# Chordality of Digraphs, Signed Graphs, and Signed Bigraphs

by

YingYing (Fay) Ye
B.Sc. University of Victoria 2007
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We acknowledge and respect the Lək$^{w}$əŋən (Songhees and X$^{w}$sepsəm/Esquimalt) Peoples on whose territory the university stands, and the Lək$^{w}$əŋən and W̱SÁNEĆ Peoples whose historical relationships with the land continue to this day.

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# ABSTRACT

Chordal graphs and chordal bigraphs enjoy beautiful characterizations, in terms of forbidden subgraphs, vertex or edge orderings, and tree-like representations. A digraph analogue of chordal graphs was introduced by Haskins and Rose. Unfortunately, a forbidden subdigraph characterization of chordal digraphs is not known and finding such a characterization seems to be a difficult problem. The study of chordal digraphs has been restricted to various classes of digraphs, such as semicomplete digraphs, quasi-transitive digraphs, extended semicomplete digraphs and locally semicomplete digraphs.

In this dissertation, we introduce the new class of weakly quasi-transitive digraphs as a common generalization of semicomplete digraphs, quasi-transitive digraphs and symmetric digraphs. We show that weakly quasi-transitive digraphs can be obtained from these three classes of digraphs by substitutions. As a consequence, weakly quasi-transitive digraphs admit a recursive construction. This construction theorem allows us to find a forbidden subdigraph characterization of weakly quasi-transitive chordal digraphs. In addition, we use it to prove that the small quasi-kernel conjecture holds for weakly quasi-transitive digraphs.

We extend the notion of chordality to signed graphs and signed bigraphs. Interestingly, chordal signed graphs are equivalent to strict chordal digraphs studied by Hell and Hernandez-Cruz. The forbidden subdigraph characterization of strict chordal digraphs can be translated to a forbidden subgraph characterization of chordal signed graphs. We give a forbidden subgraph characterization of chordal signed bigraphs. The forbidden subgraphs for chordal signed bigraphs are analogous to those for chordal signed graphs but the proofs are much more complicated and intriguing.

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# DEDICATION

To the memory of my beloved grandfather, Jing Liu.

# Chapter 1

# Introduction

## 1.1 Background

A graph $G$ is *perfect* if for every induced subgraph $H$ of $G$, the chromatic number of $H$ is equal to the size of a maximum clique in $H$ [11, 12, 31]. Berge [11, 12] conjectured that a graph is perfect if and only if it does not contain an odd cycle of length $\geq 5$ or its complement as an induced subgraph. This conjecture was known as strong perfect graph conjecture and later proved by Chudnovsky, Robertson, Seymour and Thomas [15, 16].

One of the first classes of graphs recognized as being perfect was the class of chordal graphs. A vertex $u$ in a graph is *simplicial* if its neighbours induce a complete subgraph. A graph $G$ is *chordal* if every induced subgraph of $G$ contains a simplicial vertex [23, 51]. Hajnal and Surányi [35] showed that every chordal graph is perfect.

Chordal graphs admit elegant characterizations, in terms of forbidden subgraphs, vertex orderings, and tree representations [23, 29, 61]. These characterizations enable many optimization problems to be solvable in linear time on chordal graphs [31]. A *perfect elimination ordering* of a graph $G$ is a vertex ordering $v_1, v_2, \ldots, v_n$ such that

$v_i$ is a simplicial vertex in the subgraph of $G$ induced by $v_i, v_{i+1}, \ldots, v_n$. Fulkerson [26] proved that a graph is chordal if and only if it has a perfect elimination ordering. It is proved that a graph is chordal if and only if it does not contain a cycle of length greater than three as an induced subgraph [61]. Chordal graphs can also be represented by subtrees in trees [29]. Rose and Tarjan [60] obtained a simple linear time recognition algorithm for chordal graphs.

In 1973, Haskins and Rose [38] introduced a digraph analogue of chordal graphs. A vertex $v$ in a digraph $D$ is called a *di-simplicial vertex* if every in-neighbour of $v$ dominates every out-neighbour of $v$. A digraph $D$ is *chordal* if every induced subdigraph of $D$ contains a di-simplicial vertex. It follows that every chordal digraph $D$ has a vertex ordering $v_1, v_2, \ldots, v_n$ such that $v_i$ is a di-simplicial vertex in the subdigraph of $D$ induced by $v_i, v_{i+1}, \ldots, v_n$ for each $i \geq 1$. Such a vertex ordering is called a *perfect elimination ordering* of $D$. Graphs can be viewed as symmetric digraphs, and so the di-simplicial vertices of a symmetric digraph coincide with the simplicial vertices of its underlying graph. Thus, chordal graphs are equivalent to the symmetric chordal digraphs and hence the class of chordal graphs is a subclass of chordal digraphs.

Chordal digraphs arise naturally from consideration of sparse matrices whose sparseness can be preserved by Gaussian eliminations [38]. Chordal digraphs can be recognized in polynomial time [56]. But, unlike chordal graphs, little is known about the structure of chordal digraphs. In particular, there is no known forbidden subdigraph characterization of chordal digraphs. According to [47], finding forbidden subdigraph characterizations of chordal digraphs may be a difficult problem. Research has been focusing on characterizing chordal digraphs with additional properties. Meister and Telle [56] characterized semicomplete chordal digraphs by forbidden subdigraphs. Locally semicomplete digraphs, quasi-transitive digraphs and extended semicomplete digraphs (to be defined in section 2) are three classes of digraphs that generalize the class of semicomplete digraphs. The forbidden subdigraph characterizations for the chordal digraphs in these three digraph classes have been obtained in [65].

In this dissertation, we introduce the class of weakly quasi-transitive digraphs. This class of digraphs contains quasi-transitive digraphs, extended semicomplete digraphs and symmetric digraphs. We show that every weakly quasi-transitive digraph can be obtained from these digraphs by a substitution operation. This allows us to obtain a forbidden subdigraph characterization of weakly quasi-transitive chordal digraphs (Theorem 3.5).

Although chordal digraphs are a natural digraph analogue of chordal graphs, they seem to be too broad to recover the elegance of chordal graphs. In [41], a subclass of chordal digraphs was introduced. A vertex $v$ in a digraph $D$ is called *strict di-simplicial* if every pair of neighbours of $v$ are joined by symmetric arcs. A digraph $D$ is called *strict chordal* if every induced subdigraph of $D$ contains a strict di-simplicial vertex. Since strict di-simplicial vertices are di-simplicial, every strict chordal digraph is chordal. Surprisingly, a forbidden subdigraph characterization of strict chordal digraphs has been obtained in [41]. The forbidden subdigraph characterization of strict chordal digraphs makes use of the fact that the underlying graph of a strict chordal digraph is a chordal graph [41]. This fact is also critical for extensions of the minimal separators property of chordal graphs to strict chordal digraphs [41, 55]. There is a polynomial time recognition algorithm for strict chordal digraphs [41].

We propose an intermediate notion between chordal digraphs and strict chordal digraphs. A vertex $v$ in a digraph $D$ is called *semi-strict di-simplicial* if every in-neighbour of $v$ adjacent to every out-neighbour of $v$ by symmetric arcs. We call a digraph $D$ *semi-strict chordal* if every induced subdigraph of $D$ contains a semi-strict di-simplicial vertex. Note that strict di-simplicial vertices are semi-strict di-simplicial vertices which are di-simplicial vertices, and so every strict chordal digraph is a semi-strict chordal digraph which is chordal. In Figure 1.1, the digraph on the left is semi-strict chordal (and hence chordal) but not strict chordal, the digraph on the right is chordal but not semi-strict chordal (and hence not strict chordal). We obtain forbidden subdigraph characterizations of semi-strict chordal digraphs within the classes of locally semicomplete digraphs and weakly quasi-transitive digraphs.

Figure 1.1: Chordal digraphs

Let $G$ be a bipartite graph with bipartition $(X, Y)$. A subgraph $H$ of $G$ is called a *biclique* if every vertex in $V(H) \cap X$ is adjacent to all vertices in $V(H) \cap Y$. An edge $uv$ in $G$ is *simplicial* if the neighbours of $u$ and the neighbours of $v$ induce a biclique in $G$. A *perfect edge-without-vertex elimination ordering* of $G$ is an edge ordering $e_1, e_2, \ldots, e_m$ such that each $e_i$ is a simplicial edge in $G - \{e_1, e_2, \ldots, e_{i-1}\}$ [32]. A bipartite graph is *chordal* if it has a perfect edge-without-vertex elimination ordering. Kloks and Kratsch [48] provided an efficient algorithm of computing a perfect edge-without-vertex elimination ordering. In fact, chordal bipartite graphs are exactly the bipartite graphs which do not contain an induced cycle of length $\geq 6$ [32]. This implies in particular that a bipartite graph is chordal if and only if every induced subgraph has a simplicial edge.

A *signed graph* $\widehat{G}$ is a graph $G$ in which each edge is signified with either a positive sign "+" or a negative sign "−" [36, 66, 67, 68]. Edges with positive signs are called *positive* and edges with negative signs are called *negative*. Signed graphs were introduced in 1953 by Harary [66] to model social balances. Signed graphs also have applications in geometry, matroid theory, and social science [66]. We introduce the notion of chordal signed graphs. Let $\widehat{G}$ be a signed graph. A *positive clique* in $\widehat{G}$ is a clique in which all edges are positive. A vertex $v$ is called *signed simplicial* if the neighbours of $v$ induce a positive clique in $\widehat{G}$. We call $\widehat{G}$ *chordal* if every induced subgraph of $\widehat{G}$ has a signed simplicial vertex. Like chordal graphs, the vertices of a signed graph $\widehat{G}$ can be ordered $v_1, v_2, \ldots, v_n$ such that each $v_i$ is a signed simplicial vertex in $\widehat{G} - \{v_1, v_2, \ldots, v_{i-1}\}$.

Let $D$ be a digraph, and $G$ be the underlying graph of $D$. Define a signed graph $\widehat{G}$ where an edge $uv$ in $\widehat{G}$ is positive if both $uv$ and $vu$ are arcs in $D$, and $uv$ is negative if only one of $uv$ or $vu$ is an arc in $D$. Then a digraph $D$ can be viewed as a signed graph $\widehat{G}$. It is easy to check that if $v$ is a strict di-simplicial vertex of $D$, then it is a signed simplicial vertex of $\widehat{G}$. It turns out that chordal signed graphs are an equivalent notion to strict chordal digraphs, and we conclude that $D$ is strict chordal if and only if $\widehat{G}$ is chordal.

A *signed bipartite graph* is a signed graph $\widehat{G}$ where $G$ is bipartite. If $G$ is a chordal bipartite graph, then $\widehat{G}$ is a *signed chordal bipartite graph*. We extend the concept of chordal signed graph to chordal signed bipartite graph. An edge $uv$ of $\widehat{G}$ is called a *signed simplicial edge* if the neighbours of $u$ and the neighbours of $v$ induce a positive biclique in $\widehat{G}$. Note that a signed simplicial edge may have negative edges incident with its end vertices. A *signed simplicial perfect edge-without-vertex elimination* ordering of $\widehat{G}$ is an edge ordering $e_1, e_2, \ldots, e_m$ of $\widehat{G}$ such that each $e_i$ is a signed simplicial edge in $\widehat{G} - \{e_1, e_2, \ldots, e_{i-1}\}$. We call $\widehat{G}$ a *chordal signed bipartite graph* if it has a signed simplicial perfect edge-without-vertex elimination ordering. Note the difference between chordal signed bipartite graphs and signed chordal bipartite graphs in which case every chordal signed bipartite graph is a signed chordal bipartite graph, but a signed chordal bipartite graph may not be a chordal signed bipartite graph. For instance, a $C_4$ with all negative edges is signed chordal but not chordal signed. We obtain the forbidden subgraph characterization of chordal signed bipartite graphs. Our characterization implies that a signed bipartite graph is chordal if and only if every induced subgraph has a signed simplicial edge, in particular, every induced subgraph is chordal.

## 1.2 Terminology and notation

In this section, we give definitions and notations that will be used throughout this dissertation.

A *graph* $G$ is a pair $(V, E)$ of sets, where the elements in $V$ are called the *vertices* of $G$, and the elements in $E$ are called the *edges* of $G$ which are unordered pairs of elements in $V$. If $uv \in E$, then we say that $u$ and $v$ are *adjacent*. The *neighbourhood* of a vertex $v$, denoted by $N(v)$, is the set $\{u \in V : uv \in E\}$. The *closed neighbourhood* $N[v]$ is the set $\{v\} \cup N(v)$.

A *path* of *length* $k$ in a graph $G$ is a sequence of distinct vertices $v_0, v_1, \ldots, v_k$ where $v_i$ is adjacent to $v_{i+1}$ for each $0 \leq i < k$. An *independent set* of a graph is a set of vertices such that no two of them are adjacent. A graph is a *bipartite graph* (or *bigraph* in short) if its vertices can be partitioned into independent sets $X$ and $Y$, where $X$ and $Y$ are referred to as the partite sets of the bipartite graph. Equivalently, a bigraph is a graph that does not contain any odd cycles.

A graph $G$ is *complete* if there is an edge between any pair of vertices. A *clique* of a graph $G$ is a complete subgraph of $G$. A *complete bigraph* is a bipartite graph where every vertex of one partite set is adjacent to every vertex of the other partite set. If a subgraph of a bigraph is a complete bigraph, then it is called a *biclique* of the bigraph.

An *intersection graph* is a graph $G$ that each vertex $v$ of $G$ is associated with a set $S_v$ such that two vertices $u$ and $v$ are adjacent in $G$ if and only if $S_u$ and $S_v$ intersect.

A *digraph* $D = (V, A)$ consists of a *vertex set* $V$ and an *arc set* $A$ where each arc is an ordered pair of vertices in $V$. Let $u, v$ be two vertices of a digraph $D$. We say that $u, v$ are *adjacent* if there is an arc between $u$ and $v$, otherwise we say that they are *not* adjacent. If $uv$ or $vu$ is an arc but not both, then we say $u$ is adjacent to $v$ by a *non-symmetric* arc. If both $uv$ and $vu$ are arcs, then we call them *symmetric arcs*. In a drawing of a digraph, adjacency between two vertices is depicted by a solid line, non-adjacency by dashed line, symmetric arcs by a solid line and having arrows at both ends, and non-symmetric arc by a solid line and marked by a short bar (e.g. see Figure 4.2). If $uv$ is an arc, then we say that $u$ *dominates* $v$ (or $v$ is *dominated* by $u$), moreover, $u$ is an *in-neighbour* of $v$ and $v$ is an *out-neighbour* of $u$. The set of all in-neighbours of $v$ is denoted by $N^-(v)$ and the set of all out-neighbours of $v$ is denoted by $N^+(v)$. We use $N^{\pm}(v)$ to denote the intersection of $N^-(v)$ and $N^+(v)$. A digraph which does not contain symmetric arcs is called an *oriented graph*. A digraph which contains only symmetric arcs is called a *symmetric digraph*. We use $S(D)$ to denote the spanning subdigraph of $D$ whose arc set consists of the symmetric arcs in $D$. Graphs may be viewed as symmetric digraphs.

An *oriented path* of *length* $k$ in $D$ is a sequence of distinct vertices $v_0, v_1, \ldots, v_k$ such that $v_i$ and $v_{i+1}$ are joined by a non-symmetric arc for each $i = 0, 1, \ldots, k-1$. If $v_i v_{i+1}$ is an arc for each $0 \leq i < k$, then it is called *directed path* (or *dipath* in short). A digraph $D$ is *strongly connected* (*strong* for short) if for every pair of distinct vertices $u$ and $v$, there exists a directed path from $u$ to $v$ and a directed path from $v$ to $u$.

A *directed cycle* $C$ of length $k$ is a cyclic sequence of vertices $v_0, v_1, \ldots, v_{k-1}$ such that $v_i v_{i+1}$ is an arc for each $i = 0, 1, \ldots, k-1 \pmod{k}$. If there is no arc between

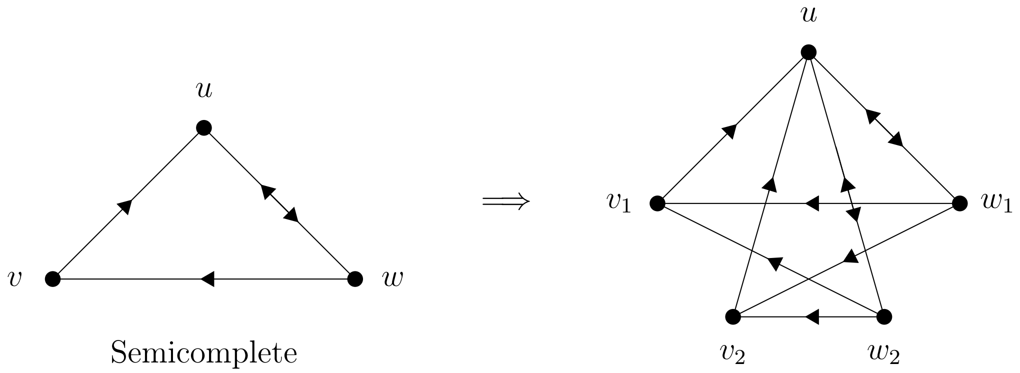
$v_i$ and $v_j$ for any $i, j$ such that $i - j \neq \pm 1 \pmod{k}$, then $C$ is called *induced directed cycle*. A digraph is *acyclic* if it does not contain any directed cycle.

A *sink* is a vertex that has no out-neighbour and a *source* is a vertex that has no in-neighbour. A digraph $D$ is said to be *sink-free* if it has no sink. An *independent set* of a digraph $D$ is a vertex set $I$ such that any $u, v \in I$, $u$ and $v$ are not adjacent. A *kernel* of a digraph $D$ is an independent set $K \subseteq V$ such that for every vertex $v \in V \setminus K$, there exists an arc from $v$ to a vertex $u$ of $K$. A *quasi-kernel* of a digraph $D$ is an independent set $Q \subseteq V$ such that for every vertex $v \in V \setminus Q$, there exists a directed path of length $\leq 2$ from $v$ to a vertex $u$ in $Q$.

The *reversal* of a digraph $D$ is the digraph obtained from $D$ by reversing all arcs. The *underlying graph* $U(D)$ of a digraph $D$ is a simple graph consisting of vertex set $V(D)$ and edges $uv$ where $u$ and $v$ are adjacent in $D$. The *complementary digraph* $\overline{D}$ of a digraph $D$ is a digraph such that $V(\overline{D}) = V(D)$, and $uv$ is an arc of $\overline{D}$ if and only if it is not an arc of $D$.

Let $D$ be a digraph with vertices $v_1, v_2, \ldots, v_n$ and let $H_1, H_2, \ldots, H_n$ be vertex disjoint digraphs. A *substitution* of the digraph $H_i$ for the vertex $v_i$ in $D$ for each $i$ is a new digraph $D^*$ obtained from the disjoint union of $H_1, H_2, \ldots, H_n$ by adding all possible arcs $xy$ where $x \in V(H_i)$ and $y \in V(H_j)$ for each arc $v_iv_j$ in $D$.

A digraph $D$ is *semicomplete* if there is at least one arc between any two vertices. A *locally semicomplete* digraph is a digraph $D$ such that for every vertex $v \in V(D)$, $N^-(v)$ and $N^+(v)$ each induces a semicomplete subdigraph in $D$. Locally semicomplete digraphs are a popular generalization of semicomplete digraphs and have been extensively studied [4, 5, 6, 43]. An *extended semicomplete* digraph is a digraph obtained from a semicomplete digraph by substituting each vertex with an independent set [6]. (See Figure 1.2 for an example.) A *transitive oriented graph* is an oriented graph such that for any three vertices $u, v$ and $w$, both $uv, vw$ are arcs imply $uw$ is an arc. Quasi-transitive digraphs are another well-studied class of digraphs generalizing semicomplete digraphs [7, 8, 9, 27]. A digraph $D$ is *quasi-transitive* if for any three vertices $u, v$ and $w$ such that $u \in N^-(v)$ and $w \in N^+(v)$, then $u$ and $w$ are adjacent. The class of quasi-transitive digraphs contains all semicomplete digraphs and transitive oriented graphs [7].



Semicomplete

Extended semicomplete

Figure 1.2: An extended semicomplete digraph blown up from a semicomplete digraph

In this dissertation, we study both graphs and digraphs all of which are assumed to be finite and do not contain multiple edges or multiple arcs, but may contain self-loops and symmetric arcs.

We will use edge-coloured graphs to represent signed graphs where positive edges are coloured *blue*, negative edges are coloured *red*, and edges whose signs are not specified are coloured *black*. Let $\widehat{G}$ be a signed graph. A subgraph of $\widehat{G}$ is a signed graph $\widehat{H}$ obtained from $\widehat{G}$ by deleting vertices and/or edges. If $\widehat{H}$ is obtained from $\widehat{G}$ by vertex deletions only, then it is called an *induced subgraph* of $\widehat{G}$ or a subgraph of $\widehat{G}$ induced by the vertex set $V(\widehat{H})$ of $\widehat{H}$. We call a subgraph of $\widehat{G}$ positive if its edges are all *positive*. For an edge $uv$ of $\widehat{G}$, we use $N(uv)$ to denote $(N(u) \cup N(v)) - \{u, v\}$.

Figure 1.3: A graph (left) and a signed graph (right)

Suppose that $F$ is an edge-coloured graph whose edges are coloured with blue, red and black. If $F$ has no black edge, then it represents a unique signed graph. If $F$ has black edges, then by replacing each black edge of $F$ with a blue or a red edge we obtain a signed graph. In this way, $F$ represents a set of signed graphs. We say that a signed graph $\widehat{G}$ *contains* a graph in $F$ if at least one of the graphs in $F$ is a subgraph of $\widehat{G}$, otherwise we say that $\widehat{G}$ does not contain $F$. The set of signed graphs in Figure 1.3 (right) contains a positive $C_3$ but does not contain a negative $C_3$.

A signed graph $\widehat{G}$ is called a *signed complete bigraph* when $G$ is a complete bigraph. A bigraph is *separable* if it contains an induced $2K_2$, otherwise it is *non-separable*. A signed bigraph $\widehat{G}$ is separable if $G$ is separable, otherwise it is non-separable.

A *separating set* in a bigraph $G$ is a set of vertices $S$ such that $G - S$ contains at least two non-trivial components. Clearly, a bigraph is separable if and only if it has a separating set. Similarly, a *separating set* in a signed separable bigraph $\widehat{G}$ is a set of vertices $S$ such that $\widehat{G} - S$ contains at least two non-trivial components. A separating set is *minimal* if no proper subset of the set is separating.

## 1.3 Outline of results

This dissertation studies the structure of digraphs, signed graphs and signed bigraphs in connection to chordality.

In Chapter 2, we introduce a new class of digraphs called weakly quasi-transitive digraphs. We show that weakly quasi-transitive digraphs admit a recursive construction. As a by-product, we prove that the small quasi-kernel conjecture holds for weakly quasi-transitive digraphs.

In Chapter 3, we apply the recursive construction of weakly quasi-transitive digraphs to characterize weakly quasi-transitive chordal digraphs by forbidden subdigraphs.

We turn to semi-strict chordal digraphs in Chapter 4. The notion of semi-strict digraphs was introduced by the author in [65] where forbidden subdigraph characterizations were obtained for semi-strict digraphs within several classes of digraphs. We extend the results from [65] to find a forbidden subdigraph characterization of weakly quasi-transitive semi-strict digraphs.

In Chapter 5, we introduce the concept of chordal signed graphs and chordal signed bigraphs. We observe that chordal signed graphs are equivalent to strict chordal digraphs previously studied in [41]. The forbidden subdigraph characterization of strict chordal digraphs from [41] yields immediately a forbidden subgraph characterization of chordal signed graphs. We obtain a forbidden subgraph characterization of chordal signed bigraphs.

Finally, we conclude in Chapter 6 with some remarks on the results obtained in this dissertation. We also propose several open problems for future research.

# Chapter 2

# Weakly Quasi-transitive Digraphs

## 2.1 Introduction

The definition of transitive oriented graphs ensures that they are all acyclic. Every vertex of a transitive oriented graph is a di-simplicial vertex. So each transitive oriented graph is chordal.

Quasi-transitive digraphs were initially studied in [30]. Interestingly, they share the same underlying graphs with transitive oriented graphs, which are known as comparability graphs [30]. In 1995, Bang-Jensen and Huang [7] provided a structure theorem for this digraph class, which led to polynomial time solutions of many problems. The class of quasi-transitive digraphs contains all transitive oriented graphs and all semicomplete digraphs.

Let $D$ be a digraph with vertices $v_1, v_2, \ldots, v_n$ and let $H_1, H_2, \ldots, H_n$ be vertex-disjoint digraphs. Recall that the substitution of the digraph $H_i$ for the vertex $v_i$ in $D$ for each $i$ is a new digraph $D^*$ obtained from $H_1, H_2, \ldots, H_n$ by adding all possible arcs $xy$ where $x \in V(H_i)$ and $y \in V(H_j)$ for each arc $v_iv_j$ in $D$. We use $D[H_1, H_2, \ldots, H_n]$ to denote the new digraph $D^*$ and also say that it is obtained from

$D$ by substituting $H_i$ for $v_i$ for each $i$.

**Theorem 2.1.** *([7]) Let $D$ be a quasi-transitive digraph. Then the following statements hold:*

1. *If $D$ is non-strong, then $D = T[H_1, H_2, \ldots, H_n]$ where $T$ is a transitive oriented graph and each $H_i$ is a strong quasi-transitive digraph.*
2. *If $D$ is strong, then $D = S[H_1, H_2, \ldots, H_n]$ where $S$ is a strong semicomplete digraph and each $H_i$ is either a single-vertex or a non-strong quasi-transitive digraph.* □

Theorem 2.1 implies that quasi-transitive digraphs can be obtained from transitive oriented graphs and semicomplete digraphs recursively by substitution.

The class of extended semicomplete digraphs contains all semicomplete digraphs. However, there is no containment relationship between quasi-transitive digraphs and extended semicomplete digraphs. We introduce a new class of digraphs as a common generalization of several classes of digraphs including quasi-transitive digraphs, extended semicomplete digraphs, and symmetric digraphs.

Given a digraph $D$ and vertices $u, v, w$ of $D$ such that $u, w \in N(v)$, we say that $u$ and $w$ are synchronous neighbours of $v$ if they are both in $N^-(v) \setminus N^+(v)$, or in $N^+(v) \setminus N^-(v)$, or in $N^-(v) \cap N^+(v)$. Otherwise, they are *asynchronous neighbours* of $v$. A digraph $D$ is *weakly quasi-transitive* if for each vertex $v$ in $D$, any two asynchronous neighbours of $v$ are adjacent. Clearly, every symmetric digraph is weakly quasi-transitive. This is because symmetric digraphs have the property that the neighbours of each vertex are synchronous and any digraph having this property is weakly quasi-transitive. Since graphs can be viewed as symmetric digraphs, this new class of digraphs contains all graphs. The class of weakly quasi-transitive digraphs also contains all quasi-transitive digraphs and hence all semicomplete digraphs, and all transitive oriented digraphs. Indeed, suppose that $D$ is not weakly quasi-transitive. Then some vertex $v$ has two non-adjacent asynchronous neighbours $u, w$. Since $u, w$ are asynchronous neighbours of $v$, one of $u, w$ is in $N^{-}(v)$ and the other is in $N^{+}(v)$. Hence $D$ is not a quasi-transitive digraph.

If a digraph $D$ is weakly quasi-transitive, then any digraph obtained from $D$ by substituting an independent set for each vertex of $D$ is also weakly quasi-transitive. It follows that the class of weakly quasi-transitive digraphs also contains extended semicomplete digraphs. Figure 2.1 depicts a containment hierarchy of the digraph classes relevant to this chapter.

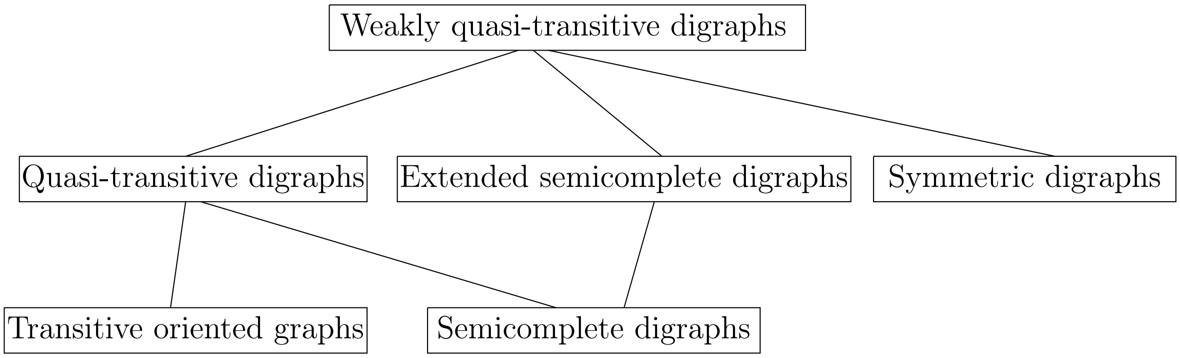


Figure 2.1: A containment hierarchy.

In this chapter, we show that weakly quasi-transitive digraphs can be constructed recursively from transitive oriented graphs, symmetric digraphs, and semicomplete digraphs by substitutions (see Theorem 2.2). As a by-product of this recursive construction, we will prove in Chapter 3 that the forbidden subdigraphs for weakly quasi-transitive chordal digraphs are exactly those for semicomplete chordal digraphs. The forbidden subdigraph characterization of weakly quasi-transitive chordal digraphs generalizes not only the results of [65] on quasi-transitive chordal digraphs and extended semicomplete chordal digraphs but also the classical results on chordal graphs.

Kernels in a digraph were first introduced by von Neumann and Morgenstern in the context of games and they have applications in economy and logic [64]. It is proved in [2] that the complement of a perfect digraph has a kernel. Not every digraph has a kernel. Richardson [58] proved that every digraph with no odd directed cycle has a kernel. Chvátal [17] proved that it is an NP-complete problem to decide if a digraph has a kernel. Hell and Hernández-Cruz showed that this problem is polynomial time solvable for quasi-transitive digraphs [40].

In contrast, Chvátal and Lovász [18] proved that every digraph has a quasi-kernel. Quasi-kernels have been studied in [1, 18, 20, 27, 33, 39, 63, 46, 49, 50]. In 1976, Erdős and Székely [24] stated the small quasi-kernel conjecture in digraphs. We will use the structure theorem of weakly quasi-transitive digraphs to prove that the small quasi-kernel conjecture holds for the class of digraphs in this Chapter.

## 2.2 Structure theorem

Transitive oriented graphs and semicomplete digraphs are basic building blocks for quasi-transitive digraphs [7]. Using these blocks one can form a class $\mathcal{Q}$ of digraphs as follows:

- Each transitive oriented graph is in $\mathcal{Q}$.
- Each semicomplete digraph is in $\mathcal{Q}$.
- If $D$, $H_1, H_2, \ldots, H_n \in \mathcal{Q}$, then $D[H_1, H_2, \ldots, H_n] \in \mathcal{Q}$, provided that $H_i$ is a single-vertex digraph whenever the vertex $v_i$ for which $H_i$ is substituted is incident with a symmetric arc for each $i$.

Since transitive oriented graphs and semicomplete digraphs are quasi-transitive and the substitution operation maintains the property of being quasi-transitive, the digraphs in $\mathcal{Q}$ are all quasi-transitive. Theorem 2.1 ensures that quasi-transitive digraphs are all in $\mathcal{Q}$. Therefore we have the following:

**Corollary 2.1.** *The class $\mathcal{Q}$ consists of quasi-transitive digraphs, that is, a digraph is quasi-transitive if and only if it is in $\mathcal{Q}$.* $\square$

Weakly quasi-transitive digraphs can also be constructed in a similar way as quasi-transitive digraphs.

Let $\mathcal{W}$ be the class of digraphs defined as follows:

- Each transitive oriented graph is in $\mathcal{W}$;
- Each semicomplete digraph is in $\mathcal{W}$;
- Each symmetric digraph is in $\mathcal{W}$;
- If $D, H_1, H_2, \ldots, H_n \in \mathcal{W}$, then $D[H_1, H_2, \ldots, H_n] \in \mathcal{W}$.

A *module* in a digraph $D$ is an induced subdigraph $H$ of $D$ such that for any vertex $x$ not in $H$, either $x$ is adjacent to no vertex in $H$ or the vertices in $H$ are synchronous neighbours of $x$. A module is called *trivial* if it has only one vertex or is the entire digraph $D$ and *non-trivial* otherwise.

**Theorem 2.2.** *The class $\mathcal{W}$ consists of weakly quasi-transitive digraphs, that is, a digraph is weakly quasi-transitive if and only if it is in $\mathcal{W}$.*

*Proof.* Transitive oriented graphs and semicomplete digraphs are quasi-transitive, so they are weakly quasi-transitive. Symmetric digraphs are also weakly quasi-transitive because any vertex in a symmetric digraph has only synchronous neighbours. In order to prove the rest of the digraphs in $\mathcal{W}$ are all weakly quasi-transitive, let $D^* = D[H_1, H_2, \ldots, H_n]$ where $D$, $H_1, H_2, \ldots, H_n$ are weakly quasi-transitive digraphs. Consider three vertices $u, v, w$ in $D^*$ where $u, w$ are asynchronous neighbours of $v$. Assume that $u \in V(H_i)$, $v \in V(H_j)$ and $w \in V(H_k)$. If $i = j = k$ then $u, w$ are adjacent as $H_i$ is weakly quasi-transitive. Suppose that $i = j \neq k$. Since $v$ and $w$ are adjacent, each vertex of $H_i$ is adjacent to all vertices of $H_k$ and in particular, $u$ is adjacent to $w$. Similarly, if $i \neq j = k$, then $u$ and $w$ are adjacent. Suppose that $i \neq j \neq k$. Then $i \neq k$ because $u$ and $w$ are asynchronous neighbours of $v$. Since $D$ is weakly quasi-transitive, the two vertices of $D$ corresponding to $H_i$ and $H_k$ are adjacent and so $u$ and $w$ are adjacent. Hence all digraphs in $\mathcal{W}$ are weakly quasi-transitive.

We prove by induction on the number of vertices that every weakly quasi-transitive digraph is in $\mathcal{W}$. Let $D$ be a weakly quasi-transitive digraph with $n$ vertices. Assume that every weakly quasi-transitive digraph with fewer than $n$ vertices is in $\mathcal{W}$. If $D$ is quasi-transitive or symmetric then it is in $\mathcal{W}$. Hence assume that $D$ is neither quasi-transitive nor symmetric. Since $D$ is not quasi-transitive, there exist vertices $u, v, w$ in $D$ with $u \in N^-(v)$ and $w \in N^+(v)$ such that $u$ and $w$ are not adjacent in $D$. Thus $u$ and $w$ are non-adjacent neighbours of $v$. Since $D$ is weakly quasi-transitive, any two asynchronous neighbours of $v$ are adjacent. Hence $u$ and $w$ are synchronous neighbours of $v$, which implies $u$ and $w$ are both in $N^{\pm}(v)$.

Suppose that $H$ is a non-trivial module in $D$. Let $D'$ be the digraph obtained from $D$ by deleting all vertices of $H$ except one. Then $D = D'[H'_1, H'_2, \ldots, H'_k]$ where $H'_1 = H$ and each $H'_i$ with $i \geq 2$ is a single-vertex digraph. The digraphs $D', H'_1, \ldots, H'_k$ each has vertices fewer than $n$ and is weakly quasi-transitive and hence in $\mathcal{W}$. This means that $D$ is obtained from digraphs in $\mathcal{W}$ by substitution and by definition $D$ is also in $\mathcal{W}$. Thus, it suffices to show that there is a non-trivial module in $D$.

Let $R$ be the subdigraph of $D$ induced by $N^{\pm}[v]$. Then $u$ and $w$ are a pair of non-adjacent vertices in $R$. Let $M_1$ be the subdigraph of $R$ induced by the vertices which are connected to $u$ by paths in $\overline{U(R)}$. Clearly, $M_1$ contains $u$ and $w$ but not $v$. Suppose that $x$ is a vertex in $N^+[v] \cup N^-[v]$ but not in $M_1$. We claim that $x$ is completely adjacent to $M_1$. Indeed, if $x \in N^+[v] \cap N^-[v]$, then the definition of $M_1$ implies that $x$ is completely adjacent to $M_1$. On the other hand, if $x \in N^+(v) \oplus N^-(v)$ (note that $A \oplus B$ is $(A \setminus B) \cup (B \setminus A)$), then $x$ and any vertex of $M_1$ are asynchronous neighbours of $v$ so $x$ is also completely adjacent to $M_1$. By definition any two vertices of $M_1$ are connected by a path in $\overline{U(M_1)}$. In such path any two consecutive vertices are not adjacent in $D$ and hence are synchronous neighbours of $x$. It follows that the vertices of $M_1$ are synchronous neighbours of $x$.

Suppose that $x \notin N^+[v] \cup N^-[v]$. If $x$ is adjacent to a vertex $y$ in $M_1$, then $x$ and $v$ are non-adjacent neighbours of $y$ and hence they must be synchronous neighbours of $y$. The fact that $v$ is joined to $y$ by symmetric arcs implies that $x$ is joined to $y$ by symmetric arcs. Thus if $x$ is completely adjacent to $M_1$ then the vertices of $M_1$ are synchronous neighbours of $x$. It follows that $M_1$ is a non-trivial module if for each $x \notin N^+[v] \cup N^-[v]$, either $x$ is adjacent to no vertex in $M_1$ or completely adjacent to $M_1$. We may assume $M_1$ is not a module as otherwise we are done. This means that there exist vertices $x, y, y'$ with $x \notin N^+[v] \cup N^-[v]$ and $y, y' \in M_1$ such that $x$ is adjacent to $y$ but not to $y'$. These three vertices $x, y, y'$ along with $M_1$ will be referred to in the rest of proof.

Suppose that $N^+(v) \oplus N^-(v) \neq \emptyset$. Then any vertex in $N^+(v) \oplus N^-(v)$ is a neighbour of $v$ asynchronous to those of $v$ in $N^{\pm}(v)$. Hence every vertex in $N^+(v) \oplus N^-(v)$ is completely adjacent to the vertices in $N^{\pm}(v)$ and in particular adjacent to $M_1$. Suppose that the arcs between $N^+(v) \oplus N^-(v)$ and $M_1$ are all symmetric. Let $M_2$ be the subdigraph of $D$ induced by the vertices which are connected to $v$ by oriented paths. Clearly, $M_2$ contains $v$ and all vertices in $N^+(v) \oplus N^-(v)$. We show that $x$ is not a vertex in $M_2$. Suppose not; there is an oriented path connecting $x$ and $v$.

Then there must exist an oriented path connecting $x$ and a vertex in $N^+(v) \oplus N^-(v)$. Let $a_1, a_2, \ldots, , a_s$ be such an oriented path where $a_1 = x$ and $a_s \in N^+(v) \oplus N^-(v)$. Note that $a_s$ is joined to each vertex of $M_1$ by symmetric arcs and $a_1$ $(= x)$ is not adjacent to $y'$ (in $M_1$). Let $j$ be the largest subscript such that $a_j$ is not adjacent to some vertex $y''$ of $M_1$. Then $1 \leq j < s$ and $a_j \notin N^+[v] \cup N^-[v]$. Since $a_j$ and $a_{j+1}$ are joined by a non-symmetric arc, $a_{j+1} \notin N^+[v] \cap N^-[v]$. Either $a_{j+1} \in N^+(v) \oplus N^-(v)$ or $a_{j+1} \notin N^+[v] \cup N^-[v]$. In either case $a_{j+1}$ is joined to each vertex of $M_1$ by symmetric arcs. Thus $a_j$ and $y''$ are non-adjacent asynchronous neighbours of $a_{j+1}$, contradicting the assumption that $D$ is weakly quasi-transitive. So $x$ is not a vertex of $M_2$. We show that $M_2$ is a module. Let $z$ be a vertex not in $M_2$. By definition $z$ cannot be joined to any vertex of $M_2$ by a non-symmetric arc. Suppose $z$ is joined to some vertex $h$ of $M_2$ by symmetric arcs. Since $h$ can reach every other vertex of $M_2$ by an oriented path, following such a path we see that $z$ is joined to every vertex in the path by symmetric arcs. Hence the vertices of $M_2$ are synchronous neighbours of $z$. Therefore $M_2$ is a non-trivial module in $D$.

Suppose now that the arcs between $N^+(v) \oplus N^-(v)$ and $M_1$ are not all symmetric. Let $M_3$ be a subdigraph of $D$ induced by the vertices defined recursively as follows:

- $u$ is a vertex in $M_3$;
- if $h$ is a vertex in $N^{\pm}(v)$ that is not adjacent to a vertex in $M_3$, then $h$ is a vertex in $M_3$;
- if $h$ is a vertex not in $N^{\pm}(v)$ that is joined to a vertex in $M_3$ by symmetric arcs, then $h$ is a vertex in $M_3$.

It is easy to see that $M_3$ contains $u, v, w, x$ and all vertices of $M_1$. Let $b$ be a vertex in $N^+(v) \oplus N^-(v)$ which is joined to a vertex in $M_1$ by a non-symmetric arc. Assume that $b \in N^-(v) \setminus N^+(v)$. From the above we know that the vertices of $M_1$ are synchronous neighbours of $b$. In particular, $y, y'$ are synchronous neighbours of $b$. The vertex $y$ is joined to $b$ by a non-symmetric arc and joined to $x$ by symmetric arcs. Thus $b$ and $x$ are asynchronous neighbours of $y$ and hence they must be adjacent. So $x$ and $v$ are neighbours of $b$. Since $x$ and $v$ are not adjacent, they are synchronous neighbours of $b$. Since $b \in N^-(v) \setminus N^+(v)$, $bv$ is a non-symmetric arc, so $bx$ is also a non-symmetric arc. Since $bx$ is a non-symmetric arc and $x, y'$ are non-adjacent neighbours of $b$, $by'$ is also a non-symmetric arc. The fact that the vertices of $M_1$ are synchronous neighbours of $b$ ensures there is a non-symmetric arc from $b$ to every vertex in $M_1$. Similarly, if $b \in N^+(v) \setminus N^-(v)$ is joined to a vertex in $M_1$ by a non-symmetric arc then $xb$ is a non-symmetric arc and there is a non-symmetric arc from every vertex of $M_1$ to $b$.

We claim that $b$ is not a vertex in $M_3$. Suppose not; $b$ is in $M_3$. By the definition of $M_3$ there exists a sequence of vertices $h_0, h_1, \ldots, h_t$ where $h_0 = y$ and $h_t = b$ such that for each $i > 0$, $h_i \in N^{\pm}(v)$ implies that $h_i$ is not adjacent to $h_{i-1}$, and $h_i \notin N^{\pm}(v)$ implies that $h_i$ is joined to $h_{i-1}$ by symmetric arcs. We choose such a vertex $b$ so that the sequence is as short as possible. Assume that $b \in N^-(v) \setminus N^+(v)$. Then $b(= h_t)$ is joined to $h_{t-1}$ by symmetric arcs. We claim $h_{t-1} \in N^{\pm}(v)$. Indeed, since $b$ is joined to $h_{t-1}$ by symmetric arcs, $h_{t-1} \in N^+(v) \cup N^-(v)$. Suppose $h_{t-1} \in N^-(v) \setminus N^+(v)$. The choice of $b$ implies that there can only be symmetric arcs between $h_{t-1}$ and $M_1$. Since $h_{t-1}$ and $x$ are asynchronous neighbours of $b$, they are adjacent. In particular, $h_{t-1}x$ is a non-symmetric arc. Thus $x, y'$ are non-adjacent asynchronous neighbours of $h_{t-1}$, a contradiction. So $h_{t-1} \notin N^-(v) \setminus N^+(v)$. A similar proof shows $h_{t-1} \notin N^+(v) \setminus N^-(v)$. So $h_{t-1} \in N^{\pm}(v)$. Since $b$ is joined to $h_{t-1}$ by symmetric arcs and joined to each vertex of $M_1$ by a non-symmetric arc, $h_{t-1} \notin M_1$ and thus $t > 2$. Hence $h_{t-1}$ is not adjacent to $h_{t-2}$ and is completely adjacent to $M_1$. If $h_{t-2} \in N^+[v] \cup N^-[v]$, then $h_{t-2}$ must be in $N^{\pm}(v)$ and hence adjacent to $b$. Thus $h_{t-1}, h_{t-2}$ are neighbours of $b$. Since $h_{t-1}, h_{t-2}$ are not adjacent, they are synchronous neighbours of $b$, which implies $b$

is joined to $h_{t-2}$ by symmetric arcs. This contradicts the choice of the sequence as $h_0, h_1, \ldots, h_{t-2}, b$ is a shorter sequence. So $h_{t-2} \notin N^+[v] \cup N^-[v]$. Let $l$ be the largest integer such that $h_{t-2}, \ldots, h_{t-l}$ are not in $N^+[v] \cup N^-[v]$. Then $h_{t-i}$ is joined to $h_{t-i-1}$ by symmetric arcs for each $i = 2, \ldots, l$. We must have $h_{t-l-1} \in N^{\pm}(v)$. The vertex $b$ is not adjacent to $h_{t-2}$ as otherwise $h_{t-1}, h_{t-2}$ are non-adjacent asynchronous neighbours of $b$, a contradiction. For the same reason, we see that $b$ is not adjacent to $h_{t-i}$ for each $i = 2, \ldots, l$. Since $b, h_{t-l-1}$ are asynchronous neighbours of $v$, they are adjacent. They must be joined by symmetric arcs, as otherwise $b, h_{t-l}$ are non-adjacent asynchronous neighbours of $h_{t-l-1}$, a contradiction. But this contradicts the choice of the sequence because $h_0, h_1, \ldots, h_{t-l-1}, b$ is a shorter sequence. Therefore $b$ is not a vertex in $M_3$. So if $b \in N^-(v) \setminus N^+(v)$ is joined to a vertex in $M_1$ with a non-symmetric arc then $b \notin M_3$ and there is a non-symmetric arc from $b$ to every vertex in $M_1$. A similar proof shows that if $b \in N^+(v) \setminus N^-(v)$ is joined to a vertex in $M_1$ with a non-symmetric arc then $b \notin M_3$ and there is a non-symmetric arc from each vertex of $M_1$ to $b$.

We show that $M_3$ is a module. Let $z$ be a vertex that is not in $M_3$. For each vertex $h \in M_3$, there is a sequence of vertices $h_0, h_1, \ldots, h_t$ where $h_0 = y$ and $h_t = h$ such that for each $i > 0$, $h_i \in N^{\pm}(v)$ implies that $h_i$ is not adjacent to $h_{i-1}$, and $h_i \notin N^{\pm}(v)$ implies $h_i$ is joined to $h_{i-1}$ by symmetric arcs. Suppose first that $z \in N^-(v) \setminus N^+(v)$. We know from the above that $zx$ is a non-symmetric arc and $zh$ is a non-symmetric arc for all $h \in M_1$. In particular, $zy$ $(= zh_0)$ is a non-symmetric arc. Suppose that $k > 0$ and $zh_{k-1}$ is a non-symmetric arc. If $h_k \in N^{\pm}(v)$, then $h_{k-1}, h_k$ are non-adjacent neighbours of $z$ so $zh_k$ is a non-symmetric arc. If $h_k \notin N^{\pm}(v)$, then $z, h_k$ are asynchronous neighbours of $h_{k-1}$ so they are adjacent. There are two cases. Either $h_k \in N^+(v) \oplus N^-(v)$ or $h_k \notin N^+(v) \cup N^-(v)$. If $h_k \notin N^+(v) \cup N^-(v)$, then clearly $zh_k$ is a non-symmetric arc. Assume $h_k \in N^+(v) \oplus N^-(v)$. Since $h_k \notin M_1$, $h_k$ is joined to each vertex in $M_1$ by symmetric arcs. In particular, $h_k$ is joined to $y'$ by symmetric arcs. Since $y'$ is not adjacent to $x$, $h_k$ and $x$ are not adjacent as otherwise $y'$ and $x$ are non-adjacent asynchronous neighbours of $h_k$, a contradiction. Hence $h_k$ and $x$ are synchronous neighbours of $z$. Since $zx$ is a non-symmetric arc, $zh_k$ is a non-symmetric arc. Therefore $zh$ is a non-symmetric arc for all $h \in M_3$. A similar proof shows that if $z \in N^+(v) \setminus N^-(v)$ then $hz$ is a non-symmetric arc for all $h \in M_3$. Suppose next that $z \in N^{\pm}(v)$. Since $z$ is not in $M_3$, $z$ is adjacent to every vertex in $M_3$. In particular, $z$ is adjacent to $x$. Note that $z$ and $x$ are joined by symmetric arcs. Since $x$ and $y'$ are not adjacent, $z$ is adjacent to $y'$ by symmetric arcs. This implies $z$ is also joined to $y$ by symmetric arcs. Suppose that $k > 0$ and $z$ is joined to $h_{k-1}$ by symmetric arcs. If $h_k \notin N^+(v) \cup N^-(v)$, then clearly $z$ is joined to $h_k$ by symmetric arcs. If $h_k \in N^{\pm}(v)$, then $h_k$ is not adjacent to $h_{k-1}$ and thus $h_k, h_{k-1}$ are non-adjacent neighbours of $z$. Since $z$ is joined to $h_{k-1}$ by symmetric arcs, $z$ is joined to $h_k$ by symmetric arcs. If $h_k \in N^+(v) \oplus N^-(v)$, then $h_k$ is joined to $y'$ by symmetric arcs. Since $y'$ and $x$ are not adjacent, $h_k$ and $x$ are not adjacent. Thus $h_k$ and $x$ are non-adjacent neighbours of $z$, which implies $z$ is joined to $h_k$ by symmetric arcs. Suppose now that $z \notin N^+[v] \cup N^-[v]$. Since $z$ is not in $M_3$, it is not adjacent to any vertex in $M_1$. In particular, $z$ is not adjacent to $y$. Suppose that $k > 0$ and $z$ is not adjacent to $h_{k-1}$. If $h_k \in N^{\pm}(v)$, then $z$ is not adjacent to $h_k$ as otherwise $z$ is joined to $h_k$ by symmetric arcs, which implies $z \in M_3$, a contradiction to assumption. If $h_k \notin N^{\pm}(v)$, then $h_k$ is joined to $h_{k-1}$ by symmetric arcs. Since $z$ is not adjacent to $h_{k-1}$, $z$ cannot be joined to $h_k$ by a non-symmetric arc. Since $z \notin M_3$ and $h_k \in M_3$, $z$ cannot be joined to $h_k$ by symmetric arcs. Hence $z$ is not adjacent to $h_k$.

The only case remaining is that $N^+(v) \oplus N^-(v) = \emptyset$. Since $D$ is not a symmetric digraph, it has a non-symmetric arc. Suppose $fg$ is a non-symmetric arc in $D$. Let $M_4$ be the subdigraph induced by the vertices which are connected to $f$ by oriented paths. Then any two vertices in $M_4$ are connected by an oriented path. Since $N^+(v) \oplus N^-(v) = \emptyset$, there is no oriented path connecting $f$ and $v$. So $v$ is not a vertex in $M_4$. Suppose $z$ is not in $M_4$ but is adjacent to a vertex $h$ in $M_4$. Then $z$ is joined to $h$ by symmetric arcs. Each vertex of $M_4$ is connected to $h$ by an oriented path. Following these oriented paths we see that $z$ is joined to each vertex of $M_4$ by symmetric arcs and hence the vertices of $M_4$ are synchronous neighbours of $z$. Therefore $M_4$ is a non-trivial module.

$\square$

## 2.3 Small quasi-kernel conjecture for weakly quasi-transitive digraphs

In 1976, Erdős and Székely [24] made the following conjecture on the size of quasi-kernels in sink-free digraphs:

**Conjecture 2.1.** *[24] (Small Quasi-kernel Conjecture) Every sink-free digraph $D$ has a quasi-kernel of size at most $\frac{|V(D)|}{2}$.*

This conjecture has been verified for several classes of digraphs. It is easy to see that the conjecture holds for symmetric digraphs. Heard and Huang [39] showed that every sink-free digraph $D$ has two vertex disjoint quasi-kernels if $D$ is semicomplete multipartite, quasi-transitive, or locally semicomplete. This implies that the Conjecture 2.2 holds for semicomplete multipartite, quasi-transitive, and locally semicomplete digraphs. Kostochka [49] proved that the conjecture holds for orientations of 4-colorable graphs (in particular, for orientations of all planar graphs). Ai et.al. [1] proved that the conjecture holds for claw-free digraphs. (Here, a *claw* is the digraph obtaining from $K_{1,3}$ by orienting the edges towards the center.) Moreover, Hulst [63] proved that the small quasi-kernel conjecture holds for the digraphs that have kernels.

In this section, we prove that Conjecture 2.1 holds for weakly quasi-transitive digraphs.

For convenience, call a quasi-kernel $Q$ in a digraph $D$ *small* if $|Q| \leq \frac{|V(D)|}{2}$.

**Theorem 2.3.** *Every sink-free weakly quasi-transitive digraph $D$ has a small quasi-kernel.*

*Proof.* We prove the theorem by induction on the number of vertices of $D$. It is easy to check that it is true when $D$ has at most 4 vertices. Let $D$ be a sink-free weakly quasi-transitive digraph with $|V(D)| \geq 5$. Suppose that the theorem holds for sink-free weakly quasi-transitive digraphs of order less than $|V(D)|$. If $D$ is a transitive oriented graph or semicomplete digraph or symmetric digraph, then the small quasi-kernel conjecture holds ([63, 39]). Hence we may assume that $D$ is not a transitive oriented graph, or a semicomplete digraph, or a symmetric digraph. By Theorem 2.2, $D$ is obtained from a weakly quasi-transitive digraph $W$ by substituting weakly-quasi digraphs $D_1, D_2, \ldots, D_l$ for the vertices $v_1, v_2, \ldots, v_l$ of $W$. We must have $|W| < n$ and $|D_i| < n$ for all $i$ because $D$ is not a transitive oriented graph, a symmetric digraph or a semicomplete digraph. Since every digraph has a quasi-kernel, so does $W$. Without loss of generality, let $Q_W = \{v_1, \ldots, v_k\}$ be a quasi-kernel of $W$.

Suppose that $D_i$ is sink-free for each $i = 1, 2, \ldots, k$. Then by the inductive hypothesis, each $D_i$ has a small quasi-kernel $Q_i$. Let $Q = Q_1 \cup Q_2 \cup \cdots \cup Q_k$. Clearly, $|Q| \leq \frac{|V(D)|}{2}$. We claim that $Q$ is a quasi-kernel of $D$. Indeed, each vertex $v$ of $D$ is in $D_i$ for some $i = 1, 2, \ldots, l$. If $1 \leq i \leq k$, then $v$ can reach $Q_i$ (and hence $Q$) within two steps; if $i > k$, since $v_i$ can reach $Q_W$ within two steps in $W$, $v$ can reach $Q$ within two steps.

Suppose now that $D_i$ is not sink-free for some $i = 1, 2, \ldots, k$. Note that $W$ may or may not be sink-free. Consider first the case when $W$ is not sink-free and let $S_W$ be the set of all sinks of $W$. Since every sink of a digraph must in a quasi-kernel, we have $S_W \subseteq Q_W$. Without loss of generality, let $S_W = \{v_1, v_2, \ldots, v_j\}$. We claim that each of $D_1, D_2, \ldots, D_j$ is sink-free. Suppose to the contrary that $D_i$ has a sink $v$ for some $i = 1, 2, \ldots, j$. Since $v_i$ is a sink vertex in $W$, $v$ is a sink vertex in $D$, which contradicts the assumption that $D$ is sink-free. Thus, for each $1 \leq i \leq j$, $D_i$ is sink-free and by the induction hypothesis has a small quasi-kernel $Q_i$. Note that for any two vertices $v_a, v_b$ with $v_a \in S_W$ and every $v_b \neq v_a$, either $v_a$ and $v_b$ are not adjacent in $W$ or $v_b v_a$ is a non-symmetric arc in $W$. Let $S = V(D_1) \cup V(D_2) \cdots \cup V(D_j)$ and let $D' = D \setminus N^-[S]$. Since every induced subdigraph of a weakly quasi-transitive digraph is weakly quasi-transitive, $D'$ is weakly quasi-transitive. Clearly, no vertex in $D'$ is adjacent to any vertex in $S$ and every vertex in $N^-(S) \setminus S$ is adjacent to a vertex in $S$ by a non-symmetric arc. We claim that $D'$ is sink-free. Suppose not; $u$ is a sink of $D'$. Since $D$ is sink-free, $u$ has an out-neighbour $u'$ in $D \setminus D'$. It is easy to see that $u' \in N^-[S]$. Clearly, $u' \notin S$ as otherwise $u$ is in $N^-[S]$ but not in $D'$. Since $D$ is a weakly quasi-transitive digraph and no vertex in $D'$ is adjacent to any vertex in $S$, for any $u'' \in S \cap N^+(u')$, $u$ and $u''$ must be synchronous neighbours of $u'$. This contradicts the fact that $u'u''$ is a non-symmetric arc. Hence $D'$ is sink-free. By the induction hypothesis, $D'$ has a small quasi-kernel $Q'$. Let $Q = Q' \cup Q_1 \cup Q_2 \cdots \cup Q_j$. we show $Q$ is a small quasi-kernel of $D$. Clearly, $Q$ is an independent set in $D$ of size $\leq \frac{|V(D)|}{2}$. Every vertex in $S$ can reach $Q$ within two steps, every vertex in $N^-(S) \setminus S$ can reach $Q$ in one step, and every vertex in $D'$ can reach $Q$ within two steps. Hence $Q$ is a small quasi-kernel of $D$.

Consider now the case when $W$ is sink-free. Then every vertex of $W$ has an out-neighbour. By the induction hypothesis, $W$ has a small quasi-kernel, and we may assume that it is $Q_W$. We use $N^-(Q_W)$ to denote the set of in-neighbours of $Q_W$

in $W$, and $N^{--}(Q_W) = V(W) \setminus \{Q_W \cup N^-(Q_W)\}$. Let $N^{\pm}(Q_W)$ be the subset of $N^-(Q_W)$ that consists of the vertices adjacent to vertices in $Q_W$ by symmetric arcs. Let $Q_{W_1}$ consists of the vertices $v_1, v_2, \ldots, v_k$ in $Q_W$ adjacent to a vertex in $N^{\pm}(Q_W)$ by symmetric arcs and let $Q_{W_2} = Q_W \setminus Q_{W_1}$. Since $W$ is weakly quasi-transitive, none of the vertices in $Q_{W_2}$ is adjacent to any vertex in $N^{\pm}(Q_W)$. Let $N^-(Q_{W_2})$ consists of the vertices in $N^-(Q_W)$ which have out-neighbours in $Q_{W_2}$. Note that the arcs between $N^-(Q_{W_2})$ and $Q_{W_2}$ are non-symmetric from $N^-(Q_{W_2})$ to $Q_{W_2}$. Let $N^-(Q_{W_1}) = N^-(Q_W) \setminus (N^{\pm}(Q_W) \cup N^-(Q_{W_2}))$. Every vertex in $N^-(Q_{W_1})$ must have an out-neighbours in $Q_{W_1}$, and none of the vertices in $Q_{W_2}$ is adjacent to any vertex in $N^-(Q_{W_1})$. Since $W$ is sink-free, every vertex in $Q_{W_2}$ has at least one out-neighbour in $N^{--}(Q_W)$. Denote the set of out-neighbours of $Q_{W_2}$ by $N^+(Q_{W_2})$. Since none of the vertices in $Q_{W_2}$ is adjacent to $N^{\pm}(Q_W)$, the arcs between $N^{\pm}(Q_W)$ and $N^+(Q_{W_2})$ are non-symmetric from $N^{\pm}(Q_W)$ to $N^+(Q_{W_2})$. Let $N^{--}(Q_{W_1}) = N^{--}(Q_W) \setminus N^+(Q_{W_2})$. By the definition of $N^+(Q_{W_2})$, there is no arc between $Q_{W_2}$ and $N^{--}(Q_{W_1})$. Therefore the arcs between $N^{--}(Q_{W_1})$ and $N^+(Q_{W_2})$ are non-symmetric from $N^{--}(Q_{W_1})$ to $N^+(Q_{W_2})$. Since $Q_W$ is a quasi-kernel in $W$, every vertex in $N^{--}(Q_{W_1})$ must have an out-neighbour in $N^{\pm}(Q_W) \cup N^-(Q_{W_1})$ but not in $N^-(Q_{W_2})$ (See Figure 2.2 and the dotted edge means not adjacent).

Let $J \subseteq \{1, 2, \ldots, k\}$ be maximal such that it contains no pair $a, b$ with $v_a$ and $v_b$ having a common neighbour in $N^{\pm}(Q_W)$. Let $Q_1$ be a set of vertices in $D$ which consists of one vertex $u_i$ in $D_i$ for each $i \in J$. Since $Q_{W_1}$ is an independent set, $Q_1$ is also an independent set. It is easy to see that every vertex in $D_i \setminus \{u_i\}$ for which $v_i \in Q_{W_1}$ can reach $u_i$ in two steps through a vertex in $N^{\pm}(Q_W)$. Let $Q_2$ be the union of quasi-kernels of subdigraphs $D_i$ for which $v_i \in Q_{W_2}$. Let $Q'_2$ be a quasi-kernel of the subdigraph of $D$ induced by $N^+(Q_{W_2})$. Let $Q_A = Q_1 \cup Q_2$ and let $Q_B$ consists of all vertices in $Q'_2$ and vertices in $Q_1$ not adjacent to any vertex in $Q'_2$. Note that if there exists a vertex in $Q_1$ but not in $Q_B$, then there exists a vertex in $N^{\pm}(Q_W)$ that is adjacent to a vertex in $Q_2'$ by a non-symmetric arc (from the vertex in $N^{\pm}(Q_W)$ to a vertex in $Q_2'$). Clearly, both $Q_A$ and $Q_B$ are independent sets. The definition of $Q_1$ guarantees that $|Q_1| \leq |N^{\pm}(Q_W)|$. Note that $N^{\pm}(Q_W) \cap (Q_A \cup Q_B) = \emptyset$. Note also that $N^{\pm}(Q_W)$ contains a subset of size $|Q_A \cap Q_B|$. It follows that either $|Q_A| \leq \frac{|V(D)|}{2}$ or $|Q_B| \leq \frac{|V(D)|}{2}$.

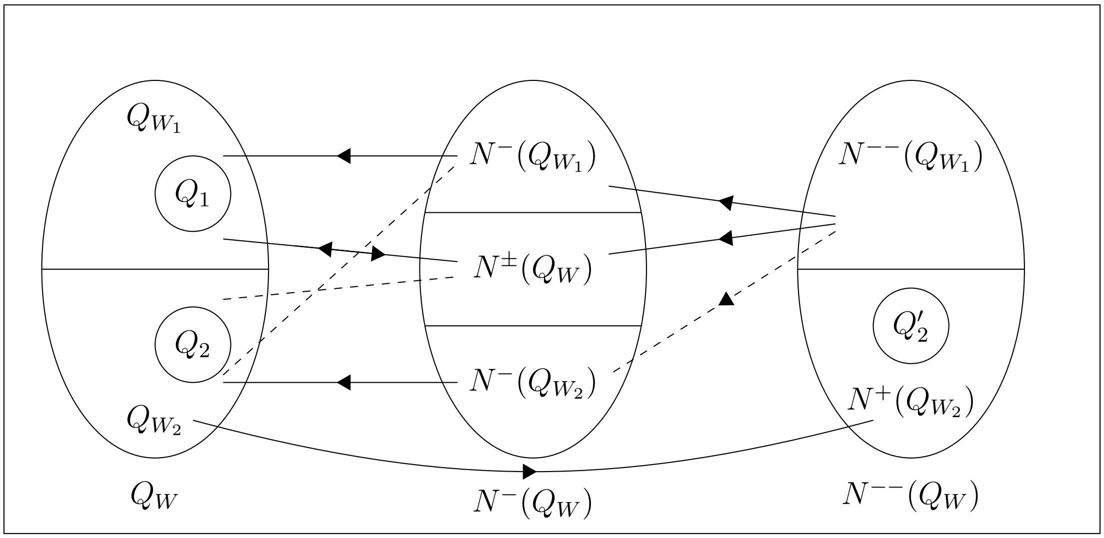


Figure 2.2: A weakly quasi-transitive digraph $W$

It suffices to show that $Q_A$ and $Q_B$ are both quasi-kernels of $D$. We show first that $Q_A$ is a quasi-kernel. Each vertex in $Q_{W_1}$ is either in $Q_1$ so in $Q_A$, or can reach $Q_1$ in two steps through a vertex in $N^{\pm}(Q_W)$. Each vertex in $N^{\pm}(Q_{W_1})$ can reach $Q_1$ in one step. Suppose that some two vertices $x_1, x_2 \in Q_{W_1}$ adjacent to a vertex $y_1 \in N^{\pm}(Q_W)$ by symmetric arcs. If there exists a vertex $y_2 \in N^{-}(Q_{W_1})$ such that $y_2x_2$ is a non-symmetric arc, then either $y_2x_1$ is a non-symmetric arc, or $y_2$ and $y_1$ are joined by symmetric arcs in which $x_1$ and $y_2$ are not adjacent. In either case, $y_2$ can reach $x_1$ within two steps. Hence every vertex in $N^{-}(Q_{W_1})$ can reach $Q_A$ in one step or two steps through $N^{\pm}(Q_W)$. By the definition of $Q_2$, every vertex in $Q_{W_2}$ is either in $Q_2$, or can reach $Q_2$ within two steps. Since $W$ is weakly quasi-transitive, every vertex in $N^-(Q_{W_2})$ can reach $Q_2$ in one step (by following the oriented path from the out-neighbour in $Q_{W_2}$ to a vertex in $Q_2$). Let $z_1 \in N^+(Q_{W_2})$. If $z_1$ has an out-neighbour in $N^-(Q_{W_2}) \cup N^{\pm}(Q_W)$, then it can reach $Q_A$ in two steps. If $z_1$ has no out-neighbour in $N^-(Q_{W_2}) \cup N^{\pm}(Q_W)$, then it must have an out-neighbour in $N^-(Q_{W_1})$. Suppose that $z_1$ cannot reach $Q_A$ in two steps. Then any out-neighbour of $z_1$ cannot reach $Q_1$ in one step. Let $y_1 \in N^+(z_1)$, it has an out-neighbour $x_1$ in $Q_{W_1}$. Since $z_1$ cannot reach $Q_A$ within two steps, $x_1$ is not in $Q_1$. Hence there exist $x_2 \in Q_1$ and $y_2 \in N^{\pm}(Q_W)$ such that $x_1y_2, x_2y_2$ are both symmetric arcs. Since $W$ is weakly quasi-transitive, $x_1$ and $z_1$ are adjacent and in fact $x_1z_1$ is a non-symmetric arc. In addition, $y_1$ and $x_2$ are not adjacent as otherwise $y_1x_2$ is a non-symmetric arc, which contradicts the assumption that $z_1$ cannot reach $Q_A$ within two steps. So $y_1$ and $y_2$ are adjacent by symmetric arcs. Since $z_1$ and $y_2$ are asynchronous neighbours of $x_1$, they are adjacent, and since $z_1$ cannot reach $Q_A$ within two steps, $y_2z_1$ must be a non-symmetric arc. Hence $x_2$ and $z_1$ are adjacent and $x_2z_1$ is a non-symmetric arc, which contradicts the facts that $y_1$ is an out-neighbour of $z_1$ and $y_1, x_2$ are not adjacent. Therefore, $z_1$ can reach $Q_A$ within two steps. Similarly, we can show that every vertex in $N^{--}(Q_{W_1})$ can reach $Q_A$ within two steps. Therefore, $Q_A$ is a quasi-kernel of $D$.

We show that $Q_B$ is also a quasi-kernel of $D$. By the definition of $Q_2'$, every vertex in $N^+(Q_{W_2})$ can reach $Q_2'$ within two steps. Since $D$ is a weakly quasi-transitive digraph and every vertex in $Q_{W_2}$ has an out-neighbour in $N^+(Q_{W_2})$, each vertex in $Q_{W_2}$ can reach $Q_2'$ in one step. As a consequence, each vertex in $N^-(Q_{W_2})$ can reach $Q_2'$ within two steps. If every vertex in $Q_1$ is also in $Q_B$, then each vertex in $Q_{W_1} \cup N^{\pm}(Q_W) \cup N^-(Q_{W_1}) \cup N^{--}(Q_{W_1})$ can reach $Q_1$ within two steps. Hence assume there exists a vertex $x_1 \in Q_1$ but not in $Q_B$. Then there exist vertices $y_1 \in N^{\pm}(Q_W)$ and $z_1 \in Q_2'$ such that $x_1z_1$, $y_1z_1$ are non-symmetric arcs, and $x_1$ and $y_1$ are joined by symmetric arcs. Moreover, every vertex in $N^{-}(x_1)$ should reach $z_1$ in one step. It is not hard to check that every vertex in $Q_{W_1}$ can reach $Q_1$ or $Q'_2$ through $N^{\pm}(Q_W)$ within two steps. Every vertex in $N^{\pm}(Q_W)$ can either reach $Q_1$ in one step, or reach $z_1$ through $x_1$ in two steps. Similarly, every vertex in $N^{-}(Q_{W_1})$ can reach $Q_B$ within two steps. Form above we know that every vertex in $N^{--}(Q_{W_1})$ can reach $Q_1$ within two steps. Suppose that $v$ is a vertex in $N^{--}(Q_{W_1})$ that can reach $x_1$ with two steps. Then $v$ can also reach $z_1$ with two steps through one of the in-neighbour of $x_1$. Hence, every vertex in $N^{--}(Q_{W_1})$ can reach $Q_B$ within two steps and $Q_B$ is a quasi-kernel of $D$.

□

**Corollary 2.2.** *If $D$ is a sink-free quasi-transitive digraph or a sink-free extended semicomplete digraph, then $D$ has a small quasi-kernel.* □

# Chapter 3

# Chordal Digraphs

## 3.1 Introduction

Recall that a digraph $D$ is chordal if every induced subdigraph of $D$ has a di-simplicial vertex. Clearly, if $D$ is chordal, then $S(D)$ is a chordal graph. Moreover, every di-simplicial vertex of $D$ is a simplicial vertex of $S(D)$. There is no known characterization of chordal digraphs by forbidden subdigraphs. In 2012, Meister and Telle [56] found the forbidden subdigraph characterization for semicomplete chordal digraphs. The following theorem is proved in [56].

**Theorem 3.1.** *[56] A semicomplete digraph $D$ is chordal if and only if $S(D)$ is chordal and $D$ does not contain any of the digraphs in Figure 3.1 as an induced subdigraph.* □

Quasi-transitive digraphs and extended semicomplete digraphs both generalize semicomplete digraphs. The chordality of these two classes have been studied in [65]. It turns out they share the same forbidden subdigraphs as semicomplete digraphs for being chordal.

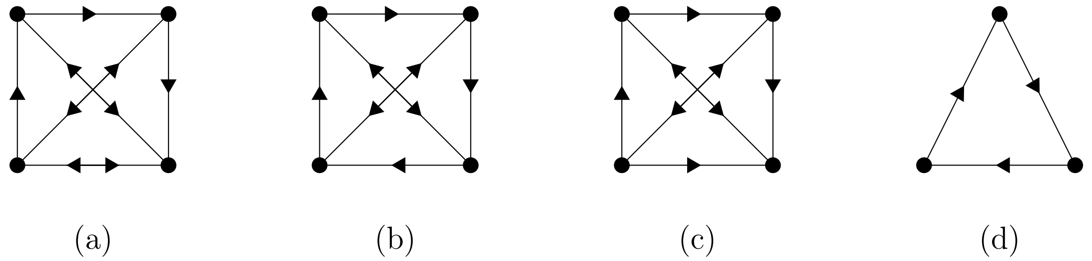


Figure 3.1: Semicomplete digraphs which are not chordal.

**Theorem 3.2.** *[65] A quasi-transitive digraph $D$ is chordal if and only if $S(D)$ is chordal and $D$ does not contain any of the digraphs in Fig 3.1 as an induced subdigraph.* □

**Theorem 3.3.** *[65] An extended semicomplete digraph $D$ is chordal if and only if $S(D)$ is chordal and $D$ does not contain any of the digraphs in Fig 3.1 as an induced subdigraph.* □

Locally semicomplete digraphs are another generalization of semicomplete digraphs. Many properties for semicomplete digraphs hold for locally semicomplete digraphs [4]. There are locally semicomplete digraphs which are neither semicomplete nor chordal. Any directed cycle consisting of non-symmetric arcs is locally semicomplete but not chordal, and it is not semicomplete if it has four or more vertices. We have proved that chordless directed cycles with four or more vertices consisting of non-symmetric arcs (Figure 3.2) are the only minimal locally semicomplete digraphs which are not chordal and which are not semicomplete [65].

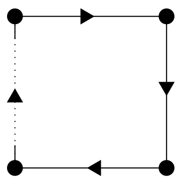


Figure 3.2: Induced directed cycles consisting of non-symmetric arcs

**Theorem 3.4.** *[65] A locally semicomplete digraph $D$ is chordal if and only if $S(D)$ is chordal and $D$ does not contain as an induced subdigraph a chordless directed cycle consisting of non-symmetric arcs (as shown in Figure 3.2) or a digraph in Figure 3.1.* $\square$

## 3.2 Weakly quasi-transitive digraphs

In this section, we prove that weakly quasi-transitive digraphs share the same forbidden subdigraphs with semicomplete digraphs for being chordal.

**Theorem 3.5.** *A weakly quasi-transitive digraph $D$ is chordal if and only if $S(D)$ is chordal and $D$ does not contain any digraph in Figure 3.1 as an induced subdigraph.*

*Proof.* The necessity follows from Theorem 3.1. For the other direction assume that $D$ is not chordal but $S(D)$ is chordal, and $D$ does not contain any digraph in Figure 3.1 as an induced subdigraph. We choose such a $D$ to be minimal. By the choice of $D$, any induced subdigraph of $D$ with fewer vertices than $D$ is chordal and hence has a di-simplicial vertex. Since $D$ is not chordal but $S(D)$ is, $D$ cannot be a transitive oriented graph or a symmetric digraph. In view of Theorem 3.1, $D$ cannot be a semicomplete digraph.

Since $D$ is weakly quasi-transitive, by Theorem 2.2, $D = D'[H_1, H_2, \ldots, H_l]$, that is, $D$ is obtained from a weakly quasi-transitive digraph $D'$ by substituting weakly-quasi digraphs $H_1, H_2, \ldots, H_l$ for the vertices $v_1, v_2, \ldots, v_l$ of $D'$. Moreover, $D'$ and one of $H_i$'s have at least two vertices. Hence, $D'$ and $H_i$ are induced subdigraphs of $D$ with fewer vertices than $D$ for each $i$, and by the choice of $D$ each of them has a di-simplicial vertex.

Since $S(D)$ is chordal, it has a simplicial vertex $u$. Assume that $u \in H_j$. For any $v_i, i \neq j$, if $v_i$ and $v_j$ are joined by symmetric arcs in $D'$, then $H_i$ is a semicomplete digraph with all symmetric arcs. For any $v_i$ and $v_k$ with $i \neq k, j \neq i$ and $j \neq k$, if both $v_jv_i$ and $v_jv_k$ are symmetric arcs in $D'$, then the arcs between $H_i$ and $H_k$ are all symmetric.

We claim that $v_j$ is a di-simplicial vertex in $D'$. Suppose by the contrary that $v_j$ is not a di-simplicial vertex in $D'$. Then there exists two vertices $v_i, v_k$ in $D'$ such that both $v_iv_j$ and $v_jv_k$ are non-symmetric arcs, but $v_iv_k$ is not an arc. Since $v_i, v_k$ are asynchronous neighbours of $v_j$ and $D'$ is weakly quasi-transitive, $v_i, v_k$ are adjacent. Since $v_iv_k$ is not an arc, $v_kv_i$ is a non-symmetric arc. But then by taking one vertex each from $H_i, H_j, H_k$, they induce a digraph in Figure 3.1 (d), a contradiction. Therefore, $v_j$ is a di-simplicial vertex in $D'$.

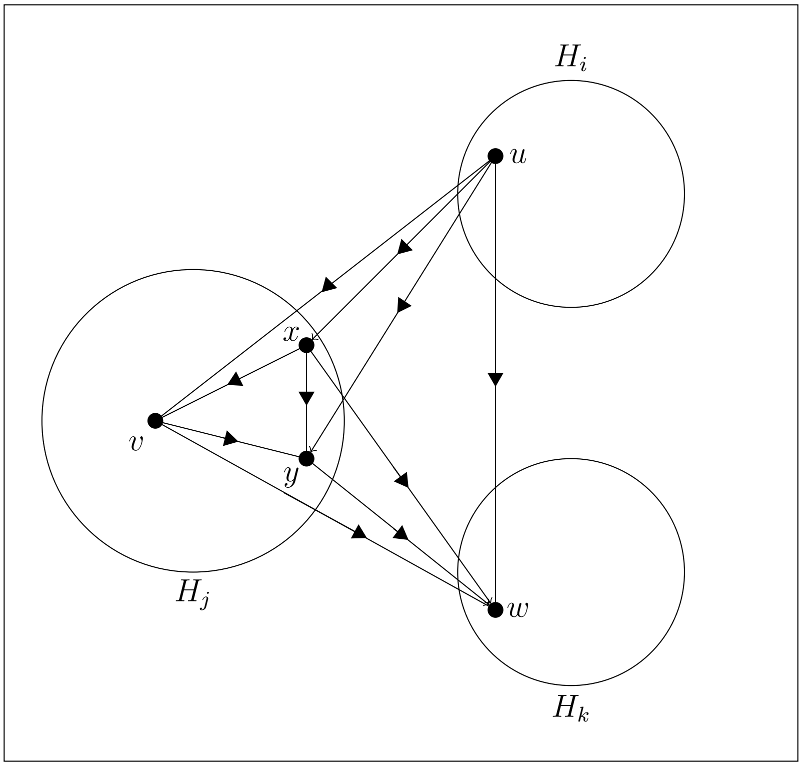


Figure 3.3: A di-simplicial vertex $v$ of a weakly quasi-transitive digraph $D$

Let $v$ be a di-simplicial vertex of $H_j$. We claim that $v$ is a di-simplicial vertex of $D$. Since $v_j$ is a di-simplicial vertex of $D'$, for every $v_i \in N^-_{D'}(v_j) \setminus N^+_{D'}(v_j)$ and

$v_k \in N^+_{D'}(v_j) \setminus N^-_{D'}(v_j)$ with $i \neq k$, $v_iv_k$ is an arc in $D'$. This means that for every vertex $u, w \in D \setminus H_j$ and for every vertex $v' \in H_j$, if $u \in N^-(v') \setminus N^+(v')$ and $w \in N^+(v') \setminus N^-(v')$, then $uw$ is an arc $(u \neq w)$. In particular, if $u \in N^-(v) \setminus N^+(v)$ and $w \in N^+(v) \setminus N^-(v)$, then $uw$ is an arc. Since $v$ is a di-simplicial vertex of $H_j$, for every in-neighbour $x \in N^-_{H_j}(v)$ and every out-neighbour $y \in N^+_{H_j}(v)$ with $x \neq y$, $xy$ is an arc in $H_j$. Now suppose that $x \in H_j, x \in N^-(v)$ and $w \in H_k(k \neq j), w \in N^+(v) \setminus N^-(v)$. Then by the structure theorem of weakly quasi-transitive digraph, $v_jv_k$ is an arc in $D'$ and so $xw$ should be an arc in $D$. Similarly, if $u \in H_i(k \neq j), u \in N^-(v) \setminus N^+(v)$ and $y \in H_j, y \in N^+(v)$, then $v_iv_j$ is an arc in $D'$ and so $uy$ is an arc in $D$ (see Figure 3.3) . Therefore, $D$ has a di-simplicial vertex $v$, which contradicts our assumption.

This completes the proof. □

The next corollary follows Theorems 2.2 and 3.5 immediately:

**Corollary 3.1.** *[56, 65] Let $D$ be a semicomplete digraph, a quasi-transitive digraph or an extended semicomplete digraph. Then $D$ is chordal if and only if $S(D)$ is chordal and $D$ does not contain any digraph in Figure 3.1 as an induced subdigraph.* □

## 3.3 Adjusted interval digraphs

An *interval graph* [31] is a graph $G$ whose vertices $v$ can be associated with intervals $I_v$ in the real line $\mathbf{R}$ such that $uv \in E(G)$ if $I_u$ and $I_v$ intersect. The class of interval graphs is a well known subclass of chordal graphs [51].

A digraph $D$ is called an *interval digraph* if its vertices can be associated with pairs of intervals $(I_v, J_v)$, $v \in V(D)$ such that $uv$ is an arc of $D$ if $I_u$ and $J_v$ intersect. Interval digraphs seem to be a natural digraph analogue of interval graphs but, unlike interval graphs, not every interval digraph is chordal. For instance, the digraph (d) in Figure 3.1 is an interval digraph but not chordal. In addition, the class of interval digraphs lacks forbidden structure characterizations. For these reasons, a new digraph analogue of interval graphs called adjusted interval digraph was introduced by Feder, Hell, Huang and Rafiey [25].

An *adjusted interval digraph* is an interval digraph $D$ whose vertices are associated with pairs of intervals $(I_v, J_v)$, $v \in V(D)$ with the additional property that $I_v$ and $J_v$ have the same left endpoint for each $v$. Note that adjusted interval digraphs are reflexive, i.e., there is a loop at each vertex. In contract, adjusted interval digraphs admit a forbidden structure characterization [25]. Let $P = x_1, x_2, \ldots, x_k$ and $Q = y_1, y_2, \ldots, y_k$ be two walks of the same length in a digraph $D$. We say that $P$ and $Q$ are *congruent* if they follow the same pattern of arcs, i.e., $x_ix_{i+1}$ (respectively, $x_{i+1}x_i$) is an arc, if and only if $y_iy_{i+1}$ (respectively, $y_{i+1}y_i$) is an arc. We say that $P$ *avoids* $Q$ if $x_iy_{i+1}$ is not an arc when $x_ix_{i+1}$ is, and $y_{i+1}x_i$ is not an arc when $x_{i+1}x_i$ is for each $i$. An *invertible pair* [25] of a digraph $D$ is a pair of distinct vertices $u, v$ such that

- there exist congruent walks $P$ from $u$ to $v$ and $Q$ from $v$ to $u$ such that $P$ avoids $Q$;
- there exist congruent walks $P'$ from $v$ to $u$ and $Q'$ from $u$ to $v$ such that $P'$ avoids $Q'$.

**Theorem 3.6.** *A reflexive digraph $D$ is an adjusted interval digraph if and only if it has no invertible pairs.* □

Another nice property of adjusted interval digraph is that they are chordal digraphs.

**Theorem 3.7.** *Every adjusted interval digraph is a chordal digraph.*

*Proof.* Let $D$ be an adjusted interval digraph whose vertices are associated with pairs of intervals $(I_v, J_v)$. Order the vertices of $D$, $u_1, u_2, \ldots, u_n$, in such a way that $i < j$

if the left end of $I_{u_i}$ precedes the left end of $I_{u_j}$. We claim that $u_n, u_{n-1}, \ldots, u_1$ is a perfect elimination ordering of $D$, that is, $u_i$ is a di-simplicial vertex of the subdigraph of $D$ induced by $u_1, u_2, \ldots, u_{i-1}, u_i$. Indeed, let $u_j$ and $u_k$ be two vertices with $j < i$ and $k < i$ such that $u_j u_i$ and $u_i u_k$ are arcs. Since $u_j u_i$ and $u_i u_k$ are arcs, $I_{u_j} \cap J_{u_i} \neq \emptyset$ and $I_{u_i} \cap J_{u_k} \neq \emptyset$. We see that $I_{u_j} \cap J_{u_k} \neq \emptyset$, and $u_j u_k$ is an arc. Therefore, $u_i$ is a di-simplicial vertex of the subdigraph of $D$ induced by $u_1, u_2, \ldots, u_{i-1}, u_i$. This completes the proof.

□

**Corollary 3.2.** *Every digraph that has no invertible pair is chordal.*

A chordal digraph may contain invertible pair and hence is not an adjusted interval digraph. For example, the reflexive digraph in Figure 3.4 is a chordal digraph but not an adjusted interval digraph as $a, d$ form an invertible pair.

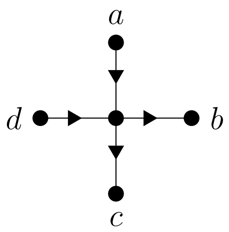


Figure 3.4: A reflexive chordal digraph that is not a adjusted interval digraph (loops are ignored)

A *subtree graph* is an intersection graph of subtrees in a tree. A well known result asserts that chordal graphs are exactly subtree graphs [29].

Let $T$ be a rooted tree with root $r$. Each subtree of $T$ can be viewed as a rooted tree whose root is the vertex closest to $r$ in $T$. A digraph $D$ is called an *adjusted subtree digraph* if its vertices can be associated with pairs of subtrees $(R_v, S_v)$, $v \in V(D)$, of $T$ such that

- $R_v$ and $S_v$ have the same root for all $v \in V(D)$, and
- $uv$ is an arc if $R_u$ and $S_v$ intersect for all $u, v \in V(D)$.

Suppose that $D$ is an adjusted subtree digraph represented by pairs of subtrees $(R_v, S_v)$, $v \in V(D)$, of a rooted tree $T$. Order the vertices of $D$ according to the non-increasing order of the distances of the roots of their corresponding subtrees from the root of $T$. A similar proof as for Theorem 3.7 shows that this is a perfect elimination ordering of $D$.

**Theorem 3.8.** *Every adjusted subtree digraph is chordal.* $\square$

# Chapter 4

# Semi-strict Chordal Digraphs

## 4.1 Introduction

In Chapter 3, we give a forbidden subdigraph characterization for weakly quasi-transitive chordal digraphs. As a by-product, the forbidden subdigraphs for weakly quasi-transitive digraphs are the same for semicomplete chordal digraphs, quasi-transitive chordal digraphs and extended semicomplete chordal digraphs. Moreover, the forbidden subdigraphs for locally semicomplete chordal digraphs was given in [44]. However, chordal digraphs in general lack structural properties such as forbidden structural characterizations. In [41], strict chordal digraphs are introduced and a forbidden subdigraph characterization of these digraphs is obtained.

In [65], an intermediate notion between chordal digraphs and strict chordal digraphs was proposed. A vertex $v$ in a digraph $D$ is *semi-strict di-simplicial* if for any $u \in N^-(v)$ and $w \in N^+(v)$ with $u \neq w$, both $uw$ and $wu$ are arcs in $D$. A digraph $D$ is *semi-strict chordal* if every induced subdigraph of $D$ contains a semi-strict di-simplicial vertex. By definition, every strict di-simplicial vertex is a semi-strict di-simplicial vertex which in turn is a di-simplicial vertex. Thus, semi-strict chordal digraphs form a class of digraphs between the class of strict chordal digraphs and the class of chordal digraphs. Unlike strict chordal digraphs, the underlying graphs of semi-strict chordal digraphs are not necessarily chordal (see Figure 4.1).

Suppose that $D$ is a semi-strict chordal digraph. Since every induced subdigraph of $D$ has a semi-strict di-simplicial vertex, the vertices of $D$ can be ordered $v_1, v_2, \ldots, v_n$ in such a way that $v_i$ is a semi-strict di-simplicial vertex of the subdigraph of $D$ induced by $v_i, v_{i+1}, \ldots, v_n$ for each $i \geq 1$. Such an ordering of a semi-strict chordal digraph $D$ is called the *semi-strict elimination ordering* (e.g., $v_1, v_5, v_3, v_2, v_4$ is a semi-strict elimination ordering of the digraph in Figure 4.1). Conversely, a digraph has a semi-strict elimination ordering then it is a semi-strict chordal digraph. A semi-strict elimination ordering of a semi-strict digraph can be obtained by successively finding a semi-strict di-simplicial vertex, which can be done in polynomial time. Therefore semi-strict chordal digraphs can be recognized in polynomial time.

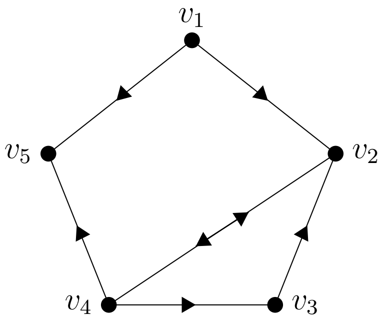


Figure 4.1: A semi-strict chordal digraph

In this chapter, we study the structure of semi-strict chordal digraphs by considering their knotting graphs. The *knotting* graph of a graph is an instrumental concept introduced by Gallai [28] for the study of *comparability* graphs (i.e., transitively orientable graphs). Gallai [28] proved that a graph is a comparability graph if and only if its knotting graph is bipartite. In [65], Ye introduced knotting graph of chordal digraphs (the concept is similar but different from knotting graph of semi-strict chordal digraphs). The following two results are obtained in [65].

**Proposition 4.1.** *[65] Let $D$ be a digraph and $\bar{K}$ be the knotting graph of $D$. Then $D$ has a di-simplicial vertex if and only if $\bar{K}$ has a group such that every vertex of that group has degree at most one.* $\square$

**Proposition 4.2.** *[65] Let $D$ be a digraph. Then for every induced subdigraph $D_s$ of $D$, the knotting graph $\bar{K}_s$ of $D_s$ has a group such that every vertex of that group has degree at most one if and only if $D$ is chordal.* $\square$

In section 4.2 of this chapter, we will give the definition of knotting graphs of digraphs and will characterize semi-strict chordal digraphs in terms of their knotting graphs (see Theorem 4.1). Unfortunately, the characterization of semi-strict chordal digraphs does not lead to a forbidden subdigraph characterization for semi-strict chordal digraphs. It appears challenging to find a forbidden subdigraph characterization for semi-strict chordal digraphs in genenal. In section 4.3, we will show that semi-strict chordal digraphs can be characterized by forbidden subdigraphs within the class of weakly quasi-transitive digraphs (see Theorem 4.2).

## 4.2 Semi-strict chordal digraphs and their knotting graphs

Let $D$ be a semi-strict chordal digraph. Then by the definition every induced subdigraph of $D$ is also semi-strict chordal. If $v$ is a semi-strict di-simplicial vertex of $D$, we claim that $v$ is a di-simplicial vertex of $S(D)$. Suppose by contradiction that $v$ is a semi-strict di-simplicial vertex of $D$ but not a di-simplicial vertex of $S(D)$. Then there exist vertices $u$ and $w$ such that both $uv$ and $vw$ are symmetric arcs but $uw$

is either not adjacent or adjacent by a single arc, which is a contradiction with the assumption that $v$ is a semi-strict di-simplicial vertex of $D$. It follows that $S(D)$ is a chordal digraph (or equivalently, the underlying graph of $S(D)$ is a chordal graph). Hence we have the following:

**Lemma 4.1.** *Suppose that $D$ is a semi-strict chordal digraph. Then $S(D)$ is chordal. Moreover, every semi-strict di-simplicial vertex of $D$ is a semi-strict di-simplicial vertex of $S(D)$.* □

**Lemma 4.2.** *If $D$ is a semi-strict chordal digraph, then $D$ does not contain an induced directed cycle consisting of non-symmetric arcs.* □

*Proof.* Suppose that $C$ is an induced directed cycle consisting of non-symmetric arcs. Then the subdigraph of $D$ induced by the vertices of $C$ has no semi-strict di-simplicial vertex, which means $D$ is not semi-strict chordal. □

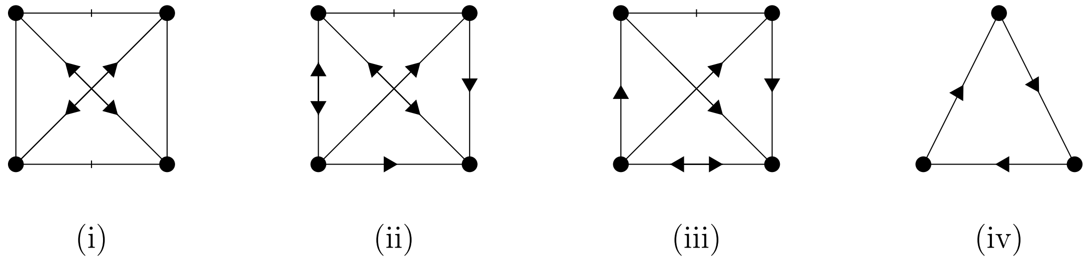

(i) (ii) (iii) (iv)

Figure 4.2: Digraphs which are not semi-strict chordal

Figure 4.2 depicts four types of semicomplete digraphs. It is easy to check that none of these digraphs has a semi-strict di-simplicial vertex. Moreover by deleting any vertex, the digraphs in Figure 4.2 become semi-strict chordal. Therefore, they are minimal not semi-strict chordal digraphs hence we have the following lemma:

**Lemma 4.3.** *Suppose that $D$ is a semi-strict chordal digraph. Then $D$ does not contain any of the digraphs in Figure 4.2 as an induced subdigraph.* □

Indeed, Figure 4.2 (i) and (iv) together forms a super class of the digraphs in Figure 3.1. This again shows that the class of chordal digraphs is a super class of semi-strict chordal digraphs.

We now introduce the knotting graphs for digraphs.

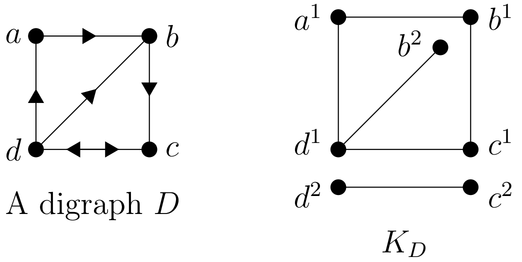
Let $D$ be a digraph and let $v$ be a vertex of $D$. Denote by $V(v)$ the set of arcs incident with $v$ (i.e., having $v$ as an endvertex). If $e = vu$ is an arc of $D$, then $e \in E(v)$ and is called *out-going*; if $e = uv$ is an arc of $D$, then $e$ is called *in-coming.* For $e, f \in E(v)$, we say that $e$ and $f$ are *related* and denoted the relation by $e\Gamma f$ if either $e = f$, or $e$ and $f$ are not both in-coming or out-going, and the two endvertices of $e$ and $f$ distinct from $v$ are not joined by symmetric arcs (which means they are either not adjacent or adjacent by a non-symmetric arc).

Clearly, $e\Gamma e$. If $e\Gamma f$ for $e \neq f$, then $f\Gamma e$. Hence, the relation $\Gamma$ is a reflexive and symmetric relation on $E(v)$. Thus the transitive closure $\Gamma^*$ of $\Gamma$ is an equivalence relation on $E(v)$, which partitions $E(v)$ into equivalence classes. Two arcs $e$ and $f$ of $E(v)$ are *knotted* if they are in the same equivalence class. We denote this relation by $e\Gamma^* f$. It follows that $e$ and $f$ of $E(v)$ are knotted if and only if there exists a sequence of arcs $e_1, e_2, \ldots, e_k$ in $E(v)$ such that $e = e_1, f = e_k$ and $e_1\Gamma^* e_2\Gamma^* \ldots \Gamma^* e_k$.

Denote by $E_1(v), E_2(v), \ldots, E_{l_v}(v)$ the equivalence classes of $\Gamma^*$ for each $v \in V(D)$. Note that if $uv \in A(D)$, then there exist unique $i$ and $j$ with $1 \leq i \leq l_u$ and $1 \leq j \leq l_v$ such that $uv \in E_i(u) \cap E_j(v)$.

The *knotting graph* $K_D$ of a digraph $D$ is the graph having vertices $v_1, v_2, \ldots, v_{l_v}$ for all $v \in V(D)$ (when $v$ is an isolated vertex, we have $l_v = 1$) and the edges $u_i v_j$ for all arcs $uv \in A(D)$ with $uv \in E_i(u) \cap E_j(v)$. Figure 4.3 depicts a digraph and its knotting graph $K_D$ where $E_1(a) = \{ab, da\}$, $E_1(b) = \{ab, bc\}$, $E_2(b) = \{db\}$, $E_1(c) = \{bc, cd\}$, $E_2(c) = \{dc\}$, $E_1(d) = \{da, cd, db\}$ and $E_2(d) = \{dc\}$.

Note that each vertex $v$ of $D$ gives rise to $l_v$ vertices $v_1, v_2, \ldots, v_{l_v}$ in $K_D$, which



A digraph $D$

$K_D$

Figure 4.3: A digraph $D$ and its knotting graph $K_D$

we call the *split* vertices of $v$. Each arc $uv$ of $D$ uniquely corresponds to an edge $u_iv_j$ of $K_D$ for some $i, j$ with $1 \leq i \leq l_u$ and $1 \leq j \leq l_j$. Moreover, for each vertex $v$ of $D$, the knotting graph of the digraph $D - v$ can be obtained from $K_D$ by deleting the split vertices of $v$. Thus the knotting graph of an induced subdigraph of $D$ can be computed efficiently form $K_D$.

Knotting graphs of digraphs can be used to determine whether a vertex in a digraph is semi-strict di-simplicial.

**Lemma 4.4.** *Let $D$ be a digraph and $K_D$ be the knotting graph of $D$. Then a vertex $v$ of $D$ is semi-strict di-simplicial if and only if the split vertices of $v$ all have degrees either zero or one in $K_D$.* □

*Proof.* Suppose that $v$ is a semi-strict di-simplicial vertex of a digraph $D$. If $v$ is an isolated vertex its unique split vertex $v_1$ has degree zero. If $v$ is not an isolated vertex, then each arc in $E(v)$ is knotted only to itself. Hence each split vertex of $v$ in $K_D$ is incident with exactly one edge and has degree one.

On the other hand, suppose that $v$ is not a semi-strict di-simplicial vertex. Then for some $u \in N^-(v)$ and $w \in N^+(v)$ with $u \neq w$, $u$ and $w$ are not joined by symmetric arcs. It follows that the two arcs $uv$ and $vw$ are knotted, that is, $u$ and $w$ are in the same equivalence class $E_j(v)$ for some $1 \leq j \leq l_v$. Hence the split vertex $v_j$ of $v$ in $K_D$ has degree at least two. □

For a subgraph $S$ of $K_D$, a *split group* in $S$ consists of the split vertices of $v$ in $S$ for some vertex $v$ of $D$.

**Theorem 4.1.** *A digraph $D$ is semi-strict chordal if and only if every induced subgraph $S$ of $K_D$ has a split group whose vertices have degrees either zero or one in $S$.*

*Proof.* Suppose that $D$ is semi-strict chordal. Let $S$ be an induced subgraph of $K_D$ and let $H$ be the subdigraph of $D$ induced by the vertices which have at least one split vertex in $S$. Let $D_H$ be the knotting graph of $H$, then $S$ is a subgraph of $D_H$. Since $D$ is semi-strict chordal, $H$ has a semi-strict di-simplicial vertex $v$. By lemma 4.4 the split vertices of $v$ in $K_H$ each has degree either zero or one. Hence the split vertices of $v$ in $S$ have degrees zero or one in $S$.

Conversely, suppose that every induced subgraph of $K_D$ has a split group whose vertices have degrees zero or one. We show that every induced subdigraph of $D$ has a semi-strict di-simplicial vertex. Let $H$ be an induced subdigraph of $D$. Then $K_H$ is an induced subgraph of $K_D$. If $H$ has isolated vertices then any isolated vertex of $H$ is a semi-strict di-simplicial vertex. So assume that $H$ has no isolated vertex. Then each vertex of $K_H$ has degree at least one in $K_H$. Thus $K_H$ has a split group whose vertices have degree one in $K_H$. By lemma 4.4 the vertex of $H$ which corresponds to this split group is a semi-strict di-simplicial vertex of $H$. Since every such induced subdigraph $H$ of $D$ has a semi-strict di-simplicial vertex, $D$ is a semi-strict chordal digraph. □

For the knotting graph $K_D$ in Figure 4.3, each split group has a vertex of degree two or more. By Theorem 4.1 the corresponding digraph $D$ is not semi-strict chordal. In Figure 4.4, the knotting graph $K_D$ satisfies the property that every induced sub-

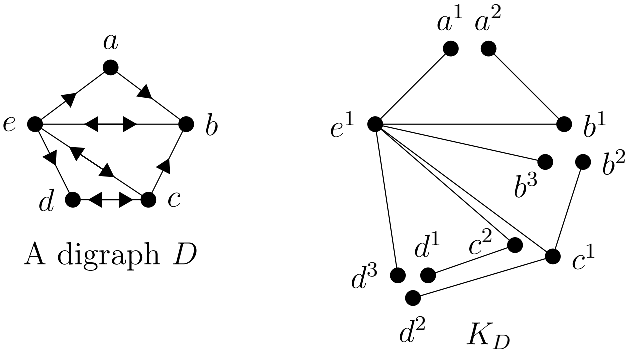


Figure 4.4: A semi-strict chordal digraph $D$ and its knotting graph

graph has a split group whose vertices each has degree zero or one and hence by Theorem 4.1 the digraph $D$ in Figure 4.4 is semi-strict chordal.

## 4.3 Weakly quasi-transitive semi-strict chordal digraphs

In this section, we investigate a list of forbidden subdigraph characterizations of weakly quasi-transitive semi-strict chordal digraphs. Recall that a digraph is weakly quasi-transitive if for each vertex $v \in V(D)$, any two asynchronous neighbours of $v$ are adjacent.

**Lemma 4.5.** *Let $D$ be a weakly quasi-transitive digraph that does not contain a digraph in Figure 4.2 (i) or (iv) as an induced subdigraph. Then $D$ does not contain a directed cycle consisting of non-symmetric arcs.* □

*Proof.* Suppose to the contrary that $D$ contains a directed cycle consisting of non-symmetric arcs. Let $C = v_1v_2\ldots v_kv_1$ be such a cycle of the minimum length $k$. The choice of $C$ implies that $C$ has no chord that is a non-symmetric arc. Clearly, $k \geq 4$ as otherwise $C$ is the digraph in Figure 4.2 (iv), contradicting the assumption. Since $v_1$ and $v_3$ are asynchronous neighbours of $v_2$ and $D$ is a weakly quasi-transitive digraph, there is an arc between $v_1$ and $v_3$. The choice of $C$ implies that $v_1$ and $v_3$ are joined by symmetric arcs. Similarly, $v_2$ and $v_4$ are joined by symmetric arcs because they are asynchronous neighbours of $v_3$. We see now that $v_1$ and $v_4$ are asynchronous neighbours of $v_2$ and so they must be adjacent (by symmetric arcs if $k > 4$, or by single arc if $k = 4$). The subdigraph of $D$ induced by $v_1, v_2, v_3, v_4$ is a digraph in Figure 4.2 (i), a contradiction to our assumption. □

Suppose that $v$ is a di-simplicial vertex of $S(D)$ but not a semi-strict di-simplicial vertex of $D$. Then there exist $u \in N^-(v)$ and $w \in N^+(v)$ such that $u$ and $w$ are not joined by symmetric arcs in $D$ ($u$ and $w$ might not be adjacent, or be adjacent by a non-symmetric arc). Since $v$ is di-simplicial in $S(D)$, $uv$ and $vw$ cannot both be edges in $S(D)$ and so at least one of them is non-symmetric. We call such an ordered triple $(u, v, w)$ a *violating triple* for $v$. We say that $v$ is *of type 1* (or a *type 1 vertex*) if for every violating triple $(u, v, w)$, $uv$ and $vw$ are both non-symmetric; otherwise, we say that $v$ is *of type 2* (or a *type 2 vertex*). Note that for any violating triple $(u, v, w)$, $u$ and $w$ are asynchronous neighbours of $v$ and thus, when the digraph $D$ is weakly quasi-transitive and $v$ is not semi-strict di-simplicial, $u$ and $w$ are joined by a non-symmetric arc.

The statements in the next lemma follows directly from the definition:

**Lemma 4.6.** *Let $D$ be a digraph and $D'$ be the reversal of $D$. Then the following hold:*

1. *$v$ is a semi-strict di-simplicial vertex of $D$ if and only if it is a semi-strict di-simplicial vertex of $D'$;*
2. *$D$ is semi-strict chordal if and only if $D'$ is semi-strict chordal;*
3. *$(u, v, w)$ is a violating triple in $D$ if and only if $(w, v, u)$ is a violating triple in $D'$;*

4. *$D$ is weakly quasi-transitive if and only if $D'$ is weakly quasi-transitive.*

**Lemma 4.7.** *Let $D$ be a weakly quasi-transitive digraph which does not contain any digraph in Figure 4.2 as an induced subdigraph. Suppose that $S(D)$ is chordal. If $(u, v, w)$ is a violating triple, then at least one of the following holds:*

1. *$u$ is a di-simplicial vertex of $S(D)$ and $uv$ is a non-symmetric arc;*
2. *$w$ is a di-simplicial vertex of $S(D)$ and $vw$ is a non-symmetric arc;*

□

*Proof.* In view of Lemma 4.6 and the fact that at least one of $uv$ and $vw$ is non-symmetric, we can assume that $uv$ is a non-symmetric arc. If $u$ is a di-simplicial vertex of $S(D)$, then we are done. Hence assume that $u$ is not a di-simplicial vertex of $S(D)$. From the above we know that the arc between $u$ and $w$ is non-symmetric.

Since $u$ is not a di-simplicial vertex of $S(D)$, $u$ has two non-adjacent neighbours $u'$ and $u''$ in $S(D)$. These two vertices $u', u''$ can not both be neighbours of $v$ in $S(D)$ as otherwise $\{u, u', u'', v\}$ induced a chordless 4-cycle in $S(D)$, contradicting the assumption that $S(D)$ is chordal. By symmetry, we assume that $u'$ is not a neighbour of $v$ in $S(D)$. However, $u'$ is a neighbour of $v$ in $D$ since $u'$ and $v$ are asynchronous neighbours of $u$. Thus $u'$ and $v$ are joined by a non-symmetric arc. Note that $u$ and $w$ are joined by a non-symmetric arc. Since $u'$ and $w$ are asynchronous neighbours of $u$, they are adjacent in $D$. If $v$ and $w$ are joined by symmetric arcs, then $\{u, v, w, u'\}$ induce a digraph in Figure 4.2 (i), a contradiction to the assumption. So $vw$ is a non-symmetric arc. This implies that $uw$ is a non-symmetric arc as otherwise $\{u, v, w\}$ induce the digraph in Figure 4.2 (iv), a contradiction to assumption. If $u'$ and $w$ are joined by symmetric arcs, then $\{u, v, w, u'\}$ induce a digraph in Figure 4.2 (ii), a contradiction. Therefore, $u'$ and $w$ are joined by a non-symmetric arc.

If $w$ is a di-simplicial vertex of $S(D)$ then the statement holds. Hence suppose that $w$ is not a di-simplicial vertex in $S(D)$. Then $w$ has two non-adjacent neighbours $w'$ and $w''$ in $S(D)$. At least one of $w', w''$ is not adjacent to $v$ in $S(D)$ as otherwise $\{v, w, w', w''\}$ induce a chordless 4-cycle in $S(D)$, contradicting the assumption that $S(D)$ is chordal. By symmetry, assume that $w'$ is not adjacent to $v$ in $S(D)$. However $w'$ and $v$ are adjacent in $D$ as they are asynchronous neighbours of $w$. Therefore $w'$ and $v$ are joined by a non-symmetric arc. Since $u$ and $w'$ are asynchronous neighbours of $w$, they are adjacent in $D$. Similarly, $u'$ and $w'$ are adjacent as they are asynchronous neighbours of $w$. If $u$ and $w'$ are joined by a non-symmetric arc, then $\{u, w, u'w'\}$ induce a digraph in Figure 4.2 (i), a contradiction to the assumption. So $u$ and $w'$ are joined by symmetric arcs. But then $\{u, v, w, w'\}$ induce a digraph in Figure 4.2 (ii), a contradiction. □

The following theorem is about the forbidden subdigraph characterizations of weakly quasi-transitive semi-strict chordal digraphs.

**Theorem 4.2.** *A weakly quasi-transitive digraph $D$ is semi-strict chordal if and only if $S(D)$ is chordal and $D$ does not contain any digraph in Figure 4.2 as an induced subdigraph.*

*Proof.* The necessity follows from Lemma 4.1 and Lemma 4.3. Hence, we only prove the sufficiency: Assume that $S(D)$ is chordal and $D$ does not contain any digraph in Figure 4.2 as an induced subdigraph. It suffices to show that $D$ has a semi-strict di-simplicial vertex. Since $S(D)$ is chordal, it has di-simplicial vertices. If any di-simplicial vertex of $S(D)$ is a semi-strict di-simplicial vertex of $D$, then $D$ has a semi-strict di-simplicial vertex and we are done. Hence assume that none of the di-simplicial vertices of $S(D)$ is a semi-strict di-simplicial vertex of $D$. Therefore each di-simplicial vertices of $S(D)$ is either a type 1 or a type 2 vertex.

We claim first that if $v$ is a type 1 vertex and $(u, v, w)$ is a violating triple for $v$, then $u$ or $w$ is a type 1 vertex. Suppose that $v$ is a type 1 vertex. Then $uv, vw$ are both non-symmetric arcs and $u, w$ are joined by a non-symmetric arc. Since $D$ does not contain the digraph in Figure 4.2 (iv) as an induced subdigraph, $uw$ is a non-symmetric arc. By Lemma 4.7, $u$ or $w$ is a di-simplicial vertex of $S(D)$.

Consider first the case when $u$ is a di-simplicial vertex of $S(D)$ but $w$ is not. If $u$ is a semi-strict di-simplicial vertex of $D$ then the prove is complete. Hence assume that $u$ is not a semi-strict di-simplicial vertex of $D$ and so there exist violating triple for $u$. We show by contradiction that $u$ is a type 1 vertex. Hence assume that $u$ is not a type 1 vertex. Then there exists a violating triple $(u_1, u, u_2)$ for $u$ such that exactly one of $u_1u$ or $uu_2$ is a non-symmetric arc. Moreover, $u_1$ and $u_2$ are joined by a non-symmetric arc. By considering the reversal of $D$, we assume that $u_1u$ is a non-symmetric arc. Since $w$ and $u_2$ are asynchronous neighbours of $u$, they are adjacent. If $w$ and $u_2$ are adjacent by symmetric arcs, then $v$ and $u_2$ are joined by symmetric arcs as otherwise $u, v, w, u_2$ induces Figure 4.2 (ii), a contradiction. Since $v$ and $u_1$ are asynchronous neighbours of $u$, they are adjacent. If $v$ and $u_1$ are joined by symmetric arcs, then $u_1$ and $u_2$ are non-adjacent neighbours of $v$ in $S(D)$, which contradicts the assumption that $v$ is a di-simplicial vertex of $S(D)$. Therefore $v$ and $u_1$ are joined by a non-symmetric arc. If $vu_1$ is a non-symmetric arc, then $u, v, u_1$ induce the digraph in Figure 4.2 (iv), a contradiction. Hence $u_1v$ is a non-symmetric arc. But then $(u_1, v, u_2)$ is a violating triple where $v$ and $u_2$ are adjacent by symmetric arcs, showing that $v$ is not a type 1 vertex, a contradiction to our assumption. Therefore $u_2$ and $w$ are joined by a non-symmetric arc.

Since $w$ is not a di-simplicial vertex of $S(D)$, $w$ has non-adjacent neighbours $w_1$ and $w_2$ in $S(D)$. Note that $u$ is adjacent to both $w_1$ and $w_2$ in $D$ as it is a neighbour of $w$ asynchronous form $w_1$ and $w_2$. Since $u$ is a semi-strict di-simplicial vertex of

$S(D)$, it cannot be adjacent to both $w_1$ and $w_2$ in $S(D)$. But then $u, u_1, w, w_1$ induce a digraph in Figure 4.2 (i), a contradiction to the assumption. Therefore $u$ is a type 1 vertex. By symmetric, if $w$ is a semi-strict di-simplicial vertex of $S(D)$ but $u$ is not, then $w$ is a type 1 vertex.

Consider now the case when both $u$ and $w$ are semi-strict di-simplicial vertices of $S(D)$. Suppose to the contrary that $u$ and $w$ are both type 2 vertices. Then there exists a violating triple $(u_1, u, u_2)$ where exactly one of $u_1u$ or $uu_2$ is non-symmetric. Similarly, there exists a violating triple $(w_1, w, w_2)$ where exactly one of $w_1w$ or $ww_2$ is non-symmetric. Suppose that $u_1u$ and $w_1w$ are non-symmetric arcs. (The proofs for other cases are similar.) Then $uu_2$ and $ww_2$ are symmetric arcs. We can see that $u_2$ is adjacent to both $v$ and $w$, and $w_2$ is adjacent to both $v$ and $u$. If $u_2$ and $w$ are joined by symmetric arcs, then a similar proof as above shows that either $D$ contains Figure 4.2 (ii) as an induced subdigraph or $v$ is not a type 1 vertex, which are contradictions. Therefore, $u_2$ and $w$ are joined by a non-symmetric arc. Similarly, $w_2$ and $u$ are joined by a non-symmetric arc. Then $u_2$ and $w_2$ are adjacent as they are asynchronous neighbours of $u$, but then $u, w, u_2, w_2$ induce a subdigraph in Figure 4.2 (i), a contradiction to assumption. Therefore, at least one of $u$ or $w$ is a type 1 vertex.

We then claim that $D$ does not contain a type 1 vertex. Suppose to the contrary that $v$ is a type 1 vertex and $(u, v, w)$ is a violating triple for $v$. Assume that $w$ is not a type 1 vertex. Then by our previous claim $u$ is a type 1 vertex and there exists a violating triple $(u_1, u, x_1)$ for $u$. Since $u$ is of type 1, both $u_1u$ and $ux_1$ are non-symmetric arcs. Since $v$ and $u_1$ are asynchronous neighbours of $u$, they are adjacent in $D$. If $v$ and $u_1$ are joined by symmetric arcs, then $(u, v, u_1)$ is a violating triple for v with $vu_1$ being symmetric arcs, certifying that $v$ is not a type 1 vertex, a contradiction to our assumption. Hence $v$ and $u_1$ are joined by a non-symmetric arc. If $vu_1$ is a non-symmetric arc, then $D$ contains Figure 4.2 (iv) as a subdigraph induced by $u, v, u_1$, a contradiction. Therefore, $u_1v$ is a non-symmetric arc. Moreover, $u_1$ and $w$ are adjacent in $D$ as they are asynchronous neighbours of $v$. If $u_1$ and $w$ are joined by symmetric arcs, then $D$ contains Figure 4.2 (iii) induced by $u, v, w, u_1$ as a subdigraph, a contradiction. If $wu_1$ is a non-symmetric arc, then $\{u, w, u_1\}$ induce a digraph in Figure 4.2 (iv), which is also a contradiction. Hence $u_1w$ is a non-symmetric arc and so $(u_1, v, w)$ is a violating triple for $v$. Since $w$ is not a type 1 vertex by our assumption, $u_1$ is a type 1 vertex by the previous claim. Thus there exists a violating triple $(u_2, u_1, x_2)$ for $u_1$ where $u_2u_1$ and $u_1x_2$ are both non-symmetric arcs. A similar proof as above shows that $(u_2, v, w)$ is a violating triple for $v$. Since $w$ is not a type 1 vertex, $u_2$ must be a type 1 vertex. Continuing this pattern we obtain a sequence of vertices $v, u, u_1, u_2, \ldots$, where $uv, u_1u, u_2u_1, u_3u_2, \ldots$ are non-symmetric arcs. Since $D$ is finite, this implies that $D$ contains a directed cycle consisting for non-symmetric arcs. By Lemma 4.5, $D$ contains a digraph in Figure 4.2 (i) or (iv) as an induced subdigraph, which is a contradiction. Therefore $w$ must also be a type 1 vertex.

Let $w_1$ be a vertex such that $ww_1$ is a non-symmetric arc; know that such a vertex $w_1$ can be obtained from a violating triple of $w$. Then $v$ and $w_1$ must be adjacent as they are asynchronous neighbours of $w$. If $v$ and $w_1$ are joined by symmetric arcs, then $(w_1, v, w)$ is a violating triple for $v$ of type 2, contradicting our assumption. Hence $v$ and $w_1$ are joined by a non-symmetric arc, and the non-symmetric arc must be $vw_1$ as otherwise $\{v, w, w_1\}$ induce the digraph in Figure 4.2 (iv), a contradiction. We see now that $u$ and $w_1$ are adjacent as they are asynchronous neighbours of $v$. If $u, w_1$ are joined by symmetric arcs, then $u, v, w, w_1$ induce a digraph in Figure 4.2 (iii), a contradiction. So $u$ and $w_1$ are joined by a non-symmetric arc and it must be $uw_1$. Thus $(u, v, w_1)$ is a violating triple. If $w_1$ is not a type 1 vertex, then we are back to the previous case. Hence, $w_1$ is a type 1 vertex. Then there exists a vertex $w_2$ such that $w_2w_1$ is a non-symmetric arc. Continuing in this way, we obtain a sequence of vertices $v, w, w_1, w_2, w_3, \dots$ where $vw, ww_1, w_1w_2, w_2w_3, \dots$ are non-symmetric arcs. Since $D$ is finite, this again implies that $D$ contains a directed cycle consisting of non-symmetric arcs, which end up with a contradiction. Hence $D$ does not contain any type 1 vertex. It follows that every di-simplicial vertex of $S(D)$ is a type 2 vertex.

Suppose that $v$ is a type 2 vertex. Then there is a violating triple $(u, v, w)$ such that exactly one of $uv$ or $vw$ is a non-symmetric arc. If $uv$ is the non-symmetric arc, then Lemma 4.7 implies that $u$ is a di-simplicial vertex of $S(D)$ and hence must be a type 2 vertex. On the other hand, if $vw$ is the non-symmetric arc, then Lemma 4.7 implies that $w$ is a di-simplicial vertex of $S(D)$ and so it is of type 2. Hence, for each type 2 vertex $v$, there is a type 2 vertex $v'$ such that $v$ and $v'$ are part of a violating triple for $v$ and either $vv'$ or $v'v$ is a non-symmetric arc. It follows that there exist $z_1, z_2, z_3, \dots, z_l$ along with $z'_1, z'_2z'_3 \dots, z'_l$ such that for each $i = 1, 2, \dots, l$:

- $z_i$ is a type 2 vertex;
- either $(z_{i+1}, z_i, z'_i)$ or $(z'_i, z_i, z_{i+1})$ is a violating triple;
- $z_i$ and $z_{i+1}$ are joined by a non-symmetric arc; and
- $z_i$ and $z'_i$ are joined by symmetric arcs.

where the subscripts are modulo $l$. Choose such sequence so that the length $l$ is minimum. By Lemma 4.5, the sequence $z_1, z_2, \dots, z_l$ cannot form a directed cycle in either direction. Assume without loss of generality that $z_1z_2$ and $z_1z_l$ are both non-symmetric arcs. Hence $(z'_1, z_1, z_2)$ and $(z_1, z_l, z'_l)$ are violating triples and each of the pairs $z_2, z'_1$ and $z_1, z'_l$ is joined by a non-symmetric arc. Since $z'_1$ and $z_l$ are asynchronous neighbours of $z_1$, they are adjacent in $D$. If $z'_1$ and $z_l$ are adjacent by a non-symmetric arc, then $D$ contains Figure 4.2 (i) induced by $z_1, z'_1, z_l, z'_l$ as a subdigraph, a contradiction. Hence $z_1'$ and $z_l$ are joined by symmetric arcs. Since $z_2$ and $z_l$ are asynchronous neighbours of $z_1'$, they are adjacent in $D$. If $z_2$ and $z_l$ are joined by symmetric-arcs, then $D$ contains Figure 4.2 (i) as a subdigraph induced by $z_1, z_2, z_l, z_1'$, a contradiction. Therefore, $z_2$ and $z_l$ are joined by a non-symmetric arc. If $z_2 z_l$ is a non-symmetric arc, then $(z_2, z_l, z_1')$ is a violating triple and we end up with a shorter sequence, which contradicts the minimality of $l$. On the other hand, if $z_l z_2$ is a non-symmetric arc, then $(z_1', z_l, z_2)$ is a violating triple which again results in a shorter sequence, a contradiction. Therefore, $D$ must contain a semi-strict di-simplicial vertex. This completes the proof. □

Since quasi-transitive digraphs and extended semicomplete digraphs are subclasses of weakly quasi-transitive digraphs (see Theorem 2.2), we have the following:

**Corollary 4.1.** *Let $D$ be a quasi-transitive digraph or an extended semicomplete digraph. Then $D$ is semi-strict chordal if and only if $S(D)$ is chordal and $D$ does not contain any digraph in Figure 4.2 as an induced subdigraph.* □

A perfect elimination ordering of a digraph, if one exists, can be obtained by recursively finding a di-simplicial vertex, which can be recognized in $O(n^2m)$ time [60]. In a similarly way, semi-strict chordal digraphs can also be recognized in time $O(n^2m)$.

# Chapter 5

# Chordal Signed Graphs and Bigraphs

In this chapter, we extend the concept of chordality of graphs to signed graphs and bigraphs. We characterize by forbidden subgraphs of chordal signed graphs and bigraphs.

## 5.1 Chordal signed graphs

A vertex $v$ in a signed graph $\widehat{G}$ is *signed simplicial* if $N(v)$ induces a positive clique in $\widehat{G}$, and a signed graph $\widehat{G}$ is chordal if every induced subgraph of $\widehat{G}$ has a signed simplicial vertex. In [41], an equivalent form of chordal signed graphs has been studied by Hell and Hernández-Cruz.

Let $D$ be a digraph. Two adjacent vertices in $D$ are joining by either a non-symmetric arc or symmetric arcs (but not both). By replacing each non-symmetric arc in $D$ as an negative edge and each pair of symmetric arcs as a positive edge, we obtain a signed graph $\widehat{G}$. Thus, a strict di-simplicial vertex in $D$ corresponds to a signed simplicial vertex of $\widehat{G}$. Hence $D$ is a strict chordal digraph if and only if $\widehat{G}$ is a chordal signed graph.

The following theorem is a direct translation of the results from [41].

**Theorem 5.1.** *[41] Let $\widehat{G}$ be a signed graph. Then $\widehat{G}$ is chordal if and only if it does not contain any signed graph in Figure 5.1 as an induced subgraph.* □

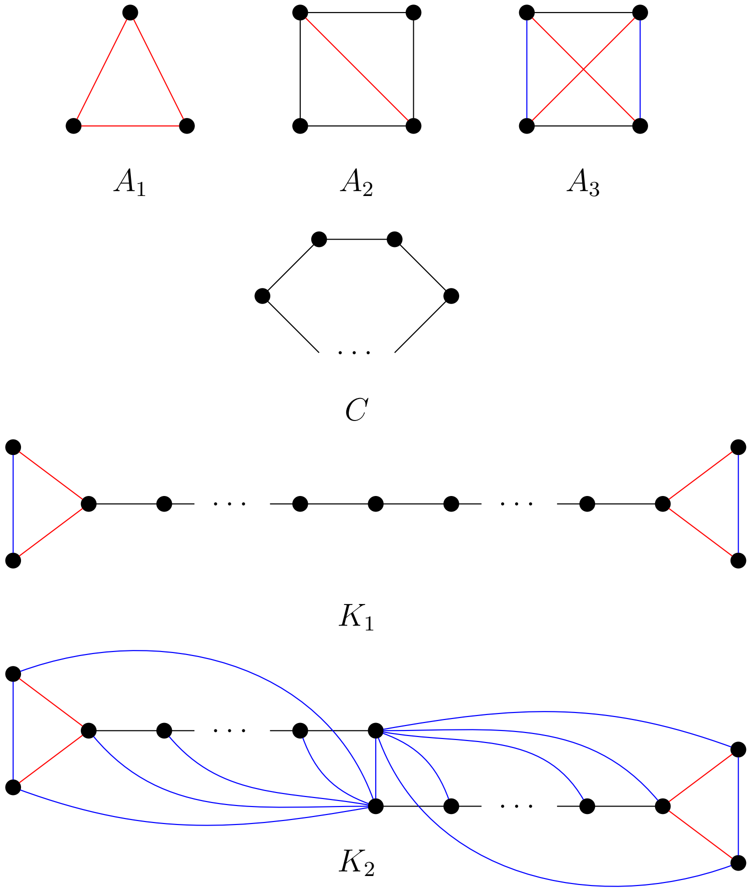


Figure 5.1: The forbidden subgraphs for chordal signed graphs

Figure 5.1 depicts six sets of signed graphs. It is easy to verify that none of them has a signed simplicial vertex and hence none of them are chordal signed graph. Not all of these graphs are minimal non-chordal signed graphs. For instance, $A_2$ contains a signed graph with $A_1$ being a proper subgraph. The signed graphs listed in Figure 5.2 are all minimal non-chordal signed graphs in $A_1 \cup A_2 \cup A_3$.

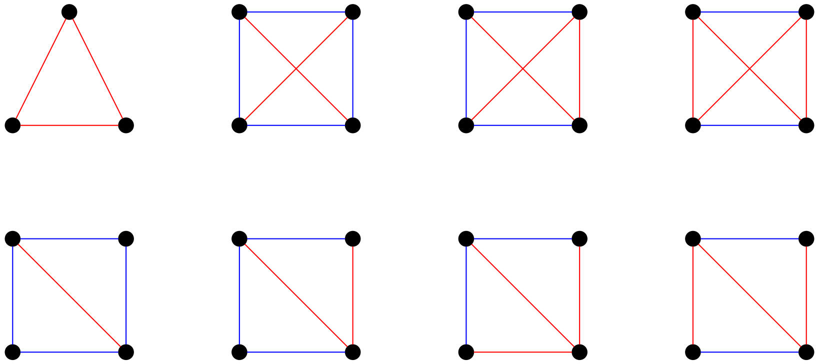


Figure 5.2: The minimal non-chordal signed graphs in $A_1 \cup A_2 \cup A_3$

## 5.2 Chordal signed bigraphs

An edge $uv$ of a bigraph $G$ is *simplicial* if $N(uv)$ induced a biclique. Every induced subgraph of a chordal bigraph contains a simplicial edge [32]. An edge $uv$ of a signed bigraph $G$ is *signed simplicial* if $N(uv)$ induced a positive biclique. A signed bigraph $\widehat{G}$ is chordal if it has a signed perfect edge-without-vertex elimination ordering $e_1, e_2, \ldots, e_m$, that is, each $e_i$ is a signed simplicial edge in $\widehat{G} - \{e_1, e_2, \ldots, e_{i-1}\}$.

It follows from definition that an edge in a signed bigraph is signed simplicial if one of its endvertices has degree one. This implies that every signed tree has a signed simplicial edge-without-vertex ordering and therefore is a chordal signed bigraph. On the other hand, since no signed cycle of length $\geq 6$ has a signed simplicial edge, the underlying unsigned graph of a chordal signed bigraph is a chordal bigraph. Hence the class of chordal signed bigraphs contains all signed tree and it is a subclass of signed chordal bigraphs. However, a signed chordal bigraph may not be chordal signed bigraph (any graph in Figure 5.4 is a signed chordal bigraph but not a chordal signed bigraph). (Note the difference between chordal signed bigraphs and signed chordal bigraphs.)

**Lemma 5.1.** *Suppose that $\widehat{G}$ is a signed bigraph and $e$ is a signed simplicial edge of $\widehat{G}$. If $\widehat{G} - e$ is chordal, then $\widehat{G}$ is chordal.*

*Proof.* Since $\widehat{G} - e$ is chordal, it has a signed simplicial edge-without-vertex ordering $e_1, e_1, \ldots, e_m$. Then $e, e_1, e_1, \ldots, e_m$ is a signed simplicial edge-without-vertex ordering of $\widehat{G}$, showing that $\widehat{G}$ is chordal. □

We give in this chapter a forbidden subgraph characterization of chordal signed bigraphs. It is easy to verify that every induced subgraph of a chordal signed bigraph is chordal. Our characterization of chordal signed bigraphs implies that if every induced subgraph of a signed bigraph $\widehat{G}$ is chordal, then $\widehat{G}$ is chordal.

Recall that we use edge-coloured graphs to represent signed graphs where positive edges are coloured blue, negative edges are coloured red, and edges whose signs are not specified are coloured black. Suppose that $F$ is an edge-coloured graph whose edges are coloured with blue, red, and black colours. If $F$ has no black edge then it represents a unique signed graph. If $F$ has black edges then by replacing each black edge of $F$ with a blue or a red edge we obtain a signed graph. This way $F$ represents a set of signed graphs. We say that a signed graph $\widehat{G}$ *does not contain* $F$ (*as an induced subgraph*) if none of the graphs in $F$ is a subgraph (an induced subgraph) of $\widehat{G}$.

### 5.2.1 Complete bigraphs

In this section we consider signed complete bigraphs. We will show that there are only five minimal signed complete bigraphs which are not chordal (see Figure 5.4).

**Lemma 5.2.** *Let $\widehat{G}$ be a signed complete bigraph with bipartition $(X, Y)$ where $X = \{x_1, x_2, \ldots, x_\alpha\}$ and $Y = \{y_1, y_2, \ldots, y_\beta\}$. Suppose that $x_1y_1$ is a signed simplicial edge in $\widehat{G}$. Then $x_1y_i$ is a signed simplicial edge in $\widehat{G} - \{x_1y_1, x_1y_2, \ldots, x_1y_{i-1}\}$ for each $i = 1, 2, \ldots, \beta$. Moreover, if $\widehat{G} - \{x_1\}$ is chordal then $\widehat{G}$ is chordal.*

*Proof.* Denote $\widehat{G_i} = \widehat{G} - \{x_1y_1, x_1y_2, \ldots, x_1y_{i-1}\}$ for each $i = 1, 2, \ldots, \beta$. Since $\widehat{G}$

is complete, $N_{\widehat{G_i}}(x_1) = \{y_i, y_{i+1}, \ldots, y_\beta\}$ and $N_{\widehat{G_i}}(y_i) = X$. Since $x_1y_1$ is a signed simplicial edge of $\widehat{G}$, $\{x_2, x_3, \ldots, x_\alpha\} \cup \{y_2, y_3, \ldots, y_\beta\}$ induces a positive biclique in $\widehat{G}$. In particular, $\{x_2, x_3, \ldots, x_\alpha\} \cup \{y_{i+1}, y_{i+2}, \ldots, y_\beta\}$ induces a positive biclique in $\widehat{G_i}$, which means that $x_1y_i$ is a signed simplicial edge in $\widehat{G_i}$ for each $i = 1, 2, \ldots, \beta$.

If $\widehat{G} - x_1$ is chordal, then $\widehat{G} - x_1$ has a signed simplicial edge-without-vertex elimination ordering $e_1, e_2, \ldots, e_m$. Hence $x_1y_1, x_1y_2, \ldots, x_1y_\beta, e_1, e_2, \ldots, e_m$ is a signed simplicial edge-without-vertex elimination ordering of $\widehat{G}$, and so $\widehat{G}$ is chordal. □

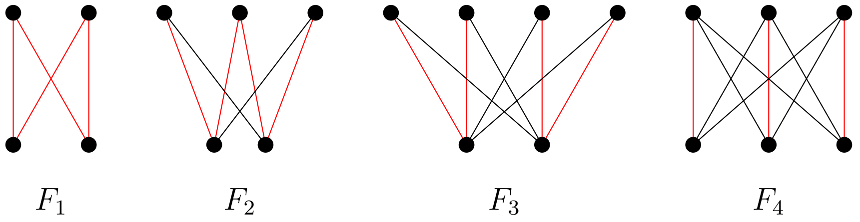


Figure 5.3: The forbidden subgraphs for chordal signed complete bigraphs

Figure 5.3 depicts four sets $F_1, \ldots, F_4$ of signed complete bigraphs. It is easy to verify that none of them has a signed simplicial edge and hence cannot be contained in a chordal signed bigraph.

**Proposition 5.1.** *Let $\widehat{G}$ be a signed complete bigraph. Then $\widehat{G}$ is chordal if and only if it does not contain any bigraph in $F_1, \ldots, F_4$ in Figure 5.3 as an induced subgraph.*

*Proof.* The necessity of the statement follows from the remark above. For the sufficiency suppose that there exists a signed complete bigraph which contains none graph in $F_1, \ldots, F_4$ and it is not chordal. Let $\widehat{G}$ be a minimal such signed complete bigraph. Then Lemma 5.2 implies that no edge of $\widehat{G}$ is signed simplicial.

Let $X = \{x_1, x_2, \ldots, x_\alpha\}$ and $Y = \{y_1, y_2, \ldots, y_\beta\}$. We claim that $\widehat{G}$ contains no negative $P_4$. Suppose to the contrary that $x_1y_1x_2y_2$ is a negative $P_4$ in $\widehat{G}$. Since $x_2y_1$ is not a signed simplicial edge, there is a negative edge $x_sy_t$ in $N(x_2y_1)$. If $s = 1$ and

$t = 2$, then $\widehat{G}$ contains $F_1$ as an induced subgraph. If $s = 1$ and $t \neq 2$, or $s \neq 1$ and $t = 2$, then $\widehat{G}$ contains a graph in $F_2$ as an induced subgraph. If $s \neq 1$ and $t \neq 2$ then $\widehat{G}$ contains a graph in $F_4$ as an induced subgraph. All of these contradict the assumption. So $\widehat{G}$ contains no negative $P_4$.

Since no edge of $\widehat{G}$ is signed simplicial, there is a negative edge in the graph induced by $N(x_i, y_j)$ for each $1 \leq i \leq \alpha$ and $1 \leq j \leq \beta$. This implies that there are two negative edges which share no endvertex. Without loss of generality assume that $x_1y_1$ and $x_2y_2$ are negative. Let $x_sy_t$ be a negative edge in the graph induced by $N(x_1y_2)$. If $s \geq 3$ and $t \geq 3$, then $\widehat{G}$ contains a graph in $F_4$ as an induced subgraph, a contradiction to the assumption. Thus $s = 2$ or $t = 1$. Since $\widehat{G}$ contains no negative $P_4$, $s \neq 2$ or $t \neq 1$. Hence we have either $s = 2$ and $t \neq 1$ or $s \neq 2$ and $t = 1$. By symmetry we assume that $s = 2$ and $t \neq 1$. Let $x_py_q$ be a negative edge in the graph induced by $N(x_2, y_1)$. Since $\widehat{G}$ does not contain a negative $P_4$, $q \neq 2$ and $q \neq t$. If $p \neq 1$ then $\widehat{G}$ contain a graph in $F_4$ induced by $x_1, x_2, x_p, y_1, y_2, y_q$, a contradiction to the assumption. Thus $p = 1$ and hence $\widehat{G}$ contains a graph in $F_3$ induced by $x_1, x_2, y_1, y_2, y_t, x_q$, which again contradicts the assumption. $\square$

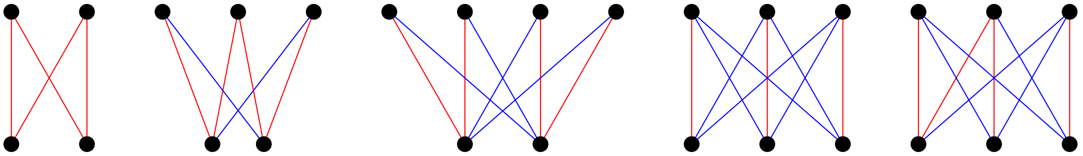


Figure 5.4: The minimal forbidden subgraphs for chordal signed complete bigraphs

Not all bigraphs in Figure 5.3 are minimal non-chordal bigraphs. For instance, if any of the black edges in $F_2$ is replaced by a red edge then it contains $F_1$. On the other hand, replacing both black edges in $F_2$ by blue edges results in a minimal non-chordal bigraph. All five bigraphs in Figure 5.4 are obtained in this way from the bigraphs in Figure 5.3. By Proposition 5.1 they are not chordal. Deleting a vertex in any of these bigraphs results in a chordal signed bigraph. So they are all minimal non-chordal signed bigraphs. Moreover, every signed bigraph in $F_1, \ldots, F_4$ contains a graph in Figure 5.4 as a subgraph. Hence by Proposition 5.1, if a signed complete bigraph is not chordal then it contains one of the graphs in Figure 5.4 as a subgraph. Therefore we have the following:

**Theorem 5.2.** *A signed complete bigraph is chordal if and only if it does not contain any of the signed bigraphs in Figure 5.4 as a subgraph.* □

### 5.2.2 Non-separable bigraphs

Recall that a bigraph is separable if it contains an induced $2K_2$, otherwise it is non-separable. Since an induced cycle of length $\geq 6$ contains an induced $2K_2$, it cannot be an induced subgraph of a non-separable bigraph. So non-separable bigraphs are all chordal. However, not all signed non-separable bigraphs are chordal (e.g., a negative $C_4$ is not chordal). In this section we characterize chordal signed non-separable bigraphs by their forbidden subgraphs.

We first take a look at some basic properties of non-separable bigraphs. Suppose that a bigraph $G$ with bipartition $(X, Y)$ is non-separable. Since $G$ does not contain an induced $2K_2$, the neighbourhoods of the vertices in $X$ are comparable and the neighbourhoods of the vertices in $Y$ are also comparable. That is, the vertices of $X$ and $Y$ can be ordered $x_1, x_2, \ldots, x_\alpha$ and $y_1, y_2, \ldots, y_\beta$ respectively in such a way that

$$N(x_1) \supseteq N(x_2) \supseteq \cdots \supseteq N(x_\alpha) \text{ and } N(y_1) \subseteq N(y_2) \subseteq \cdots \subseteq N(y_\beta).$$

We call such a vertex ordering $x_1, x_2, \ldots, x_\alpha, y_1, y_2, \ldots, y_\beta$ a *canonical ordering* of $G$.

**Lemma 5.3.** *Let $G$ be a non-separable bigraph with bipartition $(X, Y)$ without isolated vertices, and let $x_1, x_2, \ldots, x_\alpha, y_1, y_2, \ldots, y_\beta$ be a canonical ordering of $G$. Then the* *following statements hold:*

1. *$x_1$ is adjacent to all vertices in $Y$;*
2. *Each vertex in $N(y_1)$ is adjacent to all vertices in $Y$;*
3. *For each $x_i \in N(y_1)$, $x_iy_1$ is a simplicial edge; in particular $x_1y_1$ is a simplicial edge;*
4. *If $x_iy_j$ is a simplicial edge then $G - x_iy_j$ is non-separable;*
5. *If $x_iy_j$ is a non-simplicial edge, then there exist $x_k \in N(y_j)$ and $y_l \in N(x_i)$ with $k > i$ and $l < j$ such that $x_k$ and $y_l$ are not adjacent.*

*Proof.* For each $y_j \in Y$, since $G$ has no isolated vertex, $y_j$ is adjacent to some $x_i \in X$. The canonical ordering ensures that $N(x_i) \subseteq N(x_1)$. Hence $y_j \in N(x_i) \subseteq N(x_1)$ and so $x_1$ is adjacent to $y_j$.

For each $y_j \in Y$, the canonical ordering ensures that $N(y_1) \subseteq N(y_j)$. Thus each vertex in $N(y_1)$ is adjacent to $y_j$.

For each $x_i \in N(y_1)$, since $x_i$ is adjacent to all vertices in $Y$, $x_iy_1$ is an edge and $x_i$ is adjacent to all vertices in $N(x_1)$. So $x_iy_1$ is a simplicial edge and in particular $x_1y_1$ is a simplicial edge.

If $G - x_iy_j$ is separable then it contains an induced $2K_2$ consisting of edges $x_ay_b$ and $x_cy_d$. Since G is non-separable, the edges $x_ay_b$ and $x_cy_d$ do not form an induced $2K_2$ in $G$. So we must have $x_i = x_a$ and $y_j = y_d$ or $x_a = x_c$ and $y_j = y_b$. But in either case $x_iy_j$ is not a simplicial edge, which is a contradiction to our assumption.

If $x_iy_j$ is a non-simplicial edge, then there are non-adjacent vertices $x_k \in N(y_j)$ and $y_l \in N(x_i)$. This implies that $N(x_i) \nsubseteq N(x_k)$ and $N(y_j) \nsubseteq N(y_l)$. The canonical ordering ensures that $k > i$ and $l < j$. □

Due to the symmetry of non-separable bigraphs, the following statement holds:

**Corollary 5.1.** *Let $G$ be a non-separable bigraph with bipartition $(X, Y)$ without isolated vertices, and let $x_1, x_2, \ldots, x_\alpha, y_1, y_2, \ldots, y_\beta$ be a canonical ordering of $G$. Then:*

1. *$y_\beta$ is adjacent to all vertices in $X$;*
2. *Each vertex in $N(x_\alpha)$ is adjacent to all vertices in $X$;*
3. *For each $y_j \in N(x_\alpha)$, $x_\alpha y_j$ is a simplicial edge, in particular, $x_\alpha y_\beta$ is a simplicial edge.*

□

Since signed complete bigraphs are non-separable, the graphs in $F_1, \ldots, F_4$ in Figure 5.3 are also forbidden for chordal signed non-separable bigraphs. Two additional sets of forbidden subgraphs $F_5$ and $F_6$ for chordal signed non-separable bigraphs are depicted in Figure 5.5. Thus no chordal signed non-separable bigraph contains any graph in $F_1, \ldots, F_6$ as an induced subgraph. We will show these are all the forbidden subgraphs for chordal signed non-separable bigraphs.

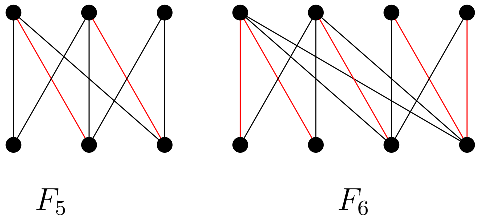


Figure 5.5: Additional forbidden subgraphs for chordal signed non-separable bigraphs

**Proposition 5.2.** *Let $\widehat{G}$ be a signed non-separable bigraph. Then $\widehat{G}$ is chordal if and only if it does not contain any graph in $F_1, \ldots, F_6$ in Figure 5.3 and in Figure 5.5 as an induced subgraph.*

Before proving this proposition, we need two lemmas:

**Lemma 5.4.** *Let $\widehat{G}$ be a signed non-separable bigraph with bipartition $(X, Y)$ without isolated vertices and let $x_1, x_2, \ldots, x_\alpha, y_1, y_2, \ldots, y_\beta$ be a canonical ordering of $\widehat{G}$. Suppose that $x_i, x_j, x_k, y_l, y_r$ induce a graph in $Z_1$ in Figure 5.6 where $x_i, x_j \in N(y_1)$. If no edge in $\widehat{G}$ is signed simplicial then $\widehat{G}$ contains a graph in $F_2, \ldots, F_5$ as an induced subgraph.*

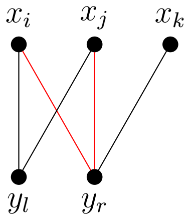


Figure 5.6: $Z_1$

*Proof.* Since $x_i, x_j \in N(y_1)$, by Lemma 5.3 (2), $N(x_i) = N(x_j) = Y$. Since $x_k$ is adjacent to $y_r$ but not adjacent to $y_l$, $N(y_r) \nsubseteq N(y_l)$. The canonical ordering ensures that $l < r$. Since $x_k$ is not adjacent to all vertices in $Y$ , $N(x_k) \subset N(x_i) = N(x_j) = Y$ and hence $k > i$, $k > j$.

Choose such a $Z_1$ so that $k$ is as large as possible. We claim that $x_k y_r$ is a simplicial edge in $G$ where $G$ is the unsigned bigraph of $\widehat{G}$. Indeed, if not then by Lemma 5.3(5) there exists $x_{k'} \in N(y_r)$ with $k' > k$ and $y_{\ell'} \in N(x_k)$ with $\ell' < r$ such that $x_{k'}$ and $y_{\ell'}$ are not adjacent. Then $x_i, x_j, x_{k'}, y_{l'}, y_r$ induce a copy of $Z_1$ with $k' > k$, contradicting the choice of $Z_1$. Hence $x_k y_r$ is a simplicial edge in $G$. By assumption $x_k y_r$ is not a signed simplicial edge in $\widehat{G}$. Hence the subgraph of $\widehat{G}$ induced by $N(x_k y_r)$ has negative edge.

Let $x_a y_b$ be a negative edge in the subgraph induced by $N(x_k y_r)$. If $x_a$ is adjacent to $y_l$ then $\widehat{G}$ contains a graph in $F_5$ induced by $x_i, x_a, x_k, y_l, y_r, y_b$ when $x_a \neq x_i$ or by $x_j, x_a, x_k, y_l, y_r, y_b$ when $x_a \neq x_j$. So we may assume that $x_a$ is not adjacent to $y_l$, which means that $x_a \neq x_i$ and $x_a \neq x_j$. The choice of $Z_1$ implies that $a < k$.

Consider the subgraph induced by $N(x_a y_r)$. Since $x_a y_r$ is not signed simplicial, either it is not a biclique or it is a biclique that contains a negative edge. Suppose first that the subgraph induced by $N(x_a y_r)$ is not a biclique. Then there exist $x_c \in N(y_r)$ with $c > a$ and $y_d \in N(x_a)$ with $d < r$ such that $x_c$ and $y_d$ are not adjacent. We must have $c \leq k$ as otherwise $x_i, x_j, x_c, y_d, y_r$ induce a graph in $Z_1$ with $c > k$, contradicting the choice of $Z_1$. Thus $y_b \in N(x_k) \subseteq N(x_c)$. Hence $x_j, x_a, x_c, y_d, y_r, y_b$ induce a graph in $F_5$. Suppose now that the subgraph induced by $N(x_a y_r)$ is a biclique that contains a negative edge $x_c y_d$. Note that $x_c$ is adjacent to $y_b$ as otherwise $c > k$ and $x_i, x_j, x_c, y_b, y_r$ induce a graph in $Z_1$, a contradiction. If $y_d = y_b$ then $x_i, x_j, x_c, x_a, y_r, y_b$ induce a graph in $F_3$ when $x_c \neq x_i$ and $x_c \neq x_j$, and $x_i, x_j, x_a, y_r, y_b$ induce a graph in $F_2$ when $x_c = x_i$ or $x_c = x_j$. If $y_d \neq y_b$ then $\widehat{G}$ contains a graph in $F_4$ induced by $x_i, x_c, x_a, y_d, y_r, y_b$ when $x_c \neq x_i$ or by $x_j, x_c, x_a, y_d, y_r, y_b$ when $x_c \neq x_j$. □

**Lemma 5.5.** *Let $\widehat{G}$ be a signed non-separable bigraph with bipartition $(X, Y)$ without isolated vertices and let $x_1, x_2, \ldots, x_\alpha, y_1, y_2, \ldots, y_\beta$ be a canonical ordering of $\widehat{G}$. Suppose that $x_i, x_j, x_k, y_l, y_q, y_r$ induce a graph in $Z_2$ in Figure 5.7 where $x_i, x_j \in N(y_1)$. If no edge in $\widehat{G}$ is signed simplicial then $\widehat{G}$ contains one of a graph in $F_3, \ldots, F_6$ as an induced subgraph.*

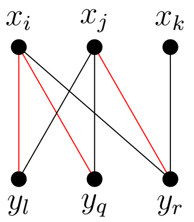


Figure 5.7: $Z_2$

*Proof.* Since $x_i, x_j \in N(y_1)$, by Lemma 5.3 (2), $N(x_i) = N(x_j) = Y$. Choose such a $Z_2$ so that $k$ is as large as possible. We claim that the subgraph induced by $N(x_k y_r)$

is a biclique. Suppose that it is not a biclique. Then by Lemma 5.3 (5) there exist non-adjacent vertices $x_a \in N(y_r)$ with $a > k$ and $y_b \in N(x_k)$ with $b < r$. Since $x_k$ is adjacent to $y_b$ but not to $y_l$ or $y_q$, $N(y_b) \supset N(y_l)$ and $N(y_b) \supset N(y_q)$. This implies that $x_a$ is not adjacent to $y_l$ or $y_q$ as it is not adjacent to $y_b$. Hence $x_i, x_j, x_a, y_l, y_q, y_r$ induce a graph in $Z_2$ with $a > k$, a contradiction to the choice of $Z_2$. So the subgraph induced by $N(x_k y_r)$ is a biclique and therefore contains a negative edge $x_a y_b$ as $x_k y_r$ is not signed simplicial.

Since $x_k$ is adjacent to $y_b$ but not adjacent to $y_l$ or $y_q$, $y_b \neq y_l, y_b \neq y_q$. If $x_a = x_i$ then $\widehat{G}$ contains a graph in $F_5$ induced by $x_i, x_j, x_k, y_\ell, y_b, y_r$. If $x_a = x_j$ then $\widehat{G}$ contains an $F_3$ induced by $x_i, x_j, y_\ell, y_q, y_b, y_r$. So assume that $x_a \neq x_i$ or $x_j$. If $x_a$ is adjacent to $y_q$ then $\widehat{G}$ contains a graph in $F_5$ induced by $x_j, x_a, x_k, y_q, y_b, y_r$. If $x_a$ is adjacent to $y_l$ then $\widehat{G}$ contains a graph in $F_5$ induced by $x_j, x_a, x_k, y_l, y_b, y_r$. Hence we further assume that $x_a$ is not adjacent to $y_q$ or $y_l$. Consider the subgraph induced by $N(x_a y_r)$. Since $x_a y_r$ is not signed simplicial, it is either not a biclique or is a biclique that containing a negative edge. If the subgraph induced by $N(x_a y_r)$ is not a biclique, then by Lemma 5.3(5) there exist non-adjacent $x_c \in N(y_r)$ and $y_d \in N(x_a)$. If $x_c y_b$ is an edge, then $\widehat{G}$ contains a graph in $F_5$ induced by $x_j, x_a, x_c, y_d, y_b, y_r$. If $x_c$ and $y_b$ are not adjacent, then $c > k$ and $x_i, x_j, x_c, y_l, y_q, y_r$ induce a graph in $Z_2$, a contradiction to the choice of $Z_2$. So assume that the subgraph induced by $N(x_a y_r)$ is a biclique and contains a negative edge $x_c y_d$. If $x_c = x_i$ then $\widehat{G}$ contains a graph in $F_5$ induced by $x_i, x_j, x_a, y_\ell, y_d, y_r$. If $x_c = x_j$ then $\widehat{G}$ contains a graph in $F_3$ induced by $x_i, x_j, y_\ell, y_q, y_d, y_r$. So assume that $x_c$ is not equal to $x_i$ or $x_j$. If $x_c$ is adjacent to $y_q$ then $\widehat{G}$ contains a graph in $F_5$ induced by $x_j, x_a, x_c, y_q, y_d, y_r$. If $x_c$ is not adjacent to $y_q$ and $y_b \neq y_d$, then $\widehat{G}$ contains a graph in $F_4$ induced by $x_j x_a, x_c, y_b, y_d, y_r$. Finally, if $x_c$ is not adjacent to $y_q$ and $y_b = y_d$ then $\widehat{G}$ contains a graph in $F_6$ induced by $x_i, x_j, x_a, x_c, y_l, y_q, y_b, y_r$. $\square$

*Proof of Proposition 5.2.* We only prove the if-part of the statement as the only-if part is discussed above. So assume that $\widehat{G}$ does not contain any graph in $F_1, \dots, F_6$ as an induced subgraph. Suppose to the contrary that $\widehat{G}$ is not chordal. According to Lemma 5.3 (4), deleting a signed simplicial edge from $\widehat{G}$ maintains the property of being non-separable. Also, deleting any edge from $\widehat{G}$ does not results in a graph containing any a graph in $F_1, \dots, F_6$. Thus, we can further assume that $\widehat{G}$ has no signed simplicial edge and has no isolated vertex.

Let $(X, Y)$ be the bipartition and $x_1, x_2, \dots, x_\alpha, y_1, y_2, \dots, y_\beta$ be a canonical ordering of $\widehat{G}$. Since no edge in $\widehat{G}$ is signed simplicial, for any edge $x_iy_j$ the subgraph induced by $N(x_iy_j)$ is not a positive biclique, that is, either it is not a biclique or is a biclique but contains a negative edge. By Lemma 5.3 (3), the subgraph induced by $N(x_1y_1)$ is a biclique so it contains a negative edge $x_ay_b$. Choose such an edge $x_ay_b$ so that $b$ is as large as possible. By Lemma 5.3 (2), we know that the vertices in $N(y_1)$ have the same neighbourhood. Thus by re-ordering the vertices of $N(y_1)$ if necessary we can assume that $x_a$ is the vertex with the largest subscript which is incident with a negative edge in the subgraph induced by $N(x_1y_1)$. Hence the subgraph induced by $N(x_1y_1)$ contains no negative edge $x_{a'}y_{b'}$ with $a' > a$ or $b' > b$.

Consider the subgraph induced by $N(x_ay_1)$. It is a biclique according to Lemma 5.3(3) and thus contains a negative edge $x_cy_d$. Note that $N(x_1) = Y$ by Lemma 5.3 (1). So $x_cy_d$ is a negative edge in the subgraph induced by $N(x_1y_1)$. The choice of $x_ay_b$ implies that $1 \leq c < a$ and $1 < d \leq b$. Moreover, $N(x_c) = Y$.

**Case 1.** $d < b$.

If some vertex $x$ is adjacent to $y_d$ (and hence to $y_b$ as $N(y_d) \subseteq N(y_b)$) but not to a vertex $y \in Y$, then $\widehat{G}$ contains $F_5$ induced by $x_c, x_a, x, y, y_d, y_b$, contradicting the assumption. So any vertex adjacent to $y_d$ is adjacent to all vertices in $Y$. This implies that the subgraph induced by $N(x_ay_d)$ is a biclique and hence contains a negative edge $x_ey_f$. Since $x_e$ is adjacent to $x_d$, it is adjacent all vertices in $Y$ and in particular to $y_1$. So the edge $x_ey_f$ is a negative edge in the subgraph induced by $N(x_1y_1)$. The choice of $x_ay_b$ implies $e < a$ and $f \leq b$. If $x_e \neq x_c$ and $y_f \neq y_b$ then $x_e, x_c, x_a, y_f, y_d, y_b$ induce an $F_4$, contradicting the assumption. Hence $x_e = x_c$ or $y_f = y_b$.

**Subcase 1.1.** $x_e = x_c$.

If some vertex $x$ is adjacent to $y_b$ but not to $y_d$ (and hence not to $y_1$ as $N(y_1) \subseteq N(y_d)$) then $x_c, x_a, x, y_1, y_d, y_b$ induce a graph in $Z_2$. By Lemma 5.5 $\widehat{G}$ contains one of $F_3, \ldots, F_6$ as an induced subgraph, a contradiction to the assumption. So every vertex adjacent to $y_b$ is adjacent to $y_d$ and hence to all vertices in $Y$. This implies that the subgraph induced by $N(x_cy_b)$ is a biclique and thus contains a negative edge $x_gy_h$. Since $x_g$ is adjacent to $y_b$, it is adjacent to all vertices in $Y$. Thus $x_gy_h$ is in the subgraph induced by $N(x_1y_1)$. If $x_g \neq x_a$ and $y_h \neq y_d$, then $x_c, x_a, x_g, y_d, y_b, y_h$ induce a graph in $F_4$, a contradiction. If $x_g = x_a$ and $y_h \neq y_d$, then $x_c, x_a, y_d, y_b, y_h$ induce a graph in $F_2$ when $y_f = y_b$ and $x_c, x_a, x_g, y_f, y_h$ induce a graph in $F_3$ when $y_f \neq y_b$, contradicting the assumption. If $x_g = x_a$ and $y_h = y_d$, then $x_c, x_a, y_d, y_b$ induce an $F_1$ when $y_f = y_b$ and $x_c, x_a, y_d, y_f, y_b$ induce a graph in $F_2$ when $y_f \neq y_b$, contradicting the assumption. If $x_g \neq x_a$ and $y_h = y_d$, then $x_c, x_a, x_g, y_d, y_b$ induce an $F_2$ when $y_f = y_b$ and $x_c, x_a, x_g, y_d, y_f, y_b$ induce a graph in $F_4$ when $y_f \neq y_b$, again contradicting the assumption.

**Subcase 1.2.** $x_e \neq x_c$.

We must have $y_f = y_b$. As in Subcase 1.1 we have that every vertex adjacent to $y_b$ is adjacent to $y_d$ and hence to all vertices in $Y$. This together with Lemma 5.3(3) imply that the subgraph induced by $N(x_cy_b)$ is a biclique and thus contains a negative edge $x_iy_j$. Since $x_i$ is adjacent to $x_b$ it is adjacent to all vertices in $Y$. Thus $x_iy_j$ is in the subgraph induced by $N(x_1y_1)$. If $y_j \neq y_d$ then $x_c, x_e, x_i, y_d, y_j, y_b$ induce a graph in $F_4$ when $x_i \neq x_e$ and $x_c, x_i, x_a, y_d, y_j, y_b$ induce a graph in $F_4$ when $x_i \neq x_a$, contradicting the assumption. If $y_j = y_d$, then $x_c, x_e, x_i, x_a, y_d, y_b$ induce a graph in $F_3$ when $x_i \neq x_e$ and $x_i \neq x_a$ and $x_c, x_e, x_a, y_d, y_b$ induce a graph in $F_2$ when $x_i = x_e$ or $x_a$, contradicting the assumption.

**Case 2.** $d = b$.

If some vertex $x$ is adjacent to $y_b$ but not to a vertex $y \in Y$ then $x_c, x_a, x, y, y_b$ induce a $Z_1$ and by Lemma 5.4 $\widehat{G}$ contains one of $F_2, \ldots, F_5$ as an induced subgraph, a contradiction to the assumption. So any vertex adjacent to $y_b$ is adjacent to all vertices of $Y$. This together with Lemma 5.3(3) imply that the subgraph induced by $N(x_a y_b)$ is a biclique and hence contains a negative edge $x_e y_f$. Note that $x_e y_f$ is an edge in the subgraph induced by $N(x_1 y_1)$. The choice of $x_a y_b$ implies that $e < a$ and $f < b$.

**Subcase 2.1.** $x_e = x_c$.

Consider the subgraph induced by $N(x_c y_b)$. By Lemma 5.3(3) and the fact that every vertex adjacent to $y_b$ is adjacent to all vertices in $Y$, it is a biclique and hence contains a negative edge $x_g y_h$. Again $x_g y_h$ is in the subgraph induced by $N(x_1 y_1)$. If $x_g = x_a$ and $y_h = y_f$ then $x_c, x_a, y_f, y_b$ induce an $F_1$. If $x_g \neq x_a$ and $y_h \neq y_f$ then $x_c, x_g, x_a, y_f, y_h, y_b$ induce a graph in $F_4$. If $x_g = x_a$ and $y_h \neq y_f$ then $x_c, x_a, y_f, y_h, y_b$ induce a graph in $F_2$. If $x_g \neq x_a$ and $x_h = y_f$ then $x_c, x_g, x_a, y_f, y_d$ induce a graph in $F_2$. All these contradict the assumption.

**Subcase 2.2.** $x_e \neq x_c$.

Consider the subgraph induced by $N(x_e y_b)$. A similar argument as in Subcase 2.1 shows that it is a biclique and hence contains a negative edge $x_g y_h$ which is in the subgraph induced by $N(x_1 y_1)$. If $y_h \neq y_f$ then $\widehat{G}$ contains a graph in $F_4$ induced by $x_e, x_g, x_a, y_f, y_h, y_b$ when $x_g \neq x_a$, and by $x_c, x_g, x_e, y_f, y_b, y_b$ when $x_g \neq x_c$. If $y_h = y_f$ and $x_g \neq x_c$ or $x_a$ then $x_c, x_e, x_g, x_a, y_f, y_b$ induce a graph in $F_3$. If $y_h = y_f$ and $x_g = x_c$ and $x_g \neq x_a$ then $\widehat{G}$ contains a graph in $F_2$ induced by $x_c, x_e, x_a, y_f, y_b$. $\square$

### 5.2.3 Separable bigraphs

Recall that a separating set in a bigraph $G$ is a set of vertices $S$ such that $G - S$ contains at least two non-trivial components. Clearly, a bigraph has a separating set if and only if it is separable. A separating set is minimal if no proper subset of the set is separating. It follows from the definitions that a set $S$ is a minimal separating set if and only if there exist two non-trivial components $H$ and $H'$ in $G - S$ such that every vertex of $S$ has a neighbour in $H$ and in $H'$. In this case we also say that the minimal separating set $S$ *separates* $H$ and $H'$.

Our goal in this section is to find all minimal signed separable bigraphs which are not chordal. As we know signed cycles in $C_{2k}$ with $k \geq 3$ (see Figure 5.8) are separable but not chordal, and deleting any vertex results in a chordal signed bigraph. So they are minimal signed separable bigraphs which are not chordal. The signed bigraphs in $D$ as depicted in Figure 5.8 are another collection of such bigraphs.

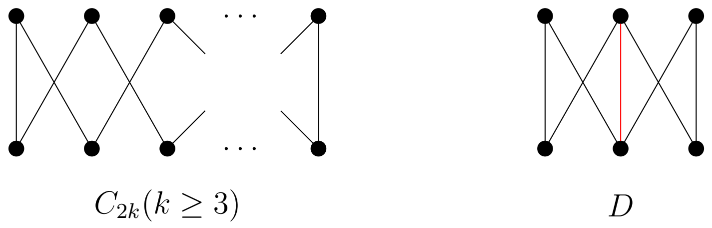


Figure 5.8: Forbidden subgraphs $C_{2k}$ and $D$ for chordal signed separable bigraphs

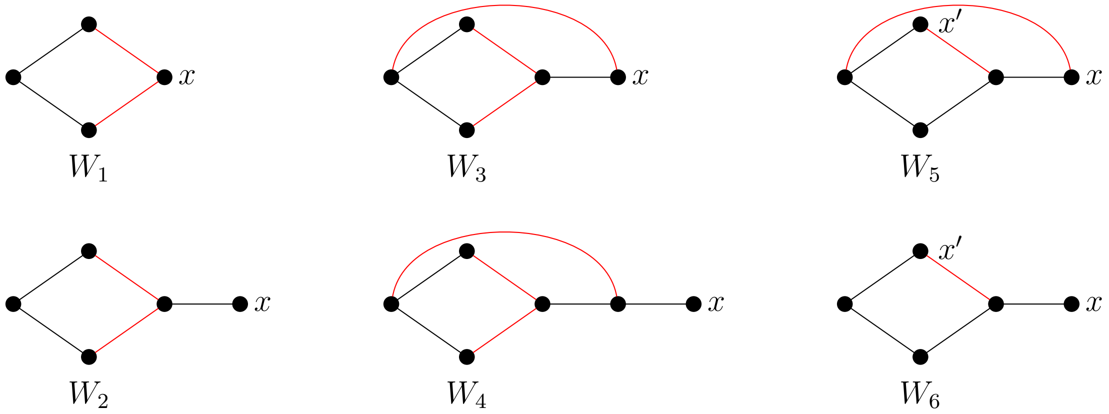
Denote $\mathcal{C} = \cup_{k \geq 3} C_{2k}$, that is, $\mathcal{C}$ consists of signed cycles of length $\geq 6$.

**Lemma 5.6.** *Let $\widehat{G}$ be a signed separable bigraph which does not contain a signed graph in $\mathcal{C} \cup D$ in Figure 5.8 as an induced subgraph. Suppose that $S$ is a minimal separating set which separates $H$ and $H'$. Then*

- *$S$ induces a positive biclique in $\widehat{G}$, and*

- *any two vertices in $S$ from the same partite set of $\widehat{G}$ have a common neighbour in $H$ and a common neighbour in $H'$.*

*Proof.* Let $u, u'$ be two vertices in $S$. Since $S$ is a minimal separating set that separates $H$ and $H'$, each of $u, u'$ has a neighbour in $H$ and a neighbour in $H'$. Let $P$ (respectively, $Q$) be a shortest path in $H$ (respectively, $H'$) from a vertex in $N(u)$ (respectively, $N(u')$) to a vertex in $N(u')$ (respectively, $N(u)$). If $u, u'$ are from different partite sets of $\widehat{G}$, then $u, u'$ must be adjacent and the lengths of $P, Q$ are both equal to one, as otherwise $uPu'Qu$, $uPu'u$, or $u'Quu'$ is an induced cycle of length $\geq 6$ in $\widehat{G}$, contradicting the assumption. Since $\widehat{G}$ does not contain a graph in $D$ as an induced subgraph, the edge $uu'$ must be positive. Hence $S$ induces a positive biclique in $\widehat{G}$. Similarly, if $u, u'$ are from the same partite set of $\widehat{G}$, then $uPu'Qu$ is an induced cycle of length $\geq 6$, unless $P$ and $Q$ each has length 0, that is, $u, u'$ have a common neighbour in $H$ and a common neighbour in $H'$. □



$W_1$ $W_3$ $W_5$

$W_2$ $W_4$ $W_6$

Figure 5.9: Graphs $W_i (1 \leq i \leq 6)$

**Lemma 5.7.** *Let $\widehat{G}$ be a chordal signed non-separable bigraph with bipartition $(X, Y)$. Suppose that $S$ is a set of vertices such that*

- *$S$ induces a positive biclique in $\widehat{G}$,*

- *every vertex of $S$ has a neighbour in $\widehat{G} - S$, and*
- *$\widehat{G} - S$ is connected and contains at least one edge but none of them is a signed simplicial edge of $\widehat{G}$.*

*Then $\widehat{G}$ contains one of the graphs in $W_1, \ldots, W_6$ in Figure 5.9 as an induced subgraph where vertices $x, x'$ are the only ones in $S$ (and rest are in $\widehat{G} - S$). In particular, when $\widehat{G}$ is a complete bigraph, it contains one of the graphs in $W_1, W_3, W_5$ as an induced subgraph.*

*Proof.* Consider first the case when $G$ is complete. Since there are edges which are not signed simplicial in $\widehat{G}$, there is at least one negative edge. We claim that there is a negative edge between $S$ and $\widehat{G} - S$. Indeed, there is no negative edge has its both endvertices in $S$ as $S$ induces a positive biclique in $\widehat{G}$. Since $\widehat{G}$ is chordal it has a signed simplicial edge. Any signed simplicial edge must be adjacent to all negative edges. Since $\widehat{G} - S$ contains no signed simplicial edge of $\widehat{G}$, every signed simplicial edge has an endvertex in $S$. If there is no negative edge between $S$ and $\widehat{G} - S$ then any signed simplicial edge must have an endvertex in $\widehat{G} - S$ incident with all negative edges. It follows that any negative edge is a signed simplicial edge in $\widehat{G}$ and lies in $\widehat{G} - S$, which contradicts the assumption. Hence $\widehat{G}$ has a negative edge between $S$ and $\widehat{G} - S$.

We know from above that some vertex $x \in S$ is incident with a negative edge. Suppose that $x$ is incident with two negative edges, that is, there are vertices $y', y''$ in $\widehat{G} - S$ such that $xy', xy''$ are both negative. Since $\widehat{G} - S$ is connected it contains a vertex $x'$ is the opposite partite set of $y', y''$. Then $x, x', y', y''$ induce a copy of $W_1$ in $\widehat{G}$ with $x$ being the only vertex in $S$. So assume that no vertex in $S$ incident with two negative edges. If some vertex $y \in S$ in the opposite partite set of $x$ which is adjacent to a negative edge, then the two negative edges incident with $x$ and $y$ must be adjacent to any signed simplicial edge of $\widehat{G}$. Thus $xy$ is signed simplicial. Together with the fact each of $x, y$ is incident with exactly one negative edge we conclude that the negative edges incident with $x, y$ are the only negative edges in $\widehat{G}$. This means that the edge in $\widehat{G} - S$ adjacent to the two negative edges is a signed simplicial edge in $\widehat{G}$, contradicting the assumption. Hence no vertex of $S$ in the opposite partite set of $x$ is incident with a negative edge. Suppose that $x$ is the only vertex in $S$ that is incident with a negative edge. Denote this edge by $xy$. Let $x'$ be any vertex in $\widehat{G} - S$ and in the same partite set as $x$. Since $x'y$ is not a signed simplicial edge there is a negative edge $x''y''$ in the graph induced by $N(x'y)$. Then $xy''$ is a signed simplicial edge and thus all negative edges other than $xy$ are incident with $y''$. Since $x''y$ lies in $\widehat{G} - S$, it is not a signed simplicial edge in $\widehat{G}$. There must be a vertex $x'''$ (which may be $x'$ but not $x''$) such that $x'''y''$ is negative. We see that $x, x'', x''', y, y''$ induce a copy of $W_3$ in $\widehat{G}$ with $x$ being the only vertex in $S$. Finally suppose that $x$ is not the only vertex in $S$ that is incident with a negative edge. If all negative edges between $S$ and $\widehat{G} - S$ are incident with the same vertex $y$ then $y$ must be incident with any signed simplicial edge. It follows that $y$ is incident with all negative edges. But then any edge of $\widehat{G} - S$ incident with $y$ is a signed simplicial edge in $\widehat{G}$, a contradiction. Hence there exist vertices $x, x' \in S$ incident with negative edges $xy, x'y'$ where $y \neq y'$. Therefore $x, y, x', y'$ together with any vertex of $\widehat{G} - S$ in the same partite set as $x$ induce a copy of $W_5$ in $\widehat{G}$ with $x, x'$ being the only vertices in $S$.

Consider next the case when $G$ is non-separable but not complete. Let

$$\prec\colon\ x_1, x_2, \ldots, x_\alpha, y_1, y_2, \ldots, y_\beta$$

be a canonical ordering of $G$. Since $G$ is not complete, $x_\alpha$ and $y_1$ are not adjacent and hence they are not both in $S$. By renaming the vertices if necessary we assume that $y_1$ is in $\widehat{G} - S$. Since $\widehat{G} - S$ is connected and has at least one edge, it contains edge $x_ky_1$ for some $k$. Choose such an edge with $k$ as small as possible. The canonical ordering $\prec$ ensures that $N(x_i) \supseteq N(x_k)$ for all $i < k$. Thus $x_iy_1$ is an edge for all $i < k$. The choice of $k$ implies that $x_i \in S$ for all $i < k$. According to Lemma 5.3(3) $x_ky_1$ is a simplicial edge in $G$ but by assumption is not a signed simplicial edge in $\widehat{G}$. So the subgraph of $\widehat{G}$ induced by $N(x_ky_1)$ contains a negative edge $x_ay_b$.

**Case 1.** $x_a \in S$ and $y_b \notin S$.

The edge $x_ky_b$ is an edge in $\widehat{G} - S$ and by assumption is not signed simplicial. That is, the subgraph of $\widehat{G}$ induced by $N(x_ky_b)$ is either not a biclique or is a biclique and contains a negative edge.

**Subcase 1.1.** The subgraph induced by $N(x_ky_b)$ is a biclique.

Let $x_cy_d$ be a negative edge where $x_c \in N(y_b)$ and $y_d \in N(x_k)$. Since the subgraph induced by $N(x_ky_b)$ is a biclique and $x_a \in N(y_b)$, $N(x_k) \subseteq N(x_a)$ which implies that $x_ay_d$ is an edge. Suppose that $x_c \in S$. Since $S$ induces a positive biclique and $x_cy_d$ is a negative edge, $y_d \notin S$. When $x_c = x_a$, $\widehat{G}$ contains a graph in $W_1$ induced by $x_a, x_k, y_b, y_d$ with $x_a$ being the only vertex in $S$; when $x_c \neq x_a$, $\widehat{G}$ contains a graph in $W_5$ induced by $x_a, x_c, x_k, y_b, y_d$ with $x_a, x_c$ being the only vertices in $S$.

Suppose now that $x_c \notin S$. Since the subgraph induced by $N(x_ky_b)$ is a biclique, $x_c \in N(y_b)$ and $y_1 \in N(x_k)$, we have $N(x_k) \subseteq N(x_c)$ and $x_cy_1$ is an edge. The choice of $x_k$ thus implies that $k < c$. Moreover, the canonical ordering $\prec$ implies that $N(x_c) \subseteq N(x_k)$. The edge $x_cy_b$ is in $\widehat{G} - S$ and by assumption is not signed simplicial. So the subgraph graph induced by $N(x_cy_b)$ is either not a biclique or a biclique and contains a negative edge. That is, there exist $x_e \in N(y_b)$ and $y_f \in N(x_c)$ such that $x_ey_f$ is either not an edge or is a negative edge. We claim that $x_ey_f$ is an edge (and hence a negative edge). Indeed, using the fact $N(x_ky_b)$ is a biclique, we have $y_f \in N(x_c) \subseteq N(x_k) \subseteq N(x_e)$, which $x_ey_f$ is an edge and hence a negative edge. Note that $x_ay_f$ is also an edge as $y_f \in N(x_k) \subseteq N(x_a)$.

If $x_e = x_a$ then $y_f \notin S$ and $\widehat{G}$ contains a graph in $W_1$ induced by $x_a, x_k, y_b, y_f$ with $x_a$ being the only vertex in $S$. So assume that $x_e \neq x_a$. Suppose that $x_e = x_k$. If $y_f \neq y_d$ then $\widehat{G}$ contains a graph in $F_4$ in Figure 5.3 induced by $x_a, x_c, x_e, y_b, y_d, y_f$. This means that $\widehat{G}$ is not chordal signed, a contradiction to assumption. So $y_f = y_d$. If $y_d \in S$ then $\widehat{G}$ contains a graph in $W_1$ induced by $x_c, x_e, y_d, y_b$ with $y_d$ being the only vertex in $S$; if $y_d \notin S$ then $\widehat{G}$ contains a graph in $W_3$ induced by $x_a, x_c, x_e, y_b, y_d$ with $x_a$ being the only vertex in $S$.

Suppose that $x_e \neq x_k$. If $y_f \neq y_d$, $\widehat{G}$ contains a graph in $F_4$ in Figure 5.3 induced by $x_a, x_c, x_e, y_b, y_d, y_f$, a contradiction to the assumption that $\widehat{G}$ is chordal. So $y_f = y_d$. If $y_d \in S$ then $\widehat{G}$ contains a graph in $W_1$ induced by $x_c, x_e, y_1, y_d$ with $y_d$ being the only vertex in $S$; if $y_d \notin S$ and $x_e \notin S$ then $\widehat{G}$ contains a graph in $W_3$ induced by $x_a, x_c, x_e, y_b, y_d$ with $x_a$ being the only vertex in $S$; if $y_d \notin S$ and $x_e \in S$ then $\widehat{G}$ contains a graph in $W_5$ induced by $x_a, x_k, x_e, y_b, y_d$ with $x_a, x_e$ being the only vertices in $S$.

**Subcase 1.2.** The subgraph induced by $N(x_k y_b)$ is not a biclique.

Let $x_c \in N(y_b)$ and $y_d \in N(x_k)$ be chosen so that $x_c, y_d$ are not adjacent and $c$ is as large as possible. By Lemma 5.3(5), $c > k$ and $d < b$. Since $y_d \in N(x_k) \subseteq N(x_a)$, $x_a y_d$ is an edge. The canonical ordering $\prec$ ensures that $c > a$.

If $x_c \in S$ then $\widehat{G}$ contains a graph in $W_6$ induced by $x_a, x_k, x_c, y_1, y_b$ with $x_a, x_c$ being the only vertices in $S$. So assume that $x_c \notin S$. Then the edge $x_c y_b$ is in $\widehat{G} - S$ and by assumption is not signed simplicial. If there exist $x_e \in N(y_b)$ and $y_f \in N(x_c)$ such that $x_e, y_f$ are not adjacent then $e > c$ and $f < b$ by Lemma 5.3 (5). Since $y_f \in N(x_c) \subseteq N(x_k)$, $x_k y_f$ is an edge. Thus $x_e, y_f$ is a pair of non-adjacent vertices with $x_e \in N(y_b)$, $y_f \in N(x_k)$ and $e > c$, which contradicts to the choice of the pair $x_c, y_d$. Hence the subgraph induced by $N(x_c y_b)$ is a biclique and contains a negative edge.

Let $x_ey_f$ be a negative edge where $x_e \in N(y_b)$ and $y_f \in N(x_c)$. Suppose that $x_ey_d$ is an edge. If $x_e = x_a$ then $\widehat{G}$ contains a graph in $W_1$ induced by $x_a, x_k, y_b, y_f$ with $x_a$ being the only vertex in $S$; if $x_e \neq x_a$ then $\widehat{G}$ contains a graph in $F_5$ in Figure 5.3 induced by $x_a, x_c, x_e, y_b, y_d, y_f$, a contradiction to the assumption that $\widehat{G}$ is chordal. So assume now that $x_ey_d$ is not an edge. We must have $e < c$ by the choice of the pair $x_c, y_d$. If $x_e \in S$ then $\widehat{G}$ contains a graph in $W_6$ induced by $x_a, x_k, x_e, y_1, y_b$ with $x_a, x_e$ being the only vertices in $S$. So assume that $x_e \notin S$. Then the edge $x_ey_b$ is in $\widehat{G} - S$ and by assumption is not signed simplicial. Then there exist $x_g \in N(y_b)$ and $y_h \in N(x_e)$ such that $x_gy_h$ is either not an edge or a negative edge.

Suppose that $x_gy_h$ is not an edge. Note that $x_ky_h$ is an edge as $y_h \in N(x_e) \subseteq N(x_k)$. By the choice of $x_c, y_d$, $g \leq c$. Note also that $x_gy_f$ is an edge (as $N(x_cy_b)$ is a biclique). So $y_f \neq y_h$. Then $\widehat{G}$ contains a graph in $F_5$ in Figure 5.3 induced by $x_a.x_g, x_e, y_b, y_f, y_h$, a contradiction to the assumption that $\widehat{G}$ is chordal.

Suppose that $x_gy_h$ is a negative edge. Note that $x_gy_f$ is an edge (as $N(x_cy_b)$ is a biclique). If $x_g = x_a$ then $\widehat{G}$ contains a graph in $W_1$ induced by $x_a, x_k, y_b, y_h$ with $x_a$ being the only vertex in $S$. So assume that $x_g \neq x_a$. If $y_h \neq y_f$ then $\widehat{G}$ contains a graph in $F_4$ in Figure 5.3 induced by $x_a, x_e, x_g, y_b, y_f, y_h$, a contradiction. Thus $y_h = y_f$. If $y_f \in S$ then $\widehat{G}$ contains a graph in $W_1$ induced by $x_g, x_e, y_b, y_f$ with $y_f$ being the only vertex in $S$; if $y_f \notin S$ and $x_g \notin S$ then $\widehat{G}$ contains a graph in $W_3$ induced by $x_a, x_e, x_g, y_b, y_f$ with $x_a$ being the only vertex in $S$; if $y_f \notin S$ and $x_g \in S$, then $\widehat{G}$ contains a graph in $W_5$ induced by $x_a, x_c, x_g, y_b, y_f$ with $x_a, x_g$ being the only vertices in $S$.

**Case 2.** $x_a \notin S$ and $y_b \in S$.

Then $x_ay_1$ is an edge in $\widehat{G} - S$ and thus not a signed simplicial edge. By Lemma 5.3(3), $N(x_ay_1)$ is a biclique in $G$. Hence there is a negative edge $x_cy_d$ with $x_c \in N(y_1)$ and $y_d \in N(x_a)$. If $y_b = y_d$ then $\widehat{G}$ contains a graph in $W_1$ induced by $x_a, x_c, y_1, y_b$

with $y_b$ being the only vertex in $S$. So assume that $y_b \neq y_d$. If $y_d \in S$ then $x_c \notin S$ and $\widehat{G}$ contains a graph in $W_5$ induced by $x_a, x_c, y_1, y_b, y_d$ with $y_b, y_d$ being the only vertices in $S$. Hence we can assume from now on that $y_d \notin S$. This means that $x_a y_d$ is in $\widehat{G} - S$ and by assumption is not a signed simplicial edge. Then there exist $x_e \in N(y_d)$ and $y_f \in N(x_a)$ such that either $x_e y_f$ is not an edge or a negative edge when the subgraph induced by $N(x_a y_d)$ is a biclique.

Suppose that $x_e y_f$ is a negative edge in it. If $y_f = y_b$ then $x_e \notin S$ as $x_e y_b$ is a negative edge. We see that $\widehat{G}$ contains a graph in $W_1$ induced by $x_a, x_e, y_1, y_b$ with $y_b$ being the only vertex in $S$. So assume that $y_f \neq y_b$. If $x_c \neq x_e$ then $\widehat{G}$ contains a graph in $F_4$ in Figure 5.3 induced by $x_a, x_c, x_e, y_b, y_d, y_f$, a contradiction to the assumption that $\widehat{G}$ is chordal. Hence $x_c = x_e$. If $x_c \in S$ then $\widehat{G}$ contains a graph in $W_1$ induced by $x_a, x_c, y_d, y_f$ with $x_c$ being the only vertex in $S$; if $x_c \notin S$ and $y_f \notin S$ then $\widehat{G}$ contains a graph in $W_3$ induced by $x_a, x_c, y_b, y_d, y_f$ with $y_b$ being the only vertex in $S$; if $x_c \notin S$ and $y_f \in S$ (note $y_f \neq y_1$) then $\widehat{G}$ contains a graph in $W_5$ induced by $x_a, x_c, y_1, y_b, y_f$ with $y_b, y_f$ being the only vertices in $S$.

Suppose now $x_e y_f$ is not an edge. We may assume such a pair of non-adjacent vertices $x_e, y_f$ in $N(x_a y_d)$ is chosen so that $e$ is the largest. Note that $x_e y_b$ cannot be an edge as otherwise $\widehat{G}$ contains a graph in $F_5$ induced by $x_a, x_c, x_e, y_b, y_d, y_f$, which is a contradiction. Since $x_e, y_b$ are not adjacent and $y_b \in S$, $x_e \notin S$. Then $x_e y_d$ is in $\widehat{G} - S$ and not a signed simplicial edge. We claim that $N(x_e y_d)$ is a biclique. Indeed, for any $x_g \in N(y_d)$, we must have $g < e$ as otherwise $x_g, y_f$ would be a pair in $N(x_a y_d)$ with $g > e$, contradicting the choice of $x_e, y_f$. Combining this with Lemma 5.3(5) we conclude that $N(x_e y_d)$ is a biclique containing a negative edge $x_g y_h$.

Suppose that $x_g y_b$ is an edge. If $x_g \neq x_c$ then $\widehat{G}$ contains a graph in $F_5$ induced by $x_c, x_e, x_g, y_b, y_d, y_h$, a contradiction. So $x_g = x_c$. If $x_c \in S$ then $\widehat{G}$ contains a graph in $W_1$ induced by $x_a, x_c, y_d, y_h$. Hence assume that $x_c \notin S$. If $y_h \notin S$ then $\widehat{G}$ contains a graph in $W_2$ induced by $x_c, x_e, y_b, y_d, y_h$ with $y_b$ being the only vertex in $S$; if $y_h \in S$ then $\widehat{G}$ contains a graph in $W_5$ induced by $x_a, x_c, y_1, y_b, y_h$ with $y_b, y_h$ being the only vertices in $S$.

Suppose $x_g, y_b$ are not adjacent. Then $x_g \notin S$ (as $y_b \in S$). Thus $x_g y_d$ is in $\widehat{G} - S$ and is not a signed simplicial edge. Let $x_i, y_j$ be a pair of vertices in $N(x_g y_d)$ such that either they are not adjacent or forming a negative edge. Note that $y_j \neq y_b$ as $x_g y_b$ is not an edge but $x_g y_j$ is an edge. Suppose that $x_i = x_a$. Then $x_a y_j$ is a negative edge and $\widehat{G}$ contains a graph in $F_5$ induced by $x_a, x_c, x_g, y_b, y_d, y_j$, a contradiction. So $x_i \neq x_a$.

Suppose that $x_i = x_c$. If $x_c \in S$ then $\widehat{G}$ contains a graph in $W_1$ induced by $x_a, x_c, y_d, y_j$ with $x_c$ being the only vertex in $S$; if $x_c \notin S$ and $y_j \notin S$ then $\widehat{G}$ contains a graph in $W_2$ induced by $x_g, x_c, y_b, y_d, y_j$ with $y_b$ being the only vertex in $S$; if $x_c \notin S$ and $y_j \in S$ then $\widehat{G}$ contains a graph in $W_5$ induced by $x_a, x_c, y_b, y_1, y_j$ with $y_b, y_j$ being the only vertices in $S$. Hence we can assume that $x_i \neq x_c$.

Suppose that $x_i y_b$ is an edge. If $y_j \neq y_b$ then $\widehat{G}$ contains a graph in $F_4$ or in $F_5$ induced by $x_a, x_c, x_i, y_b, y_d, y_j$ (depending whether or not $x_i, y_j$ are adjacent), a contradiction. So $y_j = y_b$ and $\widehat{G}$ contains a graph in $W_1$ induced by $x_a, x_i, y_b, y_d$ with $y_b$ being the only vertex in $S$. So we can assume that $x_i, y_b$ are not adjacent.

Suppose that $x_i \neq x_e$. The choice of $x_e, y_f$ implies that $i < e$ (as otherwise $x_i, y_f$ would have been chosen). Note that $x_i y_h$ is an edge as $y_h \in N(x_e) \subseteq N(x_i)$. Hence $\widehat{G}$ contains a graph in $F_4$ or in $F_5$ induced by $x_c, x_g, x_i, y_d, y_h, y_j$ (depending whether or not $x_i, y_j$ are adjacent), a contradiction.

Suppose that $x_i = x_e$. If $y_j \neq y_h$ then $\widehat{G}$ contains a graph in $F_4$ or in $F_5$ induced by $x_c, x_i, x_g, y_d, y_j, y_h$ (depending whether or not $x_i, y_j$ are adjacent), a contradiction. So $y_j = y_h$ and $\widehat{G}$ contains a graph in $W_4$ induced by $x_c, x_g, x_i, y_b, y_d, y_h$ with $y_b$ being the only vertex in $S$.

**Case 3.** $x_a \notin S$ and $y_b \notin S$.

By Lemma 5.3(3), the subgraph induced by $N(x_{k'}y_1)$ is a biclique in $G$ for any $k'$ (including the case when $k' = k$). In view of Cases 1 and 2, we can assume that no negative edge in this biclique has an endvertex in $S$.

Since $x_a y_1$ is in $\widehat{G} - S$, it is not signed simplicial. Since the subgraph induced by $N(x_a y_1)$ is a biclique in $G$, it contains a negative edge $x_c y_d$. By our assumption neither $x_c$ nor $y_d$ is in $S$.

**Subcase 3.1** $x_c = x_k$.

Since $x_k y_b$ is not signed simplicial, there exist $x_e \in N(y_b)$ and $y_f \in N(x_k)$ such that either $x_e y_f$ is not an edge or a negative edge when the subgraph induced by $N(x_k y_b)$ is a biclique.

Suppose that the subgraph induced by $N(x_k y_b)$ is a biclique and thus $x_e y_f$ is a negative edge. Since $x_e \in N(y_1)$, by Lemma 5.3(2), $x_e$ is adjacent to every vertex in $Y$. By our assumption above, $x_e$ is not in $S$. Assume first that $x_e \neq x_a$ and $y_b = y_d$.

Since $x_e y_b$ is an edge in $\widehat{G} - S$ and hence not signed simplicial, there exist $x_g \in N(y_b)$ and $y_h \in N(x_e)$ such that either $x_g y_h$ is not an edge or a negative edge in the biclique induced by $N(x_e y_b)$. We claim that the subgraph induced by $N(x_e y_b)$ is not a biclique; otherwise, $\widehat{G}$ contains a graph in $F_2$ induced by $x_a, x_c, x_e, y_f, y_b$ when $y_h = y_f$ and $x_g \in \{x_a, x_k\}$, or in $F_3$ induced by $x_a, x_c, x_e, x_g, y_b, y_f$ when $y_h = y_f$ and $x_g \notin \{x_a, x_k\}$, or in $F_4$ induced by $x_a, x_c, x_e, y_b, y_f, y_h$ when $y_h \neq y_f$ and $x_g \in \{x_a, x_k\}$, or in $F_4$ induced by $x_a, x_e, x_g, y_b, y_f, y_h$ when $y_h \neq y_f$ and $x_g \notin \{x_a, x_k\}$. So $x_g, y_h$ are not adjacent.

Denote $A = N(y_d) \setminus N(y_h)$. Then $A \neq \emptyset$ as $x_g \in A$. If $S$ contains a vertex $x_t \in A$ then $\widehat{G}$ contains $W_2$ induced by $x_k, x_a, x_t, y_1, y_d$ with $x_t$ being the only vertex in $S$. Otherwise $S$ contains no vertex of $A$. That is, for any $x_t \in A$, $x_t y_d$ is not a signed simplicial edge. Then $\widehat{G}$ contains a graph in $Z_1$ induced by $x_k, x_a, x_t, y_1, y_d$. Applying

Lemma 5.4 to the subgraph of $\widehat{G}$ induced by $A \cup \{x_k, x_a, y_1, y_d\}$ we conclude that $\widehat{G}$ contains one of the graphs in $F_2, \ldots, F_5$, a contradiction.

Assume now that $x_e \neq x_a$ and $y_b \neq y_d$. Then $y_f = y_d$ as otherwise $\widehat{G}$ contains a graph in $F_4$ induced by $x_a, x_e, x_k, y_b, y_d, y_f$, a contradiction. Since $x_a y_d$ is in $\widehat{G} - S$ it is not signed simplicial. We claim that the subgraph induced by $N(x_a y_d)$ is not a biclique. Indeed, if it is then it contains a negative edge $x_g y_h$. If $y_h \neq y_b$ and $x_k = x_g$ then $\widehat{G}$ contains a graph in $F_4$ induced by $x_a, x_e, x_g, y_b, y_f, y_h$; if $y_h \neq y_b$ and $x_k \neq x_g$ then $\widehat{G}$ contains a graph in $F_4$ induced by $x_a, x_k, x_g, y_b, y_d, y_h$; if $y_h = y_b$, and $x_g = x_k$ (or $x_g = x_e$), then $\widehat{G}$ contains a graph in $F_2$ induced by $x_a, x_k, x_e, y_b, y_d$; if $y_h = y_b$, and $x_g \neq x_k$ (and $x_g \neq x_e$) then $\widehat{G}$ contains a graph in $F_3$ induced by $x_a, x_k, x_e, x_g, y_b, y_d$, contradictions. So there are non-adjacent vertices $x_g \in N(y_d)$ and $y_h \in N(x_a)$.

If $S$ contains a vertex $x_t \in A$ then $\widehat{G}$ contains a graph in $W_2$ induced by $x_k, x_e, x_t, y_1, y_d$ with $x_t$ being the only vertex in $S$. Otherwise $S$ contains no vertex of $A$. That is, for any $x_t \in A$, $x_t y_d$ is not a signed simplicial edge. Then $\widehat{G}$ contains a graph in $Z_1$ induced by $x_k, x_e, x_t, y_1, y_d$. Applying Lemma 5.4 to the subgraph of $\widehat{G}$ induced by $A \cup \{x_k, x_e, y_1, y_d\}$ we conclude that $\widehat{G}$ contains one of the graphs in $F_2, \ldots, F_5$, a contradiction.

Assume now that $x_e = x_a$. Note that $x_a y_d$ is not signed simplicial. We claim that the subgraph induced by $N(x_a y_d)$ is not a biclique. Indeed, if it is then it contains a negative edge $x_g y_h$. If $y_b = y_d$, $x_g = x_k$ and $y_f = y_h$, then $\widehat{G}$ contains $F_1$ induced by $x_a, x_k, y_b, y_h$; if $y_b = y_d$, $x_g = x_k$ and $y_f \neq y_h$, then $\widehat{G}$ contains a graph in $F_2$ induced by $x_a, x_k, y_b, y_f, y_h$; if $y_b = y_d$, $x_g \neq x_k$ and $y_h = y_f$, then $\widehat{G}$ contains a graph in $F_2$ induced $x_a, x_g, x_k, y_b, y_f$; if $y_b = y_d$, $x_g \neq x_k$ and $y_h \neq y_f$, then $\widehat{G}$ contains a graph in $F_4$ induced by $x_a, x_k, x_g, y_b, y_f, y_h$; if $y_b \neq y_d$ and $x_g \neq x_k$, then $\widehat{G}$ contains a graph in $F_4$ induced by $x_a, x_k, x_g, y_b, y_d, y_h$; if $y_b \neq y_d$, $x_g = x_k$ and $y_h = y_b$ (or $y_h = y_f$), then $\widehat{G}$ contains a graph in $F_2$ induced by $x_a, x_k, y_b, y_d, y_f$; if $y_b \neq y_d$, $x_g = x_k$ and

$y_h \neq y_b$ (and $y_h \neq y_f$), then $\widehat{G}$ contains a graph in $F_3$ induced by $x_a, x_k, y_b, y_d, y_f, y_h$, contradictions. So there exist non-adjacent vertices $x_g \in N(y_d)$ and $y_h \in N(x_a)$.

Suppose that $y_b = y_d$. A similar proof as above (with the same definition of the set $A$) shows that $\widehat{G}$ contains a graph in $W_2$ induced by $x_a, x_c, x_t, y_1, y_d$ when $S$ contains a vertex $x_t \in A$ with $x_t$ being the only vertex in $S$; otherwise, $\widehat{G}$ contains one of the graphs in $F_2, \ldots, F_5$ as an induced subgraph, a contradiction.

Suppose that $y_b \neq y_d$. Then $x_g$ is adjacent to neither of $y_b, y_f$ as otherwise $\widehat{G}$ would contain a graph in $F_5$ induced by $x_a, x_k, x_g, y_b, y_d, y_h$ or by $x_a, x_k, x_g, y_d, y_f, y_h$. If there exists a vertex $x_t \in N(y_d) \setminus N(y_h)$ that is adjacent to $y_f$ or $y_b$ then $\widehat{G}$ contains a graph in $F_5$ induced by $x_a, x_k, x_t, y_d, y_f, y_h$ or by $x_a, x_k, x_t, y_d, y_b, y_h$. So assume that none of vertices in $N(y_d) \setminus N(y_h)$ is adjacent to $y_f$ or $y_b$. Again, a similar proof as above (with the same definition of the set $A$ but using Lemma 5.5) shows that $\widehat{G}$ contains $W_4$ induced by $x_a, x_k, x_t, y_b, y_d, y_f$ when $S$ contains $x_t \in A$ with $x_t$ being the only vertex in $S$; otherwise, $\widehat{G}$ contains one of the graphs in $F_3, \ldots, F_6$, a contradiction. Here we apply Lemma 5.5 (instead of Lemma 5.4) to the subgraph of $\widehat{G}$ induced by $A \cup \{x_a, x_k, y_b, y_d, y_f\}$ and verify that it contains a graph in $Z_2$.

Suppose now that $x_e y_f$ is not an edge. Since $x_a y_1$ is an edge the canonical ordering implies that $x_a y_f$ is an edge. It follows that $x_e \neq x_a$.

Suppose that $y_b = y_d$. We must have $x_e \in S$ as otherwise a similar proof as above with the set $A = N(y_d) \setminus N(y_f)$ shows that $\widehat{G}$ contains a graph in $Z_1$ induced by $x_a, x_c, x_e, y_f, y_b$. By Lemma 5.4 $\widehat{G}$ contains one of the graphs in $F_2, \ldots, F_5$ as an induced subgraph, a contradiction. Hence $\widehat{G}$ contains a graph in $W_2$ induced by $x_a, x_k, x_e, y_1, y_b$ with $x_e$ being the only vertex in $S$.

Suppose now that $y_b \neq y_d$. Then $x_e y_d$ is not an edge as otherwise $\widehat{G}$ contains a graph in $F_5$ induced by $x_a, x_k, x_e, y_b, y_d, y_f$, a contradiction. Since $x_a y_d$ is an edge in $\widehat{G} - S$, it is not signed simplicial, Then there exist $x_g \in N(y_d)$ and $y_h \in N(x_a)$ such that either $x_g y_h$ is not an edge or a negative edge when the subgraph induced by $N(x_a y_d)$ is a biclique. Suppose that $x_g y_h$ is not an edge. Note that $x_g y_d$ is an edge but $x_e y_d$ is not so, by the canonical ordering, $g < e$ and hence that $x_g y_b$ is an edge but $x_e y_h$ is not an edge. Clearly, $x_g \neq x_a$ or $x_k$. We see that $\widehat{G}$ contains a graph in $F_5$ induced by $x_a, x_k, x_g, y_b, y_d, y_h$, a contradiction. Hence the subgraph induced by $N(x_a y_d)$ is a biclique (and thus $x_g y_h$ is a negative edge). Since $x_g \in N(y_d)$ and $y_1 \in N(x_a)$, $x_g y_1$ is an edge. The canonical ordering then implies $x_g$ is adjacent to every vertex in $Y$. In the case when $x_g = x_k$, we must have that $x_e, y_h$ are not adjacent as otherwise $\widehat{G}$ contains a graph in $F_5$ induced by $x_a, x_k, x_e, y_b, y_d, y_h$, a contradiction. Moreover, we must also have $x_e \in S$, as otherwise $\widehat{G}$ contains a graph in $Z_2$ induced by $x_a, x_k, x_e, y_b, y_d, y_h$. (Here we use a similar proof as above with $A = N(y_b) \setminus N(y_f)$.) By Lemma 5.5 $\widehat{G}$ contains one of the graphs in $F_3, \ldots, F_6$ as an induced subgraph, again a contradiction. We see now that $\widehat{G}$ contains a graph in $W_4$ induced by $x_a, x_e, x_k, y_b, y_d, y_h$ with $x_e$ being the only vertex in $S$. In the case when $x_g \neq x_k$, we must have $y_h = y_b$ as otherwise $\widehat{G}$ contains a graph in $F_4$ induced by $x_a, x_k, x_g, y_b, y_d, y_h$, a contradiction. Further, we must have $x_e \in S$ as otherwise $\widehat{G}$ contains a graph in $Z_1$ induced by $x_a, x_g, x_e, y_f, y_b$, and by Lemma 5.4 $\widehat{G}$ contains one of the graphs in $F_2, \ldots, F_5$ as an induced subgraph, again a contradiction. Therefore $\widehat{G}$ contains a graph in $W_2$ induced by $x_a, x_e, x_g, y_1, y_b$ with $x_e$ being the only vertex in $S$.

**Subcase 3.2.** $x_c \neq x_k$.

We know from above that $x_a y_d$ is not signed simplicial. Thus there exist $x_e \in N(y_d)$ and $y_f \in N(x_a)$ such that either $x_e y_f$ is not an edge or is a negative edge in the case when the subgraph induced by $N(x_a y_d)$ is a biclique.

Suppose that the latter case occurs and thus $x_e y_f$ is a negative edge. So $x_e, y_f \notin S$ according to the assumption at the beginning of Case 3. Since the subgraph induced by $N(x_a y_d)$ is a biclique and $x_e \in N(y_d)$, $N(x_a) \subseteq N(x_e)$ which implies that $x_e$ is adjacent to every vertex in $Y$.

Note that $x_e y_b$ is not a signed simplicial edge as it is in $\widehat{G} - S$. There exist $x_g \in N(y_b)$ and $y_h \in N(x_e)$ such that either $x_g y_h$ is not an edge or a negative edge in the biclique induced by $N(x_e y_b)$.

Assume that $y_b = y_d$. We show by contradiction that the subgraph induced by $N(x_e y_b)$ is not a biclique. So assume that the subgraph induced by $N(x_e y_b)$ is a biclique and $x_g y_h$ is a negative edge. Clearly, $x_g$ is adjacent to every vertex in $Y$. If $y_h \neq y_f$ and $x_e \neq x_c$ then $\widehat{G}$ contains a graph in $F_4$ induced by $x_a, x_e, x_g, y_b, y_f, y_h$ when $x_g = x_c$ or by $x_c, x_e, x_g, y_b, y_f, y_h$ when $x_g \neq x_c$; if $y_h \neq y_f$ and $x_e = x_c$ then $\widehat{G}$ contains a graph in $F_2$ induced by $x_a, x_c, y_f, y_b, y_h$ when $x_g = x_a$ or contains a graph in $F_4$ induced by $x_a, x_e, x_g, y_b, y_f, y_h$ when $x_g \neq x_a$; if $y_h = y_f$ and $x_e = x_c$ then $\widehat{G}$ contains $F_1$ induced by $x_a, x_c, y_b, y_f$ when $x_g = x_a$ or contains a graph in $F_2$ induced by $x_a, x_c, x_g, y_b, y_f$ when $x_g \neq x_a$; if $y_h = y_f$ and $x_e \neq x_c$ then $\widehat{G}$ contains a graph in $F_2$ induced by $x_a, x_c, x_e, y_b, y_f$ when $x_g = x_a$ or $x_g = x_c$, or contains a graph in $F_3$ induced by $x_a, x_c, x_e, x_g, y_b, y_f$ when $x_g \neq x_a$ and $x_g \neq x_c$. Hence $x_g y_h$ is not an edge. Clearly, $x_g \notin \{x_a, x_c, x_k\}$. Therefore $\widehat{G}$ contains a graph in $W_2$ induced by $x_a, x_c, x_t, y_1, y_b$ with $x_t$ being the only vertex in $S$.

Assume now that $y_b \neq y_d$. We separate this in two cases. First we consider the case when $x_c = x_e$. Since $x_e y_b$ is not signed simplicial, there exist $x_g \in N(y_b)$ and $y_h \in N(x_e)$ such that either $x_g y_h$ is not an edge or is a negative edge in the biclique induced by $N(x_e y_b)$. We claim that the subgraph induced by $N(x_e y_b)$ is not a biclique. Suppose not. Then $x_g y_h$ is a negative edge. If $y_f = y_b$ and $y_h = y_d$ then $\widehat{G}$ contains $F_1$ induced by $x_a, x_c, y_b, y_d$ when $x_g = x_a$ or a graph in $F_2$ induced by $x_a, x_e, x_g, y_b, y_d$ when $x_g \neq x_a$; if $y_f = y_b$ and $y_h \neq y_d$ then $\widehat{G}$ contains a graph in $F_2$ induced by $x_a, x_c, y_b, y_d, y_h$ when $x_g = x_a$ or in $F_4$ induced by $x_a, x_c, x_g, y_b, y_d, y_h$ when $x_g \neq x_a$;

if $y_f \neq y_b$ and $y_h = y_d$ then $\widehat{G}$ contains a graph in $F_2$ induced by $x_a, x_c, y_b, y_f, y_h$ when $x_g = x_a$ or in$F_4$ induced by $x_a, x_e, x_g, y_b, y_f, y_h$ when $x_g \neq x_a$; if $y_f \neq y_b$ and $y_h \neq y_d$ then $\widehat{G}$ contains a graph in $F_4$ induced by $x_a, x_c, x_g, y_b, y_d, y_h$ when $x_g \neq x_a$ or in $F_3$ induced by $x_a, x_c, y_b, y_d, y_f, y_h$ when $x_g = x_a$ and $y_f \neq y_h$ or in $F_2$ induced by $x_a, x_c, y_b, y_d, y_f$ when $x_g = x_a$ and $y_f = y_h$. Hence the subgraph induced by $N(x_e y_b)$ is not a biclique, that is, $x_g, y_h$ are not adjacent.

We claim that $x_g$ is in $S$ and adjacent to neither of $y_d, y_f$. Indeed, $\widehat{G}$ contains a graph in $F_5$ induced by $x_a, x_c, x_g, y_b, y_d, y_h$ when $x_g$ is adjacent to $y_d$ or induced by $x_a, x_c, x_g, y_b, y_f, y_h$ when $x_g$ is adjacent to $y_f$, contradicting the assumption. If $x_g \notin S$ then a similar proof as above (with $A = N(y_b) \setminus (N(y_d) \cup N(y_f)))$ shows that $\widehat{G}$ contains a graph in $Z_2$ and hence by Lemma 5.5 one of the graphs in $F_3, \ldots, F_6$ as an induced subgraph, a contradiction. Therefore $\widehat{G}$ contains a graph in $W_4$ induced by $x_a, x_c, x_g, y_b, y_d, y_f$ with $x_g$ being the only vertex in $S$.

We consider now the case when $x_c \neq x_e$. We must have $y_f = y_b$ as otherwise $\widehat{G}$ contains a graph in $F_4$ induced by $x_a, x_c, x_e, y_b, y_d, y_f$, a contradiction. Note that $x_c y_b$ is not a signed simplicial edge as it is in $\widehat{G} - S$. There exist $x_g \in N(y_b)$ and $y_h \in N(x_c)$ such that either $x_g y_h$ is not an edge or a negative edge in the biclique induced by $N(x_c y_b)$.

We claim that the subgraph induced by $N(x_c y_b)$ is not a biclique. Suppose that it is. Then $x_g y_h$ is a negative edge. If $y_h = y_d$ then $\widehat{G}$ contains a graph in $F_2$ induced by $x_a, x_c, x_e, y_b, y_d$ when $x_g = x_a$ or $x_g = x_e$ or in $F_3$ induced by $x_a, x_c, x_e, x_g, y_b, y_d$ when $x_g \neq x_a$ and $x_g \neq x_e$; if $y_h \neq y_d$ then $\widehat{G}$ contains a graph in $F_4$ induced by $x_c, x_e, x_g, y_b, y_d, y_h$ when $x_g = x_a$ or induced by $x_a, x_c, x_g, y_b, y_d, y_h$ when $x_g \neq x_a$. Hence the subgraph induced by $N(x_c y_b)$ is not a biclique and $x_g, y_h$ are not adjacent.

We must have $x_g \in S$, otherwise $\widehat{G}$ contains a graph in $Z_1$ and hence one of the graphs in $F_2, \ldots, F_5$ as an induced subgraph, a contradiction. Therefore $\widehat{G}$ contains a graph in $W_2$ induced by $x_a, x_e, x_g, y_1, y_b$ with $x_g$ being the only vertex in $S$.

Suppose now that $x_e, y_f$ are not adjacent. We separate this in two cases. First assume that $y_b = y_d$. Then $x_e \in S$ as otherwise $\widehat{G}$ contains a graph in $Z_1$ and hence one of the graphs in $F_2, \ldots, F_5$ as an induced subgraph, a contradiction. (Here again we use a similar proof with $A = N(y_d) \setminus N(y_f)$.) Therefore $\widehat{G}$ contains a graph in $W_2$ induced by $x_a, x_c, x_e, y_1, y_b$ with $x_e$ being the only vertex in $S$. Assume now that $y_b \neq y_d$. We must have that $x_e, y_b$ are not adjacent as otherwise $\widehat{G}$ contains a graph in $F_5$ induced by $x_a, x_c, x_e, y_b, y_d, y_f$, a contradiction. Since $x_c y_b$ is not a signed simplicial edge, there exist $x_g \in N(y_b)$ and $y_h \in N(x_c)$ such that either $x_g y_h$ is not an edge or a negative edge in the biclique induced by $N(x_c y_b)$. We claim that $N(x_c y_b)$ induces a biclique. Suppose not; $x_g, y_h$ are not adjacent. Note $g < e$ because $y_b \in N(x_g) \setminus N(x_e)$, which implies $x_g y_d$ is an edge. But then $\widehat{G}$ contains a graph in $F_5$ induced by $x_a, x_c, x_g, y_b, y_d, y_h$, a contradiction. Therefore $x_g y_h$ is a negative edge. Note that $x_g$ is adjacent to every vertex in $Y$.

If $y_h = y_d$ then $x_e \in S$ as otherwise $\widehat{G}$ contains a graph in $Z_1$ and hence one of the graphs in $F_2, \ldots, F_5$ as an induced subgraph, a contradiction. Hence $\widehat{G}$ contains a graph in $W_2$ induced by $x_c, x_e, x_g, y_1, y_d$ with $x_e$ being the only vertex in $S$. So assume $y_h \neq y_d$. If $x_e y_h$ is an edge then $\widehat{G}$ contains a graph in $F_5$ induced by $x_c, x_e, x_g, y_b, y_d, y_h$, a contradiction. So $x_e, y_h$ are not adjacent. Then we must have $x_e \in S$ as otherwise $\widehat{G}$ contains a graph in $Z_2$ and hence one of the graphs in $F_3, \ldots, F_6$ as an induced subgraph, a contradiction. Therefore $\widehat{G}$ contains a graph in $W_4$ induced by $x_a, x_c, x_e, y_b, y_d, y_h$ with $x_e$ being the only vertex in $S$. This completes the proof. □

**Lemma 5.8.** *Let $\widehat{G}$ be a chordal signed separable bigraph. Suppose that $S$ is a minimal separating set which separates $\widehat{H_1}$ and $\widehat{H_2}$. Then $\widehat{H_1}$ or $\widehat{H_2}$ must contain a signed simplicial edge of $\widehat{G}$.*

*Proof.* Since $\widehat{G}$ is chordal, the subgraph $\widehat{G'}$ of $\widehat{G}$ induced by $S \cup V(\widehat{H_1}) \cup V(\widehat{H_2})$ is also chordal and hence contains a signed simplicial edge $e$. As each vertex of $S$ has a neighbour in $\widehat{H_1}$ and a neighbour in $\widehat{H_2}$, neither of the endvertices of $e$ is in $S$. Since no edge has one endvertex in $\widehat{H_1}$ and the other in $\widehat{H_2}$, $e$ is an edge of $\widehat{H_1}$ or of $\widehat{H_2}$. Hence the neighbourhood of $e$ in $\widehat{G'}$ is equal to the neighbourhood of $e$ in $\widehat{G}$, which means that $e$ is a signed simplicial edge of $\widehat{G}$. $\square$

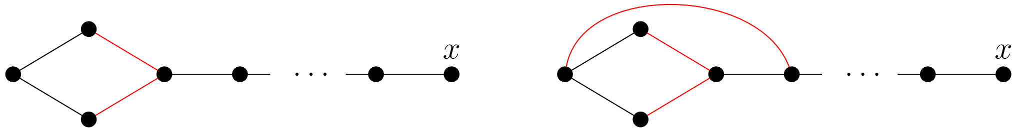


Figure 5.10: Lollipops

We shall call the graphs in Figure 5.10 *lollipops*, and refer to each graph on the left as a lollipop of type 1 and each graph on the right as a lollipop of type 2. We call the vertex $x$ the *end* of the lollipop (see Figure 5.10). Note that the graphs in $W_1 \cup W_2 \cup W_3 \cup W_4$ in Figure 5.9 are lollipops. A graph in $W_5$ is a lollipop only when it is in $W_3$ and a graph in $W_6$ is a lollipop only when it is in $W_2$.

**Lemma 5.9.** *Let $\widehat{G}$ be a connected chordal signed separable bigraph. Suppose that no two signed simplicial edges of $\widehat{G}$ induce a $2K_2$. Then, for any signed simplicial edge $e$ of $\widehat{G}$, there is an induced lollipop whose end is incident with $e$.*

*Proof.* We prove the lemma by induction on the number of vertices of $\widehat{G}$. Note that $\widehat{G}$ contains at least six vertices and one can verify easily that the lemma is true when $\widehat{G}$ has exactly six vertices. So assume that $\widehat{G}$ has at least seven vertices and the lemma is true for all connected chordal signed separable bigraphs with fewer vertices than $\widehat{G}$.

Let $S$ be a minimal separating set of $\widehat{G}$ that separates $\widehat{H_1}$ and $\widehat{H}_2$. By Lemma 5.8, a signed simplicial edge $e$ of $\widehat{G}$ is contained in $\widehat{H_1}$ or in $\widehat{H}_2$. We assume without loss of generality that $e$ is in $\widehat{H_1}$. Let $\widehat{G_1}$ (respectively, $\widehat{G_2}$) be the subgraph of $\widehat{G}$ induced by $V(\widehat{H_1}) \cup S$ (respectively, $V(\widehat{H_2}) \cup S$). Note that every signed simplicial edge of $\widehat{G_2}$ must have an endvertex in $S$, as otherwise it is a signed simplicial edge of $\widehat{G}$ which forms an induced $2K_2$ with $e$, contradicting the assumption. Since $\widehat{H_1}$ has at least two vertices, $\widehat{G_2}$ has at most $|V(\widehat{G})| - 2$ vertices. We prove that it is possible to choose such a set $S$ so that $\widehat{G_2}$ is non-separable or has at most $|V(\widehat{G})| - 3$ vertices.

For the sake of proof we assume that $\widehat{G_2}$ is separable and has $|V(\widehat{G})| - 2$ vertices. Then $\widehat{H_1}$ consists of $e$ only and $e$ is the only signed simplicial edge of $\widehat{G}$. Since $\widehat{G_2}$ is separable, it contains two edges $e'_1, e'_2$ forming an induced $2K_2$. Let $S'$ be a minimal separating set of $\widehat{G}$ that separates $\widehat{H'_1}$ and $\widehat{H'_2}$ which contain $e'_1$ and $e'_2$ respectively. By Lemma 5.8 we can assume that $\widehat{H'_1}$ contains $e$. Thus $\widehat{H'_1}$ contains both $e$ and $e'_1$, which means it has at least four vertices and hence the subgraph of $\widehat{G}$ induced by $V(\widehat{H'_2}) \cup S'$ has at most $|V(\widehat{G})| - 4$ vertices. Therefore we can assume that $\widehat{G_2}$ is non-separable or has at most $|V(\widehat{G})| - 3$ vertices.

Suppose that $\widehat{G_2}$ is non-separable. Then by Lemma 5.7, $\widehat{G_2}$ contains a graph $W \in W_1 \cup W_2 \cup \cdots \cup W_6$ (see Figure 5.9) as an induced subgraph with the vertices $x, x'$ being the only vertices in $S$. We claim that $W \notin W_5 \cup W_6$. Indeed, suppose to the contrary that $W \in W_5 \cup W_6$ with $x, x'$ being the only vertices in $S$. Then $x, x'$ are the only vertices in $W$ and by Lemma 5.6 they have a common neighbour $y$ in $\widehat{H_1}$. Hence the subgraph of $\widehat{G}$ induced by $V(W) \cup \{y\}$ is a graph in $D$ if $W \in W_5$, and a graph in $F_5$ if $W \in W_6$, contradicting the assumption that $\widehat{G}$ is chordal. Therefore $W \in W_1 \cup W_2 \cup W_3 \cup W_4$. If $W \in W_1 \cup W_2$, then $W$ together with a shortest path connecting $x$ and $e$ in $\widehat{G}_1$ induce a lollipop of type 1. If $W \in W_3 \cup W_4$, then $W$ together with a shortest path connecting $x$ and $e$ in $\widehat{G}_1$ induce a lollipop of type 2. In either case the end of the lollipop is incident with the signed simplicial edge $e$.

Suppose now that $\widehat{G_2}$ is separable. Our assumption above ensures that it has at most $|V(\widehat{G})| - 3$ vertices. Since $S$ is a minimal separating set that separates $\widehat{H_1}$ and $\widehat{H_2}$, every vertex of $S$ has a neighbour in $\widehat{H_1}$ and a neighbour in $\widehat{H_2}$. Consider two vertices $s, s'$ of $S$ from the same partite set. By Lemma 5.6, $s, s'$ have a common neighbour in $\widehat{H_2}$ and hence their neighbourhood in $\widehat{H_1}$ are comparable, as otherwise $\widehat{G}$ contains an induced cycle of length $\geq 6$, a contradiction to the assumption that $\widehat{G}$ is chordal. This implies that the vertices of $S$ of the same partite set have a common neighbour in $\widehat{H_1}$. Let $u, v$ be a pair of vertices in $\widehat{H_1}$ where $u$ is a common neighbour of the vertices of $S$ in one partite set and $v$ is a common neighbour of the vertices of $S$ in the other partite set. (If all vertices of $S$ are in one partite set then let $v$ be any vertex in $\widehat{H_1}$ adjacent to $u$). Let $\widehat{G^*}$ be the graph obtained from the subgraph of $\widehat{G}$ induced by $V(\widehat{G_2}) \cup \{u, v\}$ by adding an edge $uv$ (either positive or negative) if $uv$ is not an edge of $\widehat{G}$. Since the neighbourhood of $uv$ is $S$ which forms a positive biclique, $uv$ is a signed simplicial edge of $\widehat{G^*}$. The graph graph $\widehat{G^*}$ is chordal. Indeed, if $uv$ is an edge of $\widehat{G}$, then $\widehat{G^*}$ is an induced subgraph of $\widehat{G}$; if $uv$ is not an edge of $\widehat{G}$ then $\widehat{G^*} - uv$ is an induced subgraph of $\widehat{G}$. Since $\widehat{G_2}$ has at most $|V(\widehat{G})| - 3$ vertices, $\widehat{G^*}$ has fewer vertices than $\widehat{G}$. Since no two signed simplicial edges of $\widehat{G}$ induce a $2K_2$, no two signed simplicial edges of $\widehat{G^*}$ induce a $2K_2$. Hence, by the inductive hypothesis, $\widehat{G^*}$ contains an induced lollipop $L$ with its end incident with $uv$. It is easy to verify that $L - \{u, v\}$ is an induced lollipop in $\widehat{G_2}$ with its end being the only vertex in $S$. The shortest path between the end of $L - \{u, v\}$ and $e$ together with $L - \{u, v\}$ is an induced lollipop of $\widehat{G}$ with its end incident with $e$. This completes the proof. $\square$

**Corollary 5.2.** *Let $\widehat{G}$ be a signed separable bigraph and $S$ be a minimal separating set that separates $\widehat{H_1}$ and $\widehat{H_2}$. Let $\widehat{G_1}$ (respectively, $\widehat{G_2}$) be the subgraph of $\widehat{G}$ induced by $V(\widehat{H_1}) \cup S$ (respectively, $V(\widehat{H_2}) \cup S$). If either of $\widehat{G_1}$ or $\widehat{G_2}$ contains a graph in $W_5 \cup W_6$ as an induced subgraph with $x, x'$ being the only vertices in $S$ then $\widehat{G}$ contains a graph in $F_5 \cup D$ as an induced subgraph.* $\square$

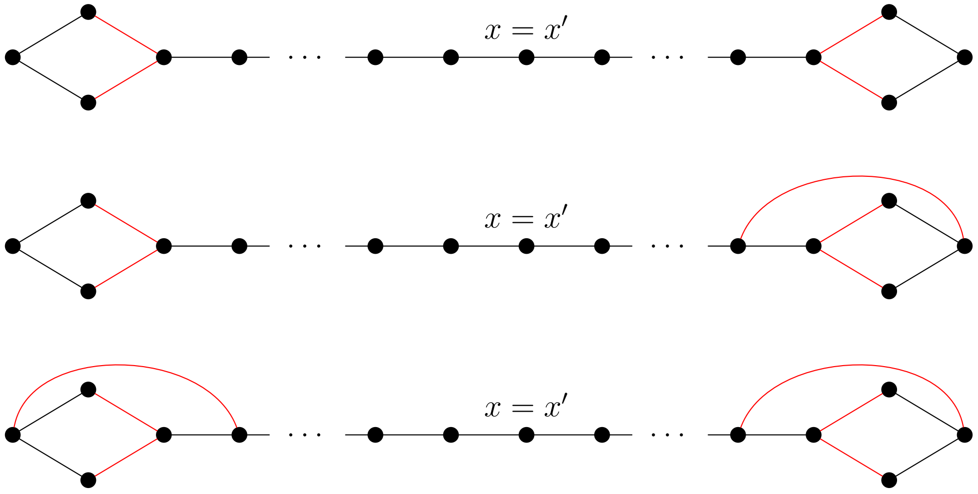


Figure 5.11: Sum of two lollipops

For a lollipop $L$ and a vertex $u$ not in $L$, we say that $u$ and $L$ are *completely adjacent* (or $u$ is *completely adjacent to* $L$) if $u$ is adjacent to all vertices of $L$ in the partite set opposite to $u$.

Let $L$ and $L'$ be two lollipops with ends $x$ and $x'$ respectively. The *sum* of $L$ and $L'$ is the signed bigraph obtained from $L$ and $L'$ by identifying $x$ and $x'$. Figure 5.11 depicts all possible sums of lollipops. The *join* of $L$ and $L'$ is the signed bigraph obtained from $L$ and $L'$ by

- adding a complete adjacency between $x$ and $L'$ with positive edges, except when $L'$ is $W_1$ in which case $xx'$ is an added edge and the other edge between $x$ and the vertex at distance 2 from $x'$ in $L'$ may be either positive or negative, and

- adding a complete adjacency between $x'$ and $L$ with positive edges, except when $L$ is $W_1$ in which case $xx'$ is an added edge and the other edge between $x'$ and the vertex at distance 2 from $x$ in $L$ may be either positive or negative.

Figure 5.12 depicts all joins of lollipops $L, L'$ of type 1 and there is a complete adjacency with positive edges between $x$ and $L'$ and between $x'$ and $L$. Figure 5.13 depicts all joins of lollipops $L$ of type 1 and $L'$ of type 2 and there is a complete adjacency with positive edges between $x$ and $L'$ and between $x'$ and $L$. Figure 5.14 depicts all joins of lollipops $L, L'$ of type 2 and there is a complete adjacency with positive edges between $x$ and $L'$ and between $x'$ and $L$. Figure 5.15 depicts all joins of $L = W_1$ and $L'$ of type 1 (which may also be $W_1$) where the edge between $x'$ and the vertex at distance 2 from $x$ in $L$ is negative (in the case when $L = W_1 = L'$ the edge between $x$ and the vertex at distance 2 from $x'$ in $L'$ may also be negative as shown in the last one).

For convenience we label the vertices of lollipops as shown in Figure 5.16. Call the vertices $w, y, z$ in a lollipop the *heads* of the lollipop.

Denote by $\mathcal{S}$ (respectively, $\mathcal{J}$) the set of sums (respectively, joins) of lollipops, and $\mathcal{F} = F_1 \cup F_2 \cup \cdots \cup F_6 \cup \mathcal{C} \cup D \cup \mathcal{S} \cup \mathcal{J}$.

**Lemma 5.10.** *Let $\widehat{G}$ be a signed bigraph which does not contain any of the bigraphs in $\mathcal{F}$ as an induced subgraph. Let $L$ be an induced lollipop in $\widehat{G}$ whose vertices are labeled as in Figure 5.16 (with $x_1 = x$ being its end). Let $x'$ be a vertex not on $L$ such that*

- *either $xx'$ is an edge, or*
- *$x, x'$ have a common neighbour adjacent to no vertex of $L$ other than $x$.*

*If $x'$ is adjacent to a head of $L$ then $x'$ is completely adjacent to $L$ by positive edges, except when $x'x$ or $x'w$ is an edge in which case either one can be positive or negative.*

*Proof.* Suppose first that $xx'$ is an edge. Consider first the case when $k$ is odd. Since $x'$ is adjacent to a head of $L$, $x'w$ is an edge. Thus the statement holds when $k = 1$. So

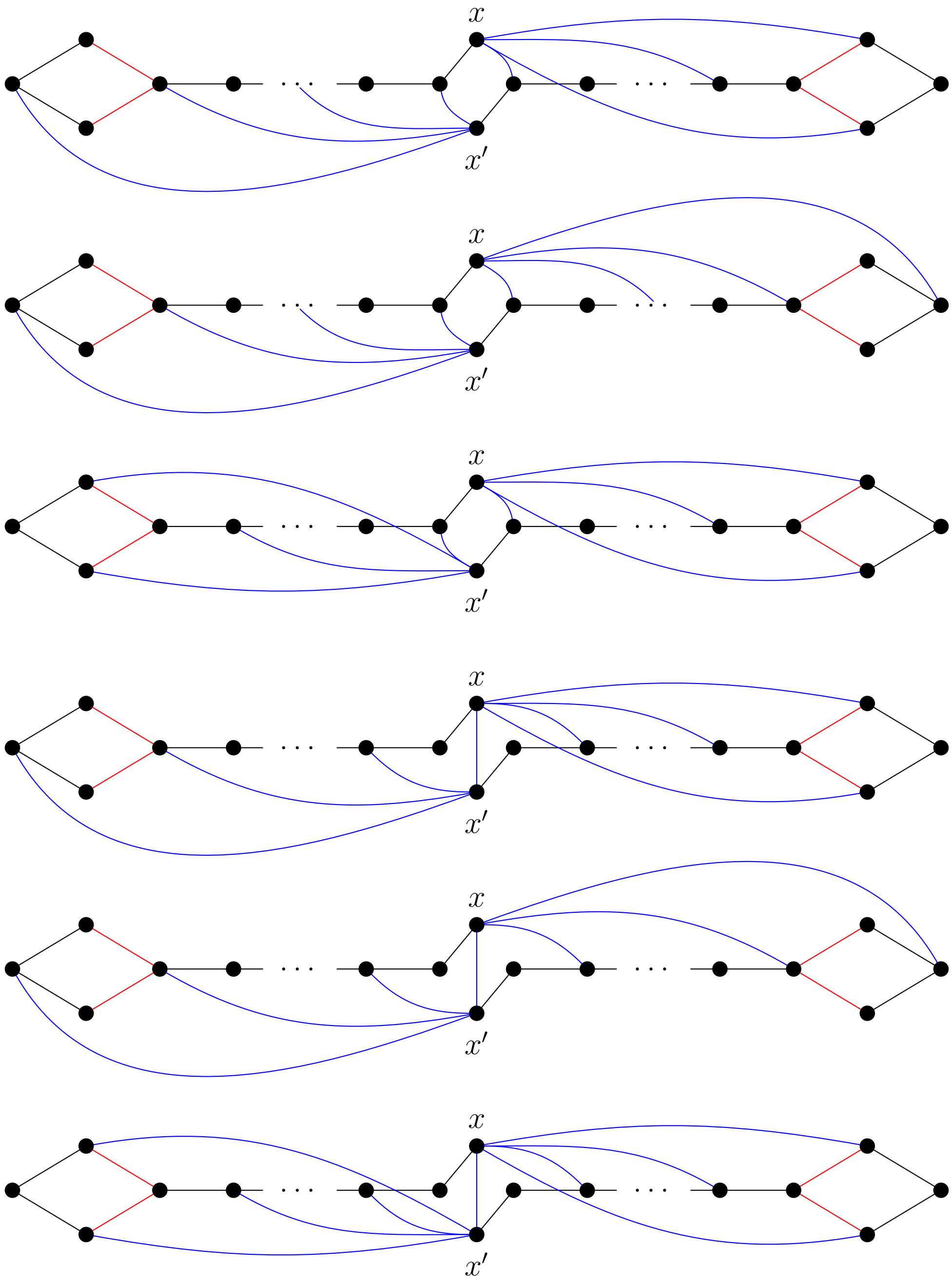


Figure 5.12: Join of two lollipops (of type 1)

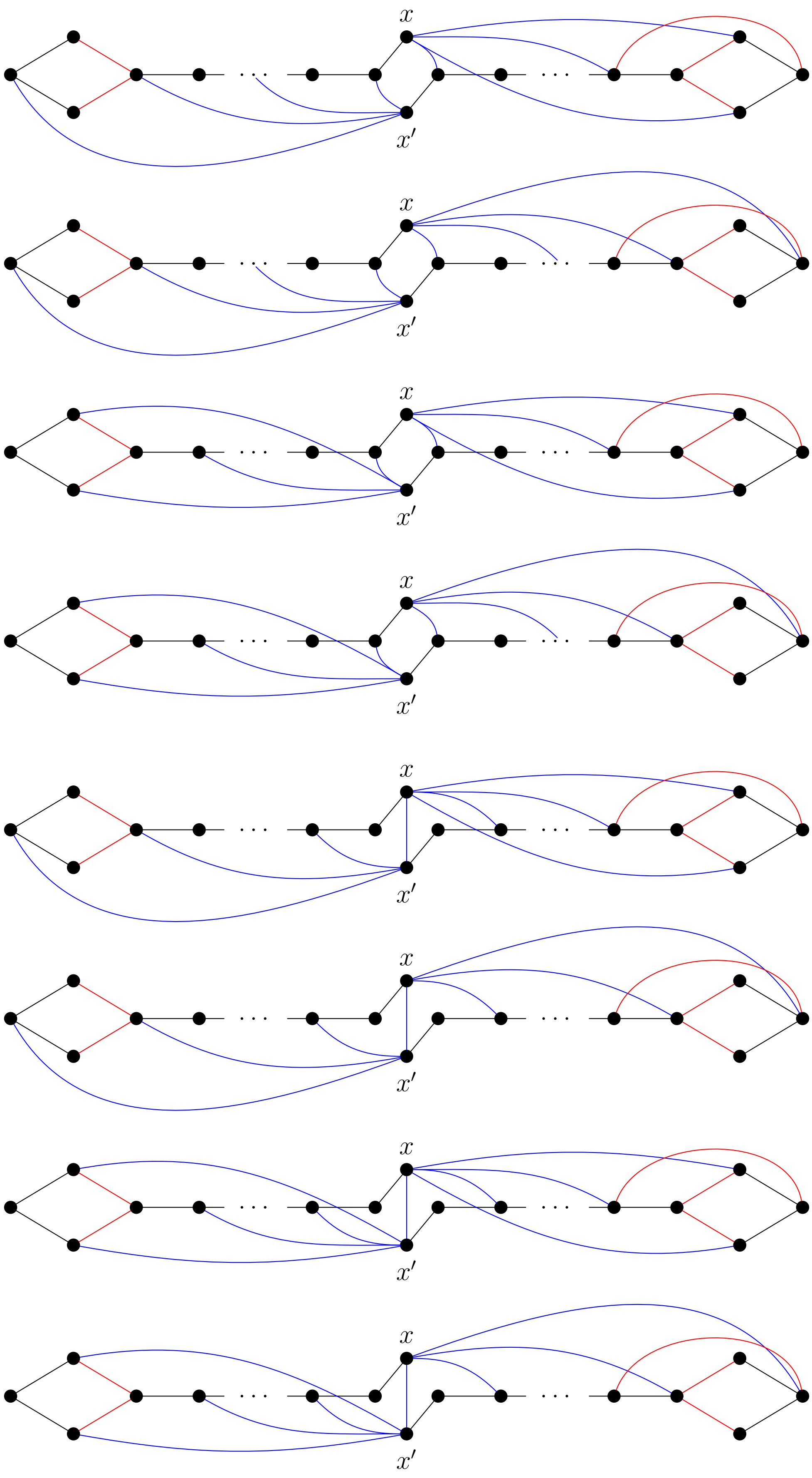


Figure 5.13: Join of two lollipops (type 1 and type 2)

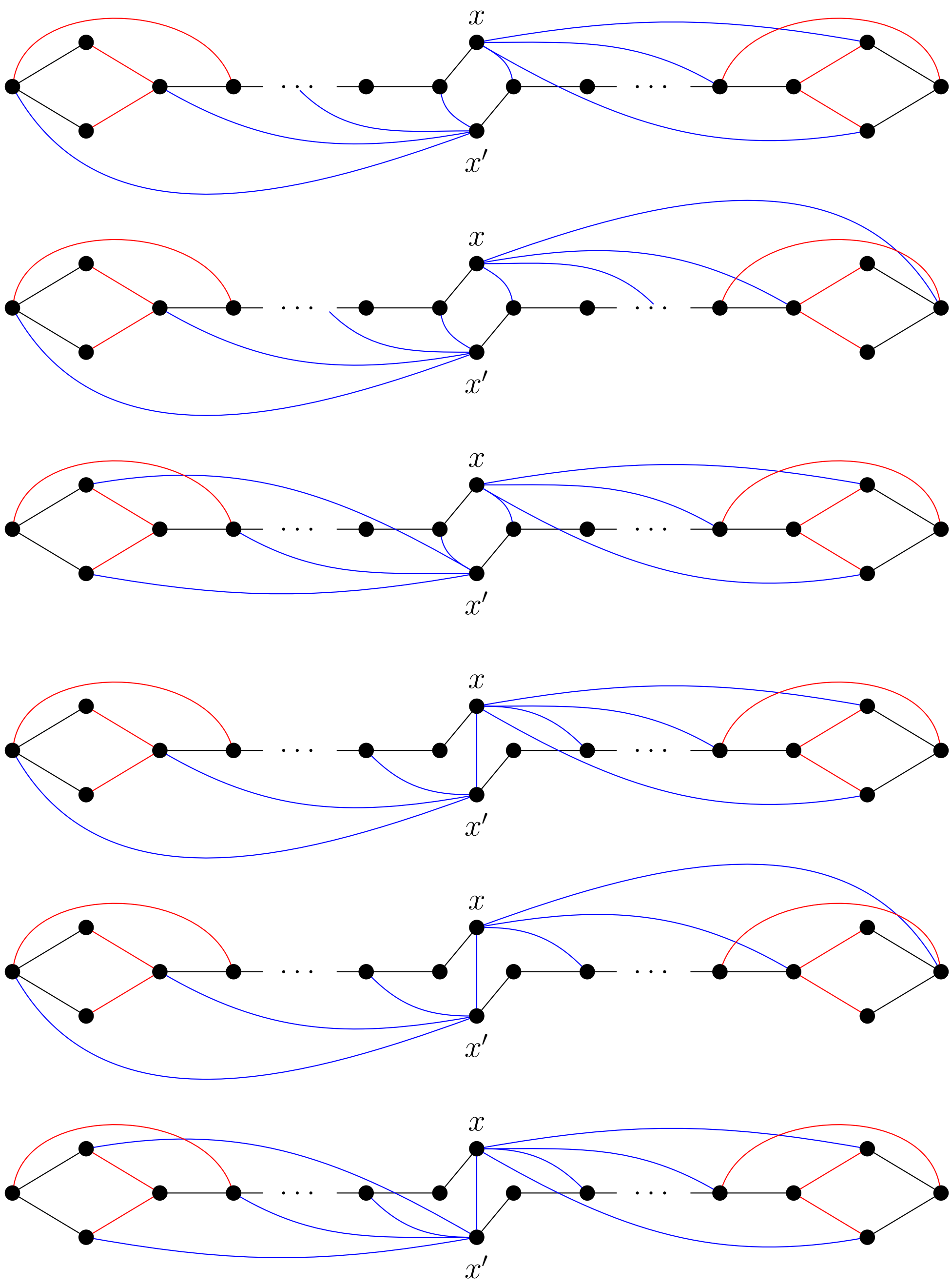


Figure 5.14: Join of two lollipops (of type 2)

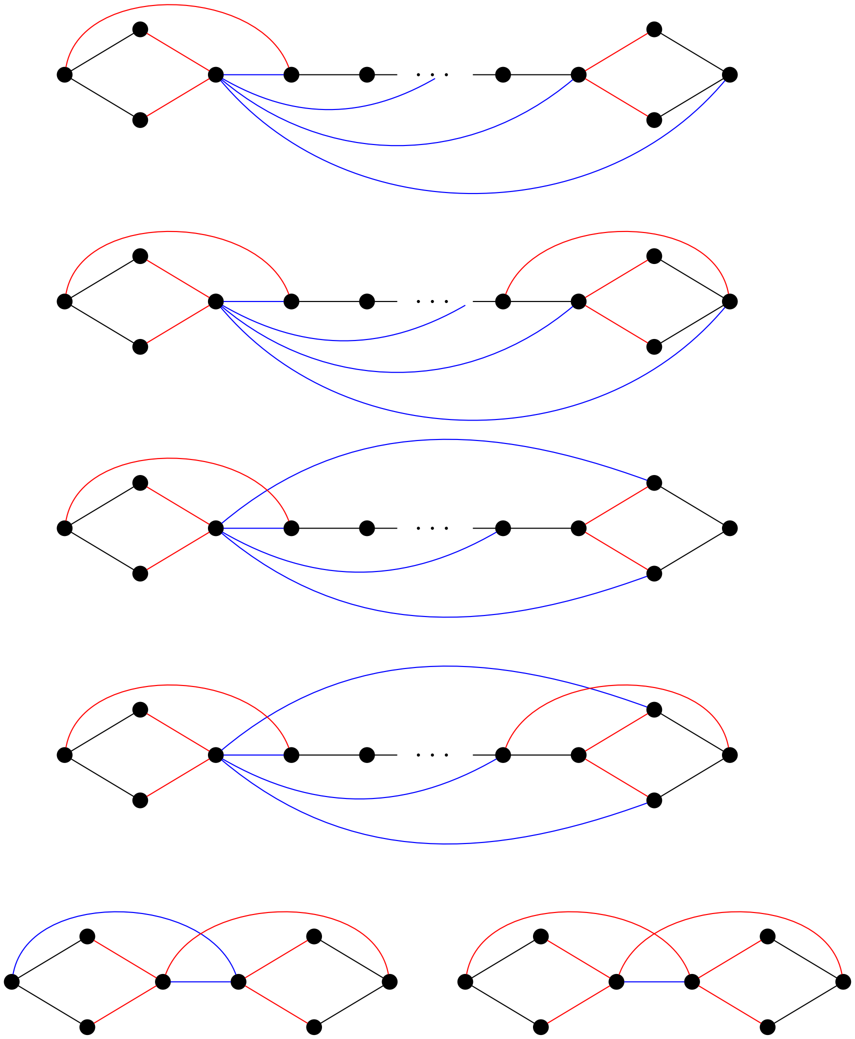


Figure 5.15: Join of lollipops when one of them is a $W_1$

assume that $k \geq 3$. Let $x'x_t$ be an edge with $t$ being the largest. We claim that $t = k$. Suppose not; $t < k$. When $L$ is of type 1, $x'x_t x_{t+1} \ldots x_k y w x'$ is an induced cycle of length $\geq 6$, a contradiction to the assumption. When $L'$ is of type 2, if $t = k-2$, then $x', x_{k-2}, x_{k-1}, x_k, y, w$ induce a graph in $D$, otherwise $t < k-2$, $x'x_t x_{t+1} \ldots x_{k-1} w x'$ is an induced cycle of length $\geq 6$, contradictions. Hence $x'x_k$ is an edge. We must have that $x'$ is completely adjacent to $L$ as otherwise there is a cycle of length $\geq 6$. We show that $x'x_i$ is positive for all $i \geq 2$. Suppose not; $x'x_i$ is negative for any $i$ with $i \geq 2$. If $i < k$ then $\widehat{G}$ contains a graph in $D$ induced by $x', x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}$. If

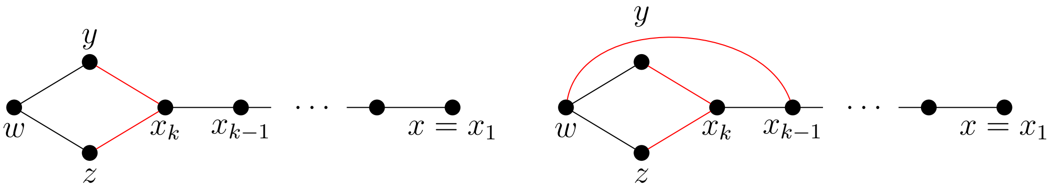


Figure 5.16: Labeled lollipops

$i = k$ then the subgraph of $\widehat{G}$ induced by $x', x_{k-2}, x_{k-1}, x_k, y, w$ is a graph in $D$ when $L$ is of type 1, or a graph in $F_5$ when $L$ is of type 2.

Consider then the case when $k$ is even. By assumption and the symmetry between $y$ and $z$ we assume that $x'$ is adjacent to $y$. Let $x'x_r$ be an edge with $r$ being the largest. We claim that $r = k - 1$ as otherwise $x'x_rx_{r+1} \ldots x_kyx'$ is an induced cycle of length $\geq 6$, a contradiction. The edge $x'z$ must be present in $\widehat{G}$ as otherwise $x', x_{k-1}, x_k, z, w, y$ induce a graph in $D$ if $L$ is of type 1, or a graph in $F_5$ if $L$ is of type 2. In fact, both $x'y, x'z$ must be positive or else the same six vertices $x', x_{k-1}, x_k, z, w, y$ induce a graph in $F_5$ when $L$ is of type 1, or a graph in $F_4$ when $L$ is of type 2. It is now easy to see that $x'$ is completely adjacent to $L$ by positive edges except possibly $x'x$.

Suppose now that $u$ is a common neighbour of $x, x'$ which is adjacent to no vertex of $L$ other than $x$. As above we consider two cases depending on the parity of $k$. Assume first that $k$ is odd. By the symmetry between $y$ and $z$ we assume that $x'$ is adjacent to $y$. In the case when $k = 1$, $x'z$ must be present and both $x'y, x'z$ are positive, as otherwise $u, x', x, y, z, w$ induce a graph in $D$ or in $F_5$. So $k \geq 3$. There must be a complete adjacency between $x'$ and $L$ by positive edges. Indeed, if $x'$ is not completely adjacent to $L$ by positive edges among the vertices $x_i$ then $\widehat{G}$ contains an induced cycle of length $\geq 6$ or an induced graph in $D$; if $x'$ is not adjacent to $z$ or either of $x'y, x'z$ is negative then $\widehat{G}$ contains an induced graph in $F_5 \cup D$ when $L$ is of type 1 and an induced graph in $F_4 \cup F_5$ when $L$ is of type 2.

Finally, assume that $k$ is even. Then the only head of $L$ adjacent to $x'$ is $w$. A similar argument as above shows that $x'$ is completely adjacent to $L$ by positive edges except possibly $x'w$ or else $\widehat{G}$ contains a graph in $\mathcal{C} \cup D \cup F_5$ as an induced subgraph. This completes the proof. $\square$

We are now ready to prove the main theorem of this chapter. Recall that

$$\mathcal{F} = F_1 \cup F_2 \cup \cdots \cup F_6 \cup \mathcal{C} \cup D \cup \mathcal{S} \cup \mathcal{J}.$$

We say that a signed graph is *$\mathcal{F}$-free* if it does not contain any graph in $\mathcal{F}$ as an induced subgraph. We make two remarks on the graphs in $\mathcal{F}$. First, since none of the graphs in $\mathcal{F}$ has a signed simplicial edge, none of them is chordal and hence can be an induced subgraph of a chordal signed bigraph. Second, if a signed bigraph $\widehat{G}$ is a $\mathcal{F}$-free signed bigraph and $e$ is a signed simplicial edge in $\widehat{G}$ then $\widehat{G} - e$ is again $\mathcal{F}$-free.

**Theorem 5.3.** *Let $\widehat{G}$ be a signed bigraph. Then $\widehat{G}$ is chordal if and only if it is $\mathcal{F}$-free.*

*Proof.* The necessity has been discussed above so we only prove the sufficiency. Suppose to the contrary that there exists an $\mathcal{F}$-free signed bigraph which is not chordal. Let $\widehat{G}$ be such a graph that is minimal with respect to vertex deletion, that is, deleting a vertex from $\widehat{G}$ results in a chordal signed bigraph. In view of the remarks above we can assume that $\widehat{G}$ contains no signed simplicial edge. Proposition 5.2 implies that $\widehat{G}$ is separable.

Let $S$ be a minimal separating set in $\widehat{G}$ that separates $\widehat{H_1}$ and $\widehat{H_2}$. Let $\widehat{G_1}$ (respectively, $\widehat{G_2}$) be the subgraph of $\widehat{G}$ induced by $V(\widehat{H_1}) \cup S$ (respectively, by $V(\widehat{H_2}) \cup S$). The minimality of $\widehat{G}$ ensures that both $\widehat{G_1}$ and $\widehat{G_2}$ are chordal (and hence each has a signed simplicial edge). Since $\widehat{G}$ does not have a signed simplicial edge, any signed simplicial edge of $\widehat{G_1}$ or $\widehat{G_2}$ must have an endvertex in $S$.

We show that $\widehat{G_1}$ and $\widehat{G_2}$ each contains an induced lollipop with its end being the only vertex in $S$. By symmetry we only prove it for $\widehat{G_1}$. If $\widehat{G_1}$ is non-separable, then, by Lemma 5.7 and Corollary 5.2, $\widehat{G_1}$ contains an induced lollipop in $W_1 \cup W_2 \cup W_3 \cup W_4$ with its end being the only vertex in $S$. Suppose that $\widehat{G_1}$ is separable. Since $\widehat{H_2}$ has an edge, it has at least two vertices. If $\widehat{H_2}$ has exactly two vertices then the (only) edge in $\widehat{H_2}$ is a signed simplicial edge of $\widehat{G}$, a contradiction to the choice of $\widehat{G}$. Thus $\widehat{H_2}$ has at least three vertices, which means that $\widehat{G_1}$ has at most $|V(\widehat{G})| - 3$ vertices. Let $u, v$ be a pair of vertices in $\widehat{H_2}$ where $u$ is a common neighbour of the vertices of $S$ in one partite set and $v$ is a common neighbour of the vertices of $S$ in the other partite set. (If all vertices of $S$ are in one partite set then let $v$ be any vertex in $\widehat{H_1}$ adjacent to $u$). Let $\widehat{G^*}$ be the graph obtained from the subgraph of $\widehat{G}$ induced by $V(\widehat{G_1}) \cup \{u, v\}$ by adding an edge $uv$ (either positive or negative) if $uv$ is not an edge of $\widehat{G}$. Since the neighbourhood of $uv$ is $S$ which forms a positive biclique, $uv$ is a signed simplicial edge of $\widehat{G^*}$. Either $\widehat{G^*}$ or $\widehat{G^*} - uv$ is an induced subgraph of $\widehat{G}$ with fewer vertices than $\widehat{G}$, $\widehat{G^*}$ is chordal and $uv$ is the only signed simplicial edge of $\widehat{G^*}$. By Lemma 5.9, $\widehat{G^*}$ contains an induced lollipop whose end is incident with $uv$. By deleting $u, v$ from the lollipop we obtain a lollipop $L$ in $\widehat{G_1}$ with its end being the only vertex in $S$. Similarly, $\widehat{G_2}$ contains an induced lollipop $L'$ with its end being the only vertex in $S$.

Let $x_1, x_2, \ldots, x_k, w, y, z$ be the vertices of $L$ as depicted in Figure 5.16. In a similar way, let $x'_1, x'_2, \ldots, x'_\ell, w', y', z'$ be the vertices of $L'$. Note that $x_1$ and $x'_1$ are the ends of $L$ and $L'$ respectively. We know from above that they are the only vertices in $S$.

We will show that $\widehat{G}$ contains a graph in $\mathcal{F}$ as an induced subgraph which contradicts the assumption that $\widehat{G}$ is $\mathcal{F}$-free. Note that any edge between $L$ and $L'$ is incident with at least one of $x_1$ and $x'_1$. If $x_1 = x'_1$, then $L$ and $L'$ together induce a graph in $\mathcal{S}$, contradicting the assumption that $\widehat{G}$ is $\mathcal{F}$-free. So $x_1 \neq x'_1$.

If $x'_1$ is not adjacent to any of $w, y, z$ and is adjacent to $x_j (j > 1)$, choose such $j$ to be the largest, then $L - \{x_1, x_2, \ldots, x_{j-1}\}$ and $L'$ together induce a graph in $\mathcal{S}$, a contradiction. So we may assume that $x'_1$ must be adjacent to one of $w, y, z$ if it is adjacent to $x_j$ for some $j > 1$. Similarly, $x_1$ must be adjacent to one of $w', y', z'$ if it is adjacent to $x'_j$ for some $j > 1$.

Suppose first that $x_1$ and $x'_1$ are in different partite set of $\widehat{G}$. Then they are joint by a positive edge as $S$ induces a positive biclique in $\widehat{G}$. If $x_1 x'_1$ is the only edge between $L$ and $L'$ then again $L$ and $L'$ induce a graph in $\mathcal{S}$, a contradiction. So there is at least one edge between $L$ and $L'$ other than $x_1 x'_1$.

Consider the case when the edges between $L$ and $L'$ are all incident with $x'_1$. From above we know that $x'_1$ is adjacent to one of $w, y, z$. By Lemma 5.10, $x'_1$ is completely adjacent to $L$ by positive edges, except possibly $x'_1 w$ (which may or may not be an edge). If $x'_1 w$ is a negative edge, then $L - \{x_1, x_2, \ldots, x_{k-1}\}$ and $L'$ induce a graph in $\mathcal{S}$, a contradiction. So $x'_1$ is completely adjacent to $L$ by positive edges.

Since $S$ is a minimal separating set that separates $\widehat{H_1}$ and $\widehat{H_2}$, $x_1$ has a neighbour in $\widehat{H_2}$. Let $u$ be a neighbour of $x_1$ in $\widehat{H_2}$. Suppose that $u$ is adjacent to a vertex in $L'$. Then we know from above that $u$ must be adjacent to one of $w', y', z'$. By Lemma 5.10, $u$ is completely adjacent to $L'$ by positive edges except possibly $uw'$. If $uw'$ is a negative edge, then $L$, $u$, and $L' - \{x'_1, x'_2, \ldots, x'_{\ell-1}\}$ form an induced a graph in $\mathcal{S}$. On the other hand if $u$ is completely adjacent to $L'$ by positive edges, then $L$, $u$, and $L'$ form an induced graph in $\mathcal{J}$. Hence no neighbour of $x_1$ in $\widehat{H_2}$ is adjacent to any vertex in $L'$. Let $u_1 u_2 \ldots u_t$ be a shortest path in $\widehat{H_2}$ from a neighbour of $x_1$ to $L' - x'_1$. Then $t \geq 3$. If $u_{t-1}$ is not adjacent to any of $w', y', z'$, then

$u_t = x'_j$ for some $j$. Choose such a $j$ to be the largest. Then $L$, $u_1, u_2, \ldots, u_{t-1}$, and $L' - \{x'_1, x'_2, \ldots, x'_{j-1}\}$ induce a graph in $\mathcal{S}$, a contradiction. Thus $u_{t-1}$ is adjacent to at least one of $w', y', z'$. Let $u'_t$ be the vertex in $L'$ closest to $x'_1$ which is a neighbour of $u_{t-1}$. Then $x_1x'_1x'_2 \ldots u'_tu_{t-1}u_{t-2} \ldots u_1x_1$ is a cycle of length $\geq 6$. This cycle cannot be induced as $\widehat{G}$ is $\mathcal{C}$-free. Since any chord in the cycle is incident with $x'_1$, $x'_1$ is completely adjacent to $u_1u_2 \ldots u_{t-1}u'_t$. In particular, the distance between $x'_1$ and $u'_t$ in $L'$ is at most 2. Since $\widehat{G}$ does not contain any graph in $D$ as a induced subgraph, all chords in the cycle are positive. Applying Lemma 5.10 to $L'$ and $u_{t-1}$ we conclude that $u_{t-1}$ is completely adjacent to $L'$ by positive edges except possibly $u_{t-1}w'$. If $u_{t-1}w'$ is a negative edge then $L$, $u_1, u_2, \ldots, u_{t-1}$, and $L' - \{x'_1, x'_2, \ldots, x'_{\ell-1}\}$ form an induced graph in $\mathcal{S}$. If $u_{t-1}$ is completely adjacent to $L'$ by positive edges then $L$, $u_1, u_2, \ldots, u_{t-1}$, and $L'$ form an induced graph in $\mathcal{J}$.

Suppose that there is an edge between $x_1$ and $L'$ (other than $x_1x'_1$) and an edge between $x'_1$ and $L$ (other than $x_1x'_1$). Again from above we know that $x_1$ is adjacent to at least one of $w', y', z'$ and that $x'_1$ is adjacent to at least one of $w, y, z$. By Lemma 5.10, $x_1$ is completely adjacent to $L'$ by positive edges except possibly $x_1w'$, and $x'_1$ is completely adjacent to $L$ by positive edges except possibly $x'_1w$. If $x_1$ is completely adjacent to $L'$ by positive edges and $x'_1$ is completely adjacent to $L$ by positive edges, or $L = W_1$ and $L' = W_1$ then $L$ and $L$ form a graph in $\mathcal{J}$ as an induced subgraph. Suppose that $L = W_1$ and $L' \neq W_1$. If $x_1w'$ is positive and $x'_1w$ is negative then $L$ and $L'$ induce a graph in $\mathcal{J}$. If $x_1w'$ is negative then $L$ and $L' - \{x'_1, x'_2, \ldots, x'_{\ell-1}\}$ induce a graph in $\mathcal{S}$. A similar discussion applies to the case when $L \neq W_1$ and $L' = W_1$. Suppose that $L \neq W_1$ and $L' \neq W_1$. If $x_1w'$ is negative, then $L$ and $L' - \{x'_1, x'_2, \ldots, x'_{\ell-1}\}$ induce a graph in $\mathcal{S}$. Similarly, if $x'_1w$ is negative, then $L - \{x_1, x_2, \ldots, x_{k-1}\}$ and $L'$ induce a graph in $\mathcal{S}$.

Suppose now that $x_1$ and $x'_1$ are in the same partite set of $\widehat{G}$. Assume first that

$x_1$ is adjacent to a vertex in $L'$ and $x_1'$ is adjacent to a vertex in $L$. We know from above that $x_1$ is adjacent to one of $w', y', z'$ and $x_1'$ is adjacent to one of $w, y, z$. By Lemma 5.10, $x_1$ is completely adjacent to $L'$ by positive edges except possibly $x_1w'$, and $x_1'$ is completely adjacent to $L$ by positive edges except possibly $x_1'w$. If $x_1$ is completely adjacent to $L'$ by positive edges and $x_1'$ is completely adjacent to $L$ by positive edges then $L$ and $L'$ induce a graph in $\mathcal{J}$. If $x_1w'$ is a negative edge then $L$ and $L' - \{x_1', x_2', \ldots, x_{\ell-1}'\}$ induce a graph in $\mathcal{S}$. Similarly, if $x_1'w$ is negative then $L - \{x_1, x_2, \ldots, x_{k-1}\}$ and $L'$ induce a graph in $\mathcal{S}$.

Assume next that $x_1$ is adjacent to a vertex in $L'$ but $x_1'$ is adjacent to no vertex in $L$. As above, we know that $x_1$ is completely adjacent to $L'$ by positive edges except possibly $x_1w'$. If $x_1w'$ is a negative edge then $L$ and $L' - \{x_1', x_2', \ldots, x_{\ell-1}'\}$ induce a graph in $\mathcal{S}$. So $x_1$ completely adjacent to $L'$ by positive edges.

Let $u$ be a neighbour of $x_1'$ in $\widehat{H_1}$. Suppose that $u$ is adjacent to a vertex in $L - x_1$. A similar proof as above shows that $u$ must be adjacent to one of $w, y, z$. Observe that $L$ and $x_1', x_2'$ form an induced lollipop with $x_1'$ being the end. Applying Lemma 5.10 to this lollipop and $u$ we conclude that $u$ is completely adjacent to this lollipop by positive edges except possibly $uw$ and $ux_1'$. If $uw$ is a negative edge then $L - \{x_1, x_2, \ldots, x_{k-1}\}$, $u$, and $L'$ form an induced graph in $\mathcal{S}$. So $u$ is completely adjacent to $L$ by positive edges. Hence $L$, $u$, and $L'$ form an induced graph in $\mathcal{J}$. Hence no neighbour of $x_1'$ in $\widehat{H_1}$ is adjacent to any vertex in $L - x_1$.

Let $u_1, u_2, \ldots, u_t$ be a shortest path in $\widehat{H_1}$ from a neighbour of $x_1'$ in $\widehat{H_1}$ to $L - x_1$. Then $t \geq 3$. A similar proof as above shows that $x_1$ is completely adjacent to $u_1u_2 \ldots u_{t-1}u_t'$ where $u_t'$ is the neighbour of $u_{t-1}$ in $L$ closest to $x_1$ by positive edges. Moreover, $u_{t-1}$ is completely adjacent to $L$ by positive edges except possibly $u_{t-1}w$ and $u_{t-1}x_1$. Since $\widehat{G}$ contains no induced graph in $D$, $u_{t-1}x_1$ is a positive edge. If $u_{t-1}w$ is a negative edge, then $L - \{x_1, x_2, \ldots, x_{k-1}\}$, $u_1, u_2, \ldots, u_{t-1}$, $L'$ form an induced graph in $\mathcal{S}$. So $u_{t-1}$ is completely adjacent to $L$ by positive edges. Then $L$, $u_1, u_2, \ldots, u_{t-1}$, and $L'$ form an induced graph in $\mathcal{J}$. A similar discussion applies when $x_1$ is adjacent to no vertex in $L'$ but $x_1'$ is adjacent to a vertex in $L$.

It remains to consider the case when $x_1$ is adjacent to no vertex in $L'$ and $x_1'$ is adjacent to no vertex in $L$. By Lemma 5.6, $x_1$ and $x_1'$ have a common neighbour $u$ in $\widehat{H_1}$ and a common neighbour $u'$ in $\widehat{H_2}$. If $u$ is not adjacent to any vertex of $L - x_1$ then $L$, $L'$ and $u$ induce a graph in $\mathcal{S}$. Similarly, if $u'$ is not adjacent to any vertex of $L' - x_1'$ then $L$, $L'$ and $u'$ induce a graph in $\mathcal{S}$. So $u$ is adjacent to some vertex in $L - x_1$ and $u'$ is adjacent to some vertex in $L' - x_1'$. We know from above that $u$ must be adjacent to one of $w, y, z$. By Lemma 5.10 $u$ is completely adjacent to $L$ by positive edges except possibly $uw$ and $ux_1$. If $uw$ is a negative edge then $L - \{x_1, x_2, \ldots, x_{k-1}\}$, $u$ and $L'$ induce a graph in $\mathcal{S}$. If $ux_1$ is a negative edge then $\widehat{G}$ contains a graph in $D$ induced by $x_1, u', x_1', u, x_3, x_2$ when $k \geq 3$, or by $x_1, u', x_1', u, y, x_2$ when $k = 2$, or by $x_1, u', x_1', u, w, y$ when $k = 1$. Hence $u$ is completely adjacent to $L$ by positive edges. A similar argument shows that $u'$ is completely adjacent to $L'$ by positive edges. Therefore $L$ and $L'$ together with $u$ and $u'$ induce a graph in $\mathcal{J}$. This completes the proof. $\square$

# Chapter 6

# Concluding Remarks and Future Research

This dissertation focuses on the chordality of various classes of digraphs. One of the digraph classes is the class of weakly quasi-transitive digraphs. We gave a structure theorem for this class of digraphs. We applied this structure theorem to derive forbidden subdigraph characterizations of weakly quasi-transitive chordal digraphs and the semi-strict chordal digraphs. In addition, we showed that the small quasi-kernel conjecture holds for the weakly quasi-transitive digraphs.

Unfortunately, forbidden subdigraph characterizations are not known for either of chordal digraphs or semi-strict chordal digraphs. It remains an open problem to find such a characterization.

In this dissertation, we also introduced the notion of chordal signed graph and chordal signed bigraph. Chordal signed bigraph can be recognized in $O(m^2)$ time. This can be done by recursively finding a signed simplicial edge if one exists. It would be interesting to find a linear time recognition algorithm for these classes of graphs. Chordal signed graphs coincide with strict chordal digraphs perviourly studied by Hell and Hernández-Cruz. The forbidden subdigraph characterization of strict chordal digraphs obtained in [45] translates directly to a forbidden subgraph characterization of chordal signed graphs. We characterize chordal signed bigraphs by forbidden subgraphs.

Switching is an operation that can be applied to any vertex of a signed graph. More precisely, given a signed graph $\widehat{G}$ and a vertex $v$, *switching* at $v$ negates the signs of all edges incident with $v$ in $\widehat{G}$ [13]. We say that two signed graphs are *switching equivalent* if one can be obtained from the other by a sequence of switchings. Switching operation can be applied in various areas such as social psychology, data clustering, neuroscience and complex systems [3, 21, 57].

A signed graph $\widehat{G}$ is *chordally switchable* if it is switching equivalent to a chordal signed graph. *Chordally switchable signed bigraphs* can be defined similarly. It is easy to see that none of signed cycles of length at least 4 are chordally switchable signed graphs. Every signed graph in $A_1 \cup A_2 \cup A_3 \cup K_1 \cup K_2$ (in Figure 5.1) is chordally switchable signed graph. Similarly, none of signed even cycles of length at least 6 are chordally switchable signed bigraphs. All signed bigraphs in $F_1 \cup F_2 \cup F_3 \cup F_4 \cup F_5 \cup F_6 \cup D \cup \mathcal{S} \cup \mathcal{J}$ in chapter 5 are chordally switchable signed bigraphs.

The following proposition can be easily verified.

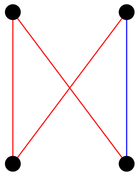


Figure 6.1: $M_0$

**Proposition 6.1.** *Let $\widehat{G}$ be a signed bigraph and $xy$ be an edge of $\widehat{G}$. Then $\widehat{G}$ can be switched to a graph in which $xy$ is a signed simplicial edge if and only if the subgraph* *of $\widehat{G}$ induced by $N(xy)$ is a biclique and does not contain a subgraph that is switching equivalent to $M_0$ in Figure 6.1.*

One can apply the proposition above to verify that none of the graphs in the Figures 6.2, 6.3 and 6.4 can be switch to a chordal signed bigraph.

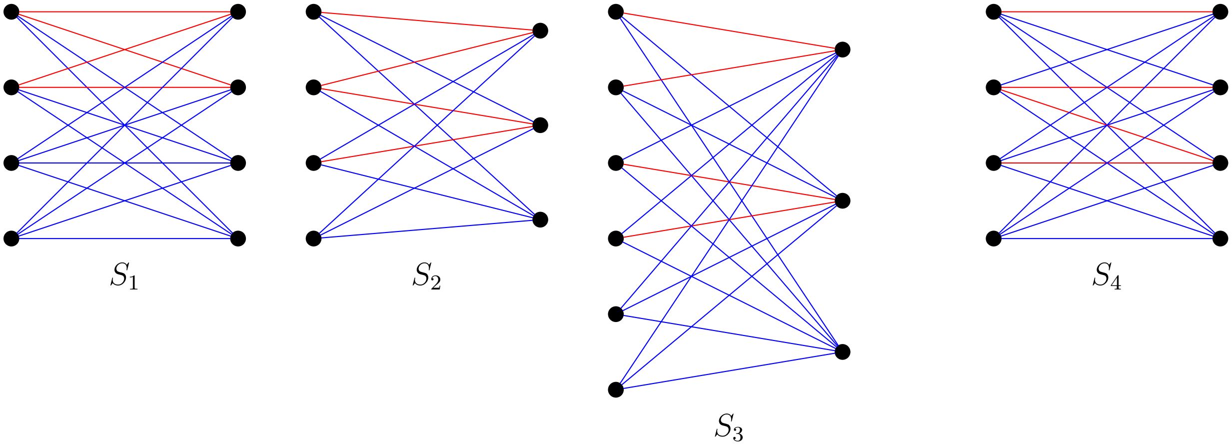


Figure 6.2: Signed complete bigraphs which are not chordal under switching operation

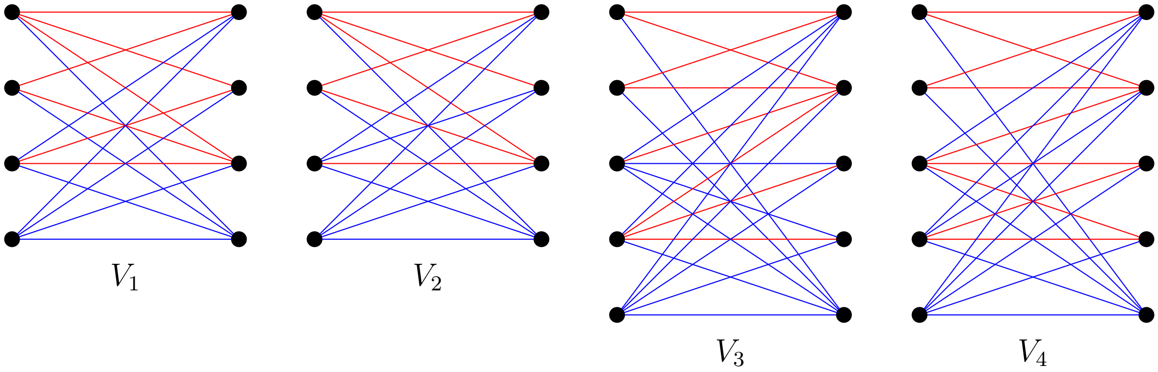
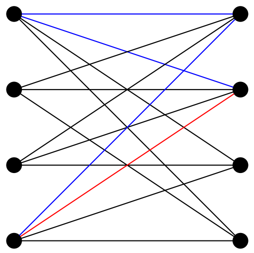


Figure 6.3: Signed non-separable bigraphs which are not chordal under switching operation

Another class of signed bigraphs which are not switching equivalent to chordal signed bigraphs can be obtained as follows. Let $\widehat{H}$ be either a sum or a join of



$D_1$

Figure 6.4: Signed separable bigraphs which are not chordal under switching operation

lollipops. Let $U$ be the signed bigraph obtained from $\widehat{H}$ by adding a positive edge $uv$ completely adjacent to $\widehat{H}$ with positive edges connecting to the endvertices of the negative edges shown in the drawings of Figures 5.11, 5.12, 5.13, 5.14 and 5.15.

We suspect the signed bigraphs described above are the only minimal obstructions to chordally switchable signed bigraphs. It remains a challenging problem to show this is the case.

# Bibliography

[1] J. Ai, S. Gerke, G. Gutin, A. Yeo and Y. Zhou, Results on the small quasi-kernel conjecture, Discrete Math. **346** (2023), no. 7, Paper No. 113435, 9 pp.

[2] S. D. Andres, W. Hochstättler, Perfect digraphs, Journal of Graph Theory (2014) 21-28.

[3] T. Antal, P. L. Krapivsky and S. Redner, Social balance on networks: the dynamics of friendship and enmity, Phys. D **224** (2006), no. 1-2, 130–136.

[4] J. Bang-Jensen, Locally semicomplete digraphs: a generalization of tournaments, J. Graph Theory **14** (1990), no. 3, 371–390.

[5] J. Bang-Jensen, Y. Guo, G. Gutin and L. Volkmann, A classification of locally semicomplete digraphs, Discrete Math. **167/168** (1997), 101–114.

[6] J. Bang-Jensen and G. Gutin, *Classes of directed graphs*, Springer Monographs in Mathematics, Springer, Cham, 2018.

[7] J. Bang-Jensen and J. Huang, Quasi-transitive digraphs, J. Graph Theory **20** (1995), no. 2, 141–161.

[8] J. Bang-Jensen and J. Huang, Kings in quasi-transitive digraphs, Discrete Math. **185** (1998), no. 1-3, 19–27.

[9] J. Bang-Jensen, J. Huang and A. Yeo, Strongly connected spanning subdigraphs with the minimum number of arcs in quasi-transitive digraphs, SIAM J. Discrete Math. **16** (2003), no. 2, 335–343.

[10] S. Benzer, On the topology of the genetic fine structure, Proc. Natl. Acad. Sci, USA **45** (1959), 1607-1620.

[11] C. Berge, Färbung von Graphen deren sämtliche bzw. deren ungerade Kreise starr sind, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe. (1961), 10: 114.

[12] C. Berge, Perfect graphs, in *Studies in graph theory, Part I*, pp. 1–22, Studies in Mathematics, Vol. 11, Math. Assoc. America, Washington, DC, 1975.

[13] J. Bok, R. Brewster, T. Feder, P. Hell and N. Jedličková, List homomorphism problems for signed graphs, in *45th International Symposium on Mathematical Foundations of Computer Science*, Art. No. 20, 14 pp., LIPIcs. Leibniz Int. Proc. Inform., 170, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2020.

[14] A. Brandstädt, V. B. Le and J. P. Spinrad, *Graph classes: a survey*, SIAM Monographs on Discrete Mathematics and Applications, SIAM, Philadelphia, PA, 1999.

[15] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, The strong perfect graph theorem, Ann. of Math. (2) **164** (2006), no. 1, 51–229.

[16] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, Progress on perfect graphs, Math. Program. **97** (2003), no. 1-2, Ser. B, 405–422.

[17] V. Chvátal, On the computational complexity of finding a kernel, Technical report CRM-300, Centre de recherches mathématiques, Université de Montréal, 1973.

[18] V. Chvátal and L. Lovász, Every digraph has a semi-kernel, in: Lecture Notes in Mathematics. 411 (1974), 175-175.

[19] J.E. Cohen, Food Webs and Niche Space, Princeton University Press, Princeton (1978).

[20] C. Croitoru, A note on quasi-kernels in digraphs, Inform. Process. Lett. **115** (2015), no. 11, 863–865.

[21] J. M. Dambacher, H. W. Li, P. A. Rossignol, Relevance of Community Structure in Assessing Indeterminacy of Ecological Predictions, Ecology, vol. 83, no. 5, 2002, pp. 1372–85. JSTOR, https://doi.org/10.2307/3071950. Accessed 2 Feb. 2025.

[22] S. Das, M. K. Sen, A. B. Roy and D. B. West, Interval digraphs: an analogue of interval graphs, J. Graph Theory **13** (1989), no. 2, 189–202.

[23] G. A. Dirac, On rigid circuit graphs, Abh. Math. Sem. Univ. Hamburg **25** (1961), 71–76.

[24] P. L. Erdős and L. A. Székely, Two conjectures on quasi-kernels, Open Problems No. 4. in Fete of Combinatorics and Computer Science. Bolyai Society Mathematical Studies, 2010.

[25] T. Feder, P. Hell, J. Huang and A. Rafiey, Interval graphs, adjusted interval digraphs, and reflexive list homomorphisms, Discrete Appl. Math. **160** (2012), no. 6, 697–707.

[26] D. R. Fulkerson and O. A. Gross, Incidence matrices and interval graphs, Pacific J. Math. **15** (1965), 835–855.

[27] H. Galeana-Sánchez and R. Rojas-Monroy, Kernels in quasi-transitive digraphs, Discrete Math. **306** (2006), no. 16, 1969–1974.

[28] T. Gallai, Transitiv orientierbare Graphen, Acta Math. Acad. Sci. Hungar. **18** (1967), 25–66.

[29] F. Gavril, The intersection graphs of subtrees in trees are exactly the chordal graphs, J. Combinatorial Theory Ser. B **16** (1974), 47–56.

[30] A. Ghouila-Houri, Caractérisation des graphes non orientés dont on peut orienter les arĕtes de manière à obtenir le graphe d'une relation d'ordre, C. R. Acad. Sci. Paris **254** (1962), 1370–1371.

[31] M. C. Golumbic, *Algorithmic graph theory and perfect graphs*, Computer Science and Applied Mathematics, Academic Press, New York-London-Toronto, 1980.

[32] M. C. Golumbic and C. F. Goss, Perfect elimination and chordal bipartite graphs, J. Graph Theory **2** (1978), no. 2, 155–163.

[33] G. Z. Gutin, K. M. Koh, E. G. Tay and A. Yeo, On the number of quasi-kernels in digraphs, J. Graph Theory **46** (2004), no. 1, 48–56.

[34] W. Gutjahr, E. Welzl and G. J. Woeginger, Polynomial graph-colorings, Discrete Appl. Math. **35** (1992), no. 1, 29–45.

[35] A. Hajnal and J. Surányi, Über die Auflösung von Graphen in vollständige Teilgraphen, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **1** (1958), 113–121.

[36] F. Harary, On the notion of balance of a signed graph, Michigan Math. J. **2** (1953/54), 143–146 (1955).

[37] F. Harary, J. A. Kabell and F. R. McMorris, Subtree acyclic digraphs, Ars Combin. **34** (1992), 93–95.

[38] L. Haskins and D. J. Rose, Toward characterization of perfect elimination digraphs, SIAM J. Comput. **2** (1973), 217–224.

[39] S. Heard and J. Huang, Disjoint quasi-kernels in digraphs, J. Graph Theory **58** (2008), no. 3, 251–260.

[40] P. Hell and C. Hernández-Cruz, On the complexity of the 3-kernel problem in some classes of digraphs, Discuss. Math. Graph Theory **34** (2014), no. 1, 167–185.

[41] P. Hell and C. Hernández-Cruz, Strict chordal and strict split digraphs, Discrete Appl. Math. **216** (2017), 609–617.

[42] P. Hell and J. Nešetřil, *Graphs and homomorphisms*, Oxford Lecture Series in Mathematics and its Applications, 28, Oxford Univ. Press, Oxford, 2004.

[43] J. Huang, On the structure of local tournaments, J. Combin. Theory Ser. B **63** (1995), no. 2, 200–221.

[44] J. Huang and Y. Y. Ye, Chordality of locally semicomplete and weakly quasi-transitive digraphs, Discrete Math. **344** (2021), no. 6, Paper No. 112362, 9 pp..

[45] J. Huang and Y. Y. Ye, Semi-strict chordality of digraphs, Graphs Combin. **40** (2024), no. 3, Paper No. 56, 15 pp..

[46] H. Jacob and H. Meyniel, About quasi-kernels in a digraph, Discrete Math. **154** (1996), no. 1-3, 279–280.

[47] D. J. Kleitman, A note on perfect elimination digraphs, SIAM J. Comput. **3** (1974), 280–282.

[48] T. Kloks and D. Kratsch, Computing a perfect edge without vertex elimination ordering of a chordal bipartite graph, Inform. Process. Lett. **55** (1995), no. 1, 11–16.

[49] A. V. Kostochka, R. Luo and S. Shan, Towards the small quasi-kernel conjecture, Electron. J. Combin. **29** (2022), no. 3, Paper No. 3.49, 6 pp..

[50] H. Langlois, F. Meunier, R.Rizzi and S. Vialette, Algorithmic aspects of small quasi-kernels, in *Graph-theoretic concepts in computer science*, 370–382, Lecture Notes in Comput. Sci., 13453, Springer, Cham.

[51] C. G. Lekkerkerker and J. C. Boland, Representation of a finite graph by a set of intervals on the real line, Fund. Math. **51** (1962/63), 45–64.

[52] L. Lovász, Normal hypergraphs and the perfect graph conjecture, Discrete Math. **2** (1972), no. 3, 253–267.

[53] L. Lovász, A characterization of perfect graphs, J. Combinatorial Theory Ser. B **13** (1972), 95–98.

[54] F. R. McMorris and H. M. Mulder, Subpath acyclic digraphs, Discrete Math. **154** (1996), no. 1-3, 189–201.

[55] T. A. McKee, Strict chordal digraphs viewed as graphs with distinguished edges, Discrete Appl. Math. **247** (2018), 122–126.

[56] D. Meister and J. A. Telle, Chordal digraphs, Theoret. Comput. Sci. **463** (2012), 73–83.

[57] C. J. Puccia and R. Levins, Qualitative Modeling of Complex Systems: An Introduction to Loop Analysis and Time Averaging, Harvard University Press, Cambridge, MA. (1986).

[58] M. Richardson, On weakly ordered systems, Bull. Amer. Math. Soc. **52** (1946), 113–116.

[59] D. J. Rose, Triangulated graphs and the elimination process, J. Math. Anal. Appl. **32** (1970), 597–609.

[60] D. J. Rose and R. E. Tarjan, Algorithmic aspects of vertex elimination on directed graphs, SIAM J. Appl. Math. **34** (1978), no. 1, 176–197.

[61] D. J. Rose, R. E. Tarjan and G. S. Lueker, Algorithmic aspects of vertex elimination on graphs, SIAM J. Comput. **5** (1976), no. 2, 266–283.

[62] E. Marczewski, Sur deux propriétés des classes d'ensembles, Fund. Math. **33** (1945), 303–307.

[63] A. van Hulst, Kernels and Small Quasi-Kernels in Digraphs, arXiv:2110.00789, 2021.

[64] J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton Univ. Press, Princeton, NJ, 1944.

[65] Y.Y. Ye, On Chordal Digrpahs and Semi-Strict Chordal Digraphs (M.Sc. Thesis), University of Victoria. 2019.

[66] T. Zaslavsky, Signed graphs, Discrete Appl. Math. **4** (1982), no. 1, 47–74.

[67] T. Zaslavsky, Orientation of signed graphs, European J. Combin. **12** (1991), no. 4, 361–375.

[68] T. Zaslavsky, A mathematical bibliography of signed and gain graphs and allied areas, Electron. J. Combin. **5** (1998), Dynamic Surveys 8, 124.