If any two vertices of a digraph D are joined by symmetric arcs then D is a *complete digraph*. A digraph D such that any two vertices are joined by at least one arc, which may or may not be symmetric, is said to be *semicomplete*. A digraph D is *in-semicomplete* if $N^-(v)$ induces a semicomplete subdigraph for every $v \in V$. A *clustered* digraph D is in-semicomplete and for any two vertices u, v in the same strong component of D, $N^-(u) = N^-(v)$ and either $N^+(u) \subseteq N^+(v)$ or $N^+(v) \subseteq N^+(u)$.

We shall use $\mathbb{Z}$ to denote the set of integers and $\mathbb{Z}_m$ to denote the set $\{0, 1, 2, \ldots, m-1\}$. The real numbers are given by $\mathbb{R}$, and the n-dimensional real coordinate space by $\mathbb{R}^n$.

Chapter 2

Preliminaries

We begin this chapter by exploring some properties of chronological rectangle digraphs. This allows us to derive two important necessary conditions for a digraph to be chronological rectangle. Afterwards, we develop characterizations of chronological rectangle digraphs by functions and vertex orderings. To conclude this chapter, we use the ordering characterization to derive some additional properties of chronological rectangle digraphs, and identify several digraphs which are not chronological rectangle.

Recall that digraph $D = (V, A)$ is a chronological rectangle digraph if we can associate every vertex $v \in V$ with an axis-parallel rectangle R_v in the plane $\mathbb{R}^2$, in such a way that $uv \in A$ if and only if the lower-left corner of R_v is contained in R_u. We call the R_v, $v \in V$, a *chronological rectangle model* for D.

We will use $[a, b] \times [c, d]$ to denote a rectangle R with lower-left corner (a, c) and upper-right corner (b, d). When we wish to refer to the rectangle corresponding to a particular vertex $v \in V$, the rectangle is denoted by $R_v = [a_v, b_v] \times [c_v, d_v]$. Using this notation, we make an observation that will be used throughout this thesis:

$$uv \in A \iff a_u \leq a_v \leq b_u \text{ and } c_u \leq c_v \leq d_u.$$

2.1 Weakly-Clustered Property

Let $D = (V, A)$ be a digraph with chronological rectangle model $R_v, v \in V$. Given two vertices u, v in the same strong component of D, there exist directed paths $P : u = u_1, u_2, \ldots, u_p = v$ and $Q : v = v_1, v_2, \ldots, v_q = u$. The existence of P implies that $a_u \le a_v \le b_u$ and $c_u \le c_v \le d_u$ while the existence of Q gives $a_v \le a_u \le b_v$ and $c_v \le c_u \le d_v$, so $a_u = a_v$ and $c_u = c_v$. Thus $uv \in S(D)$ and all strong components are complete in any chronological rectangle digraph. Furthermore if $wu \in A$ then $a_w \le a_u = a_v \le b_w$ and $c_w \le c_u = c_v \le d_w$ so that $wv \in A$. It follows that given a complete subdigraph C, if $u, v \in C$ then $N^-(u) = N^-(v)$.

Recall that chronological interval digraphs are clustered, so that the strong components of chronological interval digraphs have out-neighbourhoods which are comparable. We introduce a concept to help describe the out-neighbourhoods of vertices in strong components. A bigraph H with bipartition (X, Y) is a *Ferrers bigraph* if the neighbourhoods of the vertices in X can be linearly ordered by inclusion; that is, for every $u, u' \in X$ either $N(u) \subseteq N(u')$ or $N(u') \subseteq N(u)$. Ferrers bigraphs are bipartite analogues of Ferrers digraphs.

A complete bipartite graph with a single edge removed is a Ferrers bigraph. As a consequence, every bigraph is the intersection of finitely many Ferrers bigraphs. This motivates defining the *Ferrers dimension* of a bigraph H to be the minimum number of Ferrers bigraphs whose intersection is H. If there exist k Ferrers bigraphs whose intersection is H, we say that H has Ferrers dimension at most k. Theorem 2.1 appeared in [36] as an extension of a result from [17, 58].

Theorem 2.1. *A bipartite graph H with partititon (X, Y) is of Ferrers dimension at most k if and only if there exist k functions $f_i : V(H) \to \mathbb{R}, i = 1, 2, \ldots, k$ such that for $u \in X$ and $w \in Y$, $uw \in E(H)$ if and only if $f_i(u) \le f_i(w)$ for each $i = 1, 2, \ldots, k$.*

We use the Ferrers dimension to describe a necessary condition on the out-neighbourhoods of strong components for a digraph to be chronological rectangle. This result also appeared in [37].

Proposition 2.2. *If D is a chronological rectangle digraph then the underlying bigraph H induced by the arcs from any strong component C to $N^+(C)$ has Ferrers dimension at most 2.*

Proof Let $[a_v, b_v] \times [c_v, d_v], v \in V$ be a chronological rectangle digraph for D. We have seen previously that $a_u = a_w$ and $c_u = c_w$ for all $u, w \in C$. If v is in $N^+(C)$ then for all $u \in C$ we have both $a_u \leq a_v$ and $c_u \leq c_v$, where at least one of the inequalities is strict.

Define two functions $f_1, f_2 : C \cup N^+(C) \to \mathbb{R}$ by:

$$f_1(t) = \begin{cases} b_t & \text{if } t \in C \\ a_t & \text{if } t \in N^+(C) \end{cases} \qquad\qquad f_2(t) = \begin{cases} d_t & \text{if } t \in C \\ c_t & \text{if } t \in N^+(C) \end{cases}$$

Since D is a chronological rectangle digraph, we have

$$\begin{aligned} vu \in E &\iff uv \in A \\ &\iff a_u \leq a_v \leq b_u \text{ and } c_u \leq c_v \leq d_u \\ &\iff f_1(v) \leq f_1(u) \text{ and } f_2(v) \leq f_2(u). \end{aligned}$$

Now $vu \in E(H)$ if and only if $f_1(v) \leq f_1(u)$ and $f_2(v) \leq f_2(u)$, so H has Ferrers dimension at most 2 by Theorem 2.1. ■

Proposition 2.3 summarizes the local structure of strong components in chronological rectangle digraphs.

Proposition 2.3. *Let $D = (V, A)$ be a chronological rectangle digraph and C be a strong component. Then the following hold:*

(1) C is a complete subdigraph;

(2) any two vertices in C have the same in-neighbourhood in D;

(3) the underlying bipartite graph induced by the arcs from C to $N^+(C)$ has Ferrers dimension at most 2.

We say that a digraph is *weakly-clustered* if its strong components satisfy properties (1), (2), and (3) of Proposition 2.3. The definitions of clustered and weakly-clustered illustrate two differences between chronological interval and chronological rectangle digraphs. First, chronological interval digraphs are in-semicomplete, whereas chronological rectangle digraphs need not be. Secondly, in chronological interval digraphs, if u, u' belong to the same strong component C, then either $N^+(u) \subseteq N^+(u')$ or $N^+(u') \subseteq N^+(u)$ so that the underlying bigraph induced by arcs from C to $N^+(C)$ is a Ferrers bigraph. Property (3) of Proposition 2.3 shows that for chronological rectangle digraphs the underlying bigraphs induced by the arcs from a strong component C to $N^+(C)$ has Ferrers dimension at most 2.

2.2 Umbrella Path Property

A digraph D has the *umbrella path property* if for every directed path $p_1, p_2, \ldots, p_m$, $p_1 p_m \in A$ implies that $p_1 p_i \in A$ for each $i = 2, 3, \ldots, m$. It was observed in [22] that all chronological interval digraphs satisfy the umbrella path property. This property also holds for chronological rectangle digraphs.

Proposition 2.4. *Every chronological rectangle digraph satisfies the umbrella path property.*

Proof Let D be a chronological rectangle digraph, and let $p_1, p_2, \ldots, p_m$, be a directed path. Denote by (a_i, c_i) and (b_i, d_i) the lower-left and upper-right corners respectively of the rectangle corresponding to p_i in a chronological rectangle model for D. Since $p_ip_{i+1} \in A$ for $1 \le i \le m-1$, $a_i \le a_{i+1} \le b_i$ and $c_i \le c_{i+1} \le d_i$ for $1 \le i \le m-1$. As a consequence, $p_1p_m \in A$ implies $a_1 \le a_m \le b_1$ and $c_1 \le c_m \le d_1$. Now we have both $a_1 \le a_2 \le \ldots \le a_m \le b_1$ and $c_1 \le c_2 \le \ldots \le c_m \le d_1$. We conclude that $p_1p_3, p_1p_4, \ldots, p_1p_{m-1} \in A$ so that D satisfies the umbrella path property. ■

Two digraphs, U_1 and U_2, are given in Figure 2.1. Neither U_1 nor U_2 satisfies the umbrella path property, so by Proposition 2.4, neither U_1 nor U_2 is a chronological rectangle digraph.

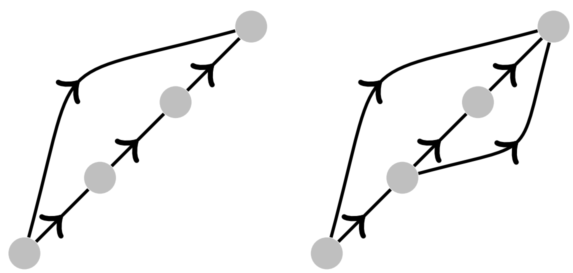

Figure 2.1: Two digraphs which do not satisfy the umbrella path property. U_1 is given on the left while U_2 is on the right.

The umbrella path property has a geometric interpretation in terms of chronological rectangle models. If there is an induced directed path on three or more vertices, then the rectangle corresponding to the first vertex are disjoint from the rectangle corresponding to the last vertex.

Proposition 2.5. *If $p_1, p_2, \ldots, p_m$ is an induced directed path with $m \ge 3$ in a chronological rectangle digraph then the rectangle corresponding to p_1 are disjoint from the rectangle corresponding to p_m.*

Proof Once again let (a_i, c_i) and (b_i, d_i) denote the lower-left and upper-right corners respectively of the rectangle R_i corresponding to p_i. Since $p_ip_{i+1} \in A$ for $1 \le i \le m-1$ we have that $a_1 \le a_2 \le \ldots a_m$ and $c_1 \le c_2 \le \ldots \le c_m$. Now the fact that $p_1p_m \notin E$ implies that either $b_1 < a_m$ or $d_1 < c_m$. In either case, R_1 and R_m are disjoint. ■

The weakly-clustered and umbrella path properties alone are not sufficient in general for a digraph to be chronological rectangle. In Section 2.6, we will prove that several digraphs which satisfy these properties are not chronological rectangle. Before we do so, we develop characterizations for chronological rectangle digraphs.

2.3 Function Characterization

We now give a concise characterization of chronological rectangle digraphs, which also appeared in [37].

Theorem 2.6. *A reflexive digraph $D = (V, A)$ is chronological rectangle if and only if there exist two functions $\phi_x, \phi_y : V \to \mathbb{R}$ which satisfy the following properties for all u, v, w, w', where w, w' are not necessarily distinct:*

(i) If $uv \in A$ then $\phi_x(u) \leq \phi_x(v)$ and $\phi_y(u) \leq \phi_y(v)$.

(ii) If $uw, uw' \in A$, $\phi_x(u) \leq \phi_x(v) \leq \phi_x(w)$ and $\phi_y(u) \leq \phi_y(v) \leq \phi_y(w')$, then $uv \in A$.

Proof Let $R_v = [a_v, b_v] \times [c_v, d_v], v \in V$. be a chronological rectangle model for D. We define $\phi_x(v) = a_v$ and $\phi_y(v) = c_v$ and show that they satisfy properties (i) and (ii). If $uv \in A$ then

$$\phi_x(u) = a_u \leq a_v = \phi_x(v) \leq b_u \text{ and}$$

$$\phi_y(u) = c_u \leq c_v = \phi_y(v) \leq d_u$$

so that property (i) is satisfied. Now suppose that $uw, uw' \in A$ where w, w' not necessarily distinct, so that $\phi_x(u) \leq \phi_x(v) \leq \phi_x(w)$ and $\phi_y(u) \leq \phi_y(v) \leq \phi_y(w')$. This implies that $a_u \leq a_v \leq a_w \leq b_u$ and $c_u \leq c_v \leq c_{w'} \leq d_u$. Hence $(a_v, c_v) \in R_u$ and since $R_v, v \in V$ is a chronological rectangle model for D we have $uv \in A$. Given the chronological rectangle digraph D, we have constructed a pair of functions which satisfy properties (i) and (ii).

Now suppose that D is a digraph for which there exist two functions ϕ_x and ϕ_y which satisfy properties (i) and (ii). Let

$$R_u = \left[\phi_x(u), \max_{w \in N^+(u)} \phi_x(w)\right] \times \left[\phi_y(u), \max_{w' \in N^+(u)} \phi_y(w')\right], u \in V.$$

Using the previous notation that $R_v = [a_v, b_v] \times [c_v, d_v]$ we have in particular that $a_v = \phi_x(v)$ and $c_v = \phi_y(v)$. We claim that $R_v, v \in V$, is a chronological rectangle model for D, so we have to establish that $uv \in A$ if and only if $(a_v, c_v) \in R_u$.

If $uv \in A$, then $\phi_x(u) \leq \phi_x(v)$ by (i) and $\phi_x(v) \leq \max_{w \in N^+(u)} \phi_x(w)$ since $v \in N^+(u)$, so that $a_u = \phi_x(u) \leq a_v = \phi_x(v) \leq b_u = \max_{w \in N^+(u)} \phi_x(w)$ and $a_v \in [a_u, b_u]$. Similarly, $uv \in A$ implies $\phi_y(u) \leq \phi_y(v)$ by (i) and $\phi_y(v) \leq \max_{w' \in N^+(u)} \phi_y(w')$ since $v \in N^+(u)$, and we have $c_u = \phi_y(u) \leq c_v = \phi_y(v) \leq d_u = \max_{w' \in N^+(u)} \phi_y(w')$ and $c_v \in [c_u, d_u]$. We conclude that if $uv \in A$ then $(a_v, c_v) \in R_u$.

Now suppose that $(a_v, c_v) \in R_u$. We have $\phi_x(u) \leq \phi_x(v) \leq \max_{w \in N^+(u)} \phi_x(w)$ and $\phi_y(u) \leq \phi_y(v) \leq \max_{w' \in N^+(u)} \phi_y(w')$. Since $uw, uw' \in A$ by definition, property (ii) implies that $uv \in A$. ■

2.4 Ordering Characterization

In this section, we derive a characterization of chronological rectangle digraphs in terms of two linear orderings of the vertices. We begin with a brief survey of ordering characterizations for related graph and digraph classes. Theorem 2.7 is the classic ordering characterization for interval graphs, independently observed by many researchers including Olariu [47], Ramalingam and Pandu [50], and Raychaudhuri [51]. Theorem 2.8 is an analogous characterization for interval catch digraphs [44, 48].

Theorem 2.7. [47, 50, 51] *A graph G is interval if and only if there exists an ordering $\prec$ of V such that for all u, v, w with $u \prec v \prec w$, $uw \in E$ implies that $uv \in E$.*

Theorem 2.8. [44, 48] *A digraph D is an interval catch digraph if and only if there is an ordering $\prec$ of V such that for $u \prec v \prec w$, $uw \in A$ implies $uv \in A$, and $wu \in A$ implies $wv \in A$.*

Recall that chronological interval digraphs are interval catch digraphs where the distinguished point of every interval is the corresponding left endpoint. An ordering characterization specific to chronological interval digraphs was presented in [22], which we reproduce in Theorem 2.9.

Theorem 2.9. [22] *A digraph D is chronological interval if and only if there exists an ordering $\prec$ of V which satisfies the following properties for all $u \prec v$ (for $P1$) and for any $u \prec v \prec w$ (for $P2 - P4$).*

(P1) $vu \notin A - S$

(P2) $uw \in S$ *implies* $uv, vw \in S$

(P3) $uw \in A - S$ *implies* $uv \in A - S$ *or both* $uv \in S$ *and* $vw \in A - S$

(P4) $uw \notin A$ *implies* $uv \notin A$ *or* $vw \notin S$

To give an ordering characterization for chronological rectangle digraphs, we define a linear ordering $\prec$ of V to be *fundamental* if it satisfies condition (R1) for every $u \prec v$, and satisfies (R2), (R3), and (R4) for every $u \prec v \prec w$:

(R1) $vu \notin A - S$

(R2) $uw \in S$ implies $uv, vw \in S$

(R3) $uw \in A - S$ implies $uv \in A - S$ or $vw \notin S$

(R4) $uw \notin A$ implies $uv \notin A$ or $vw \notin S$

Conditions (R1), (R2), and (R4) for a fundamental ordering are the same as (P1), (P2), and (P4) of Theorem 2.9. Condition (P3) of the chronological interval ordering characterization implies condition (R3), but there are numerous cases where (R3) holds but (P3) does not.

It is clear that (R2) implies that the vertices of any strong component are consecutive with respect to $\prec$. If u and w belong to a strong component then $uw \in S$ so for any v such that $u \prec v \prec w$ it is true that $uv, vw \in S$ so v belongs to the strong component. Lemma 2.10 gives a further property of fundamental orderings.

Lemma 2.10. *If $D = (V, A)$ is a digraph for which there exists a fundamental ordering $\prec$, then the strong components of D are complete and have the same in-neighbourhood.*

Proof Let D be a digraph and let $\prec$ be a fundamental ordering of V. First we argue that the strong components of D are complete by induction on the number of vertices in the strong component. A maximal strong component with exactly two vertices v_1 and v_2 has $v_1v_2, v_2v_1 \in A$, so $v_1v_2 \in S$.

Suppose by induction that every maximal strong component on k vertices is complete. Let $v_1 \prec v_2 \prec \ldots \prec v_k \prec v_{k+1}$ be a strong component on $k+1$ vertices. By induction, $v_1, v_2, \ldots, v_k$ is complete. There exists some i, $1 \leq i \leq k$, with $v_{k+1}v_i \in A$, and (R1) implies that $v_iv_{k+1} \in S$. Repeatedly applying (R2) guarantees that $v_jv_{k+1} \in S$ for all $i \leq j \leq k$, since we have $v_i \prec v_j \prec v_{k+1}$ with $v_iv_{k+1} \in S$. Now consider a vertex v_0 such that $v_0 \prec v_i \prec v_j$ and $v_0v_i, v_iv_j \in S$ for some $i < j \leq k$. Condition (R4) implies that $v_0v_j \in A$, while (R3) guarantees that $v_0v_j \in S$. Hence by induction, all strong components are complete.

We wish to argue that every vertex v in a strong component C has the same in-neighbourhood. Let $v_1 \prec \ldots \prec v_k$ be a maximal strong component C which is complete. Hence, $C \subseteq N^+(v)$ for all $v \in C$. Now suppose that $uv_i \in A$ for some $1 \leq i \leq k$ and $u \notin C$. Since the strong component is maximal, $uv_i \notin S$, so $uv_i \in A-S$ and $u \prec v_i$. Furthermore by (R2) we have $u \prec v_1$, since otherwise $v_1 \prec u \prec v_i$ and $v_1u, uv_i \in S$. Now for every $1 \leq j < i$ we have $u \prec v_j \prec v_i$ with $uv_i \in A - S$ and $v_jv_i \in S$ so (R3) implies that $uv_j \in A - S$. For $i < j \leq k$ we have $u \prec v_i \prec v_j$ with $uv_i \in A - S$ and $v_iv_j \in S$ so (R4) implies that $uv_j \in S$. Hence $uv_i \in A$ for all $1 \leq i \leq k$. ■

Having a fundamental ordering is not sufficient for a digraph to be chronological rectangle. In fact, two fundamental orderings with an additional property on the pair of orderings are required.

A pair of linear orderings $\prec_x$ and $\prec_y$ of V are *compatible* if they satisfy (R5) for every $u \prec_x v \prec_x w$ and $u \prec_y v \prec_y w'$, where possibly $w = w'$:

(R5) $uw, uw' \in A$ implies $uv \in A$.

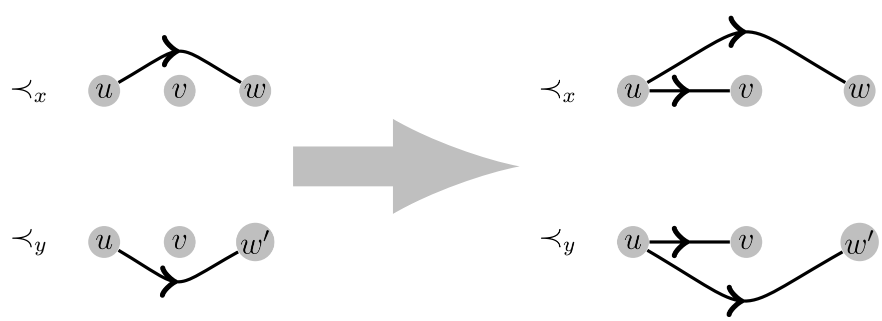

Figure 2.2: An illustration of the compatible property for a pair of orderings of the vertices of a digraph.

The compatible property is reminiscent of the ordering characterizations for interval graphs and interval catch digraphs in Theorems 2.7 and 2.8 respectively. The fact that the compatible property depends on two vertex orderings is a significant contrast.

Compatibility is equivalent to the statement that if $u \prec_x v \prec_x w$ where $uw \in A$ and $uv \notin A$ then either $v \prec_y u$ or $w' \prec_y v$ for all $w' \in N^+(u)$, where the roles of $\prec_x$ and $\prec_y$ may be exchanged.

Our next goal is to prove that the reflexive digraphs which admit a pair of compatible, fundamental orderings are exactly the chronological rectangle digraphs. First, we show that every chronological rectangle digraph has a model in which no two vertices are assigned identical rectangles.

Lemma 2.11. *If $D = (V, A)$ is a chronological rectangle digraph then there exists a chronological rectangle model $R_v = [a_v, b_v] \times [c_v, d_v], v \in V$ such that if $u, v \in V$, $u \neq v$ then $R_u \neq R_v$.*

Proof Let $u_1, u_2, \ldots, u_m$ be a set of vertices which have the same rectangle $R_u = [a_u, b_u] \times [c_u, d_u]$. All of these vertices have the same in- and out-neighbourhoods. Let s be the smallest lower endpoint of any rectangle which is strictly larger that b_u. For each u_i, we construct a rectangle $R_i' = \left[a_u, b_u + \dfrac{s - b_u}{i+1}\right]$. Now each of the rectangles has a distinct right endpoint, but none of the R_i' contains the lower-left corner of a rectangle that R_u did not, so it is still a chronological rectangle model for the same digraph. ■

We now state and prove an ordering characterization for chronological rectangle digraphs. This characterization is our primary tool for deciding whether or not digraphs are chronological rectangle.

Theorem 2.12. *A digraph D is chronological rectangle if and only if there exists a pair of compatible, fundamental orderings $\prec_x$ and $\prec_y$.*

Proof First suppose that $D = (V, A)$ is a chronological rectangle digraph, and let $R_v = [a_v, b_v] \times [c_v, d_v], v \in V$ be a chronological rectangle model for D. We construct two orderings $\prec_x$ and $\prec_y$ according to the following rules. Set $u \prec_x v$ if and only if $a_u < a_v$; or $a_u = a_v$ and $c_u < c_v$; or $a_u = a_v$, $c_u = c_v$ and $b_u < b_v$; or $a_u = a_v$, $c_u = c_v$, $b_u = b_v$, and $d_u < d_v$. Set $u \prec_y v$ if and only if $c_u < c_v$; or $c_u = c_v$ and $a_u < a_v$; or $c_u = c_v$, $a_u = a_v$ and $d_u < d_v$; or $c_u = c_v$, $a_u = a_v$, $d_u = d_v$, and $b_u < b_v$.

Constructing $\prec_x$ in this way corresponds to ordering the vertices based on the left endpoints of the corresponding rectangles from left to right, where we break ties using (in order) the lower endpoints (also from lowest to highest), the right endpoints (from left to right) and the upper endpoints (from lowest to highest). Using Lemma 2.11 the vertices may be assumed to have rectangles where at least one endpoint is distinct, so this is sufficient to order all of the vertices. Similarly, $\prec_y$ is constructed by ordering the vertices based on the lower endpoints, then breaking ties using the left endpoints, upper endpoints, then right endpoints. A pair of compatible, fundamental orderings could be defined without considering the upper-right corner. The additional rules are used to determine a fixed pair of orderings determined by the chronological rectangle model.

We will argue that $\prec_x$ and $\prec_y$ satisfy conditions (R1), (R2), (R3), (R4), and (R5).

Suppose that $u \prec_x v$. If $a_u < a_v$ or $c_u < c_v$ then certainly the rectangle corresponding to v does not contain the lower-left corner of the rectangle corresponding to u so $vu \notin A - S$. Otherwise $u \prec_x v$ implies that $a_u = a_v$ and $c_u = c_v$ which would guarantee $uv \in S$, so that $vu \notin A - S$. A similar argument holds for $\prec_y$, and therefore (R1) is satisfied.

For the remainder we assume that $u \prec_x v \prec_x w$, as an analogous argument holds for $\prec_y$.

Assume that $uw \in S$. We have $a_u = a_w$ and $c_u = c_w$, so in order for $u \prec_x v \prec_x w$ we have $a_u = a_v = a_w$ as well as $c_u = c_v = c_w$, which implies that $uv, vw \in S$, so (R2) is satisfied.

Consider $uw \in A - S$, so that $a_u \leq a_w \leq b_u$ and $c_u \leq c_w \leq d_u$ where at least one of $a_u < a_w$ or $b_u < b_w$. Suppose by way of contradiction that $uv \notin A - S$ and $vw \in S$. We have $a_u \leq a_w = a_v \leq b_u$ and $c_u \leq c_w = c_v \leq d_u$, where at least one of $a_u < a_v \leq b_u$ or $c_u < c_v \leq d_u$. This contradicts the fact that $uv \notin A - S$ since $(a_v, c_v) \in R_u$ and $(a_u, c_u) \notin R_v$ in the chronological rectangle model. We conclude that $uv \in A - S$ or $vw \notin S$ so (R3) is satisfied.

Next assume that $uw \notin A$, and suppose by way of contradiction that $uv \in A$ and $vw \in S$. This implies that $a_u \leq a_v = a_w \leq b_u$ and $c_u \leq c_v = c_w \leq d_u$. Now $(a_u, c_u) \in R_w$ in the chronological rectangle model contradicts the fact that $uw \notin A$. We conclude that $uv \notin A$ or $vw \notin S$ so (R4) is satisfied.

Finally suppose that $u \prec_x v \prec_x w$, $u \prec_y v \prec_y w'$, and $uw, uw' \in A$ where w and w' are not necessarily distinct. We have $a_u \leq a_v \leq a_w \leq b_u$ and $c_u \leq c_v \leq c_{w'} \leq b_u$ so that $(a_v, c_v) \in R_u$. Since $R_v, v \in V$ is a chronological rectangle model for D, we have $uv \in A$ and (R5) is satisfied.

To prove the other implication, we suppose that there exist two orderings $\prec_x$ and $\prec_y$ which satisfy conditions (R1), (R2), (R3), (R4), and (R5). Let $x_1, x_2, \dots x_n$ and $y_1, y_2, \dots, y_n$ be the orderings of V with respect to $\prec_x$ and $\prec_y$ respectively. We construct a family of rectangles

$$R_u = \left[\min_{x_i \in N^+(u)} i, \max_{x_i \in N^+(u)} i\right] \times \left[\min_{y_i \in N^+(u)} i, \max_{y_i \in N^+(u)} i\right]$$

and argue that $R_u, u \in V$ constitutes a chronological rectangle model for D.

We make an observation about these rectangle models. For each $u = x_i$, either $a_u = i$ or $a_u = j$ is the smallest j such that $ux_j \in S$, since for $x_j \prec x_i$, (R2) implies that $x_i x_j \notin A - S$. A similar property holds for the lower endpoint of the rectangle corresponding to u.

We start by arguing that $uv \in A$ implies that $(a_v, c_v) \in R_u$. First suppose that $uv \in S$. We apply the previous observation about the chronological rectangle models and Lemma 2.10. The conclusion is that $a_u = a_v = i$ where i is the smallest index such that x_i is in the same strong component as u and v while $c_u = c_v = j$ where j is the smallest index such that y_j is in the same strong component as u and v. Finally, $(a_v, c_v) = (a_u, c_u) \in R_u$. Now suppose that $uv \in A - S$. First, if $v = x_i$ has $a_v = i$ then $a_v \in [a_u, b_u]$ by definition. So suppose that there is some j such that $x_j v \in S, j < i$. Since strong components are consecutive by (R2), we have $u \prec_x x_j \prec_x v$, and $ux_j \in A$ by (R3). Now $j = a_v \in [a_u, b_u]$ by construction. Applying the same argument to the lower endpoints, we find that $(a_v, c_v) \in R_u$.

Now assume that $(a_v, c_v) \in R_u$. Then both $u \prec_x v \prec_x w$ and $u \prec_y v \prec_y w'$ for some not necessarily distinct $w, w' \in N^+(u)$. This implies $uv \in A$ by (R4); or at least one of $z \prec_x v \prec_x u$, $z \prec_y v \prec_y u$ for $uz \in S$. Combined with (R2), we have $uv \in S$. This property still holds in the case that $v = x_i$ and $a_v = j$, where $x_j v \in S$ and $j < i$, since the strong component containing v and x_j is consecutive with respect to both orderings. ■

In the case of digraphs with no symmetric arcs, conditions (R2), (R3), and (R4) are vacuously satisfied, so we have the following corollary.

Corollary 2.13. *A digraph with no symmetric arcs is chronological rectangle if and only if there exists a pair of compatible orderings $\prec_x$ and $\prec_y$ such that $uv \in A$ implies $u \prec_x v$ and $u \prec_y v$.*

Note that this characterization can be restated in terms of permutations, especially in the case of reflexive digraphs with no symmetric arcs.

Corollary 2.14. *A reflexive digraph with no symmetric arcs and n vertices is chronological rectangle if and only if there exist two permuations ϕ_1 and ϕ_2 of the vertices* *of D and $2n$ constants $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$, such that $v_i v_j \in A$ if and only if $\phi_1(v_i) < \phi_1(v_j) < a_i$ and $\phi_2(v_i) < \phi_2(v_j) < b_i$.*

2.5 Additional Structural Properties

In this section, we examine some properties of the chronological rectangle models for particular digraph configurations. First, we show how the proof of Theorem 2.12 can be used to modify Lemma 2.11.

Proposition 2.15. *If $D = (V, A)$ is a chronological rectangle digraph then there exists a chronological rectangle model $R_v = [a_v, b_v] \times [c_v, d_v], v \in V$ such that for all distinct $u, v \in V$ their rectangles satisfy $a_u \neq b_v$, $c_u \neq d_v$, $b_u \neq b_v$, and $d_u \neq d_v$. Furthermore, if $uv \notin S$ then $a_u \neq a_v$ and $c_u \neq c_v$.*

Proof Construct chronological rectangle models as in the proof of Theorem 2.12, except that we set

$$b_u = \max_{x_i \in N^+(u)} \left\{ i + \frac{j}{k+1} \right\}$$

where u is the jth vertex in an arbitrary order of the k in-neighbours of x_i and

$$d_u = \max_{y_i \in N^+(u)} \left\{ i + \frac{j}{k+1} \right\}$$

where again u is the jth vertex in an arbitrary order of the k in-neighbours of y_i.

We may also apply an observation made during the proof of Theorem 2.12. The left- and lower-endpoints of the rectangle corresponding to u in this construction are associated with the first x_j in each ordering for which $ux_j \in S$. Since strong components are complete in any chronological rectangle digraph, if $uv \notin S$ then these vertices and hence the corresponding endpoints are not equal. ■

In Proposition 2.16 we show that Proposition 2.2 can be extended beyond strong components.

Proposition 2.16. *Let $D = (V, A)$ be a chronological rectangle digraph with chronological rectangle model $R_v = [a_v, b_v] \times [c_v, d_v], v \in V$.*

If $X, Y \subseteq V$ such that $a_u < a_w$ and $c_u < c_w, \forall u \in X, w \in Y$ then the underlying bigraph induced by the arcs from X to Y has Ferrers dimension at most 2.

The proof of Proposition 2.16 is identical to proof of Proposition 2.2 except that the facts that $a_u < a_w$ and $c_u < c_w, \forall u \in X, w \in Y$ are assumed rather than a consequence of the fact that X is a strong component.

The contrapositive of Proposition 2.16 can be useful in proving that digraphs are not chronological rectangle. Let (A, B) be a bipartition of a bigraph G of Ferrers dimension strictly greater than 2. If D is a chronological rectangle digraph with compatible, fundamental orderings $\prec_x, \prec_y$ such that the edges of G are oriented from A to B, then there exist $a \in A$ and $b \in B$ such that either $b <_x a$ or $b <_y a$.

Proposition 2.17 follows from Proposition 2.16 because having a directed path from x to y in a chronological rectangle digraph guarantees that $a_x \leq a_y$ and $c_x \leq c_y$.

Proposition 2.17. *Let $D = (V, A)$ be a chronological rectangle digraph. If $X, Y \subseteq V$ such that for every $x \in X$ and for every $y \in Y$ there is directed path from x to y, then the underlying bigraph induced by the arcs from X to Y has Ferrers dimension at most 2.*

As an application of Proposition 2.17, we discuss structures where these conditions of Proposition 2.3 are sufficient for a digraph to be chronological rectangle.

Proposition 2.18. *Let D be a connected digraph whose vertices can be partitioned into a complete digraph X, an independent set Y, with some arcs oriented from X to*

Y. Then D is chronological rectangle if and only if the bigraph induced by the edges underlying the arcs from X to Y has Ferrers dimension at most 2.

Proof The necessity of the condition is guaranteed by Proposition 2.17, since the fact that X is complete gives directed paths from all vertices of X to all vertices of Y.

Conversely, suppose that H is a bigraph of Ferrers dimension at most 2, and that $f_1, f_2 : X \cup Y \to \mathbb{R}$ are two functions such that $xy \in E(H)$ if and only if both $f_1(y) \leq f_1(x)$ and $f_2(y) \leq f_2(x)$. We define $a_0 = \min_{v \in X \cup Y} f_1(v)$ and $c_0 = \min_{v \in X \cup Y} f_2(v)$.

We define a chronological rectangle model by:

$$R_v = \begin{cases} [a_0 - 1, f_1(v)] \times [c_0 - 1, f_2(v)] & \text{if } v \in X \\ (f_1(v), f_2(v)) & \text{if } v \in Y. \end{cases}$$

Note that the vertices of Y are assigned degenerate rectangles (single points) since they have no out-neighbours.

We observe that this is a chronological rectangle model for D since for $x \in X$ and $y \in Y$, $(f_1(y), f_2(y)) \in [a_0 - 1, f_1(x)] \times [c_0 - 1, f_2(x)]$ if and only if $f_1(y) \leq f_1(x)$ and $f_2(y) \leq f_2(x)$, and hence $xy \in E(H)$ and $xy \in A(D)$. ■

Proposition 2.19. *Let D be a digraph whose vertices can be partitioned into a transitive tournament X, an independent set Y, a vertex c which is an out-neighbour of all vertices of X as well as an in-neighbour of all vertices of Y, and some arcs oriented from X to Y. Then D is chronological rectangle if and only if the underlying bigraph induced by the arcs from X to Y has Ferrers dimension at most 2.*

Proof Proceed as in the proof of Proposition 2.18 but construct the rectangle for the vertex c as $[0, |Y|+1] \times [0, |Y|+1]$ and change the lower-left corners of vertices in X as follows. Let the vertices of X be $x_1, \ldots, x_m$, where $x_i x_j \in A$ if and only $j \leq i$. Set the lower-left corner of x_i to be $(-i, -i)$. ■

Proposition 2.20. *Let D be a digraph whose vertices can be partitioned into two independent sets X and Y, a vertex c which is an in-neighbour of all vertices of X as well as an out-neighbour of all vertices of Y, with some arcs oriented from X to Y. Then D is chronological rectangle if and only if the bigraph induced by the arcs from X to Y has Ferrers dimension at most 2.*

Proof We again modify the construction from Proposition 2.18. If X is an independent set $x_1, \ldots, x_m$ then set the lower-left corner of x_i to be $(-i, -m+i)$. ■

We consider some structural lemmas inspired by analogous work on boxicity in [9]. Note that the loops have been omitted from all of the following configurations.

Proposition 2.21. *If $R_v, v \in V$ is a chronological rectangle model for the digraph D with $V = \{a, b, c, d\}$ and $A = \{ac, bc, cd\}$ then $R_c \not\subseteq R_a \cup R_b$.*

Proof The vertex d is an out-neighbour of c, so there is a region of R_c which contains the lower-left corner of R_d. However, d is not an out-neighbour of a or b, so this point is not contained in the rectangles corresponding to a or b. ■

Proposition 2.22. *If $R_v, v \in V$ is a chronological rectangle model for the digraph D with $V = \{a, b, c, d, e\}$ and $A = \{ac, bc, ad, be, cd, ce\}$ then $R_c \cap (R_a - R_b) \neq \emptyset$ and $R_c \cap (R_b - R_a) \neq \emptyset$.*

Proof Since d is an out-neighbour of c and a but not b, there is a point corresponding to the lower-left corner of R_d which is contained in a and c but b. ■

2.6 Ordering Properties and Obstructions

In this section, we show that the weakly-clustered and umbrella path properties together are not sufficient for a digraph to be chronological rectangle, by describing three digraphs which satisfy these properties and are not chronological rectangle. First we give a pair of lemmas describing compatible, fundamental orderings of chronological rectangle digraphs. These results are interesting in their own right.

Proposition 2.23 says that if two non-adjacent vertices in a chronological rectangle digraph have a common out-neighbour, then they appear in oppposite order in two compatible, fundamental orderings. In other words if u and v have a common out-neighbour but $uv, vu \notin A$, then $u \prec_x v$ implies $v \prec_y u$.

Proposition 2.23. *Let D be the digraph with $V = \{a, b, c\}$ and $A = \{ac, bc\}$. If D is a chronological rectangle digraph with an induced subdigraph isomorphic to D, then a and b appear in opposite order in any pair of compatible, fundamental orderings $\prec_x$ and $\prec_y$.*

Proof We prove the statement by contradiction. Suppose that $\prec_x$ and $\prec_y$ are a pair of compatible, fundamental orderings of D. By the fundamental property, a and b are before c in both orderings. Now having both $a \prec_x b \prec_x c$ and $a \prec_y b \prec_y c$ with $ac \in A$, $ab \notin A$ would contradict compatibility. Similarly, $b \prec_x a \prec_x c$ and $b \prec_y a \prec_y c$ together with $bc \in A$ and $ba \notin A$ would contradict compatibility. ■

In general, two vertices with a common in-neighbour need not appear in opposite order in a pair of compatible, fundamental orderings. Consider for example the digraph H with $V(H) = \{a, b, c\}$ and $A(H) = \{ab, ac\}$. One pair of fundamental, compatible orderings for H are $a \prec_x b \prec_x c$ and $a \prec_y b \prec_y c$.

However, there is at least one other configuration which forces two vertices in opposite order in any pair of compatible, fundamental orderings.

Proposition 2.24. *Let D be a digraph with at least four distinct vertices a_1, a_2 and b_1, b_2. If $a_1b_1, a_2b_2 \in A$ and $a_1b_2, a_2b_1 \notin A$ but there are directed paths from a_1 to b_2*

and from a_2 to b_1, then b_1 and b_2 appear in opposite order in any pair of compatible, fundamental orderings $\prec_x$ and $\prec_y$.

Proof We prove the statement by contradiction. Suppose that $\prec_x$ and $\prec_y$ are a pair of compatible, fundamental orderings for D such that $b_1 \prec_x b_2$ and $b_1 \prec_y b_2$. Since there is a directed path from a_2 to b_1, the fundamental property implies that both $a_2 \prec_x b_1$ and $a_2 \prec_y b_1$. However, $a_2 \prec_x b_1 \prec_x b_2$ and $a_2 \prec_y b_1 \prec_y b_2$ with $a_2b_2 \in A$ and $a_2b_1 \notin A$ would contradict compatibility.

Consider instead the case when $\prec_x$ and $\prec_y$ are a pair of fundamental, compatible orderings such that $b_2 \prec_x b_1$ and $b_2 \prec_y b_1$. Again the fundamental property and the fact that there is a directed path from a_1 to b_2 implies that $a_1 \prec_x b_2$ and $a_1 \prec_y b_2$. We would conclude that $a_1 \prec_x b_2 \prec_x b_1$ and $a_1 \prec_y b_2 \prec_y b_1$ with $a_1b_1 \in A$ and $a_1b_2 \notin A$, contradicting compatibility. ■

We now proceed to show that three digraphs which satisfy the weakly-clustered and umbrella path properties are not chronological rectangle. The strategy for each digraph is to use Proposition 2.23 or Proposition 2.24 to prove that a pair of compatible, fundamental orderings does not exist. Note that all of the digraphs are again reflexive, but loops are omitted from the figures for convenience.

Proposition 2.25. *The digraph O_1 given in Figure 2.3 is not chronological rectangle.*

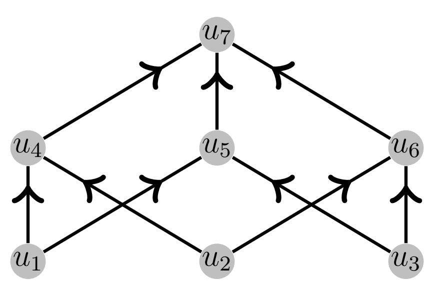

Figure 2.3: The digraph O_1 which is not chronological rectangle by Proposition 2.25.

Proof We prove the statement by contradiction. Suppose that there exists a pair of compatible, fundamental orderings $\prec_x, \prec_y$ of the vertices of O_2. Consider the vertices labelled u_4, u_5, u_6. They pairwise have a common out-neighbour, and hence appear in opposite order in $\prec_x$ and $\prec_y$ by Proposition 2.23. We may suppose without loss of generality that $u_4 \prec_x u_5 \prec_x u_6$ and $u_6 \prec_y u_5 \prec_y u_4$. Now since $\prec_x$ and $\prec_y$ are fundamental orderings, $u_2 \prec_x u_5 \prec_x u_6$ and $u_2 \prec_y u_5 \prec_x u_4$, contradicting the fact that the two orderings are compatible. ■

Proposition 2.26. *The digraph O_2 given in Figure 2.4 is not chronological rectangle.*

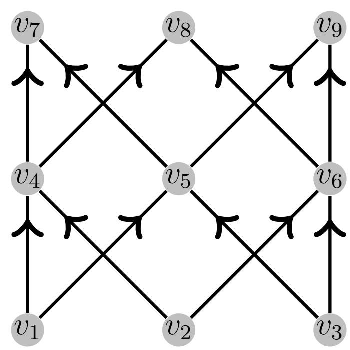

Figure 2.4: The digraph O_2 which is not chronological rectangle by Proposition 2.26.

Proof The proof is similar to the proof of Proposition 2.25, but with the three vertices v_7, v_8, and v_9 playing a role analogous to u_7.

We prove the statement by contradiction. Suppose that there exists a pair of compatible, fundamental orderings $\prec_x, \prec_y$ of the vertices of O_2. Since the vertices v_4, v_5, and v_6 pairwise have common out-neighbours, Proposition 2.23 implies that they appear in opposite order in $\prec_x$ and $\prec_y$. Suppose without loss of generality that $v_4 \prec_x v_5 \prec_x v_6$ and $v_6 \prec_y v_5 \prec_y v_4$. Now $v_2 \prec_x v_4 \prec_x v_5 \prec_x v_6$ and $v_2 \prec_y v_6 \prec_y v_5 \prec_y v_4$, which contradicts the fact that $\prec_x$ and $\prec_y$ are compatible. ■

Proposition 2.27. *The digraph O_3 given in Figure 2.5 is not chronological rectangle.*

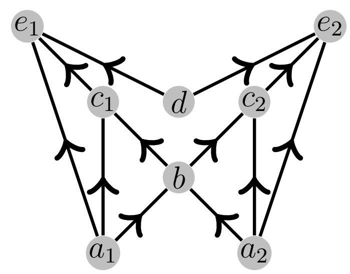

Figure 2.5: The digraph O_3 which is not chronological rectangle by Proposition 2.27.

Proof We prove the statement by contradiction. Suppose that $\prec_x$ and $\prec_y$ are a pair of compatible, fundamental orderings of O_3.

By Proposition 2.24, the vertices c_1 and e_2 appear in opposite order, so without loss of generality, $c_1 \prec_x e_2$ and $e_2 \prec_y c_1$. The fundamental property then implies that $c_2 \prec_y e_2 \prec_y c_1 \prec_y e_1$. We may now apply Proposition 2.24 again to c_2 and e_1 so that $e_1 \prec_x c_2$. The fundamental property of $\prec_x$ now gives that $c_1 \prec_x e_1 \prec_x c_2 \prec_x e_2$.

Now consider the placement of the vertex d in both orderings. By the fundamental property, $d \prec_x e_1 \prec_x c_2$ and $d \prec_y e_2 \prec_y c_1$. However, since d has common out-neighbours with c_1 and c_2, Proposition 2.23 implies that d appears in opposite order to both c_1 and c_2 in $\prec_x$ and $\prec_y$. As a consequence, $c_1 \prec_x d \prec_x c_2$ and $c_2 \prec_y d \prec_y c_1$.

Finally, consider ordering the vertex b. By the fundamental property, $b \prec_x c_1 \prec_x d \prec_x c_2$ and $b \prec_y c_2 \prec_y d \prec_y c_1$. However, $b \prec_x d \prec_x c_2$ and $b \prec_y d \prec_y c_1$ with $bc_2, bc_1 \in A$ and $bd \notin A$ contradicts compatibility. ■

In this chapter we have established properties of, and characterizations for, chronological rectangle digraphs. In Chapters 4 and 5, we study some particular classes of digraphs for which we can show that the weakly-clustered and umbrella path properties are sufficient for a digraph to be chronological rectangle.

Chapter 3

Chronological Interval Dimension

In the discussion prior to Theorem 2.1, we established that every bigraph is the intersection of a finite number of Ferrers bigraphs. This provides a simple argument that the Ferrers dimension is a defined for every bipartite graph. Analogously, it is well-known that the Ferrers dimension is defined for every digraph as well.

A complete graph with a single edge removed is an interval graph. Every graph is the intersection of a finite number of complete graphs with a single edge removed, so every graph is the intersection of a finite number of interval graphs. The *boxicity* of a graph G is the minimum integer k such that there exist k interval graphs whose intersection is G.

An analogous argument does not apply to intersections of chronological interval digraphs. A complete digraph with one arc removed is not chronological interval because it has strong components which are not complete. In fact, not all digraphs are the intersection of a finite number of chronological interval digraphs. For example, consider the digraph U_1 on the right of Figure 2.1 which does not satisfy the umbrella path property. Every chronological interval digraph H containing D satisfies the umbrella path property, so every H would include the arc v_1v_3, which is not in D. As a consequence, D is not the intersection of chronological interval digraphs.

The primary goal of this chapter is to characterize digraphs that are the intersection of some number of chronological interval digraphs. We then use this result to construct digraphs which are intersections of different numbers of chronological interval digraphs. Given a digraph D, we also explore the contrast between the number of chronological interval digraphs whose intersection is D and the boxicity of the underlying graph. To begin, we show that chronological rectangle digraphs are the intersections of at most two chronological interval digraphs.

3.1 Relation to Chronological Interval Digraphs

Roberts [54] defined graphs of boxicity at most 2 so that they are both the intersection graphs of axis-parallel rectangles, as well as the intersections of at most two interval graphs. Proposition 3.1 shows that the definition of chronological rectangle digraphs is analogous in that chronological rectangle digraphs are the intersections of at most two chronological interval digraphs.

Proposition 3.1. *A digraph $D = (V, A)$ is a chronological rectangle digraph if and only if there exist two chronological interval digraphs $D_x = (V, A_x)$ and $D_y = (V, A_y)$ such that $A_x \bigcap A_y = A$.*

Proof First assume that $D = (V, A)$ has a chronological rectangle model $R_v = [a_v, b_v] \times [c_v, d_v], v \in V$. Let $[a_v, b_v], v \in V$ be a chronological interval model for D_x, so that $uv \in A_x$ if and only if $a_v \in [a_u, b_u]$. Similarly take $[c_v, d_v], v \in V$ to be a chronological interval model for D_y, so that $uv \in A_y$ if and only if $c_v \in [c_u, d_u]$. Now $uv \in A$ if and only if $(a_v, c_v) \in [a_u, b_u] \times [c_u, d_u]$, which is precisely when both $a_v \in [a_u, b_u]$ and $c_v \in [c_u, d_u]$ so $uv \in A_x \bigcap A_y$.

Similarly, given that $A_x \bigcap A_y = A$, let $[a_v, b_v], v \in V$ be a chronological interval model for D_x while $[c_v, d_v], v \in V$ is a chronological rectangle model for D_y. Define $R_v = [a_v, b_v] \times [c_v, d_v]$, the Cartesian product of the intervals, to be the chronological rectangle model. Now $uv \in A$ if and only if both $a_v \in [a_u, b_u]$ and $c_v \in [c_u, d_u]$, so $(a_v, c_v) \in [a_u, b_u] \times [c_u, d_u]$. ■

Proposition 3.2 follows from the fact that D is the intersection of two chronological interval digraphs. Furthermore chronological interval digraphs are interval digraphs, which are known to be the intersections of at most two Ferrers digraphs whose intersection is complete [56].

Proposition 3.2. *If D is a chronological rectangle digraph, then D has Ferrers dimension at most 4. Furthermore, $D = F_1 \cap F_2 \cap F_3 \cap F_4$ where each $F_i, 1 \leq i \leq 4$ is a Ferrers digraph and $F_1 \cup F_2$ and $F_3 \cup F_4$ are complete digraphs.*

Proof If D is a chronological rectangle digraph then by Proposition 3.1 there exist chronological interval digraphs D_1, D_2 such that $D = D_1 \cap D_2$. Furthermore, every chronological interval digraph is an interval digraph. As discussed in the introduction, interval digraphs are the intersection of two Ferrers digraphs whose union is complete [56]. Hence there exist Ferrers digraphs such that $F_1 \cup F_2$ is complete and $D_1 = F_1 \cap F_2$, while at the same time $F_3 \cup F_4$ is complete and $D_2 = F_3 \cap F_4$. Hence $D = (F_1 \cap F_2) \cap (F_3 \cap F_4)$ so D has Ferrers dimension at most 4. ■

3.2 Chronological Interval Dimension Theorem

The purpose of this section is to prove Theorem 3.3, which is main result of this chapter. The theorem characterizes the digraphs which are the intersection of some number of chronological interval digraphs.

Theorem 3.3. *A digraph $D = (V, A)$ is the intersection of a finite number of chronological interval digraphs if and only if D satisfies the umbrella path property, all strong components of D are complete, and any two vertices in the same strong component have the same in-neighbourhood.*

Proof First suppose that $D = H_1 \cap H_2 \cap \cdots \cap H_k$ where H_i is a chronological interval digraph for all $1 \leq i \leq k$.

Let $v_1, v_2, \ldots, v_m$ be a directed path in D. If $v_1 v_m \in A(D)$ then the umbrella path property for chronological interval digraphs implies that $v_1 v_j \in A(H_i)$ for all $3 \leq j \leq m-1$ and $1 \leq i \leq k$. Therefore, $v_1 v_j \in A(D)$ for all $3 \leq j \leq m-1$ and D satisfies the umbrella path property.

Now suppose that $v_1, v_2, \ldots, v_m$ is a strong component C in the digraph D. Strong components are complete in each chronological interval digraph H_i. Hence, there is an arc between every pair of vertices of C in the intersection D. Similarly, if $uv_j \in A(D)$ for any $1 \leq j \leq m$ then since strong components have the same in-neighbourhood in any chronological interval digraph, $uv_i \in A(D)$ for all $1 \leq i \leq m$. Therefore, vertices of strong components have the same in-neighbourhood in D.

To prove the other implication, assume that D satisfies the umbrella path property, all strong components of D are complete and any two vertices in the same strong component have the same in-neighbourhood. Our strategy is to construct a digraph D_u for each vertex $u \in V(D)$ such that $V(D_u) = V(D)$, $A(D) \subseteq A(D_u)$, and for the fixed vertex u we have that $uv \in A(D)$ if and only if $uv \in A(D_u)$. Hence we will be able to conclude that $D = \cap_{u \in V} D_u$.

We define the following sets:

$$X_u = \{v \in V \mid v \text{ cannot be reached by a directed path starting at } u.\}$$

$$S_u = \{v \in V \mid uv \in S\}$$

$$T_u = \{v \in N^+(u)\} \backslash S_u$$

$$Y_u = \{v \in V \mid d(u,v) \geq 2\}$$

We note that the vertices reachable by directed path starting at u is exactly $V \backslash X_u = \{u\} \cup S_u \cup T_u \cup Y_u$.

We make some observations about the structure of these sets within the digraph D. First, there are no arcs from $v \in \{u\} \cup S_u \cup T_u \cup Y_u$ to $w \in X_u$. Suppose by way of contradiction that $vw \in A$. Following the directed path from u to v, the arc from v to w would give a path from u to w, a contradiction.

Secondly, there are no arcs from vertices $v \in T_u \cup Y_u$ to $w \in \{u\} \cup S_u$. Suppose by way of contradiction that $vw \in A$. We have a directed path from u to v, and now either an arc from v to u or a path from v to w to u (if w is in the same strong component as u). But then u and v are in the same strong component, contradicting the fact that $v \notin \{u\} \cup S_u$.

Finally, there are no arcs from $v \in Y_u$ to $w \in T_u$. Suppose by way of contradiction $vw \in A$. If $wv \in A$, then the fact that strong components have the same in-neighbourhood and $uw \in A$ implies that $uv \in A$, which contradicts $v \in Y_u$. Otherwise, $wv \notin A$ but there exists a directed path $u, t, y_1, \ldots, v, w$ and $uw \in A$ and by the umbrella path property on D, $uv \in A$, which again contradicts $v \in Y_u$.

Returning to the construction of the digraphs D_u, we now define a set of arcs A_u by:

$$A_u = \{vw | v \in X_u, w \in V; v \in S_u \cup T_u, w \in T_u; v \in T_u \cup Y_u, w \in Y_u\}$$

and use it to define the digraph D_u with $V(D_u) = V(D)$ and $A(D_u) = A(D) \cup A_u$.

The digraph D_u may be obtained from the digraph H with $V(H) = \{x, u, t, y\}$ and $A(H) = \{xu, xt, xy, ut, ty\}$ by replacing each vertex with a complete digraph (corresponding to $X_u, S_u \cup \{u\}, T_u, Y_u$ respectively) of the correct size. Since H is chronological interval, so is each D_u.

We claim that $D = \cap_{u \in V(D)} D_u$. Certainly $D \subseteq D_u$ for all $u \in V(D)$, so $D \subseteq \cap_{u \in V(D)} D_u$. Now for a fixed vertex u, $uv \in A(D)$ if and only if $uv \notin A(D_u)$. Therefore, $\cap_{u \in V(D)} D_u \subseteq D$. ∎

Corollary 3.4. *Given a digraph with no symmetric arcs, the chronological interval dimension of D is defined if and only if D satisfies the umbrella path property.*

Define a digraph $D = (V, A)$ to have *chronological interval dimension at most k* if there exist k chronological interval digraphs $D_1, D_2, \ldots, D_k$ whose intersection is D.

Corollary 3.5. *If $D = (V, A)$ is a digraph such that all strong components are complete, have the same in-neighbourhood, and the umbrella path property is satisfied, then D has chronological interval dimension at most $|V| = n$.*

Proof In the proof of Theorem 3.3 we showed that D is the intersection of at most $|V|$ digraphs, one for each vertex of D. ∎

We have not observed any digraphs which achieve this bound of n, so finding a tight bound on the chronological interval dimension of a digraph remains an open problem.

Theorem 2.6 can be extended to characterize all digraphs of chronological interval dimension $k \geq 1$.

Theorem 3.6. *Digraph $D = (V, A)$ has chronological interval dimension at most k if and only if there exist k functions $\phi_1, \phi_2, \ldots, \phi_k : V \to \mathbb{R}$ such that*

1. *If $uv \in A(D)$ then $\phi_i(u) \leq \phi_i(v)$ for all $1 \leq i \leq k$.*

2. *If there exist* w_i*,* $1 \leq i \leq k$ *such that* $uw_i \in A(D)$ *and* $\phi_i(u) \leq \phi_i(v) \leq \phi_i(w_i)$ *for all* $1 \leq i \leq k$ *then* $uv \in A(D)$*.*

The proof of Theorem 3.6 is similar to the function characterization for chronological rectangle digraphs, but extended from 2 to k dimensions (including the chronological interval case).

Similarly, Corollary 2.13 can be generalized to characterize digraphs of chronological interval dimension $k \geq 1$ which have no symmetric arcs.

Theorem 3.7. *A digraph* $D = (V, A)$ *with no symmetric arcs has chronological interval dimension at most* k *if and only if there exist* k *orderings* $\prec_1, \prec_2, \ldots, \prec_k$: $V \to \mathbb{R}$ *such that*

1. *if* $uv \in A(D)$ *then* $u \prec_i v$ *for all* $1 \leq i \leq k$*;*

2. *if there exist* w_i*,* $1 \leq i \leq k$ *such that* $uw_i \in A(D)$ *and* $u \prec_i v \prec_i w_i$ *for all* $1 \leq i \leq k$ *then* $uv \in A(D)$*.*

We now show that in general, a digraph of chronological interval dimension at most k with no symmetric arcs is the intersection of k chronological interval digraphs with no symmetric arcs.

Proposition 3.8. *If* D *is a digraph of chronological interval dimension at most* k *with no symmetric arcs, then there exist chronological interval graphs* $D_1, D_2, \ldots, D_k$ *with no symmetric arcs such that* $D = \cap_{i=1}^{k} D_i$*.*

Proof Let $C_1, C_2, \ldots, C_k$ be a collection of k chronological interval digraphs whose intersection is D. Strong components are complete in every chronological interval digraph, so we partition the vertices of each digraph C_i into complete subdigraphs $T_1, T_2, \ldots, T_m$ such that $uv \in S(D)$ if and only if $u, v \in T_j$.

For each C_i, we construct a digraph M_i which is a chronological interval digraph with no symmetric arcs such that $D = \cap_{i=1}^{k} M_i$.

Let $I_v = [a_v, b_v], v \in V(D)$ be a chronological interval representation for C_i. Consider a complete component T_j of C_i. If $u, v, w \in T_j$, then since D has no symmetric arcs either $uv \notin A(D)$ or $vu \notin A(D)$. Furthermore, since strong components are complete in any chronological interval digraph, if $uv, vw, wu \in A(D)$ then $uv, vu \in A(D)$, a contradiction. Hence we conclude that if $uv, vw \in A(D)$ then $wu \notin A(D)$. As a result, we can fix an ordering $v_1, v_2, \ldots, v_{|T_j|}$ such that if $v_p v_q \in A(D)$ then $p < q$ for each T_j.

Let a_v be the left endpoint of the intervals corresponding to the vertices in T_j. Let $d_w = \min_{u \in V(D)} \{a_u \text{ or } b_u | a_u > a_v, b_u > a_v\}$. Given $v_p \in T_j$, we construct a chronological interval model $J_{v_p}, v_p \in V(D)$ corresponding to a chronological interval digraph M_i such that

$$J_{v_p} = \begin{cases} I_{v_p} & \text{if } |T_j| = 1 \\ \left[a_{v_p} + k\dfrac{d_w - a_v}{|T_j|}, b_{v_p}\right] & \text{otherwise.} \end{cases}$$

The simple idea is to separate the left endpoints of vertices in complete components by using the ordering above to shift some of them a small distance to the right in such a way that the intersection of intervals does not change.

Let M_i be the chronological interval digraph obtained from $J_v, v \in V(D)$ by setting $uv \in A(M_i)$ if and only if $a_v \in J_u$. It follows that each M_i has no symmetric arcs, so we need only argue that $D = \cap_{i=1}^{k} M_i$. This holds since by construction, $(M_i/C_i) \cap D = \emptyset$ for all i, so $uv \in A(D)$ if and only if $uv \in \cap_{i=1}^{k} C_i$ if and only if $D = \cap_{i=1}^{k} M_i$. ■

Corollary 3.9. *If D is a chronological rectangle digraph with no symmetric arcs, then there exists a chronological rectangle model in which no two rectangles have the same left endpoint or the same lower endpoint.*

Using Proposition 3.8, to prove that a digraph D with no symmetric arcs is not chronological rectangle it is sufficient to prove that there do not exist chronological interval digraphs with no symmetric arcs whose intersection is D.

We may also generalize the weakly-clustered property from Proposition 2.3. We say that a digraph D is *weakly-k-clustered* if any two vertices in the same strong component have the same in-neighbourhood and the underlying bipartite graph induced by the arcs from C to $N^+(C)$ has Ferrers dimension at most k.

Proposition 3.10. *If D has chronological interval dimension at most k then D is weakly-k-clustered.*

The proof of Proposition 3.10 is identical to the proof of Proposition 2.3 generalized to k dimensions.

3.3 Digraphs of Each Chronological Interval Dimension

We conclude this chapter with Proposition 3.11, in which we construct digraphs of every possible chronological interval dimension.

Proposition 3.11. *For every positive integer m there exists a digraph of chronological interval dimension exactly m.*

Proof Any chronological interval digraph (for example a transitive tournament) has chronological interval dimension exactly one, so we assume that $m \geq 2$.

Let B^{2m-1} be the digraph with

$$V(B^{2m-1}) = \mathbb{Z}_{2m-1} \times \mathbb{Z}_2 \text{ and } A(B^{2m-1}) = \{(a_1, 0)(a_2, 1) \mid a_1 \neq a_2; a_1, a_2 \in \mathbb{Z}_{2m-1}\}.$$

This digraph is obtained from a complete bipartite graph $K_{2m-1,2m-1}$ with a perfect matching removed, by orienting from one part to the other.

First we argue that the chronological interval dimension of B^{2m-1} is at most m by constructing a family of digraphs which are chronological interval digraphs which contain B^{2m-1}.

Define the following sets of arcs $0 \leq i \leq m-2$:

$$\begin{aligned} Q_i &= \{\text{Arcs of a transitive tournament on the vertices} \\ &\quad (2i+1,0),(2i+2,0),\ldots,(2m-1),(0,0),\ldots,(2i,0)\} \end{aligned}$$

$$R_i = \{(2i,0)(2i,1),(2i+1,0)(2i+1,1)\}$$

$$P = \{(a,0)(a,1) \mid a \in \mathbb{Z}_{2m-1}\}$$

Let B_{2i}^{2m-1}, $0 \leq i \leq m-2$, be the digraph on $V(B^{2m-1})$ with $A(B_{2i}^{2m-1}) = Q_i \cup P \backslash R_i$. Furthermore take B_{2m-2}^{2m-1} to be the digraph on $V(B^{2m-1})$ with $A(B_{2m-2}^{2m-1}) = Q_{m-1} \cup P \backslash \{(2m-2,0)(2m-2,1),(0,0)(0,1)\}$. B_{2i}^{2m-1} is a digraph with a chronological order $(2i+1,0),(2i+2,0),\ldots,(2m-1,0),(1,0),(2,0),\ldots,(2i-1,0),(2i,1),(2i,0),(2i-1,1),(2i-2,1),\ldots,(1,1),(2m-1,1),(2m-2,1),\ldots,(2i+1,1)$. Since B_{2i}^{2m-1} has no symmetric arcs, we need only check that the ordering satisfies $uv \in A(B_{2i}^{2m-1})$ implies $u < v$ and $u < v < w$ with $uw \in A(B_{2i}^{2m-1})$ implies $uv \in A(B_{2i}^{2m-1})$. Both properties hold by inspection.

We have that $\cap_{i=0}^{m-1} B_{2i}^{2m-1} = B^{2m-1}$ since $A(B^{2m-1}) \subset B_{2i}^{2m-1}$ for all i; $(2i,0)(2i,1)$, $(2i+1,0)(2i+1,1) \notin A(B_{2i}^{2m-1})$; and $(j,0)(k,0) \notin A(B_{2i}^{2m-1})$ for $i \leq k$ if $k < j$ or $k \leq i$ if $j < k$.

Finally, we argue that the chronological interval dimension of B^{2m-1} is at least m. By Proposition 3.8, since B^{2m-1} is a digraph with no symmetric arcs, if B^{2m-1}

is the intersection of k chronological interval digraphs, then there exist chronological interval digraphs $D_i, i \in \mathbb{Z}_k$ with no symmetric arcs such that $B^{2m-1} = \cap_{i \in \mathbb{Z}_k} D_i$.

Consider the chronological orderings of such digraphs D_i. Each of the vertices $(j, 1)$, occurs after call of their in-neighbours $(k, 0), j \neq k$. Hence we order at least $2m - 2$ vertices $(j, 1)$ after all of the vertices $(k, 0)$. Note that we can also order one vertex $(j, 1)$ after all of the out-neighbours of $(j, 0)$. Each chronological interval digraph containing B^{2m-1} thus contains at least $2m - 3$ arcs of the form $(j, 0)(j, 1)$ which are not in $A(B^{2m-1})$. In particular, at most 2 arcs of the form $(j, 0)(j, 1)$ can be missing so we need a minimum of m digraphs so that each of the $2m - 1$ arcs are missing from some D_i. ■

3.4 Boxicity and Chronological Interval Dimension

Consider once again the digraph U_1 on the left side of Figure 2.1. This digraph does not satisfy the umbrella path property, so the chronological interval dimension of this digraph is not defined by Theorem 3.3. On the other hand, the underlying graph is a cycle on four vertices, which is known to have boxicity exactly two.

We construct a digraph Q such that the chronological interval dimension is strictly larger than the boxicity of the underlying graph of Q. To start, Proposition 3.12 shows that the digraph Q in Figure 3.1 has chronological interval dimension at least 3.

Proposition 3.12. *The digraph Q is not chronological rectangle.*

Proof Let the vertices be named as in Figure 3.1. The subdigraph of Q induced by v_1, v_2, v_3, w_1, w_2, and w_3 is an orientation of a 6-cycle from $X = \{v_1, v_2, v_3\}$ to $Y = \{w_1, w_2, w_3\}$. Note that a 6-cycle is a bipartite graph with Ferrers dimension 3. Furthermore, all vertices of Y are out-neighbours of either v_4. Since there is a directed path from v_1 to v_3 and v_4, all vertices of Y appear after all vertices of X in

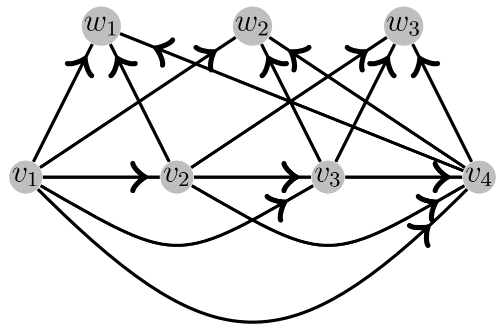

Figure 3.1: The digraph Q which is not chronological rectangle by Proposition 3.12.

any fundamental ordering. Hence, a pair of compatible, fundamental orderings for Q does not exist by Proposition 2.16. We conclude that Q is not chronological rectangle by Theorem 2.12. ■

On the other hand consider the following three orderings of the vertices of Q:

- $\prec_x$: $v_1, v_2, v_3, v_4, w_1, w_2, w_3$
- $\prec_y$: $v_1, v_2, v_3, v_4, w_2, w_3, w_1$
- $\prec_z$: $v_1, v_2, v_3, v_4, w_3, w_1, w_2$.

It is not difficult to verify that these orderings satisfy the hypotheses of Corollary 3.7. This proves that Q has chronological interval dimension 3.

At the same time, the rectangle model given in Figure 3.2 shows that the underlying graph of Q has boxicity at most 2. We conclude that the chronological interval dimension of a digraph may be strictly larger than the boxicity of the underlying graph.

To conclude this chapter, we show that given a chronological rectangle digraph, the boxicity of the underlying graph may in fact be arbitrarily large.

Proposition 3.13. *For any positive integer k there exists a chronological rectangle digraph whose underlying graph has boxicity exactly k.*

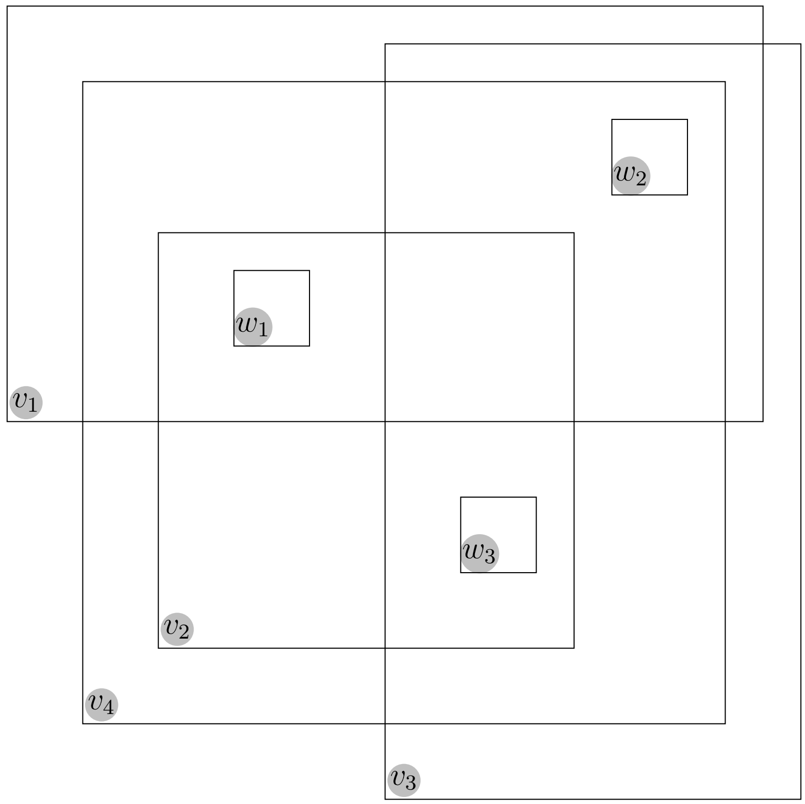

Figure 3.2: A boxicity at most 2 model for the underlying graph of Q.

Proof Consider the graph G_{2k} with vertex set $\mathbb{Z}_2 \times \mathbb{Z}_k$ and edges $(i_1, j_1)(i_2, j_2)$ where $j_1 \neq j_2$. This graph is obtained from a complete graph on $2k$ vertices by removing a perfect matching. Roberts [54] argued that such a graph has boxicity exactly k, because each pair of disconnected vertices must be separated in a different dimension than each other pair.

We prove that for any positive integer k, there is a digraph obtained by orienting G_{2k} and adding loops to every vertex which is chronological rectangle. Consider the digraph D_{2k} obtained from G_{2k} by adding loops, and orienting the edges so that $(i_1, j_1)(i_2, j_2) \in A$, where $j_1 < j_2$. Since there are no edges with $j_1 = j_2$, this completely determines an orientation.

To prove that D_{2k} is chronological rectangle, we consider the following pair of orderings:

$\prec_x$: $(1,1),(2,1),(1,2),(2,2),(1,3),(2,3),\ldots,(1,k),(2,k)$

$\prec_y$: $(2,1),(1,1),(2,2),(1,2),(2,3),(1,3),\ldots,(2,k),(1,k)$

The orderings $\prec_x$ and $\prec_y$ are fundamental because $(i_1,j_1)(i_2,j_2) \in A$ implies that $j_1 < j_2$, so that $(i_1,j_1) \prec_x (i_2,j_2)$ and $(i_1,j_1) \prec_y (i_2,j_2)$ by inspection.

Furthermore the pair of orderings $\prec_x, \prec_y$ are compatible, because all non-adjacencies are of the form $(1,i)(2,i)$, and these pairs of vertices appear in opposite order in $\prec_x$ and $\prec_y$. Hence the digraph D_{2k} is a chronological rectangle digraph by Theorem 2.12, whose underlying graph G_{2k} has boxicity exactly k. ■

In summary, we have exhibited that chronological rectangle digraphs are a natural generalization of chronological interval digraphs, and further extended the concept to digraphs with chronological interval dimension at most k. This idea has parallels to the generalization of interval graphs to boxicity at most k. However, the digraphs on which the chronological interval dimension of a digraph is defined have distinct and interesting structures of their own.

Chapter 4

Series-Parallel Digraphs

The primary goal of Chapter 4 is to prove Theorem 4.11, a characterization of the series-parallel digraphs which are chronological rectangle. We begin with some definitions and background to motivate our study of this problem.

4.1 Background

A *two-terminal graph* is a connected graph $(G; s, t)$ with a distinguished source vertex s and a distinguished sink vertex t such that $s \neq t$. We define the class of *two-terminal series-parallel graphs* recursively as follows:

- $(K_2; s, t)$ is a two-terminal series-parallel graph, where K_2 is the graph with $V(K_2) = \{s, t\}$ and $E(K_2) = \{st\}$.

- (series composition) Let $(G_1; s_1, t_1)$ and $(G_2; s_2, t_2)$ be disjoint two-terminal series-parallel graphs. Define *the series composition of* G_1 *with* G_2 to be the graph $(G; s_1, t_2)$ obtained from the union of G_1 and G_2 by identifying the sink t_1 of G_1 with the source s_2 of G_2 into a single vertex. The graph $(G; s_1, t_2)$ is a two-terminal series-parallel graph.

- (parallel composition) Let $(G_1; s_1, t_1)$ and $(G_2; s_2, t_2)$ be disjoint two-terminal series-parallel graphs. Define *the parallel composition of* G_1 *and* G_2 to be the graph $(G; s, t)$ obtained from the union of G_1 and G_2 by identifying s_1 and s_2 into a single source vertex s and identifying t_1 and t_2 into a single sink vertex t. The graph $(G; s, t)$ is a two-terminal series-parallel graph.

- There are no other two-terminal series-parallel graphs.

Two-terminal series-parallel graphs can be characterized in terms of subdivisions and homeomorphisms. Define an *elementary subdivision* of a graph G to be a graph obtained by removing an edge $uv \in E$ and adding a new vertex w together with edges uw, wv. A *subdivision* of a graph G is any graph obtained by a (possibly empty) series of elementary subdivisions. Two graphs G_1 and G_2 are *homeomorphic* if there exists an isomorphism from some subdivision of G_1 to a subdivision of G_2. Using this vocabulary, the work of Duffin [26] gives a characterization of two-terminal series-parallel graphs.

Theorem 4.1. [26] *A two-terminal graph is a two-terminal series-parallel graph if and only if it does not contain a subgraph homeomorphic to* K_4.

Bohra, Chandran, and Raju [9] showed that there exist two-terminal series-parallel graphs of boxicity 3. However, planar graphs have boxicity at most 3 [59], so two-terminal series-parallel graphs have boxicity at most 3. We will define a natural digraph analogue of two-terminal series-parallel graphs and determine precisely which digraphs in this class are chronological rectangle. In addition, we show that there are two-terminal digraphs in this class with chronological interval dimension at least 3.

A *two-terminal digraph* $(D; s, t)$ is a connected digraph with a unique source vertex s of in-degree 0 and a unique sink vertex t of out-degree 0 where $s \neq t$. We define the class of *two-terminal series-parallel digraphs*, recursively as follows:

- $(P_2; s, t)$ is a two-terminal series-parallel digraph, where P_2 is the digraph with $V(P_2) = \{s, t\}$ and $A(P_2) = \{st\}$.

- (series composition) Let $(D_1; s_1, t_1)$ and $(D_2; s_2, t_2)$ be disjoint two-terminal series-parallel digraphs. Define *the series composition of* D_1 *with* D_2 to be the digraph $(D; s_1, t_2)$ obtained from the union of D_1 and D_2 by identifying the sink t_1 of D_1 with the source s_2 of D_2 into a single vertex. The digraph $(D; s_1, t_2)$ is a two-terminal series-parallel digraph.

- (parallel composition) Let $(D_1; s_1, t_1)$ and $(D_2; s_2, t_2)$ be disjoint two-terminal series-parallel digraphs. Define *the parallel composition of* D_1 *and* D_2 to be the digraph $(D; s, t)$ obtained from the union of D_1 and D_2 by identifying s_1 and s_2 into a single source vertex s and identifying t_1 and t_2 into a single sink vertex t. The digraph $(D; s, t)$ is a two-terminal series-parallel digraph.

- There are no other two-terminal series-parallel digraphs.

Two-terminal series-parallel digraphs are obtained by orienting series-parallel graphs from the source to the sink and adding loops to every vertex. They are connected digraphs with no symmetric arcs by definition. We will simply use D to denote a two-terminal series-parallel digraph $(D; s, t)$ where the identities of the source and sink are either clear in the context or not relevant.

Let D be the series composition of D_1 with D_2. If $uv \in A(D)$, then either $uv \in A(D_1)$ or $uv \in A(D_2)$, but not both. On the other hand, take $(D; s, t)$ to be parallel composition of $(D_1; s, t)$ with $(D_2; s, t)$. In this case $uv \in A(D)$ implies that either $uv \in A(D_1)$ or $uv \in A(D_2)$ where $uv \in A(D_1) \cap A(D_2)$ is only possible when $uv = st$. This simple observation will be used in some of the proofs later in this chapter.

To state the characterization of two-terminal series-parallel digraphs, we provide the definitions for subdivisions and homeomorphisms of digraphs. An *elementary subdivision* of a digraph D is a digraph obtained by removing an arc $uv \in A$ and adding a new vertex w together with arcs uw, wv. A *subdivision* of a digraph D is any digraph obtained by a (possibly empty) series of elementary subdivisions. Two digraphs D_1 and D_2 are *homeomorphic* if there exists an isomorphism from some subdivision of D_1 to a subdivision of D_2. Tarjan and Lawler [60] remark that Duffin's proof of the characterization of two-terminal series-parallel graphs can be modifed to prove Theorem 4.2.

Theorem 4.2. *A two-terminal digraph D is a two-terminal series-parallel digraph if and only if D has no subdigraph homeomorphic to the digraph W given in Figure 4.1.*

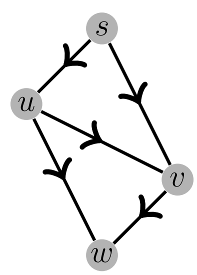

Figure 4.1: The two-terminal series-parallel digraph obstruction W.

4.2 Forbidden Induced Subdigraphs

Now that we have stated the characterization of two-terminal series-parallel digraphs, we pivot to determining which digraphs in this class are chronological rectangle digraphs. By Corollary 3.4, we know that the chronological interval dimension of a digraph D with no symmetric arcs is defined if and only if D satisfies the umbrella path property. As a consequence, there are two-terminal series-parallel digraphs for which the chronological interval dimension is not defined. For example, consider the digraph U_1 given in Figure 4.2 which does not satisfy the umbrella path property.

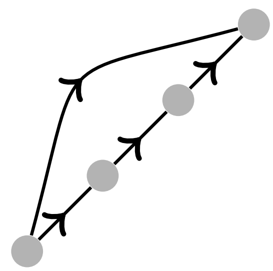

Figure 4.2: The two-terminal series-parallel digraph U_1.

We can obtain U_1 by taking the parallel composition of a directed path on two vertices with a directed path on four vertices, so it is a two-terminal series-parallel digraph for which the chronological interval dimension is not defined.

In general, the umbrella path property is preserved by series composition. On the other hand, the construction of U_1 illustrates that series composition does not preserve the umbrella path property. We thus consider a restricted parallel operation and the corresponding subclass of two-terminal series-parallel digraph. Define the class of *two-terminal series-path-parallel digraphs* $(D; s, t)$ recursively as follows:

- $(P_2; s, t)$ is a two-terminal series-path-parallel digraph.

- (series composition) Let $(D_1; s_1, t_1)$ and $(D_2; s_2, t_2)$ be disjoint two-terminal series-path-parallel digraphs. The series composition $(D; s_1, t_2)$ of D_1 with D_2 is a two-terminal series-path-parallel digraph.

- (path-parallel composition) Let $(D_1; s_1, t_1)$ and $(D_2; s_2, t_2)$ be disjoint two-terminal series-path-parallel digraphs, such that if at least one of them has an arc from the source to the sink, then the other has arcs from the source to all vertices except possibly the sink. Define *the path-parallel composition of* D_1 *and* D_2 to be the digraph D obtained from the union of D_1 and D_2 by identifying s_1 and

s_2 into a single source vertex s and identifying t_1 and t_2 into a single sink vertex t. The digraph $(D; s, t)$ is a two-terminal series-path-parallel digraph.

- There are no other two-terminal series-path-parallel digraphs.

The digraph produced by path-parallel composition does not depend on the order of the two digraphs being composed, so the path-parallel composition is commutative. The constraint on parallel composition that if at least one digraph has an arc from the source to the sink, then the other has arcs from the source to all vertices except possibly the sink implies the umbrella path property for two-terminal series-parallel digraphs. Furthermore if a two-terminal series-parallel digraph satisfies the umbrella path property then parallel composition involving at least one digraph with an arc from the source to the sink produces a digraph with arcs from the source to all other vertices, so the parallel composition is a path-parallel composition. As a consequence, the two-terminal series-path-parallel digraphs are precisely the two terminal series-parallel digraphs which satisfy the umbrella path property.

The two-terminal series-path-parallel digraphs are digraphs with no symmetric arcs which satisfy the umbrella path property. By Corollary 3.4 the chronological interval dimensions of the digraphs in this class are defined. However, we will show that not all two-terminal series-path-parallel digraphs are chronological rectangle.

As the first step in constructing a two-terminal series-path-parallel digraph which is not chronological rectangle, consider the digraph D_b given in Figure 4.3 below.

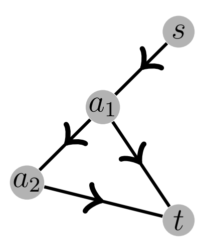

Figure 4.3: The two-terminal series-path-parallel digraph D_b.

This digraph is obtained by series composition of two copies of P_2, path-parallel composition of the result with another copy of P_2, and then series composition of the result with one more copy of P_2. Therefore, D_b is a two-terminal series-path-parallel digraph.

Let D_{3b} be the digraph obtained by path-parallel composition of three copies of D_b, which is illustrated in Figure 4.4. Since D_b is a two-terminal series-path-parallel digraph, D_{3b} is a two-terminal series-path-parallel digraph as well.

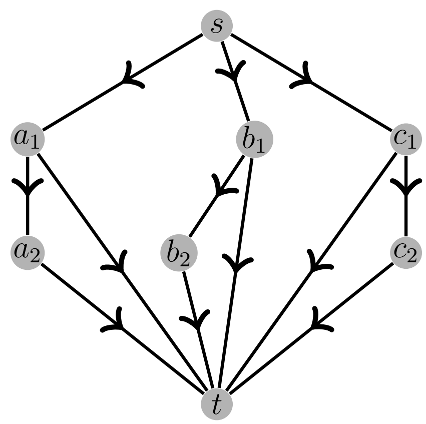

Figure 4.4: The two-terminal series-path-parallel digraph D_{3b} which is not chronological rectangle by Proposition 4.3.

Proposition 4.3. *The digraph D_{3b} is not chronological rectangle.*

Proof Suppose the vertices of D_{3b} are named as in Figure 4.4. We prove the statement by contradiction. Assume that D_{3b} is a chronological rectangle digraph. By Theorem 2.12, there exists a pair of compatible, fundamental orderings $\prec_x$ and $\prec_y$ of D_{3b}.

Since the orderings $\prec_x$ and $\prec_y$ are fundamental, they both begin with s and end with t. The fundamental property also guarantees that both $b_1 \prec_x b_2$ and $b_1 \prec_y b_2$. Due to symmetry we may assume without loss of generality that $a_1 \prec_x b_1 \prec_x c_1$. Since $a_1t, b_1t, c_1t \in A$, by Proposition 2.23 we have that $c_1 \prec_y b_1 \prec_y a_1$.

Now consider the position of b_2 relative to a_1 and c_1 in both orderings. If $c_1 \prec_x b_2 \prec_x t$ then the previous assumption that $c_1 \prec_y b_1 \prec_y b_2 \prec_y t$ would give a pair of orderings which are not compatible. Applying the same reasoning gives a contradiction when $a_1 \prec_y b_2 \prec_y t$. However, the remaining possibility that both $s \prec_x b_2 \prec_x c_1$ and $s \prec_y b_2 \prec_y a_1$ with $sc_1, sa_1 \in A$ but $sb_2 \notin A$ would contradict the compatibility of $\prec_x$ and $\prec_y$. We conclude that a pair of compatible, fundamental orderings of D_{3b} does not exist. ■

By Corollary 3.4 and Lemma 4.3, D_{3b} has chronological interval dimension at least 3. In fact, we can determine that the chronological interval dimension of D_{3b} is exactly three. Consider the three orderings $\prec_1, \prec_2, \prec_3$ given below:

- $\prec_1$: $s, a_1, a_2, b_1, b_2, c_1, c_2, t$
- $\prec_2$: $s, b_1, b_2, c_1, c_2, a_1, a_2, t$
- $\prec_3$: $s, c_1, c_2, a_1, a_2, b_1, b_2, t$

Using Theorem 3.7, the ordering characterization for digraphs of chronological interval dimension $k \geq 1$ with no symmetric arcs, we conclude that the chronological interval dimension of D_{3b} is less than or equal to 3. Together with Proposition 4.3 this proves that the chronological interval dimension of D_{3b} is exactly 3.

Our next goal is to define an infinite family of two-terminal series-path-parallel digraphs other than D_{3b} which are not chronological rectangle digraphs.

First, define D_{2b} to be the digraph obtained by path-parallel composition of two copies of D_b so that D_{2b}, given in Figure 4.5, is a two-terminal series-path-parallel digraph.

Consider the structure of a pair of compatible, fundamental orderings $\prec_x, \prec_y$ of D_{2b}. Without loss of generality, there are two cases for the relative order of u_1 and v_2

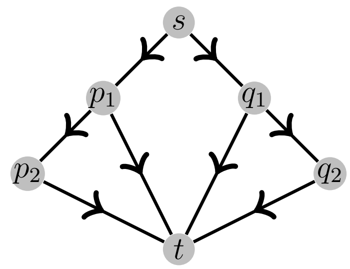

Figure 4.5: The two-terminal series-path-parallel digraph D_{2b}.

in $<_x$. If $u_1 \prec_x v_2$, then since u_1 and v_2 have a common out-neighbour r_1, we apply Proposition 2.23 to conclude $v_2 \prec_y u_1$. Together with the fundamental property of $\prec_y$ this guarantees that $s \prec_y v_1 \prec_y v_2 \prec_y u_1 \prec_y u_2 \prec_y t$. Now v_1 and u_2 also have a common out-neighbour so by Proposition 2.23 we have $u_2 \prec_x v_1$. Now the fundamental property of $\prec_x$ gives $s \prec_x u_1 \prec_x u_2 \prec_x v_1 \prec_x v_2 \prec_x t$. Note that the other case for the relative order for u_1 and v_2 in $\prec_x$ gives the same two orderings with the roles of $\prec_x$ and $\prec_y$ exchanged.

Let B_n be the digraph obtained by series composition of D_{2b} with a directed path on $n \geq 2$ vertices. The digraph B_4 is given in Figure 4.6.

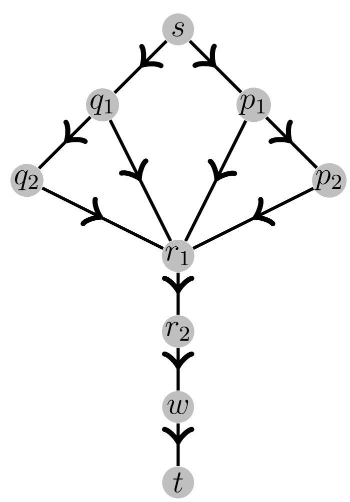

Figure 4.6: The digraph B_4 formed by series composition of D_{2b} with a directed path on 4 vertices.

We define $D_{p(n)}$ to be the digraph obtained by path-parallel composition of B_n with a directed path on exactly 3 vertices. For example, $D_{p(4)}$ is shown in Figure 4.7. We prove that the two-terminal series-path-parallel digraphs $D_{p(n)}$ for $n \geq 2$ form an infinite family of digraphs which are not chronological rectangle.

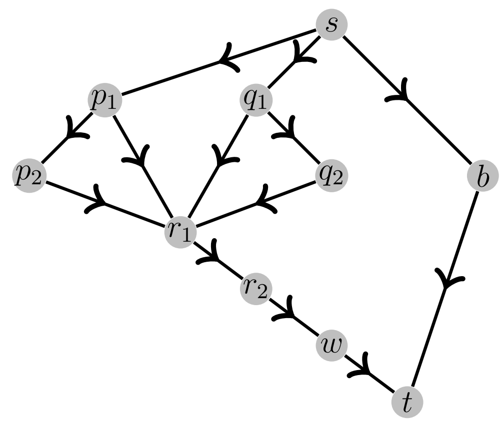

Figure 4.7: The two-terminal series-path-parallel digraph $D_{p(4)}$ obtained by path-parallel composition of B_4 with a directed path on 3 vertices. This is a member of the family of digraphs $D_{p(n)}$ with $n = 4$.

Proposition 4.4. *The digraph $D_{p(n)}$ is not chronological rectangle for $n \geq 2$.*

Proof Let the vertices be defined as in Figure 4.7, where r_1, r_2 will generally be replaced by a directed path on $n-2$ vertices $r_1, r_2, \ldots, r_{n-2}$. We prove the statement by contradiction. Assume that $D_{p(n)}$ is a chronological rectangle digraph. By Theorem 2.12 there exists a pair of compatible, fundamental orderings $\prec_x$ and $\prec_y$ of D_{3b}.

By the argument following the definition of D_{2b}, we may use the symmetry of the digraph to assume that $p_1 \prec_x p_2 \prec_x q_1 \prec_x q_2$ and $q_1 \prec_y q_2 \prec_y p_1 \prec_y p_2$.

Observe that w and b have a common out-neighbour t, so by Proposition 2.23 we may assume without loss of generality that $w \prec_x b$ and $b \prec_y w$. Since $\prec_x$ is fundamental, we have $q_2 \prec_x w \prec_x b$. Now $s \prec_x q_2 \prec_x b$ and $s \prec_y q_2 \prec_y p_1$ with $sb, sp_1 \in A$ but $sq_2 \notin A$, contradicting the compatibility of $\prec_x$ and $\prec_y$. We conclude that a pair of compatible, fundamental orderings of $D_{p(n)}$ does not exist. ■

Once again we can establish that the chronological interval dimension of $D_{p(n)}$ is exactly 3 for all $n \geq 2$. Consider the three orderings $\prec_1, \prec_2, \prec_3$ given below:

- $\prec_1$: $s, u_1, u_2, v_1, v_2, r_1, r_2, \ldots, w, b, t$
- $\prec_2$: $s, b, u_1, u_2, v_1, v_2, r_1, r_2, \ldots, w, t$
- $\prec_3$: $s, b, v_1, v_2, u_1, u_2, r_1, r_2, \ldots, w, t$

It is not difficult to verify that these three orderings satisfy the hypotheses of Theorem 3.7. Combined with Proposition 4.4, this shows that the chronological interval dimension of $D_{p(n)}$ is exactly 3 for each $n \geq 2$.

We thus conclude that a two-terminal series-path-parallel digraph which is chronological rectangle has no induced subdigraph isomorphic to $D_{p(n)}$ for $n \geq 2$ or D_{3b}. The challenge for the rest of this chapter is to prove the other implication of Theorem 4.1.

4.3 Partitioning Vertices

Before stating and proving the main theorem, we introduce some additional notation and give a series of lemmas to describe a special partition of the vertices of any two-terminal series-path-parallel digraph. This partition will be used to construct pairs of compatible, fundamental orderings of two-terminal series-path-parallel digraphs

We start with Lemma 4.5 which uses Theorem 4.2 to describe some structure of vertices which are out-neighbours of the source vertex and in-neighbours of the sink vertex.

Lemma 4.5. *If $(D; s, t)$ is a two-terminal series-path-parallel digraph and $y_1, y_2 \in N^+(s) \cap N^-(t)$, then $y_1y_2, y_2y_1 \notin A$. Furthermore, there are no directed paths from y_1 to y_2 or from y_2 to y_1.*

Proof We prove the statement by contradiction. Suppose that D is a two-terminal series-path-parallel digraph with $y_1y_2 \in A$. Let H be the subdigraph of D consisting of the vertices s, y_1, y_2, t and the arcs $sy_1, sy_2, y_1y_2, y_1t, y_2t$. This subdigraph is isomorphic to the digraph W in Figure 4.1, contradicting the fact that D is a two-terminal series-parallel digraph by Theorem 4.2. The same argument applies when $y_2y_1 \in A$.

Now suppose that there is a directed path $y_1, v_1, v_2, \ldots, y_2$. Now since $y_1t, y_2t \in A$, the umbrella path property implies that $y_1y_2 \in A$, which we already argued is a contradiction. If there is a directed path from y_1 to y_2 then an analogous arguments applies. ■

As a consequence of Lemma 4.5, we assume that all two-terminal series-path-parallel digraphs $(D; s, t)$ satisfy the property that there are no directed paths between vertices of $N^+(s) \cap N^-(t)$.

Given a two-terminal series-path-parallel digraph $(D; s, t)$, let Y denote the vertices of $N^+(s) \cap N^-(t)$, not including s or t; we also let $Y_i = \{y_i\}$. In addition, X denotes the vertices of D which lie on directed paths from s to vertices of Y, not including s or the vertices of Y. In Lemma 4.6, we show that the vertices of X lie on directed paths to exactly one vertex of Y.

Lemma 4.6. *If $(D; s, t)$ is a two-terminal series-path-parallel digraph and $y_1, y_2 \in Y$, then for every vertex $x \neq s$ for which there exists a directed path from x to y_1, there is no directed path from x to y_2.*

Proof We prove the statement by contradiction. Suppose that there are directed paths from x to y_1 and y_2. In particular, define $P = x, p_1, p_2, \ldots, p_\alpha, y_1$ and $Q = x, q_1, q_2, \ldots, q_\beta, y_2$ to be directed paths from x to y_1 and y_2 respectively. Note that since there is a directed path from s to y_1 through x and $sy_1 \in A$, the umbrella path property implies that $sx \in A$. Define a subdigraph H of D whose vertex set is the vertices of P and Q as well as s and t. For the arcs of H, take sx, xp_1, $p_1p_2, \ldots$, $p_\alpha y_1$, y_1t, xq_1, $q_1q_2, \ldots$, $q_\beta y_2$, sy_2, and y_2t. The digraph H is a subdivision of a digraph H' with vertices s, x, y_2, and t as well as arcs sx, sy_2, xt, xy_2, and y_2t. Now H is isomorphic to the digraph W, contradicting Theorem 4.2. ■

Let Z denote the vertices which lie on directed paths from Y to t, not including vertices of Y or t. Note that if $y \in Y$ and there is directed path from y to z, then $yt \in A$ and the umbrella path property implies that $yz \in A$. Thus it suffices to consider out-neighbours of y other than t.

Lemma 4.7. *If $(D; s, t)$ is a two-terminal series-path-parallel digraph and $y_1, y_2 \in Y$, then for every vertex $z \neq t$ for which $y_1z \in A$, we have $y_2z \notin A$.*

Proof We prove the statement by contradiction using the same strategy as the proof of Lemma 4.6. Suppose that $y_1z, y_2z \in A$, and let $P : z, p_1, p_2, \ldots, p_\alpha, t$ be the path from z to t. Consider a subdigraph H with the vertices of P together with s, y_2, and y_1. Let the arcs of H be $sy_1, y_1z, sy_2, y_2z, y_2t, zp_1, p_1p_2, \ldots, p_\alpha t$. Now H is a subdigrah of D which is isomorphic to a subdivision of W, contradicting Theorem 4.2. ■

Note that there are no directed paths from vertices of Z to vertices of X or Y, since this would produce either a directed cycle or a directed path between two distinct vertices of Y, contradicting Lemma 4.6.

Lemma 4.8. *If $(D; s, t)$ is a two-terminal series-path-parallel digraph with $v \in X \cup Y$ and $z \in Z$, then $vz \in A$ implies $v \in Y$.*

Proof We prove the lemma by contradiction. Suppose that $v \notin Y$ with $vz \in A$. We consider two cases.

First suppose there is a directed path $P : v, p_1, p_2, \ldots, p_\alpha, y, z$ from v to z through a vertex $y \in Y$. Consider the subdigraph H of D with the vertices of P together with

s. Define the arcs of H to be sv, sy, vp_1, p_1p_2, …, $p_\alpha y$, yz, vz. This subdigraph H is a subdivision of a digraph isomorphic to W, in contradiction of Theorem 4.2.

The other case is when v lies on a directed path $P : s, v, p_1, p_2, \ldots, p_\alpha, y_1$ from s to $y_1 \in Y$ while z lies on a directed path $Q : y_2, z, q_1, q_2, \ldots, q_\beta, t$ from $y_2 \in Y$ to t. Define a subdigraph H with the vertices of P and Q together with the arcs sv, sy_2, vz, y_2z, vp_1, $p_1p_2, \ldots, p_\alpha y_1$, y_1t, zq_1, $q_1q_2, \ldots, q_\beta t$. Now H is a subdigraph of D which is isomorphic to a subdivision of W, a contradiction. ■

Let L denote the out-neighbours of s which are neither elements of X nor Y. Denote the remaining vertices of D by $M = V - (X \cup Y \cup Z \cup L \cup \{s, t\})$. Note that all vertices of M lie on directed paths from s to t through vertices of L.

Lemma 4.9. *If $(D; s, t)$ is a two-terminal series-path-parallel digraph $\ell \in L$ and $m \in M$ then there are no directed paths from m to ℓ.*

Proof If there is a directed path from ℓ to t through m then there would a directed cycle on at least three vertices through ℓ and m, a contradiction. Otherwise there are directed paths $P : s, \ell, p_1, p_2, \ldots, p_\alpha, t$, $Q : s, \ell', q_1, q_2, \ldots, q_i, m, q_{i+1}, \ldots, q_\beta, t$, and $R : m, r_1, r_2, \ldots, r_\gamma, \ell$. Define a subdigraph H of D which is induced by the vertices of P, Q, and R. The digraph H will have a subdigraph isomorphic to a subdivision of W, a contradiction. ■

Lemma 4.10. *If $(D; s, t)$ is a two-terminal series-path-parallel digraph with $u \in X \cup Y \cup Z$ and $v \in L \cup M$ then there are no directed paths between u and v.*

Proof Consider the subdigraph H of D consisting of the directed paths from s to t through u and v. Note that H is a subdivision of a digraph isomorphic to W, using the vertices s, u, v, t. ■

We denote the sets of the vertex partition of the digraph D by $X(D)$, $Y(D)$, $Z(D)$, $L(D)$, and $M(D)$ respectively. We particularly need to distinguish between these sets when we have two digraphs D_1 and D_2. Whenever we have a digraph D_i, we use the notation $X^i = X(D_i)$, $Y^i = Y(D_i)$, $Z^i = Z(D_i)$, $L^i = L(D_i)$, $M^i = M(D_i)$.

As a consequence of Lemma 4.6, we can partition $X(D)$ into $X_1, X_2, \ldots, X_c$, where X_i are vertices other than s which have directed paths to a particular $y_i \in Y$. Similarly Lemma 4.7 shows that we can partition $Z(D)$ into $Z_1, Z_2, \ldots, Z_c$ where Z_i are the out-neighbours of a particular vertex $y_i \in Y$, other than t. Furthermore, having no induced subdigraph isomorphic to D_{3b} implies that at most 2 of the Z_i are nonempty; we assume that these are Z_1 and Z_2. Figure 4.8 illustrates a partition of the vertices of a two-terminal series-path-parallel digraph.

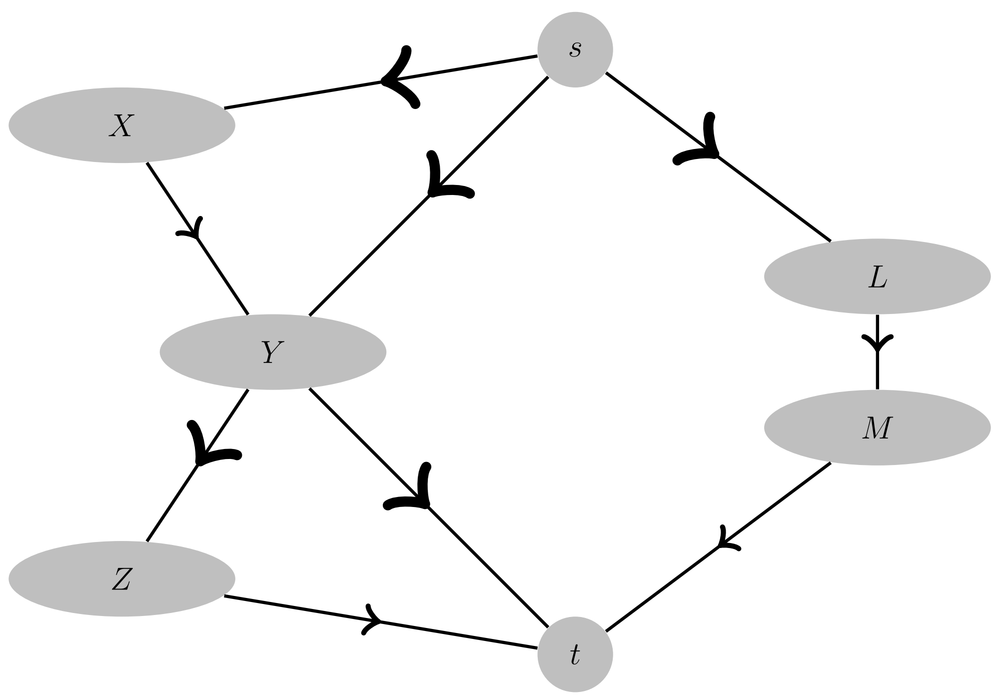

Figure 4.8: The partition of a two-terminal series-path-parallel digraph.

4.4 Main Theorem

The main result of this chapter is Theorem 4.11, which says that D_{3b} and the family $D_{p(n)}$ are the only minimal induced subdigraphs of two-terminal series-path-parallel digraphs which are not chronological rectangle digraphs.

Theorem 4.11. *A two-terminal series-path-parallel digraph is chronological rectangle if and only if it has no induced subdigraph isomorphic to $D_{p(n)}$ for $n \geq 2$ or D_{3b}.*

Proof We have already proved in Propositions 4.3 and 4.4 that a two-terminal series-path-parallel digraph does not have any induced subdigraph isomorphic to $D_{p(n)}$ for $n \geq 2$ or D_{3b}.

To prove the other direction of the theorem, we construct pairs of compatible, fundamental orderings for two-terminal series-path-parallel digraphs recursively from P_2. In general, if the last operation in the construction of D is series composition, then we construct a pair of orderings by concatenating a pair of compatible, fundamental orderings for the digraphs D_1 and D_2.

Now suppose that D is formed by path-parallel composition of two digraphs D_1 and D_2. We prove that if D_1 and D_2 each has a pair of compatible, fundamental orderings then D does as well by considering three cases for the structure of D. Case 1 is when D is induced by $\{s, t\}$ and vertices of X, Y, and Z; in other words, D has no vertices of L or M. Case 2 is when D is induced by $\{s, t\}$ and vertices of L, M, so that D has no vertices of X, Y or Z. In the most general case we suppose that D has vertices of X, Y, Z, L, M, and $\{s, t\}$.

First we establish the result in case 1 by considering three subcases: when D has no induced subdigraph isomorphic to D_b, D_{2b}, or D_{3b} respectively.

If D has no induced subdigraph isomorphic to D_b and no vertices of L or M then D is formed by path-parallel composition of two-terminal series-path-parallel digraphs D_1 and D_2 with no induced subdigraph isomorphic to D_b and no vertices of L or M. Suppose that D_i has a pair of compatible, fundamental orderings which satisfy $s \prec_1 X^i \cup Y^i \prec_1 t$ and $s \prec_2 X^i \cup Y^i \prec_2 t$ for $i = 1, 2$. We can construct a pair of compatible, fundamental orderings for D which satisfy the same property by setting $s \prec_x X^1 \cup Y^1 \prec_x X^2 \cup Y^2 \prec_x t$ and $s \prec_y X^2 \cup Y^2 \prec_y X^1 \cup Y^1 \prec_y t$ where the vertices follow the same relative ordering as $\prec_1$ in $\prec_x$ and $\prec_2$ in $\prec_y$.

To argue that the orderings we have constructed are fundamental, recall the fact that $uv \in A(D)$ implies that either $uv \in A(D_1)$ or $uv \in A(D_2)$. In any case, the fact that $\prec_x$ and $\prec_y$ are fundamental follows from the fact that $\prec_1$ and $\prec_2$ are fundamental orderings of D_1 or D_2.

We also argue that $\prec_x$ and $\prec_y$ are a pair of compatible orderings. Note that if $u \prec_x v$ and $u \prec_y v$, then u and v are both vertices of either D_1 or D_2 by construction of the orderings. Furthermore, all out-neighbours of u belong to the same digraph. Therefore, compatibility follows from the compatibility of $\prec_1$ and $\prec_2$.

Next suppose D has no induced subdigraph isomorphic to D_{2b} and no vertices of L or M. We may assume without loss of generality that D is formed by path-parallel composition of D_1 and D_2, where D_1 is induced by the vertices of $\{s, t\}$, X_1, Y_1, and Z_1. Given a chronological rectangle digraph with this structure, any pair of compatible, fundamental orderings satisfy $s \prec_1 X^1 \cup Y^1 \prec_1 Z^1 \prec_1 t$ and $s \prec_2 X^1 \cup Y^1 \prec_1 Z^1 \prec_1 t$. We construct a pair of compatible, fundamental orderings for D by taking $s \prec_x X^2 \cup Y^2 \prec_x X^1 \cup Y^1 \prec_x Z^1 \prec_x t$ and $s \prec_y X^1 \cup Y^1 \prec_y Z^1 \prec_y X^2 \cup Y^2 \prec_y t$, again where the vertices follow the same relative ordering as $\prec_1$ in $\prec_x$ and $\prec_2$ in $\prec_y$. The proof that these orderings are compatible and fundamental is analogous to the previous subcase.

Suppose that D has no induced subdigraph isomorphic to D_{3b} and no vertices of L or M. We once again assume that D is formed by path-parallel composition of D_1

and D_2 where D_1 is induced by the vertices of $\{s,t\}$, X_1, Y_1, and Z_1. Furthermore, D_1 has a pair of compatible, fundamental orderings satisfying $s \prec_1 X^1 \cup Y^1 \prec_1 Z^1 \prec_1 t$ and $s \prec_2 X^1 \cup Y^1 \prec_1 Z^1 \prec_1 t$. As a result of the previous argument, D_2 has a pair of compatible, fundamental orderings which satisfy $s \prec_1 X^2 \cup Y^2 - (X_1^2 \cup Y_1^2) \prec_1 X_1^2 \cup Y_1^2 \prec_1 Z_1^2 \prec_1 t$ and $s \prec_2 X_1^2 \cup Y_1^2 \prec_2 Z_1^2 \prec_2 X^2 \cup Y^2 - (X_1^2 \cup Y_1^2) \prec_2 t$. A pair of compatible, fundamental orderings for D can be constructed as follows:

$$\prec_x\colon s, X^1 \cup Y^1, Z^1, X^2 \cup Y^2 - (X_1^2 \cup Y_1^2), X_1^2 \cup Y_1^2, Z_1^2, t$$

$$s, X_1^2 \cup Y_1^2, Z_1^2, X^2 \cup Y^2 - (X_1^2 \cup Y_1^2), X^1 \cup Y^1, Z^1, t$$

where the vertices follow the same relative ordering as $\prec_1$ in $\prec_x$ and $\prec_2$ in $\prec_y$. The proof that these orderings are fundamental and compatible are similar to the other subcases, so this completes the proof of Case 1.

For Case 2 we again consider two subcases. First, suppose that D_1 and D_2 have no induced subdigraph isomorphic to B_n for $n \geq 2$ which contains s and t and a pair of compatible, fundamental orderings which satisfy $s \prec_1 L^i \prec_1 M^i \prec_1 t$ and $s \prec_2 L^i \prec_2 M^i \prec_2 t$ for $i = 1, 2$. We construct a pair of fundamental, compatible orderings $s \prec_x L^1 \prec_x L^2 \prec_x M^2 \prec_x M^1 \prec_x t$ and $s \prec_y L^2 \prec_y L^1 \prec_y M^1 \prec_y M^2 \prec_y t$.

The other subcase is when the digraphs D_1 and D_2 may have induced subdigraphs isomorphic to B_n for $n \geq 2$ which contain s and t. Note that in this case, we may assume that neither D_1 nor D_2 has any vertices of X, Y or Z, since D has no induced subdigraph isomorphic to $D_{p(n)}$ for $n \geq 2$. The series composition of a digraph resulting from Case 1 or Case 3 with a second digraph results in a digraph with a partition of L and M into $L_1, L_2, L_3, L_4, L_5, M_1, M_2, M_3, M_4, M_5$ with the following properties:

- Vertices of L_1 only have out-neighbours in L_1, M_1, or M_4 while vertices of L_2 only have out-neighbours in L_2, M_2, or M_4, and vertices of L_3 only have out-neighbours in L_3, M_3, or M_4
- Vertices of L_4 only have out-neighbours in L_4 or M_4;
- Vertices of L_5 only have out-neighbours in L_5 or M_5;
- Vertices of M_1 only have out-neighbours in M_1 or M_4 while vertices of M_2 only have out-neighbours in M_2 or M_4 and vertices of M_3 only have out-neighbours in M_3 or M_4.
- Vertices of M_4 only have out-neighbours in $M_4 \cup \{t\}$ while vertices of M_5 only have out-neighbours in $M_5 \cup \{t\}$.

Digraphs with this particular structure have pairs of compatible, fundamental orderings which satisfy:

$$\prec_x\colon s, L_1, M_1, L_2, M_2, L_5, M_5, L_4, L_3, M_3, M_4, t$$

$$\prec_y\colon s, L_5, L_3, M_3, L_4, L_2, L_1, M_2, M_1, M_4, M_5, t.$$

Due to the particular nuances of this case, we construct a pair of compatible, fundamental orderings for D in the following way:

$$\prec_x\colon s, L_1^1, L_1^2, M_1^2, M_1^1, L_2^1, L_2^2, M_2^2, M_2^1, L_5^1, L_5^2,$$

$$M_5^2, M_5^1, L_4^1, L_4^2, L_3^1, L_3^2, M_3^2, M_3^1, M_4^2, M_4^1, t$$

$$\prec_y\colon s, L_5^2, L_5^1, L_3^2, L_3^1, M_3^1, M_3^2, L_4^2, L_4^1, L_2^2, L_2^1$$

$$L_1^2, L_1^1, M_2^1, M_2^2, M_1^1, M_1^2, M_4^1, M_4^2, M_5^1, M_5^2, t.$$

The orderings $\prec_x$ and $\prec_y$ are fundamental from the construction together with the fundamental property of $\prec_1$ and $\prec_2$. The pair of orderings are compatible by the definition of the sets, the construction of $\prec_x$ and $\prec_y$, as well as the compatibility of the orderings $\prec_1$ and $\prec_2$.

In Case 3, we assume that D is formed by parallel composition of D_1 and D_2 where D_1 has no vertices of L or M and D_2 has no vertices of X,, Y, or Z. Now D_2 does not have an induced subdigraph isomorphic to B_n for $n \geq 2$ which contains s and t, so D_2 has a pair of compatible, fundamental orderings which satisfy $s \prec_1 L^2 \prec_1 M^2 \prec_1 t$ and $s \prec_2 L^2 \prec_2 M^2 \prec_1 t$. We insert the vertices of D_2 into a pair of orderings for D_1 resulting from Case 1 as follows:

$$\prec_x\!: s, X_1^1 \cup Y_1^1, Z_1^1, X^1 \cup Y^1 - (X_1^1 \cup Y_1^1 \cup X_2^1 \cup Y_2^1), X_2^1 \cup Y_2^1, L, Z_2^1, M, t$$

$$\prec_y\!: s, L, M, X_2^1 \cup Y_2^1, Z_2^1, X^1 \cup Y^1 - (X_1^1 \cup Y_1^1 \cup X_2^1 \cup Y_2^1), X_1^1 \cup Y_1^1, Z_1^1, t$$

Each of these two orderings $\prec_x, \prec_y$ are fundamental. Note that if $uv \in A(D)$, then $uv \in A(D_1)$ or $uv \in A(D_2)$, where $uv \in A(D_1) \cap A(D_2)$ only if $uv = st$. Now since the orderings of D_1 and D_2 are fundamental, so are the orderings of D.

To show they are compatible, consider vertices u, v, z, z' such that $u \prec_x v \prec_x z$ and $u \prec_y v \prec_y z'$, with $uz, uz' \in A$. Note that if all four vertices belong to D_1 or D_2 then $uv \in A$ by the compatibility of the orderings of each component digraph. Noting that the out-neighbours of vertices of L belong to either L or M, the only other case to consider is when $u = s$ and $v \in X \cup Y$. But then $uv \in A$ since all vertices of $X \cup Y$ are out-neighbours of s. We conclude the pair of orderings is compatible.

We have considered all possible cases for the structure of path-parallel composition. In each case we have shown how to construct compatible, fundamental orderings for two-terminal series-path-parallel digraphs. This completes the proof. ∎

Recall that the two-terminal series-path-parallel digraphs are precisely the two-terminal series-parallel digraphs which satisfy the umbrella path property. Applying this concept, we restate Theorem 4.11 as follows.

Theorem 4.12. *A two-terminal series-parallel digraph is chronological rectangle if and only if it satisfies the umbrella path property and has no induced subdigraph isomorphic to* D_{3b} *or* $D_{p(n)}$ *for* $n \geq 2$.

Furthermore, we conclude that the two-terminal series-path-parallel digraphs with subdigraphs isomorphic to D_{3b} or $D_{p(n)}$ for $n \geq 2$ are exactly those with chronological interval dimension greater than or equal to 3.

As a closing comment on the chapter, consider the two-terminal series-path-parallel digraph D_{kb} obtained by path-parallel composition of k copies of the digraph D_b in Figure 4.3. Extending the proof of Proposition 4.3 as well as the discussion after, we see that D_{kb} has chronological interval dimension exactly k for all $k \geq 1$. This is in stark contrast with the fact that all series-parallel graphs have boxicity at most 3.

Chapter 5

Chronological Rectangle Digraphs as Graph Orientations

In this chapter we explore the problem of obtaining chronological rectangle digraphs by orienting graphs belonging to common classes and adding loops to all vertices. We begin by considering digraphs which arise from adding loops to all vertices of orientations of trees and k-trees. However, we will also construct a chordal graph G such that no digraph obtained by adding loops to every vertex of an orientation of G is a chronological rectangle digraph.

5.1 Digraphs Constructed from Trees

We begin by studying digraphs obtained by taking any orientation of a tree and adding loops to every vertex. The goal of this section is to prove Theorem 5.1.

Theorem 5.1. *If D is obtained from an orientation of a tree by adding a loop to every vertex then D is a chronological rectangle digraph.*

Define a *root* of an orientation T of a tree to be a vertex of in-degree 0. An orientation of a tree has at least one root because it has no directed cycles. An *outbranching* is an orientation of a tree with exactly one root. An example of an outbranching is given in Figure 5.1. We use the term *level* to describe the set of vertices in an outbranching at the same directed distance from the root.

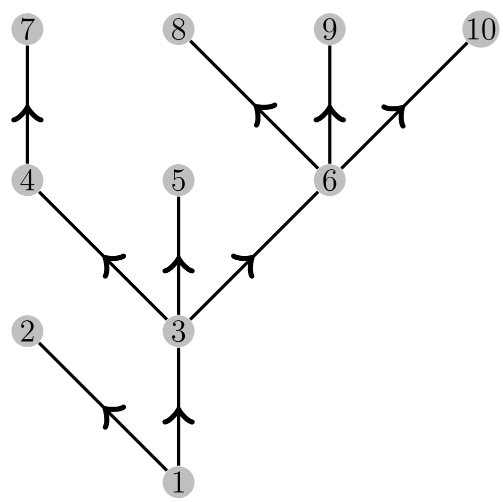

Figure 5.1: An example of an outbranching.

The proof of Theorem 5.1 proceeds by induction on the number of roots in the orientation. In Lemma 5.2 we prove that the theorem holds in the base case, when the orientation is an outbranching.

Lemma 5.2. *If D is obtained from an outbranching by adding a loop to every vertex then D is a chronological rectangle digraph.*

Proof We proceed by induction on $n = |V(D)|$. We will prove that D has a chronological rectangle model in which rectangles corresponding to vertices in the same level have a common lower endpoint which is distinct from the common lower endpoint associated with any other levels.

When $n = 1$ we construct a rectangle with corners at $(0, 0), (3, 0), (0, 3)$, and $(3, 3)$, so the lemma is true for the base case.

Now suppose that the lemma holds for $1 \leq n < k$. Let D be the digraph obtained by adding loops to every vertex of an orientation of a tree with exactly k vertices, including a single root r.

Consider the components $D_1, \ldots, D_m$ of $D - r$. Since the underlying graph of D is a tree, each D_j for $1 \leq j \leq m$ contains exactly one out-neighbour of r. As a result we find that each D_j is an outbranching with a smaller number of vertices than D. Hence by the induction hypothesis each D_j has a chronological rectangle representation in which rectangles corresponding to vertices in the same level have a common lower endpoint which distinguishes them from other levels. Note that by applying Proposition 2.15 we may assume that the endpoints of rectangles are otherwise distinct from one another. Furthermore, we assume that the lower-left corners of the roots in the chronological rectangle representations for each D_j are at the origin. We translate and stretch rectangles vertically while maintaining the chronological rectangle model so that there is a distance of 2 in the plane between lower endpoints of rectangles corresponding to consecutive levels. Denote the length of the entire chronological rectangle representation of D_j by L_j.

We construct a rectangle for r with corners located at $(0, 0)$, $(m+1+\sum_{i=1}^{m} L_i, 0)$, $(0, 3)$, and $(m + 1 + \sum_{i=1}^{m} L_i, 3)$. We then translate the chronological rectangle representations of each D_j up by 2 and left by $j + 1 + \sum_{i=1}^{j-1} L_i$, so that the lower-left corner of the rectangle corresponding to the root of D_j is located at $(j+1+\sum_{i=1}^{j-1} L_i, 2)$.

We claim that this gives a chronological rectangle representation for D. By the induction hypothesis we have chronological rectangle representations for the D_js. The corresponding rectangles are disjoint since the rectangles for D_j have x-values from $j + \sum_{i=1}^{j-1} L_i$ to $j + \sum_{i=1}^{j} L_i$. The rectangle for r contains of the lower-left corners of its out-neighbours, the roots of all the D_js. However r avoids the lower-left corners of all other rectangles, since their lower endpoints are at distance 2 from the lower endpoint of their in-neighbour. For example, vertices of directed distance 2 from r have y-values of 4 while the rectangle corresponding to r ends at $y = 3$. ■

A chronological rectangle model constructed using Lemma 5.2 is illustrated in Figure 5.2 for the outbranching from Figure 5.1.

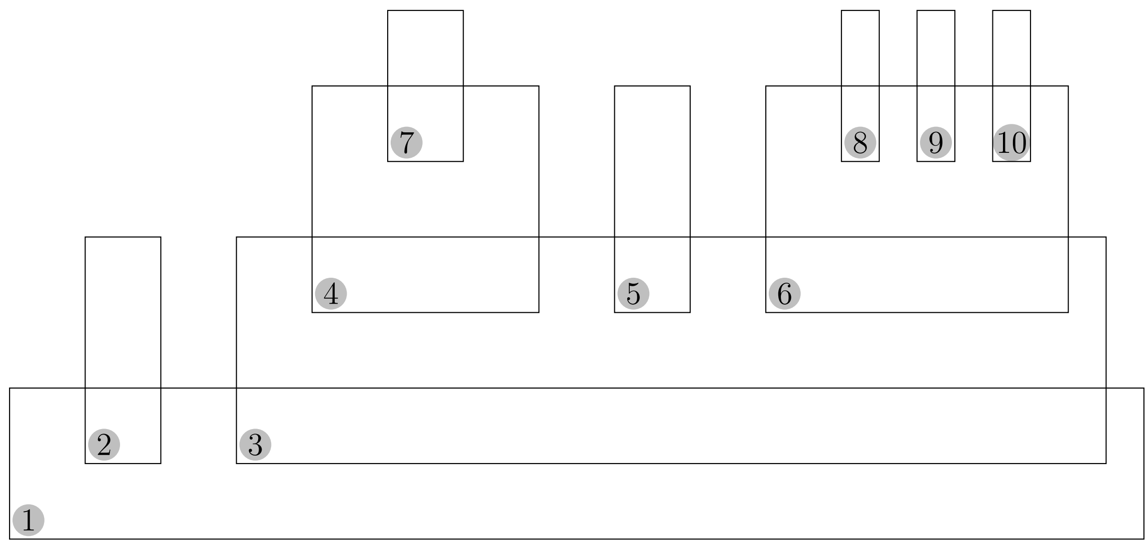

Figure 5.2: A chronological rectangle model for the outbranching given in Figure 5.1.

Before proceeding with the induction step of the proof of Theorem 5.1, we make a structural observation about orientations of trees. Let T be an orientation of a tree, and let r be any root of T. Denote the maximal outbranching rooted at r by R, then take U_R to be the vertices of R which have in-degree 2 or more in T. We say that a root r of T is *good* if either U_R is empty or there is a vertex $u \in U_R$ such that there are directed paths from u to all other vertices of U_R. Note that since the underlying graph of T is connected, U_R is empty if and only if T is a directed path.

Lemma 5.3. *Every orientation of a tree has a good root.*

Proof Start by choosing any root and labelling it r_0. If the root is good then the lemma holds; otherwise there are at least two vertices of in-degree at least 2 with no directed path between them in the maximal outbranching rooted at r_0. Choose one of these vertices of in-degree at least 2 and label it u_0. We can find a new root with a directed path to u_0 other than r_0 because T has no directed cycles; label this root r_1. Again if this root is good the lemma holds; otherwise the maximal outbranching rooted at r_1 has a vertex u_1 of in-degree at least 2 with no directed path to u_0. We can then find another root r_2 with a directed path to u_1. Each time the process is repeated, a new vertex of in-degree at least 2 and a new root can be chosen. Since D is an orientation of a tree, there are no cycles in the underlying graph and the process terminates by finding a good root. ■

As a consequence of Lemma 5.3, we can always find a good root r of any orientation T of a tree. Let u be the vertex of in-degree at least 2 in the maximal outbranching R rooted at r such that there is a directed path from u to every other vertex of in-degree at least 2 in R. Let R' be the maximal outbranching rooted at u and u^- be the in-neighbour of u in R. We will exploit a partition of any orientation of a tree T into an outbranching $T_2 = R - R'$, an orientation $T_1 = T - T_2$ of a tree with fewer roots than T, and a single arc u^-u between T_2 and T_1.

Proof of Theorem 5.1 We proceed by induction on the number s of roots in the orientation. We know that the theorem holds for the base case $s = 1$ by Lemma 5.2.

Now suppose that the theorem holds for $1 \leq s < k$ and consider the digraph D obtained from an orientation of a tree T with exactly k roots by adding loops to every vertex. By Lemma 5.3 we can find a good root r and an outbranching T_2 rooted at r such that $T_1 = T - T_2$ is an outbranching with exactly $k - 1$ roots of the orientation. Let $u \in V(T)$ be the unique vertex in T_1 which has an in-neighbour in T_2. Since T_2 is a outbranching, u has a unique in-neighbour u^- in T_2. Let T_3 be the maximum outbranching in T_2 rooted at u^-. By construction, none of the vertices in T_3 have in-neighbours in $T - T_3$ except for u^-.

Let D_1 and D_2 be the digraphs obtained by adding loops to all vertices of T_1 and T_2 respectively. By the induction hypothesis D_1 and D_2 each have a chronological rectangle model. Once again, we assume without loss of generality that all chronological rectangle models have smallest x- and y-values of 0. We show how these can be combined into a chronological rectangle model for D.

Let $R_v, v \in V(D_1)$, be a chronological rectangle model for D_1 constructed via the induction hypthesis. We also construct a chronological rectangle model $S_v, v \in V(D_2)$ for D_2 via the induction hypothesis, and then we translate it so that the top of this model is below the model for D_1. Now consider a_u, the left-hand endpoint of the rectangle corresponding to u. Since we may assume that all of the left- and right-endpoints of rectangles in the chronological rectangle model for D_1 are distinct by Proposition 2.15, no other rectangle has an endpoint at a_u, and there exist x-values x_1 and x_2 which are the closest endpoints of rectangles to the left and right respectively of a_u. We modify S_v subject to two constraints: $a_{u^-} < a_u \leq b_{u^-}$, and the left endpoint for the entire model for D_2 is between $(x_1 + 1)/x_1$ and a_u while the right endpoint of the entire model for D_2 is between a_u and $(x_2 - 1)/x_2$. Now we extend S_{u^-} up so that it contains (a_u, c_u); we need to do this in a way that places the rectangles for the maximal outbranching in D_2 rooted at u^- horizontally between $(x_1 + 1)/x_1$ and a_u as well as vertically above the top endpoint of the entire model for D_1.

By construction, we have chronological rectangle models for D_1 and D_2. The modifications mean that the rectangle corresponding to u^- contains the lower-left corner of the rectangle corresponding to u while otherwise maintaining the chronological rectangle models. Therefore, the modified rectangles give a chronological rectangle model for D. ■

5.2 Digraphs Constructed from k-trees

Recall that we can define the class of k-trees recursively as follows. The complete graph on $k+1$ vertices is a k-tree and so is any graph obtained from a k-tree by adding a vertex which is adjacent to exactly k vertices which form a clique.

Suppose that a k-tree G is constructed by recursively adding the vertices $v_1, v_2, \ldots v_n$. For each i, $k+2 \leq i \leq n$, let G_i denote the subgraph of G induced by the vertices $v_1, v_2, \ldots, v_i$. For each v_i, there is at least one vertex v_j, $1 \leq j < i$ such that $N(v_i) \subseteq N(v_j)$ in G_i. This follows from the fact that all maximal cliques in a k-tree have $k+1$ vertices, so the k neighbours of v_i have a common neighbour not adjacent to v_i.

Theorem 5.4. *For every k-tree G there exists an orientation D of G such that the digraph obtained by adding a loop to every vertex of D is a chronological rectangle digraph.*

Proof Given any k-tree G, we recursively construct an orientation D such that the digraph obtained by adding loops to all vertices of D is chronological rectangle.

Let $v_1, v_2, \ldots, v_{k+1}$ be the initial clique from a recursive construction of G. Start with an orientation of this clique as a transitive tournament. For each vertex v_i, $k+2 \leq i \leq n$, we know by the discussion preceding the theorem that there is a vertex v_j with $1 \leq j < i$ such that $N(v_i) \subseteq N(v_j)$ in G_i. We orient the edges incident with v_i in the same direction as the orientations of the corresponding edges incident with v_j. In other words if v_k is a common neighbour of v_i and v_j in G, and the edge $v_k v_j$ is oriented from v_k to v_j then we also orient the edge $v_k v_i$ from v_k to v_i. Let D denote the digraph obtained by adding loops to all vertices of the orientation obtained from G in this way, while D_i is the subdigraph induced by the vertices $v_1, v_2, \ldots, v_i$.

We claim that there is a pair of compatible, fundamental orderings for D. The proof proceeds by induction on the number of vertices.

The base case is when $i-1=k+1$, so that $v_1, v_2, \ldots, v_{k+1}$ is a transitive tournament with $v_a v_b \in A$ if $a \leq b$. We take both $\prec_x, \prec_y$: $v_1, v_2, \ldots, v_{k+1}$ to obtain a pair of compatible, fundamental orderings. Therefore, the transitive tournament induced by $v_1, v_2, \ldots, v_{k+1}$ is chronological rectangle by Theorem 2.12 and the claim holds when $i-1=k+1$.

For the induction step, we suppose that $\prec_1$ and $\prec_2$ are a pair of compatible, fundamental orderings for D_{i-1}. Again, let v_j be a vertex such that $N(v_i) \subseteq N(v_j)$ in D_i. We construct a pair of orderings $\prec_x, \prec_y$ for D_i by placing v_i immediately preceding v_j in $\prec_1$ and placing v_i immediately after v_j in $\prec_2$.

First, we show $\prec_x$ and $\prec_y$ are fundamental. If $v_a v_b \in A$ with $1 \leq a \leq b < i$, then both $v_a \prec_x v_b$ and $v_a \prec_y v_b$ because $\prec_1$ and $\prec_2$ are fundamental orderings of D_{i-1}. If $v_k v_i \in A$ then $v_k v_j \in A$, so $v_k \prec_x v_j$ and $v_k \prec_y v_j$. Since v_i and v_j are consecutive in both orderings, we also have $v_k \prec_x v_i$ and $v_k \prec_y v_i$. A similar argument applies if $v_i v_k \in A$.

Now we argue that $\prec_x$ and $\prec_y$ are a pair of compatible orderings. If $u \prec_x v \prec_x z$ and $u \prec_y v \prec_y z'$ with $u, v, z, z' \in V(D_{i-1})$, then $uv \in A$ because $\prec_1$ and $\prec_2$ are a pair of compatible orderings for D_{i-1}. Therefore, we need only consider configurations involving v_i.

Suppose that $v_i \prec_x v \prec_x z$ and $v_i \prec_y v \prec_y z'$ with $v_i z, v_i z' \in A$, noting that v_i and v_j are in opposite order, so that none of v, z, z' is v_j. We have $v_j \prec_x v \prec_x z$ and $v_j \prec_y v \prec_y z'$ with $v_j z, v_j z' \in A$, so compatibility of the orderings of D_{i-1} implies that $v_j v \in A$, which guarantees that $v_i v \in A$.

Similarly, suppose $u \prec_x v_i \prec_x z$ and $u \prec_y v_i \prec_y z'$ with $uz, uz' \in A$. Again note that v_i and v_j are in opposite order, so that none of u, z, z' is v_j. By construction of the two orderings we have $u \prec_x v_j \prec_x z$ and $u \prec_y v_j \prec_y z'$, so $uz, uz' \in A$ and by compatibility of the orderings of D_{i-1} we have $uv_j \in A$ so also $uv_i \in A$.

Finally, consider $u \prec_x v \prec_x v_i$, $u \prec_y v \prec_y z'$, where possibly $z' = v_i$ or $z' = v_j$, and $uv_m, uz \in A$. Note that u and v are distinct from v_j because v_i is before v_j in $\prec_x$. We have $u \prec_x v \prec_x v_j$ and $uv_j \in A$ by construction. If $z' \neq v_i$, then the compatibility of the orderings of D_{i-1} would imply that $uv \in A$. If instead we have $z' = v_i$, then $u \prec_y v \prec_y v_j$ with $uv_j \in A$, and once again the compatibility of the orderings of D_{i-1} implies that $uv \in A$. This concludes the argument that $\prec_x$ and $\prec_y$ are a pair of compatible, fundamental orderings for D_i.

Therefore, we can construct a pair of compatible, fundamental orderings $\prec_x$ and $\prec_y$ for D by induction, so that D is a chronological rectangle digraph by Theorem 2.12. ■

Notice that not every orientation of a k-tree gives a chronological rectangle digraph after adding loops to all vertices. For example, the digraph U_2 on the right side of Figure 2.1 is not a chronological rectangle digraph because it does not satisfy the umbrella path property, but its underlying graph is a 2-tree. Furthermore, the digraph Q given in Figure 3.1 is a 3-tree which satisfies the umbrella path and weakly-clustered properties but is not chronological rectangle by Lemma 3.12.

5.3 A Graph Which Cannot be Oriented as Chronological Rectangle

In Proposition 5.5 we construct a split graph (and hence a chordal graph) such that for every orientation D the digraph obtained by adding loops to every vertex of D is not a chronological rectangle digraph. The strategy is to construct a graph such that every orientation with no directed cycles which satisfies the umbrella path property contains an induced subdigraph isomorphic to the digraph Q given in Figure 3.1.

Proposition 5.5. *There exists a split graph G such that for every orientation of G, the digraph obtained by adding loops to every vertex is not chronological rectangle.*

Proof We define a graph G as follows. Let $T = \{v_1, \ldots, v_6\}$ while T' is the collection of all four element subsets of T. We associate the vertices of G with the elements of $T \cup T'$. We define edges between every pair of vertices corresponding to elements of T, as well as between vertices corresponding to elements of T and vertices associated with the subsets of T to which the element belongs.

Now suppose that D is a chronological rectangle digraph obtained from an orientation of G by adding loops to every vertex. As a consequence, D has no directed cycles (except for loops) and satisfies the umbrella path property.

The subgraph of G induced by vertices associated with elements of T is complete. The edges of this complete subgraph must be oriented as a transitive tournament because D has no directed cycles except for loops. Due to the symmetry of the vertices of edges, we may assume without of generality that the edges are oriented so that $v_iv_j \in A$ if $i \leq j$.

For $i = 1, 2, 3$ let s_i denote the vertex which is not adjacent to v_i or v_5. For example, s_1 is the vertex corresponding to the subset $\{v_2, v_3, v_4, v_6\}$ of T.

Consider the orientation of the edges v_6s_i. If $s_iv_6 \in A$, then the subdigraph induced by v_4, s_i, v_5, and v_6 would fail the umbrella path property, a contradiction. Therefore $v_6s_i \in A$ for $i = 1, 2, 3$. Now since D has no directed cycles (except for loops), all edges of the form v_js_i with $1 \leq j \leq 4$, $j \neq i$ are oriented from v_j to s_i.

As a consequence, the subdigraph of D induced by the vertices v_1, v_2, v_3, v_4, s_1, s_2, s_3 is isomorphic to the digraph Q from Figure 3.1. However Q is not chronological rectangle by Lemma 3.12, contrary to the assumption. ■

In this chapter, we explored how to orient trees and k-trees to produce chronological rectangle digraphs. In contrast, we have also shown that there exist chordal graphs such that no orientations can produce chronological rectangle digraphs.

Chapter 6

Proper and Unit Chronological Rectangle Digraphs

6.1 Background

Recall that G is an *interval graph* if there is a family of intervals on the real line such that two vertices are adjacent if and only if the corresponding intervals intersect. An interval graph is *proper* if the family of intervals can be chosen to be inclusion free. An interval graph is *unit* if every interval can be chosen to have unit length. The classes of unit interval graphs and proper interval graphs are known to be equivalent, among numerous other characterizations [32, 33, 38, 43, 45].

Both interval graphs and proper interval graphs have been extensively studied, and there exist linear-time algorithms to find a representation of a given input graph by a family of inclusion free intervals if one exists [25, 32]. A well-known result states that the proper interval graphs are exactly the interval graphs with no induced subgraph isomorphic to the claw $K_{1,3}$ given in Figure 6.1 [53, 61].

Graphs of boxicity at most k were introduced together with the graphs of *cubicity*

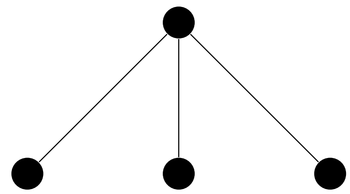

Figure 6.1: The claw $K_{1,3}$.

at most k, which are the intersection graphs of axis-parallel k-dimensional cubes [54]. Graphs of cubicity at most k are thus higher dimensional analogues of unit interval graphs. Although cubicity did not initially attract much interest, it has recently been the subject of several papers [2, 3, 14, 15].

We previously introduced a few related concepts for directed graphs. A chronological interval digraph D is called *proper* if it has a chronological interval model in which no interval is contained in another. A *straight enumeration* of a digraph is an ordering $v_1, v_2, \ldots, v_n$ of its vertices such that for each i there exist nonnegative integers ℓ and k (which depend on i) such that the in-neighbours of v_i are $v_i, v_{i-1}, v_{i-2}, \ldots, v_{i-\ell}$ and the out-neighbours of v_i are $v_i, v_{i+1}, v_{i+2}, \ldots, v_{i+k}$.

We define a digraph D to be a *unit chronological interval digraph* if it has a corresponding family of unit intervals $I_v = [a_v, a_v+1]$, $v \in V(D)$ such that $uv \in A(D)$ if and only if $a_v \in I_u$.

Theorem 6.1. *The following are equivalent for any reflexive digraph D with no symmetric arcs:*

(1) D has a straight enumeration;

(2) D is a proper chronological interval digraph;

(3) D is a unit chronological interval digraph.

Proof The equivalence of (1) and (2) was proved in [25].

To prove that (2) implies (3), we explain how a short proof of Bogart and West [8] can be applied to transform a proper chronological interval model of D into a unit chronological interval model for D.

Note that in any proper interval model of a graph G, the right endpoints of the intervals appear in the same sequence as the left endpoints of the intervals. Furthermore, any proper chronological interval model of D satisfies this same property.

The Bogart and West algorithm transforms a proper interval model of a graph into a unit interval model while maintaining the relative orders of all of the endpoints. Hence if we apply their algorithm to a family of inclusion free chronological intervals $I_v = [a_v, b_v], v \in V$, we obtain a family of unit intervals $I'_v = [a'_v, a'_v + 1], v \in V$ such that $a'_v \in I'_u$ if and only if $a_v \in I_u$. This shows that every proper chronological interval digraph is also a unit chronological interval digraph.

To show that (3) implies (1), first suppose that D is a unit chronological interval digraph with no symmetric arcs. For each $v_i \in V$, let $[a_i, a_i + 1]$ be the unit interval corresponding to v_i.

Let $v_1, v_2, \ldots, v_n$ be an ordering of V in increasing order of the left-endpoints of the corresponding vertices. We show that $v_1, v_2, \ldots, v_n$ is a straight enumeration. For each v_i, let k be the smallest index such that $a_k \leq a_i - 1$; if no such k exists then set $k = 0$. Similarly, for each v_i let ℓ be the largest index with $a_\ell \leq a_i + 1$. ■

In the remainder of the chapter, we define and investigate the classes of proper and unit chronological rectangle digraphs. These classes are inspired by the definitions of proper and unit interval graphs, but exhibit some distinguishing characteristics.

6.2 Unit Chronological Rectangle Digraphs

Define a chronological rectangle digraph to be *unit* if the family of rectangles can be chosen to have sides of unit length. A unit chronological rectangle model $R_v = [a_v, a_v+1] \times [c_v, c_v+1]$, $v \in V$, thus consists of squares with unit area. In this section, we develop some fundamental properties of unit chronological rectangle digraphs.

Let u and v be vertices with directed paths from u to v and from v to u. The rectangles corresponding to u and v have the same lower-left corner. Two unit rectangles with the same lower-left corners would be identical, so that the corresponding vertices would have the same closed in- and out-neighbourhoods. We may conclude that strong components in unit chronological rectangle digraphs are complete subdigraphs with identical in- and out-neighbourhoods.

The essential relationship between chronological interval digraphs and chronological rectangle digraphs was described by Lemma 3.1. We now show that the same relationship holds between unit chronological interval digraphs and unit chronological rectangle digraphs.

Proposition 6.2. *A digraph is unit chronological rectangle if and only if there exist two unit chronological interval digraphs D_1 and D_2 such that $D = D_1 \cap D_2$.*

Proof We give an argument which follows the same strategy as the proof of Proposition 3.1. First suppose that D has a unit chronological rectangle model. Such a model can be projected onto the x- and y-axes respectively to obtain two unit chronological interval models. The intersection of the unit chronological interval digraphs corresponding to these models is D.

On the other hand, suppose that D is the intersection of two unit chronological interval digraphs D_1 and D_2. Taking the Cartesian product of the unit chronological interval models for D_1 and D_2 gives a unit chronological rectangle model for D. ■

It is interesting to note that unit chronological rectangle digraphs satisfy a stronger condition than the umbrella path property. We say that a digraph $D = (V, A)$ satisfies the *special umbrella path property* if $p_1p_2, p_2p_3, \ldots, p_{m-1}p_m, p_1p_m \in A$ implies that $p_ip_j \in A$ for all $1 \leq i \leq j \leq m$. In other words, a digraph with the special umbrella path property is such that every directed path $p_1, p_2, \ldots, p_m$ with the additional arc $p_1p_m \in A$ forms a transitive tournament where $p_ip_j \in A$ for all $1 \leq i < j \leq m$.

Proposition 6.3. *If D is a unit chronological rectangle digraph then D satisfies the special umbrella path property.*

Proof Let $p_1, p_2, \ldots, p_m$ be a directed path in D. Denote by (a_i, c_i) the lower-left corner of the rectangle corresponding to p_i in a unit chronological rectangle model for D. Remark that the upper-right corner of the rectangle corresponding to p_i is $(a_i + 1, c_i + 1)$. Since $p_ip_{i+1} \in A$ for $1 \leq i \leq m - 1$, $a_i \leq a_{i+1} \leq a_i + 1 \leq a_{i+1} + 1$ and $c_i \leq c_{i+1} \leq c_i + 1 \leq c_{i+1} + 1$, also for $1 \leq i \leq m - 1$. Furthermore, $p_1p_m \in A$ implies $a_1 \leq a_m \leq a_1 + 1$ and $c_1 \leq c_m \leq c_1 + 1$. Now we have both $a_1 \leq a_2 \leq \ldots \leq a_m \leq a_1+1 \leq a_2+1 \leq \ldots \leq a_m+1$ and $c_1 \leq c_2 \leq \ldots \leq c_m \leq c_1+1 \leq c_2+1 \leq \ldots \leq c_m+1$. We conclude that $p_ip_j \in A$ for all $1 \leq i < j \leq m$ so that D satisfies the special umbrella path property. ■

In Theorem 6.4, we give an ordering characterization which describes unit chronological rectangle digraphs in the same way that Theorem 2.12 characterizes chronological rectangle digraphs. Before stating the result, we give some relevant definitions.

Recall from Chapter 2 that a linear ordering $\prec$ of V is *fundamental* if it satisfies condition (R1) for every $u \prec v$, and satisfies (R2), (R3), and (R4) for every $u \prec v \prec w$:

(R1) $vu \notin A(D) - S$

(R2) $uw \in S$ implies $uv, vw \in S$

(R3) $uw \in A(D) - S$ implies $uv \in A(D) - S$ or $vw \notin S$

(R4) $uw \notin A(D)$ implies $uv \notin A(D)$ or $vw \notin S$.

We now define a pair of linear orderings $\prec_x$ and $\prec_y$ of V to be *strongly compatible* if they satisfy condition (Q6) for every $z \prec_x u \prec_x v \prec_x w$ and $z' \prec_x u \prec_y v \prec_y w'$:

(Q6) $zw, z'w' \in A$ implies $uv \in A$;

where z, z', u are not necessarily distinct and w, w', v are not necessarily distinct.

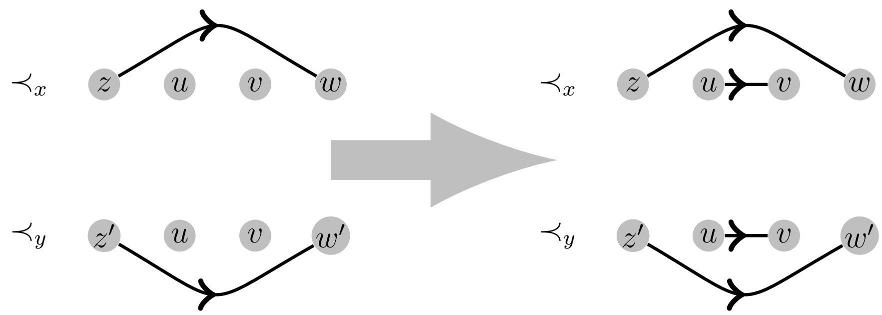

Figure 6.2: An illustration of the strongly compatible property for a pair of orderings of the vertex set of a digraph.

An important application of a pair of strongly compatible orderings $\prec_x$, $\prec_y$ is the case of vertices a, b, c, c' with $a \prec_x b \prec_x c$ and $a \prec_y b \prec_y c'$ with $ac, ac' \in A$. If we take $a = z = u = z'$, $b = v$, $c = w$ and $c' = w'$ then strong compatibility gives $ab \in A$. Note that this shows that any pair of strongly compatible orderings is compatible. At the same time if we consider $a = z = z'$, $b = u$, $c = w = v = w'$ then strong compatibility implies that $bc \in A$, so that a, b, c form a transitive tournament. This observation will be useful in the proof of Theorem 6.4.

Theorem 6.4. *A digraph D is unit chronological rectangle if and only if there exists a pair of fundamental orderings $\prec_x$, $\prec_y$ of V which are strongly compatible.*

Proof First, suppose that D has a unit chronological rectangle model

$R_v = [a_v, a_v + 1] \times [c_v, c_v + 1], v \in V$. We construct $\prec_x$ so that either $a_u < a_v$, or both

$a_u = a_v$ and $c_u < c_v$ implies that $u \prec_x v$. Vertices u, v corresponding to rectangles with $a_u = a_v$ and $c_u = c_v$ are ordered consecutively, but arbitrarily with respect to one another. Similarly, we construct $\prec_y$ so that either $c_u < c_v$ or both $c_u = c_v$ and $a_u < a_v$ implies $u \prec_y v$ while vertices with $a_u = a_v$ and $c_u = c_v$ are ordered consecutively in an arbitrary way.

The proof that these orderings are fundamental is identical to the argument in Theorem 2.12. To show that $\prec_x$ and $\prec_y$ are strongly compatible, suppose that $w \prec_x u \prec_x v \prec_x z$ and $w' \prec_y u \prec_y v \prec_y z'$ with $wz, w'z' \in A(D)$. We have $a_w \leq a_u \leq a_v \leq a_z \leq a_w + 1 \leq a_u + 1$ and $c_{w'} \leq c_u \leq c_v \leq c_{z'} \leq c_{w'} + 1 \leq c_u + 1$, where in both cases we use the fact that the rectangles have unit length. Now since this is a unit chronological rectangle model for D, we find that $uv \in A(D)$, so $\prec_x$ and $\prec_y$ are strongly compatible.

To prove the other direction of the theorem, suppose that there exists a pair of strongly compatible, fundamental orderings $\prec_x$ and $\prec_y$ of D. Let D^* be the digraph obtained from D deleting all but one vertex from every strong component. We construct two oriented graphs D_1 and D_2 whose intersection is D^*, then construct straight enumerations of D_1 and D_2. By Theorem 6.1 and Proposition 6.2 this shows that D^* is unit chronological rectangle. Now take any unit chronological rectangle model for D^*. We may obtain a unit chronological rectangle model for D by adding unit rectangles for vertices in strong components which are identical to their sole representative in D^*.

First, define two sets of arcs by:

$$A_x = \{uv, vw | u \prec_x v \prec_x w \text{ and } uw \in A(D^*); u, v, w \text{ all distinct}\}$$

$$A_y = \{uv, vw | u \prec_y v \prec_y w \text{ and } uw \in A(D^*); u, v, w \text{ all distinct}\}$$

Construct two oriented graphs D_1 and D_2 with $V(D^*) = V(D_1) = V(D_2)$, $A(D_1) = A(D^*) \cup A_x$, and $A(D_2) = A(D^*) \cup A_y$. It is easy to check that $\prec_x$ is a straight enumeration of D_1 and $prec_y$ is a straight enumeration of D_2.

Now we argue that $D_1 \cap D_2 = D$. Since $A(D^*) \subseteq A(D_1)$ and $A(D^*) \subseteq A(D_2)$, we have $A(D^*) \subseteq A(D_1) \cap A(D_2)$. Suppose instead that $uv \in A_x \cap A_y$. There are four different cases: $w \prec_x u \prec_x v$ and $w' \prec_y u \prec_y v$; $w \prec_x u \prec_x v$ and $u \prec_y v \prec_y z'$; $u \prec_x v \prec_x z$ and $w' \prec_y u \prec_y v$; $u \prec_x v \prec_x z$ and $u \prec_y v \prec_y z'$. In any of the cases, strong compatibility would imply that $uv \in A(D^*)$, as discussed prior to the theorem. We conclude that $A(D_1) \cap A(D_2) \subseteq A(D^*)$. This completes the proof that $A(D_1) \cap A(D_2) = A(D^*)$, so that D^* is the intersection of two straight digraphs D_1 and D_2. ■

Recall that Proposition 2.23 states that if non-adjacent vertices u, v have a common out-neighbour, then they appear in opposite order in any pair of compatible, fundamental orderings. Lemma 6.5 is a companion result which applies only to strongly compatible orderings.

Proposition 6.5. *If $\prec_x$ and $\prec_y$ are a pair of strongly compatible, fundamental orderings, then $uv, uw \in A(D)$ and $vw, wv \notin A(D)$ imply that v and w are in the opposite order in $\prec_x$ and $\prec_y$. In other words, any pair of non-adjacent vertices which have a common in-neighbour appear in the opposite order.*

Proof We prove the statement by contradiction. Suppose that $\prec_x$ and $\prec_y$ are a pair of strongly compatible, fundamental orderings such that both $v \prec_x w$ and $v \prec_y w$. By the fundamental property, $u \prec_x v \prec_x w$ and $u \prec_y v \prec_y w$ would follow. The strongly compatible property of $\prec_x$ and $\prec_y$ would then imply that $vw \in A$, a contradiction. On the other hand if we suppose instead that $w \prec_x v$ and $w \prec_y v$, the fundamental property gives $u \prec_x w \prec_x v$ and $u \prec_y w \prec_y v$. Now the strongly compatible property of $\prec_x$ and $\prec_y$ would give $wv \in A$, again a contradiction. ■

Given a digraph D, let D' be the digraph obtained by reversing the orientations of all arcs of D. Note that this operation preserves loops and symmetric arcs. If D is a chronological rectangle digraph, then D' may not be chronological rectangle. Figure 6.3 gives one such pair of examples.

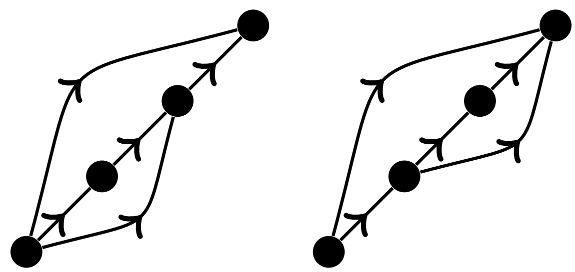

Figure 6.3: On the left is a chronological rectangle digraph. The digraph on the right obtained by reversing the directions of all of the arcs is not a chronological rectangle digraph since it does not satisfy the umbrella path property.

In Proposition 6.7 we prove that if D is a unit chronological rectangle digraph, then so is D'. We begin by establishing a lemma about the structure of unit chronological rectangle models.

Lemma 6.6. *If $R_v, v \in V$ is a unit chronological rectangle model for D, then R_u contains the lower-left corver of R_w if and only if R_w contains the upper-right corner of R_u.*

Proof Let $R_v = [a_v, a_v + 1] \times [c_v, c_v + 1]$, $v \in V$, denote the unit rectangle corresponding to v. Then R_u contains the lower-left corner of R_w if and only if $a_u \leq a_w \leq a_u + 1 \leq a_w + 1$ and $c_u \leq c_w \leq c_u + 1 \leq c_w + 1$, if and only if R_w contains the upper-right corner of R_u. ■

Proposition 6.7. *An digraph D is a unit chronological rectangle digraph if and only if the digraph D' obtained by reversing all arcs of D is a unit chronological rectangle digraph.*

Proof Take any unit chronological rectangle model $R_v = [a_v, a_v + 1] \times [c_v, c_v + 1]$, $v \in V$, for D. Let $m_x = \max_{v \in V(D)} (a_v + 1) - \min_{v \in V(D)} (a_v)$ and $m_y = \max_{v \in V(D)} (c_v + 1) - \min_{v \in V(D)} (c_v)$ be the length and height respectively of the entire model for D. We claim that $R'_v = [m_x - a_v - 1, m_x - a_v] \times [m_y - c_v - 1, m_y - c_v]$, $v \in V$, is a unit chronological rectangle model for D'. This construction maps the upper-right corners of the rectangle model for D to the lower-left corners of a rectangle model for D'. By Lemma 6.6, R_u contains the lower-left corner of R_v if and only if R_v contains the upper-right corner of R_u, so R_u contains the lower-left corner of R_v if and only if R'_v contains the lower-left corner of R'_u. We can prove the other direction by repeating the same process starting with a unit chronological rectangle model for D'. ■

6.3 Proper Chronological Rectangle Digraphs

We define a chronological rectangle digraph to be *proper* if the rectangle model can be chosen to be inclusion free. This section is dedicated to presenting an ordering characterization for proper chronological rectangle digraphs. In Section 6.4 we contrast this class with the classes of chronological interval digraphs, unit chronological interval digraphs, chronological rectangle digraphs, and unit chronological rectangle digraphs.

In general, proper chronological rectangle digraphs need not be unit chronological rectangle digraphs. For example, the digraph in Figure 6.4 is a proper chronological rectangle digraph but not unit chronological rectangle because there exists a strong component whose vertices do not all have the same out-neighbourhood.

We say that a pair of orderings $\prec_x, \prec_y$ are *properly compatible* if they satisfy condition (Q7) for every $z \prec_x u \prec_x v \prec_x w, u \prec_x v', z \prec_y u \prec_y v' \prec_y w'$, $u \prec_y v$, where possibly $z = u$, $w = w'$, or $v = v'$:

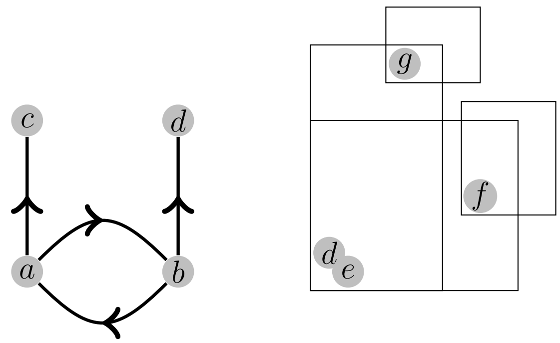

Figure 6.4: On the left is a proper chronological rectangle digraph with a corresponding proper chronological rectangle model on the right.

(Q7) $zv, zv', uw, uw' \in A(D)$ implies either $uv \in A(D)$ or $uv' \in A(D)$.

Theorem 6.8. *A digraph $D = (V, A)$ is proper chronological rectangle if and only if there exists a pair of properly compatible, fundamental orderings $\prec_x$ and $\prec_y$ of $V(D)$.*

Proof First suppose that D has a proper chronological rectangle model. We construct a pair of orderings $\prec_x, \prec_y$ as in the proof of Theorem 2.12. The fact that these orderings are fundamental follows from the argument in that proof, so we need only show that the orderings are properly compatible. Assume that we have $w \prec_x u \prec_x v \prec_x z$, $u \prec_x v'$, and $w \prec_y u \prec_y v' \prec_y z'$, $u \prec_y v$ with $wv, wv', uz, uz' \in A(D)$. Since the orderings were constructed from a proper chronological rectangle model, the rectangle for u extends past the rectangle for w in at least one direction. However, in order to do so and contain the lower-left corners of the rectangles corresponding to z and z', it extends past lower-left corner of either v or v', so that either $uv \in A(D)$ or $uv' \in A(D)$.

To prove the other direction let $\prec_x$: $x_1, x_2, \ldots, x_n$ and $\prec_y$: $y_1, y_2, \ldots, y_n$ be a pair of properly compatible, fundamental orderings. For a vertex $u = x_i = y_j$, we define $\phi_x(u) = i$ and $\phi_y(u) = j$.

We construct proper chronological rectangles as follows:

$$R_u = \left[\phi_x(u), \max_{v \in N^+(u)} \phi_x(v) + \frac{\phi_x(u)}{n}\right] \times \left[\phi_y(u), \max_{v \in N^+(u)} \phi_y(v) + \frac{\phi_y(u)}{n}\right]$$

except in the case where there exist vertices w, t, t' with $w \prec_x u \prec_x t$, $w \prec_y u \prec_y t'$, $wt, wt' \in A(D)$, $ut, ut' \notin A(D)$. In this case, proper compatibility implies that in at least one of the two orderings, u is adjacent to all out-neighbours of w before the last out-neighbour of u. We use this idea to extend the rectangle corresponding to u further than the rectangle corresponding to w in that direction. For example if that property holds for a vertex u in $\prec_y$ then we adapt the rectangle for u as follows:

$$R_u = \left[\phi_x(u), \max_{v \in N^+(u)} \phi_x(v) + \frac{\phi_x(u)}{n}\right] \times \left[\phi_y(u), \max_{w \in N^-(u)} \left\{\max_{v \in N^+(w)} \phi_y(v)\right\} + \frac{\phi_y(u)}{n}\right].$$

In particular, this modified construction is used in the case where u has no out-neighbours to guarantee that the rectangle corresponding to u is not contained in any other.

The fact that the R_v, $v \in V$, are a chronological rectangle model for D follows from the construction in the proof of Theorem 2.12, and the fact that the adjustments to rectangles do not cause them contain any additional lower-left corners. It remains to show that the collection of rectangles is inclusion free.

Certainly if $u \prec_x v$ and $v \prec_y u$ then $R_u \not\subseteq R_v$ and $R_v \not\subseteq R_u$. Consider pairs of vertices such that $w \prec_x u$ and $w \prec_y u$. By construction, if the last neighbour of u is equal to the last neighbour of w with respect to one ordering, then u extends further than w in that direction.

Suppose that there exist $t, t' \in N^+(w)$ such that $w \prec_x u \prec_x t$ and $w \prec_y u \prec_y t'$. By proper compatibility, we know that in one of the two orderings, u dominates all neighbours of w which are between u and its last neighbour in that direction. In this case, we use the modified construction to extend u past w in that direction; by the property, u is adjacent to all of the vertices necessary to do that. ■

6.4 Relationships Between Classes

This chapter concludes with a discussion of the relationship between all of the digraph classes discussed in this chapter.

Let $\mathcal{S}, \mathcal{I}, \mathcal{U}, \mathcal{P}, \mathcal{R}$ denote the families of straight, chronological interval, unit chronological rectangle, proper chronological rectangle, and chronological rectangle digraphs respectively.

Consider the digraph H_1 with $V(H_1) = \{a, b, c, d\}$ and $A(H_1) = \{ab, ac, ad, bc, cd\}$ which is given on the left of Figure 6.3. This digraph H_1 is chronological interval since $a \prec b \prec c \prec d$ is a chronological ordering of $V(H_1)$. However, H_1 does not satisfy the strong umbrella path property and thus cannot be unit chronological interval by Proposition 6.3. Consequently, we also have that $\mathcal{I} \not\subseteq \mathcal{U}$.

On the other hand, the digraph H_2 with $V(H_2) = \{a, b, c\}$ and $A(H_2) = \{ac, bc\}$. Now H_2 is unit chronological rectangle because $a \prec_x b \prec_y c$ and $b \prec_y a \prec_y c$ are a pair of fundamental orderings of $V(H_2)$ which are strongly compatible. However, H_2 is not chronological interval because a chronological ordering of $V(H_2)$ does not exist. Therefore we may conclude that $\mathcal{U} \not\subseteq \mathcal{I}$.

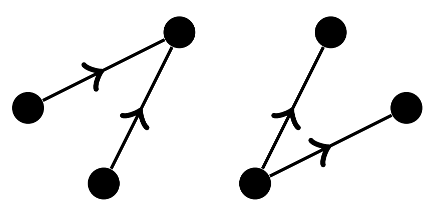

Figure 6.5: The digraph H_2 is on the left while the digraph H_3 is on the right.

The fact that $\mathcal{S} \subseteq \mathcal{U} \subseteq \mathcal{P} \subseteq \mathcal{R}$ follows from the definitions of the classes. In fact, we show that all of these inclusions are strict.

Take first the digraph H_3 with $V(H_3) = \{a, b, c\}$ and $A(H_3) = \{ab, ac\}$. The fundamental orderings $a \prec_x b \prec_x c$ and $a \prec_y c \prec_y b$ are a pair of strongly compatible orderings so that H_3 is unit chronological rectangle. However, a straight enumeration of $V(H_3)$ does not exist so it is not straight and $\mathcal{S} \subset \mathcal{U}$.

A digraph H_4 which is proper chronological rectangle but not unit chronological rectangle is an orientation of C_6 from one part to the other, as given in Figure 6.6. Let the parts be $\{s_1, s_2, s_3\}$ and $\{t_1, t_2, t_3\}$ where $s_i t_i \notin A(H_4)$ for $i = 1, 2, 3$.

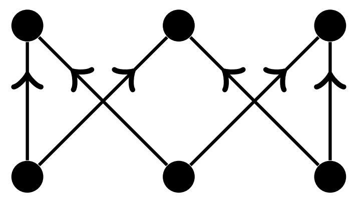

Figure 6.6: The digraph H_4, which is an orientation of C_6 from one part to the other.

The pair of orderings $s_1, s_2, t_3, s_3, t_1, t_2$ and $s_3, s_2, t_1, s_1, t_3, t_2$ are properly compatible, so that the digraph H_4 is proper chronological rectangle. To show that H_4 is not a unit chronological rectangle digraph, consider the fact that any pair of vertices in the same part have an in-neighbour (or out-neighbour) in common, so they appear in the opposite order in both orderings by Lemma 6.5. Now without loss of generality suppose that $t_1 \prec_x t_2 \prec_x t_3$ and $t_3 \prec_y t_2 \prec_y t_3$ are the orderings of the vertices in the part with in-neighbours but not out-neighbours. There is a vertex s which is a common in-neighbour of t_1 and t_3 but is not an in-neighbour of t_2. Now allowing $w = w' = u = s$, $v = t_2$, $t_1 = z$ and $t_3 = z'$ we have that a pair of strongly compatible orderings does not exist. As a consequence, $\mathcal{U} \subset \mathcal{P}$.

To construct a digraph which is chronological rectangle but not proper chronological rectangle, we construct a digraph H_5 as follows: let s_1, s_2, s_3, s_4 be a transitive tournament oriented so that $s_i s_j \in A(H_5)$ if and only if $i < j$. Let t_1 be a common out-neighbour of s_1 and s_3; t_2 be a common out-neighbour of s_1 and s_4; t_3 be a common out-neighbour of s_2 and s_3; let t_4 be a common out-neighbour of s_2 and s_4; and let u be an in-neighbour of $s_3, s_4, t_1, t_2, t_3, t_4$. The digraph H_5 in illustrated in Figure 6.7.

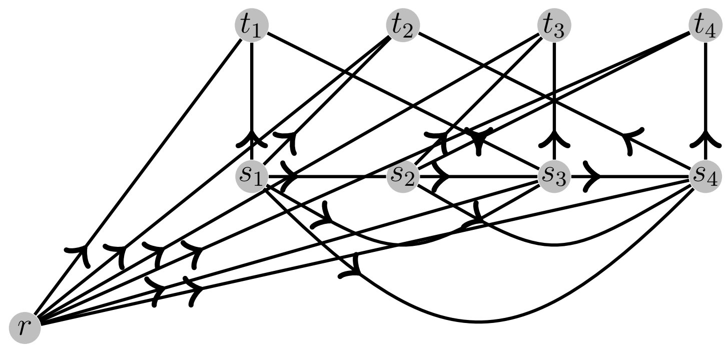

Figure 6.7: The digraph H_5.

Consider the following pair of orderings:

- $\prec_x$: $s_1 \prec_x s_2 \prec_x r \prec_x s_3 \prec_x t_1 \prec_x s_4 \prec_x t_3 \prec_x t_2 \prec_x t_4$
- $\prec_y$: $r \prec_y s_1 \prec_y s_2 \prec_y s_3 \prec_y t_2 \prec_y s_4 \prec_y t_4 \prec_y t_1 \prec_y t_3$

It is not difficult to verify that $\prec_x$ and $\prec_y$ are a pair of compatible, fundamental orderings for H_5, guaranteeing that H_5 is chronological rectangle.

We prove by contradiction that a pair of properly compatible, fundamental orderings of H_5 does not exist. Suppose that $\prec_x$ and $\prec_y$ are a pair of properly compatible, fundamental orderings.

Proposition 2.24 states that given vertices a_1, a_2, b_1, b_2 where $a_1b_1, a_2b_2 \in A$, $a_1b_2, a_2b_1 \notin A$ and there are directed paths from a_1 to b_2 and from a_2 to b_1, then b_1 and b_2 appear in opposite order in any pair of compatible, fundamental orderings. Since every pair of properly compatible are compatible, the result applies in this context.

Consider the vertices s_1, s_2, t_2, and t_4, which satisfy the hypotheses of Proposition 2.24. We may conclude without loss of generality that $t_4 \prec_x t_2$ and $t_2 \prec_y t_4$.

Now taking instead the vertices s_1, s_2, t_2, and t_3, we find that t_2 and t_3 appear in opposite order. However, having $t_4 \prec_x t_2 \prec_x t_3$ and $t_3 \prec_y t_2 \prec_y t_4$ would contradict compatibility. This implies that both $t_2 \prec_y t_3$ and $t_2 \prec_y t_4$.

Next we may determine that t_1 and t_4 are in opposite order by considering s_1, s_2, t_1, and t_4. In this case, setting $t_1 \prec_x t_4 \prec_x t_2$ and $t_2 \prec_y t_4 \prec_y t_1$ would contradict compatibilty. Thus $t_4 \prec_x t_1$ and $t_4 \prec_x t_2$.

Now suppose that $t_2 \prec_x t_1$. By the fundamental property of $\prec_x$ we have $s_3 \prec_x s_4 \prec_x t_2 \prec_x t_1$ and $s_3 \prec_y s_4 \prec_y t_2 \prec_y t_3$, contradicting compatibilty. As a consequence, our compatible, fundamental orderings satisfy $t_4 \prec_x t_1 \prec_x t_2$. Suppose instead that $t_4 \prec_x t_3$. This time the fundamental property on $\prec_y$ would guarantee $s_3 \prec_x s_4 \prec_x t_4 \prec_x t_1$ and $s_3 \prec_y s_4 \prec_y t_4 \prec_y t_3$, contradicting compatibilty. Hence, our compatible, fundamental orderings also satisfy $t_2 \prec_y t_3 \prec_x t_4$.

To summarize, without loss of generality the orderings are such that $r \prec_x s_3 \prec_x t_4 \prec_x t_1 \prec_x t_2$ and $r \prec_y s_3 \prec_y t_2 \prec_y t_3 \prec_y t_4$ with $s_3t_2, s_3t_4 \notin A(H_5)$. This contradicts the fact that the two orderings are properly compatible. In conclusion, H_5 has a pair of compatible, fundamental orderings but no pair of properly compatible, fundamental orderings. This shows that $\mathcal{P} \subset \mathcal{R}$.

This gives relationships between the classes of digraphs discussed in this chapter. Of particular note is the fact that the classes of unit chronological rectangle digraphs and proper chronological rectangle digraphs are not equivalent, in contrast with unit interval graphs and proper interval graphs. Furthermore, the digraph H_5 shows that the intersection of two proper chronological interval digraphs need not be a proper chronological rectangle digraph.

Chapter 7

Further Research

The purposes of this chapter are two-fold. First, we would like to discuss implications of the most important results presented in this dissertation. Second, we would like to discuss some open problems and address possible directions for future reasearch.

As shown in Chapter 2, every chronological rectangle digraph satisfies the umbrella path property as well as the weakly-clustered property. Not every digraph with these two properties is chronological rectangle (see Section 2.6). However, these two properties are sufficient for certain digraphs to be chronological rectangle. It would be interesting to identify classes of digraphs for which these two conditions are sufficient.

Chronological rectangle digraphs are characterized by functions in Theorem 2.6 and by pairs of compatible, fundamental vertex orderings in Theorem 2.12. Some additional properties of these orderings are also established in Sections 2.5 and 2.6. Compatible, fundamental orderings are the most powerful tool to prove whether or not various digraph classes are chronological rectangle.

The most important problem this dissertation does not address is the recognition problem, that is, to decide whether or not a digraph is chronological rectangle. Unfortunately our characterizations of chronological rectangle digraphs in terms of functions and in terms of compatible, fundamental vertex orderings do not immediately lead to a solution to this problem.

Problem 7.1. *What is the complexity of recognizing chronological rectangle digraphs?*

A related problem is to characterize chronological rectangle digraphs in terms of forbidden structures. Both interval graphs and chronological interval digraphs can be recognized in linear time. The key for devising such algorithms is the existence of an appropriate structural characterization.

Problem 7.2. *Find a forbidden structure characterization of chronological rectangle digraphs.*

The concept of chronological interval dimension is introduced in Chapter 3. Theorem 3.3 gives necessary and sufficient conditions for a reflexive digraph to be the intersection of some finite number of chronological interval digraphs, i.e., having a chronological interval dimension. The proof of Theorem 3.3 implies that if a digraph which satisfies the conditions then its chronological interval dimension bounded above by the number of vertices.

Problem 7.3. *Determine a tight bound on the chronological interval dimension of a digraph D which satisfies the conditions listed in Theorem 3.3.*

As discussed in the introduction, determining whether or not a graph has boxicity at most k is NP-complete for each $k \geq 2$ [41, 63]. The analogous problem for chronological interval dimension generalizes the recognition problem for chronological rectangle digraphs.

Problem 7.4. *What is the complexity of determining whether a reflexive digraph which satisfies the conditions in Theorem 3.3 has chronological interval dimension at most k for each $k \geq 2$?*

A characterization of the two-terminal series-parallel digraphs which are chronological rectangle is the main result of Chapter 4. We are able to identify precisely which two-terminal series-parallel digraphs are not chronological rectangle, and recursively construct pairs of compatible, fundamental orderings for the digraphs which are chronological rectangle. A possible generalization of our result would be to consider series-parallel digraphs which do not necessarily have a single source and sink.

We began Chapter 5 by showing that the digraphs obtained by adding a loop to each vertex of any orientation of a tree are chronological rectangle. However, an analogous result does not hold for k-trees with $k \geq 2$. There are numerous digraphs obtained from orientations of k-trees by adding loops to all vertices with $k \geq 2$ which do not satisfy the weakly-clustered and umbrella path properties. Furthermore, consider the digraph Q given in Figure 3.1 which is obtained from an orientation of a 3-tree by adding loops to all vertices. This digraph Q satisfies the weakly-clustered and umbrella path properties but is not chronological rectangle by Lemma 3.12. A natural question would be determine which orientations of a k-tree for $k \geq 2$ lead to chronological rectangle digraphs.

On the other hand, Proposition 5.5 showed that there exists a split graph such that no orientation leads to a chronological rectangle digraph. The graph constructed in the proof of Proposition 5.5 is the only known example which satisfies this property.

Problem 7.5. *Determine which graphs have orientations which lead to chronological rectangle digraphs.*

Our final topic involved the introduction of the classes of unit and proper chronological rectangle digraphs. We made some observations about these classes and gave ordering characterizations for them. Unit chronological rectangle digraphs in particular satisfied several stronger properties than chronological rectangle digraphs. However, forbidden substructure characterizations and the complexity of the recognition problems remain unsolved for either class of digraphs.

Problem 7.6. *Find a forbidden structure characterization of unit (resp. proper) chronological rectangle digraphs.*

Problem 7.7. *What is the complexity of recognizing unit (resp. proper) chronological rectangle digraphs?*

Throughout our work, many properties of chronological rectangle digraphs are established by synthesizing together diverse ideas from various sources. These include several powerful theorems as well as some useful tools which can be used for further study of these digraphs. It is our hope that other interesting results on chronological rectangle digraphs will be discovered in the near future.

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