# TOPOLOGICAL ORBIT EQUIVALENCE AND $C^*$-CROSSED PRODUCTS

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## Introduction.

The present paper has one foot within the theory of topological dynamical systems and the other within $C^*$-algebra theory. The link between the two is provided by $K$-theory — via the crossed product construction. The key concept is that of a *dimension group* (Definition 1.9), which first appeared in connection with the classification of approximately finite dimensional (AF) $C^*$-algebras [11]. The original definition of a dimension group was closely linked with the embedding scheme for an ascending sequence of finite dimensional algebras called a *Bratteli diagram* [3]. Subsequently its $K$-theoretic underpinning was realized, namely as $K_0$ of the associated $AF$-algebra — endowed with a natural ordering. (Simple) dimension groups may also be defined in terms of dynamical concepts (Definition 1.11 and Theorem 1.12) — being order isomorphic to $K_0$ of the associated $C^*$-crossed products (Cor. 1 to Theorem 1.17). We may summarize this paper by saying that we show that $K$-theory — in conjunction with a dynamical

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interpretation of the Bratteli diagram — will yield complete information about the orbit structure of the dynamical systems we are exploring. On the $C^*$-algebra side $K$-theory yields complete isomorphism invariants of the associated $C^*$-crossed products [12]. Combining the two we get a "Krieger-type" theorem.

Our original motivation was to obtain results in the topological/$C^*$-algebra setting similar to those of W. Krieger in the measure-theoretic/von Neumann algebra setting. Krieger's theorem [20], [21] says that if $\Theta_1 = (X_1, \mathcal{B}_1, \mu_1, T_1)$ and $\Theta_2 = (X_2, \mathcal{B}_2, \mu_2, T_2)$ are two ergodic non-singular systems, then the associated von Neumann crossed product factors (also called *Krieger factors*) are isomorphic if and only if the systems $\Theta_1$ and $\Theta_2$ are orbit equivalent, i.e. there exists a bimeasurable bijection $F : X_1 \to X_2$, preserving the measure class, so that for almost all $x \in X_1, F(\text{orbit}_{T_1}(x)) = \text{orbit}_{T_2}(F(x))$. A precursor of the development leading up to Krieger's result is a celebrated theorem of H. Dye from 1963 [8]. Henry Dye, whose name is inextricably linked to *the* basic result in the theory of orbit equivalence of ergodic (finite) measure preserving transformations, proved that any two such transformations are orbit equivalent. An immediate offspring of his theorem is that the associated von Neumann factors are isomorphic and equal to the unique approximately finite dimensional ("hyperfinite") $II_1$ factor of Murray-von Neumann. Thanks to A. Connes' deep analysis [7] (the $III_1$ case being finally settled by U. Haagerup [16]) we know that the family of approximately finite dimensional von Neumann factors coincides with the family of Krieger factors. (Recall that $M$ is approximately finite dimensional if $M$ is the weak operator closure of the union of an ascending sequence of finite dimensional algebras.)

In the topological setting a *minimal* dynamical system is the natural analogue of an ergodic system. (Recall that a dynamical system $(X, \varphi)$, where $X$ is compact metric and $\varphi : X \to X$ is a homeomorphism, is minimal if $\varphi$ has no non-trivial closed invariant subspace — equivalently, each $\varphi$-orbit is dense. Also, the associated $C^*$-crossed product $C(X) \times_{\varphi} \mathbf{Z}$ is *simple*, i.e. it has no proper two-sided ideals.) For various reasons it will be fruitful to restrict the study to *Cantor systems*, i.e. minimal dynamical systems $(X, \varphi)$, where $X$ is a compact (perfect) zero-dimensional space, or, equivalently,

$X$ is a totally disconnected compact metric space with no isolated points. Cantor showed that all such spaces are homeomorphic and we name them *Cantor sets.* The family consisting of all Cantor systems is vast — in fact, every minimal dynamical system is a *factor* of a Cantor system (cf. Section 1). For Cantor systems we do indeed have an analogue of Krieger's theorem (Theorem 2.1) — the relevant notion being (topological) *strong orbit equivalence* (Definition 1.3). Furthermore, analogous to the case for Krieger factors in the measure-theoretic setting, the simple $C^*$-algebras associated to Cantor systems form a distinguished class of $C^*$-algebras, characterized by being inductive limits of a certain kind and by having $K_1$-groups equal to $\mathbf{Z}$ (Theorem 1.15).

It is interesting to compare the proofs of Dye's and Krieger's theorems with the proof we give in the topological setting. The proof of Dye's theorem is achieved by application of an approximation result (Rohlin's lemma) which is successively refined so as to mimic the given transformation as an odometer action. As for Krieger's theorem it is shown that the ratio set — a notion introduceed in ergodic theory by Krieger, motivated by work that had been done on the classification of von Neumann algebras — is a complete invariant for orbit equivalence in the type $III_\lambda$ $(0 < \lambda \leq 1)$ case. This may be proved by a similar procedure as in the proof of Dye's theorem by introducing special measures equivalent to the given ones, cf. [19]. (The $III_0$ case must be handled differently, namely by introducing the so called Krieger flow and showing that two systems are orbit equivalent if and only if the associated Krieger flows are isomorphic [19].) In the topological setting the crucial tool in proving Theorem 2.1 is the model theorem (Theorem 3.6): every Cantor system is conjugate to a Bratteli-Vershik system, i.e. for $(X, \varphi)$ a Cantor system there exists a (simple) ordered Bratteli diagram $B = (V, E, \geq)$ so that the associated path space $X_B$ with the induced "lexicographic" order engenders a (Vershik) map $\lambda_B : X_B \to X_B$ conjugate to $\varphi : X \to X$. Furthermore, the $K_0$-group of the system, $K^0(X, \varphi)$ (Definition 1.11) — endowed with the natural ordering — is order isomorphic to the dimension group $K_0(V, E)$ associated to $B$ (Theorem 3.7), and every simple dimension group arises in this way (Theorem 1.12). The ordered Bratteli diagram is obtained by constructing a succession of Kakutani-Rohlin towers, where at each level the number of vertices of the diagram equals the number of towers, while the edges between successive levels are determined and ordered according to how the towers are traversed. (In the measure-theoretic case, specifically for type II and type $III_\lambda$ ($\lambda \neq 0$), the approximation is achieved by one single tower at each level.) It turns out that $K^0(X, \varphi)$ (with order unit) is a complete invariant for (topological) strong orbit equivalence and also a complete isomorphism invariant for the associated $C^*$-crossed product (Theorem 2.1). The notion of strong orbit equivalence — involving, as it does, orbit equivalence plus continuity properties of the orbit cocycles — is put into perspective by Theorem 2.4 and Theorem 2.5.

How is the basic notion of (topological) *orbit equivalence* (Definition 1.2) related to $K$-theory? The answer to that question is given in Theorem 2.2, and the proof occupies the main part of this paper — involving techniques and results from a wide variety of sources, including some homological algebra. The complete invariant for orbit equivalence of Cantor systems turns out to be the $K_0$-group modulo the infinitesimal subgroup (Definition 1.10, cf. also Theorem 1.13). Put in dynamical terms: two Cantor systems $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ are orbit equivalent if and only if they have the same invariant probability measures — more precisely, there is a homeomorphism $F : X_1 \to X_2$ carrying the $\varphi_1$-invariant probability measures onto the $\varphi_2$-invariant probability measures. Since this formulation only involves dynamical concepts, it is natural to ask whether this equivalence could be proved directly, i.e. without recourse to the machinery we use to prove Theorem 2.2. This seems problematic even at first glance — given $F$, how could one recover orbit information? In fact, there exists an example of two Cantor systems and an $F$ as above with the following property: if $x$ and $y$ are any two distinct points in the same $\varphi_1$-orbit, then the points $F(x)$ and $F(y)$ are in different $\varphi_2$-orbits [5]! We mention a remarkable consequence of Theorem 2.2 which has the flavour of a "Dye-type" theorem in the topological setting: a uniquely ergodic Cantor system is either orbit equivalent to an odometer system or to a Denjoy system (Cor. 2 to Theorem 2.2). (A few comments about the proof of Theorem 2.2: By application of a theorem of Cartan-Eilenberg in homological algebra the problem is reduced to the case where the infinitesimal subgroups in question are isomorphic to the free abelian group $\mathbf{Z}^\infty$ of countable rank. The crucial ingredient of the proof is the dynamical "realization" of extensions of $\mathbf{Z}^\infty$ by a given simple dimension group (Cor. 3 to Theorem 1.17 and Theorem 10.1). Finally, we proceed along the same lines as in the proof of Theorem 2.1 — an extended version of the Bratteli-Vershik model for Cantor systems being used.)

There is a curious consequence of Theorems 2.1, 2.2 and 2.3 which we mention briefly. Let $X$ be a Cantor set and suppose $E \subset X \times X$ is an equivalence relation on $X$ with countable equivalence classes, i.e. $E$ is a principal groupoid in the sense of [31]. If one wants to construct the groupoid $C^*$-algebra of $E$, then $E$ must be equipped with a topology in which it is locally compact and Hausdorff. This must also satisfy some other conditions — the two canonical projection maps to $X$ must be continuous and open, and the counting measure on the equivalence classes must be a Haar system. If the equivalence classes of $E$ are given as the orbits of a minimal homeomorphism then there is a natural choice for such a topology, cf. [31]. The same is true if they are given as the orbits of a minimal $AF$-action (cf. the discussion preceding Theorem 2.3). A consequence of Theorem 2.1 and Theorem 2.2 is the following: There is an equivalence relation $E$ on the Cantor set $X$ with two distinct topologies $\sigma_1$ and $\sigma_2$, both compatible with the topology of $X$ as described above, such that the associated (simple) groupoid $C^*$-algebras $C^*(E, \sigma_1)$ and $C^*(E, \sigma_2)$ are non-isomorphic. (Recall that the $C^*$-algebra $C^*(E, \sigma)$, where $(E, \sigma)$ is associated to the Cantor system $(X, \varphi)$, is isomorphic to the crossed product $C(X) \underset{\varphi}{\times} \mathbf{Z}$, and so by Corollary 1 to Theorem 1.17 we have $K_0(C^*(E, \sigma)) \cong K^0(X, \varphi)$.) In view of Theorem 2.3, we may choose $\sigma_1$ and $\sigma_2$ so that $C^*(E, \sigma_1)$ is an $AF$-algebra while $C^*(E, \sigma_2)$ is not $AF$. (On the other hand, contrasting the above example, by Theorem 2.1 it is easy to find examples $(E, \sigma_1)$ and $(E, \sigma_2)$, with the topologies $\sigma_1$ and $\sigma_2$ distinct, so that $C^*(E, \sigma_1)$ *is* isomorphic to $C^*(E, \sigma_2)$.)

Theorem 2.1 begs the question of what happens if the $K_0$-groups of two Cantor systems are order-isomorphic — dropping the requirement that the distinguished order units are mapped to each other by the isomorphism. This is answered in Theorem 2.6. (The corresponding question for Theorem 2.2 is dealt with in Proposition 2.7.) The answer is that the two systems are *Kakutani strong orbit equivalent* (Definition 1.8) and that the associated $C^*$-crossed products are *strong Morita equivalent* (Definition 1.19). The proof of Theorem 2.6 (and of Proposition 2.7) relies on a result of independent interest, namely that two Cantor systems are *Kakutani equivalent* (Definition 1.7) if and only if the Bratteli-Vershik model of one is obtained from the other by making a finite change of the associated ordered Bratteli diagram (Theorem 3.8).

In summary we may say that $K$-theory yields complete information about the orbit structure of Cantor systems. Thus we have a convenient way to subdivide Cantor systems into equivalence classes of (strong) orbit equivalent systems — in fact, the $K_0$-group is fairly simple to compute for concrete systems. It is interesting to investigate how this subdivision compares with the subdivision resulting from entropy considerations. The answer is that orbit equivalence has nothing to do with entropy! In fact, the results of [6] show that, basically, entropy and orbit equivalence are independent (although the most general conjecture is not settled, namely: within each strong orbit equivalence class all entropies in $[0, \infty]$ occur — and, within a fixed entropy all strong orbit equivalence classes occur). We should also mention that M. Boyle and D. Handelman have recently investigated the ordered group $K^0$ and many other related ideas for more general (i.e. non-minimal) systems on the Cantor set [39].

Part of this work was done during a visit by the second author to the University of Trondheim in the fall of 1991; he would like to thank the Department of Mathematics and Statistics, University of Trondheim, for their kind hospitality during his stay.

## 1. Basic concepts and definitions.

As general references for the notions of topological dynamical nature we refer to [26] and [38]. A (topological) dynamical system consists of a compact Hausdorff space $X$ and a homeomorphism $\varphi : X \to X$. We use the notation $(X, \varphi)$. We say that $\varphi$ is *minimal* if all orbits are dense in $X$, i.e. if $x \in X$ then $\text{orbit}_\varphi(x)^- = \{\varphi^n(x) | n \in \mathbf{Z}\}^- = X$, where $^-$ denotes closure and $\varphi^n$ denotes the $n$'th iterate of $\varphi$ if $n \geq 1$, $\varphi^0 = id$, and $\varphi^n$ is the $(-n)$'th iterate of the inverse homeomorphism $\varphi^{-1}$ if $n < 0$. Minimality is equivalent to $\varphi$

having no non-trivial closed invariant subset $Y$, i.e. $\varphi(Y) \subset Y$ implies $Y$ is $X$ or $\emptyset$. A minimal dynamical system $(X, \varphi)$ is *uniquely ergodic* if there is a unique $\varphi$-invariant probability measure $\mu$ — the support of $\mu$ being then necessarily all of $X$.

In this paper we shall be studying minimal dynamical systems $(X, \varphi)$, where $X$ is a Cantor set, i.e. $X$ has a countable basis of closed and open ("clopen") sets (i.e. $X$ is 0-dimensional) and $X$ has no isolated points. (Equivalently, a Cantor set may be described as a totally disconnected metrizable compact space with no isolated points.) A theorem of Cantor says that all Cantor sets are homeomorphic.

The Cantor set $X$ possesses the following "universal" property (proved by Aleksandrov and Urysohn): Let $Y$ be a compact metrizable space. Then there exists a continuous surjection $F : X \to Y$.

Using this fact one can show that if $(Y, \psi)$ is a minimal system, $Y$ compact metrizable, then there exists a minimal system $(Z, \varphi)$, $Z$ Cantor, and a continuous surjection $G : Z \to Y$ so that $G \circ \varphi = \psi \circ G$; in other words , $(Y, \psi)$ is a *factor* of $(Z, \varphi)$. In fact, if $F : X \to Y$ is a continuous surjection, $X$ Cantor, we define the subset $K = \{(x_n) | x_n \in X, F(x_{n+1}) = \psi(F(x_n)), n \in \mathbf{Z}\}$ of the countable Cartesian product of $X$ endowed with the product topology. $K$ is a closed and $\sigma$-invariant subset of the Cartesian product, where $\sigma$ denotes the two-sided shift. Let $(Z, \varphi)$ be a minimal subsystem of $(K, \sigma)$, where $\varphi$ denotes the restriction of $\sigma$ to $Z$. Then $Z$ is a Cantor set and if $G : Z \to Y$ is the evaluation map at the zero'eth coordinate followed by F, we easily see that $(Z, \varphi)$ has the desired property.

So minimal dynamical systems on the Cantor set have a "universal" property.

**Terminology.** We will use the generic term *Cantor system* about a minimal dynamical system $(X, \varphi)$, where $X$ is the Cantor set.

**Definition 1.1.** (Conjugacy and flip conjugacy.) The dynamical systems $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ are *conjugate* if there exists a homeomorphism $F : X_1 \to X_2$ so that $F \circ \varphi_1 = \varphi_2 \circ F$. $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ are *flip conjugate* if $(X_1, \varphi_1)$ is conjugate either to $(X_2, \varphi_2)$ or to $(X_2, \varphi_2^{-1})$.

Obviously both conjugacy and flip conjugacy are equivalence relations.

**Definition 1.2.** (Orbit equivalence.) The dynamical systems $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ are *(topologically) orbit equivalent* if there exists a homeomorphism $F : X_1 \to X_2$ so that $F\ (\text{orbit}_{\varphi_1}(x)) = \text{orbit}_{\varphi_2}(F(x))$ for all $x \in X_1$. We will use the generic term *orbit map* about a map like $F$.

Obviously orbit equivalence is an equivalence relation. Observe that minimality is a property preserved by orbit equivalence. It is also easily seen that flip conjugacy (and hence conjugacy) implies orbit equivalence.

Let $(X_1, \varphi_1)$, $(X_2, \varphi_2)$ and $F$ be as in Definition 1.2. For each point $x$ in $X_1$ there exists an integer $n(x)$ so that $F \circ \varphi_1(x) = \varphi_2^{n(x)} \circ F(x)$. Likewise, there exists an integer $m(x)$ so that $F \circ \varphi_1^{m(x)}(x) = \varphi_2 \circ F(x)$. If $(X_1, \varphi_1)$ (and hence $(X_2, \varphi_2)$) is minimal it is easily seen that $m$ and $n$ are uniquely defined integer-valued functions on $X_1$. We call $m$ and $n$ the *orbit cocycles* associated to the orbit map $F$.

**Definition 1.3** (Strong orbit equivalence.) Let $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ be minimal systems that are (topologically) orbit equivalent. We say that $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ are *strong (topologically) orbit equivalent* if there exists an orbit map $F : X_1 \to X_2$ so that the associated orbit cocycles $m, n : X_1 \to \mathbf{Z}$ each have at most one point of discontinuity.

**Remarks.** That strong orbit equivalence really is an equivalence relation is a consequence of Theorem 2.1. Obviously flip conjugacy implies strong orbit equivalence.

One can show that if one of the orbit cocycles, say $m$, has a single discontinuity in $\text{orbit}_{\varphi_1}(x), x \in X_1$, then $n$ must have (at least) one discontinuity in the same orbit. (Cf. the proof of (i) $\Rightarrow$ (ii) of Theorem 2.5.)

It is worthwhile pointing out that between strong orbit equivalent systems (in fact, even between conjugate systems) one may find orbit maps so that the associated orbit cocycles each have more than one point of discontinuity.

To put Definition 1.3 in perspective we cite the following theorem (unpublished) of M. Boyle [4], a special case of which will be part of our Theorem 2.4. By kind permission we will present a proof there.

**Theorem 1.4.** (M. Boyle). Let $(X, \varphi_1)$ and $(X, \varphi_2)$ be two dynamical systems on the compact metric space $X$ having the same orbits, one orbit being dense (i.e. the systems are transitive). Assume that one of the orbit cocycles $m$ and $n$ is continuous everywhere. Then the two systems are flip conjugate.

**Remark.** Boyle's theorem is the topological analogue of Belinskaya's theorem in the ergodic measure-preserving setting where the integrability of either $m$ or $n$ implies (flip) metrical isomorphism.

We shall need the concept of *induced transformation* and *Kakutani equivalence.* These notions were originally introduced by Kakutani in ergodic theory but they can be carried over to topological dynamics (cf. [27]).

**Definition 1.5.** Let $(X,\varphi)$ be a Cantor system. Let $A$ be a clopen subset of $X$ (hence $A$ is again a Cantor set). Let $\varphi_A : A \to A$ be the "first return map", i.e. if $x \in A$ then

$$\varphi_A(x) = \varphi^{r_A(x)}(x), \quad \text{where } r_A(x) = \inf\{m \in \mathbf{Z}^+ | \varphi^m(x) \in A\}.$$

We call $\varphi_A$ the *induced transformation* on $A$, and say $(A,\varphi_A)$ is the *induced* (or *derivative) system* of $(X,\varphi)$ with respect to $A$.

**Remark.** By [27] $r_A$ is continuous and thus takes only a finite set of values. It is easily seen that $(A,\varphi_A)$ is again a minimal dynamical system.

To recover $\varphi$ from $\varphi_A$ we introduce the *tower* (or *skyscraper)* construction.

**Definition 1.6.** Let $(X,\varphi)$ be a Cantor system. Let $N$ be a positive integer and let $h : X \to \{1,2,\ldots,N\}$ be a continuous function ($\{1,2,\ldots,N\}$ has the discrete topology). Let $X^h = \{(x,i)|1 \le i \le h(x)\}$ and make $X^h$ in a natural fashion a Cantor set. Define $\varphi^h : X^h \to X^h$ by

$$\varphi^h(x,i) = \begin{cases} (x,i+1) & \text{if } \; i+1 \le h(x) \\ (\varphi(x),1) & \text{if } \; i+1 > h(x) \end{cases}$$

The transformation $\varphi^h$ on $X^h$ is called a *tower* (or *skyscraper*) built over $\varphi$ by $h$, and $(X^h,\varphi^h)$ is called the *primitive* of $(X,\varphi)$ with respect to $h$.

**Remark.** It is easy to see that $\varphi^h$ is a minimal homeomorphism and that the following relations hold:

If we set $X^1 = \{(x,i)|i=1\}$ (clearly $X^1$ may be identified with $X$), then

$$(\varphi^h)_{X^1} = \varphi, \quad r_{X^1} = h, \quad (\varphi_A)^{r_A} = \varphi,$$

where $A$ is a clopen subset of $X$. So inducing and tower building are dual (or converse) constructions. In Section 3 we shall give a concrete interpretation of these constructions in terms of ordered Bratteli diagrams (Theorem 3.8).

**Definition 1.7.** (Kakutani equivalence.) Let $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ be Cantor systems. We say that the systems are *Kakutani equivalent* if (up to conjugacy) they have a common derivative — or, what is the same, a common primitive.

**Remark.** That this really is an equivalence relation is proved as in [23; Ch. 1]. (In [23] ergodic measure-preserving systems are considered, but the proof readily translates to our topological setting.)

**Definition 1.8.** (Kakutani orbit (resp. Kakutani strong orbit) equivalence.) The Cantor systems $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$, are *Kakutani orbit* (resp. *Kakutani strong orbit) equivalent* if they are orbit (resp. strong orbit) equivalent to the systems $(Y_1, \psi_1)$ and $(Y_2, \psi_2)$, respectively, where $(Y_1, \psi_1)$ and $(Y_2, \psi_2)$ are Kakutani equivalent.

**Remark.** That these really are equivalence relations will be a consequence of Theorem 2.6 and Proposition 2.7. Observe that Kakutani orbit (resp. Kakutani strong orbit) equivalence is generated by Kakutani equivalence and orbit (resp. strong orbit) equivalence.

We now turn to an entirely different realm of concepts — those associated with ordered groups. (As general references we refer to [9] and [15].) In Section 3 the correspondence between the dynamical concepts and the concepts related to ordered groups will be illuminated via Bratteli diagrams.

By an *ordered* group we shall mean a countable abelian group $G$ together with a subset $G^+$, called the positive cone, so that

(i) $G^+ + G^+ \subseteq G^+$, (ii) $G^+ - G^+ = G$, (iii) $G^+ \cap (-G^+) = \{0\}$.

We shall write $a \le b$ (resp. $a < b$) if $b-a \in G^+$ (resp. $b-a \in G^+ \setminus \{0\}$). We say that the ordered group $(G, G^+)$ is *unperforated* if $a \in G$ and $na \in G^+$

for some $n \in \mathbf{Z}^+$ implies $a \in G^+$. We observe that an unperforated ordered group is torsion free — in fact, $na = 0$ implies that $a \geq 0$ and $a \leq 0$, and so $a = 0$. By an *order unit* for $(G, G^+)$ we mean an element $u$ in $G^+$ so that, for every $a \in G$, $a \leq nu$ for some $n \in \mathbf{Z}^+$.

**Definition 1.9.** (Dimension group.) An unperforated ordered group $(G, G^+)$ is a *dimension group* if $G$ satisfies the *Riesz interpolation property*, i.e. given $a_1, a_2, b_1, b_2 \in G$ with $a_i \leq b_j (i, j = 1, 2)$, there exists a $c \in G$ with $a_i \leq c \leq b_j$. (If we may get strict inequality provided $a_i < b_j$ we say $G$ satisfies the *strict* Riesz interpolation property.)

**Remark.** Dimension groups were introduced by Elliott [11] and defined by him as the inductive limit of a system of groups of the form $\mathbf{Z}^r$ (with the usual ordering) and positive homomorphisms. This definition was motivated by Bratteli's [3] introduction of the diagrams subsequently named after him (cf. Section 3). The equivalence of Elliott's definition and the abstract definition given above is due to Effros, Handelman and Shen [10].

In this paper we shall exclusively encounter *simple* dimension groups $(G, G^+)$, i.e. $G$ contains no non-trivial order ideal. An order ideal is a subgroup $J$ so that $J = J^+ - J^+$ (where $J^+ = J \cap G^+$) and $0 \leq a \leq b \in J$ implies $a \in J$. Simple dimension groups have a nice interpretation in terms of Bratteli diagrams, cf. Section 3. If $G$ is a simple dimension group it is easily seen that any $a \in G^+ \setminus \{0\}$ is an order unit. Letting $G$ be a simple dimension group with fixed order unit $u \in G^+ \setminus \{0\}$, we say that a homomorphism $p : G \to \mathbf{R}$ is a *state* if $p$ is positive (i.e. $p(G^+) \geq 0$), and $p(u) = 1$. Denote the collection of all states by $S_u(G)$. Then it is a fact that $S_u(G)$ determines the order on $G$. In fact, by [9; Cor. 4.2],

$$G^+ = \{a \in G | p(a) > 0 \text{ for all } p \in S_u(G)\} \cup \{0\}$$

This motivates the following definition.

**Definition 1.10.** (Infinitesimal subgroup.) Let $G$ be a simple dimension group and let $u \in G^+ \setminus \{0\}$. We say that $a \in G$ is *infinitesimal* if $-\varepsilon u \leq a \leq \varepsilon u$ for all $0 < \varepsilon \in \mathbf{Q}$. (If $\varepsilon = p/q, p, q \in \mathbf{Z}^+$, then $a \leq \varepsilon u$ means that $qa \leq pu$.) An equivalent definition is: $a \in G$ is infinitesimal if $p(a) = 0$ for all $p \in S_u(G)$. (It is evident that the infinitesimal elements do not depend on the particular order unit $u$.) The collection of infinitesimal elements of $G$ form a subgroup, the *infinitesimal subgroup* of $G$, which we denote by $Inf(G)$.

**Remark.** The quotient group $G/Inf(G)$ has a natural induced ordering, i.e. $\dot{a} > 0$ if $a > 0$, where $\dot{}$ denotes the quotient map. It is then easy to see that $G/Inf(G)$ becomes a simple dimension group with no infinitesimal elements except 0. If $G$ has distinguished order unit $u$ then $G/Inf(G)$ inherits the distinguished order unit $\dot{u}$. Note that an order isomorphism $\alpha : G_1 \to G_2$, i.e. $\alpha(G_1^+) = G_2^+$, where $(G_1, G_1^+)$ and $(G_2, G_2^+)$ are dimension groups, maps $Inf(G_1)$ onto $Inf(G_2)$.

The link between dimension groups and dynamical systems is established by the next theorem. First we need a definition.

**Definition 1.11.** Let $(X, \varphi)$ be a Cantor system. Let $C(X, \mathbf{Z})$ denote the continuous functions on $X$ with values in $\mathbf{Z}$ — so $C(X, \mathbf{Z})$ is a countable abelian group under addition. Let $B_\varphi$ denote the (coboundary) subgroup $Im(id - \varphi_*) = \{f - f \circ \varphi^{-1} | f \in C(X, \mathbf{Z})\}$. Let $K^0(X, \varphi)$ be the quotient group $C(X, \mathbf{Z})/B_\varphi$ and define

$$K^0(X, \varphi)^+ = \{\dot{f} | f \geq 0, f \in C(X, \mathbf{Z})\},$$

where $\dot{}$ denotes the quotient map.

Let $\mathbf{1} = \mathbf{1}_X$ denote the element of $K^0(X, \varphi)^+$ that the constant function 1 maps to. By abuse of notation we will also let $\mathbf{1}$ denote the element of $K^0(X, \varphi)/Inf(K^0(X, \varphi))$ that $\mathbf{1}$ maps to.

**Theorem 1.12.** ([28 ; Thm 4.1], [17 ; Cor. 6.3].) Let $(X, \varphi)$ be a Cantor system. Then $K^0(X, \varphi)$ with positive cone $K^0(X, \varphi)^+$ is a simple dimension group with (canonical) distinguished order unit $\mathbf{1}$. Furthermore, if $(G, G^+)$ is a simple dimension group with distinguished order unit $u$, there exists a Cantor system $(X, \varphi)$ so that $(G, G^+, u) \cong (K^0(X, \varphi), K^0(X, \varphi)^+, \mathbf{1})$, meaning that there exists an order isomorphism $\alpha : G \to K^0(X, \varphi)$ so that $\alpha(u) = \mathbf{1}$.

**Remark.** The group $C(X, \mathbf{Z})/B_\varphi$, as an abstract group without order, has appeared before in the theory of dynamical systems — it is the first

Čech cohomology group $H^1(\hat{X}, \mathbf{Z})$ of the suspension $\hat{X}$ of $(X, \varphi)$, where $\hat{X}$ is obtained from $X \times [0,1]$ by identifying $(x, 1)$ and $(\varphi(x), 0)$, cf. [24 ; Ch. IV, Sect. 3]. The reason for our notation $K^0(X, \varphi)$ is the connection with $K$-theory (see below).

We next give a characterization of the infinitesimal subgroup $Inf(K^0(X, \varphi))$, which we state as a theorem.

**Theorem 1.13.** ([17; Thm. 5.5], [9; Cor. 4.2].) Let $(X, \varphi)$ be a Cantor system. Then

(i) every $\varphi$-invariant probability measure $\mu$ on $X$ induces a state $T(\mu)$ on $(K^0(X, \varphi), K^0(X, \varphi)^+, \mathbf{1})$ by $f \to \int_X f d\mu$, $f \in C(X, \mathbf{Z})$.

(ii) The map $T$ is a bijective correspondence between the set of $\varphi$-invariant probability measures on $X$ and the set of states on $(K^0(X, \varphi),\ K^0(X, \varphi)^+, \mathbf{1})$.

Hence, $Inf(K^0(X, \varphi)) = Z_\varphi / B_\varphi$, where $Z_\varphi = \{f \in C(X, \mathbf{Z}) | \int_X f d\mu = 0$ for all $\varphi$-invariant prob. measures $\mu\}$. Thus $K^0(X, \varphi)/Inf(K^0(X, \varphi))$ is naturally isomorphic to the quotient group $C(X, \mathbf{Z})/Z_\varphi$, and this isomorphism is an order isomorphism preserving the distinguished order unit $\mathbf{1}$ if $C(X, \mathbf{Z})/Z_\varphi$ is given the induced order, i.e. $\dot{f} \geq 0$ if $f \geq 0$ in $C(X, \mathbf{Z})$.

We now make the connection between dynamical systems and $C^*$-algebras. We briefly summarize some basic general facts about $C^*$-crossed products associated to the dynamical system $(X, \varphi)$ and refer the reader to [25] and [36] for a complete treatment.

The homeomorphism $\varphi$ gives rise to a $*$-automorphism of $C(X)$, also denoted by $\varphi$, by setting $\varphi(f) = f \circ \varphi^{-1}, f \in C(X)$. The associated $C^*$-crossed product, which we denote by $C(X) \underset{\varphi}{\times} \mathbf{Z}$, is the universal $C^*$-algebra generated by $C(X)$ and a unitary element $u$ satisfying $ufu^* = \varphi(f)$, for all $f$ in $C(X)$. (The full and reduced crossed products coincide in this case.) Assume that $(X, \varphi)$ is minimal. Then we may realize the crossed product as follows: Let $\mu$ be any $\varphi$-invariant probability measure on $X$ (which always exists). Then $C(X) \underset{\varphi}{\times} \mathbf{Z}$ is isomorphic to the $C^*$-algebra acting on the Hilbert space $H = L^2(X, \mu)$ and generated by the multiplication operators $m_f : g \to fg$ and the unitary operator $u = u_\varphi : g \to g \circ \varphi^{-1}$, where $f \in C(X), g \in H$. Also, $C(X)$ is naturally embedded in $C(X) \underset{\varphi}{\times} \mathbf{Z}$ as a maximal abelian $C^*$-subalgebra. The minimality of $\varphi$ implies that $C(X) \underset{\varphi}{\times} \mathbf{Z}$ is a simple $C^*$-algebra, i.e. it has no proper two-sided ideals. There is a $1-1$ correspondence between the set of normalized traces on $C(X) \underset{\varphi}{\times} \mathbf{Z}$ and the set of $\varphi$-invariant probability measures on $X$.

Let us now assume that $(X, \varphi)$ is a Cantor system. We will make use of $K$-theory for $C^*$-algebras and we refer to [1] and [9] for the appropriate definitions and basic properties. In [28] it is shown that $K_0(C(X) \underset{\varphi}{\times} \mathbf{Z})$ is order isomorphic to $K^0(X, \varphi)$, and so is a simple dimension group. (Cf. Corollary 1 to Theorem 1.17 below.) Also $K_1(C(X) \underset{\varphi}{\times} \mathbf{Z}) = \mathbf{Z}$, and so, in particular, $C(X) \underset{\varphi}{\times} \mathbf{Z}$ is not an $AF$-algebra (cf. Section 3).

We list some further properties of $C(X) \underset{\varphi}{\times} \mathbf{Z}$ which follow by combining results from [2], [11] and [29]:

(i) $C(X) \underset{\varphi}{\times} \mathbf{Z}$ has stable rank one, in other words , the invertibles are dense.

(ii) $C(X) \underset{\varphi}{\times} \mathbf{Z}$ has real rank zero, in other words, the invertible self-adjoints are dense in all the self-adjoints.

(iii) $C(X) \underset{\varphi}{\times} \mathbf{Z}$ is an inductive limit of a sequence $\{A_n\}$, where $A_n$ is a finite direct sum of $C^*$-algebras, each summand being a $k \times k$ matrix over either $\mathbf{C}$ or $C(\mathbf{T}), \mathbf{T}$ denoting the unit circle. We will call $C^*$-algebras of this form *circle algebras.*

Combining Elliott's [12] remarkable classification theorem for circle algebras of real rank zero with the above properties we state the following theorem which will be part of our Theorem 2.1.

**Theorem 1.14.** A complete isomorphism invariant for the family of $C^*$-crossed products $C(X) \underset{\varphi}{\times} \mathbf{Z}$, where $(X, \varphi)$ is a Cantor system, is the simple dimension group $K^0(X, \varphi)$ with distinguished order unit $\mathbf{1}$.

Combining the above results with Theorem 1.12 we get the following theorem.

**Theorem 1.15.** ([17 ; Th. 8.6, Cor. 8.7].) The family $\mathcal{C} = \{C(X) \underset{\varphi}{\times} \mathbf{Z}) | (X, \varphi)$ Cantor system$\}$ coincides with the family $\mathcal{F}$ of simple circle algebras of real rank zero with $K_1$ equal to $\mathbf{Z}$ (excluding the case when $K_0 = \mathbf{Z}$).

Of crucial importance for proving our results will be an exact sequence of $K_0$–groups of $C^*$-algebras that are associated with the Cantor system $(X, \varphi)$ — a penetrating analysis of which was done in [28]. We summarize the main results of [28] by stating three theorems adapted to our situation and use.
Let $Y$ be a non-empty closed subset of $X$. Define the $C^*$-subalgebra $A_Y^\varphi$ of $C(X) \underset{\varphi}{\times} \mathbf{Z}$ which is generated by $C(X)$ and $u_\varphi C_0(X \setminus Y)$, where $C_0(X \setminus Y)$ denotes the continuous functions vanishing on $Y$.

**Theorem 1.16.** ([28; Th. 3.3 and Cor. 5.6.].) $A_Y^\varphi$ is an AF-algebra which is simple if and only if $Y$ meets each $\varphi$-orbit at most once.

**Theorem 1.17.** ([28; Th. 4.1].) Let $i : A_Y^\varphi \to C(X) \underset{\varphi}{\times} \mathbf{Z}$ denote the inclusion map. Then there is an exact sequence

$$0 \to \mathbf{Z} \xrightarrow{\alpha} C(Y, \mathbf{Z}) \to K_0(A_Y^\varphi) \xrightarrow{i_*} K_0(C(X) \underset{\varphi}{\times} \mathbf{Z}) \to 0$$

where $\alpha$ is the map taking $n \in \mathbf{Z}$ to the constant function $n$. Moreover, for every $a \in K_0(C(X) \underset{\varphi}{\times} \mathbf{Z})^+$, there is $b \in K_0(A_Y^\varphi)^+$ so that $i_*(b) = a$.

**Corollary 1.** Let $Y = \{y\}$. Then $i_* : K_0(A_{\{y\}}^\varphi) \to K_0(C(X) \underset{\varphi}{\times} \mathbf{Z})$ is an isomorphism of ordered groups preserving the distinguished order units, i.e. the elements of $K_0$ corresponding to the identity operators. Hence $K_0(C(X) \underset{\varphi}{\times} \mathbf{Z})$ is a simple dimension group. Also $K_0(C(X) \underset{\varphi}{\times} \mathbf{Z})$ is order isomorphic to $K^0(X, \varphi) = C(X, \mathbf{Z})/Im(id - \varphi_*)$ by a map preserving the distinguished order units.

Combining Theorem 1.17 with Corollary 5.7 of [28] we may state the following two corollaries which will be pivotal results in proving Theorems 2.2, 2.3 and 2.5.

**Corollary 2.** Let $Y = \{y_1, \dots, y_{k+1}\}$ consist of $k+1$ points lying in distinct orbits. Then $A^{\varphi}_{\{y_1,\dots,y_{k+1}\}}$ is a simple AF-algebra and

$$0 \to \mathbf{Z}^k \to K_0(A^{\varphi}_{\{y_1,\dots,y_{k+1}\}}) \xrightarrow{i_*} K_0(C(X) \underset{\varphi}{\times} \mathbf{Z}) \to 0$$

where $\mathbf{Z}^k$ embeds into $Inf(K_0(A^{\varphi}_{\{y_1,\dots,y_{k+1}\}}))$. Furthermore, $i_*$ is strict positivity preserving, i.e. $i_*^{-1}(K_0(C(X) \underset{\varphi}{\times} \mathbf{Z})^+ \setminus \{0\}) = K_0(A^{\varphi}_{\{y_1,\dots,y_{k+1}\}})^+ \setminus \{0\}$.

**Corollary 3.** Let $Y = \{y_1, \dots, y_n, \dots, y_\infty\}$ be a countable set of points lying in distinct orbits so that $y_n \to y_\infty$. Then $A^{\varphi}_{\{y_1,\dots,y_n,\dots,y_\infty\}}$ is a simple AF-algebra and

$$0 \to \mathbf{Z}^\infty \to K_0(A^{\varphi}_{\{y_1,\dots,y_n,\dots,y_\infty\}}) \xrightarrow{i_*} K_0(C(X) \underset{\varphi}{\times} \mathbf{Z}) \to 0$$

where $\mathbf{Z}^\infty$ denotes the direct sum of countable copies of $\mathbf{Z}$, and $\mathbf{Z}^\infty$ embeds into $Inf(K_0(A^{\varphi}_{\{y_1,\dots,y_n,\dots,y_\infty\}}))$. Furthermore, $i_*$ is strict positivity preserving.

Let $\Gamma^\varphi$ denote the quotient group $\mathcal{N}(C(X), C(X) \underset{\varphi}{\times} \mathbf{Z})/\mathcal{U}(C(X))$, where $\mathcal{N}(C(X), C(X) \underset{\varphi}{\times} \mathbf{Z})$ denotes the normalizer group of $C(X)$ in $C(X) \underset{\varphi}{\times} \mathbf{Z}$ and $\mathcal{U}(C(X))$ denotes the unitaries of $C(X)$. Then $\Gamma^\varphi$ acts on $X$ as homeomorphisms preserving the $\varphi$-orbits. Likewise, let $\Gamma^\varphi_Y$ denote $\mathcal{N}(C(X), A^\varphi_Y)/\mathcal{U}(C(X))$. So $\Gamma^\varphi_Y$ is a subgroup of $\Gamma^\varphi$.

**Theorem 1.18.** ([28; Th. 5.4 and Cor. 5.5].) Let $Y = \{y_1, \dots, y_m\}$ consist of $m$ points lying in disjoint orbits. Let $\gamma \in \Gamma^\varphi$. Then $\gamma \in \Gamma^\varphi_Y$ iff $\gamma(\{\varphi^l(y_i) | l \geq 1\}) = \{\varphi^l(y_i) | l \geq 1\}$ for $i = 1, \dots, m$ — in other words, $\gamma$ preserves the half orbits of $y_1, \dots, y_m$.

Finally we shall need the concept of *strong Morita* equivalence of $C^*$-algebras. Rieffel [32] defined strong Morita equivalence of two $C^*$-algebras in the sense of having an imprimitivity bimodule. For separable $C^*$-algebras $B$ and $C$ this was subsequently shown to be equivalent to stable isomorphism, i.e. $B \otimes \mathcal{K} \cong C \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators on a separable infinite-dimensional Hilbert space. We will adopt an equivalent definition of strong Morita equivalence which is analogous to Definition 1.7 of Kakutani equivalence for dynamical systems. We state the definition for simple, unital and separable $C^*$-algebras. (We refer to [32] for a survey article on Morita equivalence for operator algebras.)

**Definition 1.19** (Strong Morita equivalence.) The simple, unital and separable $C^*$-algebras $B$ and $C$ are *strongly Morita equivalent* if $B$ and $C$ are corners of a simple, unital and separable $C^*$-algebra $A$, i.e. there exist projections $p$ and $q$ in $A$ so that $B \cong pAp$, $C \cong qAq$.

**Remark.** Two simple, unital $AF$-algebras $B$ and $C$ are strongly Morita equivalent if and only if the simple dimension groups $(K_0(B), K_0(B)^+)$ and $(K_0(C), K_0(C)^+)$ are order isomorphic, cf. [32]. Theorem 2.6 states an analogous result in the setting of dynamical systems and associated $C^*$-crossed products.

## 2. Main results.

Recall that a Cantor system is a minimal dynamical system $(X, \varphi)$, where $X$ is a Cantor set.

**Theorem 2.1.** Let $(X_i, \varphi_i)$ be Cantor systems ($i = 1, 2$). The following are equivalent:

(i) $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ are strong orbit equivalent.

(ii) $K^0(X_1, \varphi_1)$ is order isomorphic to $K^0(X_2, \varphi_2)$ by a map preserving the distinguished order units.

(iii) $C(X_1) \times_{\varphi_1} \mathbf{Z}$ is isomorphic to $C(X_2) \times_{\varphi_2} \mathbf{Z}$.

**Theorem 2.2.** Let $(X_i, \varphi_i)$ be Cantor systems ($i = 1, 2$). The following are equivalent:

(i) $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ are orbit equivalent.

(ii) The dimension groups $K^0(X_i, \varphi_i)/Inf(K^0(X_i, \varphi_i)), i = 1, 2$, are order isomorphic by a map preserving the distinguished order units.

(iii) There exists a homeomorphism $F : X_1 \to X_2$ carrying the $\varphi_1$-invariant probability measures onto the $\varphi_2$-invariant probability measures.

**Corollary 1.** Let $(X_i, \varphi_i)$ be uniquely ergodic Cantor systems $(i = 1, 2)$. Then $(X_1, \varphi_1)$ is orbit equivalent to $(X_2, \varphi_2)$ if and only if $\{\mu_1(E) | E$ clopen subset of $X_1\} = \{\mu_2(F) | F$ clopen subset of $X_2\}$, where $\mu_i$ is the unique $\varphi_i$-invariant probability measure on $X_i (i = 1, 2)$.

**Proof.** By Theorem 1.13, $(K^0(X_i, \varphi_i), K^0(X_i, \varphi_i)^+, \mathbf{1})$ has a unique state induced by $\mu_i, i = 1, 2$. By [9; Cor. 4.2.] condition (ii) of the theorem is equivalent to the stated equality. □

Recall that the dynamical system $(X, \varphi)$, $X$ compact and metrizable, is *distal* if for every $x_1 \neq x_2$ in $X$ there exists $\varepsilon > 0$ so that $d(\varphi^n(x_1), \varphi^n(x_2)) > \varepsilon$ for all $n \in \mathbf{Z}$. Here $d$ is a metric that yields the topology of $X$. (Observe that the concept of distality does not depend on the metric.) It is a fact that the family of distal Cantor systems coincides with the *odometer systems*, i.e. the family of minimal rotations on compact 0-dimensional groups ("a-adic groups"), cf. [13], [18; Chapters II, IV]. In particular, these systems are uniquely ergodic, the (normalized) Haar measures being the unique invariant probability measures.
Recall that a *Denjoy homeomorphism* is an aperiodic homeomorphism of the circle, $\varphi : \mathbf{T} \to \mathbf{T}$, which is not conjugate to a pure rotation. Denjoy proved that such $\varphi$ can not be of class $C^2$, i.e. $\varphi$ and $\varphi^{-1}$ are twice diffentiable with continuous derivatives. By a *Denjoy system* we shall mean a Denjoy homeomorphism restricted to its unique invariant Cantor set. A Denjoy system is uniquely ergodic. We refer to [30] for a survey and $K$-theoretic properties of such systems.

We have the following remarkable corollary of Theorem 2.2.

**Corollary 2.** Let $(X, \varphi)$ be a uniquely ergodic Cantor system. Then $(X, \varphi)$ is either orbit equivalent to an odometer system or to a Denjoy system. Specifically, $(X, \varphi)$ is orbit equivalent to an odometer system if and only if $\{\mu(E) | E$ clopen subset of $X\}$ is a subset of the rational numbers $\mathbf{Q}$, where $\mu$ is the unique $\varphi$-invariant probability measure.

**Proof.** The simple ordered Bratteli diagrams yielding the Bratteli-Vershik models (Theorem 3.6) for odometer systems are easily seen to be the so-called UHF-diagrams, characterized by the property that the vertex sets

$V_0, V_1, V_2, \ldots$ are all one-point sets. The associated dimension groups are characterized by being order isomorphic to subgroups of $\mathbf{Q}$ (with natural order) containing $\mathbf{Z}$ — the distinguished order units being mapped to 1. Moreover, all subgroups of $\mathbf{Q}$ containing $\mathbf{Z}$ (except $\mathbf{Z}$ itself) occur in this manner [9; Chapters 4, 7]. So by Corollary 1 we get the second statement. Assume now that $\{\mu(E) | E \text{ clopen subset of } X\}$ contains an irrational number $\alpha$. By [30; Theorem 5.3] and Corollary 1 there exists a Denjoy system with rotation number $\alpha$ which is orbit equivalent to $(X, \varphi)$. □

**Remark.** The equivalence of (i) and (iii) in Theorem 2.2 is not true if we drop the condition that the $X_i$'s be Cantor sets. In fact, if $X_1 = X_2 = \mathbf{T}$ (the unit circle) and $\varphi_1$ and $\varphi_2$ are irrational rotations by $\alpha$ and $\beta$ that are linearly independent over $\mathbf{Q}$, then $F = id : \mathbf{T} \to \mathbf{T}$ satisfies the condition of the corollary, but the two minimal systems are not orbit equivalent. (It is easy to show that orbit equivalence coincides with flip conjugacy in this case. More generally, by a theorem of Sierpiński [40; Theorem 6, Ch. V, §47, III] one may conclude that the same holds true for minimal systems on *connected* compact metric spaces.)

**Remark.** To get an analogy to the equivalence of (i) and (iii) of Theorem 2.2 we may formulate the Dye theorem in the ergodic measure-preserving case as follows:

The ergodic measure-preserving systems $(X_i, \mathcal{B}_i, \mu_i, \varphi_i)$, where $(X_i, \mathcal{B}_i, \mu_i)$ are Lebesgue spaces with (non-atomic) probability measures $\mu_i (i = 1, 2)$, are orbit equivalent (in the measure-theoretic sense) if and only if there exists a bimeasurable bijection $F : X_1 \to X_2$ mapping $\mu_1$ to $\mu_2$.
(In fact, it is a classical result dating back to Carathéodory and Souslin that for all systems $(X_1, \mathcal{B}, \mu_1, \varphi_1)$ and $(X_2, \mathcal{B}_2, \mu_2, \varphi_2)$ such an $F$ exists [33; Ch. 15].)

We will extend Theorem 2.2 to actions of groups $\Gamma$ that were considered by Krieger [22] and by Strătilă and Voiculescu [35]. Specifically, we suppose $\Gamma$ is a locally finite, countable group acting on the Cantor set $X$ so that the action of $\Gamma$ is ample (in the sense of [22]), and so that each element of $\Gamma$ has a fix-point set in $X$ which is clopen. Such groups arise as a countable union of finite groups $\{\Gamma_n\}$ each of which permutes the initial segments of paths

associated to a Bratteli diagram $(V, E)$. (Cf. Section 3 for definitions.) This will be explained and utilized later. For this reason such systems $(X, \Gamma)$ will be referred to as AF-systems. We will restrict our attention to minimal AF-systems, i.e. systems $(X, \Gamma)$, where $X$ is a Cantor set and each $\Gamma$-orbit is dense. As described in [22] there is associated with such a system a simple dimension group which we denote by $K^0(X, \Gamma)$.
We may now state the following theorem which will extend Theorem 2.2 to also include AF-systems. For convenience in stating the theorem we will alter our notation so that $K^0(X, \varphi)$ becomes $K^0(X, \mathbf{Z})$. (We must take care that it is clear *which* integer action we consider.)

**Theorem 2.3.** Let $(X_1, \Gamma_1), (X_2, \Gamma_2)$, where $X_1$ and $X_2$ are Cantor sets, be two minimal dynamical systems, each of which is either an AF-system or is singly generated, i.e. $\Gamma_i = \mathbf{Z}$.
The following are equivalent:

i) $(X_1, \Gamma_1)$ and $(X_2, \Gamma_2)$ are orbit equivalent.

ii) The dimension groups $K^0(X_i, \Gamma_i)/Inf K^0(X_i, \Gamma_i)$, $i = 1, 2$, are order isomorphic by a map preserving the distinguished order units.

iii) There exists a homeomorphism $F : X_1 \to X_2$ carrying the $\Gamma_1$-invariant probability measures onto the $\Gamma_2$-invariant probability measures.

The next theorem states what extra condition we must impose on an isomorphism between crossed products to get flip conjugacy of the dynamical systems. Also, the equivalence of (i) and (ii) of the theorem is a special case of Boyle's theorem (Theorem 1.4) and we present a proof of this equivalence here by kind permission.

**Theorem 2.4.** Let $(X_i, \varphi_i)$ be Cantor systems ($i = 1, 2$). The following are equivalent:

(i) $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ are flip conjugate.

(ii) There exists an orbit map $F : X_1 \to X_2$ so that one (hence the other) of the associated orbit cocycles is continuous everywhere.

(iii) There exists an isomorphism $\alpha : C(X_1) \underset{\varphi_1}{\times} \mathbf{Z} \to C(X_2) \underset{\varphi_2}{\times} \mathbf{Z}$ so that $\alpha$ maps $C(X_1)$ onto $C(X_2)$.

**Remark.** Tomiyama [37] has recently obtained a generalization of Theorem 2.4, whereby the Cantor condition may be dropped.

**Remark.** There is an abundance of Cantor systems that are strong orbit equivalent but not flip conjugate. In fact, Boyle and Handelman [6] have recently constructed examples of dynamical systems of all possible (topological) entropies which are strongly orbit equivalent to the dyadic adding machine, and they conjecture that an analogous result is true in general. In fact, they conjecture something much stronger: if $(X, \varphi)$ is a Cantor system and $(Y, \mathcal{D}, \mu, \psi)$ is an ergodic measure-preserving system ($\mu$ non-atomic probability measure), then there should exist a Cantor system $(\tilde{X}, \tilde{\varphi})$ that is strongly orbit equivalent to $(X, \varphi)$ and a $\tilde{\varphi}$-invariant probability measure $\nu$ on $\tilde{X}$ so that $(\tilde{X}, \mathcal{B}, \nu, \tilde{\varphi}), \mathcal{B}$ the Borel sets, is metrically isomorphic to $(Y, \mathcal{D}, \mu, \psi)$. The conjecture would generalize both Dye's theorem as well as the Jewett-Krieger theorem [25; Ch. 4, Sect. 4] (i.e. every $(Y, \mathcal{D}, \mu, \psi)$ is metrically isomorphic to some minimal uniquely ergodic homeomorphism of the Cantor set). (Recall that orbit equivalent systems (and so strong orbit equivalent systems) have the same space of invariant probability measures, cf. Theorem 2.2.) However, the authors of this paper have found an obstruction for the above conjecture to be true, and this concerns the *rational* part of the discrete spectrum. (Details of this will be published elsewhere.) In fact, the conjecture should be modified by adding the condition that the rational discrete spectrum of $(Y, \mathcal{D}, \mu, \psi)$ should contain the rational eigenvalues of $(X, \varphi)$.

Returning to Theorem 2.4: As entropy is an invariant under flip conjugacy the above theorem in conjunction with Theorem 2.1 implies that the Cartan subalgebras (or "diagonals") (cf. [31]) of the crossed product $C(X) \underset{\varphi}{\times} \mathbf{Z}$ that are isomorphic to $C(X)$ are not conjugate. (Here $(X, \varphi)$ is a Cantor system.) In fact, if they were conjugate any isomorphism of Theorem 2.1 will satisfy condition (iii) of Theorem 2.4 by composing with an automorphism. This contrasts with the situation for AF-algebras, cf. [31; Ch. III, Cor. 1.16].

The next theorem considers the continuity properties of the orbit cocycles $m$ and $n$ and throws new light on the notion of strong orbit equivalence (Definition 1.3).

**Theorem 2.5.** Let $(X_i, \varphi_i)$ be Cantor systems ($i = 1, 2$). The following are equivalent:

(i) $(X_1, \varphi_1)$ is orbit equivalent to $(X_2, \varphi_2)$, and there exists an orbit map $F : X_1 \to X_2$ so that the associated orbit cocycles $m, n : X_1 \to \mathbf{Z}$ each have finitely many points of discontinuity in (necessarily the same) disjoint $\varphi_1$-orbits. Assume $k + 1$ is the least possible number of discontinuity points by considering all such maps $F$. (If there are no discontinuity points we set $k = 0$.)

(ii) There exist subgroups, both isomorphic to $\mathbf{Z}^k$, of $Inf(K^0(X_1, \varphi_1))$ and $Inf(K^0(X_2, \varphi_2))$, respectively, so that the quotient groups

$$K^0(X_i, \varphi_i)/\mathbf{Z}^k \quad (i = 1, 2)$$

(with the induced order) are order isomorphic by a map preserving the distinguished order units (more precisely, the quotient image of the distinguished order units), where $k$ is the least natural number with this property.

**Remark.** The groups $K^0(X_i, \varphi_i)/\mathbf{Z}^k$ ($i = 1, 2$) may have torsion and so are not necessarily dimension groups. However, the positive cones corresponding to the induced order do satisfy conditions (i), (ii), (iii) that are listed in Section 1 for ordered groups. And so it makes sense to talk about order isomorphism in this case.

**Corollary.** Let $(X_i, \varphi_i)$ be Cantor systems ($i = 1, 2$). Assume $K^0(X_1, \varphi_1)$ is order isomorphic to $K^0(X_2, \varphi_2)$ by a map carrying the distinguished order unit of $K^0(X_1, \varphi_1)$ to the distinguished order unit of $K^0(X_2, \varphi_2)$ modulo $Inf(K^0(X_2, \varphi_2))$. There exists an orbit map $F : X_1 \to X_2$ so that each of the associated orbit cocycles has at most 2 points of discontinuity in disjoint orbits.

**Proof.** Let $\alpha : K^0(X_1, \varphi_1) \to K^0(X_2, \varphi_2)$ be the order preserving map so that $\alpha(\mathbf{1}) = \mathbf{1} + e$, where $e \in Inf(K^0(X_2, \varphi_2))$. So by Theorem 2.2

$(X_1, \varphi_1)$ is orbit equivalent to $(X_2, \varphi_2)$. Also, $e$ generates a subgroup of $Inf(K^0(X_2, \varphi_2))$ isomorphic to $\mathbf{Z}$. Likewise $f = \alpha^{-1}(e)$ is an element of $Inf(K^0(X_1, \varphi_1))$ that generates a subgroup of $Inf(K^0(X_1, \varphi_1))$ isomorphic to $\mathbf{Z}$. Dividing out by $\mathbf{Z}$ in both $K^0(X_1, \varphi_1)$ and $K^0(X_2, \varphi_2)$ and considering the induced map

$$\tilde{\alpha} : K^0(X_1, \varphi_1)/\mathbf{Z} \to K^0(X_2, \varphi_2)/\mathbf{Z},$$

we see that condition (ii) of the theorem is satisfied for $k \leq 1$. □

**Remark.** There do exist systems where the scenario of the corollary arises and gives rise to exactly 2 points of discontinuity. The converse is not true however: one may have systems with 2 points of discontinuity not arising from the above scenario. Examples can be obtained by using the fact that $Ext(\mathbf{Q}, \mathbf{Z}) \neq 0$. In fact, let $G$ be a (acyclic) simple dimension group with order unit $u$, and let $G_1 = G \oplus Inf(G_1)$, $G_2 = G \oplus Inf(G_2)$, with order units $(u, 0)$, so that $Inf(G_1)/\mathbf{Z} \cong Inf(G_2)/\mathbf{Z} \cong \mathbf{Q}$ (for subgroups of $Inf(G_i), i = 1, 2$, isomorphic to $\mathbf{Z}$), but $Inf(G_1) \not\cong Inf(G_2)$. Then $G_1$ is not order isomorphic to $G_2$, but condition (ii) of Theorem 2.5 is satisfied for $k = 1$ (cf. Theorem 1.12).

The next theorem relates Kakutani strong orbit equivalence of dynamical systems to Morita equivalence of the crossed products — the link being the associated dimension groups *without* distinguished order units.

**Theorem 2.6.** Let $(X_i, \varphi_i)$ be Cantor systems ($i = 1, 2$). The following are equivalent:

(i) $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ are Kakutani strong orbit equivalent.

(ii) $K^0(X_1, \varphi_1)$ is order isomorphic to $K^0(X_2, \varphi_2)$ (by a map not necessarily preserving the distinguished order units).

(iii) $C(X_1) \underset{\varphi_1}{\times} \mathbf{Z}$ is strongly Morita equivalent to $C(X_2) \underset{\varphi_2}{\times} \mathbf{Z}$.

**Proposition 2.7.** Let $(X_i, \varphi_i)$ be as in Theorem 2.6 ($i = 1, 2$). Then $(X_1, \varphi_1)$ is Kakutani orbit equivalent to $(X_2, \varphi_2)$ if and only if the simple dimension groups

$$K^0(X_i, \varphi_i)/Inf(K^0(X_i, \varphi_i)) \ ; \ i = 1, 2,$$

are order isomorphic (by a map not necessarily preserving the distinguished order units).

## 3. Ordered Bratteli diagrams and associated dynamical systems. The dynamical significance of the order unit – Kakutani equivalence.

We first recall some basic concepts and results which also will give us the opportunity of introducing some relevant notation. For details we refer to [9], [17]. (Cf. also [34].)

**Definition 3.1.** A *Bratteli diagram* $(V, E)$ consists of a vertex set $V$ and an edge set $E$, where $V$ and $E$ can be written as a countable disjoint union of non-empty finite sets:

$$V = V_0 \cup V_1 \cup V_2 \cup \cdots \text{ and } E = E_1 \cup E_2 \cup \cdots$$

with the following properties:

i) $V_0 = \{v_0\}$ is a one-point set.

ii) There exist a range map $r$ and a source map $s$ from $E$ to $V$ so that $r(E_n) \subset V_n$ and $s(E_n) \subset V_{n-1}$. Also, we assume that $s^{-1}(v) \neq \emptyset$ for all $v \in V$ and $r^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_0$.

There is an obvious notion of isomorphism between Bratteli diagrams $(V, E)$ and $(V', E')$ ; namely, there exist a pair of bijections between $V$ and $V'$ and between $E$ and $E'$ preserving the gradings and intertwining the respective source and range maps.

A Bratteli diagram can be given a diagrammatic presentation with $V_n$ the vertices at level $n$ and $E_n$ the edges between $V_{n-1}$ and $V_n$, and with the obvious range and source maps. Also, if $|V_{n-1}| = t_{n-1}$ and $|V_n| = t_n$ then the edge set $E_n$ is described by a $t_n \times t_{n-1}$ incidence matrix. (See Fig. 1).

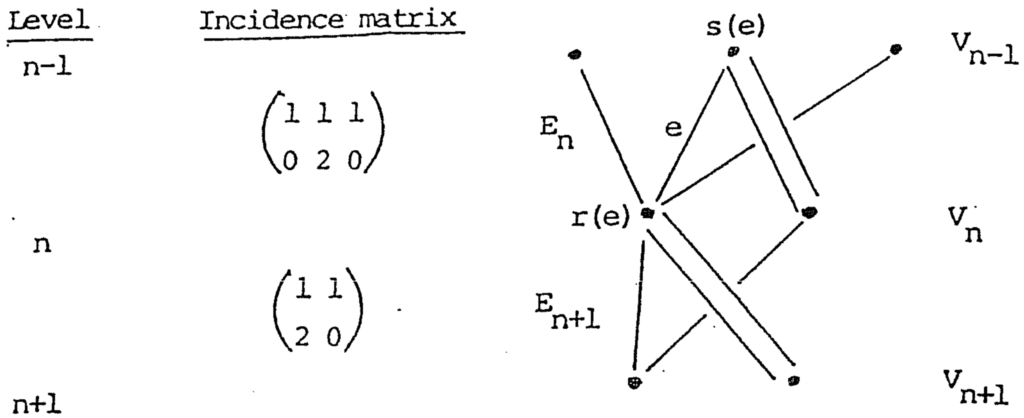



Figure 1.

Now let $k, l \in \mathbf{Z}^+$ with $k < l$ and let $E_{k+1} \circ E_{k+2} \circ \cdots \circ E_l$ denote all paths from $V_k$ to $V_l$. Specifically,

$$\begin{aligned} E_{k+1} \circ \cdots \circ E_l &= \{(e_{k+1}, \cdots, e_l) | e_i \in E_i, i = k+1, \cdots, l; \\ & \quad r(e_i) = s(e_{i+1}), i = k+1, \cdots, l-1\}. \end{aligned}$$

We define $r((e_{k+1}, \cdots, e_l)) = r(e_l)$ and $s((e_{k+1}, \cdots, e_l)) = s(e_{k+1})$ .

**Definition 3.2.** Given a Bratteli diagram $(V, E)$ and a sequence $m_0 = 0 < m_1 < m_2 < \cdots$ in $\mathbf{Z}^+$, we define the *telescoping* (called *contraction* in [17]) of $(V, E)$ to $\{m_n\}$ as $(V', E')$, where $V'_n = V_{m_n}$ and $E'_n = E_{m_{n-1}+1} \circ \cdots \circ E_{m_n}$ and the range and source maps are as above.

For example, if we remove level $n$ of Figure 1 we get a telescoping to levels $n-1$ and $n+1$ as indicated in Figure 2. Note that the new incidence matrix is the product of the two incidence matrices of Figure 1.

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 2 \end{pmatrix}$$

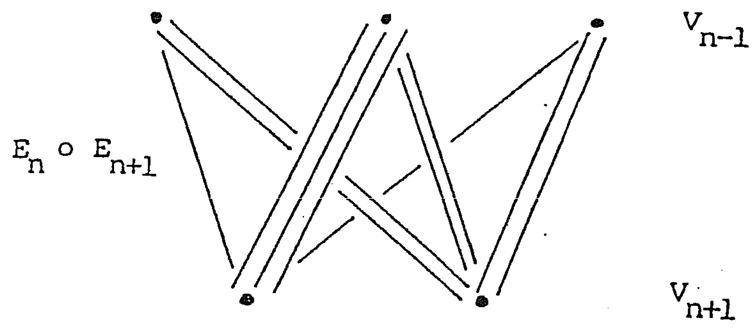



Figure 2.

The converse operation of telescoping a Bratteli diagram is *microscoping*, i.e. filling in new levels – thus making the diagram more "detailed" – so that by telescoping to the old levels we get the original diagram back.

The Bratteli diagram $(V, E)$ gives rise to an approximately finite-dimensional $(AF)$ $C^*$-algebra $AF(V, E)$ as follows: at the vertex $v \in V$ we put $M_{k(v)}(\mathbf{C})$, the $k(v) \times k(v)$ matrix algebra over $\mathbf{C}$, where $k(v)$ is the number of paths from the top vertex $v_0 \in V_0$ to $v$. (We put $\mathbf{C}$ at $v_0$.) At level $n$ we get a (finite-dimensional) multi-matrix algebra $A_n$ of the form

$$A_n = M_{k(v_1^{(n)})} \oplus \cdots \oplus M_{k(v_{t_n}^{(n)})}$$

where $V_n = \{v_1^{(n)}, \ldots, v_{t_n}^{(n)}\}$. $A_n$ has a unique norm in which it is a $C^*$-algebra. The Bratteli diagram encodes the various (unital) inclusion maps $i_{n+1} : A_n \to A_{n+1}$. By definition $AF(V, E)$ is the $C^*$-algebraic limit of the system

$$\mathbf{C} = A_0 \xrightarrow{i_1} A_1 \xrightarrow{i_2} A_2 \longrightarrow \cdots$$

We let $\sim$ denote the equivalence relation on Bratteli diagrams generated by isomorphism and telescoping. It is not hard to show that $(V^1, E^1) \sim (V^2, E^2)$ if and only if there exists a Bratteli diagram $(V, E)$ so that telescoping $(V, E)$ to odd levels $0 < 1 < 3 < \cdots$ yields a telescoping of either $(V^1, E^1)$ or $(V^2, E^2)$, and telescoping $(V, E)$ to even levels $0 < 2 < 4 < \cdots$

yields a telescoping of the other. The $AF$-algebra $AF(V,E)$ associated to $(V,E)$ only depends on the equivalence class of $(V,E)$. In fact, by Bratteli's fundamental paper [3; Thm. 2.7] we have that $AF(V,E) \cong AF(V',E')$ if and only if $(V,E) \sim (V',E')$. We remark that to the Bratteli diagram $(V,E)$ is also associated a *dimension group* which we denote by $K_0(V,E)$ — the notation is motivated by the connection to $K$-theory, see below. In fact, to the Bratteli diagram $(V,E)$ is associated a system of ordered groups

$$\mathbf{Z}^{|V_0|} \xrightarrow{\varphi_1} \mathbf{Z}^{|V_1|} \xrightarrow{\varphi_2} \mathbf{Z}^{|V_2|} \xrightarrow{\varphi_3} \mathbf{Z}^{|V_3|} \rightarrow \cdots$$

where the positive homomorphism $\varphi_n$ is given by matrix multiplication with the incidence matrix between levels $n-1$ and $n$. By definition $K_0(V,E)$ is the inductive limit of the system above endowed with the induced order. $K_0(V,E)$ has a distinguished order unit, namely the element of $K_0(V,E)^+$ corresponding to the element $1 \in \mathbf{Z}^{|V_0|} = \mathbf{Z}$. Elliott [11] showed that $(V,E) \sim (V',E')$ if and only if $K_0(V,E)$ is order isomorphic to $K_0(V',E')$ by a map sending the distinguished order unit of $K_0(V,E)$ to the distinguished order unit of $K_0(V',E')$.

**Definition 3.3.** An *ordered Bratteli diagram* $(V,E,\geq)$ is a Bratteli diagram $(V,E)$ together with a partial order $\geq$ on $E$ so that edges $e,e'$ in $E$ are comparable if and only if $r(e) = r(e')$ ; in other words, we have a linear order on each set $r^{-1}(v), v \in V \setminus V_0$.

Note that if $(V,E,\geq)$ is an ordered Bratteli diagram and $k < l$ in $\mathbf{Z}^+$, then the set $E_{k+1} \circ E_{k+2} \circ \cdots \circ E_l$ of paths from $V_k$ to $V_l$ may be given an induced (lexicographic) order as follows:

$$(e_{k+1}, e_{k+2}, \cdots, e_l) > (f_{k+1}, f_{k+2}, \cdots, f_l)$$

if and only if for some $i$ with $k+1 \leq i \leq l$, $e_j = f_j$ for $i < j \leq l$ and $e_i > f_i$. It is a simple observation that if $(V,E,\geq)$ is an ordered Bratteli diagram and $(V',E')$ is a telescoping of $(V,E)$ as defined in Definition 3.2, then with the induced order $(V',E',\geq)$ is again an ordered Bratteli diagram. We say that $(V',E',\geq)$ is a *telescoping* of $(V,E,\geq)$.

Again there is an obvious notion of isomorphism between ordered Bratteli diagrams. We let $\approx$ denote the equivalence relation on ordered Bratteli diagrams generated by isomorphism and by telescoping. One can show that $B^1 \approx B^2$, where $B^1 = (V^1, E^1, \geq^1)$, $B^2 = (V^2, E^2, \geq^2)$, if and only if there exists an ordered Bratteli diagram $B = (V, E, \geq)$ so that telescoping $B$ to odd levels $0 < 1 < 3 < \cdots$ yields a telescoping of $B^1$ (or $B^2$), and telescoping $B$ to even levels $0 < 2 < 4 < \cdots$ yields a telescoping of the other. This is analogous to the situation for the equivalence relation $\sim$ on Bratteli diagrams as we discussed above.

In contrast to the situation for Bratteli diagrams not every microscoping procedure will render $\approx$ equivalent ordered Bratteli diagrams. However, the special microscoping procedure that we will call *symbol splitting* does give $\approx$ equivalent diagrams. Symbol splitting between two consecutive levels, say level $n-1$ and level $n$, of the ordered Bratteli diagram $B = (V, E, \geq)$ is done by filling in one new level between the two so that the number of new vertices equals the number of edges in $E_n$ , and so that there is exactly one path from level $n-1$ to level $n$ going through each of the new vertices. It is easy to see how to introduce an order on the edge set of the new diagram so that by telescoping one gets the original ordered diagram back. Figure 3 illustrates symbol splitting (the ordering of the edges are indicated).

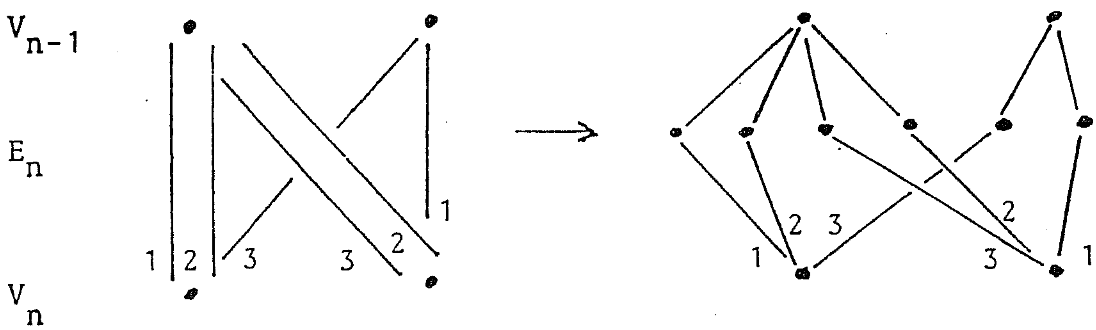



Figure 3

Let $B = (V, E, \geq)$ be an ordered Bratteli diagram. Let $X_B$ denote the associated infinite path space, i.e.

$$X_B = \{(e_1, e_2, \cdots) |\ e_i \in E_i, r(e_i) = s(e_{i+1}) \text{ for } i = 1, 2, \cdots\} .$$

We will exclude trivial cases and assume henceforth that $X_B$ is an infinite set. Two paths in $X_B$ are said to be *cofinal* if they have the same tails, i.e.

the edges agree from a certain stage. We topologize $X_B$ by postulating a basis of open sets, namely the family of cylinder sets

$$U(e_1, e_2, \cdots, e_k) = \{(f_1, f_2, \cdots) \in X_B | f_i = e_i, 1 \leq i \leq k\}$$

Each $U(e_1, \cdots, e_k)$ is also closed, as is easily seen, and so we observe that $X_B$ becomes a compact Hausdorff space with a countable basis of clopen sets, i.e. a zero-dimensional space. We call $X_B$ with this topology the *Bratteli compactum* associated to $B = (V, E, \geq)$. (Note that we do not need the order of $B$ to define the space $X_B$ — in fact, $X_B$ is determined solely by $(V, E)$.)

Recall that $(V, E)$ is a *simple* Bratteli diagram if there exists a telescoping $(V', E')$ of $(V, E)$ so that the incidence matrices of $(V', E')$ have only non-zero entries at each level. This is equivalent to $AF(V, E)$ being a simple $C^*$-algebra, and also equivalent to the dimension group $K_0(V, E)$ associated to $(V, E)$ being simple. Let $B = (V, E, \geq)$ be an ordered Bratteli diagram and let $E_{\max}$ and $E_{\min}$ denote the maximal and minimal elements of the partially ordered set $E$, respectively.

**Definition 3.4.** A *simple* ordered Bratteli diagram is an ordered Bratteli diagram $B = (V, E, \geq)$ so that

(a) $(V, E)$ is a simple Bratteli diagram.

(b) There is a unique path $y_{\max} = (d_1, d_2, \cdots)$ in $X_B$ so that each $d_i \in E_{\max}, i = 1, 2, \cdots$. Likewise, there is a unique path $y_{\min} = (h_1, h_2, \cdots)$ in $X_B$ so that each $h_i \in E_{\min}, i = 1, 2, \cdots$.

**Remarks.**

(i) In [17] the term *essentially simple* ordered Bratteli diagram was used about ordered Bratteli diagrams satisfying condition (b). A slightly more general family of dynamical systems was studied there.

(ii) Observe that condition (a) implies that the topological space $X_B$ has no isolated points, and so $X_B$ is a Cantor set.

(iii) We stress that it is the *uniqueness* part of (b) that is important. In fact, one can show that for any ordered Bratteli diagram $B$ there exist paths in $X_B$ so that each edge lies in $E_{\max}$ (respectively, $E_{\min}$).

(iv) If $(V,E)$ is a simple Bratteli diagram with only non-zero entries in the incidence matrices associated to $(V,E)$, one may introduce an order $\geq$ on $(V,E)$ to make it into a simple ordered Bratteli diagram $B = (V,E,\geq)$, cf. [34; Section 1]. Actually, it is not hard to see that one can do the following: if $x$ and $z$ are two paths of the infinite path space associated to the simple Bratteli diagram $(V,E)$ such that the set of edges making up $x$, respectively $z$, are disjoint, one may introduce an order $\geq$ so that $x$, respectively $z$, is the unique max path $y_{\max}$, respectively unique min path $y_{\min}$, of $(V,E,\geq)$.

We now define a homeomorphism $\lambda_B : X_B \to X_B$ , where $B = (V,E,\geq)$ is a simple ordered Bratteli diagram. We let $\lambda_B(y_{\max}) = y_{\min}$. For $x = (e_1, e_2, \cdots) \neq y_{\max}$, we define $\lambda_B(x)$ as follows: since $x \neq y_{\max}$ at least one of the edges $\{e_i\}$ is not in $E_{\max}$ ; let $k$ be the smallest natural number so that $e_k \notin E_{\max}$. Let $f_k$ denote the successor of $e_k$ in $E$ (and so $r(f_k) = r(e_k)$). Let $(f_1, f_2, \cdots, f_{k-1})$ denote the unique least element in $E_1 \circ E_2 \circ \cdots \circ E_{k-1}$ from the top vertex $v_0 \in V_0$ to $s(f_k) \in V_{k-1}$. Then we define

$$\lambda_B((e_1, e_2, \cdots)) = (f_1, f_2, \cdots, f_k, e_{k+1}, e_{k+2}, \cdots)$$

noting that $r(f_k) = r(e_k) = s(e_{k+1})$.

It is a simple observation that the map $\lambda_B : X_B \to X_B$ is a minimal homeomorphism. We will refer to $\lambda_B$ as the *Vershik* map and to $(X_B, \lambda_B)$ as the *Bratteli-Vershik* dynamical system associated to $B$. It will be a crucial fact that every Cantor system is conjugate to a Bratteli-Vershik system (Theorem 3.6).

In the proof of Theorem 2.2, Theorem 2.3 and Theorem 2.5 we shall need to consider ordered Bratteli diagrams $B = (V,E,\geq)$ where only condition (a) of Definition 3.4 is satisfied. With $X_B^{\max} = \{(e_1, \cdots, e_n, \cdots) \in X_B | e_n \in E_{\max}\}$, $X_B^{\min} = \{(f_1, \cdots, f_n, \cdots) \in X_B | f_n \in E_{\min}\}$, we first observe that condition (a) implies that $X_B^{\min} \cap X_B^{\max} = \emptyset$. We define the map $\lambda_B$ on $X_B \setminus X_B^{\max}$ as above and we retain the name Vershik map for $\lambda_B$. A simple argument shows that $\lambda_B$ maps $X_B \setminus X_B^{\max}$ homeomorphically onto $X_B \setminus X_B^{\min}$ (cf. [17; Sect. 3]). The problem of when $\lambda_B$ may be extended continuously to $X_B$ is rather subtle and will be commented on later. If $\lambda_B$ can be extended to yield a homeomorphism (which we still will denote by $\lambda_B$) of $X_B$ we say that $B$ is *compatible*. If $B$ is compatible then $\lambda_B : X_B \to X_B$ is uniquely determined and $(X_B, \lambda_B)$ is a minimal dynamical system (which we again will call a Bratteli-Vershik system). In fact, the simplicity of the Bratteli diagram $(V, E)$ implies that $X_B \setminus X_B^{\max}$ is dense in $X_B$ and also that the $\lambda_B$-orbits are dense — facts that are easily established. Clearly $\lambda_B$ maps $X_B^{\max}$ onto $X_B^{\min}$. Also, if $B' = (V', E', \geq)$ is a telescoping of a compatible $B = (V, E, \geq)$ then $B'$ is compatible and $(X_B, \lambda_B)$ is conjugate to $(X_{B'}, \lambda_{B'})$ via the natural map $X_B \to X_{B'}$. Consequently, if $B = (V, E, \geq$ ) is compatible and $B \approx B' = (V', E', \geq')$ then $B'$ is compatible and the associated Bratteli-Vershik systems are conjugate.

**Definition 3.5.** Let $(X_1, \varphi_1, x_1)$ and $(X_2, \varphi_2, x_2)$ be two (pointed) dynamical systems, where $x_1 \in X_1, x_2 \in X_2$. A *pointed* topological conjugacy between the two systems is a homeomorphism $h : X_1 \to X_2$ so that $h(x_1) = x_2$ and $h \circ \varphi_1 = \varphi_2 \circ h$. (In particular, the dynamical systems $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ are conjugate.)

To a simple ordered Bratteli diagram $B = (V, E, \geq)$ with unique maximal path $y_{\max}$ we associate the (pointed) minimal dynamical system $(X_B, \lambda_B, y_{\max})$, $X_B$ being a Cantor set. We can now state the model theorem of [17].

**Theorem 3.6.** ([17; Th. 4.7].) Let $(X, \varphi, x)$ be a (pointed) minimal dynamical system, where $X$ is a Cantor set. Then there exists a simple ordered Bratteli diagram $B = (V, E, \geq)$, with unique maximal path $y_{\max}$ , so that $(X, \varphi, x)$ is pointedly topologically conjugate to $(X_B, \lambda_B, y_{\max})$. Moreover, this correspondence establishes a bijection of equivalence classes. Specifically, if $(X_i, \varphi_i, x_i)$ corresponds to $B^i = (V^i, E^i, \geq^i), i = 1, 2$, then $(X_1, \varphi_1, x_1)$ is pointedly conjugate to $(X_2, \varphi_2, x_2)$ if and only if $B^1 \approx B^2$.

**Remarks.**

(i) Change of base points will in general yield $\approx$-inequivalent simple ordered Bratteli diagrams. It is noteworthy, however, that the Bratteli diagrams themselves (i.e. strip the order structure) are $\sim$-equivalent. This will be an immediate consequence of Th. 3.7 below.

(ii) We observe that the orbits of the dynamical system $(X_B, \lambda_B)$ are determined by the cofinal relation, i.e. paths having the same tails, with the proviso that the paths that are cofinal with the unique maximal path $y_{\max}$ are "glued" to the paths that are cofinal with the unique minimal path $y_{\min}$.

**Theorem 3.7.** ([17; Th. 5.4].) Let $(X, \varphi)$ be a Cantor system. Let $B = (V, E, \geq)$ be the associated simple ordered Bratteli diagram according to Theorem 3.6 (having chosen a base point of $X$). Let $K_0(V, E)$ be the simple dimension group associated to $B$ with distinguished order unit. Then

$$K^0(X, \varphi) \cong K_0(V, E)$$

as ordered groups with distinguished order units.

Observe that if $(V, E)$ is a Bratteli diagram with associated dimension group $G = K_0(V, E)$, then any *finite* change of $(V, E)$, i.e. adding and/or removing a finite number of edges, thus turning $(V, E)$ into a new Bratteli diagram $(V', E')$, does not change the isomorphism class of $G$ but does change the order unit. In fact, $G' = K_0(V', E')$ is order isomorphic to $G$ but the distinguished order units are not necessarily preserved by the isomorphism. Clearly any change of order unit of $G$ may be obtained by such a procedure.

Likewise, if $B = (V, E, \geq)$ is a simple ordered Bratteli diagram we may change $B$ into a new simple ordered Bratteli diagram $B' = (V', E' \geq')$ by making a finite change, i.e. adding and/or removing any finite number of edges and then making arbitrary choices of linear orderings of the edges meeting at the same vertex for a finite number of vertices. So $B$ and $B'$ are cofinally identical, i.e. they only differ on finite initial portions. (Observe that this defines an equivalence relation on the family of simple ordered Bratteli diagrams.) We shall show that the Bratteli-Vershik systems $(X_B, \lambda_B)$ and $(X_{B'}, \lambda_{B'})$ associated to $B$ and $B'$, respectively, are Kakutani equivalent. Conversely, any dynamical system Kakutani equivalent to $(X_B, \lambda_B)$ is obtained in this way. This is the content of the next theorem.

**Theorem 3.8.** Let $(X_B, \lambda_B)$ be the Bratteli-Vershik system associated to the simple ordered Bratteli diagram $B = (V, E, \geq)$. Then the Cantor system $(Z, \psi)$ is Kakutani equivalent to $(X_B, \lambda_B)$ if and only if $(Z, \psi)$ is conjugate to $(X_{B'}, \lambda_{B'})$, where $B' = (V', E' \geq')$ is obtained from $B$ by a finite change as described above.

**Proof.** Assume first that $B' = (V', E' \geq')$ is obtained from $B = (V, E, \geq)$ by a finite change. Let $B'$ and $B$ be identical from level $n'$ and level $n$, respectively. Telescope $B'$ to level $n'$ and $B$ to level $n$ from the top vertices. We may as well work with the telescoped diagrams. Hence we may assume that $B$ and $B'$ are identical from level 1.

Now the simple ordered Bratteli diagram $\tilde{B}$ that is obtained from $B$ and $B'$ by being identical with the two from level 1 and having exactly one edge for each vertex at level 1 connected to the top vertex, has the following property: The associated Bratteli-Vershik system $(X_{\tilde{B}}, \lambda_{\tilde{B}})$ is an induced system (or derivative, cf. Definition 1.5) of both $(X_B, \lambda_B)$ and $(X_{B'}, \lambda_{B'})$, which is an immediate consequence of the way the Vershik map is defined. Hence $(X_B, \lambda_B)$ and $(X_{B'}, \lambda_{B'})$ are Kakutani equivalent according to Definition 1.7.

Conversely, assume $(Z, \psi)$ is Kakutani equivalent to $(X_B, \lambda_B)$ and let $(Y, \theta)$ be a common derivative. So $(Y, \theta)$ is the induced system of $(X_B, \lambda_B)$ with respect to a clopen set $F$ in $X_B$. Now $F$ is a finite union of cylinder sets. We may telescope $B = (V, E, \geq)$ from level 0 to a new level 1 so that $F$ in the telescoped diagram is a finite union of cylinders, each of which is defined by a single edge from level 0 to level 1. We may also assume by the simplicity of $(V, E)$ (telescoping further if necessary) that the defining edges will connect the top vertex to each of the vertices at level 1. In the telescoped diagram let $(V'', E'')$ be the Bratteli diagram corresponding to $F$ in an obvious way and let $B'' = (V'', E'', \geq'')$ be the simple ordered diagram that is induced from (the telescoped) $B$ by restricting the ordering of the edge set. We may henceforth assume that $B$ and $B''$ agree from level 1. The situation between level 0 and level 1 is illustrated in Figure 4, where

$1 \leq a_i \leq k_i$, $i = 1, 2, \ldots, m$, denote the number of edges.

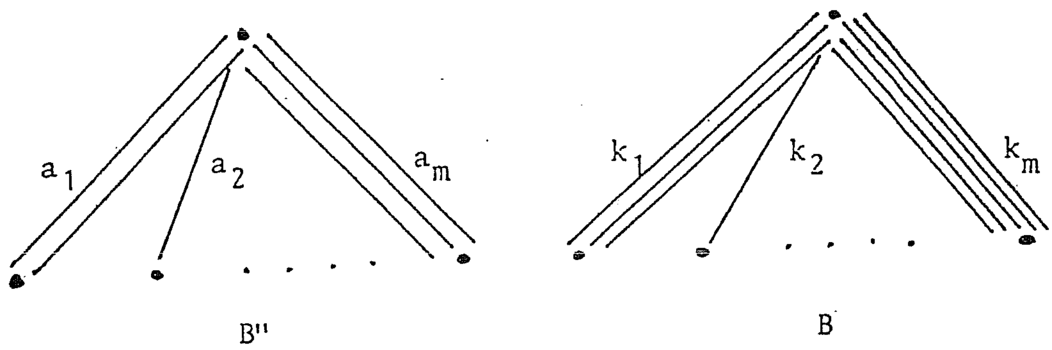
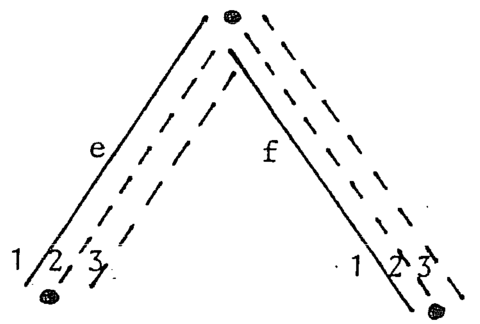



Figure 4.

We observe as we did above that the Bratteli-Vershik system $(X_{B''}, \lambda_{B''})$ associated to $B''$ is the derivative of $(X_B, \lambda_B)$ with respect to $F$, and hence $(X_{B''}, \lambda_{B''})$ is conjugate to $(Y, \theta)$. So $(Z, \psi)$ is a primitive of $(X_{B''}, \lambda_{B''})$, cf. Definition 1.6.

We will now construct a simple ordered Bratteli diagram $B' = (V', E', \geq')$, which will be obtained by a finite change of $B''$, hence by a finite change of $B$, so that $(Z, \psi)$ is conjugate to $(X_{B'}, \lambda_{B'})$, thus completing the proof.

In fact, let $h : X_{B''} \rightarrow \{1, 2, \ldots, N\}$ be the continuous function that determines the tower $(X^h_{B''}, \lambda^h_{B''})$ built over $\lambda_{B''}$ as described in Definition 1.6 so that $(Z, \psi)$ is conjugate to $(X^h_{B''}, \lambda^h_{B''})$. Let $F_i = h^{-1}(i), i = 1, \ldots, N$. Then $X_{B''} = F_1 \cup \cdots \cup F_N$ is a partition of $X_{B''}$ into clopen sets. Each $F_i$ is a finite union of cylinder sets. Telescope $B''$ from level 0 to a new level 1 so that every one of the cylinder sets in question is defined by a single edge between level 0 and level 1 of the telescoped diagram. Now do a symbol splitting between these two levels, thereby introducing a new level 1, so that there are no multiple edges between the top vertex and the vertices at level 1 of the ensuing diagram $B'''$. $B'''$ is $\approx$-equivalent to $B''$ and hence the respective Bratteli-Vershik systems are conjugate (cf. Theorem 3.6). Now define $B'$ from $B'''$ by adding $i - 1$ edges to each of the edges between level 0 and level 1 of $B'''$ representing $F_i (\neq \emptyset)$, in each case letting the original edge be the smallest among the $i$ edges meeting at the same vertex, for $i = 1, 2, \ldots, N$. The procedure is illustrated in Figure 5 where the ordering of the edges is indicated.




$F_3 = U(e) \cup U(f)$

Figure 5.

It is easily verified that the Bratteli-Vershik system associated to $B'$ is conjugate to $(X^h_{B''}, \lambda^h_{B''})$, hence conjugate to $(Z, \psi)$. Since obviously $B'$ is obtained from $B''$ by a finite change we are done. $\square$

We now want to establish the effect Kakutani equivalence has with respect to crossed products. The following proposition will address that question.

**Proposition 3.9.** Let $(X, \varphi)$ be a Cantor system. Let $Z$ be a clopen subset of $X$ and let $(Z, \psi)$ denote the induced (derivative) system of $(X, \varphi)$ with respect to $Z$. Then $C(Z) \underset{\psi}{\times} \mathbf{Z} \cong p(C(X) \underset{\varphi}{\times} \mathbf{Z})p$, where $p$ is the projection $\chi_Z$ in $C(X) \underset{\varphi}{\times} \mathbf{Z}$, $\chi_Z$ denoting the characteristic function of $Z$. Conversely, if $q$ is a projection in $C(X) \underset{\varphi}{\times} \mathbf{Z}$, then $q(C(X) \underset{\varphi}{\times} \mathbf{Z})q \cong C(Z) \underset{\psi}{\times} \mathbf{Z}$ for some induced system $(Z, \psi)$ of $(X, \varphi)$, where $Z$ is a clopen subset of $X$.

**Proof.** Let $Z$ be a clopen subset of $X$ and let $\psi = \psi_Z : Z \to Z$ be the first return map as described in Definition 1.5. $C(X) \underset{\varphi}{\times} \mathbf{Z}$ is generated by $C(X)$ and by $u = u_\varphi$ acting on $L^2(X, \mu)$, where $\mu$ is a $\varphi$-invariant probability measure, as described in Section 1. Likewise, $C(Z) \underset{\psi}{\times} \mathbf{Z}$ is generated by $C(Z)$ and $u_0 = u_\psi$ acting on $L^2(Z, \nu)$, where $\nu$ is the $\psi$-invariant probability measure $\frac{1}{\mu(Z)}\mu$ (restricted to $Z$). $L^2(Z, \nu)$ is naturally isomorphic to $L^2(Z, \mu) = p(L^2(X, \mu))$, where $p$ is the projection $\chi_Z$ in $C(X) \underset{\varphi}{\times} \mathbf{Z}$. Now $C(Z) = \chi_Z \cdot C(X) = C(X) \cdot \chi_Z$ and so $C(Z) \subset p(C(X) \underset{\varphi}{\times} \mathbf{Z})p$. Let $\lambda = r_Z : Z \to \mathbf{Z}^+$ be the map described in Definition 1.5 and let $J_1, \ldots, J_K$

be the values $\lambda$ attains. Since $\psi|_{\lambda^{-1}(J_k)} = \varphi^{J_k}|_{\lambda^{-1}(J_k)}$, we get that $u_0|_{\lambda^{-1}(J_k)} = u^{J_k}|_{\lambda^{-1}(J_k)} = pu^{J_k}|_{\lambda^{-1}(J_k)}$ for $k = 1, \ldots, K$. Hence $u_0 \in p(C(X) \underset{\varphi}{\times} \mathbf{Z})p$ and so $C(Z) \underset{\psi}{\times} \mathbf{Z} \subset p(C(X) \underset{\varphi}{\times} \mathbf{Z})p$. To show equality it is sufficient to show that $\chi_Z u^i \chi_Z \in C(Z) \underset{\psi}{\times} \mathbf{Z}$ for all $i \in \mathbf{Z}^+$. So fix $i \in \mathbf{Z}^+$. Now $\varphi^i|_A = \psi^l|_A$, where $A \subset Z$ is the clopen set

$$\begin{aligned} A = A_{J_{i_1}, \ldots, J_{i_l}} = \; & \lambda^{-1}(J_{i_1}) \cap \varphi^{-J_{i_1}}(\lambda^{-1}(J_{i_2})) \cap \cdots \\ & \cap \varphi^{-(J_{i_1} + \cdots + J_{i_{l-1}})}(\lambda^{-1}(J_{i_l})) \end{aligned}$$

with $i = J_{i_1} + \cdots + J_{i_l}$. So $u_0^l|_A = u^i|_A = \chi_Z u^i|_A$. Hence $\chi_Z u^i \chi_A = u_0^l \chi_A$. So if $B$ is the (finite) union of the sets $\{A_{J_{i_1}, \ldots, J_{i_l}} | i = J_{i_1} + \cdots J_{i_l}\}$, then $\chi_Z u^i \chi_B \in C(Z) \underset{\psi}{\times} \mathbf{Z}$. As $\chi_Z u^i \chi_Z = \chi_Z u^i \chi_B$ this establishes that $\chi_Z u^i \chi_Z \in C(Z) \underset{\psi}{\times} \mathbf{Z}$ and we are done.

Conversely, let $q$ be a projection in $C(X) \underset{\varphi}{\times} \mathbf{Z}$. By [29; Thm. 2.1] $q$ is unitarily equivalent (in $C(X) \underset{\varphi}{\times} \mathbf{Z}$) to a projection $p = \chi_Z$ in $C(X)$, where $Z$ is a clopen subset of $X$. So $q(C(X) \underset{\varphi}{\times} \mathbf{Z})q \cong p(C(X) \underset{\varphi}{\times} \mathbf{Z})p$, and by the first part we get the desired conclusion. □

**Corollary.** Let $(X, \varphi)$ and $(Z, \psi)$ be as in the proposition. Let $Y$ be a closed subset of $Z$. Then $A_Y^\psi \cong pA_Y^\varphi p$, where $p = \chi_Z$ and the isomorphism is given by the same map as in the proposition. In particular, $f \in C(Z) \subset A_Y^\psi$ is mapped to $\tilde{f} \in pC(X)p \subset pA_Y^\varphi p$, where $\tilde{f}(x) = f(x)$ for $x \in Z$ and 0 otherwise.

**Proof.** Recall that $A_Y^\psi$ is the $C^*$-subalgebra of $C(Z) \underset{\psi}{\times} \mathbf{Z}$ generated by $C(Z)$ and $u_0 C_0(Z \setminus Y)$ — similarly for $A_Y^\varphi$. Now $u^* C_0(X \setminus F)u = C_0(X \setminus \varphi^{-1}(F))$, where $F$ is a closed subset of $X$. Using this we proceed as in the proof of the proposition to show first that $u_0 C_0(Z \setminus Y) \subset pA_Y^\varphi p$, hence $A_Y^\psi \subset pA_Y^\varphi p$. To get the opposite inclusion it is sufficient to show that for $i \in \mathbf{Z}^+$ and $f_1, \ldots, f_i \in C_0(X \setminus Y)$ we have $puf_1uf_2 \cdots uf_ip \in A_Y^\psi$. Now $uf_1uf_2 \cdots uf_i = u^i f$ for some $f \in C_0(X \setminus (Y \cup \varphi^{-1}(Y) \cup \cdots \cup \varphi^{-i+1}(Y)))$. Proceding as in the proof of the proposition one establishes $pu^i fp \in A_Y^\psi$. □

## 4. Proof of Theorem 2.1.

(ii) ⇔ (iii). This is Theorem 1.14.

(ii) $\Rightarrow$ (i). By Theorem 3.6 and Theorem 3.7 we may assume that $(X_i, \varphi_i)$ is the Bratteli-Vershik system $(X_{B_i}, \lambda_{B_i})$ associated to the simple ordered Bratteli diagram $B^i = (V^i, E^i, \geq^i), i = 1, 2$, and with $K_0(V^1, E^1) \cong K_0(V^2, E^2)$ as ordered groups with distinguished order units. Hence $(V^1, E^1) \sim (V^2, E^2)$ and so there exists a simple (not ordered!) Bratteli diagram $(V, E)$ with the property that by telescoping $(V, E)$ to even levels $0 < 2 < 4 < \cdots$ we get a telescoping of $(V^2, E^2)$, and by telescoping $(V, E)$ to odd levels $0 < 1 < 3 < \cdots$ we get a telescoping of $(V^1, E^1)$, cf. Section 3. We may assume that all entries of the various incidence matrices of $(V, E)$ are $\geq 2$. In fact, this is achieved by telescoping $(V, E)$ appropriately to alternate odd and even levels. Thus we may assume at start that $(V, E)$, which we will call the *aggregate* Bratteli diagram of $(V^1, E^1)$ and $(V^2, E^2)$, has the property of having multiple edges between every pair of vertices from consecutive levels, and that telescoping $(V, E)$ to even levels yields $(V^2, E^2)$ and telescoping to odd levels yields $(V^1, E^1)$. (Recall that by telescoping a simple ordered Bratteli diagram the associated Bratteli-Vershik systems are conjugate, cf. Theorem 3.6.) We want to construct two homeomorphisms $F_1 : X_{B^1} \to X$, $F_2 : X_{B^2} \to X$, where $X$ is the path space associated to $(V, E)$, topologized as described in Section 3, with the properties of preserving the cofinal relation and so that $F_1(y^1_{\max}) = F_2(y^2_{\max})$, $F_1(y^1_{\min}) = F_2(y^2_{\min})$. (Cf. Remark (ii) to Theorem 3.6.) This is done in the following manner:

Using the notation introduced in Section 3 we will establish for each even $n$ bijections

$$(*) \qquad E_n \circ E_{n+1} \leftrightarrow E^1_{\frac{n}{2}+1}, \quad E_{n-1} \circ E_n \leftrightarrow E^2_{n/2}$$

respecting the range and source maps, recalling that $V_m = V^1_{\frac{m+1}{2}}$ for $m$ odd and $V_m = V^2_{\frac{m}{2}}$ for $m$ even. In addition we have $E_1 = E^1_1$ and $V_0 = V^1_0 = V^2_0$. The bijections will be chosen successively. Let $y^i_{\max} = (e^i_1, e^i_2, \cdots) \in X_{B^i}, y^i_{\min} = (f^i_1, f^i_2, \cdots) \in X^i_B$, $i = 1, 2$. Assume we have defined the various bijections up to level $n$ of $(V, E)$, starting with $E_1$ identified with $E^1_1$. For the sake of argument let us assume $n$ is even.

$$\begin{aligned}\text{Let } v' &= s(e^2_{n/2}) \in V^2_{\frac{n}{2}-1} = V_{n-2},\\ v'' &= r(e^1_{n/2}) = s(e^2_{\frac{n}{2}+1}) \in V^1_{n/2} = V_{n-1},\\ v''' &= r(e^2_{n/2}) = s(e^2_{\frac{n}{2}+1}) \in V^2_{n/2} = V_n,\\ v'''' &= r(e^1_{\frac{n}{2}+1}) \in V^1_{\frac{n}{2}+1} = V_{n+1}.\end{aligned}$$

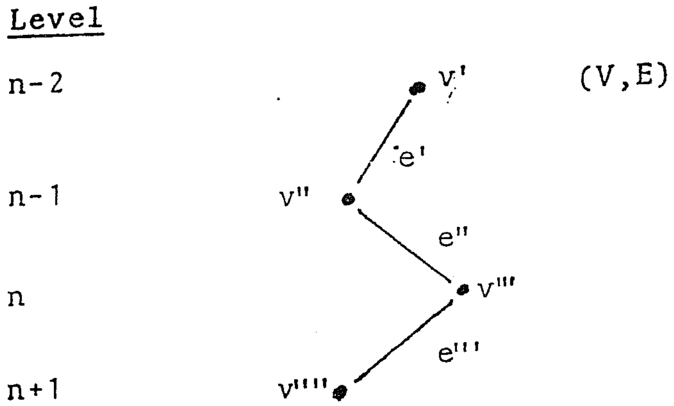



Figure 6.

Let $(e', e'') \in E_{n-1} \circ E_n$ correspond to $e^2_{n/2} \in E^2_{n/2}$ by the (already defined) bijection $E_{n-1} \circ E_n \leftrightarrow E^2_{n/2}$. (See Figure 6.) We postulate that $(e'', e''') \in E_n \circ E_{n+1}$ shall correspond to $e^1_{\frac{n}{2}+1} \in E^1_{\frac{n}{2}+1}$, where $e'''$ is any edge in $E_{n+1}$ with $s(e''') = v'''$, $r(e''') = v''''$. Similarly we handle the min edge $f^1_{\frac{n}{2}+1} \in E^1_{\frac{n}{2}+1}$, with the proviso that $f''' \in E_{n+1}$ is chosen different from $e'''$, which is possible by our assumption on $(V, E)$. Let $E_n \circ E_{n+1} \leftrightarrow E^1_{\frac{n}{2}+1}$ be any bijection extending the above correspondences and respecting the range and source maps. (If $n$ is odd we proceed similarly.)
The various bijections $(*)$ give rise in an obvious way to homeomorphisms

$$F_1 : X_{B^1} \to X, \; F_2 : X_{B^2} \to X,$$

with the asserted properties. In fact, if $x^i \in X_{B^i}$ is cofinal with $z^i \in X_{B^i}$, then $F_i(x^i)$ is cofinal with $F_i(z^i)$, $i = 1, 2$, and the bijections $(*)$ are rigged so that

$$F_1(y^1_{\max}) = F_2(y^2_{\max}), \quad F_1(y^1_{\min}) = F_2(y^2_{\min}).$$

Let $F = F_2^{-1} \circ F_1 : X_{B^1} \to X_{B^2}$. Then $F$ is an orbit map between $(X_{B^1}, \lambda_{B^1})$ and $(X_{B^2}, \lambda_{B^2})$ according to Remark (ii) to Theorem 3.6. Let $m, n : X_{B^1} \to$

$\mathbf{Z}$ be the orbit cocycles associated to $F$ and defined by

$$F \circ \lambda_{B^1}(x) = \lambda_{B^2}^{n(x)} \circ F(x), \; F \circ \lambda_{B^1}^{m(x)}(x) = \lambda_{B^2} \circ F(x), \; x \in X_{B^1}.$$

We will show that $n(x)$ is continuous for all $x$ except possibly for $x = y^1_{\text{max}}$. (A similar argument works for $m(x)$.) This will complete the proof of (ii) $\Rightarrow$ (i).

Let $x = (g_1, g_2, \ldots) \in X_{B_1}$, $x \neq y^1_{\text{max}} = (e^1_1, e^1_2, \ldots)$, and let $k$ be the smallest natural number so that $g_k \neq e^1_k$. Consider the (clopen) cylinder set

$$U = U(g_1, g_2, \ldots, g_k, g_{k+1}) = \{(h_1, h_2, \ldots) \in X_{B^1} | h_i = g_i, 1 \leq i \leq k+1\}$$

containing $x$. Let $z \in U$. By one of the characteristic properties of the Vershik map $z$ and $\lambda_{B^1}(z)$ will be cofinal from level $k$, i.e. the edges making up $z$ and $\lambda_{B^1}(z)$ coincide from level $k$ starting with the edge $g_{k+1}$. Consequently, bearing in mind how $F$ is defined, $F(z)$ and $F(\lambda_{B^1}(z))$ are cofinal from level $k$. In particular, $F(z)$ and $F(\lambda_{B^1}(z))$ pass through the same vertex $v$ at level $k$. Since $x, z \in U$ we get that $\lambda_{B^1}(x)$ and $\lambda_{B^1}(z)$ have the same initial segment from the top vertex (i.e. level 0) to level $k+1$. Consequently, $F(\lambda_{B^1}(x))$ and $F(\lambda_{B^1}(z))$ have the same initial segment from level 0 to level $k$. The same is true for $F(x)$ and $F(z)$. Putting all this together, bearing in mind that iterates of $\lambda_{B^2}$ applied to $F(z)$ "sweeps out" the initial segments from the top vertex to vertex $v$, we conclude that there exists an integer $N$ so that

$$F \circ \lambda_{B^1}(z) = \lambda_{B^2}^{N} \circ F(z)$$

for all $z \in U$.

This proves that $n : X_{B^1} \to \mathbf{Z}$ is continuous at $x$, thus finishing the proof of (ii) $\Rightarrow$ (i).

(We remark that in general $n(x)$ and $m(x)$ will not be continuous at $x = y^1_{\text{max}}$ as the simple example of figure 7 shows. The two Bratteli-Vershik systems with the indicated ordering are orbit equivalent by the identity map. Yet it is easily seen that the orbit cocycles are discontinuous at the max path $y^1_{\max}$. In this case the two systems are actually conjugate!)

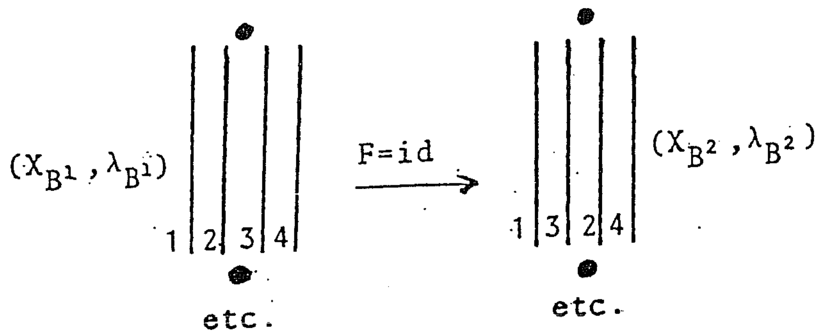



Figure 7.

**(i) ⇒ (ii).** Assume $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ are strong orbit equivalent and let $F : X_1 \to X_2$ be an orbit map so that the associated orbit cocycles $m, n : X_1 \to \mathbf{Z}$ defined by

$$F \circ \varphi_1(x) = \varphi_2^{n(x)} \circ F(x), \quad F \circ \varphi_1^{m(x)}(x) = \varphi_2 \circ F(x), \; x \in X_1,$$

have at most one point of discontinuity each. By considering $\tilde{\varphi_2} = F^{-1} \circ \varphi_2 \circ F : X_1 \to X_1$, which is conjugate to $\varphi_2$ and has the same orbits as $\varphi_1$, we get that

$$\varphi_1(x) = \tilde{\varphi_2}^{n(x)}(x), \; \tilde{\varphi_2}(x) = \varphi_1^{m(x)}(x), x \in X_1.$$

So we may assume to start with that $\varphi_1$ and $\varphi_2$ act on the same space $X$ and has the same orbits, and that the orbit cocycles $m, n : X \to \mathbf{Z}$ defined by

$$\varphi_1(x) = \varphi_2^{n(x)}(x), \; \varphi_2(x) = \varphi_1^{m(x)}(x), \; x \in X,$$

have at most one point of discontinuity each. By Definition 1.11 it is enough to show that $Im(id - \varphi_{1*}) = Im(id - \varphi_{2*})$. Assume first that $n(x)$ is continuous for all $x \in X$ except possibly for $x = x_0$. We shall deduce from this that $Im(id - \varphi_{1*}) \subset Im(id - \varphi_{2*})$. Since a similar argument for $m(x)$ will yield the other inclusion, this will complete the proof. Clearly it is enough to show that $(id - \varphi_{1*})(\chi_E) = \chi_E - \chi_{\varphi_1(E)} \in Im(id - \varphi_{2*})$ for a clopen set $E \subset X$, since any $f \in C(X, \mathbf{Z})$ can be written as a finite linear combination of characteristic functions of clopen sets. In the sequel we will use the notation $[f]_i$, where $f \in C(X, \mathbf{Z})$, to denote the element of $K^0(X, \varphi_i) \cong K_0(C(X) \underset{\varphi_i}{\times} \mathbf{Z})$, $i = 1, 2$, that $f$ maps into. Recall from Section 1 that for $i = 1, 2$, $C(X) \underset{\varphi_i}{\times} \mathbf{Z} \cong C^*(C(X), u_i = u_{\varphi_i})$, i.e. $C(X) \underset{\varphi_i}{\times} \mathbf{Z}$ is isomorphic to the $C^*$-algebra generated by $C(X)$ (acting as multiplication operators) and the unitary operator $u_i : g \to g \circ \varphi_i^{-1}$ acting on the Hilbert space $H = L^2(X, \mu)$, where $\mu$ is a $\varphi_i$-invariant probability measure. (Since $\varphi_1$ and $\varphi_2$ have the same orbits the $\varphi_1$-invariant and $\varphi_2$-invariant probability measures coincide, a fact that is easily established, cf. the proof of (i) $\Rightarrow$ (iii) of Theorem 2.2.) Now $u_i \chi_E u_i^* = \chi_{\varphi_i(E)}$ and so $[\chi_E]_i = [\chi_{\varphi_i(E)}]_i$ for $i = 1, 2$, where $E$ is a clopen set in $X$. If $\varphi_1(x) = \varphi_2^k(x)$, $x \in E$, for some $k \in \mathbf{Z}$, we get that $u_1 \chi_E = u_2^k \chi_E$ by the definition of $u_1$ and $u_2$. Now let $E$ be any clopen set not containing $x_0$. Then

$$u_1 \chi_E = \sum_k u_2^k \chi_{E \cap n^{-1}(k)}$$

where the sum is finite. Hence $u_1 \chi_E \in C(X) \underset{\varphi_2}{\times} \mathbf{Z}$, and consequently $A^{\varphi_1}_{\{x_0\}} \subset C(X) \underset{\varphi_2}{\times} \mathbf{Z}$, where $A^{\varphi_1}_{\{x_0\}}$ is the simple AF-algebra $C^*(C(X), u_1 C_0(X \backslash \{x_0\}))$, cf. Section 1. Now $[\chi_E]_1 = [\chi_{\varphi_1(E)}]_1$ is in $K_0(A^{\varphi_1}_{\{x_0\}})$ (which by Corollary 1 to Theorem 1.17 is isomorphic to $K_0(C(X) \underset{\varphi_1}{\times} \mathbf{Z})$ by the injection map) and so there exists a unitary $v \in A^{\varphi_1}_{\{x_0\}}$ so that $v \chi_E v^* = \chi_{\varphi_1(E)}$. By the preceeding $v \in C(X) \underset{\varphi_2}{\times} \mathbf{Z}$ and so $[\chi_E]_2 = [\chi_{\varphi_1(E)}]_2$. Thus $[\chi_E]_2 - [\chi_{\varphi_1(E)}]_2 = 0$ in $K^0(X, \varphi_2) = C(X, \mathbf{Z})/Im(id - \varphi_{2*})$ and so $\chi_E - \chi_{\varphi_1(E)} \in Im(id - \varphi_{2*})$. This completes the proof. □

## 5. Proof of Theorem 2.2.

The proof of the implication (ii) $\Rightarrow$ (i) is lengthy and will require the application of techniques and results from a wide variety of sources, including some homological algebra. On the other hand, the proofs of the implications (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii) are straightforward and will be dealt with first.

(i) $\Rightarrow$ (iii). Let $F : X_1 \to X_2$ be an orbit map between $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ with associated orbit cocycles $n, m : X_1 \to \mathbf{Z}$ defined by

$$F \circ \varphi_1(x) = \varphi_2^{n(x)} \circ F(x), \quad F \circ \varphi_1^{m(x)}(x) = \varphi_2 \circ F(x); \quad x \in X_1.$$

Define $\tilde{\varphi}_2 = F^{-1} \circ \varphi_2 \circ F : X_1 \to X_2$. Then $\varphi_1(x) = \tilde{\varphi}_2^{n(x)}(x)$, $\tilde{\varphi}_2(x) = \varphi_1^{m(x)}(x)$ and $\text{orbit}_{\varphi_1}(x) = \text{orbit}_{\tilde{\varphi}_2}(x)$ for all $x \in X_1$. Hence we may assume at start that $\varphi_1$ and $\varphi_2$ act on the same space $X$ with the same orbits and

$$\varphi_1(x) = \varphi_2^{n(x)}(x), \quad \varphi_2(x) = \varphi_1^{m(x)}(x); \; x \in X.$$

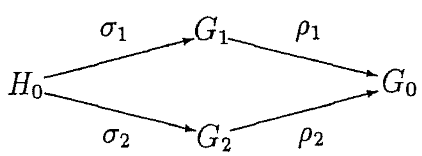
We first observe that $n$ and $m$ are Borel functions – for example, the family of sets $\{A_k\}_{k \in \mathbf{Z}}$ form a partition of $X$ into closed sets, where $A_k = \{x | n(x) = k\}$. Let $\mu$ be a $\varphi_2$-invariant probability measure and let $E$ be a Borel subset of $X$. Then

$$\begin{aligned} & \mu(\varphi_1(E)) = \mu(\cup_k \varphi_1(E \cap A_k)) \\ = & \textstyle\sum_k \mu(\varphi_1(E \cap A_k)) = \sum \mu(\varphi_2^k(E \cap A_k)) \\ = & \textstyle\sum_k \mu(E \cap A_k) = \mu(E), \end{aligned}$$

and so $\mu$ is $\varphi_1$-invariant. Likewise we show that a $\varphi_1$-invariant probability measure is $\varphi_2$-invariant. So we have proved (i) $\Rightarrow$ (iii).

**(iii) $\Rightarrow$ (ii).** This implication is an immediate consequence of Theorem 1.13.

Before embarking on proving the many and varied lemmas needed to prove the implication (ii) $\Rightarrow$ (i) we will sketch the strategy of our proof. (All the groups we consider in the sequel will be torsion free, countable, abelian groups and the maps will be group homomorphisms.) Let $G_1 = K^0(X_1, \varphi_1)$, $G_2 = K^0(X_2, \varphi_2)$ and let $G_0 = G_1/Inf(G_1) \cong G_2/Inf(G_2)$. Then $G_0$ is a simple dimension group with distinguished order unit (cf. Remark to Definition 1.10). Let $\rho_1 : G_1 \to G_0, \rho_2 : G_2 \to G_0$ be the quotient maps and let $H_0$ be the associated pullback defined by the pullback diagram




Specifically, $H_0 = \{(g_1, g_2) \in G_1 \times G_2 | \rho_1(g_1) = \rho_2(g_2)\}$ and $\sigma_1$ and $\sigma_2$ are the projection maps. Endowed with the induced ordering $H_0$ becomes a simple dimension group. Let $\rho_0 : H_0 \to G_0$ be the map $\rho_1 \circ \sigma_1 (= \rho_2 \circ \sigma_2)$.

Then $ker(\rho_0) = Inf(G_1) \bigoplus Inf(G_2) = Inf(H_0)$ and so we have a short exact sequence

$$0 \to Inf(H_0) \xrightarrow{i_0} H_0 \xrightarrow{\rho_0} G_0 \to 0$$

Now choose a surjective map $\beta : \mathbf{Z}^\infty (= \bigoplus_1^\infty \mathbf{Z}) \to Inf(H_0)$. (Replacing $\mathbf{Z}^\infty$ by $\mathbf{Z}^\infty \bigoplus \mathbf{Z}^\infty$ and $\beta$ by $\beta \bigoplus 0$, we may assume that $ker(\beta)$ has infinite rank.) By a theorem of Cartan-Eilenberg in homological algebra [14; Thm. 51.3] there exist a group $H$ and maps $i : \mathbf{Z}^\infty \to H$, $\eta : H \to H_0$, $\rho : H \to G_0$ so that

$$\begin{array}{ccccccc} 0 \to & \mathbf{Z}^\infty & \xrightarrow{i} & H & \xrightarrow{\rho} & G_0 & \to 0 \\ & \beta \downarrow & & \eta \downarrow & & \| & \\ 0 \to & Inf(H_0) & \xrightarrow{i_0} & H_0 & \xrightarrow{\rho_0} & G_0 & \to 0 \end{array}$$

is a commutative diagram with exact rows. Since $\beta$ is surjective so is $\eta$, and since $ker(\beta)$ has infinite rank so does $ker(\eta)$. $H$ becomes a simple dimension group with positive cone

$$H^+ \setminus \{0\} = \rho^{-1}(G_0^+ \setminus \{0\}) = \eta^{-1}(H_0^+ \setminus \{0\}).$$

By considering the maps $\pi_1 = \sigma_1 \circ \eta : H \to G_1, \pi_2 = \sigma_2 \circ \eta : H \to G_2$, we get $ker(\pi_k) \cong \mathbf{Z}^\infty$, $k = 1, 2$. We thus obtain the following two short exact sequences

$$\mathcal{E}_1 : \; 0 \to \mathbf{Z}^\infty \xrightarrow{i_1} H \xrightarrow{\pi_1} G_1 \to 0$$
$$\mathcal{E}_2 : \; 0 \to \mathbf{Z}^\infty \xrightarrow{i_2} H \xrightarrow{\pi_2} G_2 \to 0$$

Letting $v$ be any element of $H$ so that $\eta(v) = (\mathbf{1}_{X_1}, \mathbf{1}_{X_2}) \in H_0$, where $\mathbf{1}_{X_1}$ and $\mathbf{1}_{X_2}$ are the distinguished order units of $G_1 = K^0(X_1, \varphi_1)$ and $G_2 = K^0(X_2, \varphi_2)$, respectively, we get $\pi_k(v) = \mathbf{1}_{X_k}$, $k = 1, 2$. Moreover, $H^+ \setminus \{0\} = \pi_k^{-1}(G_k^+ \setminus \{0\})$ and so $i_k(\mathbf{Z}^\infty) \subset Inf(H)$ for $k = 1, 2$. Now we invoke the Fundamental Extension Theorem (Theorem 10.1) which says that any element of $Ext(G_k, \mathbf{Z}^\infty)$ is realized as described in Corollary 3 of Theorem 1.17, where $G_k = K^0(X_k, \varphi_k) \cong K_0(C(X_k) \underset{\varphi_k}{\times} \mathbf{Z})$, $k = 1, 2$. In particular, to $\mathcal{E}_1$ in $Ext(G_1, \mathbf{Z}^\infty)$ and $\mathcal{E}_2$ in $Ext(G_2, \mathbf{Z}^\infty)$ there exist Cantor systems $(Z_1, \psi_1)$, $(Z_2, \psi_2)$ and countable subsets $Y_1 \subset Z_1, Y_2 \subset Z_2$, where $Y_k = \{y_1^{(k)}, \ldots, y_n^{(k)}, \ldots, y_\infty^{(k)}\}, y_n^{(k)} \to y_\infty^{(k)}$, $Y_k$ meets each $\psi_k$-orbit at most once, and isomorphisms $\eta_k : K_0(A_{Y_k}^{\psi_k}) \to H$ rendering a commutative diagram

$$\begin{array}{ccccccc} 0 \to & \mathbf{Z}^\infty & \to & K_0(A_{Y_k}^{\psi_k}) & \to & K^0(Z_k, \psi_k) & \to 0 \\ & \| & & \eta_k \downarrow & & \cong\downarrow & \\ 0 \to & \mathbf{Z}^\infty & \xrightarrow{i_k} & H & \xrightarrow{\pi_k} & K^0(X_k, \varphi_k) & \to 0 \end{array}$$

for $k = 1, 2$. (Here $\cong$ denotes isomorphism of ordered groups with distinguished order units.)

It is easy to show that actually $\eta_1$ and $\eta_2$ are order isomorphisms. By keeping track of how the order unit $v$ of $H$ is mapped by $\eta_k^{-1}$ we adjust the dynamical systems $(Z_k, \psi_k), k = 1, 2$, and (retaining the same notation for the adjusted systems) we establish an order isomorphism between $K_0(A_{Y_1}^{\psi_1})$ and $K_0(A_{Y_2}^{\psi_2})$ preserving the distinguished order units. From this data we are able to show orbit equivalence between $(Z_1, \psi_1)$ and $(Z_2, \psi_2)$. This will be achieved in an analogous manner as we showed (strong) orbit equivalence from the assumption $K^0(Z_1, \psi_1) \cong K^0(Z_2, \psi_2)$ in the proof of Theorem 2.1. (In fact, if $Y_k$ is a one-point set in $Z_k, k = 1, 2$, then $K_0(A_{Y_k}^{\psi_k}) \cong K^0(Z_k, \psi_k)$ as ordered groups with distinguished order units by Corollary 1 of Theorem 1.17.) Finally, $K^0(Z_k, \psi_k) \cong K^0(X_k, \varphi_k)$ implies (strong) orbit equivalence of $(Z_k, \psi_k)$ and $(X_k, \varphi_k)$ for $k = 1, 2$ by Theorem 2.1, thus completing the proof.

We begin the detailed proof of the implication (ii) $\Rightarrow$ (i) of Theorem 2.2 by first establishing the result just alluded to. The following two lemmas will do that.

**Lemma 5.1.** Let $(X, \varphi)$ be a Cantor system. Let $Y$ be a closed subset of $X$. There exists an ordered Bratteli diagram $B = (V, E, \geq)$ and a homeomorphism $F : X \to X_B$, so that $F(Y)$ is the set of max paths $X_B^{\max} = \{(e_1, \ldots, e_n, \ldots) \in X_B |\ e_n \in E_{\max}\}$ and $F(\varphi(Y))$ is the set of min paths $X_B^{\min} = \{(f_1, \ldots, f_n, \ldots) \in X_B | f_n \in E_{\min}\}$. The Vershik map $\lambda_B$, which is defined on $X_B \setminus X_B^{\max}$ (as described in Section 3), yields a homeomorphism $\lambda_B : X_B \setminus X_B^{\max} \to X_B \setminus X_B^{\min}$ that can be (uniquely) extended to a homeomorphism (still denoted by $\lambda_B$) $\lambda_B : X_B \to X_B$ so that $(X, \varphi)$ is conjugate to $(X_B, \lambda_B)$ via the conjugating map $F$. Moreover, $K_0(V, E) \cong K_0(A_Y^{\varphi})$ as ordered groups with distinguished order units.

**Proof.** In the proof of the model theorem (Theorem 3.6) the idea is to have a sequence of clopen sets $\{Z_n\}$ in $X$ shrink to a one-point set $\{y\}$ — the associated sequence of Kakutani-Rohlin towers will determine a (simple) ordered Bratteli diagram so that $y$, resp. $\varphi(y)$, correspond to the (unique) max path, resp. min path, of the associated path space (cf. [17; Section 4]). Now if instead of a one-point set we start with the closed subset $Y$

and let $\{Z_n\}$ be a sequence of clopen sets in $X$ that shrink to $Y$, we get by the same construction as in [17; Section 4] an ordered Bratteli diagram $B = (V, E, \geq)$ and a homeomorphism $F : X \to X_B$ so that $F(Y) = X_B^{\max}$ and $F(\varphi(Y)) = X_B^{\min}$. The associated Vershik map $\lambda_B$ gives rise to a homeomorphism between $X_B \setminus X_B^{\max}$ and $X_B \setminus X_B^{\min}$ (cf. Section 3) which may be extended to $X_B$ with the properties asserted.
Finally, by construction we get $K_0(V, E) \cong K_0(A_Y^{\varphi})$, cf. [28; Section 3]. □

**Remark.** If $Y$ meets each orbit at most once the Bratteli diagram $(V, E)$ is simple by Theorem 1.16, and so $B = (V, E, \geq)$ is compatible according to the terminology we introduced in Section 3.

**Lemma 5.2.** Let $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ be Cantor systems. For $k = 1, 2$, let $Y^k = \{y_1^{(k)}, \ldots, y_n^{(k)}, \ldots, y_\infty^{(k)}\} \subset X_k$, where $y_n^{(k)} \to y_\infty^{(k)}$ and $Y^k$ meets each $\varphi_k$-orbit at most once. Assume $K_0(A_{Y^1}^{\varphi_1}) \cong K_0(A_{Y^2}^{\varphi_2})$ as ordered groups with distinguished order units. Then $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ are orbit equivalent.

**Proof.** Let $k = 1$ or $2$. By Lemma 5.1 and the ensuing Remark we may assume that $(X_k, \varphi_k)$ is the Bratteli-Vershik system $(X_{B^k}, \lambda_{B^k})$, where $B^k = (V^k, E^k, \geq^k)$ is the ordered Bratteli diagram associated to $Y^k$ and $(X_k, \varphi_k)$, $(V^k, E^k)$ being a simple Bratteli diagram. Furthermore, $K_0(V^1, E^1) \cong K_0(V^2, E^2)$ as ordered groups with distinguished order units.
Let us for the time being denote $X = X_{B^k}, \lambda = \lambda_{B^k}$, $Y = \{y_1, \ldots, y_n, \ldots, y_\infty\}$ for $Y^k = \{y_1^{(k)}, \ldots, y_n^{(k)}, \ldots, y_\infty^{(k)}\}$, $B = (V, E, \geq)$ for $B^k = (V^k, E^k, \geq^k)$. Observe that no two distinct points of $Y \cup \lambda(Y)$ are cofinal. In fact, if two paths in $X$ are cofinal they lie in the same orbit by one of the characteristic properties of the Vershik map. Also, for $n \in \{1, 2, \ldots, \infty\}$, $y_n$ and $\lambda(y_n)$ are not cofinal since $y_n \in X^{\max}$, $\lambda(y_n) \in X^{\min}$. (Recall that $X^{\max}$, resp. $X^{\min}$, denote the set of paths in $X$ made up of edges in $E_{\max}$, resp. $E_{\min}$.) We note that the orbits of $\lambda$ are determined by the cofinal relation, with the proviso that paths that are cofinal with $y_n$ are "glued" to the paths cofinal with $\lambda(y_n)$ for $n \in \{1, 2, \ldots, \infty\}$.
Since no two distinct paths of $Y \cup \lambda(Y)$ are cofinal and $y_n \to y_\infty$, $\lambda(y_n) \to \lambda(y_\infty)$, we may telescope $B = (V, E, \geq)$ to get a new ordered Bratteli diagram (which we again will denote by $B = (V, E, \geq)$) in which the paths in $Y \cup \lambda(Y)$ may be described in the following manner:
(In the sequel we use the notation $x(i) = g_i \in E_i$, where $x = (g_1, g_2, \ldots) \in$

$X$. Also we denote by $|Z|$ the number of elements of the set $Z$.) There are two strictly increasing sequences of finite subsets of $Y$, resp. $\lambda(Y)$,

$$y_\infty \in Y_1 \underset{\neq}{\subset} Y_2 \underset{\neq}{\subset} \cdots \subset Y, \quad Y = \cup Y_i$$

$$\lambda(y_\infty) \in Y_1' \underset{\neq}{\subset} Y_2' \underset{\neq}{\subset} \cdots \subset \lambda(Y), \quad \lambda(Y) = \cup Y_i'$$

so that the set $\{y(i), y'(i) | y \in Y_i, y' \in Y_i'\}$ consists of $|Y_i| + |Y_i'|$ distinct edges in $E_i$. Furthermore, the initial part of each $y \in Y \setminus Y_i^{\infty}$, resp. $y' \in \lambda(Y) \setminus Y_i'$, coincides with the initial part of $y_\infty$, resp. $\lambda(y_\infty)$, between level 0 and level $i$. In other words, the paths in $Y_i \setminus Y_{i-1}$, resp. $Y_i' \setminus Y_{i-1}'$, splits off from the remaining paths in $Y$, resp. $\lambda(Y)$, at level $i-1$. We illustrate this in Figure 8.

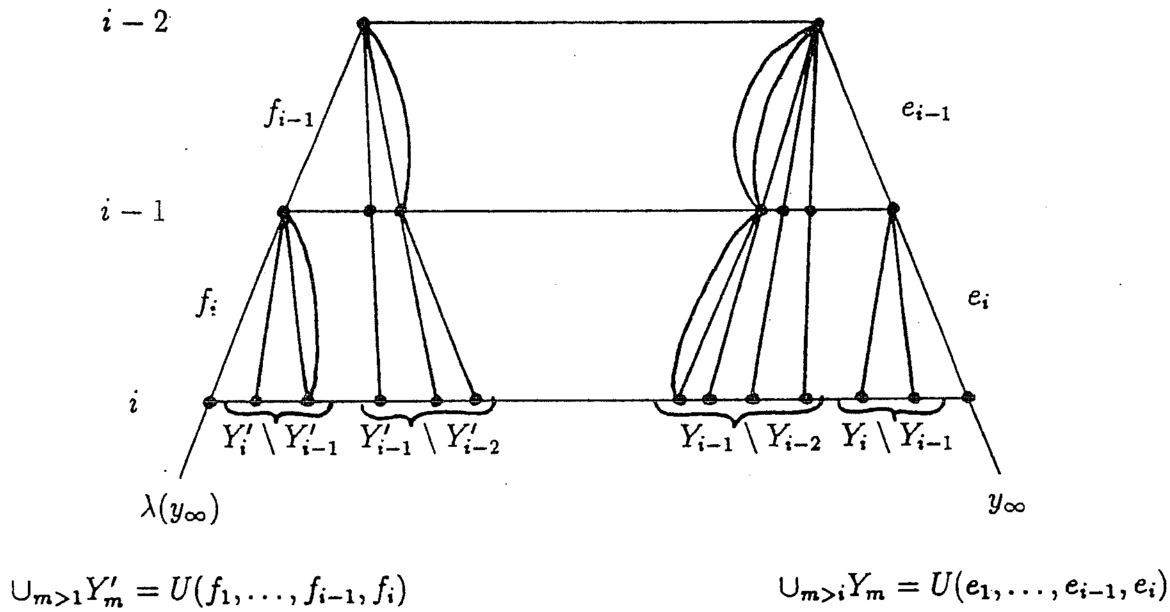



Figure 8.

Now we perform symbol splittings (cf. Section 3) between consecutive levels of the diagram, hence getting a microscoping of the ordered Bratteli diagram. By telescoping to the new levels thus introduced we wind up with an ordered Bratteli diagram (again denoted by $B = (V, E, \geq)$) equivalent to the original one — hence the associated Bratteli-Vershik system is conjugate to the original one — which has the property that at level

$i$ the set $\{r(y(i)),\ r(y'(i))|y \in Y_i, y' \in Y_i'\}$ consists of $|Y_i| + |Y_i'|$ distinct vertices in $V_i$. Furthermore, $\{r(y(j))|y \in Y \setminus Y_i\} = \{r(y_\infty(j))\}$ and $\{r(y'(j))|y' \in \lambda(Y) \setminus Y_i'\} = \{r(\lambda(y_\infty)(j))\}$ for $1 \leq j \leq i$. In other words, from level $i$ on the paths in $Y_i \setminus Y_{i-1}$ and $Y_i' \setminus Y_{i-1}'$ pass through distinct vertices splitting off from the remaining paths in $Y$, resp. $\lambda(Y)$. (Observe that this property is preserved under telescoping — the new $Y_i$'s and $(Y_i')$'s being subsequences of the original sequences.)
Henceforth we assume that the ordered Bratteli diagrams $B^k = (V^k, E^k, \geq^k), k = 1, 2$, have the property described above. Now we proceed in an analogous fashion to what we did in the proof of Theorem 2.1. Since $K_0(V^1, E^1) \cong K_0(V^2, E^2)$ as ordered groups with distinguished order units there exists an ("aggregate") Bratteli diagram $(V, E)$ (unordered!) so that telescoping $(V, E)$ to odd levels $0 < 1 < 3 < \cdots$ yields a telescoping of $(V^1, E^1)$, and telescoping $(V, E)$ to even levels $0 < 2 < 4 < \cdots$ yields a telescoping of $(V^2, E^2)$. Let $X$ denote the path space associated to $(V, E)$ topologized as before. The proof will be completed if we can find two homeomorphisms $F_1 : X_{B^1} \to X,\ F_2 : X_{B^2} \to X$, preserving the cofinal relation and so that

$$F_1(y_n^{(1)}) = F_2(y_n^{(2)}), \quad F_1(\lambda_{B^1}(y_n^{(1)})) = F_2(\lambda_{B^2}(y_n^{(2)}))$$

for $n = 1, 2, \ldots, \infty$. In fact, $F = F_2^{-1} \circ F_1 : X_{B^1} \to X_{B^2}$ will then be the desired orbit map. To achieve our goal we first telescope $(V, E)$ appropriately to alternate odd and even levels so that the resulting diagram has only non-zero entries in the associated incidence matrices. Furthermore, we may adjust this telescoping of $(V, E)$ so that the following holds: Say level $m$, resp. $m+1$, of $(V, E)$ corresponds to level $i$ of $(V^1, E^1)$, resp. $j$ of $(V^2, E^2)$. Then

$$\{l|y_l^{(1)} \in Y_i^1\} \subset \{l|y_l^{(2)} \in Y_j^2\}$$
$$\{l|\lambda_{B^1}(y_l^{(1)}) \in Y_i'^1\} \subset \{l|\lambda_{B^2}(y_l^{(2)}) \in Y_j'^2\}$$

(For $k = 1, 2$, the sets $Y_i^k$ and $Y_i'^k$ refer to the finite subsets of $Y^k$, resp. $\lambda_{B^k}(Y^k)$, that we introduced in our analysis above. Also, if instead level $m$ of $(V, E)$ corresponds to a level of $(V^2, E^2)$ we get a similar description.)
Now we do as in the proof of Theorem 2.1, establishing bijections for each even $n$

$$E_n \circ E_{n+1} \leftrightarrow E^1_{\frac{n}{2}+1}, \quad E_{n-1} \circ E_n \leftrightarrow E^2_{\frac{n}{2}}$$

respecting the range and source maps, recalling that $V_m = V^1_{\frac{m+1}{2}}$ for $m$ odd and $V_m = V^2_{\frac{m}{2}}$ for $m$ even. We let $E_1 = E^1_1$ and choose the bijections successively to make sure that the engendered maps $F_1 : X_{B^1} \to X$, $F_2 : X_{B^2} \to X$ satisfy the above stated conditions. By our setup this is easily achieved and we leave the details to the reader. $\square$

We next need some results about exact sequences involving simple dimension groups. The following lemma may be considered a folklore result. (All the maps in the sequel will be group homomorphisms.)

**Lemma 5.3.** Let

$$0 \to J \xrightarrow{i} H \xrightarrow{\pi} G \to 0$$

be a short exact sequence of torsion free and countable abelian groups. Assume first that $(G, G^+)$ is a simple dimension group, $G \not\cong \mathbf{Z}$. Then $H$ becomes a simple dimension group with the induced order, i.e. $H^+ = \pi^{-1}(G^+ \setminus \{0\}) \cup \{0\}$. Furthermore, $Inf(H) = \pi^{-1}(Inf(G))$ and so, in particular, $i(J) \subset Inf(H)$.

Conversely, assume $(H, H^+)$ is a simple dimension group with $i(J) \subset$ Inf $(H)$. Then $G$ becomes a simple dimension group with the induced order, i.e. $G^+ = \pi(H^+)$. Furthermore, $Inf(G) = \pi(Inf(H))$.

In both cases $\pi$ is order preserving.

**Proof.** The verifications are straightforward and are left to the reader. We only remark that the condition $G \not\cong \mathbf{Z}$ is necessary for the first part since the *strict* Riesz interpolation property is needed for the proof. (Cf. [10; Lemma 3.2].) $\square$

**Lemma 5.4** Let $G_1$ and $G_2$ be simple dimension groups with order units $u_1$ and $u_2$, respectively. Suppose that

$$G_1/Inf(G_1) \cong G_2/Inf(G_2)$$

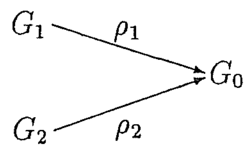
as ordered groups with order units (cf. Remark to Definition 1.10) and suppose that both are acyclic, i.e. $\not\cong \mathbf{Z}$. Then there exists a simple dimension group $H$ with order unit $v$ and two short exact sequences

$$\mathcal{E}_1 : \ 0 \to \mathbf{Z}^\infty \xrightarrow{i_1} H \xrightarrow{\pi_1} G_1 \to 0$$

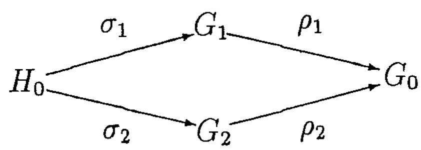
$$\mathcal{E}_2 : \ 0 \to \mathbf{Z}^\infty \xrightarrow{i_2} H \xrightarrow{\pi_2} G_2 \to 0$$

so that (a) $i_k(\mathbf{Z}^\infty) \subset Inf(H)$, (b) $\pi_k(v) = u_k$ and (c) $H^+ \setminus \{0\} = \pi_k^{-1}(G_k^+ \setminus \{0\})$ for $k = 1, 2$. ($\mathbf{Z}^\infty$ denotes $\bigoplus_1^\infty \mathbf{Z}$.)

**Proof.** Let $G_0$ denote $G_1/Inf(G_1)$ with quotient map $\rho_1 : G_1 \to G_0$ (hence $G_1^+ \setminus \{0\} = \rho^{-1}(G_0^+ \setminus \{0\})$) and let $u_0 = \rho_1(u_1)$. Let $\rho_2 : G_2 \to G_0$ be the composition of the quotient map from $G_2$ to $G_2/Inf(G_2)$ with the isomorphism $G_2/Inf(G_2) \cong G_1/Inf(G_1) = G_0$. Hence $G_2^+ \setminus \{0\} = \rho_2^{-1}(G_0^+ \setminus \{0\})$ and $\rho_2(u_2) = u_0$.
Let $H_0$ be the pullback of the diagram




in the category of abelian groups, i.e. $H_0 = \{(g_1, g_2) \in G_1 \bigoplus G_2 | \rho_1(g_1) = \rho_2(g_2)\}$. Let $\sigma_1, \sigma_2$ denote the canonical (projection) maps from $H_0$ to (actually, onto) $G_1$ and $G_2$, respectively.
We get the commutative diagram




Let $\rho_0 : H_0 \to G_0$ denote $\rho_1 \circ \sigma_1 (= \rho_2 \circ \sigma_2)$. Let $H_0^+ = \rho_0^{-1}(G_0^+ \setminus \{0\}) \cup \{0\}$. Then $(H_0, H_0^+)$ is a simple dimension group by Lemma 5.3. Clearly $H_0^+ \setminus \{0\} = \sigma_1^{-1}(G_1^+ \setminus \{0\}) = \sigma_2^{-1}(G_2^+ \setminus \{0\})$. Since $Inf(G_0) = \{0\}$ it follows by Lemma 5.3 that $Inf(H_0) = \ker \rho_0 (= Inf(G_1) \bigoplus Inf(G_2))$. So we have a short exact sequence

$$0 \to Inf(H_0) \xrightarrow{i_0} H_0 \xrightarrow{\rho_0} G_0 \to 0$$

Now choose a surjective map $\beta : \mathbf{Z}^\infty \to Inf(H_0)$. (Replacing $\mathbf{Z}^\infty$ by $\mathbf{Z}^\infty \bigoplus \mathbf{Z}^\infty$ and $\beta$ by $\beta \bigoplus 0$, we may assume $\ker \beta$ has infinite rank.) By Theorem 51.3 of [14] the map $Ext(G_0, \mathbf{Z}^\infty) \xrightarrow{\beta_*} Ext(G_0, Inf(H_0))$ is onto.

(Specifically, since $Ext^2_{\mathbf{Z}} \equiv 0$ it follows that $Ext^1_{\mathbf{Z}}$ is right exact.) This means that we may find $H$ and maps $i : \mathbf{Z}^\infty \to H$, $\eta : H \to H_0$, $\rho : H \to G_0$ so that

$$\begin{array}{ccccccc} 0 \to & \mathbf{Z}^\infty & \xrightarrow{i} & H & \xrightarrow{\rho} & G_0 & \to 0 \\ & \downarrow \beta & & \downarrow \eta & & \| & \\ 0 \to & Inf(H_0) & \xrightarrow{i_0} & H_0 & \xrightarrow{\rho_0} & G_0 & \to 0 \end{array}$$

is a commutative diagram with exact rows. (Note: $H$ is torsion free since $\mathbf{Z}^\infty$ and $G_0$ are so, $\eta$ is surjective since $\beta$ is so, and $\ker \eta$ has infinite rank since $\ker \beta$ has so.) By Lemma 5.3 $H$ becomes a simple dimension group with $H^+ = \rho^{-1}(G_0^+ \setminus \{0\}) \cup \{0\} = \eta^{-1}(H_0^+ \setminus \{0\}) \cup \{0\}$. Consider the map $\pi_1 = \sigma_1 \circ \eta : H \to G_1$. First, $\ker \pi_1 \supset \ker \eta$, and so $\ker \pi_1$ is of infinite rank. Next,

$$\ker \pi_1 \subset \ker(\rho_1 \circ \pi_1) = \ker(\rho_1 \circ \sigma_1 \circ \eta) = \ker(\rho_0 \circ \eta) = \ker \rho \cong \mathbf{Z}^\infty .$$

Thus $\ker \pi_1$ is an infinite rank subgroup of a free abelian group and is therefore isomorphic to $\mathbf{Z}^\infty$. We obtain a short exact sequence

$$0 \to \mathbf{Z}^\infty \xrightarrow{i_1} H \xrightarrow{\pi_1} G_1 \to 0$$

where $i_1$ is the composition of the isomorphism from $\mathbf{Z}^\infty$ to $\ker \pi_1$ with the natural inclusion of $\ker \pi_1$ in $H$. The short exact sequence

$$0 \to \mathbf{Z}^\infty \xrightarrow{i_2} H \xrightarrow{\pi_2} G_2 \to 0$$

is obtained in the same way.

Let $v$ be any element of $H$ so that $\eta(v) = (u_1, u_2) \in H_0$. (This uses the fact that $\rho_1(u_1) = u_0 = \rho_2(u_2)$ and that $\eta$ is surjective.) Then $\pi_k(v) = \sigma_k \circ \eta(v) = \sigma_k(u_1, u_2) = u_k$ for $k = 1, 2$, hence (b). It is clear that

$$H^+ \setminus \{0\} = \rho^{-1}(G_0^+ \setminus \{0\}) = (\rho_0 \circ \eta)^{-1}(G_0^+ \setminus \{0\}) = \eta^{-1}(H_0^+ \setminus \{0\})$$

$$= (\sigma_k \circ \eta)^{-1}(G_k^+ \setminus \{0\}) = \pi_k^{-1}(G_k \setminus \{0\}), \; k = 1, 2,$$

i.e. (c) is valid. (a) follows by Lemma 5.3. □

The next three lemmas deal with how to construct an induced (derivative) system of a given dynamical system in order to effect a change of order unit — making sure that a given closed subset is contained in the new (clopen) set.

We will use the following notation throughout: $(X, \varphi)$ is a Cantor system. For $E$ a clopen subset of $X$ we denote by $[\chi_E]$ the element in $K_0(A_Y^\varphi)$ that $\chi_E$ maps into. Here $Y$ is a closed subset of $X$ that meets each $\varphi$-orbit at most once, and so $A_Y^\varphi$ is a simple AF-algebra by Theorem 1.16. We let $\mathbf{1}_X$ and $\mathbf{1}_{X,Y}$ denote the (canonical) distinguished order units of $K^0(X,\varphi)(\cong (C(X) \underset{\varphi}{\times} \mathbf{Z}))$ and $K_0(A_Y^\varphi)$, respectively. Hence $\mathbf{1}_{X,Y} = [\chi_X]$. $\Gamma_Y^\varphi$ was defined in the section preceding Theorem 1.18 as the normalizer group of $C(X)$ in $A_Y^\varphi$ modulo the unitaries in $C(X)$. The salient property of $\Gamma_Y^\varphi$ that we shall need is: $\Gamma_Y^\varphi$ acts minimally on $X$ [28; Corollary 5.5]. Also, $[\chi_E] = [\chi_{\gamma(E)}]$ for $\gamma \in \Gamma_Y^\varphi$, $E$ a clopen subset of $X$.

**Lemma 5.5.** For any $x \in X$ and $v \in K_0(A_Y^\varphi)^+ \setminus \{0\}$ there is a clopen set $E \subset X$ so that (i) $x \in E$ and (ii) $[\chi_E] \lneq v$ in $K_0(A_Y^\varphi)$.

**Proof.** Without loss of generality we may assume $v \leq \mathbf{1}_{X,Y}$, and so $v = [\chi_F]$ for some clopen set $F \subset X$. Since $\Gamma_Y^\varphi$ acts minimally on $X$ we can find $\gamma \in \Gamma_Y^\varphi$ so that $\gamma(x) \in F$. Choose a clopen set $E$ so $x \in E \subsetneq \gamma^{-1}(F)$. □

**Lemma 5.6.** Let $Y = \{y_1, \ldots, y_n, \ldots, y_\infty\}, y_n \to y_\infty$. For $u \in K_0(A_Y^\varphi)^+ \setminus \{0\}$ with $u \leq \mathbf{1}_{X,Y}$ there is a clopen set $X_0 \subset X$ so that (i) $Y \subset X_0$ and (ii) $[\chi_{X_0}] = u$.

**Proof.** Apply Lemma 5.5 with $x = y_\infty$ to obtain a clopen set $E_\infty$, $y_\infty \in E_\infty$, $[\chi_{E_\infty}] \lneq u$. For each $y_k$ not in $E_\infty$ (finitely many!), inductively select a clopen set $E_k$ making sure that

(i) $y_k \in E_k$

(ii) $E_k$ disjoint from $E_\infty$ and previous $E_i$'s.

(iii) $[\chi_{E_k}] \lneq u - [\chi_{E_\infty \cup E_1 \cup \cdots \cup E_{k-1}}]$.

Now $E = E_\infty \cup (\bigcup_{\{k | y_k \notin E_\infty\}} E_k)$ satisfies all conditions except we have $[\chi_E] \lneq u$ rather than equal. Now $[\chi_{X \setminus E}] = [\chi_X] - [\chi_E] \gneq \mathbf{1}_{X,Y} - u$. So we can find $E' \subset X \setminus E$, $[\chi_{E'}] = \mathbf{1}_{X,Y} - u$. Let $X_0 = X \setminus E'$ and we are done. □

**Lemma 5.7** Let $(\tilde{X}, \tilde{\varphi})$ be a Cantor system. Let $X \subset \tilde{X}$ be a clopen set and let $(X, \varphi)$ be the induced system of $(\tilde{X}, \tilde{\varphi})$ with respect to $X$. Let $Y$ be a closed subset of $X$ meeting each $\varphi$-orbit at most once. (Consequently $Y$ meets each $\tilde{\varphi}$-orbit at most once.) Then there is an isomorphic map from $C(X) \underset{\varphi}{\times} \mathbf{Z}$ to $p(C(\tilde{X}) \underset{\tilde{\varphi}}{\times} \mathbf{Z})p$, respectively from $A_Y^{\varphi}$ to $pA_Y^{\tilde{\varphi}}p$, where $p = \chi_X$, mapping $f \in C(X)$ to $\tilde{f}$, where $\tilde{f}(x) = f(x)$ for $x \in X$ and 0 otherwise. This induces maps in $K$-theory so that the following diagram commutes.

$$\begin{array}{ccc} K^0(A_Y^{\varphi}) & \rightarrow & K^0(X, \varphi) \\ \cong\downarrow & & \cong\downarrow \\ K^0(A_Y^{\tilde{\varphi}}) & \rightarrow & K^0(\tilde{X}, \tilde{\varphi}) \end{array}$$

where $\cong$ denotes order isomorphism and the horizontal maps stem from the inclusions $A_Y^{\varphi} \hookrightarrow C(X) \underset{\tilde{\varphi}}{\times} \mathbf{Z}$, respectively $A_Y^{\tilde{\varphi}} \hookrightarrow C(\tilde{X}) \underset{\tilde{\varphi}}{\times} \mathbf{Z}$.

**Proof.** The first part follows from Proposition 3.9 and its corollary. Now $p(C(\tilde{X}) \underset{\tilde{\varphi}}{\times} \mathbf{Z})p \subset C(\tilde{X}) \underset{\tilde{\varphi}}{\times} \mathbf{Z}$ and $pA_Y^{\tilde{\varphi}}p \subset A_Y^{\tilde{\varphi}}$ induce isomorphisms on $K$-theory as $C(\tilde{X}) \underset{\tilde{\varphi}}{\times} \mathbf{Z}$ and $A_Y^{\tilde{\varphi}}$ are simple $C^*$-algebras. Also, $K_0(C(\tilde{X}) \underset{\tilde{\varphi}}{\times} \mathbf{Z}) \cong K^0(\tilde{X}, \tilde{\varphi})$, $K_0(C(X) \underset{\varphi}{\times} \mathbf{Z}) \cong K^0(X, \varphi)$, cf. Corollary 1 to Theorem 1.17. Combining all this we get the second part. □

We now come to the key lemma. Here the Fundamental Extension Theorem (Theorem 10.1) is the crucial tool to obtain the proof.

**Lemma 5.8.** Given a short exact sequence

$$0 \rightarrow \mathbf{Z}^{\infty} \rightarrow H \xrightarrow{\pi} G \rightarrow 0$$

where $H$ and $G$ are simple dimension groups so that $H^+ \setminus \{0\} = \pi^{-1}(G^+ \setminus \{0\})$. Also, $H$ has distinguished order unit $v$ that maps by $\pi$ to the distinguished order unit $v_0$ of $G$, i.e. $v_0 = \pi(v)$. Then there exists a Cantor system $(X, \varphi)$ and a closed subset $Y = \{y_1, \ldots, y_n, \ldots, y_{\infty}\}$, where $y_n \rightarrow y_{\infty}$ and the $y_n$'s are in distinct $\varphi$-orbits, and order isomorphic maps $\eta : K_0(A_Y^{\varphi}) \rightarrow H$, $\gamma : K^0(X, \varphi) \rightarrow G$, with $\eta(\mathbf{1}_{X,Y}) = v, \gamma(\mathbf{1}_X) = v_0$, so that the following diagram commutes:

$$\begin{array}{ccc} K_0(A_Y^{\varphi}) & \rightarrow & K^0(X, \varphi) \\ \eta\downarrow & & \gamma\downarrow \\ H & \xrightarrow{\pi} & G \end{array}$$

**Proof.** Consider $0 \to \mathbf{Z}^\infty \to H \xrightarrow{\pi} G \to 0$ with $G$ given the new order unit $2v_0$. Apply the Fundamental Extension Theorem (Theorem 10.1) to obtain a minimal system $(\tilde{X}, \tilde{\varphi}), \tilde{X}$ Cantor, and $Y = \{y_1, \ldots, y_n, \ldots, y_\infty\}$, where $y_n \to y_\infty$ and the $y_n$'s are in distinct $\tilde{\varphi}$-orbits, and order isomorphic maps $\tilde{\eta} : K_0(A_Y^{\tilde{\varphi}}) \to H,\;\; \tilde{\gamma} : K^0(\tilde{X}, \tilde{\varphi}) \to G$, with $\tilde{\gamma}(\mathbf{1}_{\tilde{X}}) = 2v_0$ and so that the diagram

$$\begin{array}{ccccccc} 0 \to & \mathbf{Z}^\infty & \to & K_0(A_Y^{\tilde{\varphi}}) & \to & K^0(\tilde{X}, \tilde{\varphi}) & \to 0 \\ & \| & & \tilde{\eta} \downarrow & & \tilde{\gamma} \downarrow & \\ 0 \to & \mathbf{Z}^\infty & \to & H & \xrightarrow{\pi} & G & \to 0 \end{array}$$

commutes. Consider $v$ and $\tilde{\eta}(\mathbf{1}_{\tilde{X},Y})$ in $H$. We claim that $\tilde{\eta}(\mathbf{1}_{\tilde{X},Y}) \underset{\neq}{>} v \underset{\neq}{>} 0$. In fact, $\pi \circ \tilde{\eta}(\mathbf{1}_{\tilde{X},Y}) = \tilde{\gamma}(\mathbf{1}_{\tilde{X}}) = 2v_0$. Since $\pi(v) = v_0$ and $2v_0 \underset{\neq}{>} v_0 \underset{\neq}{>} 0$ the claim follows by $H^+ \setminus \{0\} = \pi^{-1}(G^+ \setminus \{0\})$. Now apply Lemma 5.6 to $(\tilde{X}, \tilde{\varphi}, Y)$ and $u = \tilde{\eta}^{-1}(v)$ to get clopen $X \subset \tilde{X}$ so that (i) $Y \subset X$ and (ii) $[\chi_X] = u$ in $K_0(A_Y^{\tilde{\varphi}})$. As the final step we combine the above commutative diagram with the commutative diagram of Lemma 5.7 to get the following commutative diagram.

$$\begin{array}{ccccccc} & & & K_0(A_Y^{\varphi}) & \to & K^0(X, \varphi) & \\ & & & \cong\downarrow & & \cong\downarrow & \\ 0 \to & \mathbf{Z}^\infty & \to & K_0(A_Y^{\tilde{\varphi}}) & \to & K^0(\tilde{X}, \tilde{\varphi}) & \to 0 \\ & \| & & \tilde{\eta} \downarrow & & \tilde{\gamma} \downarrow & \\ 0 \to & \mathbf{Z}^\infty & \to & H & \xrightarrow{\pi} & G & \to 0 \end{array}$$

We get the desired maps $\eta : K^0(A_Y^{\varphi}) \to H,\; \gamma : K^0(X, \varphi) \to G$ by composing the vertical maps. By construction we see that $\eta(\mathbf{1}_{X,Y}) = v$, since first $\mathbf{1}_{X,Y}$ maps to $[\chi_X] = u \in K_0(A_Y^{\tilde{\varphi}})$ and then by $\tilde{\eta}$ to $v \in H$. We then get $\gamma(\mathbf{1}_X) = v_0 = \pi(v)$ by the commutativity of the above diagram, recalling that $\mathbf{1}_{X,Y}$ is mapped to $\mathbf{1}_X$ by the upper horizontal map. □

**(ii) ⇒ (i)** (in the proof of Theorem 2.2). By Lemma 5.4, letting $G_1 = K^0(X_1, \varphi_1)$, $G_2 = K^0(X_2, \varphi_2)$, there exists a simple dimension group $H$ with order unit $v$, and $\mathcal{E}_1$ in Ext $(K^0(X_1, \varphi_1), \mathbf{Z}^\infty)$, $\mathcal{E}_2$ in Ext $(K^0(X_2, \varphi_2), \mathbf{Z}^\infty)$ of the form

$$\mathcal{E}_1 : \;\; 0 \to \mathbf{Z}^\infty \to H \xrightarrow{\pi_1} K^0(X_1, \varphi_1) \to 0$$

$$\mathcal{E}_2 : \;\; 0 \to \mathbf{Z}^\infty \to H \xrightarrow{\pi_2} K^0(X_2, \varphi_2) \to 0$$

so that $\pi_1(v) = \mathbf{1}_{X_1}$, $\pi_2(v) = \mathbf{1}_{X_2}$. The two exact sequences $\mathcal{E}_1$ and $\mathcal{E}_2$ satisfy the hypothesis of Lemma 5.8 with $v_0 = \mathbf{1}_{X_1}$, respectively $v_0 = \mathbf{1}_{X_2}$.

Hence there exist for $k = 1, 2$, Cantor systems $(Z_k, \psi_k)$, and $Y_k \subset Z_k$ so that $Y_k = \{y_1^{(k)}, \ldots, y_n^{(k)}, \ldots, y_\infty^{(k)}\}$, where $y_n^{(k)} \to y_\infty^{(k)}$ and the $y_n^{(k)}$'s are in distinct $\psi_k$-orbits, and a commutative diagram

$$\begin{array}{ccc} K_0(A_{Y_k}^{\psi_k}) & \to & K^0(Z_k, \psi_k) \\ \eta_k \downarrow & & \downarrow \gamma_k \\ H & \to & K^0(X_k, \varphi_k) \end{array}$$

where $\eta_k$ and $\gamma_k$ are order isomorphisms, so that $\eta_k(\mathbf{1}_{Z_k, Y_k}) = v$, $\gamma_k(\mathbf{1}_{Z_k}) = \mathbf{1}_{X_k}$. Composing $\eta_1 : K_0(A_{Y_1}^{\psi_1}) \to H$ and $\eta_2^{-1} : H \to K_0(A_{Y_2}^{\psi_2})$ we get (since $\eta_1(\mathbf{1}_{Z_1, Y_1}) = v = \eta_2(\mathbf{1}_{Z_2, Y_2})$) that $K_0(A_{Y_1}^{\psi_1}) \cong K_0(A_{Y_2}^{\psi_2})$ as ordered groups with distinguished order units. By Lemma 5.2 we conclude that $(Z_1, \psi_1)$ is orbit equivalent to $(Z_2, \psi_2)$. As $K^0(Z_1, \psi_1) \cong K^0(X_1, \varphi_1)$ and $K^0(Z_2, \psi_2) \cong K^0(X_2, \varphi_2)$ as ordered groups with distinguished order units by the maps $\gamma_1$ and $\gamma_2$, respectively, we get by Theorem 2.1 that $(Z_1, \psi_1)$ and $(X_1, \varphi_1)$, respectively $(Z_2, \psi_2)$ and $(X_2, \varphi_2)$, are (strong) orbit equivalent. Hence $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ are orbit equivalent and we are done. □

## 6. Proof of Theorem 2.3.

We will reduce the proof of Theorem 2.3 to that of the proof of Theorem 2.2. This will be achieved by the ensuing lemma.

Recall that by [22] and [35] an AF-system $(X, \Gamma)$ arises in precisely the following manner: Begin with a Bratteli diagram $(V, E)$ and let $X$ be the associated infinite path space endowed with the usual topology. For each $n = 1, 2, \ldots$, let $\Gamma_n$ denote the set of permutations of $P_n = E_1 \circ E_2 \circ \cdots \circ E_n$, the paths of length $n$, which preserve the range; i.e. permutations $\gamma : P_n \to P_n$ such that $r(p) = r(\gamma(p))$ for all $p$ in $P_n$. Each element $\gamma$ of $\Gamma_n$ may be viewed as a homeomorphism of $X$, also denoted by $\gamma$, by setting

$$\gamma(e_1, e_2, \ldots) = (f_1, f_2, \ldots, f_n, e_{n+1}, e_{n+2}, \ldots)$$

where $(f_1, \ldots, f_n) = \gamma(e_1, \ldots, e_n)$. In this way it is also clear that $\Gamma_n \subset \Gamma_{n+1}$.

We let $\Gamma = \cup_n \Gamma_n$. It is easy to see that the $\Gamma$-orbit of a point $x$ in $X$ is simply all paths cofinal with $x$ in the model above.

**Lemma 6.1** Let $(X,\Gamma)$ be a minimal AF-system. Then there is a Cantor system $(X_0,\varphi_0)$ so that $(X,\Gamma)$ and $(X_0,\varphi_0)$ are orbit equivalent. Moreover, we have

$$K^0(X,\Gamma)/Inf(K^0(X,\Gamma)) \cong K^0(X_0,\varphi_0)/Inf(K^0(X_0,\varphi_0))$$

as ordered groups with distinguished order units.

**Proof.** We may assume that $(X,\Gamma)$ arises (as described above) from a Bratteli diagram $(V,E)$. Since $(X,\Gamma)$ is assumed to be minimal, $K_0(V,E)$ is a simple dimension group. As described in [22] (cf. also [35]) we have $K^0(X,\Gamma) \cong K_0(V,E)$ as ordered groups with distinguished order units. By Remark (iv) to Definition 3.4 $(V,E)$ (or a telescope of it) admits an order $\geq$ making $B=(V,E,\geq)$ a simple ordered Bratteli diagram. Let $x_1 = y_{\max}$, respectively $z_1 = y_{\min}$, denote the unique max, respectively min, path of $X = X_B$, and let $\varphi = \lambda_B$ be the associated Vershik map. Then $K^0(X,\varphi) \cong K_0(V,E)$ as ordered groups with distinguished order units according to Theorem 3.7. Now $\varphi(x_1)=z_1$, and

$$(*) \qquad \begin{cases} \{\varphi^n(x_1) | n \leq 0\} = \Gamma x_1, \quad \{\varphi^n(z_1) | n \geq 0\} = \Gamma z_1 \quad \text{and} \\ \{\varphi^n(x) | n \in \mathbf{Z}\} = \Gamma x \text{ for } x \notin \Gamma x_1 \cup \Gamma z_1 \end{cases}$$

Select any point $x_2 \notin \Gamma x_1 \cup \Gamma z_1$ and let $z_2 = \varphi(x_2)$. Let $(V',E')$ be the Bratteli diagram for the AF-algebra $A^{\varphi}_{\{x_1,x_2\}}$. Since $x_1$ and $x_2$ are in distinct $\varphi$-orbits this AF-algebra is simple, and we have a short exact sequence

$$0 \to \mathbf{Z} \to K_0(A^{\varphi}_{\{x_1,x_2\}}) \to K^0(X,\varphi) \to 0$$

where the image of $\mathbf{Z}$ is contained in $Inf(K_0(A^{\varphi}_{\{x_1,x_2\}}))$. (Corollary 1 and 2 to Theorem 1.17.)
We conclude that

$$\begin{aligned} &K_0(V',E')/Inf(K_0(V',E')) \cong K_0(V,E)/Inf(K_0(V,E)) \\ &\cong K^0(X,\Gamma)/Inf(K^0(X,\Gamma)) \end{aligned}$$

as ordered groups with distinguished order units. (Cf. also Lemma 5.3.) Now $(X,\varphi)$ is conjugate to the Bratteli-Vershik system $(X_{B'},\lambda_{B'})$ that we get by a sequence of tower constructions based on $Y=\{x_1,x_2\}$ which are described in Lemma 5.1. Also, by Lemma 5.1 we may assume $B' =$

$(V', E', \geq')$, where $(V', E')$ is the Bratteli diagram above for $A^{\varphi}_{\{x_1,x_2\}}$. Let $X_0$ denote $X_{B'}$, i.e. $X_0$ is the infinite path space associated to $(V', E')$. We transfer $\varphi$ and $\Gamma$ via the conjugating map to act on $X_0$. To simplify our notation we denote the transferred actions again by $\varphi$ and $\Gamma$. We also let $x_1, z_1, x_2, z_2$ denote the image of these points in $X_0$ by the conjugating map. We have the following scenario: The orbits of $\varphi$ are determined by the cofinal relation with the proviso that the paths cofinal with $x_1$, respectively $x_2$, are "glued" to the paths cofinal with $z_1$, respectively $z_2$. Furthermore, $x_1$ and $z_1$, respectively $x_2$ and $z_2$, are not cofinal.
Hence, since $K_0(V', E') \cong K_0(A^{\varphi}_{\{x_1,x_2\}})$ is a simple dimension group, the diagram $(V', E')$ (or a telescope of it) admits an order $\geq^0$ so that $B^0 = (V', E', \geq^0)$ is a simple ordered Bratteli diagram with $x_2$ the unique max path and $z_2$ the unique min path (cf. Remark (iv) to Definition 3.4). Let $\varphi_0$ denote the associated Vershik map $\lambda_{B^0}$ acting on $X_{B^0} = X_{B'} = X_0$. The $\varphi_0$-orbits are determined by the cofinal relation with the proviso that the paths cofinal with $x_2$ are "glued" to the paths cofinal with $z_2$. Now $K^0(X_0, \varphi_0) \cong K_0(V', E')$ as ordered groups with distinguished order units (Theorem 3.7), and so

$$K_0(X_0, \varphi_0)/Inf(K^0(X_0, \varphi_0)) \cong K^0(X, \Gamma)/Inf(K^0(X, \Gamma))$$

as ordered groups with distinguished order units. As for the $\varphi_0$-orbits we have by $(*)$:

$$\{\varphi_0^n(x_2) | n \in \mathbf{Z}\} = \{\varphi^n(x_2) | n \in \mathbf{Z}\} = \Gamma x_2,$$

$$\{\varphi_0^n(x_1) | n \in \mathbf{Z}\} = \{\varphi^n(x_1) | n \leq 0\} = \Gamma x_1,$$

$$\{\varphi_0^n(z_1) | n \in \mathbf{Z}\} = \{\varphi^n(z_1) | n \geq 0\} = \Gamma z_1 \text{ and}$$

$$\{\varphi_0^n(x) | n \in \mathbf{Z}\} = \{\varphi^n(x) | n \in \mathbf{Z}\} = \Gamma x \text{ for } x \text{ not cofinal with } x_1, z_1, x_2, z_2\,.$$

So $\varphi_0$ has the same orbits as $\Gamma$. □

**Proof of Theorem 2.3.** By the previous lemma Theorem 2.3 follows from Theorem 2.2. (By a slight modification of the proof of (i) ⇒ (iii) of Theorem 2.2 it is easy to see that the $\Gamma$-invariant probability measures map to the $\varphi_0$-invariant probability measures by the orbit map that exists by Lemma

6.1.) □

## 7. Proof of Theorem 2.4.

**(i) ⇒ (iii).** Obvious.

**(iii) ⇒ (ii).** Let $u_1 = u_{\varphi_1}$ be the unitary operator in $C(X_1) \underset{\varphi_1}{\times} \mathbf{Z}$ associated to $\varphi_1$, cf. Section 1. Hence $u_1 f u_1^* = f \circ \varphi_1^{-1}, f \in C(X_1)$. As $u_1 \in \mathcal{N}(C(X_1), C(X_1) \underset{\varphi_1}{\times} \mathbf{Z})$ we get $v = \alpha(u_1) \in \mathcal{N}(C(X_2), C(X_2) \underset{\varphi_1}{\times} \mathbf{Z})$, since $\alpha$ maps $C(X_1)$ onto $C(X_2)$. Now $v$ implements, via the map $g \to vgv^*, g \in C(X_2)$, a minimal homeomorphism $\theta$ of $X_2$ that is conjugate to $\varphi_1$ — a fact that is easily verified. By [28; Lemma 5.1 and Corollary 5.3] we conclude that $\theta$ and $\varphi_2$ have the same orbits and that the orbit cocycle $k : X_2 \to \mathbf{Z}$ is continuous, where $\theta(x) = \varphi_2^{k(x)}(x), x \in X_2$. From this we deduce quite simply that there is an orbit map $F : X_1 \to X_2$ so that the associated orbit cocycle $n : X_1 \to \mathbf{Z}$ is continuous, where $F \circ \varphi_1(x) = \varphi_2^{n(x)} \circ F(x)$.

**(ii) ⇒ (i).** By considering $\tilde{\varphi}_2 = F^{-1} \circ \varphi_2 \circ F$ and $\varphi_1$ we may assume that $X = X_1 = X_2$ and that $(X, \varphi_1)$ and $(X, \varphi_2)$ have the same orbits so that

$$\varphi_1(x) = \varphi_2^{n(x)}(x), \quad \varphi_2(x) = \varphi_1^{m(x)}(x),$$

where $n$ and $m$ are the orbit cocycles. Assume $n : X \to \mathbf{Z}$ is continuous. (It is easy to show, and, in fact, it is implicity shown in our proof of Theorem 2.5, that then $m$ is continuous.) Let $\psi$ denote $\varphi_1$ and $\varphi$ denote $\varphi_2$. From $\psi(x) = \varphi^{n(x)}(x)$, we deduce that

$$\psi^k(x) = \varphi^{f_k(x)}(x)\,, \text{ where}$$

$$(*) \qquad \begin{cases} f_k(x) & = n(x) + n(\psi(x)) + \cdots + n(\psi^{k-1}(x))\ ;\ k \geq 1 \\ f_0(x) & = 0 \\ f_k(x) & = -[n(\psi^{-1}(x)) + n(\psi^{-2}(x)) + \cdots + n(\psi^{k}(x))]\ ;\ k < 0 \\ f_k(\psi^j(x)) & = f_{k+j}(x) - f_j(x) \end{cases}$$

The map $k \to f_k(x)$ is clearly a bijection of the integers for each $x$ since $\varphi$ and $\psi$ are aperiodic.
Now, given a positive integer $M$, there is a positive integer $\overline{M}$ so that for

all $x, \{m| - M \leq m \leq M\}$ is contained in $\{f_k(x)| - \overline{M} \leq k \leq \overline{M}\}$. In fact, for each $x$ some $\overline{M}$ exists. Since the $f_k$'s are continuous $\overline{M}$ applies in a neighbourhood of $x$. By compactness we find an $\overline{M}$ that applies for all $x$. For $m$ a positive integer let

$$A_m = \{x|\forall n \geq m, f_n(x) > 0 \text{ and } f_{-n}(x) < 0\}$$

$$B_m = \{x|\forall n \geq m, f_n(x) < 0 \text{ and } f_{-n}(x) > 0\}.$$

Clearly $A_m \cap B_m = \emptyset$. We claim there exists a positive integer $K$ so that either $X = A_K$ og $X = B_K$; in other words, the "orientations" of $\varphi$ and $\psi$ are the same or opposite after some initial fluctuations. To see this, let $n_0 = \sup_{x \varepsilon X} |n(x)|$ and choose $K$ so that $\{j| - n_0 \leq j \leq n_0\}$ is contained in $\{f_i(x)| - K < i < K\}$ for all $x$. Now $X = A_K \cup B_K$. In fact, $\{j| - N \leq j \leq N\}$ is a "roadblock" insuring that $f_n(x)$ and $f_{n\pm1}(x)$ have the same sign if $|n| \geq K$ by $(*)$. Also, $f_n(x)$ and $f_{-n}(x)$ must have opposite signs if $|n| \geq K$ as $i \longrightarrow f_i(x)$ is a bijection of $\mathbf{Z}$. As $n(x)$ is continuous both $A_K$ and $B_K$ are closed. By $(*)$ they are $\psi$-invariant, and so by minimality either $X = A_K$ or $X = B_K$.

Define

$$P_m(x) = |\{f_i(x)|f_i(x) > 0, |i| \leq m\}|$$

$$N_m(x) = |\{f_i(x)|f_i(x) < 0, |i| \leq m\}|.$$

By the above the integer-valued function $a(x)$ defined by

$$a(x) = \frac{1}{2}[N_M(x) - P_M(x)]\,;\ M \geq K$$

is independent of $M$ if $M \geq K$. Notice that $a(x)$ is continuous and so $g(x) = \psi^{a(x)}(x)$ is continuous. If $X = A_K$ we claim that $g$ is a homeomorphism that implements a conjugacy between $\varphi$ and $\psi$—specifically, $g \circ \varphi = \psi \circ g$. [If $X = B_K$, then $X = \tilde{A}_K$, where $\sim$ refers to $\tilde{\psi}(x) = \psi^{-1}(x)$. This is easily seen by observing that $\tilde{f}_k(x) = f_{-k}(x)$. So to prove the theorem we may assume that $X = A_K$.]

The claim will be established if we can prove that

$$(**) \qquad a(\varphi(x)) = a(x) - j + 1\,,$$

where $j$ is determined by $\varphi(x) = \psi^j(x), i.e. f_j(x) = 1$.
In fact,

$$\begin{aligned} g \circ \varphi(x) &= \psi^{a(\varphi(x))}(\varphi(x)) = \psi^{a(x)-j+1}(\psi^j(x)) \\ &= \psi^{a(x)+1}(x) = \psi(\psi^{a(x)}(x)) = \psi \circ g(x). \end{aligned}$$

Hence $g \circ \varphi = \psi \circ g$. We observe that $g$ is surjective since $g \circ \varphi = \psi \circ g$ implies $g \circ \varphi^n = \psi^n \circ g$, and so $g$ maps a $\varphi$-orbit onto a full $\psi$-orbit, which is dense in $X$. Also, $g$ is injective. In fact, if $g(x) = g(y)$, i.e. $\psi^{a(x)}(x) = \psi^{a(y)}(y)$, then $x$ and $y$ lie in the same $\psi$-orbit, hence in the same $\varphi$-orbit. So $y = \varphi^i(x)$ for some $i$. Hence

$$g(x) = g(y) = g(\varphi^i(x)) = \psi^i(g(x)).$$

This implies $i = 0$ and so $y = x$. So $g$ is a homeomorphism implementing a conjugacy between $\varphi$ and $\psi$.
It remains to prove (**). First we observe that $|j| < K$ since $\{m | -n_0 \leq m \leq n_o\}$ is contained in $\{f_s(x) | -K < s < K\}$. Now pick $M \geq 2K$. By (*) we get that for all $k$,

$$f_{j+k}(x) = f_k(\varphi(x)) + 1.$$

Using this we get (**) by a simple counting argument using that

$$a(x) = \frac{1}{2}[N_M(x) - P_M(x)]$$

$$a(\varphi(x)) = \frac{1}{2}[N_{M+j}(x) - P_{M+j}(x)]$$

This completes the proof of (ii) $\Rightarrow$ (i). □

**Remark.** The proof of (ii) $\Rightarrow$ (i) works with virtual no change in the case of one dense orbit and no periodic points.

## 8. Proof of Theorem 2.5.

**(i) $\Rightarrow$ (ii).** Without loss of generality we may assume that $\varphi_1$ and $\varphi_2$ act on the same space $X$ and have the same orbits. Set $\varphi_1 = \psi$, $\varphi_2 = \varphi$ , and let the orbit cocycles $m, n : X \to \mathbf{Z}$ be defined by

$$\psi(x) = \varphi^{n(x)}(x), \quad \varphi(x) = \psi^{m(x)}(x).$$

Let $x_1, \ldots, x_{k+1}$ let the discontinuities of $m$ and let $y_1, \ldots, y_{k+1}$ be the discontinuities of $n$, where $x_i$ and $y_i$ are in the same orbit, $i = 1, \ldots, k+1$. Let $\mu$ be a $\varphi$-invariant (hence $\psi$-invariant) probability measure on $X$. We may realize both $C(X) \underset{\varphi}{\times} \mathbf{Z}$ and $C(X) \underset{\psi}{\times} \mathbf{Z}$ as operators acting on the Hilbert space $H = L^2(X, \mu)$, cf. Section 1. We will show that $A^\varphi = A^\varphi_{\{x_1,\ldots,x_{k+1}\}} (\subset C(X) \underset{\varphi}{\times} \mathbf{Z})$ is equal to $A^\psi = A^\psi_{\{y_1,\ldots,y_{k+1}\}} (\subset C(X) \underset{\psi}{\times} \mathbf{Z})$ — both being simple AF-algebras by Theorem 1.16.

Assume this has been done. Then by Corollary 2 of Theorem 1.17 we have

$$\begin{array}{ccccccccc} 0 \to & \mathbf{Z}^k & \overset{\beta_\varphi}{\to} & K_0(A^\varphi) & \overset{\mu_\varphi}{\to} & K^0(X,\varphi) & \to 0 \\ & & & \| & & & \\ 0 \to & \mathbf{Z}^k & \overset{\beta_\psi}{\to} & K_0(A^\psi) & \overset{\mu_\psi}{\to} & K^0(X,\psi) & \to 0 \end{array}$$

Define

$$i_\varphi = \mu_\varphi \circ \beta_\psi : \mathbf{Z}^k \to K^0(X,\varphi), \quad i_\psi = \mu_\psi \circ \beta_\varphi : \mathbf{Z}^k \to K^0(X,\psi).$$

By Theorem 1.17 and Lemma 5.3 we may conclude that $Im(i_\varphi) \subset Inf(K^0(X,\varphi))$, $Im(i_\psi) \subset Inf(K^0(X,\psi))$. Observe that $Ker(i_\psi) = Ker(i_\varphi) = Im(\beta_\varphi) \cap Im(\beta_\psi)$. Hence

$$Im(i_\psi) \cong \mathbf{Z}^k / Ker(i_\psi) \cong \mathbf{Z}^l$$

and likewise $Im(i_\varphi) \cong \mathbf{Z}^l$ for some $l \leq k$. (Recall that $K^0(X,\varphi)$ and $K^0(X,\psi)$ are torsion free and so $Im(i_\varphi)$, resp. $Im(i_\psi)$, are torsion free.) Define a map

$$\eta : K^0(X,\varphi)/Im(i_\varphi) \to K^0(X,\psi)/Im(i_\psi)$$

by $\eta(a + Im(i_\varphi)) = \mu_\psi(b) + Im(i_\psi)$, where $b \in K_0(A^\psi)(= K_0(A^\varphi))$, $\mu_\varphi(b) = a$. It is routine to check that $\eta$ is well defined and that it defines an order isomorphism preserving the distinguished order units. This proves that (ii) holds except that $l$ may be strictly less than $k$. However, when we prove (ii) $\Rightarrow$ (i) below we show that $K^0(X_1,\varphi_1)/\mathbf{Z}^k \cong K^0(X_2,\varphi_2)/\mathbf{Z}^k$ implies that the number of discontinuity points is less or equal to $k+1$. Hence we may conclude that $l = k$.

So to complete the proof of (i) $\Rightarrow$ (ii) it is sufficient to show that $A^\varphi = A^\psi$, where $A^\varphi = A^\varphi_{\{x_1,\ldots,x_{k+1}\}}$, $A^\psi = A^\psi_{\{y_1,\ldots,y_{k+1}\}}$. We will show that $A^\varphi \subset A^\psi$. (A similar argument yields $A^\psi \subset A^\varphi$.) Since $A^\varphi$ is generated by $C(X)$ and

$u_\varphi \cdot C_0(X \setminus \{x_1, \ldots, x_{k+1}\})$ it is enough to show that $u_\varphi \cdot \chi_E \in A^\psi$, where $E$ is a clopen set disjoint from $\{x_1, \ldots, x_{k+1}\}$. By subdividing $E$ we may assume that $E \cap \varphi(E) = \emptyset$. Then

$$v = u_\varphi \cdot \chi_E + u_\varphi^{-1} \cdot \chi_{\varphi(E)} + \chi_{X \setminus (E \cup \varphi(E))}$$

is a self-adjoint unitary operator in $A^\varphi$ . (In fact, $(u_\varphi \cdot \chi_E)^* = u_\varphi^{-1} \cdot \chi_{\varphi(E)}$.) Since $v \cdot \chi_E = u_\varphi \cdot \chi_E$ we are done if we can show $v \in A^\psi$. To achieve that we first observe

$$v \in \mathcal{N}(C(X), A^\varphi) \text{ — i.e. } vC(X)v^* = C(X).$$

By Theorem 1.18 this means that $v$ preserves the half-orbits of $x_1, \ldots, x_{k+1}$, i.e.

$$(*) \qquad v(\{\varphi^l(x_i) | l \geq 1\}) = \{\varphi^l(x_i) | l \geq 1\}, \quad i = 1, \ldots, k+1,$$

where by abuse of notation we let $v$ also denote the element of $\Gamma^\varphi$ it maps to. To prove that $v \in A^\psi$ we will show that $v \in \mathcal{N}(C(X), A^\psi\}$, or, equivalently by Theorem 1.18, that $v \in C(X) \underset{\psi}{\times} \mathbf{Z}$ and

$$(**) \qquad v(\{\psi^l(y_i) | l \geq 1\}) = \{\psi^l(y_i) | l \geq 1\}, \quad i = 1, \ldots, k+1,$$

where we again by abuse of notation let $v$ denote the element of $\Gamma^\psi$ it maps to. We show first that $v \in C(X) \underset{\psi}{\times} \mathbf{Z}$. It is sufficient to show that $u_\varphi \cdot \chi_E \in C(X) \underset{\psi}{\times} \mathbf{Z}$. Now if $\varphi(x) = \psi^l(x)$ for $x \in F$, where $F$ is a clopen subset of $X$, then $u_\varphi \cdot \chi_F = u_\psi^l \cdot \chi_F$ — a fact that is easily established. From this and the fact that $m$ is continuous on $E$ we get

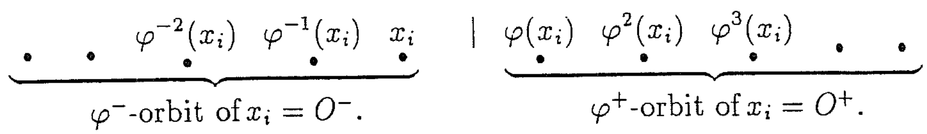
$$u_\varphi \cdot \chi_E = \sum_l u_\psi^l \cdot \chi_{E \cap m^{-1}(l)}$$

where the sum is finite. Hence we conclude that $u_\varphi \cdot \chi_E \in C(X) \underset{\psi}{\times} \mathbf{Z}$.

To show that $(**)$ holds it is sufficient by $(*)$ to show that

$$(***) \begin{cases} \text{either} & \{\varphi^l(x_i) | l \geq 1\} = \{\psi^l(y_i) | l \geq 1\}, \\ \text{or} & \{\varphi^l(x_i) | l \leq 0\} = \{\psi^l(y_i) | l \geq 1\}, \quad i = 1, \ldots, k+1. \end{cases}$$

To establish $(***)$ let us look at the orbit of $x_i$ for some $i \in \{1, \ldots, k+1\}$.



Let $y$ be any point of $X$. From $\varphi(y) = \psi^{m(y)}(y)$, we deduce that

$$\varphi^k(y) = \psi^{f_k(y)}(y), \text{ where}$$

$$(\dagger)\begin{cases} f_k(y) = m(y) + m(\varphi(y)) + \cdots + m(\varphi^{k-1}(y)); \; k > 0. \\ f_k(y) = -[m(\varphi^{-1}(y)) + m(\varphi^{-2}(y)) + \cdots + m(\varphi^k(y))]; \; k < 0. \end{cases}$$

(Cf. formula (*) in the proof of Theorem 2.4.) Setting $k = n(y)$ in ($\dagger$) we get

$$(\dagger\dagger)\begin{cases} 1 &= m(y) + m(\varphi(y)) + \cdots + m(\varphi^{n(y)-1}(y)) \quad \text{if } n(y) > 0. \\ -1 &= m(\varphi^{-1}(y)) + m(\varphi^{-2}(y)) + \cdots + m(\varphi^{n(y)}(y)) \quad \text{if } n(y) < 0. \end{cases}$$

Now let $y$ lie in the orbit of $x_i$ From ($\dagger\dagger$) we deduce that if $n(y) > 0$ and $n$ is continuous at $y$ then $\{y, \varphi(y), \ldots, \varphi^{n(y)-1}(y)\}$ does not include $x_i$. (Observe: If $m$ is discontinuous at a point $x_0$, then $\limsup_{x \to x_0} |m(x)| = \infty$. In fact, otherwise there are $k_1 \neq k_2$ and sequences $\{v_j\}, \{w_j\}$, so that $v_j \to x_0$, $m(v_j) = k_1$, and $w_j \to x_0, m(w_j) = k_2$. Hence $\varphi(v_j) = \psi^{k_1}(v_j), \varphi(w_j) = \psi^{k_2}(w_j)$, which by continuity implies $\psi^{k_1}(x_0) = \psi^{k_2}(x_0)$ — contradicting minimality of $\psi$.) Likewise, if $n(y) < 0$ and $n$ is continuous at $y$ then $\{\varphi^{n(y)}(y), \ldots, \varphi^{-1}(y)\}$ does not include $x_i$. Or equivalently (recalling $\psi(y) = \varphi^{n(y)}(y)$): If $n$ is continuous at $y$,

$$(\dagger\dagger\dagger)\begin{cases} y \in O^- & \Rightarrow \quad \psi(y) \in O^- \\ y \in O^+ & \Rightarrow \quad \psi(y) \in O^+ \end{cases}$$

Since $O^- \cup O^+$ is an orbit of $\psi$ there must be at least one point $y$ so that either $y \in O^-$ and $\psi(y) \in O^+$, or $y \in O^+$ and $\psi(y) \in O^-$. By ($\dagger\dagger\dagger$) this $y$ must necessarily be a discontinuity point for $n$, hence by our hypothesis $y = y_i$. (Incidentally, this same argument also shows that if one of the orbit cocycles have exactly one point of discontinuity in an orbit then the other cocycle must necessarily have (at least) one discontinuity in the same orbit.) We conclude from this that there are two possibilities:

$$\text{Either } O^+ = \{\varphi^l(x_i) | l \geq 1\} = \{\psi^l(y_i) | l \geq 1\},$$

$$\text{or } O^- = \{\varphi^l(x_i) | l \leq 0\} = \{\psi^l(y_i) | l \geq 1\}$$

In either case (***) holds and the proof of (i) $\Rightarrow$ (ii) is complete.

(ii) $\Rightarrow$ (i). Clearly the hypothesis implies that

$$K^0(X_1,\varphi_1)/Inf(K^0(X_1,\varphi_1) \cong K^0(X_2,\varphi_2)/Inf(K^0(X_2,\varphi_2))$$

as ordered groups with distinguished order units. So by Theorem 2.2 $(X_1,\varphi_1)$ is orbit equivalent to $(X_2,\varphi_2)$. However, by the subsequent analysis we will get orbit equivalence without direct recourse to Theorem 2.2. At the same time we establish the desired continuity properties of the orbit cocycles. As will be apparent the proof does bear a close resemblance to the proof of the implication (ii) $\Rightarrow$ (i) of Theorem 2.2, and for that reason we will be somewhat sketchy.

Let $H$ be the pullback of the diagram

$$\begin{array}{ccc} G_1 = K^0(X_1,\varphi_1) & \xrightarrow{\rho_1} & \\ & & G_0 \\ G_2 = K^0(X_2,\varphi_2) & \xrightarrow{\rho_2} & \end{array}$$

where $G_0 = K^0(X_1,\varphi_1)/\mathbf{Z}^k \cong K^0(X_2,\varphi_2)/\mathbf{Z}^k$. So $H = \{(g_1,g_2) \in G_1 \bigoplus G_2 | \rho_1(g_1) = \rho_2(g_2)\}$, and we notice that $H$ is torsion free. We get the commutative diagram

$$\begin{array}{ccccc} & \overset{\sigma_1}{\nearrow} & G_1 & \overset{\rho_1}{\searrow} & \\ H & & & & G_0 \\ & \underset{\sigma_2}{\searrow} & G_2 & \underset{\rho_2}{\nearrow} & \end{array}$$

where $\sigma_1 : H \to G_1$, $\sigma_2 : H \to G_2$, are the projection maps. We get $\sigma_1^{-1}(G_1^+\setminus\{0\}) = \sigma_2^{-1}(G_2^+\setminus\{0\})$ and we define $H^+ = \sigma_i^{-1}(G_i^+\setminus\{0\})\cup\{0\}, i = 1,2$. By Lemma 5.3 $(H,H^+)$ is a simple dimension group and we obtain the short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{Z}^k & \to & H & \xrightarrow{\sigma_1} & K^0(X_1,\varphi_1) & \to & 0 \\ 0 & \to & \mathbf{Z}^k & \to & H & \xrightarrow{\sigma_2} & K^0(X_2,\varphi_2) & \to & 0 \end{array}$$

where $\sigma_1$ and $\sigma_2$ preserve order units and are strict positivity preserving. (Clearly $H$ contains an order unit $h_0$ so that $\sigma_1(h_0) = \mathbf{1}_{X_1}, \sigma_2(h_0) = \mathbf{1}_{X_2}$.)

Let $i = 1$ or 2. Using the corollary to Theorem 10.1 (The Fundamental Extension Theorem) and arguing as in the proof of Lemma 5.8 we get that there exists a minimal system $(Z_i, \psi_i), Z_i$ Cantor, and a subset $Y_i = \{y_1^{(i)}, \ldots, y_{k+1}^{(i)}\}$, where the $y_j^{(i)}$'s are in distinct $\psi_i$-orbits, so that the following diagram commutes

$$\begin{array}{ccc} K_0(A_{Y_i}^{\psi_i}) & \rightarrow & K^0(\mathbf{Z}_i, \psi_i) \\ \eta_i \downarrow & & \gamma_i \downarrow \\ H & \xrightarrow{\sigma_i} & K^0(X_i, \varphi_i) \end{array}$$

where $\eta_i$ and $\gamma_i$ are order isomorphic maps preserving distinguished order units. So

$$\begin{array}{rccc} \eta_2^{-1} \circ \eta_1 : & K_0(A_{Y_1}^{\psi_1}) & \rightarrow & K^0(A_{Y_2}^{\psi_2}) \\ \gamma_1 : & K_0(Z_1, \psi_1) & \rightarrow & K^0(X_1, \varphi_1) \\ \gamma_2 : & K_0(Z_2, \psi_2) & \rightarrow & K^0(X_2, \varphi_2) \end{array}$$

are order isomorphisms preserving the distinguished order units. We now proceed as in the proof of Lemma 5.2 — avoiding some of the technical difficulties there due to the fact that $Y_1$ and $Y_2$ are finite sets — and apply the "aggregate Bratteli diagram" - technique (as $K_0(A_{Y_1}^{\psi_1}) \cong K_0(A_{Y_2}^{\psi_2})$) to get an orbit map $H : Z_1 \rightarrow Z_2$ mapping $Y_1$ onto $Y_2$. Now $Y_1$, respectively $Y_2$, corresponds to the max paths of the Bratteli-Vershik model we employ for $(Z_1, \psi_1)$, respectively $(Z_2, \psi_2)$. Arguing exactly as we did in the proof of Theorem 2.1 ((ii) $\Rightarrow$ (i)) we show that the associated orbit cocycles are continuous except possibly at the $k+1$ points $y_1^{(1)}, \ldots, y_{k+1}^{(1)}$ (corresponding to the max paths of the Bratteli-Vershik model for $(Z_1, \psi_1)$).
Now we must link this with the fact that $K^0(X_i, \varphi_i) \cong K^0(Z_i, \psi_i)$, $i = 1, 2$. Let $i = 1$ or 2. By the model theorem (Theorem 3.6) we may assume that $(Z_i, \psi_i)$ is the Bratteli-Vershik system associated to the simple ordered Bratteli diagram $B^i = (V^i, E^i, \geq^i)$, where $y_{k+1}^{(i)}$ is the unique max path. How do we get the preceding Bratteli-Vershik model for $(Z_i, \psi_i)$ — the one associated to $Y_i$ — from $B^i = (V^i, E^i, \geq^i)$? We will not describe this in detail here. Suffice to say that the edges (and associated vertices) making up the paths $y_1^{(i)}, \ldots, y_k^{(i)}$ are split and reordered in a prescribed way. What is important in our context is that a cylinder set defined by, say, $m$ edges in one model is mapped by the conjugating map to a cylinder set defined by $m$ edges in the other model. Now we again use the "aggregate Bratteli

diagram" - technique as in the proof of Theorem 2.1 ((ii) $\Rightarrow$ (i)) to get an orbit map $G_i : X_i \to Z_i$ with the properties described there. (We may at start assume that $(X_i, \varphi_i)$ is the Bratteli-Vershik system associated to a simple ordered Bratteli diagram.) The resulting orbit map

$$F = G_2^{-1} \circ H \circ G_1 : X_1 \to X_2$$

between $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ has the property that the associated orbit cocycles $m, n : X_1 \to \mathbf{Z}$ each have at most $k+1$ (coinciding, in fact,) discontinuities. In conjunction with what we remarked in proving (i) $\Rightarrow$ (ii) above we may conclude that there are exactly $k+1$ discontinuities. This completes the proof of (ii) $\Rightarrow$ (i). □

## 9. Proof of Theorem 2.6 and Proposition 2.7.

### Proof of Theorem 2.6.

**(i) $\Leftrightarrow$ (ii).** The equivalence of (i) and (ii) is an immediate consequence of Theorem 2.1 and Theorem 3.8 by observing the following: If $(V, E)$ is a Bratteli diagram with associated dimension group $K_0(V, E)$ and $u \in K_0(V, E)^+ \setminus \{0\}$, then there is a Bratteli diagram $(V', E')$ obtained from $(V, E)$ by a finite change (as explained in the section preceding Theorem 3.8) so that $K_0(V', E')$ is order isomorphic to $K_0(V, E)$ by a map sending the distinguished order unit of $K_0(V', E')$ to $u$. Conversely, any finite change of $(V, E)$ gives rise to a dimension group order isomorphic to $K_0(V, E)$ (but not necessarily preserving distinguished order units).

**(iii) $\Rightarrow$ (ii).** By Definition 1.19 there is a simple (unital) $C^*$-algebra $A$ and projections $p, q \in A$ so that $C(X_1) \underset{\varphi_1}{\times} \mathbf{Z} \cong pAp$, $C(X_2) \underset{\varphi_2}{\times} \mathbf{Z} \cong qAq$. Since $pAp \subset A$ and $qAq \subset A$ induce isomorphisms on $K$-theory, (ii) follows.

**(i) $\Rightarrow$ (iii).** The implication is an immediate consequence of Theorem 2.1 and Proposition 3.9. □

### Proof of Proposition 2.7.

The proof is an immediate consequence of Theorem 2.2 and Theorem 3.8 by the same observation that we stated in the proof of (i) $\Leftrightarrow$ (ii) of Theorem 2.6 above. $\square$

## 10. The Fundamental Extension Theorem.

Let us review some relevant results of [28]. (Cf. also Section 1, Theorem 1.16 and Theorem 1.17.) Let $(X, \varphi)$ be a Cantor system, and let $Y$ be a closed non-empty subset of $X$. Later, we will only be considering such triples $(X, \varphi, Y)$ satisfying:

(P1) $Y$ meets each $\varphi$-orbit at most once; i.e. $\varphi^k(Y) \cap Y = \emptyset$ for $k \neq 0$.

(P2) $Y = \{y_1, y_2, \ldots, y_\infty\}$, where $\{y_m\}_1^\infty$ is a sequence of distinct points in $X$ with limit point $y_\infty$.

As before we let $A_Y^\varphi$ denote the $C^*$-subalgebra of $C(X) \underset{\varphi}{\times} \mathbf{Z}$ generated by $C(X)$ and $u_\varphi \cdot C_0(X \setminus Y)$, cf. Section 1. We have a short exact sequence

$$\mathcal{E}_Y^\varphi : \ 0 \to C(Y, \mathbf{Z})/\mathbf{Z} \xrightarrow{\beta} K_0(A_Y^\varphi) \xrightarrow{i_*} K^0(X, \varphi) \to 0.$$

The map $i_*$ is induced by the natural inclusion of $A_Y^\varphi$ in $C(X) \underset{\varphi}{\times} \mathbf{Z}$. In the expression $C(Y, \mathbf{Z})/\mathbf{Z}$, the subgroup $\mathbf{Z}$ is understood to represent the constant functions in $C(Y, \mathbf{Z})$. The map $\beta$ is described as follows. For $f$ in $C(Y, \mathbf{Z})$, let $g : X \to \mathbf{Z}$ be any continuous extension of $f$. We regard $g - g \circ \varphi^{-1}$ as an element of $C(X, \mathbf{Z}) \cong K_0(C(X))$ and apply the map induced by the natural inclusion $C(X) \subset A_Y^\varphi$ to obtain $\beta(f)$ in $K_0(A_Y^\varphi)$. This is independent of the choice of $g$ and maps the constant functions to zero, as shown in [28]. Also in [28], it is shown that if $Y$ satisfies (P1) the map $i_*$ is strict positivity preserving, i.e. $i_*^{-1}(K^0(X, \varphi)^+ \setminus \{0\}) = K_0(A_Y^\varphi)^+ \setminus \{0\}$. Also note that if $Y$ satisfies (P2), we may identify $C(Y, \mathbf{Z})/\mathbf{Z}$ with $\bigoplus_1^\infty \mathbf{Z} = \mathbf{Z}^\infty$ — in fact, $C(Y, \mathbf{Z})/\mathbf{Z}$ is naturally isomorphic to $\{f : Y \to \mathbf{Z} | f \text{ continuous and } f(y_\infty) = 0\} \cong \mathbf{Z}^\infty$. Our aim in this section is the following "dynamic realization" or "dynamic resolution" theorem.

**Theorem 10.1.** Suppose $H$ and $G$ are acyclic simple dimension groups with distinguished order units. Also suppose we have a short exact sequence

$$0 \to \mathbf{Z}^\infty \to H \xrightarrow{\pi} G \to 0$$

where $\pi$ preserves order units and is strict positivity preserving. (I.e. if $h_0$ is the order unit for $H$ then $\pi(h_0)$ is the order unit for $G$ and $H^+ \setminus \{0\} = \pi^{-1}(G^+ \setminus \{0\})$.) Then there is a Cantor system $(X, \varphi)$ and a closed set $Y \subset X$ satisfying (P1) and (P2) so that we have a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \mathbf{Z}^\infty & \to & K_0(A_Y^\varphi) & \xrightarrow{i_*} & K^0(X,\varphi) & \to 0 \\ & \| & & \eta \downarrow & & \gamma \downarrow\cong & \\ 0 \to & \mathbf{Z}^\infty & \to & H & \xrightarrow{\pi} & G & \to 0 \end{array}$$

where $\gamma$ is an order isomorphism preserving distinguished order units and $\eta$ is an order isomorphism mapping the distinguished order unit of $K_0(A_Y^\varphi)$ to the distinguished order unit of $H$ modulo $Inf(H)$.

**Remark.** We may formulate the theorem in the following way: The given short exact sequence determines an element $\mathcal{E}$ in $Ext(G, \mathbf{Z}^\infty)$. There exist $(X, \varphi)$, $Y$ and $\gamma$ as described in the theorem so that the associated short exact sequence $\mathcal{E}_Y^\varphi$ in $Ext(K^0(X,\varphi), \mathbf{Z}^\infty)$ equals $\mathcal{E}$ after applying $\gamma$.
(In fact, this means that there exists a (group) isomorphism $\eta : K_0(A_Y^\varphi) \to H$ making the diagram of the theorem commute. It is easy to show that $\eta$ actually is an order isomorphism preserving the distinguished order units modulo infinitesimals.)

**Corollary.** Let $H$ and $G$ be as in the theorem. Suppose we have a short exact sequence

$$0 \to \mathbf{Z}^k \to H \xrightarrow{\pi} G \to 0; \quad k \in \mathbf{N},$$

where $\pi$ preserves order units and is strict positivity preserving. Then there is a Cantor system $(X, \varphi)$ and a finite subset $Y = \{y_1, \ldots, y_{k+1}\}$ consisting of $k+1$ distinct points of $X$ satisfying (P1) so that we have a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \mathbf{Z}^k & \to & K_0(A_Y^\varphi) & \xrightarrow{i_*} & K^0(X,\varphi) & \to 0 \\ & \| & & \eta \downarrow & & \gamma \downarrow\cong & \\ 0 \to & \mathbf{Z}^k & \to & H & \xrightarrow{\pi} & G & \to 0 \end{array}$$

where $\gamma$ and $\eta$ are as in the theorem.

**Proof.** The proof of the corollary proceeds exactly as the proof of the theorem except that we avoid some of the technical difficulties because $k$ is finite. Also, we identify $C(Y, \mathbf{Z}/\mathbf{Z})$ with $\{f : Y \to \mathbf{Z} | f$ continuous and $f(y_{k+1}) = 0\} \cong \mathbf{Z}^k$. □

The proof of the theorem will occupy the rest of the section and is divided into a series of lemmas. The first step is to begin with $(X, \varphi, Y)$ and compute (in a sense to be made precise shortly) the element denoted $\mathcal{E}_Y^\varphi$ of the group $Ext(K^0(X,\varphi), C(Y,\mathbf{Z})/\mathbf{Z})$ determined by the short exact sequence introduced above. We refer the reader to [14; Ch. IX] for standard facts on homological algebra.

Suppose we realize $(X, \varphi)$ as the Vershik transformation on the simple ordered Bratteli diagram $B = (V, E, \geq)$, denoting $X$, resp. $\varphi$, for $X_B$, resp. $\lambda_B$. (Cf. Theorem 3.6.) The Bratteli diagram provides us with a projective (free) resolution of $G = K_0(V, E) \cong K^0(X, \varphi)$ as follows.

**Lemma 10.2.** Let $(V, E)$ be a Bratteli diagram with associated dimension group $G$. Let $\mathbf{F}(V)$ denote the free abelian group on $V$ and define $\delta : \mathbf{F}(V) \to \mathbf{F}(V)$ by

$$\delta(v) = \sum_{\substack{e \in E \\ s(e) = v}} r(e) - v$$

for $v$ in $V$. Let $\nu$ denote the natural map of $\mathbf{F}(V)$ to $G$ (i.e. each vertex $v$ in $V_n$ determines an element of $\mathbf{F}(V_n)$ and thus in the inductive limit $G$). Then the following sequence is exact:

$$0 \to \mathbf{F}(V) \xrightarrow{\delta} \mathbf{F}(V) \xrightarrow{\nu} G \to 0.$$

**Proof.** The proof is standard in algebra and straightforward in any event so we omit it. □

As explained in [14; Ch. IX, Sect. 51], there is a natural isomorphism

$$Ext(K^0(X,\varphi), L) \cong Hom(\mathbf{F}(V), L)/\delta^* Hom(\mathbf{F}(V), L)$$

for any abelian group $L$. We wish to apply this for $L = C(Y, \mathbf{Z})/\mathbf{Z}$. Before doing this we will make some notational simplifications. We will always consider the situation when the unique max path of $X$, here denoted $y_0$, is in $Y$. (In fact, when we later restrict our attention to $Y$ which satisfies (P2), $y_0$ will be $y_\infty$. For the moment we proceed more generally). Then there is a natural isomorphism

$$C(Y,\mathbf{Z})/\mathbf{Z} \cong \{f : Y \to \mathbf{Z} | f \text{ continuous and } f(y_0) = 0\}$$

Furthermore, for any abelian group $L$, we may identify the group $Hom(\mathbf{F}(V), L)$ with the set $L^V = \{f : V \to L\}$, simply by restricting the group homomorphisms to the generating set. Combining these two together, we may identify the group $Hom(\mathbf{F}(V), C(Y,\mathbf{Z})/\mathbf{Z})$ with the set

$$\{f : V \times Y \to \mathbf{Z} | f \text{ continuous and } f|_{V\times\{y_0\}} = 0\}.$$

Of course the group operation on the latter set is simply pointwise addition of functions.

**Lemma 10.3.** Let $(X, \varphi)$ be the Bratteli-Vershik system associated with the simple ordered Bratteli diagram $(V, E, \geq)$. Let $Y$ be any closed set containing $y_0$, the unique max path of $X$. Under the isomorphism

$$Ext(K^0(X,\varphi), L) \cong Hom(\mathbf{F}(V), L)/\delta^* Hom(\mathbf{F}(V), L),$$

where $L = C(Y,\mathbf{Z})/\mathbf{Z}$, $\mathcal{E}_Y^\varphi$ is represented by $\overline{\mathcal{E}_Y^\varphi} : V \times Y \to \mathbf{Z}$ in $Hom(\mathbf{F}(V), L)$ defined by

$$\overline{\mathcal{E}_Y^\varphi}(v,y) = |\{e \in E | s(e) = v \text{ and } e \underset{\neq}{\geq} y(k+1)\}|$$

for all $v$ in $V_k$, $y$ in $Y$, where $y = (y(1), y(2), \ldots)$.

**Proof.** We will drop the $\varphi$ and $Y$ (for the proof) in $\overline{\in_Y^\varphi}$. Note that $\bar{\in}(v, y_0) = 0$ for all $v$ since $y_0(k)$ is maximal for all $k$.
Recall from [14; Ch. IX, Sect. 51] that the isomorphism we are considering is computed as follows. We begin with our free resolution of $K^0(X,\varphi)$ and our exact sequence representing $\mathcal{E}_Y^\varphi$.

$$\begin{array}{ccccccccc} 0 \to & \mathbf{F}(V) & \overset{\delta}{\to} & \mathbf{F}(V) & \to & K_0(V,E) & \to 0 \\ & & & & & \downarrow\cong & \\ 0 \to & L & \overset{\beta}{\to} & K_0(A_Y^\varphi) & \to & K^0(X,\varphi) & \to 0 \end{array}$$

We must construct a homomorphism $\mu : \mathbf{F}(V) \to K_0(A_Y^\varphi)$ making the right square commute. Then the homomorphism representing $\mathcal{E}_Y^\varphi$ is given by $\beta^{-1} \circ \mu \circ \delta$.

We define $\mu$ as follows. For $v$ in $V$, let $(e_1, \ldots e_k)$ be the unique path of minimal edges with $r(e_k) = v$. Let

$$E_v = U(e_1, \ldots, e_k) = \{x \in X | x(i) = e_i, 1 \leq i \leq k\}$$

and set $\mu(v) = [\chi_{E_v}]_0$ in $K_0(A_Y^\varphi)$. Note that there is a slight ambiguity in that we may regard $\chi_{E_v}$ as an element of $C(X) \subset A_Y^\varphi$ or as an element of $C(X, \mathbf{Z}) \cong K_0(C(X))$ and then apply the natural induced map to get to $K_0(A_Y^\varphi)$. These agree, of course, and $\mu$ is well-defined. It is easy to check that with this $\mu$ the right square of our diagram commutes.

To complete the proof it suffices to show $\beta \circ \bar{\mathcal{E}} = \mu \circ \delta$. These are, of course, group homomorphisms from $\mathbf{F}(V)$ to $K_0(A_Y^\varphi)$. However, in the definitions of both (essentially of $\mu$ and $\beta$) one actually obtains elements of $C(X, \mathbf{Z}) \cong K_0(C(X))$. These are subject to choices made and only when we apply the induced map from $C(X) \subset A_Y^\varphi$ do we get well-defined homorphisms. However, we will make careful choices so that the elements produced in $C(X, \mathbf{Z})$ will actually agree there and we will be done.

First consider $v$ in $V_k$ and compute

$$(*) \qquad \begin{cases} \mu \circ \delta(v) & = \sum\limits_{s(e)=v} \mu(r(e)) - \mu(v) \\ & = \sum\limits_{s(e)=v} \chi_{E_{r(e)}} - \chi_{E_v}. \end{cases}$$

As for $\beta \circ \bar{\mathcal{E}}(v)$, define $f : X \to \mathbf{Z}$ by

$$f(x) = |\{e \in E | s(e) = v, e \underset{\neq}{>} x(k+1)\}|.$$

Clearly $f$ is continuous — in fact constant on cylinder sets $U(e_1, \ldots, e_{k+1})$ — and for $y \in Y$, $f(y) = \bar{\mathcal{E}}(v, y)$. So let $g = f - f \circ \varphi^{-1} \in C(X, \mathbf{Z})$. The image of $g$ in $K_0(A_Y^\varphi)$ is $\beta \circ \bar{\mathcal{E}}(v)$, but we will show $g$ equals the function of $(*)$ and we will be done. For any $(e_1, \ldots, e_{k+1}) \in P_{0,k+1} = E_1 \circ \cdots \circ E_{k+1}$, where not all the $e_i$ are in $E_{\min}$, let $(e'_1, \ldots, e'_{k+1})$ be its predecessor in $P_{0,k+1}$. So we have

$$g|_{\mathrm{U}(e_1,\ldots,e_{k+1})} = f|_{\mathrm{U}(e_1,\ldots,e_{k+1})} - f|_{\mathrm{U}(e'_1,\ldots,e'_{k+1})}.$$

Now, $e'_{k+1} = e_{k+1}$ (and so $g = 0$) unless $e_1, \ldots, e_k$ are all in $E_{\min}$, in which case $e'_{k+1}$ is the predecessor of $e_{k+1}$. Still we will have $g = 0$ unless, in addition, $s(e_{k+1}) = v$ and then $g|_{U(e_1,\ldots,e_{k+1})} = -1$. Now we consider the case $e_1, \ldots e_{k+1}$ are all in $E_{\min}$. In this case, if $x$ is in $U(e_1, \ldots, e_{k+1})$ then $\varphi^{-1}(x)(k+1)$ is a maximal edge and so $f(\varphi^{-1}(x)) = 0$. So we have

$$\begin{aligned} g|_{U(e_1,\ldots,e_{k+1})} &= f|_{U(e_1,\ldots,e_{k+1})} - f \circ \varphi^{-1}|_{U(e_1,\ldots,e_{k+1})} \\ &= f|_{U(e_1,\ldots,e_{k+1})} - 0 \end{aligned}$$

$$= \begin{cases} |\{e \in E | s(e) = v, r(e) = r(e_{k+1})\}| & \text{if } s(e_{k+1} \neq v \\ |\{e \in E | s(e) = v, r(e) = r(e_{k+1})\}| - 1 & \text{if } s(e_{k+1}) = v\,. \end{cases}$$

by direct computation and the fact $e_{k+1}$ is minimal. Combining these cases, we have

$$\begin{aligned} g \;=\; & \sum_{\substack{e_1,\ldots,e_k \in E_{\min} \\ s(e_{k+1})=v}} (-\chi_{U(e_1,\ldots,e_{k+1})}) \\ + & \sum_{e_1,\ldots,e_{k+1} \in E_{\min}} |\{e \in E | s(e) = v,\ r(e) = r(e_{k+1})\}| \cdot \chi_{U(e_1,\ldots,e_{k+1})} \end{aligned}$$

In the first summation, the condition $s(e_{k+1}) = v$ is equivalent to $r(e_k) = v$. Then the path $(e_1, \ldots, e_k)$ is the unique path in $E_{\min}$ to $v$ and the first sum becomes $-\chi_{E_v}$.

As for the second sum, there is a unique $(e_1, \ldots, e_{k+1})$ in $E_{\min}$ for each vertex in $V_{k+1}$, and $U(e_1, \ldots, e_{k+1}) = E_{r(e_{k+1})}$. So this sum can be replaced by

$$\begin{aligned} & \sum_{v' \in V_{n+1}} |\{e \in E | s(e) = v,\ r(e) = v'\}| \cdot \chi_{E_{v'}} \\ &= \sum_{v' \in V_{n+1}} \sum_{\substack{e \\ s(e)=v \\ r(e)=v'}} \chi_{E_{v'}} = \sum_{\substack{e \\ s(e)=v}} \chi_{E_{r(e)}} \end{aligned}$$

and so we see $g$ agrees with the function of $(*)$ and we are done. □

Now that we have identified $\mathcal{E}_Y^\varphi$ in terms of the Bratteli diagram, we can try to construct $(X, \varphi, Y)$ which give a specified extension.

From now on, we will restrict to $Y$ satisfying (P2) and where $y_\infty$ is the unique max path of the Bratteli-Vershik model for $(X, \varphi)$. In this case, we can identify

$$\{f : V \times Y \to \mathbf{Z} | f \text{ continuous }, f(v, y_\infty) = 0 \text{ forall } v\}$$

with the group

$$\{f : V \times \mathbf{N} \to \mathbf{Z} \mid \text{for each } v \text{ in } V,\ f(v,i) = 0 \text{ for all but finitely many } i\}$$

by setting $f(v,i) = f(v, y_i)$.
Furthermore, it will be most convenient from now on to restrict our vertex sets $V$ to ones with $|V_n| \geq 3$ for all $n \geq 1$. Moreover, we will assume the elements of each $V_n$ are labelled so

$$V_n = \{v_n^+, v_n^-, v_n^1, \ldots, v_n^l\}$$

for some $l \geq 1$. Finally, we will often require our Bratteli diagrams to satisfy condition

(A0) for every $v$ in $V_n$ and $v'$ in $V_{n+1}$, $n \geq 0$, there are at least four edges from $v$ to $v'$.

Note that if $(\tilde{V}, \tilde{E})$ is a simple Bratteli diagram, with $K_0(\tilde{V}, \tilde{E}) \not\cong \mathbf{Z}$, we may by a succession of telescoping, microscoping (symbol splittings) and telescoping (in that order) of $(\tilde{V}, \tilde{E})$ obtain a Bratteli diagram $(V, E)$ that satisfies both $|V_n| \geq 3$, for all $n \geq 1$, and condition (A0). We may also speak of condition (A0) at level $n$.

The next result gives conditions on a Bratteli diagram and a map $\rho : V \times \mathbf{N} \to \mathbf{Z}$ in $Hom(\mathbf{F}(V), \mathbf{Z}^\infty)$ (representing some extension) so that we may find a simple order on $(V, E)$ and a set $Y$ satisfying (P1) and (P2) so that $\overline{\mathcal{E}_Y^\varphi} = \rho$ and, in particular, the given extension can be realized by $(X, \varphi, Y)$, where $\varphi$ is the associated Vershik map.

**Lemma 10.4.** Let $(V, E)$ be a Bratteli diagram with associated dimension group $G$. Let $\rho : V \times \mathbf{N} \to \mathbf{Z}$ be an element of $Hom(\mathbf{F}(V), \mathbf{Z}^\infty)$. Suppose that we have an increasing sequence of integers $\{r_n\}_{n=0}^\infty$ with $r_0 = 0$ so that the following hold.

(A0) as above,

(A1) for all $n$, $|V_n| \geq r_{n-1} + 2$,

(A2) for all $n$, $v$ in $V_n$ and $i > r_n$, $\rho(v,i) = 0$

(A3) for all $n$, $v$ in $V_n$ and $1 \leq i \leq r_n$, $\delta(v, v_n^+) \leq \rho(v,i) \leq N(v, v_{n+1}^i) - 1$, where $\delta(\cdot,\cdot)$ represents the Kronecker delta and $N(v,v')$ represents the number of edges from $v$ to $v'$.

Then there is a simple order on $(V, E)$ and a set $Y$ satisfying (P1) and (P2) so that, if we let $\varphi$ denote the associated Vershik transformation, we have $\rho = \overline{\mathcal{E}_Y^{\varphi}}$.

**Proof.** In the course of the proof we will make repeated use of condition (A0) without always mentioning it explicitly. Let $(e_1^+, e_2^+, \ldots)$ be any element of $X$ so that $r(e_n^+) = v_n^+$ for all $n$. Similarly choose $(e_1^-, e_2^-, \ldots)$ so that $r(e_n^-) = v_n^-$ for all $n$. These will ultimately be our unique max and min paths, respectively. We next define $y_i = (e_1^i, e_2^i, \ldots)$ as follows. Let $n$ be the largest integer so that $r_n < i$. Set $e_1^i = e_1^+, e_2^i = e_2^+, \ldots, e_{n-1}^i = e_{n-1}^+$. Choose any edge, $e_n^i$, from $v_{n-1}^+$ to $v_n^i$. Then for $k > n$ let $e_k^i$ be any edge from $v_n^i$ to $v_{n+1}^i$ (using the fact that $r_n$ is increasing). Immediately one sees that the sequence $\{y_i\}_1^\infty$ converges to $y_\infty = (e_1^+, e_2^+, \ldots)$. Also since $y_i$ and $y_j$ pass through different vertices at all levels $V_n$ with $r_n > i, j$ (and $i \neq j$) $y_i$ and $y_j$ are not cofinal. They are also not cofinal with either $(e_1^+, e_2^+, \cdots)$ or $(e_1^-, e_2^-, \cdots)$ and so $Y$ satisfies (P1) and (P2).
We now turn to the problem of choosing an order on $r^{-1}\{v\}$ for some $v$ in $V_{n+1}$. If $v$ is $v_{n+1}^+$, we must choose the order so that $e_{n+1}^+$ is maximal and so that the minimal edge has source $v_n^-$. Otherwise it is arbitrary. Similar restrictions apply to $v = v_{n+1}^-$. For any $v = v_{n+1}^i$ with $i > r_n$ (if any), the order is arbitrary so long as the maximal and minimal edges have source $v_n^+$ and $v_n^-$, respectively. (Since no $y_i$ passes through such a vertex the order is irrelevant to the computation of $\overline{\mathcal{E}_Y^{\varphi}}$). As for $v = v_{n+1}^i$, $1 \leq i \leq r_n$, we proceed as follows. Begin by choosing some edge other than $e_{n+1}^i$ from $v_n^+$ to $v$ for the maximal edge (condition (A0) is needed if $s(e_{n+1}^i) = v_n^+$) and some edge from $v_n^-$ to $v$ for the minimal edge. For each $w$ in $V_n$, the value $\rho(w,i)$ determines the number of the $N(w,v)$ edges which should be greater than $e_{n+1}^i$. If $w = v_n^+$ this must be at least one to allow for the maximal edge, if $w = v_n^-$ it must be strictly less to allow for the minimal edge and if $w = s(e_{n+1}^i)$ — either $v_n^+$ or $v_n^i$ — it must be strictly less to allow for $e_{n+1}^i$ itself. Condition (A3) asserts that these conditions can be fulfilled. This completes the construction and in view of Lemma 10.3 and the last

paragraph the desired properties follow at once. □

Now it remains to show that for any simple dimension group $G$ and $\mathcal{E}$ in $Ext(G, \mathbf{Z}^\infty)$ we can find a Bratteli diagram for $G$ and a representative for $\mathcal{E}$ satisfying (A0) - (A3). Let us adopt the notation (A3') for the condition in (A3) only involving the first inequality and (A3") for the second only. We can certainly choose a diagram satisfying (A0) as noted earlier and this will provide us with a representative for $\mathcal{E}$. We then have two things to manipulate. We can use "Bratteli moves" (i.e. telescoping and microscoping) to change the diagram and we can adjust the representative by elements of $\delta^* Hom(\mathbf{F}(V), \mathbf{Z}^\infty)$ without changing $\mathcal{E}$. The next result is that by the latter only we can obtain the sequence $\{r_n\}_{n=0}^\infty$ and (A2) and (A3').
Before stating the next result, let us make the following simple observations. For $\rho : V \times \mathbf{N} \to \mathbf{Z}$ we have

$$(\delta^*\rho)(v,k) = \sum_{\substack{e \\ r(e)=v}} \rho(s(e),k) - \rho(v,k).$$

In particular, if $\rho$ is supported on $V_n \times \mathbf{N}$ for some $n$ — we will simply say $\rho$ is supported on $V_n$ — then $\delta^*\rho$ is supported on $V_{n-1} \cup V_n$ and $\delta^*\rho|_{V_n} = -\rho|_{V_n}$.

**Lemma 10.5.** Let $(V, E)$ be a Bratteli diagram satisfying (A0). Let $\rho : V \times \mathbf{N} \to \mathbf{Z}$ be an element of $Hom(\mathbf{F}(V), \mathbf{Z}^\infty)$. Then there is an increasing sequence $\{r_n\}_0^\infty$, $r_0 = 0$, and $\sigma : V \times \mathbf{N} \to \mathbf{Z}$ in $Hom(\mathbf{F}(V), \mathbf{Z}^\infty)$ so that $\rho + \delta^*\sigma$ satisfies (A2) and (A3').

**Proof.** First set $\sigma_0 = \rho|_{V_0 \times \mathbf{N}} : V_0 \times \mathbf{N} \to \mathbf{Z}$. Set $r_0 = 0$ and $\sigma_1 = 0$ and having $r_0, \ldots r_{n-1}$ and $\sigma_1, \ldots, \sigma_n$, where $\sigma_k : V_k \times \mathbf{N} \to \mathbf{Z}$, we inductively define $r_n$, $\sigma_{n+1} : V_{n+1} \times \mathbf{N} \to \mathbf{Z}$ as follows. Consider $\rho_n = \rho|_{V_n \times \mathbf{N}} - \sigma_n$. Choose $r_n$ sufficiently large so $r_n > r_{n-1}$ and so that $\rho_n(v,i) = 0$ for all $i > r_n$ and $v$ in $V_n$. Next choose $a$ so that $1 + |\rho_n(v,i)| \leq a$ for all $v$ in $V_n$ and all $i$. Define

$$\sigma_{n+1}(v,i) = \begin{cases} a & \text{if } v = v_{n+1}^+, \quad i = 1, \ldots, r_n \\ 0 & \text{otherwise} \end{cases}$$

(There is nothing special about $v_{n+1}^+$, any vertex in $V_{n+1}$ will do.) We define $\sigma$ on $V \times \mathbf{N}$ by $\sigma|_{V_n \times \mathbf{N}} = \sigma_n$. It is routine to verify the conclusion (using property (A0)) and we omit the details. □

The final step (getting (A1) and (A3")) requires making alterations to the Bratteli diagram as well. The next result deals with how these moves affect the representative of the extension. For $v$ in $V_n$ and $v'$ in $V_m$ with $m > n$, we let $p(v, v')$ denote the number of paths from $v$ to $v'$. It will be convenient to extend the definition to $n = m$ by setting $p(v, v') = \delta(v, v')$.

**Lemma 10.6.** Let $(V, E)$ be a Bratteli diagram for the dimension group $G$ and let $(\tilde{V}, \tilde{E})$ denote its telescoping to the sequence $\{n_m\}_{m=0}^{\infty}$, $n_0 = 0$. Let $\rho : V \times \mathbf{N} \to \mathbf{Z}$ be an element of $Hom(\mathbf{F}(V), \mathbf{Z}^{\infty})$ and define $\tilde{\rho} : \tilde{V} \times \mathbf{N} \to \mathbf{Z}$ as follows. For $v$ in $\tilde{V}_m = V_{n_m}$ and $i$ in $\mathbf{N}$,

$$\tilde{\rho}(v, i) = \sum p(v, v')\rho(v', i)$$

where the sum is over $v'$ in $\cup_{k=n_m}^{n_{m+1}-1} V_k$. Then under the natural isomorphisms

$$\begin{aligned} Hom(\mathbf{F}(V), \mathbf{Z}^{\infty})/\delta^* Hom(\mathbf{F}(V), \mathbf{Z}^{\infty}) &\cong Ext(G, \mathbf{Z}^{\infty}) \\ &\cong Hom(\mathbf{F}(\tilde{V}), \mathbf{Z}^{\infty})/\tilde{\delta}^* Hom(\mathbf{F}(\tilde{V}), \mathbf{Z}^{\infty}) \end{aligned}$$

the class of $\rho$ is mapped to the class of $\tilde{\rho}$.

**Proof.** The desired isomorphism is constructed by finding homomorphisms $\alpha$ and $\beta$ as shown.

$$\begin{array}{ccccccc} 0 \to & \mathbf{F}(\tilde{V}) & \xrightarrow{\tilde{\delta}} & \mathbf{F}(\tilde{V}) & \to & G & \to 0 \\ & \downarrow \alpha & & \downarrow \beta & & \| & \\ 0 \to & \mathbf{F}(V) & \xrightarrow{\delta} & \mathbf{F}(V) & \to & G & \to 0 \end{array}$$

making the diagram commute. The map $\beta$ is induced by the inclusion $\tilde{V} \subset V$. For $v$ in $V_{n_m}$, $\alpha$ is given by $\alpha(v) = \sum_{v'} p(v, v')v'$, where the sum is over $\cup_{n_m}^{n_{m+1}-1} V_k$. The map $\alpha^* : Hom(\mathbf{F}(V), \mathbf{Z}^{\infty}) \to Hom(\mathbf{F}(\tilde{V}), \mathbf{Z}^{\infty})$ induces the desired isomorphism and we are done after observing $\alpha^* \rho = \tilde{\rho}$.
□

**Lemma 10.7.** Given a simple dimension group $G$ and an element $\mathcal{E}$ in $Ext(G, \mathbf{Z}^{\infty})$, there is a Bratteli diagram $(W, F)$ for $G$, a $\sigma : W \times \mathbf{N} \to \mathbf{Z}$ representing $\mathcal{E}$ and a sequence $\{s_m\}_{m=0}^{\infty}$ satisfying (A0) - (A3).

**Proof.** Begin by choosing any Bratteli diagram (V,E) for $G$, which we may assume satisfies (A0). Also choose a representative $\rho : V \times \mathbf{N} \to \mathbf{Z}$

representing $\mathcal{E}$. In view of Lemma 10.5, we may assume we also have a sequence $\{r_n\}_0^\infty$ satisfying (A2) and (A3').
For $m = 0, 1, 2, \ldots$, we will inductively define the following items

(a) $n_{m+1} \in \mathbf{N}$

(b) a finite vertex set $W_m$

(c) positive group homomorphisms

$$\alpha_m : \mathbf{F}(V_{n_m}) \to \mathbf{F}(W_m)$$

$$\beta_m : \mathbf{F}(W_m) \to \mathbf{F}(V_{n_{m+1}})$$

(d) $\tilde{\rho}_m : V_{n_m} \times \mathbf{N} \to \mathbf{Z}^\infty$

(e) $s_m \in \mathbf{N}$.

We begin with $n_0 = 0,\ n_1 = 1,\ W_0 = V_0,\ \alpha_0 = id,\ \beta_0 : \mathbf{F}(W_0) = \mathbf{F}(V_0) \to \mathbf{F}(V_1)$ as defined by $E_1$ (i.e. by the incidence matrix corresponding to $E_1$), $\tilde{\rho}_0 = 0$ and $s_0 = 0$. Assuming these have all been defined up to value $m-1$, we proceed as follows for step $m$. Let $s = s_{m-1}$ and let

$$R = \max\{\tilde{\rho}_{m-1}(v, i) | v \in V_{n_{m-1}},\ i \in \mathbf{N}\} + 1.$$

Since the group $G$ is simple, we may find $n > n_m$ so that there are at least $Rs + 1$ paths from $v^1_{n_m}$ to $v^1_n$. Let $n_{m+1} = n + 1$. Recall that, for some $l, V_{n_m} = \{v^+_{n_m}, v^-_{n_m}, v^1_{n_m}, \ldots, v^l_{n_m}\}$. We define $W_m = \{w^+_m, w^-_m, w^1_m, \ldots, w^{l+s}_m\}$ and note immediately that

$$|W_m| = l + s + 2 \geq s + 2 = s_{m+1} + 2$$

so we will have (A1) eventually. It is easiest to describe the homomorphisms $\alpha_m$ and $\beta_m$ combinatorily (i.e. via edges). The map $\alpha_m$ is as shown in Figure 9 (dropping subscripts)

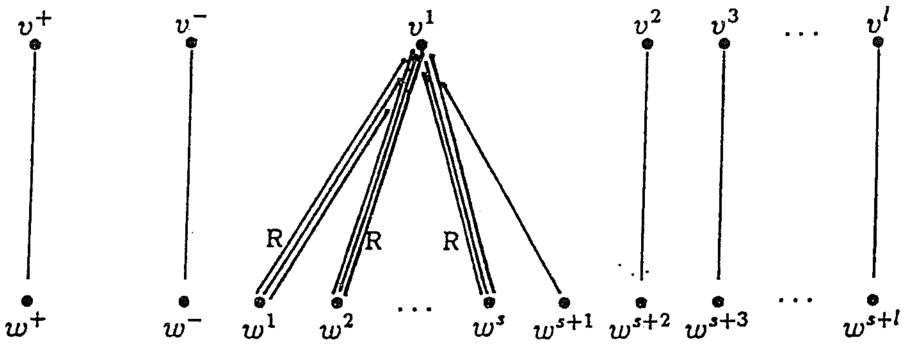



Figure 9.

where the number of edges from $v^1$ to each $w^i$, $1 \leq i \leq s$, is $R$. To describe $\beta_m$, we proceed as follows. We describe a map from $\mathbf{F}(W_m)$ to $\mathbf{F}(V_n)$ so that its composition with $\alpha_m$ gives the usual map of $\mathbf{F}(V_{n_m})$ in $\mathbf{F}(V_n)$ provided by the diagram $(V, E)$. Unfortunalety, this map we are about to give may not have enough edges so we compose it with the usual map $\mathbf{F}(V_n) \to \mathbf{F}(V_{n_{m+1}})$ (recall $n_{m+1} = n + 1$) to get $\beta_m$. As for the map to $\mathbf{F}(V_n)$, if we identify $\mathbf{F}(W_m \setminus \{w_m^1\})$ with $\mathbf{F}(V_{n_m} \setminus \{v_{n_m}^1\})$ via $\alpha_m^{-1}$, the map is just the canonical map $\mathbf{F}(V_{n_m})$ to $\mathbf{F}(V_n)$. We connect one edge from each $w_m^1, \ldots, w_m^s$ to $v_n^1$ and $p(v_{n_m}^1, v_n^1) - Rs$ edges from $w_m^{s+1}$ to $v_n^1$. There are no other edges emanating from any of the $w_m^1, \ldots, w_m^s$, and from $w_m^{s+1}$ to any $v'$ in $V_n$ there are $p(v_{n_m}^1, v')$ edges.

Next we determine $\tilde{\rho}_m$ by the formula

$$\tilde{\rho}_m(v, i) = \sum_{v'} p(v, v')\rho(v', i)$$

where the sum is over $v'$ in $\cup_{k=n_m}^{n_{m+1}-1} V_k$. Finally we set $s_m = r_{n_{m+1}-1}$.

Now, what do we do with this garbage? Telescope the diagram $(V, E)$ to the sequence $\{n_m\}_0^\infty$ to obtain $(\tilde{V}, \tilde{E})$. As defined, $\beta_m \circ \alpha_m$ agrees with the map given by $(V, E)$ so by Lemma 10.6, $\tilde{\rho}$ represents $\mathcal{E}$ in Ext $(G, \mathbf{Z}^\infty)$. We claim that $(\tilde{V}, \tilde{E})$ and $\tilde{\rho}$ satisfy (A2) and (A3') for the sequence $\{s_m\}_0^\infty$. This is clear except when considering $v = v_{n_m}^+$ in $\tilde{V}_m$. in this case, note that there is at least one path from $v$ to $v' = v_{n_{m+1}-1}^+$ and $\rho(v', i) \geq 1$ for $i = 1, \ldots, r_{n_{m+1}-1} = s_m$ and the claim follows easily.

Now we define $\sigma : W \times \mathrm{N} \to \mathbf{Z}$ as follows. On $W_m = \{w_m^+, w_m^-, w_m^1, \ldots, w_m^{s+l}\}$

set

$$\begin{aligned} &\sigma(w_m^i, i) = 0 \quad \text{for } 1 \le j \le s, \\ &\sigma(w_m^+, i) = \tilde{\rho}_m(v_{n_m}^+, i) \\ &\sigma(w_m^-, i) = \tilde{\rho}_m(v_{n_m}^-, i) \\ \text{and} \quad &\sigma(w_m^{j+s}, i) = \tilde{\rho}_m(v_{n_m}^j, i); \quad i \le j \le l \text{ and for all } i \in \mathbf{N}. \end{aligned}$$

Consider the diagram of the combined sequence

$$\mathbf{F}(V_0) = \mathbf{F}(W_0) \to \mathbf{F}(\tilde{V}_1) \to \mathbf{F}(W_1) \to \mathbf{F}(\tilde{V}_2) \to$$

with $\bar{\sigma}$ to be zero on $\tilde{V}$ and equal to $\sigma$ on $W$. If we telescope to the odd levels we get $(\tilde{V}, \tilde{E})$. Moreover, applying the formula of Lemma 10.6 and the definition of $\alpha_m$, $\bar{\sigma}$ becomes $\tilde{\rho}$. On the other hand, telescoping to the even levels yields $(W, F)$ and $\sigma$. (The edges for $F$ are described by the homomorphisms $\alpha_{m+1} \circ \beta_m$.) Thus $(W, F)$ is a Bratteli diagram for $G$ and $\sigma$ represents the desired element $\mathcal{E}$.

Recall that $\beta_m$ began by mapping $\mathbf{F}(W_m)$ into some $\mathbf{F}(V_n)$ and followed this by the map $\mathbf{F}(V_n)$ to $\mathbf{F}(V_{n+1})$ given by $E_{n+1}$. This second map has the "2-edge between all pairs of vertices property" and so $\beta_m$ does also. It follows that $(W, F)$ satisfies (A0). Property (A1) was noted earlier. Properties (A2) and (A3') follows easily from the definitions. Finally, for $(A3'')$, we consider $w$ in $W_m$, $1 \le i \le s_m$. Recall the map $\mathbf{F}(W_m)$ to $\mathbf{F}(W_{m+1})$ factors through $\mathbf{F}(\tilde{V}_{m+1}) = \mathbf{F}(V_{n_{m+1}})$. As noted above, there is at least 2 edges between each vertex in $W_m$ and each vertex in $\tilde{V}_{m+1}$. Also recall that from $v^1_{n_{m+1}}$ to $w^i_{m+1}$ there were exactly $R$ edges (definition of $\alpha_{m+1}$), where

$$R = \max\{\tilde{\rho}_m(v, i') | v \in V_{n_m},\ i' \in \mathbf{N}\} + 1$$

— note we have shifted from $m$ to $m+1$ from the original definition. This means that $\alpha_{m+1} \circ \beta_m$ has at least $2R$ edges from $w$ to $w^i_{m+1}$. Also recall that $\tilde{\rho}_m$ was used to define $\sigma$ on $W_m$ and so

$$R = \max\{\sigma(w'', i') | w'' \in W_m, i' \in \mathbf{N}\} \ge \sigma(m, i) + 1$$

and then finally

$$N(w, w^i_{m+1}) \ge 2R \mathrel{\underset{\neq}{>}} \sigma(m, i) + 1$$

and (A3) is satisfied. □

**Proof of Theorem 10.1.** The short exact sequence determines an element $\mathcal{E}$ in $Ext(G, \mathbf{Z}^\infty)$. By Lemma 10.7, Lemma 10.4 and Lemma 10.3 there is $(X, \varphi)$ and $Y \subset X$, satisfying the properties listed in the theorem, so that the associated element $\mathcal{E}_Y^\varphi$ in $Ext(K^0(X,\varphi), \mathbf{Z}^\infty)$ equals $\mathcal{E}$ after applying the isomorphism $\gamma : K^0(X,\varphi) \to G$. (By construction $\gamma$ is an order isomorphism preserving distinguished order units.) By the standard results on Ext [14; Ch. IX], this means we can find the isomorphism $\eta : K_0(A_Y^\varphi) \to H$ as claimed. To see that $\eta$ is an order isomorphism preserving the distinguished order units modulo infinitesimals, it suffices to note that $\pi$ is strict positivity preserving by hypothesis, that $i_*$ is strict positivity preserving by Theorem 1.17 (Corollary 3) and that $\gamma$ is an order isomorphism — each of them preserving distinguished order units. (Cf. also Lemma 5.3.) □

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