

A NOTE ON SUBORDINATION BETWEEN  
ANALYTIC FUNCTIONS

by

SHIGEYOSHI OWA and KATSUYUKI NISHIMOTO

DM-367-IR

JUNE 1985

## A NOTE ON SUBORDINATION BETWEEN ANALYTIC FUNCTIONS

Shigeyoshi Owa AND Katsuyuki Nishimoto

communicated by H.M. Srivastava

T.J. Suffridge proved a result for subordinate functions. The object of the present paper is to prove two subordinate theorems with the aid of the result given by Suffridge.

### 1. INTRODUCTION

Let  $f(z)$  and  $g(z)$  be analytic in the unit disk  $\mathcal{U} = \{z: |z| < 1\}$ . Then a function  $f(z)$  is said to be subordinate to  $g(z)$  if there exists a function  $w(z)$  analytic in the unit disk  $\mathcal{U}$  satisfying  $w(0) = 0$  and  $|w(z)| < 1$  such that  $f(z) = g(w(z))$  for  $z \in \mathcal{U}$ . We denote by  $f(z) \prec g(z)$  this relation. In particular, if  $g(z)$  is univalent in the unit disk  $\mathcal{U}$  the subordination is equivalent to  $f(0) = g(0)$  and  $f(\mathcal{U}) \subset g(\mathcal{U})$ .

The concept of subordination can be traced back to Lindelöf [3], but Littlewood ([4], [5]) and Rogosinski ([9], [10]) introduced the term and discovered the basic relations. Recently, Suffridge [15] and Hallenbeck and Ruschewyh [2] studied the subordination between analytic functions.

Let  $\mathcal{A}$  denote the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk  $\mathcal{U}$ . Further, let  $\mathcal{S}$  be the subclass of  $\mathcal{A}$  consisting of analytic and univalent functions in the unit disk  $\mathcal{U}$ . Then a function  $f(z)$  of  $\mathcal{S}$  is said to be starlike of order  $\alpha$  if and only if

$$(1.2) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in \mathcal{U})$$

for some  $\alpha$  ( $0 \leq \alpha < 1$ ). We denote by  $\mathcal{S}^*(\alpha)$  the class of all starlike functions of order  $\alpha$ . Further a function  $f(z)$  of  $\mathcal{S}$  is said to be convex of order  $\alpha$  if and only if

$$(1.3) \quad \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in \mathcal{U})$$

for some  $\alpha$  ( $0 \leq \alpha < 1$ ). And we denote by  $\mathcal{K}(\alpha)$  the class of all convex functions of order  $\alpha$ . We note that  $f(z) \in \mathcal{K}(\alpha)$  if and only if  $zf'(z) \in \mathcal{S}^*(\alpha)$ , and that  $\mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*(0) \equiv \mathcal{S}^*$ ,  $\mathcal{K}(\alpha) \subseteq \mathcal{K}(0) \equiv \mathcal{K}$ , and  $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S}$ .

The classes  $\mathcal{S}^*(\alpha)$  and  $\mathcal{K}(\alpha)$  were first introduced by Robertson [8], and later studied by Schild [12], MacGregor [6], and Pinchuk [7].

Some classes defined by using the extremal function for  $\mathcal{S}^*(\alpha)$  were recently studied by Ruscheweyh [11], Sheil-Small, Silverman and Silvia [13], Silverman and Silvia [14], and Ahuja and Silverman [1].

Our main tool in this paper is the following result given by Suffridge [15].

LEMMA. *Let the function*

$$(1.4) \quad f(z) = \sum_{n=1}^{\infty} a_n z^n$$

*be analytic in the unit disk  $\mathcal{U}$  and the function  $g(z)$  be in the class  $\mathcal{S}^*$ . If  $f(z) \prec g(z)$ , then*

$$(1.5) \quad \int_0^z \frac{f(t)}{t} dt < \int_0^z \frac{g(t)}{t} dt$$

*for  $z \in \bar{\mathcal{U}}(r) = \{z: |z| \leq r < 1\}$ .*

## 2. SUBORDINATE THEOREMS

In this section we prove the following subordinate theorems with the aid of the lemma in the preceding section.

**THEOREM 1.** *Let the function  $f(z)$  defined by (1.1) be in the class  $\mathcal{H}(\alpha)$ . Then  $f'(re^{i\theta})$  ( $0 \leq r < 1$ ) is contained in the image domain of the closed disk  $\bar{\mathcal{U}}(r)$  under the function  $(1-z)^{2(\alpha-1)}$ . Further, it lies (for  $r \neq 0$ ) on the boundary of the image domain if and only if*

$$(2.1) \quad f(z) = \int_0^z (1-\varepsilon t)^{2(\alpha-1)} dt, \quad (z \in \mathcal{U})$$

where  $|\varepsilon| = 1$ .

**PROOF.** Since  $f(z) \in \mathcal{H}(\alpha)$ ,  $f'(z)$  satisfies that

$$(2.2) \quad \operatorname{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > \alpha - 1 \quad (z \in \mathcal{U}).$$

We note that  $zf''(z)/f'(z) = 2\alpha_2 z + \dots$  is analytic in the unit disk  $\mathcal{U}$ , and that  $z/(1-z)$  belongs to the class  $\mathcal{H}$  which is a subclass of  $\mathcal{S}^*$  and  $\operatorname{Re}\{z/(1-z)\} > -1/2$ .

Consequently, we see that

$$(2.3) \quad \frac{zf''(z)}{f'(z)} < \frac{2(1-\alpha)z}{1-z} \quad (z \in \mathcal{U}).$$

Hence, by using the lemma, it follows that  $\log f'(re^{i\theta})$  is contained in the image domain of  $\bar{\mathcal{U}}(r)$  under the function  $\log(1-z)^{2(\alpha-1)}$ , hence further,  $f'(re^{i\theta})$  is contained in the image domain of  $\bar{\mathcal{U}}(r)$  under  $(1-z)^{2(\alpha-1)}$ . Further,  $f'(re^{i\theta})$  lies (for  $r \neq 0$ ) on the boundary of the image domain of  $\bar{\mathcal{U}}(r)$  under  $(1-z)^{2(\alpha-1)}$  if and only if

$$(2.4) \quad \frac{zf''(z)}{f'(z)} = \frac{2(1-\alpha)\varepsilon z}{1-\varepsilon z} \quad (|\varepsilon| = 1).$$

This shows that  $f(z)$  is a function of the form (2.1). Thus we have the theorem.

COROLLARY. Let the function  $f(z)$  defined by (1.1) be in the class  $\mathcal{H}(\alpha)$ .

Then

$$(2.5) \quad \left| \arg\{f'(re^{i\theta})\} \right| \leq 2(1-\alpha) \sin^{-1} r$$

for  $0 \leq r < 1$ , where  $\arg\{f'(0)\} = 0$  and  $\sin^{-1} 0 = 0$ . Further, the equality in (2.5) is attained for  $r \neq 0$  by and only by the function  $f(z)$  given by (2.1).

THEOREM 2. Let the function  $f(z)$  defined by (1.1) be in the class  $\mathcal{S}^*(\alpha)$ . Then  $f(re^{i\theta})/re^{i\theta}$  ( $0 \leq r < 1$ ) is contained in the image domain of the closed disk  $\bar{\mathcal{U}}(r)$  under the function  $(1-z)^{2(\alpha-1)}$ . Further, it lies (for  $r \neq 0$ ) on the boundary of this image domain if and only if

$$(2.6) \quad f(z) = z(1-\varepsilon z)^{2(\alpha-1)},$$

where  $|\varepsilon| = 1$ .

PROOF. Since  $f(z) \in \mathcal{S}^*(\alpha)$ ,  $f(z)$  satisfies that

$$(2.7) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} - 1 \right\} > \alpha - 1 \quad (z \in \mathcal{U}).$$

Further,  $zf'(z)/f(z) - 1 = a_2z + \dots$  is analytic in the unit disk  $\mathcal{U}$ . Thus  $zf'(z)/f(z) - 1$  takes on values in the image domain of the unit disk  $\mathcal{U}$  under the function  $2(1-\alpha)z/(1-z) \in \mathcal{H}$ , that is,

$$(2.8) \quad \frac{zf'(z)}{f(z)} - 1 < \frac{2(1-\alpha)z}{1-z} \quad (z \in \mathcal{U}).$$

By virtue of the lemma, we observe that  $\log f(re^{i\theta})/re^{i\theta}$  ( $0 \leq r < 1$ ) is contained in the image domain of  $\bar{\mathcal{U}}(r)$  under  $\log(1-z)^{2(\alpha-1)}$ , that is, that  $f(re^{i\theta})/re^{i\theta}$  is contained in the image domain of  $\bar{\mathcal{U}}(r)$  under  $(1-z)^{2(\alpha-1)}$ . Further, it lies (for  $r \neq 0$ ) on the boundary of this image domain if and only if

$$(2.9) \quad \frac{zf'(z)}{f(z)} - 1 = \frac{2(1-\alpha)\varepsilon z}{1-\varepsilon z} \quad (|\varepsilon| = 1);$$

hence, further,  $f(z) = z(1-\varepsilon z)^{2(\alpha-1)}$ . This completes the proof of Theorem 2.

#### ACKNOWLEDGEMENTS

This research was completed at the University of Victoria while the first author was on study leave from Kinki University, Osaka, Japan. The authors would like to thank Professor H.M. Srivastava for his thoughtful encouragement and numerous helpful advices.

#### REFERENCES

- [1] O.P. Ahuja and H. Silverman, Convolutions of prestarlike functions, *Internat. J. Math. and Math. Sci.* 6(1983), 59-68.
- [2] D.J. Hallenbeck and S. Ruscheweyh, Subordination by convex functions, *Proc. Amer. Math. Soc.* 52(1975), 191-195.
- [3] E. Lindelöf, Mémoire sur certaines inégalités dans la théorie des fonctions monogènes et sur quelques propriétés nouvelles de ces fonctions dans le voisinage d'un point singulier essentiel, *Acta Soc. Sci. Fenn.* 35(1909), 1-35.
- [4] J.E. Littlewood, On inequalities in the theory of functions, *Proc. London Math. Soc.* (2) 23(1925), 481-519.
- [5] J.E. Littlewood, *Lectures on the Theory of Functions*, Oxford University Press, London, 1944.
- [6] T.H. MacGregor, The radius of convexity for starlike functions of order  $1/2$ , *Proc. Amer. Math. Soc.* 14(1963), 71-76.
- [7] B. Pinchuk, On starlike and convex functions of order  $\alpha$ , *Duke Math. J.* 35(1968), 721-734.
- [8] M.S. Robertson, On the theory of univalent functions, *Ann. of Math.* 37(1936), 374-408.
- [9] W. Rogosinski, On subordinate functions, *Proc. Cambridge Philos. Soc.* 35(1939), 1-26.

- [10] W. Rogosinski, On the coefficients of subordinate functions, *Proc. London Math. Soc.* (2) 48(1945), 48-82.
- [11] S. Ruscheweyh, Linear operators between classes of prestarlike functions, *Comment. Math. Helv.* 52(1977), 497-509.
- [12] A. Schild, On starlike functions of order  $\alpha$ , *Amer. J. Math.* 87(1965), 65-70.
- [13] T. Sheil-Small, H. Silverman and E.M. Silvia, Convolution multipliers and starlike functions, *J. Analyse Math.* 41(1982), 181-192.
- [14] H. Silverman and E.M. Silvia, Subclasses of prestarlike functions, *Math. Japon.* 29(1984), 929-935.
- [15] T.J. Suffridge, Some remarks on convex maps of the unit disk, *Duke Math. J.* 37(1970), 775-777.

Department of Mathematics  
Faculty of Science and Technology  
Kinki University  
Higashi-Osaka, Osaka 577  
Japan

Department of Mathematics  
College of Engineering  
Nihon University  
Koriyama, Fukushima 963  
Japan

Received: March 11, 1985

Revised: May 10, 1985