SOME CHARACTERISTICS OF A CLASS OF ANALYTIC FUNCTIONS

by

H.M. Srivastava, S. Owa, F.–Y. Ren
and
H. Saitoh

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H.M. Srivastava

Department of Mathematics and Statistics, University of Victoria,
Victoria, British Columbia V8W 2Y2, *Canada*

Shigeyoshi Owa

Department of Mathematics, Kinki University,
Higashi–Osaka, Osaka 577, *Japan*

Fu–Yao Ren

Department of Mathematics, Fudan University,
Shanghai 200433, *People's Republic of China*

and

Hitoshi Saitoh

Department of Mathematics, Gunma College of Technology,
580 *Toriba–Machi, Maebashi, Gunma* 371, *Japan*

ABSTRACT

The object of the present paper is to derive an interesting property of certain analytic functions belonging to the class $\mathscr{A}(p)$ introduced below. Several useful corollaries of the main theorem are also deduced.

1. INTRODUCTION

Let $\mathcal{A}(p)$ be the class of functions of the form (*cf.* [1] and [4]):

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \qquad (p \in \mathbb{N} = \{1,2,3,\cdots\}), \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Recently, Saitoh [5] proved the following result.

Theorem 1. *If* $f(z) \in \mathcal{A}(p)$ *satisfies*

$$\mathrm{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} > \alpha \qquad \left[0 \leqq \alpha < \frac{p!}{(p-j)!}\,;\ 1 \leqq j \leqq p;\ z \in \mathcal{U}\right], \tag{1.2}$$

then

$$\mathrm{Re}\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\} > \frac{1}{(p-j+1)!}\ \frac{2\alpha\cdot(p-j+1)!\ +\ p!}{2(p-j)\ +\ 3} \qquad (z \in \mathcal{U}), \tag{1.3}$$

where $1 \leqq j \leqq p$.

In the present paper we shall first derive a similar property for functions $f(z) \in \mathcal{A}(p)$ which satisfy the condition (1.2). In order to establish our main assertion (given by Theorem 3 below), we require the following result due essentially to Miller [2] (see also Miller and Mocanu [3]).

Theorem 2. *Let* $\varphi(u,v)$ *be a complex–valued function such that*

$$\varphi : \mathscr{D} \longrightarrow \mathbb{C}, \quad \mathscr{D} \subset \mathbb{C} \times \mathbb{C} \quad (\mathbb{C} \text{ being the complex plane}),$$

and let

$$u = u_1 + i\,u_2 \quad \text{and} \quad v = v_1 + i\,v_2\,.$$

Suppose that the function $\varphi(u,v)$ *satisfies each of the following conditions :*

(i) $\varphi(u,v)$ *is continuous in* $\mathscr{D}$;

(ii) $(1,0) \in \mathscr{D}$ *and* $\mathrm{Re}\{\varphi(1,0)\} > 0$;

(iii) *For all* $(i\,u_2,\, v_1) \in \mathscr{D}$ *such that* $v_1 \leqq -\frac{1}{2}\,(1+u_2^2)$, $\mathrm{Re}\{\varphi(i\,u_2,\, v_1)\} \leqq 0$.

Also let the function $q(z)$ *given by*

$$q(z) = 1 + q_1 z + q_2 z^2 + \cdots \tag{1.4}$$

be regular in the open unit disk $\mathscr{U}$ *such that*

$$(p(z),\, zp'(z)) \in \mathscr{D} \quad \text{for all} \quad z \in \mathscr{U}\,.$$

If

$$\mathrm{Re}\{\varphi(p(z),\, zp'(z))\} > 0 \qquad (z \in \mathscr{U}\,), \tag{1.5}$$

then

$$\mathrm{Re}\{p(z)\} > 0 \qquad (z \in \mathscr{U}\,). \tag{1.6}$$

2. MAIN RESULT

By applying Theorem 2, we shall now establish our main result contained in

Theorem 3. *If* $f(z) \in \mathcal{A}(p)$ *satisfies the condition* (1.2), *then*

$$\operatorname{Re}\left\{\sqrt{\frac{f^{(j-1)}(z)}{z^{p-j+1}}}\right\} > \{B(p;j)\}^{-\frac{1}{2}}\,\gamma \qquad (z \in \mathcal{U}), \tag{2.1}$$

where, for convenience,

$$\gamma = \frac{1 + \sqrt{1 + 4(p-j+2)\alpha\ B(p;j)}}{2(p-j+2)} \tag{2.2}$$

and

$$B(p;j) = \frac{(p-j+1)!}{p!} \qquad (1 \leqq j \leqq p). \tag{2.3}$$

Proof. We begin by defining the function $q(z)$ by

$$\sqrt{B(p;j)\ \frac{f^{(j-1)}(z)}{z^{p-j+1}}} = \gamma + (1-\gamma)\ q(z), \tag{2.4}$$

where γ is given by (2.2). It follows from (2.4) that the function $q(z)$ is of the form given by (1.4) and is regular in $\mathcal{U}$, and that

$$B(p;j)\ \frac{f^{(j-1)}(z)}{z^{p-j+1}} = [\gamma + (1-\gamma)\ q(z)]^2\ . \tag{2.5}$$

Therefore, we have

$$B(p;j)\,\frac{f^{(j)}(z)}{z^{p-j}} = (p-j+1)\,[\gamma + (1-\gamma)\,q(z)]^2$$

$$+\,2(1-\gamma)\,[\gamma + (1-\gamma)\,q(z)]\,zq'(z)\,. \tag{2.6}$$

Using the condition (1.2), we see that

$$\mathrm{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}} - \alpha\right\}$$

$$= \{B(p;j)\}^{-1}\,\mathrm{Re}\Big\{(p-j+1)\,[\gamma + (1-\gamma)\,q(z)]^2$$

$$+\,2(1-\gamma)\,[\gamma + (1-\gamma)\,q(z)]\,zq'(z)\Big\} - \alpha$$

$$> 0. \tag{2.7}$$

Next we define the function $\varphi(u,v)$ by

$$\varphi(u,v) = \{B(p;j)\}^{-1}\,\Big\{(p-j+1)\,[\gamma + (1-\gamma)u]^2$$

$$+\,2(1-\gamma)\,[\gamma + (1-\gamma)u]v\Big\} - \alpha. \tag{2.8}$$

Then we readily observe that

(i) $\varphi(u,v)$ is continuous in $\mathscr{D} = \mathbb{C} \times \mathbb{C}$;

(ii) $(1,0) \in \mathscr{D}$ and

$$\mathrm{Re}\{\varphi(1,0)\} = \frac{p!}{(p-j)!} - \alpha > 0 \qquad (1 \leqq j \leqq p),$$

and

(iii) For all $(i\, u_2,\, v_1) \in \mathscr{D}$ such that $v_1 \leqq -\frac{1}{2}(1+u_2^2)$, we have

$$\begin{aligned}
&\mathrm{Re}\{\varphi(i\, u_2,\, v_1)\} \\
&= \{B(p;j)\}^{-1} \left\{(p-j+1)\left[\gamma^2 - (1-\gamma)^2\, u_2^2\right] + 2\gamma(1-\gamma)v_1\right\} - \alpha \\
&\leqq \{B(p;j)\}^{-1} \left\{(p-j+1)\left[\gamma^2 - (1-\gamma)^2\, u_2^2\right] - \gamma(1-\gamma)(1+u_2^2)\right\} - \alpha \\
&\leqq 0, \qquad (2.9)
\end{aligned}$$

for γ given by (2.2), because $0 < \gamma < 1$.

Since the function $\varphi(u,v)$ satisfies the hypothesis of Theorem 2, it follows from Theorem 2 that

$$\mathrm{Re}\left\{\left[\sqrt{\frac{f^{(j-1)}(z)}{z^{p-j+1}}}\right]\right\} > \{B(p;j)\}^{-\frac{1}{2}}\; \gamma \qquad (z \in \mathscr{U}),$$

which is precisely the assertion (2.1) of Theorem 3. This evidently completes the proof of Theorem 3.

3. SOME INTERESTING COROLLARIES

Taking $p = j$ in Theorem 3, we have

Corollary 1. *If* $f(z) \in \mathcal{A}(p)$ *satisfies*

$$\operatorname{Re}\left\{f^{(p)}(z)\right\} > \alpha \qquad (0 \leqq \alpha < p\,!\,;\ z \in \mathcal{U}), \tag{3.1}$$

then

$$\operatorname{Re}\left\{\sqrt{\frac{f^{(p-1)}(z)}{z}}\right\} > \frac{1}{4}\left(\sqrt{p!} + \sqrt{8\alpha + p!}\,\right) \qquad (z \in \mathcal{U}). \tag{3.2}$$

For $p = 1$, Corollary 1 immediately yields the inequality:

$$\operatorname{Re}\left\{\sqrt{\frac{f(z)}{z}}\right\} > \frac{1}{4}\left(1 + \sqrt{8\alpha + 1}\,\right) \tag{3.3}$$

for functions $f(z) \in \mathcal{A}(1)$ such that [*cf.* Equation (3.1) with $p = 1$]

$$\operatorname{Re}\{f'(z)\} > \alpha \qquad (0 \leqq \alpha < 1;\ z \in \mathcal{U}). \tag{3.4}$$

A slightly improved version of the assertion (3.3) is provided by an earlier result [6, p. 3, Corollary 3].

Setting $p = j$ and $\alpha = 0$, Theorem 3 yields

Corollary 2. *If* $f(z) \in \mathcal{A}(p)$ *satisfies*

$$\operatorname{Re}\left\{f^{(p)}(z)\right\} > 0 \qquad (z \in \mathcal{U}), \tag{3.5}$$

then

$$\mathrm{Re}\left\{\sqrt{\frac{f^{(p-1)}(z)}{z}}\right\} > \frac{\sqrt{p!}}{2} \quad (z \in \mathscr{U}). \tag{3.6}$$

Finally, taking $j = 1$ in Theorem 3, we have

Corollary 3. *If* $f(z) \in \mathscr{A}(p)$ *satisfies*

$$\mathrm{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha \quad (0 \leqq \alpha < p\,;\ z \in \mathscr{U}), \tag{3.7}$$

then

$$\mathrm{Re}\left\{\sqrt{\frac{f(z)}{z^p}}\right\} > \frac{1 + \sqrt{1 + 4\alpha(p+1)}}{2(p+1)} \quad (z \in \mathscr{U}). \tag{3.8}$$

For $p = 1$ and $\alpha = \frac{1}{2}$, this last inequality (3.8) assumes the elegant form:

$$\mathrm{Re}\left\{\sqrt{\frac{f(z)}{z}}\right\} > \frac{1 + \sqrt{5}}{4} \quad (z \in \mathscr{U}), \tag{3.9}$$

which follows also from (3.3) (with $\alpha = \frac{1}{2}$) for functions $f(z) \in \mathscr{A}(1)$ such that

$$\mathrm{Re}\{f'(z)\} > \frac{1}{2} \quad (z \in \mathscr{U}).$$

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