ON SOME LIMITING PERFORMANCE ISSUES OF MULTIUSER RECEIVERS IN FADING CHANNELS

by

DEJAN V. DJONIN
M.A.Sc, University of Belgrade, 1999

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of

DOCTOR OF PHILOSOPHY

in the Department of Electrical and Computer Engineering

We accept this dissertation as conforming to the required standard

---

Dr. V. K. Bhargava, Supervisor, Dept. of Elect. & Comp. Eng.

Dr. W.-S. Lu, Member, Dept. of Elect. & Comp. Eng.

Dr. T. A. Gulliver, Member, Dept. of Elect. & Comp. Eng.

Dr. S. Dost, Outside Member, Dept. of Mech. Eng.

Dr. I. F. Blake, External Examiner
University of Toronto

Supervisor: Dr. V. K. Bhargava

ABSTRACT

The problem of information-theoretic optimal resource allocation for the synchronous single-cell CDMA Gaussian multiple access channel is investigated. Several different cases are analyzed including: optimal sequence allocation without power control, optimal sequence allocation with optimal power control and optimal sequence allocation without power control with equal single user capacities. In order to simplify the mathematical description of the multiple access capacity region, a Cholesky decomposition characterization is introduced and utilized to find the optimal sequence allocation for equal single user capacities.

The case of randomly chosen spreading sequences in a large system model, i.e. when number of users and processing gain increase without bounds while maintaining their ratio fixed, is also analyzed. Using this model, the performance of a conventional decision feedback receiver in flat fading channels is analyzed.

A sequence allocation scheme that uses two sets of orthogonal users that can be decoded with a very simple decision feedback receiver is analyzed. It is shown that the spectral efficiency of this scheme is very close to the maximal possible.

Finally, the issue of imperfect channel state information available at the receiver is discussed and the spectral efficiency loss compared to the perfect channel state information case is evaluated for the optimal multiuser receiver.

Examiners:

Dr. V. K. Bhargava, Supervisor, Dept. of Elect. & Comp. Eng.

Dr. W.-S. Lu, Member, Dept. of Elect. & Comp. Eng.

Dr. T. A. Gulliver, Member, Dept. of Elect. & Comp. Eng.

Dr. S. Dost, Outside Member, Dept. of Mech. Eng.

Dr. I. F. Blake, External Examiner
University of Toronto

Table of Contents

List of Figures

List of Tables

Notation

$[.]$	delimiter that denotes a vector or a matrix
$\lfloor x \rfloor$	floor function
$\mathbf{A}_S$	submatrix formed from $\mathbf{A}$ by retaining vectors and columns from set S
$h(x)$	unit step function
$\mathbf{x}_\downarrow$	non-increasing rearrangement of the vector $\mathbf{x}$
c_i	Cholesky factors of a matrix
$x_{[i]}$	i-th largest element of the vector $\mathbf{x}$
$\mathcal{R}$	set of real numbers
$\prec$ and $\succ$	majorization operations among vectors
$b_k(i)$	i-th information data bit of user k
$d_k(i)$	i-th code symbol of user k
$\mathbf{d}$	vector of code symbols of K users
N	processing gain of the system
K	number of users in a multiple access system
α	system load
P_k	average transmit power of user k
P_{tot}	total user transmit power
$p(g)$	power control law
$\mathbf{P}$	diagonal matrix of average user transmit powers
$\mathbf{W}$	diagonal matrix of received user powers
$\bar{P}_k$	constraint on an average power of user k
$\bar{\mathbf{P}}$	diagonal matrix of constraints on average user transmit powers
$\mathcal{P}$	set of allowable power control laws under the constraints $\bar{\mathbf{P}}$
σ^2	noise variance
$g_k(i)$	fading coefficient of the user k at symbol interval i
$\bar{g}_k(i)$	estimated fading coefficient of the user k at symbol interval i
$\tilde{g}_k(i)$	estimation error of the fading coefficient of the user k at symbol interval i

$\mathbf{G}$	diagonal matrix of user fading gains
$\mathcal{G}$	set of all possible fading states
$f(g)$	probability density function of fading gains
$F(g)$	cumulative density function of fading gains
$F^{-1}(u)$	quantile of the distribution $f(g)$
$\mathcal{C}(\mathbf{P},\mathbf{G})$	capacity region of synchronous Gaussian CDMA channel
$\mathcal{C}_e(\mathbf{P})$	ergodic capacity region of synchronous Gaussian CDMA channel
$\mathcal{C}_d(\mathbf{P})$	delay-limited capacity region of synchronous Gaussian CDMA channel
Γ	spectral efficiency of a communication system
$I(\mathbf{X},\mathbf{Y})$	mutual information between input $\mathbf{X}$ and output $\mathbf{Y}$
$\mathcal{U}$	set of user indices
C_{sum}	sum capacity of a multiple-access channel
C_{sum}^{PC}	sum capacity of a multiple-access channel with power control
R_k	code rate of k-th user
$C_{opt}(\alpha,\mathrm{SNR})$	spectral efficiency of the optimal multi-user detector
$\bar{C}_{CDFR}(\alpha,\mathrm{SNR})$	average user capacity of the CDFR
C_{sym}	symmetric capacity of the multiple access region
$s_k(t)$	spreading waveform of user k
η	multiuser efficiency
s_{ij}	j-th chip of the k-th user spreading sequence
R_{ij}	crosscorrelation between sequences of user i and j
$\mathbf{R}$	$K\times K$ correlation matrix
$\mathbf{S}$	$N\times K$ dimensional matrix of spreading sequences
$\phi_i(t)$	i-th orthonormal basis vector
T	symbol interval length
$n(t)$	random noise process
L	number of code symbols in a code word
τ_k	channel time delay of user k
C_{GMAC}	capacity of a Gaussian Multiple Access Channel
κ	number of oversized users
$\mathrm{Ei}(n,x)$	exponential integral function
AWGN	Additive White Gaussian Noise

CDMA	Code Division Multiple Access
TDMA	Time Division Multiple Access
DS/CDMA	Direct Sequence Code Division Multiple Access
GMAC	Gaussian Multiple Access Channel
SNR	Signal to Noise Ratio
SIR	Signal to Interference Ratio
MMSE	Minimum Mean Square Error
MIMO	Multiple Input Multiple Output
CSI	Channel State Information
ICSI	Imperfect Channel State Information
WBE	generalized Welch Bound Equality sequences
QoS	Quality of Service
CDFR	Conventional Decision Feedback Receiver
TSOS	Two Sets Orthogonal Sequences
SAGE	Space Altering Generalized Algorithm

Acknowledgement

I would like to express my gratitude and appreciation to Professor Vijay K. Bhargava for his wise supervision, support and understanding.

I am very grateful to Professor Wu-Sheng Lu, Professor Aaron Gulliver and Professor Sadik Dost for serving on my committee. Special thanks go to Professor Ian Blake for agreeing to be the external examiner at my Ph.D. oral examination.

I am very grateful to Professor Tommy Guess who made me aware of the statement and the proof of the Theorem 1 of Chapter 2.

I would like to extend my sincere thanks to all of my colleagues from the communication lab at UVic for their friendship and cooperation.

Finally, I would like to thank my wife Daniela for her continuous support and helpful comments during the preparation of this thesis.

Chapter 1

Introduction

The material in this thesis traces its roots to the probably most referenced article in the history of communication systems, C.E.Shannon's famous work [1] which created the field of information theory. By using the probabilistic approach, this insightful article derived limit (also called channel capacity) on the attainable data rate with negligible probability of error for communication over a noisy channel. This paper proved only the existence of a communication scheme that can attain this limiting rate and left plenty of space for researchers to find practical schemes that can come close to the promised limits. Up to this date, numerous practical schemes were derived that can approach channel capacity. Probably the most prominent of these schemes are *turbo codes*, originally proposed in [2], that can approach the channel capacity within a fraction of dB while having practically feasible complexity. The unavoidable drawback of turbo coding schemes is its long decoding delay.

The concept of channel capacity can be extended to find limiting data rates supportable by a multiuser cellular wireless communication system. However, analysis of such a system is considerably more involved than the analysis of a single transmitter-receiver pair in a noisy channel discussed in the original article. There are several factors that make this analysis more involved. First, there are multiple users that share the common channel resource and supportable data rates are defined by a capacity region rather than a single scalar number that gives the capacity of a single user channel. Further on, in a wireless system certain multiple-access technique is usually used to simplify the complexity of the system and facilitate easier resolvement of the information of different users that share the common resource. The technology of choice of modern 3G cellular wireless systems is Code Division Multiple Accessing (CDMA) and this thesis will concentrate on it as a most probable technology for future improvements. Finally, the impediment that causes the most difficulty in the analysis and design of a wireless system is fading. Fading reflects the random nature of a wireless multipath propagation channel and effects considerably the optimal design and limiting performance of a wireless communication system. Fading is caused by obstructions that block the direct propagation of radio waves as well as multipath propagation that is caused by the reflections and scattering from the objects in the vicinity of the line of sight between the transmitter and the receiver. Fading can be also caused by moving of the transmitter or the receiver. If neglected, the fading process can significantly reduce the capacity region of a multiuser cellular wireless communication system compared to the case with additive Gaussian noise only.

However, it turns out that we can make use of the knowledge of the fading characteristic and optimize the wireless system for performance in strong fading environments. In delay-tolerant packet data applications, fading can be even beneficial, and the knowledge of fading coefficients at the transmitter can even increase the spectral efficiency of a multiple-access system beyond that of a non-fading channel. Some ideas borrowed from the information-theoretic analysis are already incorporated in IS-856 standard which presents an extension of the wireless 3G standard for delay-tolerant applications.

The scope of this thesis is the analysis and optimal design of multiuser CDMA based communication systems in fading environments. Special emphasis is placed on the analysis and design of low complexity receivers. Some aspects of both delay tolerant and delay sensitive applications are discussed. A prominent part of the thesis is the problem of optimal resource allocation, namely power allocation and signature sequence allocation. In order to give a more practical appeal to some of the results, the case of imperfectly known channel state information at the receiver is also discussed.

1.1 Communication Model

The following single cell CDMA channel model will be considered throughout this thesis. Let $b_k(i)$ be the i-th information bit of the k-th user. It is assumed that there are K users in the system. Their information bits are encoded using a length L block error correction code to produce code symbols $d_k(i)$ corresponding to the

i-th code symbol of the k-th user. In general, multiuser error correction codes can be used, meaning that all the users' bits influence the choice of the code symbols. This general setting is usually simplified to the case in which single user error corrections codes are used. Several applications of the single user error-correction codes will be discussed later in the thesis. Let Ψ_k be the error-correction code of the user k which maps l_k information bits $[b_1, \ldots, b_{l_k}]$ into L code symbols $[d_1, \ldots, d_L]$. We denote with $R_k = l_k/L$ the code rate corresponding to the user k. The code symbols in a CDMA system are spread using finite energy and equal duration spreading sequences $s_k(t)$. Hence, the received multiple-access signal during one code symbol duration is given by

$$r(t) = \sum_{i=1}^{L} \sum_{k=1}^{K} d_k(i) g_k^{1/2}(i) s_k(t - iT - \tau_i) + n(t) \tag{1.1}$$

where T is the symbol interval and $g_k(i)$ is the flat fading channel gain corresponding to the i-th code symbol of the k-th user. We consider coherent detection and channel with additive Gaussian noise process $n(t)$. In general, received information symbols are not perfectly aligned and different time delays τ_i characterize different users. We concentrate, however, on the case when user symbols are perfectly aligned in time i.e. $\tau_1 = \cdots = \tau_K$. This model is often called the *synchronous Gaussian CDMA multiple-access channel.* It is assumed that each waveform is linearly modulated by a sequence of real valued code symbols that satisfy average power constraint

$$\frac{1}{L} \sum_{i=1}^{L} \mathrm{E}[d_k^2(i)] \leq \bar{P}_k, k = 1, \ldots, K \tag{1.2}$$

for each code word of L transmitted code symbols. The average single user power constraints are arranged in a diagonal matrix $\bar{\mathbf{P}} = \mathrm{diag}[\bar{P}_1, \ldots, \bar{P}_K]$, while instantenous single user powers are arranged in a diagonal matrix $\mathbf{P} = \mathrm{diag}[P_1, \ldots, P_K]$. The synchronous CDMA Gaussian channel with single user coding and joint multiuser detector and decoder is shown in Figure 1.1.

Now, a discrete time model for the synchronous Gaussian CDMA multiple-access channel can be derived. By observing that each user signal is already a discrete-time process, it suffices to find a sequence of observable measurements that forms a sufficient statistics for the input code symbol sequence $\mathbf{d}(i) = [d_1(i), \ldots, d_K(i)]^T, i = 1, \ldots, L$. We assume that the spreading waveform signals $s_k(t)$ span an N-dimensional

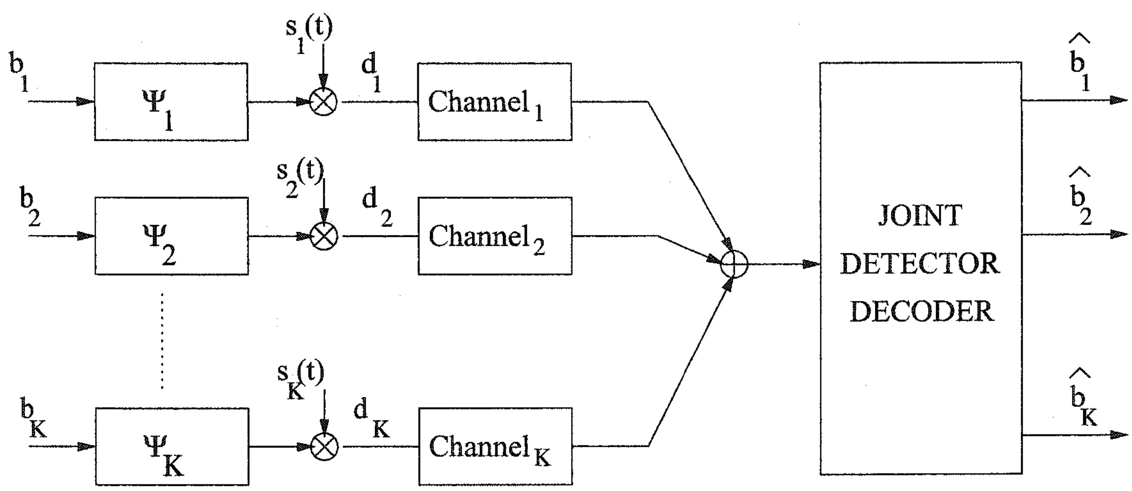

Figure 1.1. *Illustration of the single user coded CDMA channel with joint detection/decoding.*

subspace of $L_2[0, T]$ [3], and that $\phi_1(t), \ldots, \phi_N(t)$ is an orthogonal basis of this subspace. Since we assume that the additive noise process in (1.1) is white and Gaussian, the sequence of observables

$$y_k(i) = \int_{iT}^{iT+T} r(t)\phi_k(t - iT)dt \tag{1.3}$$

present a sufficient statistic for detection of code symbols $\mathbf{d}$. These observables present chip matched filter output and can be for convenience arranged in an N-dimensional vector $\mathbf{y}(i) = [y_1(i), \ldots, y_K(i)]^T$. Therefore, an equivalent discrete-time matrix model is

$$\mathbf{y} = \mathbf{S}^T\mathbf{G}^{1/2}\mathbf{d} + \mathbf{n} \tag{1.4}$$

where the $K \times N$ sequence matrix is $\mathbf{S} = [s_{kj}], k = 1, \ldots, K, j = 1, \ldots, N$ and

$$s_k(t) = \sum_{i=1}^{N} s_{ki}\phi_i(t). \tag{1.5}$$

In practice, this output can be obtained by matched filtering. The dependence on the symbol interval is dropped since the analyzed model is synchronous. Additive noise $\mathbf{n}$ is normally distributed with covariance matrix $\sigma^2\mathbf{I}$ and zero mean.

For convenience, the discrete-time version of the spreading sequence of user k can be arranged in the vector $\mathbf{s}_k = [s_{k1}, \ldots, s_{kN}]^T$. $\mathbf{d} = [d_1, d_2, \ldots, d_k]^T$ represents coded

Type of Fading	Parameter	PDF, f(g)
Rayleigh		$\frac{1}{\bar{g}}\exp\left(-\frac{g}{\bar{g}}\right)$
Nakagami-m	$\frac{1}{2} \leq m$	$\frac{m^m\, g^{m-1}}{\bar{g}^m \Gamma(m)}\exp\left(-\frac{mg}{\bar{g}}\right)$
Nakagami-n (Rice)	$0 \leq n$	$\frac{(1+n^2)e^{-n^2}}{\bar{g}}\exp\left[-\frac{(1+n^2)g}{\bar{g}}\right] \times I_0\left[2n\sqrt{\frac{(1+n^2)g}{\bar{g}}}\right]$
Nakagami-q (Hoyt)	$0 \leq q \leq 1$	$\frac{(1+q^2)}{2q\bar{g}}\exp\left[-\frac{(1+q^2)^2 g}{4q^2\bar{g}}\right] \times I_0\left[\frac{(1-q^4)g}{4q^2\bar{g}}\right]$
Log-normal shadowing	σ	$\frac{4.34}{\sqrt{2\pi}\sigma g}\exp\left[-\frac{(10\log_{10} g-\mu)^2}{2\sigma^2}\right]$

Table 1.1. *Some common fading distributions.*

data symbols. As an information-theoretic model we assume that those symbols have a Gaussian distribution, since this distribution of coded symbols maximizes the capacity of a channel with additive Gaussian noise [3]. It is known from [4] that single-user coding and optimum decision feedback decoding are sufficient to achieve the total capacity of the channel at the vertices of the capacity region. Therefore when an optimal decision feedback receiver is used we assume (in order to facilitate the simplification of the receiver structures) that single user codes are used.

We denote by $\mathbf{R} = \{R_{ij}\} = \mathbf{S}\mathbf{S}^T$ the $K \times K$ correlation matrix and by $\alpha = K/N$ the system load. Matrix $\mathbf{G}$ is a diagonal matrix of fading gains at the analyzed symbol interval $\mathbf{G} = \mathrm{diag}[g_1, \ldots, g_K]$. Let $\mathcal{G}$ be the set of all possible fading states. Now $W_k = P_k g_k$ denotes the received power of user k. For the ease of the notation we introduce the diagonal matrix $\mathbf{W} = \mathbf{PG}$ of instantenous received user powers. The knowledge of the fading gains of the channel will be called the *channel state information* CSI. Unless stated otherwise, it is assumed that perfect CSI is available at the receiver.

Let $f(g)$ denote the probability density function (pdf) of fading channel gains, with $E(g) = 1$ assumed for simplicity and $F(g)$ the respective cumulative density function (cdf) of that distribution. For example, in the case of Rayleigh fading, $f_{Ray}(g) = e^{-g}$ and $F_{Ray}(g) = 1 - e^{-g}$. Some other common fading distributions are given in Table 1.1 along with some parameters that characterize these distributions. More details on these distribution can be found in [5].

1.2 Related Results

Since the introduction of the channel capacity for single user point to point communications by C.E.Shannon in [1], its scope has been extended to numerous different cases. An important extension of the single user channel is the multiple-access channel which models the channel with several transmitters that share the same transmission medium and a single receiver. The capacity region and coding theorems for such a channel were established independently by R.Ahlswede and H.Liao in [6], [7] and [8]. They showed that the capacity region of a multiple-access channel is a convex hull of a union of pentagons. Shortly after that, A.Wyner [9] and T.Cover [10] gave a simplified version of these results for memoryless channels. Another important information-theoretic parameter, the error exponents of the multiple-access channel was analyzed in [11].

In general, the multiple-access capacity region is achievable by using multiple-access error-control codes. However, in the case of a Gaussian Multiple Access Channel (GMAC), it was shown by T.Cover [12] that the vertices of the capacity region can be achieved by using successive single user decoding and interference cancellation along with single user error-correction codes. The results on the capacity region of a multiple-access region have been extended later by S. Verdú in [13] for channels with memory, and for channels with intersymbol interference [14]. The case of symbol asynchronous multiple-access channels has been tackled in [15].

Wireless communication systems usually operate in cellular environments and there is a need to extend the usual single cell model to multiple cells. This has been accomplished in [16] where a simple model that describes the cellular environment has been introduced and respective information-theoretic parameters analyzed.

Extending the results of [6], [7], [8] to the case of CDMA the capacity region of synchronous CDMA multiple-access AWGN channel has been derived in [3] and later reformulated in [17] as

$$\mathcal{C}(\mathbf{P},\mathbf{G}) = \bigcup_{J \subset \mathcal{U}} \left\{ (R_1, R_2, \ldots, R_K) : \sum_{i \in J} R_i \leq \frac{1}{2N} \log \left| \mathbf{I}_{|J|} + \mathbf{H}_J \right|, \forall J \subseteq \{1, \ldots, K\} \right\} \tag{1.6}$$

where matrix $\mathbf{H} = \frac{1}{\sigma^2}\left(\mathbf{W}^{1/2}\right)\mathbf{S}\mathbf{S}^T\left(\mathbf{W}^{1/2}\right)$ and $\mathbf{I}_{|J|}$ is $|J| \times |J|$ dimensional unity matrix. $\mathcal{U}$ is the set of the users' indices. The code rates are expressed in [bits/chip].

This result has been derived by considering the outputs of a chip matched filter in a CDMA channel as outputs of a correlated vector multiple-access channel. This capacity region has K factorial (not necessarily distinct) vertices which correspond to the detection order of an optimum decision feedback receiver given by a permutation π. This capacity region is always contained in the GMAC capacity region.

Using the capacity region, the sum capacity (also expressed in [bits/chip]) presents an ultimate limit on total achievable rates and is defined as

$$\Gamma = C_{sum} = \max_{R \in \mathcal{C}(\mathbf{P},\mathbf{G})} \sum_{i=1}^{K} R_i. \tag{1.7}$$

Note that so defined sum capacity is equal to the spectral efficiency Γ of the system expressed in [bits/chip] and we will use both terms interchangeably. According to [3], the sum capacity in units [bits/chip] of a synchronous CDMA system is [1]

$$C_{sum} = \frac{1}{2N} \log_2 \left[\det \left(\mathbf{I} + \frac{N}{\sigma^2} \mathbf{SWS}^T \right) \right]. \tag{1.8}$$

This formula presents a starting point for the analyses of spreading sequence allocation performed in several papers. Namely, in [18] the optimal sequence allocation which maximizes the previous sum capacity has been found in the case of equal received user powers. This analysis has been later extended in [19] for the case of asymmetric received user powers.

The extension of the channel capacity with deep practical aspirations is the channel capacity of fading channels. This case has been discussed thoroughly in an excellent review paper by E.Biglieri et al. [20]. In the case of fast changing ergodic flat fading channels with perfect channel state information (CSI) at the receiver, it has been shown in [21], [22] and [20] that the capacity region of a multiple-access channel can be derived from the non-fading capacity region by averaging over all fading gains, i.e.

$$\mathcal{C}_e(\mathbf{P}) = \mathrm{E}\left[\mathcal{C}(\mathbf{P},\mathbf{G})\right] \tag{1.9}$$

where averaging is over the p.d.f. of random fading gains. This type of capacity is usually called *ergodic capacity*. We will drop the subscript e that denotes the

[1] The difference between this equation and the equation in the original paper [3] is due to the fact that we assume here that signatures have unit norm while in original paper signatures have norm N.

ergodic capacity since most derivations deal with this case. The capacity of the channel with fading can be increased if the channel state information is available at the transmitter allowing for the use of power control at the transmitter. This case has been analyzed by A.J.Goldsmith et al. in [22] for single user channels and by R.Knopp et al. in [23] for multiple-access channels. Some practical aspects of the system design that approach the predicted bound for single user fading channels have been published in [24] and [25]. The extensions of the multiple-access channels to the case of cellular environments with fading, frequently occurring in practical applications, have been analyzed in [16], [21]. All the previous results assume that perfect channel state information is known at the receiver. Some results, however, exist even when the channel state information is estimated with error: see for example [26], [27], [28] and references therein.

The notion of ergodic capacity applies to flat fading channels where the transmission time is much longer that the coherence time of the channel, i.e. the fading process can reflect its ergodic nature during the transmission. In the case where no significant variations of the fading occur during the transmission, the classical Shannon capacity might not be usable and the appropriate information-theoretic measures are *capacity versus outage* and *delay-limited capacities* [20]. The notion of the delay-limited capacities is used in this thesis and we give here its definition for the multiple-access flat fading channel with spreading (cf. [29])

$$\mathcal{C}_d(\bar{\mathbf{P}}) = \bigcup_{\mathbf{P}\in\mathcal{P}} \bigcap_{\mathbf{G}\in\mathcal{G}} \mathcal{C}(\mathbf{P}, \mathbf{G}) \tag{1.10}$$

where $\mathcal{P}$ is the set of allowed power control allocation policies under the constraint on average single user powers given by diagonal matrix $\bar{\mathbf{P}}$. This definition states that the delay-limited capacity region of the channel is maximal possible capacity region that is attainable with certain power allocation policy for all possible channels. Extending this result for single user case, G. Caire et al. in [30] has shown that delay-limited capacity is equivalent to the capacity of single-block fading channel attainable with zero outage probability.

Some other relevant results on DS-CDMA systems from the information-theoretic viewpoint can be found in [31], [32], [33] and [34].

Very similar results and mathematical models are used in the analysis of the popular Multiple Input Multiple Output (MIMO) channels which describe systems with multiple transmitting and receiving antennas. Some of the most prominent information-theoretic analyses of these channels have been reported in [35], [36] and [37].

The equation for the sum capacity of the CDMA channel (1.8) is useful only for the analysis of the particular allocation of fading gains and spreading sequences. In order to gain more general insight into the performance of the multiple-access system, random fading gains and random spreading sequence models should be used. Pursuing this basic motivation two important results [38], [39], published in March 1999 in the Transactions on Information Theory, analyzed random sequence and fading gain allocation. These results stirred considerable interest in the analysis of the performances of various receivers for large multiple-access DS-CDMA systems. Both results deal with random sequences where the number of users and processing gain increase to infinity while maintaining their ratio fixed. The large random sequence allocation model has also been analyzed in [40]. These assumptions, although theoretical in nature, provide insight into the operation of receivers for practical CDMA systems where the number of users is finite and signature sequences are pseudo-randomly chosen. Using novel results on the eigenvalue distribution of random matrices ([41], [42] and [43]), the asymptotic approach for the performance analysis of large system multiuser receivers is capable of producing analytically tractable and easily comparable results. A short review of some of these results on eigenvalue distribution of random matrices is given in Appendix A.

In [38], asymptotic spectral efficiencies of linear and optimum receivers for synchronous CDMA systems with equal user powers have been derived. The spectral efficiencies of linear multiuser receivers have been previously analyzed through simulations for finite systems in [44]. Extending the results of [38], [45] analyzes the fundamental limits on decision feedback linear receivers for large system synchronous CDMA with equal received powers and equal rates of all users. In [46], the spectral efficiency of randomly spread DS-CDMA multi-cell systems has been analyzed.

In [39], linear multiuser receivers for synchronous CDMA systems in flat fading channels have been analyzed and very useful notions of effective bandwidth and effective interference have been introduced. These results have been expanded for asynchronous CDMA [47] and for multi-path channels with imperfect channel state information [48]. Following another course of research, spectral efficiency results of [38] have been extended for flat fading channels in [49] which also covers the issues of power control and multi-antenna systems. For the case of optimal (MMSE) decision feedback receiver and synchronous CDMA it has been shown in [50] that its capacity region (and therefore the spectral efficiency) coincides with that of the optimal maximum likelihood receiver. The random sequence model has been also used in [51] where results of [39] have been extended for so called "random environments". The random sequence model has been used for the analysis of uncoded performance of linear-multiuser receivers in [52]. The asymptotic random sequence model has also been helpful in the analysis of multiple-access receivers in [53] and [54].

In the conclusion of this topic, we mention very recent results by T. Tanaka [55], [56] who has noticed that some problems in the analysis of the uncoded performance of optimal multiuser receiver for CDMA channel with random sequences arise also in the analysis of ferromagnetic materials in statistical mechanics. This result has been employed recently in [57] (and references therein) for the analysis of the capacity of the receiver for the multiple-access channel where detection and decoding are separated.

1.3 Contributions

This thesis presents several results that extend the knowledge of information-theoretic aspects of CDMA multiple-access channels. The focus is on performance analysis and optimal resource allocation of multiuser receivers in flat fading channels.

Based on the previous work by Viswanath and Anantharam [19], we characterize the optimal sequence allocation in flat fading channels for the large system model. The spectral efficiency of the optimally alocated sequences is compared with the large random sequence model of [49]. Also, an optimal sequence allocation scheme that reduces the complexity of optimal joint decoding and detection is proposed.

The non-asymptotical analysis of the optimal sequence and power allocation that maximizes the spectral efficiency is accomplished using a previous result [58]. These results are later extended for the asymptotic case and efficiency gains compared to the no power control case are evaluated.

Using the multiple-access capacity region derived by Verdú in [3], we give characterization of the vertices of the multiple-access capacity region using Cholesky decomposition. This characterization is later used to extend some results on optimal sequence allocation that maximizes the symmetric capacity. Additionally, we relate some of the results of [19] with results from a previous paper by Alsugair and Cheng [17].

Using results on the asymptotic SIR performance of linear detectors [39] and some results from the extreme value theory [59], we analyze spectral efficiency of the conventional decision feedback receiver in fading channels. The analysis is performed for a conventional decision feedback receiver with and without power ordering and with and without power control. Also, an algorithm for finding optimal power allocation is proposed.

The spectral efficiency of a simple practical receiver for two sets of orthogonal sequences is also carried out. This low complexity multiple access scheme has a very high spectral efficiency, very close to that of an optimal sequence allocation. Uncoded performance and convergence properties of such a scheme were analyzed and it was shown that analyzed receiver is a version of space altering generalized algorithm (SAGE).

We also discuss the case of imperfect channel state information at the receiver and derive the expression which gives the spectral efficiency loss of such a channel.

1.4 Thesis Outline

In Chapter 2, optimal spreading sequence allocation for the synchronous CDMA Gaussian multiple-access channel is presented. The analysis is performed for the cases when power control is allowed and when it is not allowed at the transmitter. Both cases are also analyzed in the asymptotic and non-asymptotic regime. This chapter also contains an optimal sequence allocation scheme that decreases the complexity of the joint decoder/detector when no power control is allowed.

Chapter 3 discusses the Cholesky characterization of the multiple-access capacity region and some applications of the characterization of the symmetric capacity for both optimal and random sequence allocation.

Chapter 4 presents the spectral efficiency of the conventional decision feedback receiver in flat fading channels. The analysis is carried out for the large random sequence model and with perfect CSI at the transmitter.

Chapter 5 analyzes the CDMA multiple-access scheme for two sets of orthogonal sequences and a low complexity receiver.

The case of imperfect channel state information at the receiver is discussed in Chapter 6. The spectral efficiency loss of the optimal multiuser detector due to the imperfect channel state information is given in this chapter.

Chapter 7 contains concluding remarks and suggestions for future investigations based on the results presented in this dissertation.

Appendix A complements the exposition of Chapters 2,3,4 and 6 which make use of the large random sequence model.

Chapter 2

On the Optimal Sequence Allocation in Flat Fading Channels

2.1 Introduction

Recently there has been a considerable interest in exploring the fundamental limits of coded DS/CDMA systems and their dependence on the choice of spreading sequences, coding/spreading trade-offs, the statistical effects of fading channels, power control schemes and the type of employed multi-user receiver. These analyses are conveyed in order to gain insight into the operation and design of CDMA multiple-access systems, having in mind inevitable complexity-performance trade-offs. Central to all these analyses is the notion of spectral efficiency that will be used throughout the thesis. We next give the definition of the spectral efficiency that will be used in the thesis.

Definition 1 *Spectral efficiency is a universal parameter for characterizing the performance of a certain communication scheme. It is defined as the total number of bits that can be transmitted with arbitrary reliability per second per Hertz of bandwidth under the contraints on the receiver structure and/or QoS requirements of the users.*

For a CDMA system, the total spectral efficiency of the system can be calculated using the sum of single user capacities C_i. For convenience, in simplifying our notation we express these single user rates in [bits/chip] units, as opposed to some previous papers where single user capacities were given in [bits/symbol] (see for example [38]).

The optimal unit energy sequence allocation that maximizes the previous equation was first analyzed in [18] for equal received user powers and later generalized in [19] for general asymmetric user powers. This case is equivalent to the case of no power control at the receiver. It was shown in [19] that if no users are oversized (the definition of an oversized user will be revisited in (2.2)), the sum capacity of a CDMA system with appropriate choice of signature sequences is equal to the capacity of Gaussian multiple access channel (GMAC) i.e. $C_{sum} = C_{GMAC} = \frac{1}{2}\log_2(1 + \text{SNR})$ where we have introduced for brevity of notation, $\text{SNR} = \frac{P_{tot}}{\sigma^2}$ and P_{tot} is sum of powers of users in the system. Furthermore, it was shown that the sum capacity of the system attains the sum capacity of GMAC channel if K unit norm signature sequences $\mathbf{S}$ of length $N, (K \geq N)$ have the property

$$\mathbf{S}^T (\mathbf{W}) \mathbf{S} = \frac{P_{tot}}{N}\mathbf{I}. \tag{2.1}$$

Sequences $\mathbf{S}$ that have the property (2.1) are called Generalized Welch Bound Equality Sequences (WBE). In the special case of equal power users and no fading these sequences achieve the Welch lower bound on the sum of squared coefficients of the correlation matrix or equivalently $\mathbf{S}^T\mathbf{S} = \frac{K}{N}\mathbf{I}$ [18]. An iterative centralized construction procedure for WBE sequences was given in [19]. A simple decentralized construction procedure for WBE sequences was presented in [60] and later analyzed for more general framework in [61].

In the case that some users have oversized powers, they are assigned orthogonal sequences and there is an explicit loss in sum capacity compared to the case of the Gaussian multiple access channel - GMAC (Theorem 3.1 in [19]). However, as will be demonstrated in Section 2.2.1 and 2.2.2, this loss in sum capacity can be attributed to the limiting constraint that all users have equal power.

Another approach to analyzing spectral efficiency that has received considerable attention [38, 49, 45, 46, 47, 48] is through analysis of random spreading sequences for *large systems* i.e. when the number of users K and processing gain N go to infinity, but the ratio $\alpha = K/N$ of users per chip remains constant. These results rely on powerful theory of limiting eigenvalue distribution of large matrices and produce relatively simple, mathematically tractable results. For optimal joint multi-user detectors it was shown in [38] for equal received powers and [49] for flat fading channels that there is asymptotically (as $\alpha \to \infty$) no spectral efficiency loss compared to the GMAC.

Using a result from extreme value theory [59], we will characterize the spectral efficiency of optimally allocated sequences in flat fading channels in a large system. Thus we will be able to explicitly evaluate possible gains of using optimal sequences compared to random sequences. Furthermore, we will discuss sequence allocation based on [19] which divides users in orthogonal groups in an effort to simplify the receiver structure while achieving full sum capacity. The lower bound on spectral efficiency in case of unbalanced orthogonal groups will be derived and discussed in asymptotical case.

The optimal allocation of sequences in [19] requires perfect knowledge of channel state information at the transmitter and this information might also be utilized to optimally allocate the user powers to increase the system capacity even further. Therefore, we address here the problem of joint power control law - signature sequence optimization, and show that it can be solved as a determinant optimization problem of [58]. The spectral efficiency of a system with optimal allocation of signature sequences and user powers will be derived for the general case and later extended for the asymptotic large system model.

The following definition of vector ordering will be used throughout the thesis [62].

Definition 2 *For any vector $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$, let $\boldsymbol{x}_{\downarrow} = \{x_{[1]}, x_{[2]}, \ldots, x_{[n]}\}$ denote the non-increasing rearrangement of the vector $\boldsymbol{x}$, i.e. where $x_{[1]} \geq \ldots \geq x_{[n]}$ is satisfied. Also, let the π^x be permutation vector which orders the elements of vector $\boldsymbol{x}$, i.e. $x_i = x_{[\pi_i^x]}$.*

The outline of the Chapter is as follows. Section 2.2 presents the asymptotic analysis of optimal sequence allocation in flat fading channels with and without power control. The group orthogonal sequence allocation is discussed in Section 2.3.

2.2 Optimal Sequences in Flat Fading Channels

In this section we address the issue of optimal sequence allocation in flat fading channels with and without power control at the transmitter. It is assumed that the transmitter is provided with perfect channel state information by the feedback from the receiver.

2.2.1 No Power Control at the Transmitter

In the proceeding analysis we assume that transmitted powers are equal, that is $P_i = P$ for $i = 1, \ldots, K$, and that all users are allocated unit energy sequences.

We first consider the system with more users than the processing gain (sometimes called oversaturated system) with $\alpha \geq 1$ and later comment on the case $\alpha < 1$. The problem of allocating optimal unit energy sequences with asymmetric power constraints was addressed in [19] and it was shown that a certain subset of so-called oversized users played a special role in the choice of sequences. Following our notation and the Definition 1, user i with channel gain g_i which is ordered as j-th ($j = \pi_i^g$ and $g_i = g_{[j]}$) in the vector of ordered channel gains $\mathbf{g}_\downarrow$ is defined to be oversized if

$$g_i > \frac{\sum_{l=j+1}^{K} g_{[l]}}{N-j} \tag{2.2}$$

and $j \leq N-1$. There can be at most $N-1$ oversized users and we will denote their number as κ. The sum capacity for optimally chosen equal energy sequences is according to Theorem 3.1 in [19]

$$\begin{aligned} C_{sum}(\mathbf{G}) &= \frac{N-\kappa}{2N} \log_2 \left(1 + \frac{N}{(N-\kappa)\sigma^2} \sum_{l=\kappa+1}^{K} P g_{[l]}\right) \\ &+ \frac{1}{2N} \sum_{l=1}^{\kappa} \log_2 \left(1 + \frac{N P g_{[l]}}{\sigma^2}\right) \end{aligned} \tag{2.3}$$

where the first term corresponds to $K-\kappa$ weakest users that are allocated WBE sequences and the second term corresponds to κ oversized users that are allocated orthogonal sequences. Now, the ergodic sum capacity for optimally allocated sequences in flat fading channels can be derived averaging the sum capacity of (2.3) over random channel gains i.e. $C_{sum} = E\left[C_{sum}(\mathbf{G})\right]$. The distribution function of the ordered element $g_{[l]}$ can be calculated from [63]

$$f_{[l]}(x) = \frac{K!}{(l-1)!(K-l)!} F^{K-l}(x) \left(1 - F(x)\right)^{l-1} f(x), \tag{2.4}$$

where $f(x)$ and $F(x)$ are pdf and cdf of the random channel gains respectively. In general this appears to be a complex problem since the number of oversized users is also a random variable. However, in the asymptotic case for the large number of users ($K \to \infty$) with constant ratio of number of users per processing gain $\alpha = \frac{K}{N}$, due to the weak law of large numbers this problem can be readily solved. The similar asymptotic approach was used previously for the analysis of the conventional decision feedback with receiver power ordering in [64]. By employing the asymptotic analysis we will be able to evaluate the spectral efficiency loss incurred by using unit energy sequences in the flat fading channel.

We start with the observation that with the increase in the number of samples (in this case users) the fading distribution $f_{[l]}(x)$ becomes a degenerate distribution which means that it can take only values 0 and 1 [63, 59]. Mathematically this means that for $0 < \beta < 1$, as $K \to \infty$, the cumulative distribution $F_{[\lfloor K(1-\beta)\rfloor]}(g)$ [1] converges weakly to the step function at the quantile ξ_β of the distribution f i.e.

$$\lim_{K\to\infty} F_{[\lfloor K(1-\beta)\rfloor]}(g) = h(x - \xi_\beta), \tag{2.5}$$

where function $h(x)$ is a unit step function. Alternatively formulated [65] samples of the distribution $f_{[\lfloor K(1-\beta)\rfloor]}(g)$ converge in probability to ξ_β. That is for each $\epsilon > 0$

$$\text{Prob}\left(|g_{[\lfloor K(1-\beta)\rfloor]} - \xi_\beta| > \epsilon\right) \to 0, \tag{2.6}$$

where the quantile of the distribution f is defined as $\xi_u = F^{-1}(u)$ assuming that cdf F is invertible. Note that according to this definition, $\xi_{1/2}$ is the median of the distribution. Regarding the convergence rate of (2.6), Theorem 9.2 in [63] predicts that for distributions with unbounded support and $0 < \beta < 1$ the variance of $f_{[\lfloor K(1-\beta)\rfloor]}(g)$ decreases proportionally to K^{-1} as $K \to \infty$. An illustration of the convergence of the order statistic distribution with the increase of the number of samples is given in Figure 2.1.

The following lemma will be used throughout the Chapter to evaluate asymptotic properties of sum capacity.

Lemma 1 *Let $0 \le \lambda_1 < \lambda_2 \le 1$ and $x_1, \ldots, x_K$ be samples drawn from the distribution with pdf f and cdf F and $\phi(x)$ be real valued function of a real variable, then*

$$\frac{1}{\lfloor \lambda_2 K\rfloor - \lfloor \lambda_1 K\rfloor} \sum_{i=\lfloor \lambda_1 K\rfloor}^{\lfloor \lambda_2 K\rfloor} \phi(x_{[i]}) \tag{2.7}$$

converges in probability as $K \to \infty$ to

$$\frac{1}{\lambda_2 - \lambda_1} \int_{F^{-1}(1-\lambda_2)}^{F^{-1}(1-\lambda_1)} \phi(x) f(x) dx = \frac{1}{\lambda_2 - \lambda_1} \int_{1-\lambda_2}^{1-\lambda_1} \phi(F^{-1}(u)) du. \tag{2.8}$$

[1] $\lfloor x \rfloor$ denotes the integer part of x.

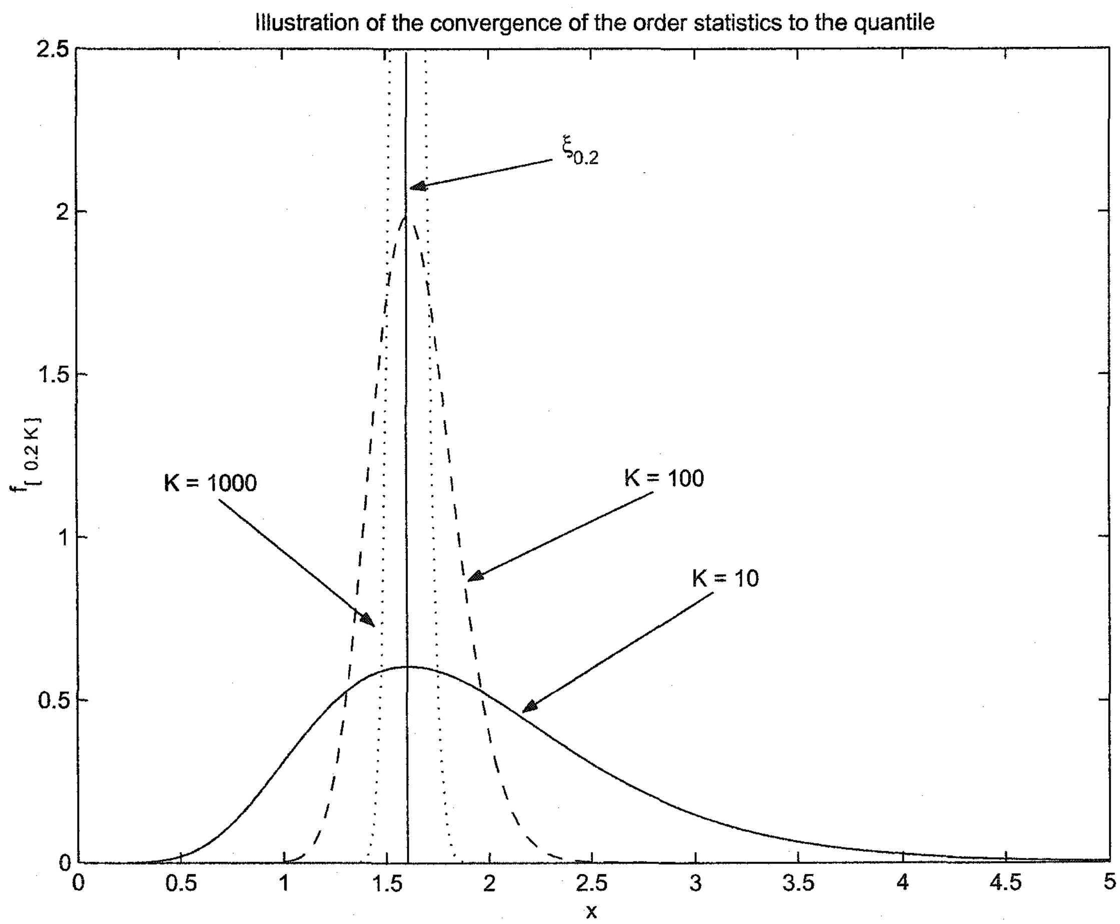

Figure 2.1. *Illustration of the convergence of the order statistic distribution to the quantile.*

Proof:

Using the asymptotic weak convergence of ordered statistics (2.5), the distribution of ordered samples $x_{[i]}$ for $i = \lfloor \lambda_1 K \rfloor, \dots \lfloor \lambda_2 K \rfloor$ drawn from the pdf $f(x)$ converges weakly to

$$f_{\lambda_1,\lambda_2}(x) = \frac{1}{\lambda_2 - \lambda_1}(h(x - F^{-1}(1-\lambda_2)) - h(x - F^{-1}(1-\lambda_1)))f(x) \tag{2.9}$$

since $x_{[\lfloor \lambda_i K \rfloor]} \to F^{-1}(1-\lambda_i)$ for $i = 1, 2$ and $K \to \infty$. The coefficient $\frac{1}{\lambda_2 - \lambda_1}$ is introduced to normalize the integral of pdf $f_{\lambda_1,\lambda_2}(x)$ to unity. Expression (2.7) then asymptotically presents the expectation $\mathrm{E}[\phi(x)]$ with respect to the pdf distribution $f_{\lambda_1,\lambda_2}(x)$. Therefore the left side of (2.8) follows immediately and the right side after substituting $u = F(x)$ in the integral on the left side (2.8). □

We denote with $\lambda = \frac{\kappa}{K}$ the fraction of oversized users, where κ is the number of oversized users and note that $\lambda < \alpha^{-1}$ since $\kappa < N$. The right hand side of (2.2) satisfies asymptotically

$$\begin{aligned} \lim_{K\to\infty} \frac{1}{N-\kappa} \sum_{l=\kappa+1}^{K} g_{[l]} &= \lim_{K\to\infty} \frac{K-\kappa}{N-\kappa} \sum_{l=\kappa+1}^{K} \frac{g_{[l]}}{K-\kappa} \\ &= \frac{1}{\alpha^{-1}-\lambda} \int_0^{1-\lambda} F^{-1}(u)du. \end{aligned} \tag{2.10}$$

where we applied Lemma 1. Therefore the fraction λ has to satisfy

$$\max_{0\le\lambda<\alpha^{-1}} \left[F^{-1}(1-\lambda) \ge \frac{1}{\alpha^{-1}-\lambda} \int_0^{1-\lambda} F^{-1}(u)du \right]. \tag{2.11}$$

We will assume that the cdf $F(x)$ and its inverse $F^{-1}(x)$ are continuous functions and show (under certain conditions on f) that the largest fraction of oversized users which still satisfy the previous inequality is simply the largest solution $0 \le \lambda < \alpha^{-1}$ of the following equation

$$F^{-1}(1-\lambda) = \frac{1}{\alpha^{-1}-\lambda} \int_0^{1-\lambda} F^{-1}(u)du. \tag{2.12}$$

For continuous distributions $f(x)$ with unbounded support the previous equation always has a solution $0 \le \lambda < \alpha^{-1}$ since the left side of (2.11) is a monotonically decreasing function and $\lim_{\lambda\to 0} F^{-1}(1-\lambda) = \infty$ while the right side of (2.11) is continuous and equal to α for $\lambda = 0$ and it generally rises to ∞ for $\lambda = \alpha^{-1}$. It can now be seen that the fraction of oversized users $0 \leq \lambda < \alpha^{-1}$ depends only on the fading distribution $F(x)$ and load α.

Continuing in the same manner and applying Lemma 1, the sum capacity of (2.3) converges in probability in the asymptotic case to

$$\begin{aligned}\lim_{K\to\infty} C_{sum} &= \frac{1-\alpha\lambda}{2}\log_2\left(1+\frac{\text{SNR}}{(1-\alpha\lambda)}\int_0^{1-\lambda}F^{-1}(u)du\right)\\ &+ \frac{\alpha}{2}\int_{1-\lambda}^{1}\log_2\left(1+\frac{\text{SNR}}{\alpha}F^{-1}(u)\right)du \qquad (2.13)\end{aligned}$$

where λ is the largest solution of equation (2.12). The previous equation can also be viewed as ergodic capacity since the distribution of ordered channel gains is degenerate in asymptotic case.

According to (2.11) and (2.12), in the particular case of Rayleigh fading that will be used in our example, $0 \leq \lambda < \alpha^{-1}$ becomes the only solution of equation

$$\alpha^{-1}\ln(\lambda) + 1 - \lambda = 0$$

and the sum capacity is equal to

$$C_{sum} = \frac{\alpha}{2\ln 2}\left[\lambda\ln(1-\text{SNR}\alpha^{-1}\ln(\lambda)) + e^{\frac{\alpha}{\text{SNR}}}\text{Ei}\left(1, -\ln(\lambda)+\frac{\alpha}{\text{SNR}}\right)\right] \qquad (2.14)$$

where the exponential integral function is defined as $Ei(n,x) = \int_1^\infty \frac{e^{-xt}}{t^n}dt$.

This derivation can also be applied for cases when all users do not have the same fading distribution. For the discussion on this topic see Remark 3 in Chapter 4.

Equation (2.13) can be applied to the case when the number of users is less than the processing gain, i.e. $\alpha < 1$, for which $\lambda = 1$ and the first term in (2.13) vanishes. In this case all users are optimally assigned orthogonal sequences (as if that all users are oversized) and the second term which denotes the contribution of the orthogonal users is equal to the sum capacity. With the increasing values of $\alpha > 1$, the fraction of oversized users λ decreases and the influence of the first term becomes significant. Ultimately, for distributions with unbounded support, as $\alpha \to \infty$ we have that $\lambda \to 0$ and the fraction of oversized users tends to zero. In this case almost all users are assigned WBE sequences and the second term of (2.13) vanishes. This behavior is illustrated in Figure 2.2 with the variation of the contribution of orthogonal users in terms of system load α for an optimal sequence allocation in Rayleigh fading channels.

Figure 2.3 compares spectral efficiencies (sum capacities) of optimally and randomly chosen sequences in Rayleigh fading and no-fading channels. The spectral efficiency of the optimal joint detector with random sequences was derived in [38] for no-fading case and in [49] for the flat fading case. From this figure it can be concluded that the spectral efficiency penalty that we have to pay for using random sequences compared to the optimal sequences in flat fading channels can be as much as $0.6 - 0.7[bit/s/Hz]$ for $E_b/N_0 = 10dB$ and smaller values of α while this spectral efficiency loss vanishes as $\alpha \to \infty$.

2.2.2 Power Control at the Transmitter

In this section we further relax the problem of finding optimal sequences by dropping the condition that all sequences have unit energy that was used to derive equations (2.2) and (2.3) in [19]. This relaxation is equivalent to the case where we allow certain power control strategy to change the radiated power of the users.

The only constraint on spreading sequences that we will use is the constraint on the total sequence energy

$$tr(\mathbf{S}\mathbf{S}^T) = K \tag{2.15}$$

which implies that the average sequence energy per user is equal to unity. Therefore the transmitted power of user i can be expressed as $P_i = P\mathbf{s}_i^T\mathbf{s}_i$, where the total radiated power that is constrained with equation (2.15) is equal to $P_{tot} = KP$. Note that matrix $\mathbf{W}$ is now the diagonal matrix with elements $Pg_1, \ldots, Pg_K$ and that power control law is solely contained in matrix $\mathbf{S}$. The following result, first proven by Witsenhausen in [58] and used later in [66] for the joint transmitter-receiver optimization for multiple input multiple output systems, will be applied to find optimal sequence and power allocation that maximizes sum capacity of (1.8). This result will be presented here without the proof, slightly simplified and adjusted to our notation.

Theorem 1 *Let $\boldsymbol{Z}$ be a Hermitian positive-definite matrix of dimension $K \times K$. The maximum of*

$$J = \det(\boldsymbol{I}_N + \boldsymbol{S}\boldsymbol{Z}\boldsymbol{S}^T) \tag{2.16}$$

over all $N \times K$ matrices $\boldsymbol{S}$ satisfying

$$tr(\boldsymbol{S}\boldsymbol{S}^T) \leq G \tag{2.17}$$

is achieved by

$$\boldsymbol{S} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T, \tag{2.18}$$

where (i) $\boldsymbol{V}$ is an $K \times K$ unitary matrix diagonalizing $\boldsymbol{Z}$, i.e.,

$$\boldsymbol{V}^T\boldsymbol{Z}\boldsymbol{V} = \boldsymbol{\Lambda}, \tag{2.19}$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix with ordered elements $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_s \geq \lambda_{s+1} = \cdots = \lambda_K = 0$, and $s = rank(\boldsymbol{Z})$ is the number of nonzero eigenvalues of $\boldsymbol{Z}$,

(ii) $\boldsymbol{U}$ is an $N \times N$ unitary matrix, and

(iii) $\boldsymbol{D}$ is a rectangular $N \times K$ matrix whose "off-diagonal" elements D_{ij} are zero, and the "diagonal" elements $d_i \equiv D_{ii}$ are calculated as

$$|d_i|^2 = \begin{cases} \frac{1}{r}\left(G + \sum_{j=1}^{r} \lambda_j^{-1}\right) - \lambda_i^{-1}, & i \leq r, \\ 0, & otherwise, \end{cases} \tag{2.20}$$

where $r \leq \min(s, N)$ is the largest integer satisfying

$$\lambda_r^{-1} \leq \frac{1}{r}\left(G + \sum_{j=1}^{r} \lambda_j^{-1}\right). \tag{2.21}$$

The maximum of (2.16) is given with

$$J_{max} = r^{-r}\left(G + \sum_{j=1}^{r} \lambda_j^{-1}\right)^r \prod_{j=1}^{r} \lambda_j. \tag{2.22}$$

To apply the previous theorem to the problem of maximizating of the sum capacity (1.8) we first note that matrix $\mathbf{Z}$ corresponds to $\mathbf{W}\frac{N}{\sigma^2}$ with rank $s = K$ and that constraint G corresponds to the number of users K. Also, ordered elements $\lambda_i = Ng_{[i]}\frac{P}{\sigma^2}$. Since matrix $\mathbf{Z}$ is a diagonal matrix, the matrix $\mathbf{V}$ is a permutation matrix which orders the elements of $\mathbf{Z}$.

To give more insight in the sequence allocation that maximizes the sum capacity predicted by the previous result we introduce the following notation. Denote the set of r strongest user indices as J. According to (2.18), (2.20) and (2.21) only users from the set J will be allocated orthogonal sequences from the unitary matrix $\mathbf{U}$ while all other users will be allocated zero energy sequences i.e. these users will not transmit for that distribution of received powers. Since

$$\mathbf{S}\mathbf{S}^T = \mathbf{V}\mathbf{D}^T\mathbf{U}^T\mathbf{U}\mathbf{D}\mathbf{V}^T = \mathbf{V}\mathbf{D}^T\mathbf{D}\mathbf{V}^T \tag{2.23}$$

is a $K \times K$ diagonal matrix with non-zero elements with indices from the set J, matrix $\mathbf{S}$ consists only of orthogonal sequences. Note that sequence matrix $\mathbf{S}$ is an $N \times K$ matrix whose columns are either equal to the zero vector or are scaled columns of a unitary matrix $\mathbf{U}$. The power allocation among the users, apart from the determination of parameter r, is now structurally similar to the so-called "water-filling" argument discussed in [12] for maximizing the capacity of transmission through parallel channels. According to (2.20) and (2.18) the power control policy that maximizes the sum capacity is

$$P_i = \frac{P\alpha}{SNR} \max\left(\nu - g_i^{-1}, 0\right) \tag{2.24}$$

where $\nu = \frac{1}{r}\left(N\mathrm{SNR} + \sum_{j=1}^{r} g_{[j]}^{-1}\right)$ and $r < \min(K, N)$ is the largest integer satisfying

$$g_{[r]}^{-1} \leq \frac{1}{r}\left(\mathrm{SNR}\; N + \sum_{j=1}^{r} g_{[j]}^{-1}\right). \tag{2.25}$$

Therefore only the $r \leq N$ strongest users will have non-zero transmitting power and only these users will be allocated orthogonal sequences. We will briefly point out an interesting parallel between optimal sequence allocation in the no-power controlled case and the power controlled case. In both cases, a fraction of $r \leq N$ of the strongest users plays a specific role. In the no power controlled case these users are assigned orthogonal sequences to decrease the interference to other users, while in the power controlled case these users are the only ones which contribute to the sum capacity of an optimized system.

The maximal value of sum capacity if both power and sequence optimization is allowed using (2.22) is given with

$$C_{sum}^{PC}(\mathbf{G}) = \sum_{j=1}^{r} \frac{1}{2N} \log_2\left(\frac{N\;\mathrm{SNR}\; g_{[j]}}{r} + \sum_{i=1}^{r} \frac{g_{[j]}}{r g_{[i]}}\right). \tag{2.26}$$

Next we derive a simple lower bound on the maximum sum capacity attainable by the sequence allocation procedure previously described. Starting from (2.26) we have

$$\begin{aligned} C_{sum}^{PC}(\mathbf{G}) &= \frac{1}{2N} \log_2\left[\prod_{j=1}^{r} g_{[j]}\left(\frac{N\;\mathrm{SNR}}{r} + \frac{1}{r}\sum_{i=1}^{r}\frac{1}{g_{[i]}}\right)\right] && (2.27)\\ &\geq \frac{r}{2N} \log_2\left(\frac{\mathrm{SNR}\frac{N}{r}}{\frac{1}{r}\sum_{i=1}^{r}\frac{1}{g_{[i]}}} + 1\right) && (2.28)\end{aligned}$$

$$\geq \frac{r}{2N} \log_2 \left(\text{SNR } g_{[r]} \frac{N}{r} + 1 \right) \tag{2.29}$$

where (2.28) follows from the fact that the geometric mean is always greater or equal to the harmonic mean. (2.29) follows from the fact that the harmonic mean $H = (\sum_{i=1}^{n} a_i^{-1})^{-1}$ of elements $a_1, \ldots, a_n$ is greater or equal to the minimal element $a_{[n]}$.

Following the ideas employed in the analysis of optimally allocated equal energy sequences of the previous Section, we present the asymptotic analysis of the sum capacity of optimally allocated sequences with power control. We use the same notation to denote the fraction of users λ that transmit with orthogonal sequences, i.e. for r users that are allowed to transmit with non-zero powers $\lambda = \frac{r}{K}$. Note that $\lambda \leq \min(1, \alpha^{-1})$. In the asymptotic case λ converges in probability to the maximal value that satisfies the following inequality

$$\max_{0 \leq \lambda \leq \min(1,\alpha^{-1})} \left[\frac{1}{F^{-1}(1-\lambda)} \leq \frac{\text{SNR}}{\alpha\lambda} + \frac{1}{\lambda} \int_{1-\lambda}^{1} \frac{du}{F^{-1}(u)} \right] \tag{2.30}$$

which follows from applying (2.6) on the left side and Lemma 1 on the right side of (2.25). Having calculated the fraction of orthogonal users with non-zero transmitting power, λ, we can determine the expression for the maximal attainable sum capacity in the asymptotic case

$$\lim_{K \to \infty} C_{max}^{PC} = \frac{\alpha}{2} \int_{1-\lambda}^{1} \log \left(F^{-1}(u) \frac{\text{SNR}}{\alpha\lambda} + \frac{F^{-1}(u)}{\lambda} \int_{1-\lambda}^{1} \frac{dx}{F^{-1}(x)} \right) du \tag{2.31}$$

which can be obtained by applying Lemma 1 twice on equation (2.26). We note that if

$$\alpha \geq \left(1 - F(\text{SNR}^{-1})\right)^{-1} \tag{2.32}$$

the maximum of inequality (2.30) is achieved for $\lambda = \alpha^{-1}$ since

$$\frac{1}{F^{-1}(1-\alpha^{-1})} \leq \text{SNR}. \tag{2.33}$$

This is illustrated in Figure 2.2 by the variation of the fraction of degrees of freedom that are assigned non-zero orthogonal sequences (i.e. $\lambda\alpha$) in terms of the load α for optimal sequence allocation with power control. As a consequence of this, it is interesting to note that for the large system model with α satisfying (2.32), the sum capacity of optimal power/sequence allocation can be lower bounded with

$$C_{sum}^{PC} \geq \frac{1}{2}\log_2\left(1 + \text{SNR}\ F^{-1}(1-\alpha^{-1})\right), \tag{2.34}$$

which follows by application of (2.29) using $r/N = \alpha\lambda = 1$. Furthermore, from (2.34) we can conclude that the sum capacity asymptotically increases without bounds for fading distributions with unbounded support. A similar fact was shown for asymptotic randomly chosen signature sequences in fading channels using optimal power control and optimal joint ML decoding of high complexity [49]. If single-user error correction codes are used, due to the orthogonality of user sequences, the optimum decision feedback detector for the optimized power/sequence allocation simplifies to a bank of conventional detectors followed by single user decoders. Therefore, using signal space partitioning of Section 2.3 with the intention of simplifying the complexity of the decoder is not necessary with optimal power/sequence allocation.

Dependence of the sum capacity (spectral efficiency) of optimal power/sequence allocation in terms of load α is presented in Figure 2.3. The optimal power control policy provides marginal improvement even for $\alpha < 1$ compared to the no power control case. To observe this we repeat in Figure 2.5 the same results for a smaller value of $E_b/N_0 = 3dB$ where this improvement is more visible. It is easily deduced from the sequence allocation procedure what price we have to pay to achieve this remarkable increase in spectral efficiency above the spectral efficiency in non-fading channels. At a given moment, only fraction λ of K users will transmit with orthogonal sequences and the rest are idle. However, since we assumed that fading is fast changing and ergodic, all users on average have the opportunity to transmit with the same average rates and average powers. This simple sequence allocation procedure can be regarded also as a CDMA slotted ALOHA system where user sequences and powers are assigned in a centralized coordinated manner. The approach of allocating powers and non-zero rates to only a fraction of strongest active users is already standardized in modern data-centric wireless systems like IS-856.

2.3 Group Orthogonal Sequence Allocation

We next discuss the *group orthogonal* sequence allocation based on [19] which is capable of approaching or attaining the sum capacity while simplifying the receiver structure. The idea of partitioning the signal space in orthogonal subspaces has been analyzed previously in different applications (see for example [67]), however we present novel results on the capacity loss that we have to pay in order to simplify the receiver through partitioning of the signal space.

We divide all K users into r subsets and embed K_i unit energy sequences of users from group i into the orthogonal subspace of dimension N_i, i.e. let K_i sequences of group i be arranged in matrix $\mathbf{S}_i^T = \mathbf{E}_i\mathbf{F}_i$ for a certain $N_i \times N_i$ dimensional matrix $\mathbf{F}_i$ and $\mathbf{E}_i$ matrix of basis vectors of i-th orthogonal subspace. Now the cross-correlation matrix of these sequences can be represented in the block diagonal form

$$\mathbf{R} = \mathrm{diag}\{\mathbf{R}_1, \ldots, \mathbf{R}_r\} \tag{2.35}$$

where $\mathbf{R}_i = \mathbf{S}_i\mathbf{S}_i^T = \mathbf{F}_i^T\mathbf{F}_i$.

The optimal multiuser detector of the data bit vector $\mathbf{b} = \{b_1, b_2, \ldots, b_K\}^T$ with exponential complexity in the number of users K simplifies to r optimum detectors whose overall complexity does not exceed the complexity which is exponential in the number of users in the largest group. This simplification can be in fact quite substantial since we can regard the group size as a design parameter that is always less than some specific threshold which denotes the maximum allowable computational complexity for the specific application.

Group orthogonal unit energy sequence allocation $\mathbf{S}_i^T = \mathbf{E}_i\mathbf{F}_i$, with K_i non-oversized users and $N_i = N/r$ degrees of freedom in group $i = 1, \ldots, r$ achieves the full capacity of CDMA system if and only if all the groups have the following property

$$\mathbf{F}_i\mathbf{W}_{(G_i)}\mathbf{F}_i^T = \frac{P_{tot}}{N}\mathbf{I}, \tag{2.36}$$

with total power in each group being equal to $P_{tot,i} = P_{tot}/r$ and where $\mathbf{W}_{(G_i)}$ denote the restriction of the diagonal matrix of user powers $\mathbf{W}$ to the set of indices G_i of the users in group i ($i = 1, \ldots, r$). The validation of this statement is that the "if" implication follows by substituting the sequence matrix of group orthogonal sequences

$\mathbf{S}^T = \left[\mathbf{S}_1^T, \ldots, \mathbf{S}_r^T\right]$ in (1.8) and the "only if" directly from [19]. A similar conclusion was derived in [60] for unit energy sequences with equal received user powers.

In practice, for an arbitrary distribution of user powers, it is not always possible to divide users in groups with exactly equal total power per group. The following equation quantifies the exact loss in sum capacity for the case of imperfect distribution of powers per group. Let the K_i group orthogonal unit energy sequences $\mathbf{S}_i^T = \mathbf{E}_i\mathbf{F}_i$ of group i satisfy the property

$$\mathbf{F}_i\mathbf{W}_{(G_i)}\mathbf{F}_i^T = \frac{P_{tot}(1+x_i)}{N}\mathbf{I}, \tag{2.37}$$

where x_i is a relative mismatch from the even group power distribution, the sum of the user powers in group i is $P_{tot,i} = P_{tot}(1+x_i)/r$, $\sum_{i=1}^r x_i = 0$.

By substituting the property (2.37) in (1.8), we can derive the expression for the sum capacity of the group orthogonal sequence allocation with group power mismatch

$$\begin{aligned} C_{sum} &= \frac{1}{2N}\log_2\left\{\det\left[\mathbf{I} + \frac{P_{tot}}{\sigma^2}\left((1+x_1)\mathbf{E}_1\mathbf{E}_1^T + \ldots + (1+x_r)\mathbf{E}_r\mathbf{E}_r^T\right)\right]\right\} \\ &= \frac{1}{2N}\log_2\left\{\det\left[\left(\mathbf{I} + \frac{P_{tot}}{\sigma^2}\left((1+x_1)\mathbf{E}_1\mathbf{E}_1^T + \ldots + (1+x_r)\mathbf{E}_r\mathbf{E}_r^T\right)\mathbf{E}\mathbf{E}^T\right)\right]\right\} \\ &= \frac{1}{2N}\log_2\left\{\det\left[\left(\mathbf{I} + \frac{P_{tot}}{\sigma^2}\left((1+x_1)\mathbf{E}^T\mathbf{E}_1\mathbf{E}_1^T\mathbf{E} + \ldots + (1+x_r)\mathbf{E}^T\mathbf{E}_r\mathbf{E}_r^T\mathbf{E}\right)\right)\right]\right\}, \end{aligned} \tag{2.38}$$

where we have used the property of determinants, $\det(\mathbf{I}+\mathbf{AB}) = \det(\mathbf{I}+\mathbf{BA})$ for square matrices $\mathbf{A}$ and $\mathbf{B}$. Now, the equation for the sum capacity of unbalanced group division (2.39) follows from the orthonormality of the matrix $\mathbf{E}$.

$$C_{sum} = \sum_{i=1}^{r} \frac{1}{2r}\log_2\left(1 + \mathrm{SNR}(1+x_i)\right). \tag{2.39}$$

The next Proposition gives the lower bound on the sum capacity with unbalanced group division.

Proposition 1 *In the case of equal number of degrees of freedom per group, and signature sequences satisfying (2.37) and $-a < x_i < a, i = 1, \ldots, r, a > 0$, the sum capacity can be lower bounded as*

$$C_{sum} \geq \frac{r-1}{4r}\log_2\left((1+SNR)^2 + a^2SNR^2\right) + \frac{1}{2r}\log_2\left(1+SNR\right). \tag{2.40}$$

Proof:

The lower bound on the sum capacity can be derived by using the theory of majorization [62]. The following definitions will be used in the course of the proof.

Definition 3 *For any $\boldsymbol{x}, \boldsymbol{y} \in \Re^n$, it is said that $\boldsymbol{y}$ majorizes $\boldsymbol{x}$, or $\boldsymbol{x} \prec \boldsymbol{y}$ if*

$$\begin{aligned} \sum_{i=1}^{k} x_{[i]} &\leq \sum_{i=1}^{k} y_{[i]},\ k = 1, \dots, n-1, \\ \sum_{i=1}^{n} x_{[i]} &= \sum_{i=1}^{n} y_{[i]}. \end{aligned} \tag{2.41}$$

As an illustration of the previous definition we state the following simple fact that can be derived for any vector $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$ for which $x_i \geq 0$ and $\sum x_i = a$

$$\{\frac{a}{n}, \dots, \frac{a}{n}\} \prec \{x_1, x_2, \dots, x_n\} \prec \{a, \dots, 0\}. \tag{2.42}$$

We will also need the following definition:

Definition 4 *A real-valued function $\phi : \Re^n \rightarrow \Re$ is said to be Schur-concave if for all $\boldsymbol{x}, \boldsymbol{y} \in \Re^n$ such that $\boldsymbol{y}$ majorizes $\boldsymbol{x}$ we have $\phi(\boldsymbol{x}) \geq \phi(\boldsymbol{y})$.*

The following fact [62] states an important class of Schur concave functions.

Lemma 2 *For any concave function $g : \Re \rightarrow \Re$ the symmetric concave function $\phi(\boldsymbol{x}) = \sum_{i=1}^{n} g(x_i)$ is Schur concave.*

Note that the expression (2.39) for sum capacity $C_{sum}(\mathbf{x})$ is Schur concave in terms of the vector $\mathbf{x} = \{x_1, \dots, x_r\}$ of relative power mismatches. This fact can be concluded applying Lemma 2 and noting that $C_{sum}(\mathbf{x})$ is a sum of concave functions $\log_2(1 + \frac{P_{tot}}{\sigma^2}(1 + x_i))$ of the components of vector $\mathbf{x}$.

Using the fact that the components of the vector $\mathbf{x}$ are bounded, there exists a vector $\mathbf{v}$ such that $\mathbf{v} \succ \mathbf{x}$ for all vectors $\mathbf{x}$ that satisfy the constraints, i.e. $-a < x_i < a, a > 0$ for $i = 1, \dots, r$. The following vectors represent extremal vectors under these conditions (see C.1., Chapter 5., [62])

$$v = \{\underbrace{-a, -a, \dots, -a}_{r/2}, \underbrace{a, a, \dots, a}_{r/2}\} \tag{2.43}$$

for even r and

$$v = \{\underbrace{-a, -a, \dots, -a}_{(r-1)/2}, 0, \underbrace{a, a, \dots, a}_{(r-1)/2}\} \tag{2.44}$$

for odd r. Now substituting the extremal vector for odd values of r (2.44) (which presents the worse case) and applying the Definition 3. of the Schur concave functions we can derive the inequality (2.40). $\square$

The loss in spectral efficiency for non-oversized users predicted by Proposition 1 is illustrated in Figure 2.4. We can note that this loss is very small even for higher values of a maximum relative power mismatch (unbalance) between the groups.

In the asymptotic case of a large system model we can always divide users in r groups with equal total power. To demonstrate this we start from noting that for the continuous pdf distribution $f(x)$ of channel gains we can find $r+1$ real numbers $u_j, j=0,\ldots,r$ such that $0=u_0<u_1<\cdots<u_r=1$ and

$$\int_{u_{i-1}}^{u_i} F^{-1}(u)du = \frac{1}{r}, \quad i=1,\ldots,r. \tag{2.45}$$

Now if we assign users with channel gains $F^{-1}(u_{i-1}) < g < F^{-1}(u_i)$ to group i, total power of each group will converge in probability to P_{tot}/r.

For completeness, it should be mentioned that in the case that some of the users of the CDMA system are oversized [19], the previous sequence allocation procedure has to be modified such that oversized users are allocated separate orthogonal degrees of freedom while the remaining users are divided into group orthogonal sequences. We note that, from the observation 1) in [19], it easily follows that there are no oversized users in a group with total power $P_{tot,i} = P_{tot}/r$ and $N_i = N/r$ degrees of freedom if no users are oversized in the complete set of users.

2.4 Conclusions

The common motif that links the results presented in this chapter is the problem of optimal resource allocation in DS/CDMA systems. To obtain mathematically tractable results that are comparable with the previous analysis of random sequences, we employed the large system model and asymptotic properties of order statistics. We will now summarize results presented in this chapter.

Asymptotical analysis of the sum capacity of optimally allocated spreading sequences without power control is presented and the gains in the sum capacity compared to the random sequence allocation in flat fading channels are discussed.

The optimal power/sequence allocation is derived in Section 2.2.2 in both non-asymptotical and asymptotical flat fading regimes. The maximization of the sum capacity in the oversaturated CDMA systems for fading channels led us to disregard the delay constraints (QoS requirements) of specific users [68]. In the delay-sensitive applications, the symmetric capacity of [18] or the delay-limited capacities of the system [20], [29] are more appropriate information-theoretic measures.

The benefits of reducing the complexity of the joint optimal decoder through allocation of sequences in orthogonal subspaces is discussed in Section 2.3 and the loss in sum capacity as a consequence of unbalanced groups is upper bounded. It is shown that asymptoticaly this loss vanishes.

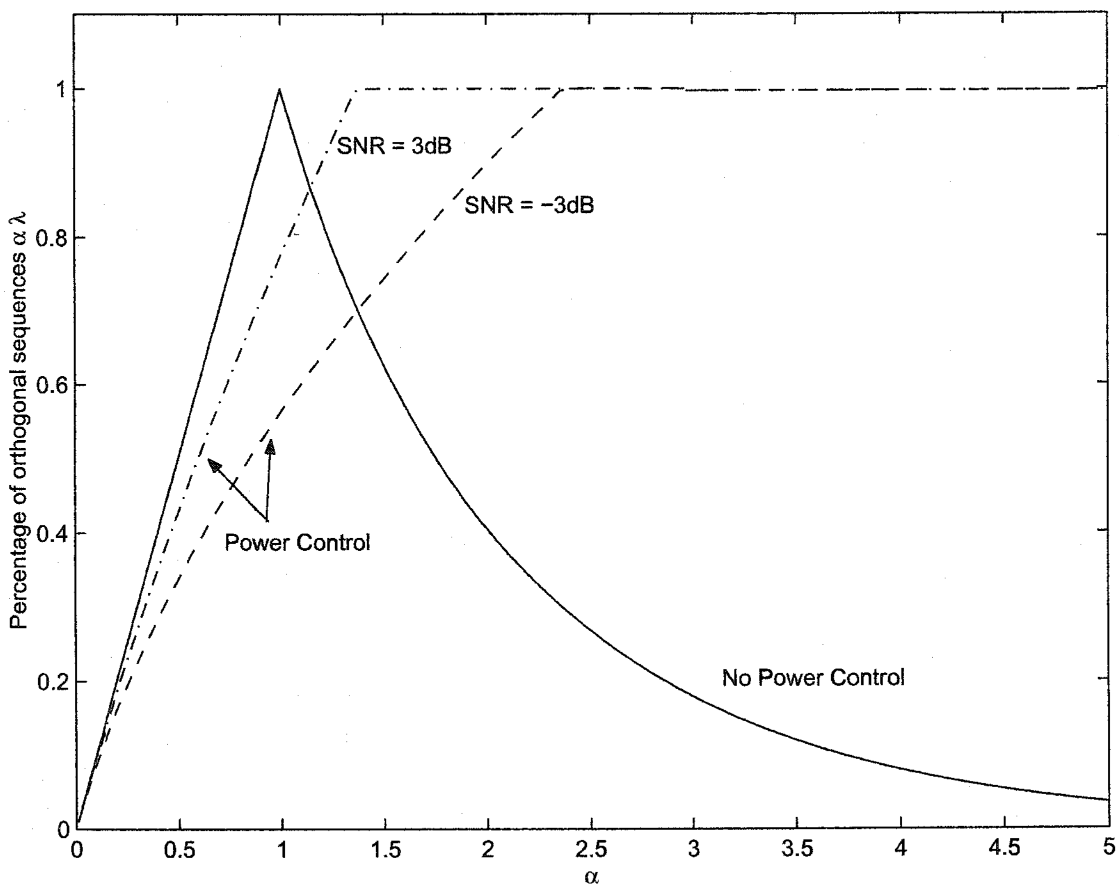

Figure 2.2. *Variation of the share of orthogonal sequences in optimal sequence allocation with no power control and power control for two values of SNR in a Rayleigh fading channel. For the no power control case the share of orthogonal users $\alpha\lambda$ rises linearly with load α for $\alpha < 1$ and monotonically decreases to 0 for $\alpha > 1$. For the power control case, the share of orthogonal users increases with α and saturates to the value of 1 when all degrees of freedom are allocated orthogonal sequences. According to the notation of Section 2.2, $\lambda\alpha = \frac{\kappa}{N}$ for the no power control case and $\lambda\alpha = \frac{\tau}{N}$ for the power control case.*

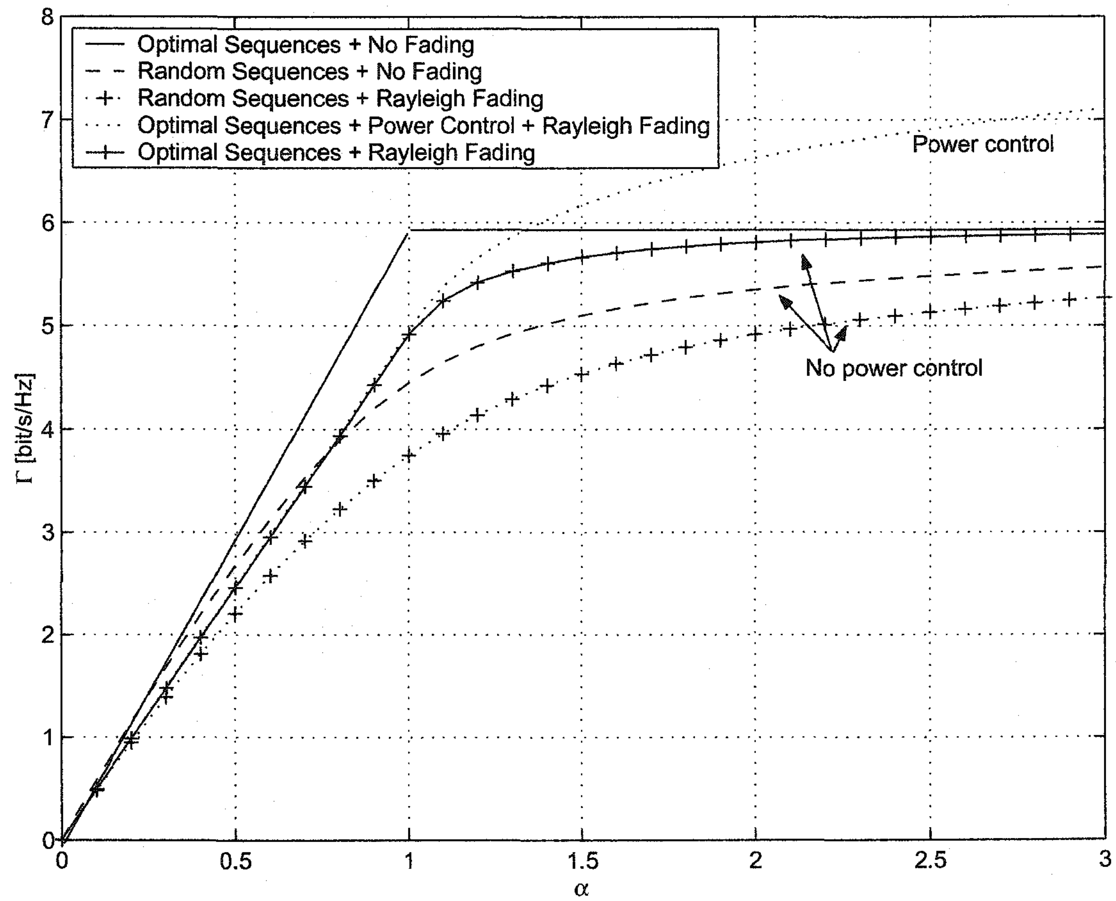

Figure 2.3. *Comparison of the spectral efficiency of the optimal sequence allocation with and without power control in Rayleigh fading for* $E_b/N_0 = 10dB$. *As a reference, we give spectral efficiencies of the optimal detector for random sequences in channels with no fading and channels with Rayleigh fading.*

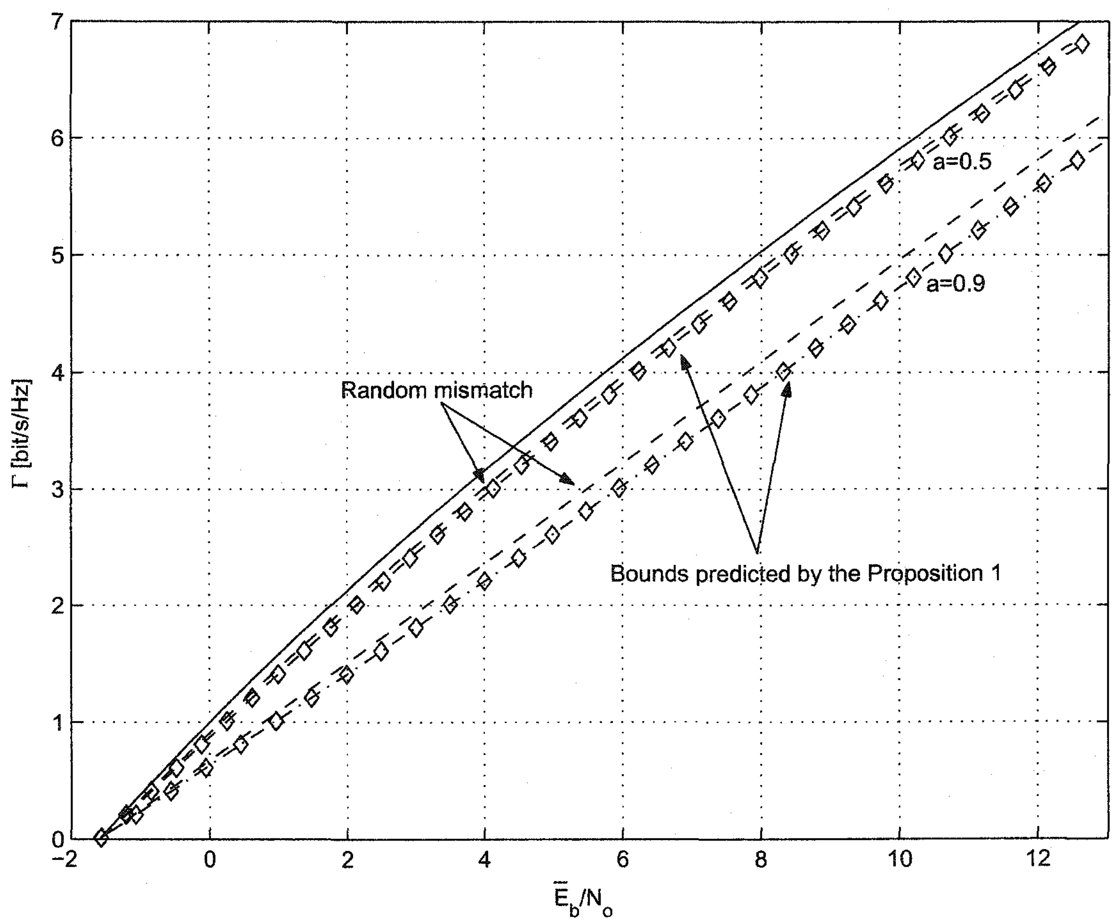

Figure 2.4. *Sum capacity in terms of E_b/N_0 as predicted by Proposition 1. for two maximum relative power imbalances ($a = 0.5$ and $a = 0.9$) between groups. Lines with $\diamond$ sign are lower bounds for certain parameter a, while dashed lines present sample sum capacities of (2.40) for randomly chosen power imbalances satisfying maximum relative power imbalances of $a = 0.5$ and $a = 0.9$. For comparison, the sum capacity of a non-fading channel is presented with full line.*

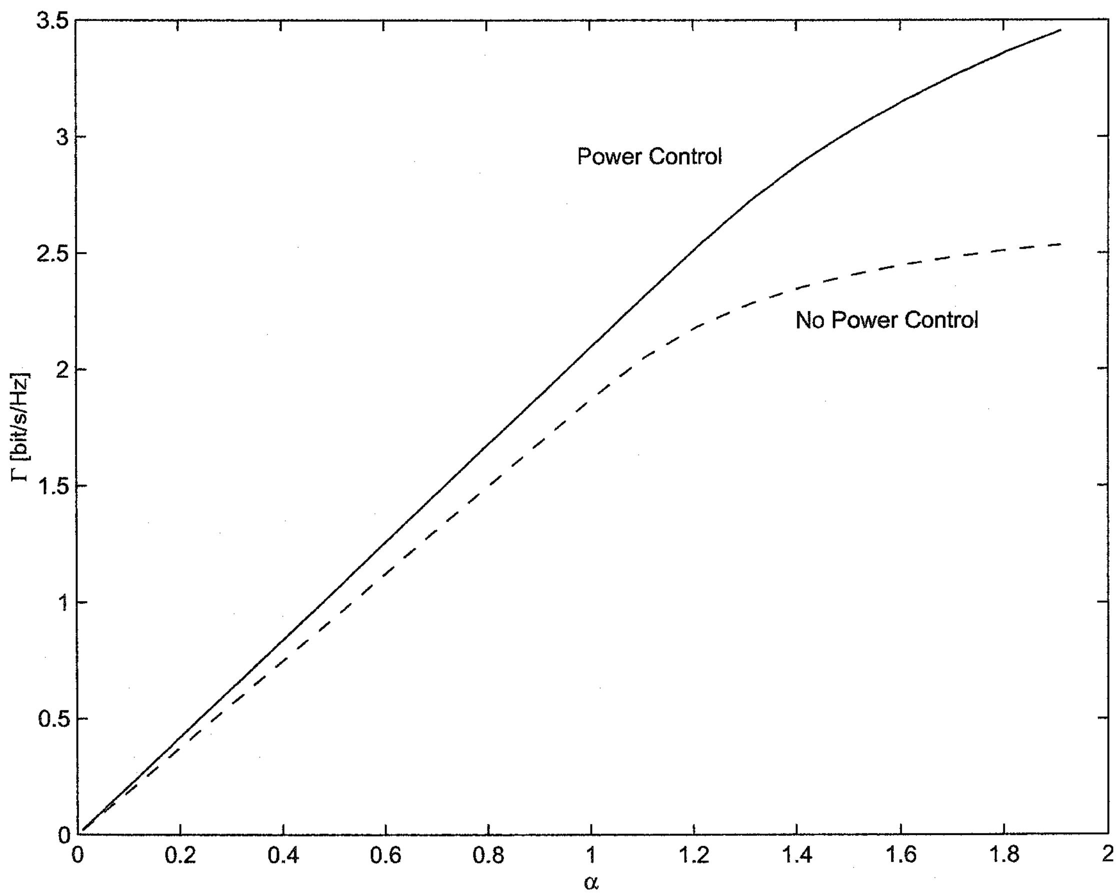

Figure 2.5. *Comparison of the spectral efficiency of the optimal sequence allocation with and without power control in Rayleigh fading for* $E_b/N_0 = 3dB$.

Chapter 3

Cholesky Characterization of the Multiple-Access Capacity Region

3.1 Introduction

Several previous articles have dealt with the issue of information theoretic optimization of the signature sequence allocation [19], [32], [60], [18] and [61]. The main idea behind all these papers is the maximization of the sum capacity (1.8) of the synchronous CDMA Gaussian multiple-access channel. As opposed to the sum capacity maximization, little has been done regarding the maximization of the whole capacity region $\mathcal{C}(\mathbf{P}, \mathbf{G})$ with respect to the signature sequence allocation. This problem can also be regarded as the maximization of the sum rate of users under QoS constraint on the rates of all users. In this setting, for every QoS constraint on user rates, a different set of signature sequences maximizes the sum capacity.

However, some results on this topic exist. In [17], the issue of symmetric capacity (equal QoS constraints on all users) has been analyzed. In this paper, upper and lower bounds on the symmetric capacity with optimal sequence allocation and equal user powers have been derived. More recently, in [34] T.Guess has analyzed the optimal sequence allocation for MMSE decision feedback receiver with different preset QoS on user rates. This paper used a constraint on total user power and allowed for power control. By using Cholesky factorization and the theory of majorizations, the author was able to explicitly find the signature sequence set that maximized the sum capacity under QoS constraints.

The material in this Chapter follows up on this article and shows that Cholesky factorization is a useful tool for characterization of the multiple-access capacity region.

Using this Cholesky characterization of the multiple-access capacity region, an algorithm that finds optimal sequences for equal QoS constraints and different received powers is proposed. This algorithm is intended for flat fading channels where power control is available at the receiver, but users have to satisfy equal rate constraints. In order to compare the numerical results of this algorithm, a previously derived upper bound on symmetric capacity [17] is employed.

The case of randomly assigned signature sequences in a large system model is also addressed in this Chapter. The capacity profile of users in a flat fading channel is evaluated for the case when optimal MMSE decision feedback receiver detects users in the decreasing and increasing order of powers. These results are also related to previous results on the sum capacity of the optimal multiuser receiver in flat fading channels.

3.2 Cholesky Characterization of the Multiple-Access Capacity Region

The following definition of a submatrix of a matrix is used in this Chapter.

Definition 5 *For any square matrix T, we let T_S be the submatrix formed by retaining only rows i, j for all $i, j \in S$.*

The following definition of the permuted matrix is also used in this Chapter.

Definition 6 *Let π be a permutation of the elements of the initial ordered set $\pi_0 = \{1, \ldots, K\}$ and let us denote with π^{-1} respective reverse order permutation. Also, we denote with $\boldsymbol{V}_{(\pi)}$ the matrix that is produced by permuting the rows of a $K \times N$ dimensional matrix $\boldsymbol{V}$ according to the permutation π.*

For simplicity of notation and unless stated otherwise, in this chapter we assume that users are arranged in non-increasing order i.e. that $W_1 \geq W_2 \geq \ldots W_K$. The next Proposition gives a simplified characterization of the multiple-access rate region using Cholesky factorization.

Proposition 2 *Let π be the permutation which characterizes one of the $K!$ vertices of CDMA multiple-access rate region. The single user capacity of user $\pi(i)$ is for that* *vertex given by*

$$C_{\pi(i)} = \frac{1}{2N}\log(c_i) \tag{3.1}$$

where $\{c_1, c_2, \ldots, c_K\}$ *are Cholesky factors of the matrix* $U_{(\pi)} = \frac{1}{\sigma^2} W_{(\pi)}^{1/2} S_{(\pi)} S_{(\pi)}^T W_{(\pi)}^{1/2} + I$*. Furthermore, these code rates are attainable with an optimal decision feedback receiver whose cancellation order is* π^{-1}*.*

Proof

We know from [4] that achievable code rate of user l with the optimal decision feedback receiver where the cancellation order is set by the permutation π^{-1} is given by

$$C_{\pi(l)} = \begin{cases} \frac{1}{2N}\log\left|\mathbf{I}_N + \frac{1}{\sigma^2}p_{\pi(l)}\mathbf{s}_{\pi(l)}\mathbf{s}_{\pi(l)}^T\right| & \text{for } l = 1 \\ \frac{1}{2N}\log\left|\mathbf{I}_N + \sum_{i=1}^{l}\frac{1}{\sigma^2}p_{\pi(i)}\mathbf{s}_{\pi(i)}\mathbf{s}_{\pi(i)}^T\right| & \\ -\frac{1}{2N}\log\left|\mathbf{I}_N + \sum_{i=1}^{l-1}\frac{1}{\sigma^2}p_{\pi(i)}\mathbf{s}_{\pi(i)}\mathbf{s}_{\pi(i)}^T\right| & \text{for } l = 2, \ldots, K. \end{cases} \tag{3.2}$$

This also follows from the polymatroid structure of the capacity region [29] and these rates also define vertices of the CDMA multiple-access rate region. We can conveniently rewrite this equation using matrix notation as follows

$$C_{\pi(l)} = \begin{cases} \frac{1}{2N}\log\left|\mathbf{U}_{J_1}\right| & \text{for } l = 1 \\ \frac{1}{2N}\log\left|\mathbf{U}_{J_l}\right| - \frac{1}{2}\log\left|\mathbf{U}_{J_{l-1}}\right| & \text{for } l = 2, \ldots, K. \end{cases} \tag{3.3}$$

where we use $|\mathbf{I} + \mathbf{AB}| = |\mathbf{I} + \mathbf{BA}|$. Telescoping sets J_l are defined as $J_l = \{1, \ldots, l\}$ for $l = 1, \ldots, K$, while matrix $\mathbf{U} = \frac{1}{\sigma^2}\mathbf{W}_{(\pi)}^{1/2}\mathbf{S}_{(\pi)}\mathbf{S}_{(\pi)}^T\mathbf{W}_{(\pi)}^{1/2} + \mathbf{I}$.

Since matrix $\mathbf{U}$ is a positive definite matrix, it admits to be factorized in the Cholesky factorization

$$\mathbf{U} = \mathbf{HCH}^T \tag{3.4}$$

for a lower diagonal matrix $\mathbf{H}$ with unit diagonal elements and diagonal matrix $\mathbf{C} = \text{diag}\{c_1, \ldots, c_K\}$ whose diagonal elements we call Cholesky factors [69], [70]. The Cholesky factorization of $\mathbf{U}_{J_l}$ can be expressed in terms of the Cholesky factorization of $\mathbf{U}$

$$\mathbf{U}_{J_l} = \mathbf{H}_{J_l}\mathbf{C}_{J_l}\mathbf{H}_{J_l}^T \tag{3.5}$$

The determinant $|\mathbf{U}_{J_l}|$ can conveniently be expressed by using its Cholesky factors as

$$|\mathbf{U}_{J_l}| = \prod_{i=1}^{l} c_i. \tag{3.6}$$

This follows from the fact that $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$ for square matrices $\mathbf{A}$ and $\mathbf{B}$. We also used the fact that the determinant of a lower diagonal matrix is equal to the product of its diagonal elements. Now, equation (3.1) in the statement of the proposition easily follows from (3.3) and (3.6).

The last statement of this Proposition follows simply from the fact that the whole multiple-access capacity region can be achieved with an optimal decision feedback receiver [4]. □

This Proposition will be used in order to formulate an optimization algorithm for the optimal allocation of signature sequences with symmetric capacity. Also, some results from the random signature sequence allocation can be reinterpreted using this result.

3.3 Optimal Sequences and Symmetric Capacity

In this section, we address the issue of finding the optimal sequence allocation in flat fading channels. We assume that all users have equal transmit power constraints ($\bar{\mathbf{P}} = \mathrm{diag}[\bar{P}_1, \ldots, \bar{P}_1]$) and that fading gains of different users are not necessarily equal. The fading gains are assumed constant or slow varying so that the optimization of the spreading sequences is feasible. The same setting has been analyzed previously in [17].

The symmetric capacity is defined as

$$C_{sym} = \max_{\{R, \ldots, R\} \in \mathcal{C}(\mathbf{P}, \mathbf{G})} R \tag{3.7}$$

or equivalently

$$C_{sym} = \min_{S \subseteq \mathcal{U}} \frac{1}{2|S|N} \log \left| \mathbf{I}_S + \frac{N}{\sigma^2} (\mathbf{W}^{1/2} \mathbf{S} \mathbf{S}^T \mathbf{W}^{1/2})_S \right| \tag{3.8}$$

where $\mathcal{U}$ is the set of indices of all users.

We begin with a remark on some known results [17] on optimal sequence allocation for symmetric capacity and then we connect these results with newer results on optimal sequence allocation for sum capacity [19].

Remark 1 *The signature sequences set, predicted in Theorem 4 of [17], that maximizes a lower bound on symmetric capacity is equivalent to the signature sequences set that maximizes the sum capacity of Theorem 3.1 in [19].*

In order to confirm this remark, we first note that the parameter $\tilde{n}$ defined in [17] as (adjusted to our own notation for received powers)

$$\tilde{n} = \arg \min_{n \in \{K-N+1,\ldots,K\}} \frac{1}{n-K+N} \sum_{i=1}^{n} W_{[K+1-i]} \tag{3.9}$$

is equal to $K - |\mathcal{K}|$, where $|\mathcal{K}|$ is the number of oversized users among K users defined as (cf. (2.2))

$$|\mathcal{K}| = \max_{l} \left\{ l \in \{1, \ldots, N-1\} : W_{[l]} > \frac{\sum_{j=l+1}^{K} W_{[j]}}{N-l} \right\} \tag{3.10}$$

in [19]. In the notation of Chapter 2, $|\mathcal{K}|$ is equivalent to κ. Now we can rewrite the equation (3.9) as

$$\begin{aligned} \tilde{n} &= \min_{n} \{ n \in \{K-N+1, \ldots, K\} : \\ &\quad \frac{1}{n-K+N} \sum_{i=1}^{n} W_{[K+1-i]} < \frac{1}{n+1-K+N} \sum_{i=1}^{n+1} W_{[K+1-i]} \right\} \end{aligned} \tag{3.11}$$

which is equivalent to

$$K - \tilde{n} = \max_{l} \left\{ l \in \{1, \ldots, N-1\} : \frac{1}{l-N} \sum_{j=l+1}^{K} W_{[j]} > \frac{1}{N+1-l} \sum_{j=l}^{K} W_{[j]} \right\} \tag{3.12}$$

that is equivalent to (3.10).

To confirm that sequences in both papers are equivalent we note that sequences in both cases are chosen such that rank N matrix $\mathbf{S}^T\mathbf{W}\mathbf{S}$ has the following eigenvalues

$$\lambda = \left\{ \frac{\sum_{j=K-|\mathcal{K}|}^{K} W_{[j]}}{N-|\mathcal{K}|}, i \in \{|\mathcal{K}|+1, \ldots, K\}; i \in \{1, \ldots, |\mathcal{K}|\}, W_{[i]} \right\}. \tag{3.13}$$

Since both sequence constructions use equal unit energy sequences, the diagonal elements of $\mathbf{S}^T\mathbf{W}\mathbf{S}$ are equal in both cases. In turn, that means that $\mathbf{S}^T\mathbf{W}\mathbf{S}$ is uniquely constructed with the same eigenvalues and diagonal elements [62] and that signature sequences sets in both cases are equal.

Let the $C_{sum}(\mathbf{w})$ denote the sum capacity of the of the optimally allocated signature sequence set and received power constraints $\mathbf{w} = [W_1, W_2, \ldots, W_K]$ derived in Theorem 3.1 of [19]. Now we state a more insightful reinterpretation of the Theorem 2 in [17] which upper bounds the symmetric capacity of a CDMA system.

Proposition 3 *The symmetric capacity of the CDMA multiple-access AWGN channel with processing gain N is upper bounded by*

$$C_{sym} \leq \min_{l=1,\ldots,K} \frac{1}{l} C_{sum}\left([W_{[K]}, W_{[K-1]}, \ldots, W_{[K-l+1]}]\right) \tag{3.14}$$

Proof:

The proof follows the same ideas as [17]

$$C_{sym} \leq \min_{S \subseteq \mathcal{U}} C_{sum}(\mathbf{p}_S) \leq \min_{l} C_{sum}([W_{[K]}, W_{[K-1]}, \ldots, W_{[K-l+1]}]) \tag{3.15}$$

where we use that C_{sum} is increasing function of its arguments i.e. $C_{sum}(\mathbf{W}_1) > C_{sum}(\mathbf{W}_2)$, where $\mathbf{W}_1 > \mathbf{W}_2$ element-wise. □

In the next part of this section, an optimization algorithm designed to find sequences that maximize the symmetric capacity is proposed. The algorithm is inspired by the maximization of the vertices of the multiple-access vector channel capacity region performed in [71]. The illustration of this approach is given in Figure 3.1. In order to perform this optimization, the permutation which defines a vertex of the capacity region is preset and the sequences are chosen in order to maximize the symmetric capacity. This permutation corresponds to the detection order of the optimum decision feedback receiver. The most plausible choice for the detection order is in the order of decreasing received powers. According to the assumption that the received powers are ordered in the order of indices, this permutation is π_0. No formal proof that validates this choice of ordering is available at this moment, however simulation results show that the symmetric capacity of the resulting sequences is almost always equal to the upper bound on symmetric capacity of [17] and Proposition 3.

Formally, the task is to find the $K \times N$ sequence matrix $\mathbf{S}$ that satisfies

$$\mathbf{S} = \arg \max_{\substack{\mathbf{S} \\ c = c_1 = c_2 = \cdots = c_K, \mathbf{s}_i \mathbf{s}_i^T \leq 1}} c \tag{3.16}$$

where $\{c_1, c_2, \ldots, c_K\}$ are Cholesky factors of the matrix $\mathbf{U} = \frac{1}{\sigma^2}\mathbf{W}_{(\pi_0)}^{1/2}\mathbf{S}\mathbf{S}^T\mathbf{W}_{(\pi_0)}^{1/2} + \mathbf{I}$. Again, as in Chapter 2, the variation of the transmitted user powers are contained in the signature sequence allocation matrix $\mathbf{S}$.

This problem is structurally similar to the problem posed in [34] where theory of majorizations and the construction of the matrix with given eigenvalues and Cholesky factors has been used. However, in this case the problem involves the construction of a matrix with given diagonal elements, Cholesky factors and eigenvalues, the case which has not yet been solved in the mathematical literature. Therefore, we have to resort to numerical optimization.

To perform this constrained optimization problem, the Sequential Quadratic Programming (SQP) algorithm has been employed [72]. The optimization has been performed for several different system loads α and processing gains N. 100 trials have been performed, each for a different randomly chosen set of fading gains. The independent identically distributed fading gains are chosen according to the Rayleigh fading distribution. It was observed that the proposed algorithm almost always converges to the upper bound on symmetric capacity derived in [17]. On average, only one in 100 trials showed small loss compared to the upper bound of Proposition 3.

3.4 Random Spreading Sequences

In this section we address some aspects of finding the capacity region of a randomly spread CDMA system. The analysis is asymptotical in the number of users, i.e. we consider the case when the number of users K and the processing gain of the system are increasing while their ratio remains fixed.

In the case considered in our model, when the number of users increases to infinity we have to adjust our way of representing user rates. Let us consider the l-th user and assume that $\frac{l}{K} \to u \in [0, 1]$ as $K \to \infty$. We denote by $C(u)$ the limiting capacity profile of the users with $NC_l \to C(u)$. The coefficient N is introduced to provide scaling such that the capacity profile is equal to the single user capacity in units [bit/symbol]. Furthermore, in order to determine this rate profile we need to know the limiting profile of Cholesky factors $c_\pi(u)$ of the matrix $\mathbf{U} = \frac{1}{\sigma^2}\mathbf{W}_{(\pi^{-1})}^{1/2}\mathbf{S}\mathbf{S}^T\mathbf{W}_{(\pi^{-1})}^{1/2} + \mathbf{I}$ for certain permutation π and random matrix $\mathbf{S}$. According to the best knowledge of the author, this problem has not yet been addressed in this form. However, as will be shown below, the solution can be fairly easily given using some known results on the limiting eigenvalues distribution [39] and [42] and asymptotic order statistics [59].

We note that the rate of (3.1) can be alternatively expressed using the effective SNR ratio at the output of the optimal decision feedback receiver [4]

$$C_l = \frac{1}{2}\log\left(1 + \mathrm{SNR}g_l\eta_l\right) \tag{3.17}$$

where η_i is the multiuser efficiency of user i [49]. η_i is a solution of the equation

$$\eta + \alpha \mathrm{E}\left[\frac{\mathrm{SNR}g\eta}{1+\mathrm{SNR}g\eta}\right] = 1 \tag{3.18}$$

where the expectation operator is taken with respect to the fading gain distribution of the users remaining after cancellation. From these equations, we see that the single user capacity of user l depends on its fading gain as well as fading gain distribution of remaining users. Similarly, according to Proposition 3 the Cholesky factors of the matrix $\mathbf{U}$ in this asymptotic case can be expressed as $c_i = 1 + \mathrm{SNR}g_l\eta_l$.

We first give the capacity profile for the vertex that corresponds to the permutation π_0, i.e. in the case when users in a optimal DF cancellation scheme are canceled from the strongest to the weakest.

The following lemma is needed in the course of the proof of Proposition 4. It follows simply from the properties of the determinants and is given without the proof.

Lemma 3 *Let π_1 and π_2 be two distinct permutations of the set $\{1,\dots,K\}$, $\boldsymbol{S}$ a certain $K\times N$ dimensional matrix and $\boldsymbol{P}$ a diagonal $K\times K$ dimensional matrix. Then the matrices $\boldsymbol{P}_{\pi_1}^{1/2}\boldsymbol{S}_{\pi_1}\boldsymbol{S}_{\pi_1}^T\boldsymbol{P}_{\pi_1}^{1/2}+\boldsymbol{I}$ and $\boldsymbol{P}_{\pi_2}^{1/2}\boldsymbol{S}_{\pi_2}\boldsymbol{S}_{\pi_2}^T\boldsymbol{P}_{\pi_2}^{1/2}+\boldsymbol{I}$ have equal eigenvalues.*

In the derivation of the following proposition which gives the average single user capacity of the CDFR detector with power ordering, we make use of the following definitions:

Definition 7 *Let the population quantile of order u, ξ_u, corresponding to the random variable of strictly increasing cdf $F(x)$ be defined as a unique solution of the equation*

$$F(\xi_u) = u. \tag{3.19}$$

Since function F is invertible, $\xi_u = F^{-1}(u)$.

Note that according to this definition, $\xi_{1/2}$ is the median of the distribution.

Proposition 4 *Let's fix α and SNR, as well as the decoding order of the users in an optimal decision feedback receiver from the strongest to the weakest. Then the capacity profile of a user, after a fraction of u users is canceled, converges in probability to*

$$\begin{aligned} C(u,\alpha,SNR,\pi_0) &= \frac{1}{2}\log(c_{\pi_0}(u)) \\ &= \frac{1}{2}\log\left(1+SNRF^{-1}(u)\eta_{\pi_0}(\alpha,SNR,u)\right) \end{aligned} \tag{3.20}$$

as $K\to\infty$. The multiuser efficiency $\eta_{\pi_0}(\alpha,SNR,u)$ is given as the unique solution of

$$\eta+\alpha\int_0^u \frac{SNR\eta F^{-1}(t)}{1+SNR\eta F^{-1}(t)}dt=1. \tag{3.21}$$

The capacity profile $C(u,\alpha,SNR,\pi_0)$ corresponds to the vertex of the capacity region for permutation π_0.

Proof: We consider user l and assume that $\frac{l}{K}\to u\in[0,1]$ in the asymptotic case as $K\to\infty$. Then $C_l(\alpha,\mathrm{SNR})\to C(u,\alpha,\mathrm{SNR},\pi_0)$ while $C(u,\alpha,\mathrm{SNR},\pi_0)$ can be expressed as in (3.17). It is important to note that the ordering of the users is no obstacle in applying the results of [39]. According to Lemma 3, rearranging of the fading gains does not alter the eigenvalues of the $\mathbf{W}^{1/2}\mathbf{S}\ \mathbf{S}^T\mathbf{W}^{1/2}$ and the limiting eigenvalue results of [42] are readily applicable.

It is known that with the increase of the number of samples (in this case users) the distribution becomes a degenerate distribution, which means that it can take only values 0 and 1. More formally (cf. Chapter 2), this means that for $0\le u\le 1$, as $K\to\infty$, the samples of the distribution $f_{[\lfloor K(1-u)\rfloor]}(g)$ converge in probability to the quantile ξ_u of the distribution f. That is, for each $\epsilon>0$

$$\mathrm{Prob}\left(|g_{[\lfloor K(1-u)\rfloor]}-\xi_u|>\epsilon\right)\to 0, \tag{3.22}$$

where the quantile of order u, ξ_u, corresponds to the random variable of strictly increasing cdf $F(x)$.

In order to evaluate the multiuser efficiency of $\eta_{\pi_0}(\alpha,\mathrm{SNR},u)$ we have to determine the empirical distribution function of fading gains of the residual interferers after the fraction of $1-u$ strongest users have been canceled. This distribution converges weakly to

$$f_u(g) = \frac{f(g)(1-h(g-\xi_u))}{u} \tag{3.23}$$

as $K \to \infty$ and where function $h(x)$ is a unit step function. The previous equation follows from equations (3.22) and (2.5). Coefficient u appears in the previous equation in order to scale the probability density function. By plugging this distribution in the equation (3.18) we can get

$$\eta + \alpha u \int_0^{F^{-1}(u)} \frac{\mathrm{SNR} g\eta}{1+\mathrm{SNR} g\eta} \frac{f(g)}{u} dg = 1 \tag{3.24}$$

and the (3.21) follows by substituting $t = F(g)$. Since $g_l \to F^{-1}(u)$ and η_l converges to the solution of (3.21) we have that $C_l \to C(u, \alpha, \mathrm{SNR}, \pi_0)$ predicted by equation (3.20) in probability. □

The following consequence can easily be derived using the previous Proposition.

Corollary 1 *In the reverse ordering case, i.e. when users are detected from the weakest to the strongest, the capacity profile converges in probability to*

$$\begin{aligned} C(u, \alpha, SNR, \pi_0^{-1}) &= \frac{1}{2}\log(c_{\pi_0^{-1}}(u)) \\ &= \frac{1}{2}\log\left(1 + SNRF^{-1}(1-u)\eta_{\pi_0^{-1}}(\alpha, SNR, u)\right) \end{aligned} \tag{3.25}$$

as $K \to \infty$. The multiuser efficiency $\eta_{\pi_0^{-1}}(\alpha, SNR, u)$ is given by

$$\eta + \alpha \int_0^u \frac{SNR\eta F^{-1}(1-t)}{1 + SNR\eta F^{-1}(1-t)} dt = 1. \tag{3.26}$$

The capacity profile $C(u, \alpha, SNR, \pi_0^{-1})$ corresponds to the vertex of the capacity region for permutation π_0^{-1}.

Note that there is an explicit loss in spectral efficiency if detection is performed in the reverse order compared to the case where detection is performed in the order from the strongest to the weakest. The capacity profile that corresponds to the capacities of the optimal decision feedback receiver with the order of detection from strongest to the weakest is shown in Figure 3.2. This capacity profile also corresponds to the vertex of the capacity region defined by the permutation π_0. This figure is given for $E_b/N_0 = 10dB$ and the Rayleigh flat fading and for the case of no fading. It can be seen that the capacity profile is the largest in Rayleigh fading case for $u = 1$: the case when first and the strongest user is being detected. For the Rayleigh fading case, the single user capacity goes to 0 as $u \to 0$ since the last user to be detected has negligible received power. In the case of no fading, the situation is opposite since all users are received with the same power and the interference is decreased as the users are being canceled and is overpowered by the noise. Obviously, the symmetric capacity can be attained only if time sharing of the users is employed, i.e. no vertex of the capacity region attains the symmetric capacity.

Now the total capacity of the system is given as an integral over all single user capacities

$$C_{sum,\pi_0}(\alpha, \mathrm{SNR}) = \int_0^1 C(u, \alpha, \mathrm{SNR}, \pi_0) du. \tag{3.27}$$

We will show that detection order from the strongest to the weakest gives the maximal sum capacity. Next we show that the previous results are equivalent to the results on sum capacity of the optimal detector in flat fading channels derived in [49]

$$\begin{aligned} C_{opt}(\alpha, \mathrm{SNR}) &= \alpha \mathrm{E}\left[\log_2(1 + \mathrm{SNR} g \eta(\alpha, \mathrm{SNR}))\right] \\ &+ \log_2 \frac{1}{\eta(\alpha, \mathrm{SNR})} + (\eta(\alpha, \mathrm{SNR}) - 1)\log_2 e \end{aligned} \tag{3.28}$$

where the expectation is with respect to the distribution of the channels gains and $\eta(\alpha, \mathrm{SNR}) = \eta_{\pi_0}(\alpha, \mathrm{SNR}, 1)$ (cf. (3.18)).

Lemma 4 *Let's fix α and SNR. The sum capacity of (3.27) where $C(u, \alpha, SNR, \pi_0)$ is given by Proposition 4. is equal to (3.28).*

Proof: In lieu of integrating (3.27) and showing that is equal to (3.28), we can demonstrate the following equivalent condition

$$\frac{\delta}{\delta u} C_{opt}(u\alpha, \mathrm{SNR}) = C(u, \alpha, \mathrm{SNR}, \pi_0). \tag{3.29}$$

We first note that

$$\frac{\delta}{\delta u} C_{opt}(u\alpha, 0) = C(u, \alpha, 0, \pi_0) = 0 \tag{3.30}$$

which follows from the fact that $\eta(\alpha, 0) = 1$ [49]. Therefore the Lemma holds for $\mathrm{SNR} = 0$. Now it is enough to prove that partial derivatives with respect to the SNR of both sides of (3.30) are equal and Lemma will hold for all SNR. Therefore we claim that

$$\frac{\delta^2}{\delta \mathrm{SNR}\, \delta u} C_{opt}(u\alpha, \mathrm{SNR}) = \frac{\delta}{\delta u}\left(\frac{\delta}{\delta \mathrm{SNR}} C_{opt}(u\alpha, \mathrm{SNR})\right) = \frac{\delta}{\delta \mathrm{SNR}} C(u, \alpha, \mathrm{SNR}, \pi_0). \tag{3.31}$$

We know from the proof of the Theorem IV.1 in [49] that

$$\frac{\delta}{\delta \mathrm{SNR}} C_{opt}(u\alpha, \mathrm{SNR}) = \frac{1-\eta(u\alpha, \mathrm{SNR})}{\mathrm{SNR}} \tag{3.32}$$

where $\eta(u\alpha, \mathrm{SNR}) = \eta_{\pi_0}(\alpha, \mathrm{SNR}, u)$ is multiuser efficiency after fraction of $1-u$ users has been canceled. By plugging the previous expression and (3.20) in (3.31) we get the following partial differential equation in terms of $\eta(u\alpha, \mathrm{SNR})$

$$-\frac{\delta}{\delta u}\frac{\eta(u\alpha, \mathrm{SNR})}{\mathrm{SNR}} = \frac{\delta}{\delta \mathrm{SNR}} \log\left(1 + \mathrm{SNR} F^{-1}(u)\eta(u\alpha, \mathrm{SNR})\right). \tag{3.33}$$

The general solution $x = \eta(u\alpha, \mathrm{SNR})$ of that partial differential equation satisfies equation

$$\int \frac{\alpha \mathrm{SNR} F^{-1}(u)}{1 + \mathrm{SNR} F^{-1}(u) x} du + 1 - \Psi(x\mathrm{SNR})\, \mathrm{SNR} = 0 \tag{3.34}$$

for any differentiable function $\Psi(y)$. In particular, for $\Psi(y) = y^{-1}$, equation (3.34) is equal to (3.21). Thus we have shown that $\eta_{\pi_0}(\alpha, \mathrm{SNR}, u)$ is indeed the solution of (3.33) and that derivatives of (3.31) coincide. This concludes the proof of the Lemma. □

The importance of this lemma lies in the fact that it bridges two different approaches in the analysis of the spectral efficiency of the optimal multiuser detector for the large random sequence model. It also validates the correctness of the asymptotic order statistic method applied in the derivation of the Proposition 4.

We note that the general characterization of all $K!$ vertices of the multiple access capacity region would involve generalization of the vector permutation operation for continuous functions of the unit interval. Due to its complexitythis analysis will remain out of the scope of this thesis and will not be addressed further.

3.5 Conclusions

In this chapter, a simplified characterization of the vertices of the multiple-access capacity region that relies on the Cholesky factorization has been proposed. This characterization simplifies the calculation and reinterprets the meaning of the single user capacities attainable at a certain vertex of the capacity region through the usage of the optimal joint detector/decoder or the optimal decision feedback detector. This characterization has been used to formulate an algorithm that finds optimal signature sequence allocation that maximize the symmetric capacity of the capacity region. Starting from the fact that this capacity is attainable in the vertex of the capacity region, this algorithm maximizes the symmetric capacity of the vertex that corresponds to single user capacities attainable by the optimal decision feedback detector with the detection order from the strongest to the weakest. Simulation results show that the proposed algorithm almost always produces signature sequences that have the symmetric capacity very close to the upper bound on symmetric capacity of [17].

The Cholesky characterization of the multiple-access capacity region is extended for the case of random sequences and large number of users. A new performance measure called capacity profile is introduced with the intention of measuring the single user capacities in such a system. The capacity profile has been derived for two orders of detection of the optimum decision feedback receiver: from the strongest to the weakest and for the reverse order from the weakest to the strongest. An illustrative example for the case of no fading has shown that equal capacities can not be attained without time-sharing of the users since capacity profile varies with the percentage of canceled users. Furthermore, through integration of the capacity profile, previous results on the sum capacity of the optimal joint detector/receiver were re-confirmed, thus validating the asymptotic order statistic approach used in the derivation of the capacity profile.

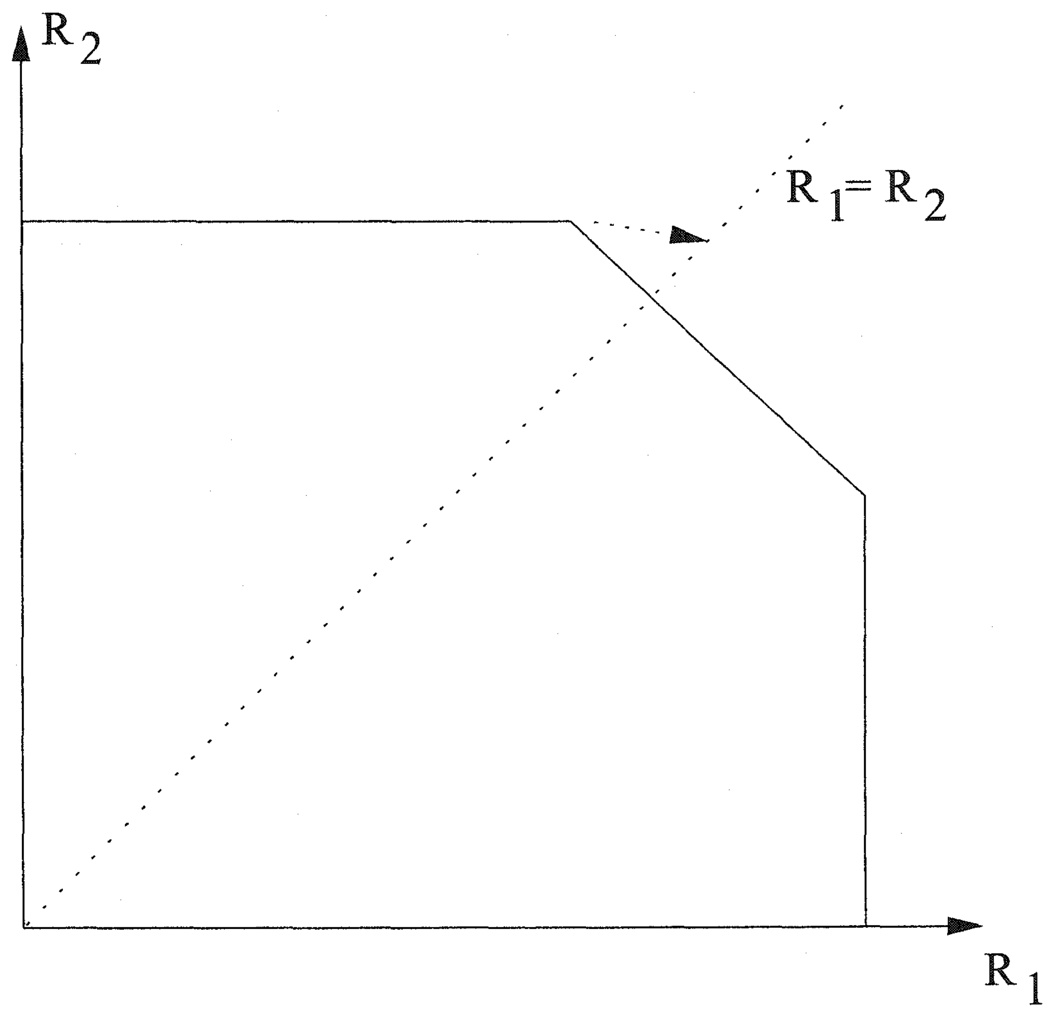

Figure 3.1. *An illustration of the two user capacity region and the principle of optimization of the vertices.*

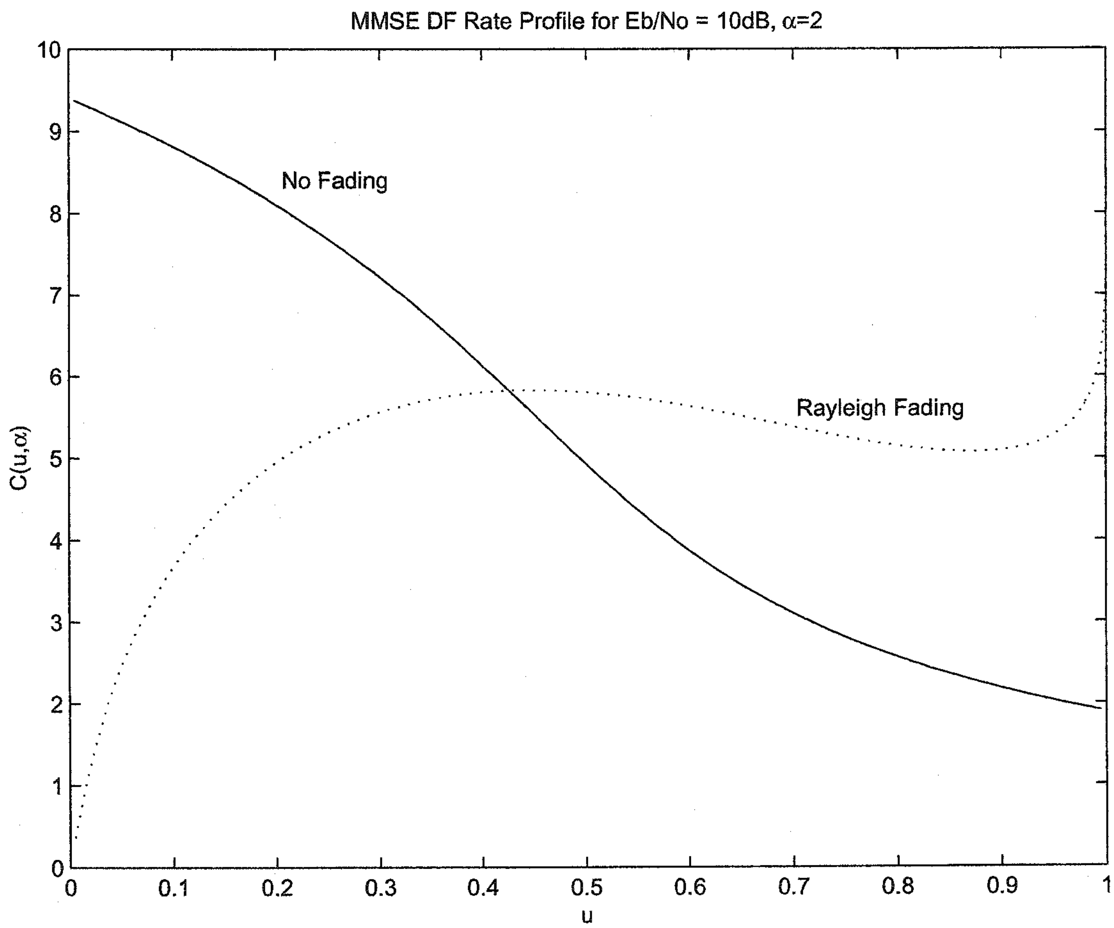

Figure 3.2. *Illustration of the capacity profile of the optimum decision feedback detector with cancellation from the strongest to the weakest user.*

Chapter 4

Asymptotic Analysis of the Conventional Decision Feedback Receiver

The simplicity of the conventional decision feedback receiver - CDFR (sometimes also called successive interference canceler) attracted significant amount of research. The idea of applying the successive interference cancellation originated in multiple-access channels without spreading (see for example [12]). It was shown that successive decoding of the user signals along with the cancellation of the re-encoded previously decoded signals is equivalent to joint optimal decoding. Later, Viterbi [73] applied this idea to spread spectrum multiple-access channels where spreading is achieved by very low rate convolutional codes. In [74] the CDFR concept was applied to multi-antenna systems and was later extended for multi-path propagation channels in [75]. An analysis of the uncoded performance of the CDFR receiver with different modulation formats was reported in [76]. Applications where signature sequences are chosen in such a way to minimize the complexity of the CDFR [77] or to achieve the specific QoS for each user in a coded system [33] were also reported.

This Chapter deals with the performance of coded DS-CDMA system with CDFR and analyzes previously overlooked topics of spectral efficiency of the CDFR with CSI at the receiver in fading channels. Two distinct cases are analyzed: CDFR with and without power ordering. For both cases spectral efficiency is derived for the flat fading channel. Issues of power control at the receiver are addressed for both cases and algorithms that find the optimal power control law are presented. The power control law that equalizes single user capacities for the CDFR with power ordering is

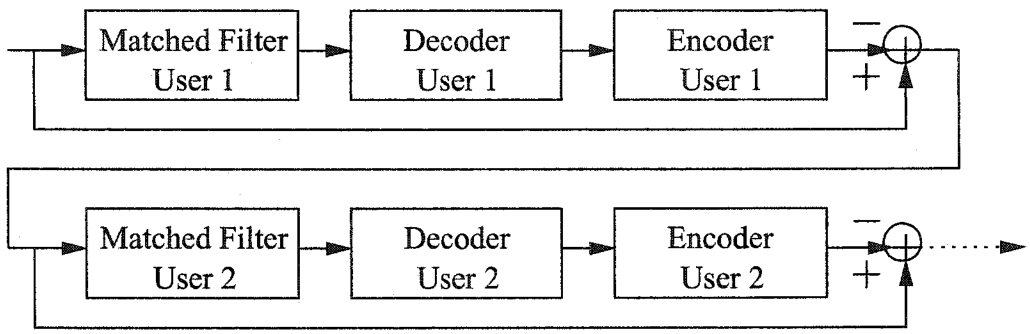

Figure 4.1. *A simplified scheme of the conventional decision feedback receiver.*

also identified.

The model analyzed in this Chapter is a synchronous DS-CDMA system with processing gain N where K users are sharing the available bandwidth resource. This model has been introduced in Chapter 1.

In this Chapter, all the results on spectral efficiency of conventional decision feedback receiver are for the asymptotic random signatures case, with the system load $\alpha = \frac{K}{N}$ constant. The notion of random signatures means that chips of all signature sequences are chosen according to the same zero mean unit variance distribution. Although randomly chosen, these sequences are assumed to be known to both the transmitter and the receiver prior to transmission. The fundamental results in [39], [47] and [48] require that pdf of the chip distribution has bounded fourth moment and this constraint will also be used here. Note that presented results also apply for the practical choice of ± 1 signature sequences.

Decisions in each step of cancellation in a CDFR receiver are based on the output of the conventional matched filter, $\mathbf{c}_k = \mathbf{s}_k$. After conventional matched filtering for a certain user the despreaded signal is decoded. The decoded data of that user is then re-encoded and respreaded and its contribution to the joint signal canceled in order to reduce the interference for subsequent users. This process is illustrated in Figure 4.1 for easy reference.

The analysis of the CDFR will be built upon user averaged channel capacity (in [bit/symbol]) of the CDFR for different user rates and the same powers derived in [45], [78]

$$\begin{aligned}\lim_{K=\alpha N\to\infty} \bar{C}_{CDFR}(\alpha, \mathrm{SNR}) &= \frac{1}{\alpha}\int_0^{\alpha} \log_2\left(1+\frac{\mathrm{SNR}}{1+\alpha'\mathrm{SNR}}\right) d\alpha' \\ &= \int_0^1 \log_2\left(1+\frac{\mathrm{SNR}}{1+\alpha u\mathrm{SNR}}\right) du. \end{aligned} \tag{4.1}$$

where the over-bar on $\bar{C}$ denotes the user average. Now, the spectral efficiency Γ can be calculated as $\Gamma = \alpha\bar{C}$. The spectral efficiency is given in [bit/s/Hz] units since we assumed unit bandwidth for the propagation of multiple-access signal. All our subsequent results will also be of asymptotical nature ($K = \alpha N \to \infty$) and the limit sign will be dropped for the brevity of notation.

4.1 Decision Feedback Without Power Ordering

In this section we analyze flat fading channel and, unless explicitly emphasized, we assume for simplicity that all channel gains have the same distribution $f(g)$ and unit variance. For the case of linear multiuser detectors, the single user channel capacity converges in probability to [39], [49]

$$C_{linear} = \log_2(1+\mathrm{SIR}), \tag{4.2}$$

where the effective signal to interference ratio SIR of that user in perfect CSI cases can be calculated using the multiuser efficiency η of the user with channel gain g as $\mathrm{SIR} = g\mathrm{SNR}\eta$. The multiuser efficiency is equal to the output SNR divided by the signal-to-noise ratio at the output of a single-user matched filter in the absence of multi-access interference. The previous equation follows from the asymptotic normality of the multiuser interference of random sequences for linear receivers like conventional and MMSE receiver.

For conventional receivers, the multiuser efficiency can be calculated as [39], [48]

$$\eta = [1+\alpha\mathrm{SNR}\,\mathrm{E}[g]]^{-1}. \tag{4.3}$$

The expectation operator in the previous equation is with respect to the limiting (for large number of users) empirical distribution of user channel gains. If all users have equal fading distribution $f(g)$ then, due to the ergodicity, this limiting distribution will also be $f(g)$.

If there is no ordering of the estimated powers prior to the successive interference cancellation, the expectation in (4.3) is equal to unity after an arbitrary number of cancellations since the remaining users retain the same power distribution. The average capacity in fading channels is thus obtained by averaging expression (4.1) with respect to the channel gain

$$\begin{aligned} \bar{C}_{CDFR}(\alpha, \mathrm{SNR}; f) &= \int_0^1 \mathrm{E}_f \left[\log_2 \left(1 + \frac{g\,\mathrm{SNR}}{1 + \alpha u \mathrm{SNR}} \right) \right] du \\ &= \mathrm{E}_f \left[\int_0^1 \log_2 \left(1 + \frac{g\,\mathrm{SNR}}{1 + \alpha u \mathrm{SNR}} \right) du \right]. \end{aligned} \tag{4.4}$$

Using Jensen's inequality and the concavity of the expression under the expectation operator in the previous equation, it can be concluded that in this case the average capacity is always less than or equal to the spectral efficiency in the case without fading (4.1). The equality is attained only if there is no fading in the channel.

4.1.1 Power Control

In situations where perfect feedback of the channel estimates from the receiver to the transmitter is possible, power control is usually introduced at the transmitter to boost the performance of transmission in fading channels. We denote the power control law which depends on the perfectly estimated channel gain g with $p(g)$. It should be noted that, for the ease of notation, the power control law $p(g)$ is normalized with noise variance σ^2. In order to constrain the radiated power, the following limitation has to be imposed

$$\int_0^\infty f(g)p(g)dg = \mathrm{SNR}. \tag{4.5}$$

For convenience we also introduce the transformed power control law $z(u) = p(F^{-1}(u))$ with $\int_0^1 z(u)du = \int_0^1 f(g)p(g)dg = \mathrm{SNR}$.

In the case when the power control law is applied on the CDFR without power ordering, the user averaged ergodic capacity converges to

$$\begin{aligned} \bar{C}_{CDFR}(\alpha, \mathrm{SNR}; f, p) &= \mathrm{E}_f \left[\int_0^1 \log_2 \left(1 + \frac{g\,p(g)}{1 + \alpha u \mathrm{E}_f[g p(g)]} \right) du \right]. \\ &= \int_0^1 \int_0^1 \log_2 \left(1 + \frac{F^{-1}(v) z(v)}{1 + \alpha u \int_0^1 F^{-1}(x) z(x) dx} \right) du dv. \end{aligned} \tag{4.6}$$

Finding of the optimal power control law is now a constrained optimization problem similar to the one solved for a single user channel in [22]. However, in this case the closed form analytical solution appears to be intractable and we resort to a relatively simple numerical procedure to identify the optimal power law. This power control law will depend on the specified fading distribution, and parameters α and SNR. The idea is to convert the functional optimization problem of maximizing (4.6) into a discrete optimization problem. In order to accomplish that we use an approximation of the right side of (4.6)

$$\bar{C}_{CDFR}(\alpha, \mathrm{SNR}; f, p) \approx \frac{1}{J}\sum_{i=1}^{J}\left[\int_0^1 \log_2\left(1 + \frac{z_i f_i}{1 + \alpha u \frac{1}{J}\sum_{j=1}^{J} z_i f_i}\right) du\right], \qquad (4.7)$$

where $f_i = F^{-1}(\frac{i-0.5}{J})$ and $z_i = z(\frac{i-0.5}{J})$ are samples of the quantile distribution and the power control law respectively. J is the number of discretization intervals. The power constraint (4.5) is now

$$\sum_{j=1}^{J} z_i \approx SNR, z_i \geq 0. \qquad (4.8)$$

Therefore the functional optimization problem is approximated by an discrete constrained optimization problem of finding the maximum of (4.7) under constraint (4.8). Note that the integral in (4.7) can be solved analytically.

The spectral efficiency of the CDFR without power ordering and optimal power control for Rayleigh fading channel is obtained by a simulation for $J = 300$ discretization intervals and is shown in Figure 4.6. The Davidon-Fletcher-Powell optimization algorithm (see for example [79]) converged very fast to the unique optimal solution.

As opposed to the CDFR with power ordering that will be analyzed in the next Section, users in CDFR without power ordering do not have the same fading-averaged capacities, even if the fading is ergodic. The reason for this fact is that cancellation order which is appointed beforehand, continually favors some users to others. Another approach to equalize user capacities or to increase the overall spectral efficiency of the system is to introduce power control law which is dependent on the cancellation order applied at the receiver. We will not discuss this possibility any further and will turn our attention to more promising CDFR with power ordering.

4.2 Decision Feedback With Power Ordering

When power ordering is used, received user powers are first estimated and ordered in non-increasing order, i.e. from the strongest to the weakest, and the successive interference cancellation is performed in that order. Assuming knowledge of the fading distribution it will be shown that this case can be regarded as a deterministic case, since we can know with high probability the power of the user that is currently being detected [64].

In the derivation of the following Proposition we will make use of the Definition 7.

Proposition 5 *Let the limiting empirical cumulative distribution function of the flat fading channels gains be $F(g)$ with pdf $f(g)$. Then the average single user capacity of the CDFR receiver with power ordering converges in probability to*

$$\bar{C}_{CDFR}(\alpha, SNR; f) = \int_0^1 \log_2\left(1 + \frac{SNRF^{-1}(u)}{1 + \alpha SNR \int_0^u F^{-1}(t)dt}\right) du \tag{4.9}$$

as number of users $K = \alpha N$ increases to infinity.

Proof:

In the case that preordering of the user gains was performed and assuming the perfect feedback, the chip matched filter output after $0 \leq u \leq 1$ fraction of users was canceled is

$$\mathbf{y} = \sum_{j=\lfloor K(1-u)\rfloor}^{K} b_{\pi_j^g}\sqrt{g_{[j]}}\mathbf{s}_{\pi_j^g} + \mathbf{n}, \tag{4.10}$$

where $\lfloor x \rfloor$ denotes the integer part of x. Now, distributions $f_{[j]}$ of the ordered gains $g_{[j]}$ are the order statistic distributions [63]. It is known that with the increase of the number of samples (in this case users) the distribution becomes a degenerate distribution, which means that it can take only values 0 and 1[1]. More formally, this means that for $0 \leq u \leq 1$, as $K \to \infty$, the cumulative distribution $F_{[\lfloor K(1-u)\rfloor]}(g)$ converges weakly to the step function at the quantile ξ_β of the distribution f i.e.

$$\lim_{K\to\infty} F_{[\lfloor K(1-u)\rfloor]}(g) = h(x - \xi_u), \tag{4.11}$$

[1]In some statistical applications it is necessary to avoid this degeneration of the limiting distribution through the appropriate scaling of the ordered samples - see for example [59].

where function $h(x)$ is a unit step function. Alternatively formulated [65], samples of the distribution $f_{[\lfloor K(1-u)\rfloor]}(g)$ converge in probability to ξ_u. That is, for each $\epsilon > 0$

$$\text{Prob}\left(|g_{[\lfloor K(1-u)\rfloor]} - \xi_u| > \epsilon\right) \to 0. \tag{4.12}$$

Using (4.2) and (4.3), in the case of power ordering, (4.1) can be rewritten as

$$\lim_{K=\alpha N\to\infty} \bar{C}_{CDFR}(\alpha, \text{SNR}) = \lim_{K\to\infty} \frac{1}{K}\sum_{i=1}^{K} \log_2\left(1 + \text{SNR} g_{[i]}\eta_{[i]}\right) \tag{4.13}$$

where, according to the model (4.10), multiuser efficiency $\eta_{[i]}$ after $i-1$ strongest users were canceled is

$$\eta_{[i]} = \left[1 + \alpha\text{SNR}\frac{1}{K}\sum_{j=i}^{K} g_{[j]}\right]. \tag{4.14}$$

Substituting $i = \lfloor K(1-u)\rfloor$ and using (4.12), as $K \to \infty$

$$\eta_{[i]} \to \left(1 + \alpha\text{SNR}\int_0^u F^{-1}(t)dt\right)^{-1} \tag{4.15}$$

in probability. Similarly, using (4.12) we have the convergence in probability of the average user capacity

$$\begin{aligned} \bar{C}_{CDFR}(\alpha, \text{SNR}) &= \frac{1}{K}\sum_{i=1}^{K} \log_2\left(1 + \text{SNR} g_{[i]}\eta_{[i]}\right) && (4.16)\\ &\to \int_0^1 \log_2\left(1 + \frac{\text{SNR}F^{-1}(u)}{1 + \alpha\text{SNR}\int_0^u F^{-1}(t)dt}\right) du. && (4.17) \end{aligned}$$

This concludes the proof. □

In the case of Rayleigh fading this expression simplifies to

$$\bar{C}_{CDFR}(\alpha, \text{SNR}; f^{Ray}) = \frac{1}{\alpha}\times \tag{4.18}$$
$$\int_0^\alpha \log_2\left(1 + \frac{\text{SNR}\ln\frac{\alpha'}{\alpha}}{1 + (\alpha - \alpha')\text{SNR} + \text{SNR}\alpha'\ln\frac{\alpha'}{\alpha}}\right) d\alpha'.$$

The comparison of CDFR with and without power ordering is given in Figure 4.2 and shows that power ordering in fading channels has larger spectral efficiency compared to the non-fading channels.

A few remarks regarding the statement and the proof of the Proposition 5 can now be stated.

Remark 2 *The Proposition 5 states that the average user capacity converges in probability to the given expression. This implies that the outage probability that the average capacity of (4.9) (and the ensuing spectral efficiency) cannot be attained by the CDFR with power ordering is asymptotically vanishing over the set of all possible fading states $\mathcal{G}$.*

This remark will be used later for the discussion on the asymptotical delay-limited of the symmetric capacity of the CDFR receiver.

Remark 3 *What happens if all the users do not have the same fading distribution? Without loss of generality, we assume that there are two groups of users with jointly ergodic fading pdf-s $f_1(g)$ and $f_2(g)$, and that p_1 and $1-p_1$ are the probabilities that a user belongs to the first or second group respectively. Then the limiting distribution of all user gains is $f(g) = p_1 f_1(g) + (1-p_1) f_2(g)$. This distribution and its cdf should be substituted in (4.9) to evaluate the user averaged capacity and the spectral efficiency. The previous discussion can be readily extended on the case of more than two classes of users.*

4.2.1 Optimal Fading Distribution

It is appropriate here to state the problem of finding the most favorable fading distribution that maximizes the spectral efficiency of the conventional decision feedback receiver. This result is of interest to evaluate how close the performance of CDFR with a certain fading distribution is to the maximal possible spectral efficiency.

Substituting $y(u) = \int_0^u F^{-1}(t)dt$ in (4.9), the problem of finding the fading distribution which maximizes the spectral efficiency is transformed to the standard problem of calculus of variations. The boundary conditions for this problem are $y(0) = 0$ and the average power constraint $y(1) = \int_0^1 F^{-1}(u)du = \int_0^\infty x f(x)dx = 1$. The objective function we want to maximize is

$$\max_{z(u)} \int_0^1 \log_2 \left(1 + \frac{\mathrm{SNR}\, y'(u)}{1 + \alpha y(u)}\right) du. \tag{4.19}$$

Substituting the previous objective function into Euler's necessary conditions that produce extremal points for problems of calculus of variations, we can get the differential equation with the following solution

$$y(u) = \int_0^u F^{-1}(t)dt = \frac{e^{c_1 \alpha \mathrm{SNR}(u+c_2)} - c_1}{c_1 \alpha \mathrm{SNR}}, \tag{4.20}$$

where the coefficients c_1 and c_2 can be calculated from the boundary conditions and are equal to $c_1 = \ln(1+\alpha\mathrm{SNR})/(\alpha\mathrm{SNR})$ and $c_2 = \ln(c_1)/\ln(1+\alpha\mathrm{SNR})$. Substituting these boundary conditions in (4.20), it follows that $y(u) = \frac{(1+\alpha\mathrm{SNR})^u - 1}{\alpha\mathrm{SNR}}$ and the pdf of the most favorable fading is SNR and load α dependent

$$f_{optim}(g) = \begin{cases} \frac{1}{g\ln(1+\alpha\mathrm{SNR})} & g > \frac{\ln(1+\alpha\mathrm{SNR})}{\alpha\mathrm{SNR}} \\ & g < \frac{\ln(1+\alpha\mathrm{SNR})(1+\alpha\mathrm{SNR})}{\alpha\mathrm{SNR}} \\ 0 & \text{elsewhere.} \end{cases} \tag{4.21}$$

In the case of most favorable fading distribution the average channel capacity converges to

$$C_{CDFR}(\alpha, \mathrm{SNR}; f_{optim}) = \log_2\left(1 + \frac{\ln(1+\alpha\mathrm{SNR})}{\alpha}\right), \tag{4.22}$$

and the E_b/N_0 can be explicitly calculated. A comparison of the optimal fading distribution and Rayleigh fading distribution is given in Figure 4.3. We remark that optimal fading cdf is not strictly increasing and that this may formally present a problem since $F(g)$ is not invertible. This problem can be easily bypassed if we can view this distribution as a limiting distribution of strictly increasing distributions that maximize the user averaged capacity of CDFR with power ordering.

4.2.2 Power Control

Following the same arguments used in the derivation of the spectral efficiency without the power control in Proposition 5, the average capacity of the decision feedback receiver with power ordering and power control law $p(g)$ converges in probability to

$$\bar{C}_{CDFR}(\alpha, \mathrm{SNR}; f, p) = \int_0^1 \log_2\left(1 + \frac{z(u)F^{-1}(u)}{1 + \alpha\int_0^u z(t)F^{-1}(t)dt}\right) du, \tag{4.23}$$

where $z(u) = p(F^{-1}(u))$ with $\int_0^1 z(u)du = \int_0^1 f(g)p(g)dg = \mathrm{SNR}$. For convenience we also introduce normalized power control law $\tilde{z}(u) = z(u)/\mathrm{SNR}$, where $\int_0^1 \tilde{z}(u)du = 1$.

Optimal power control law for decision feedback receiver with power ordering can again be obtained by using the calculus of variations. However, in this case it leads to an analytically intractable differential equation for both general fading distribution and for Rayleigh fading distribution. Instead, we use the numerical algorithm outlined in Section 4.1.1 in order to identify the optimal power control law for the specified fading distribution, and parameters α and SNR. In order to accomplish that we use an approximation of the right side of (4.23)

$$\bar{C}_{CDFR}(\alpha, \mathrm{SNR}; f, p) \approx \frac{1}{J}\sum_{i=1}^{J}\log_2\left(1+\frac{z_i f_i}{1+\alpha\frac{1}{J}\sum_{j=1}^{i} z_i f_i}\right) du, \tag{4.24}$$

where again $f_i = F^{-1}(\frac{i-0.5}{J})$ and $z_i = z(\frac{i-0.5}{J})$ are samples of the quantile distribution and the power control law respectively. The functional optimization problem is approximated by a discrete constrained optimization problem of finding the maximum of (4.24) under constraint (4.8). Computer trials show that the convergence to the unique solution of this optimization problem is very fast even when number of discretization intervals J is much greater than 100. The spectral efficiency of the CDFR with the optimal power control law obtained in this manner is shown in Figures 4.6 and 4.7 for Rayleigh and Nakagami fading respectively.

We note that the optimal power control law depends on the fading distribution and system load (unlike the optimal power control law for optimal receiver and linear receivers) which is not very convenient for practical implementations.

Next, we discuss the application of commonly used power control laws i.e. the power equalization and the truncated power equalization. It will be shown that these techniques even decrease the spectral efficiency of decision feedback receiver with power ordering. The truncated power equalization law for the threshold θ is given by

$$\begin{aligned} p_{teq}(g) &= \begin{cases} \frac{\nu\mathrm{SNR}}{g} & \text{if } g > \theta \\ 0 & \text{if } g < \theta \end{cases} \quad \text{or} \\ z_{teq}(u) &= p(F^{-1}(u)) = \begin{cases} \frac{\nu\mathrm{SNR}}{F^{-1}(u)} & \text{if } u > q \\ 0 & \text{if } u < q \end{cases}, \end{aligned} \tag{4.25}$$

where we have introduced the following notation $q = F(\theta) = \mathrm{Prob}[g \leq \theta]$. The coefficient ν can be evaluated from the power constraint $\int_0^\infty f(g)p_{teq}(g)dg = \mathrm{SNR}$. Substituting (4.25) into (4.23) we get

$$\bar{C}_{CDFR}(\alpha, \mathrm{SNR}; f, p_{teq}) = \int_q^1 \log_2\left(1+\frac{\nu\mathrm{SNR}}{1+\alpha\mathrm{SNR}\nu(u-q)}\right) du. \tag{4.26}$$

In particular, for Rayleigh fading $\nu = (\mathrm{Ei}(1, -\ln(1-q)))^{-1}$ where Ei is the exponential integral defined as $\mathrm{Ei}(n,x) = \int_1^\infty \frac{e^{-xt}}{t^n} dt$. From Figure 4.4 where spectral efficiencies of the CDFR with power ordering and truncated power equalization strategy for several values of parameter q are compared, we can see that this power control strategy is a very poor one and is worse than no power control at all. In particular, when perfect power equalization is used ($q \to 0$) with conjunction of the Rayleigh fading channel, the spectral efficiency converges to zero. This is due to the fact that Rayleigh fading is not *regular fading* [37] i.e. $E_{f_{Ray}}[1/g]$ diverges, while $\nu \to 0$.

It was mentioned previously that the problem of finding the optimal power control law for specific fading distribution, system load α and SNR is numerically difficult problem with a solution that depends on all these parameters. However, in case of increasing values of the system load $\alpha \to \infty$, we can make some useful conclusions for the choice of appropriate power control law. The spectral efficiency $\Gamma_\infty = \lim_{\alpha\to\infty} \alpha \bar{C}_{CDFR}(\alpha, \mathrm{SNR}; f, p)$ can in this case be calculated as

$$\begin{aligned}
\Gamma_\infty &= \lim_{\alpha\to\infty} \alpha \int_0^1 \log_2\left(1 + \frac{\mathrm{SNR}\ \tilde{z}(u)F^{-1}(u)}{1 + \mathrm{SNR}\ \alpha \int_0^u \tilde{z}(t)F^{-1}(t)dt}\right) du && (4.27)\\
&= \lim_{\alpha\to\infty} \alpha \int_0^1 \log_2\left(1 + \frac{\Gamma_\infty \frac{E_b}{N_o} \tilde{z}(u)F^{-1}(u)}{\alpha + \Gamma_\infty \frac{E_b}{N_o}\ \alpha \int_0^u \tilde{z}(t)F^{-1}(t)dt}\right) du && (4.28)\\
&= \int_0^1 \frac{\Gamma_\infty \frac{E_b}{N_o} \tilde{z}(u)F^{-1}(u)}{(1 + \Gamma_\infty \frac{E_b}{N_o}\ \int_0^u \tilde{z}(t)F^{-1}(t)dt)\ln 2} du && (4.29)\\
&= \log_2\left(1 + \Gamma_\infty \frac{E_b}{N_o} \int_0^1 \tilde{z}(t)F^{-1}(t)dt\right) && (4.30)
\end{aligned}$$

where equation (4.27) follows from (4.23) by substituting the normalized power control law, while (4.28) follows by substituting $\mathrm{SNR} = \frac{\Gamma}{\alpha}\frac{E_b}{N_o}$. Furthermore, equation (4.29) follows from changing the order of integration and limit operator for positive-valued sub-integrand function [65] and equation (4.30) follows by solving the integral in (4.29) using the change of variables $t(u) = \Gamma_\infty \frac{E_b}{N_o} \int_0^u \tilde{z}(t)F^{-1}(t)\ dt$. By making use of the equation (4.30) we can now draw several conclusions about the asymptotic behavior of the spectral efficiency of the CDFR detector.

In case of no power control at the transmitter, i.e. $\tilde{z}(u) = 1$, we can easily conclude that the limiting spectral efficiency of the CDFR with power ordering is equal to the spectral efficiency of the single user non-fading channel, since Γ_∞ is the solution of the equation $\Gamma_\infty = \log_2\left(1 + \Gamma_\infty \frac{E_b}{N_o}\right)$. Note that, in this large α scenario, spectral efficiency of the CDFR with power ordering in fading channels is larger than the spectral efficiency of the single user channel with fading. This is an example of the multiuser diversity effect that was reported in [49]. A similar conclusion has been made in [49] for the optimum receiver, whereas linear receivers always suffer from the spectral efficiency loss in this asymptotic regime.

The spectral efficiency in the case when the power control $\tilde{z}(u)$ is used is equal to the spectral efficiency of the single user channel with the equivalent E_b/N_o ratio increased by factor $\Phi = \int_0^1 \tilde{z}(t)F^{-1}(t)dt$. We will now show with the appropriate choice of power control law that this increase factor can be quite significant. In the case of truncated power equalization law of (4.25) this increase factor is equal to

$$\begin{aligned} \Phi &= \int_0^1 \tilde{z}_{teq}(t)F^{-1}(t)dt = (1-q)\nu \\ &= \frac{1-q}{\int_q^1 \frac{du}{F^{-1}(u)}}. \end{aligned} \tag{4.31}$$

Figure 4.5 shows how with the increasing system load, the spectral efficiency approaches the asymptotic limit discussed in this section. It can be seen that the truncated power equalization law with $q = 0.9$ eventually surpasses the single user bound on spectral efficiency and converges to the limit predicted by (4.30). The increase for larger values of truncation factor q is even slower, but the ultimate limit is higher. In case of $q \to 1$, factor Φ equals $F^{-1}(1)$, which is equal to infinity for unbounded fading distributions.

A similar conclusion has been derived in [80] (cf Chapter 2), where a power control law which amplifies only a fraction of the strongest users and assigns zero power to other users, is proven to maximize the spectral efficiency in case when signature sequences and power control law are chosen optimally. Note that this power control law is applicable only in situations of fast changing fading where all users have the same rate on average.

4.2.3 Symmetric Capacity

Instantaneous single user capacities in the CDFR with power ordering and fading channels vary with the number of canceled users. For illustration, this capacity profile that depends on the percentage of canceled users is depicted in Figure 4.8. In general it can be observed that in fading channels, the first and the strongest users to be detected and canceled have the best channel conditions and the best single user capacities. With the increase in the number of canceled users the single user capacities generally go to zero since the last users to be canceled are also the weakest ones and they are almost completely affected by the channel noise that can not be canceled. Note that all distributions (except the optimal fading distribution) depicted in Figure 4.8 have support from 0 to ∞ and the powers of the weakest users tends to zero. However, in ergodic fading channels, all users still have the same average capacities.

We can compare these conclusions with the non-fading case, where single user capacity profile steadily increases with the increase in the number of canceled users since the interference is constantly decreasing. The noise effect is now not as detrimental as in the fading case since all the users have the same received powers.

In this section we discuss the power control law that equalizes the single user capacities of users in the CDFR receiver with power ordering. A somewhat similar analysis was done in the case of the decision feedback receivers with equal user rates and no fading in [45] while the general definition of the symmetric capacity with optimal receivers was given in [18].

In order to equalize single user capacities (see equation (4.23)) we want to identify the power control law and a maximal constant SIR such that

$$\frac{z(u)F^{-1}(u)}{1+\alpha\int_0^u z(t)F^{-1}(t)dt} = \mathrm{SIR} \tag{4.32}$$

for $0 \le u \le 1$. The equation (4.32) is an integral equation in $z(u)$ with a general solution

$$z(u) = \frac{ce^{c\alpha u}}{F^{-1}(u)}. \tag{4.33}$$

With this solution we can see that $\mathrm{SIR} = c$ and single user capacities of all users are equal to $C_{sym} = \log_2(1+c)$. The maximal value of c can be obtained by satisfying the power constraint (4.5) i.e.

$$\int_0^1 \frac{ce^{c\alpha u}}{F^{-1}(u)}du = \int_0^\infty \frac{ce^{c\alpha F(g)}}{g}f(g) = \mathrm{SNR}. \tag{4.34}$$

Note that in case of fading distribution that is not regular, the previous equation does not have any solution $c > 0$ since the integral on the left side diverges. This conclusion is in accordance with the results of [37] and [29]. However, for regular fading distributions, the previous equation does have a positive solution. A numerical example of this is given in Figure 4.7 for Nakagami fading distribution

$$f^{Nakagami}(g) = \frac{m^m g^{m-1}}{\Gamma(m)} \exp(-mg) \tag{4.35}$$

with parameter $m = 2$, where $\Gamma(m) = \int_0^\infty e^{-t}\, t^{m-1} dt$ is gamma function. In order to facilitate the comparison with other results we extend the scope of the definition of the spectral efficiency (1.7) to the case of symmetric capacities. For convenience we will call it symmetric spectral efficiency. The sum rate in (1.7) is now defined as a maximal sum of equal single user rates for a specific receiver architecture.

It was shown in Section 4.2 that as $\alpha \to \infty$, the spectral efficiency of CDFR with power ordering in fading channels approaches the single user limit. We now evaluate the symmetric spectral efficiency $\Gamma_{sym,\infty} = \lim_{\alpha\to\infty} \alpha\bar{C}_{sym}(\alpha, \text{SNR}; f)$ in the asymptotic regime. Starting from (4.34) and repeating the procedure of Section 4.2, the limiting spectral efficiency can be obtained as a solution of the equation

$$\int_0^1 \frac{2^{\Gamma_{sym,\infty} u} \ln 2}{F^{-1}(u)} = \frac{E_b}{N_0}. \tag{4.36}$$

Numerically solving the previous equation in the case of Nakagami fading distribution with $m = 2$ and $\frac{E_b}{N_0} = 10dB$ shown in Figure 4.7, the asymptotic value of the symmetric spectral efficiency is $\Gamma_{sym,\infty} \approx 6.447$. We note that this value surpasses the non-fading single user spectral efficiency (capacity of a Gaussian multiple-access channel) at $\frac{E_b}{N_0} = 10dB$.

We now link the derived symmetric capacity with the discussion on the delay limited capacity of a fading channel presented in Chapter 1.

Remark 4 *Similar to Remark 2, the symmetric capacity of the CDFR with power ordering converges to its limit in probability. This implies that, as the number of users increases to infinity, the outage probability (the probability that the CDFR with the power control law (4.33) can not attain the limiting C_{sym}) goes to* 0. *Therefore, the limiting symmetric capacity of a CDFR is also delay-limited symmetric capacity since for almost all fading states there exist a power control law that satisfies power constraints and a coding scheme that attains this symmetric capacity [30], [29].*

Extensions of the symmetric capacity derivation for other preset capacity profiles is straightforward.

4.3 Other Limiting Issues

So far we have addressed only the issue of performance analysis of a CDFR detector in synchronous channels and we have neglected more realistic asynchronous channels. However, using the results of [47] for bit-synchronous symbol-asynchronous channels, we can see that results presented in previous sections directly apply for symbol-asynchronous channels. It was shown in [47] that a conventional matched filter detector does not suffer from loss in effective signal to noise ratio due to symbol asynchronicity if a long enough frame for symbol detection is used. Since the CDFR performs matched filtering in each stage of cancellation there is no loss in performance for this detector either.

For completeness of presentation we note that the material presented in previous sections can also be used to calculate the error-exponents of a CDFR. Knowledge of error exponents can help us obtain an upper bound on bit error probability of a certain user when finite length code words are used. Namely, the Gaussian error exponent of the k-th user is given in [81]

$$E_{r,k}(R_k) = \max_{\rho\in[0,1]} \left\{ \frac{\rho}{2} \log_2 \left(1 + \frac{\mathrm{SIR}_k}{1+\rho} \right) - \rho R_k \right\} \tag{4.37}$$

where $\mathrm{SIR}_k = g\mathrm{SNR}\eta$ for flat fading channels with perfect CSI and η is expressed with (4.3). R_k is the code rate of user k in bits/symbol. Similarly to the exposition of Sections 4.1 and 4.2, average error exponents of CDFR in flat fading with and without power ordering can be calculated by averaging over the distribution of channel gains.

4.4 Conclusions

Spectral efficiencies of the CDFR in frequency flat and frequency selective fading channels with perfect and imperfect CSI were evaluated for the large system model. It was shown that ordering of the user powers prior to cancellation can significantly increase the spectral efficiency of this system and even exceed that of non-fading channels. In order to gain more insight on the influence of the Rayleigh fading and other fading distributions on this receiver, the optimal fading distribution that maximizes the spectral efficiency was derived. It was shown that the spectral efficiency of Rayleigh fading is quite close to that of the optimal fading distribution.

The choice of optimal power control law for CDFR with and without power ordering was discussed. Furthermore, it was shown that truncated power equalization is not an adequate method for power control for CDFRs and that it can even decrease the spectral efficiency of the system. However, it was shown that this power control is asymptoticaly optimal for large system loads α.

The power control law which equalizes single user capacities attainable by the CDFR receiver is also identified. It was shown that such symmetric capacity is attainable for almost all fading states and that it can be interpreted as the delay-limited symmetric capacity.

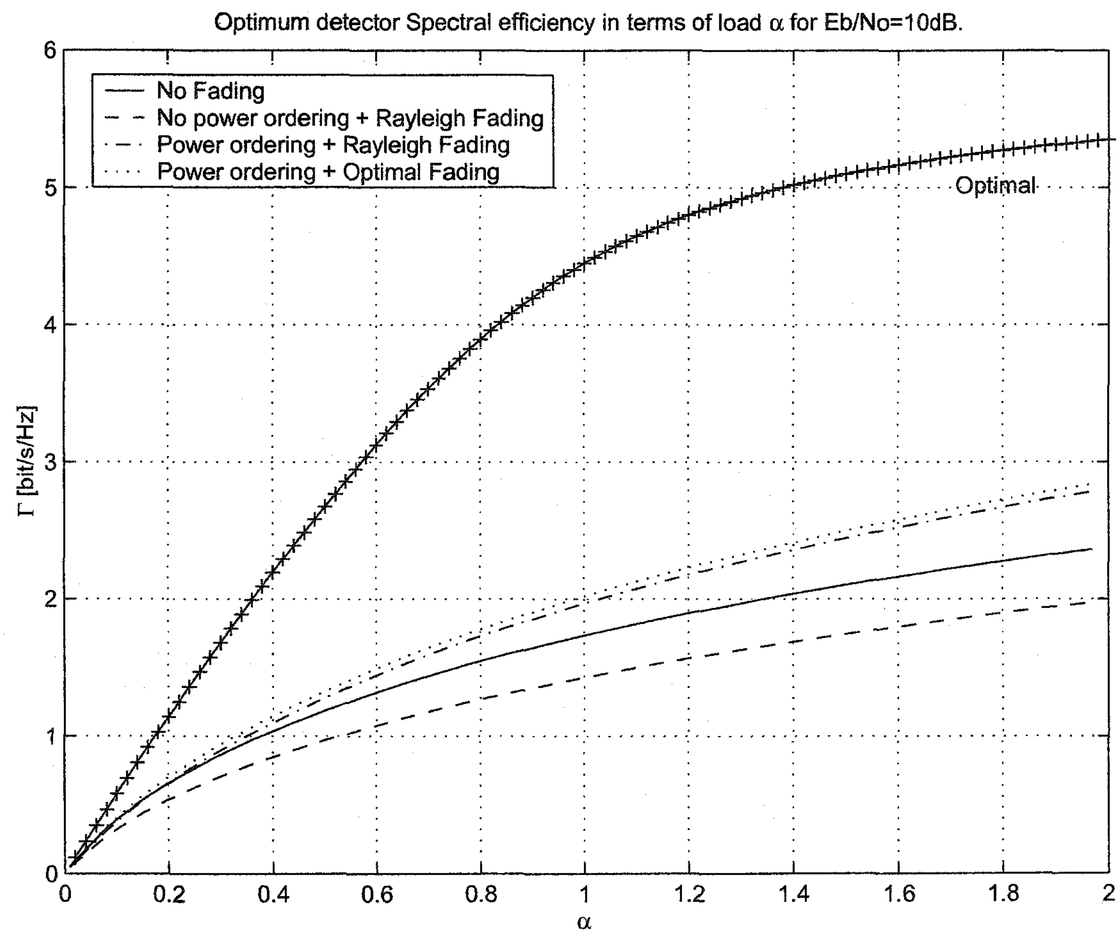

Figure 4.2. *Comparison of the spectral efficiencies of conventional decision feedback receivers (CDFR) in terms of load α. The following curves are displayed in this figure: CDFR in non-fading channel, CDFR without power ordering in Rayleigh fading channel, CDFR with power ordering in Rayleigh fading channel, CDFR with the most favorable fading. For reference, spectral efficiency of the optimal receiver with random sequences is also plotted.*

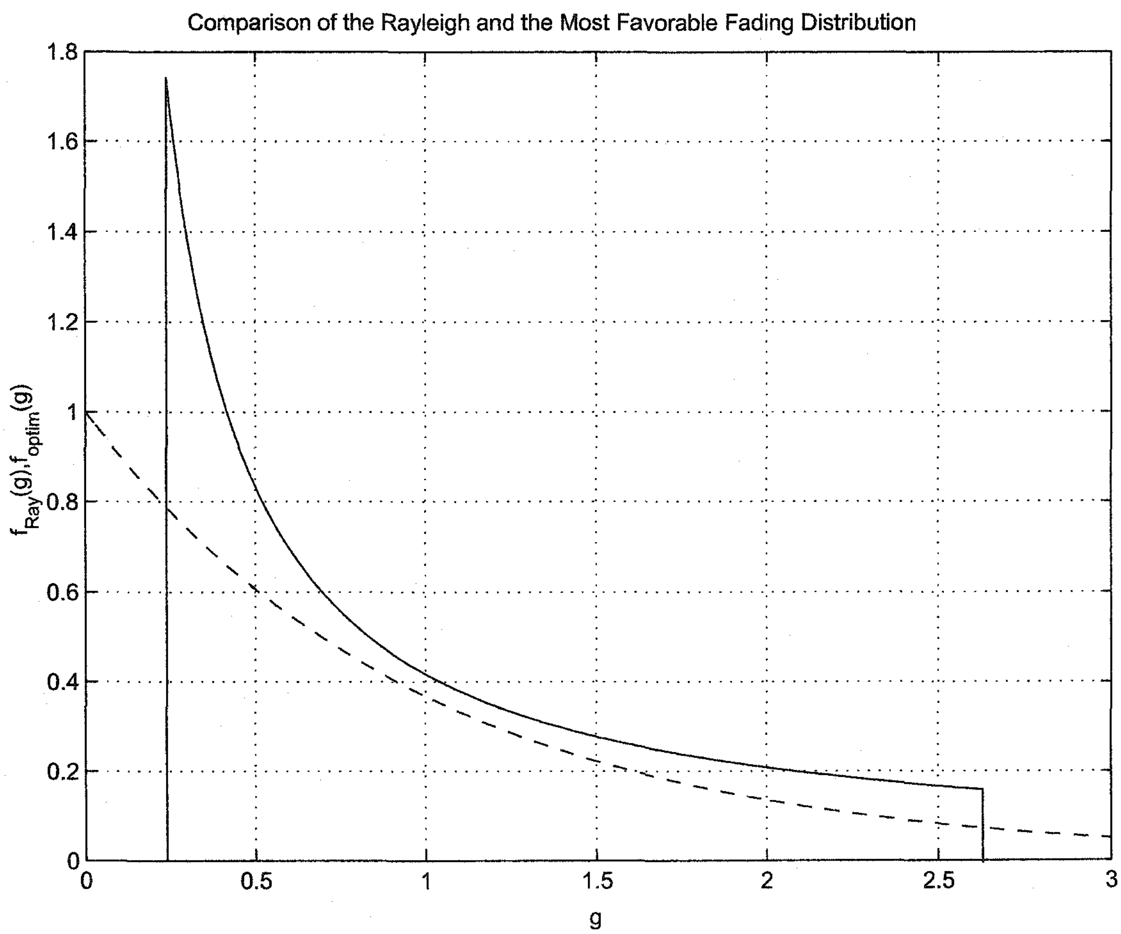

Figure 4.3. *Comparison of the probability density functions of the Rayleigh fading (dashed line) and the most favorable fading pdf (solid line) that maximizes the spectral efficiency of the CDFR with power ordering for* $SNR = 10dB$ *and* $\alpha = 1$.

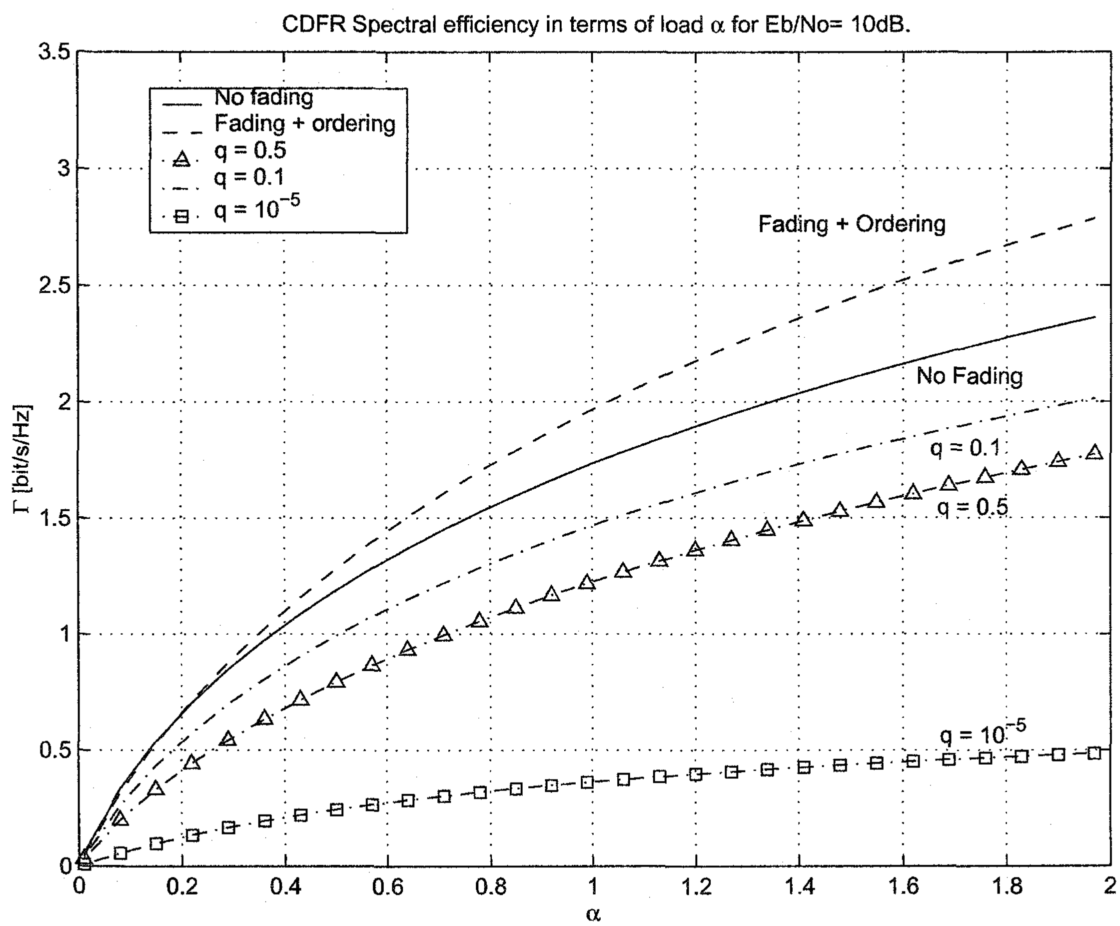

Figure 4.4. *The influence of truncated power equalization on the spectral efficiency of CDFR with power ordering for various thresholds q in Rayleigh fading channels. The threshold at $q = 10^{-10}$ shows that spectral efficiency converges to zero for small values of q i.e., when we use perfect power equalization.*

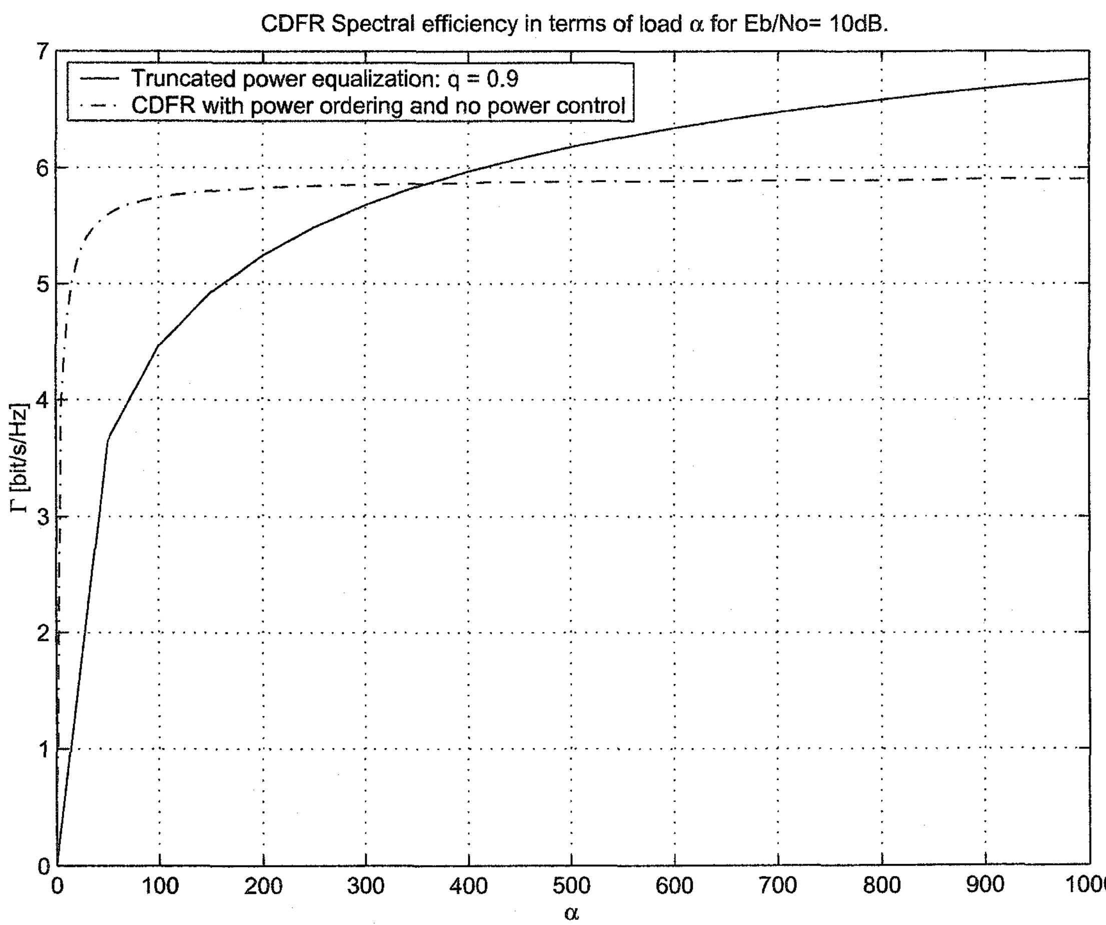

Figure 4.5. *Influence of the truncated power equalization on the spectral efficiency of CDFR with power ordering for very large values of load* α.

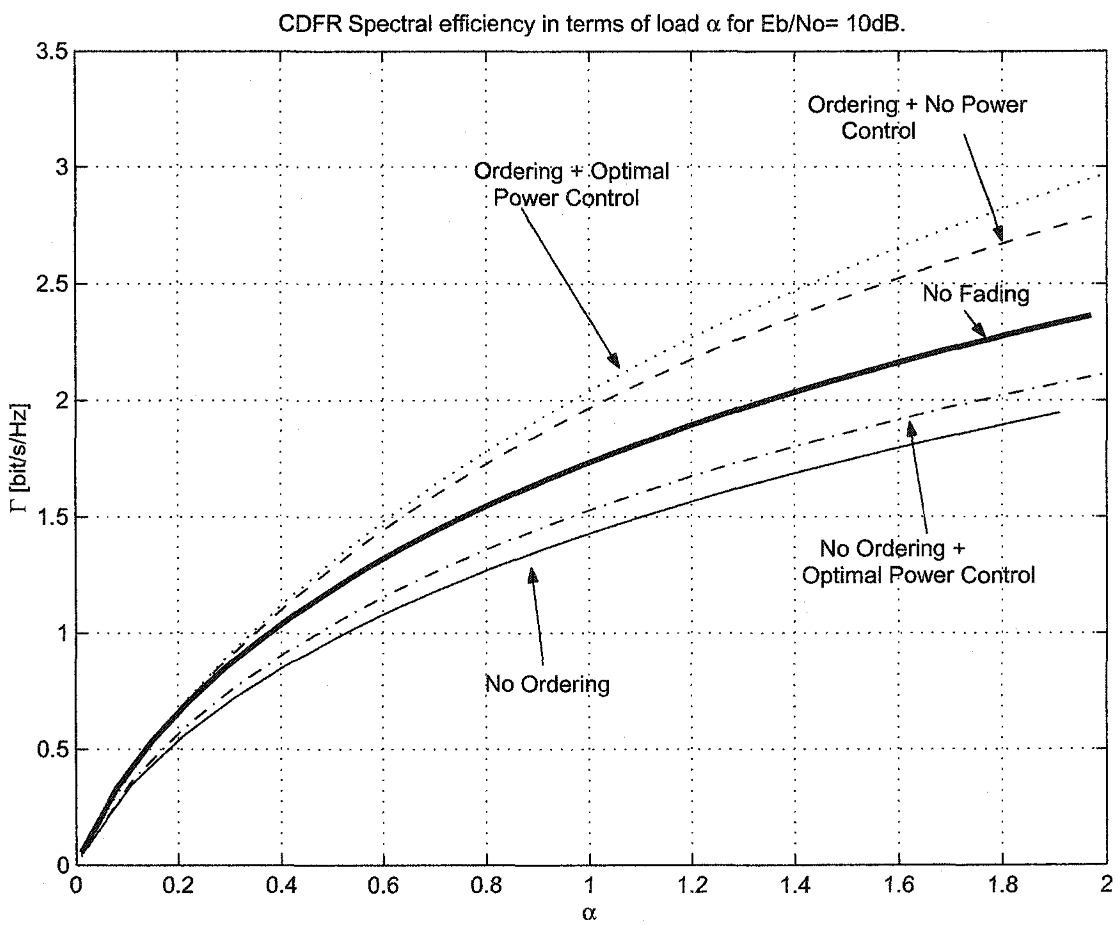

Figure 4.6. *The influence of the optimal power control law on the spectral efficiency of CDFR with and without power ordering. The fading distribution is Rayleigh.*

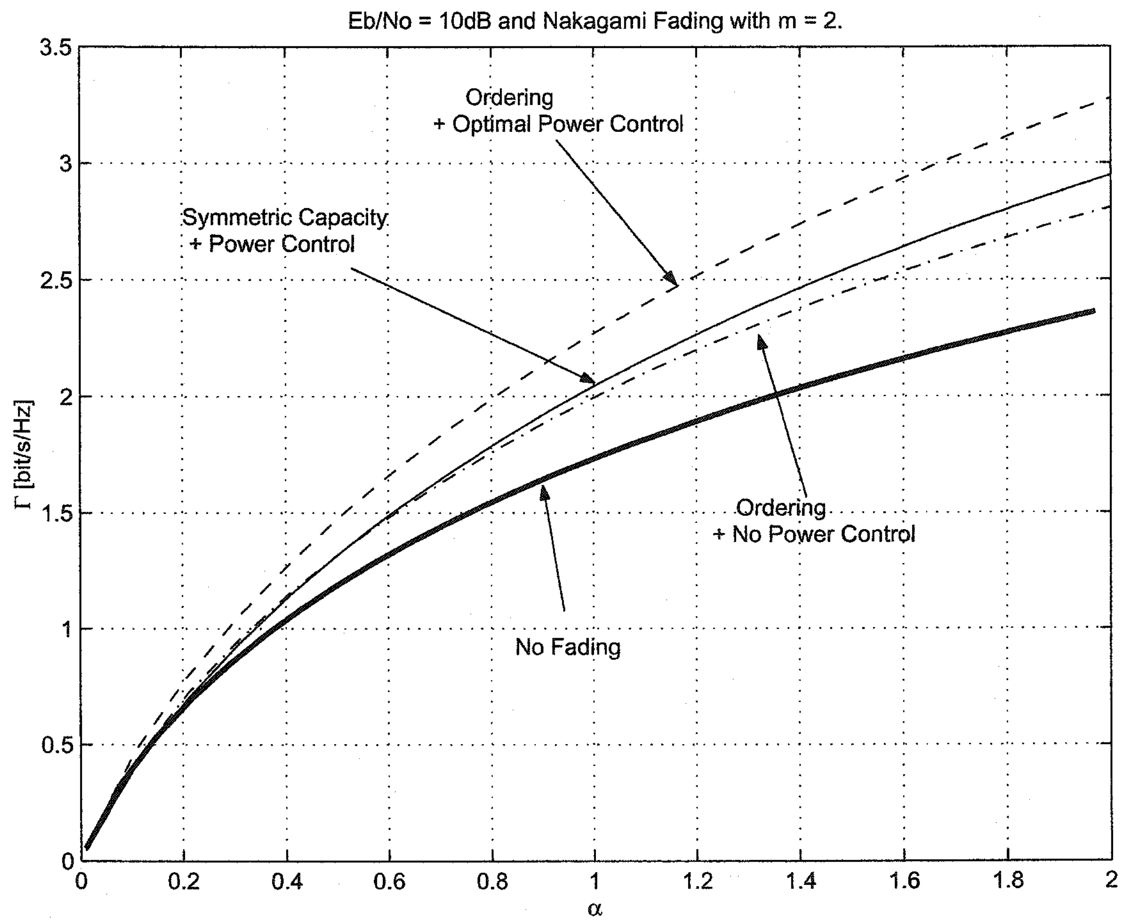

Figure 4.7. *The spectral efficiency of CDFR with power ordering in channels with Nakagami fading distribution with parameter $m = 2$. This figure illustrates the influence of the optimal power control law as well as the power control law which equalizes single user capacities on the spectral efficiency of CDFR.*

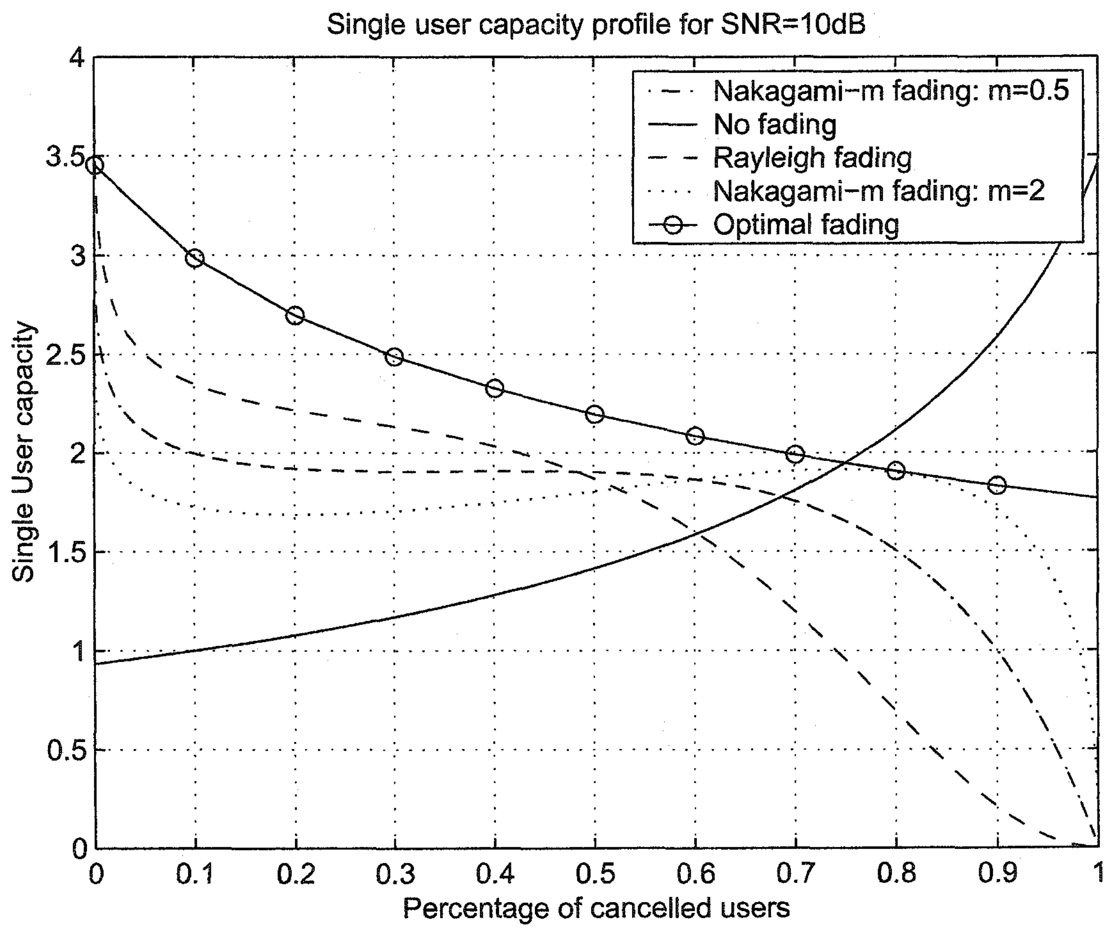

Figure 4.8. *Maximum possible code rate in terms of the percentage of the canceled users in CDFR with power ordering for SNR = 10dB and* $\alpha = 1$. *This capacity profile is plotted for no-fading case, as well as Rayleigh, Nakagami fading and optimal fading cases.*

Chapter 5

On the Feedback Receiver for Two Sets of Orthogonal Sequences

Recently, there has been a considerable interest in exploring the potential of multiuser detection to provide service to more users than the processing gain of a CDMA system. A particularly prominent technique that is analyzed in this chapter is a scheme for synchronous CDMA systems where user signatures are divided into two sets of orthogonal signature sequences (TSOS). For example, the first group of users can use TDMA time slots and the second group orthogonal CDMA sequences [77] or the groups can use complex orthogonal sequences derived from Walsh Hadamard sequences as in [82]. In these results, a very efficient iterative multistage receiver has been used as detector.

In this Chapter we extend the results of [77] and [82] for the coded case and derive the spectral efficiency analysis of the decision feedback receiver. The added complexity and detection delay of the receiver is minimal compared to the conventional matched filter. We show that in the case of a coded system, the spectral efficiency of this scheme is close to that of a Gaussian multiple-access channel if the incomplete group of orthogonal sequences is detected first. We also discuss the choice of orthogonal sequences that maximizes the spectral efficiency. This analysis is extended for the case of flat fading channels.

5.1 System Model and Mathematical Preliminaries

Users in the TSOS scheme are divided into two groups with unit energy orthogonal sequences assigned to each group which simplifies sequence allocation and detection (see [77], [82]). Sets of indices of the two groups of users will be denoted as $G_1 = \{1, \ldots, k_1\}$ and $G_2 = \{k_1 + 1, \ldots, K\}$, $|G_1| = k_1, |G_2| = k_2$ and $k_1 + k_2 = K$. We assume in this Chapter that $K \leq 2N$, but we will also briefly comment on the case $K \geq 2N$. Iterative feedback detector first detects user bits from the first group G_1 of users, reconstructs the interference and subtracts it from the received sampled sequence (1.4). In the second stage, information bits from the second group G_2 are detected and this process can be iteratively continued until convergence is achieved. In each stage, the detector uses simple conventional matched filtering. The detection principle is now the same as the conventional decision feedback receiver shown in Figure 4.1 with the only difference that detection and cancellation can be performed in parallel for all users in the same group of orthogonal users. This ensures that the decoding delay is much smaller compared to the case when each user bits are detected and canceled separately.

For completeness we mention that [83] showed that the uncoded version of the iterative decision feedback receiver algorithm was equivalent to the Space Altering Generalized Algorithm (SAGE).

For the coded case and capacity evaluation purposes, the multistage iterative detection receiver can be regarded as a single iteration conventional decision feedback receiver since we assume that no errors are made on the the bits that are fed back. It is not necessary to use distinct single user error-correction codes for all users within the group of orthogonal users since these users do not interfere with each other. Therefore it is assumed that distinct randomly chosen single user error-correction codes are used by users in different groups of orthogonal users in order to distinguish users of different groups.

5.2 Spectral Efficiency of the TSOS Scheme

In this section, the spectral efficiency of the feedback receiver for TSOS scheme in channels with and without fading is discussed. For both cases we assume that all users in each group have the same transmitted power, i.e. all users in group G_1 have power P_1 and all users in group G_2 have power P_2. In the case of non-fading channels, which we analyze first, fading coefficients are $g_i = 1, i = 1, \dots, K$.

The capacity of user k in [bit/symbol] is

$$C_k = \frac{1}{2}\log_2\left(1 + \frac{p_k}{\sigma^2 + \sum_{i=k+1}^{K} p_i R_{ki}^2}\right), \tag{5.1}$$

where the interference plus noise $\gamma_k = \sigma^2 + \sum_{i=k+1}^{K} p_i R_{ki}^2$ is modeled as the Gaussian distribution since we assume Gaussian code symbols as in [4]. In the previous equation, we have accounted only for the interference that still has not been canceled.

We discuss two different cancellation orders for TSOS scheme. In the first, that is for convenience called TSOS1, the first group consists of $k_1 = K - N$ orthogonal users and the second group consists of $k_2 = N$ orthogonal users. The average user capacity in non-fading case (equation (5.1) averaged over all users for this sequence allocation) is independent of the specific orthogonal sequence allocation and equals to

$$\bar{C} = \left(\frac{1}{2} - \frac{1}{2\alpha}\right)\log_2\left(1 + \frac{P_1}{\sigma^2 + P_2}\right) + \frac{1}{2\alpha}\log_2\left(1 + \frac{P_2}{\sigma^2}\right), \tag{5.2}$$

where over-bar on $\bar{C}$ denotes the user average. The spectral efficiency is, according to (1.7), now equal to $\Gamma = \alpha\bar{C}$. Note that the average capacity of (5.2) can also be derived by substituting the appropriately partitioned matrix $\mathbf{S}$ into the sum capacity derived in [3], since conventional decision feedback receiver is optimal decision feedback receiver [4] in this order of cancellation and equal user powers in the second group. The optimality of conventional decision feedback receiver can be shown by substituting TSOS sequences in the equation for optimal decision feedback receiver of [4]. We note that TSOS1 scheme can not achieve the full capacity of the system since the sum of squared correlation coefficients of these sequences is $3K - 2N$ which is strictly larger than the Welch lower bound for optimal sequences of K^2/N when $N < K < 2N$. For the reverse order of cancellation (TSOS2), the average capacity is sequence dependent and we illustrate this case in Figure 5.1 with the example sequences used in [77] where first $k_1 = N$ users use Walsh Hadamard sequences and the remaining $k_2 = K - N$ users use TDMA-like sequences. The average capacity in this case can be derived in a similar manner as in (5.2). In subsequent discussion we focus on the first case (TSOS1) since it provides higher spectral efficiency than TSOS2 if the sequences in two groups are distinct, i.e. the case when these two schemes actually differ.

The problem of assigning powers P_1 and P_2 to both groups can be approached in two different ways. First, we can simply assign both groups the same power $P_1 = P_2 = P$, which results in different achievable rates for both groups of users since respective SNRs for users in different groups will not be equal. Another approach is to assign different powers to each group but achieve balanced (equal) rates for both groups by equating the signal to noise ratios for users in both groups. For the TSOS1 receiver, equal rate condition simplifies to

$$\frac{P_1}{\sigma^2 + P_2} = \frac{P_2}{\sigma^2}, \tag{5.3}$$

where we impose the condition $(1 - \frac{1}{\alpha})P_1 + \frac{1}{\alpha}P_2 = P$ to maintain the same average power as in the case of equal powers of users. By solving these two conditions and substituting in (5.2) we can get the average capacity for equal rates of users in both groups

$$\bar{C}_{EqRate} = \frac{1}{2}\log_2\left(1 + \frac{\sqrt{\alpha(\alpha + 4\frac{P}{\sigma^2}(\alpha - 1))} - \alpha}{2(\alpha - 1)}\right). \tag{5.4}$$

All previous results for spectral efficiency are summarized and illustrated in Figure 5.1.

What happens if more than two groups of orthogonal sequences are used? In this case, orthogonal sequences should be assigned to a certain group until the full set of orthogonal sequences is exhausted. After that, another group of orthogonal sequences is assigned to the next group. If distinct error-correction codes are used in different groups and detection performed first in the group with incomplete set of orthogonal users the spectral efficiency results will be periodically continued results of TSOS1 scheme for higher loads α.

Regarding the flat fading channels the following Proposition will help us find lower and upper bounds on spectral efficiency of TSOS1 scheme in these situations.

Proposition 6 *The upper bound on the single user capacity of the user $k \in G_1$ of the TSOS1 receiver in a flat fading channel, with received powers given by $p_i = g_i P_1$*

for $i \in G_1$ and $p_i = g_i P_2$ for $i \in G_2$ is

$$C_k \leq \frac{1}{2} E \left[\log_2 \left(1 + \frac{g_k P_1}{\sigma^2 + P_2 g_{\phi(k)}} \right) \right] \tag{5.5}$$

which corresponds to the case where sequences from group G_1 present the subset of sequences in G_2, i.e. user $k \in G_1$ and $\phi(k) \in G_2$ have the same sequences where specific choice of one-to-one mapping ϕ is irrelevant. The lower bound is given by

$$C_k \geq \frac{1}{2} E \left[\log_2 \left(1 + \frac{g_k P_1}{\sigma^2 + \frac{P_2}{N} \sum_{i=1}^{N} g_{i+K-N}} \right) \right], \tag{5.6}$$

which corresponds to the case where the sequences in the first group produce the same amount of interference to all users in the second group.

Proof:

The average user capacity of fast changing ergodic flat fading channels can be obtained by averaging the non-fading capacity over all random channel gains $g_1, \ldots, g_K$ [49]. To bound the average user capacity for users in the first group, $k \in G_1$, we employ the following result (Chapter 12, Section G, [62]). For independent random variables $\{g_1, \ldots, g_n\}$ and convex function g, function

$$\psi(\mathbf{x}) = \mathrm{E}\left[g\left(\sum x_i g_i \right) \right] \tag{5.7}$$

is Schur convex. We first note that for $k \in G_1$, $\sum_{i=k+1}^{K} R_{ki}^2 = \sum_{i=K-N+1}^{K} R_{ki}^2 = 1$ which follows from the fact that sequences in G_2 make a full unit norm basis. Now employing the observation (2.42) and convexity of the function $g(x) = \log\left(1 + \frac{a}{b+x}\right), a > 0, b > 0$, we can conclude that fading averaged user capacity of the user k is upper bounded in the case when the ordered set of cross-correlations between the two groups $\{R_{k\,K-N+1}^2, \ldots, R_{kK}^2\}$ is $\{1, 0, \ldots, 0\}$ and lower bounded for $\{\frac{1}{N}, \ldots, \frac{1}{N}\}$. The first case corresponds to using equal sequences in both groups i.e. when certain user $\phi(k) \in G_2$ uses the same sequence as user $k \in G_1$, and the second case to spreading the cross-correlation evenly among the second group. Note that $\phi(k)$ has to be one-to-one to ensure orthogonality of both groups of sequences. By substituting these extremal vectors in (5.1) and averaging over all random fading channel gains we can get the statement of the Proposition. □

Using the fact that single user capacities of users in G_2 are independent of the choice of sequences in TSOS scheme we can easily calculate upper and lower bounds on spectral efficiency of the TSOS1 detector in Rayleigh fading channels. These results are summarized in Figure 5.2.

5.3 Conclusions

The spectral efficiency of the TSOS scheme is derived in the non-fading case and lower and upper bounded in flat fading case. It has been shown that the spectral efficiency of the TSOS scheme is higher when the group with the smaller number of users is detected first than for the opposite cancellation order which is used in [77, 82]. This is due to the fact that the bandwidth is more efficiently used when a complete set of orthogonal sequences transmits through the channel that is interference free. The power allocation in this scheme is analyzed and it is shown that there is a penalty in spectral efficiency in order to have equal transmission rates of all users. In the flat fading case, upper and lower bounds on spectral efficiency are derived. We note that the upper bound on spectral efficiency (5.5) of TSOS1 scheme is achieved when sequences in group G_1 are repeated in G_2 - the case which is hardest to implement in practice since the separation of the users is achieved only through different error-correction codes used by users in different groups.

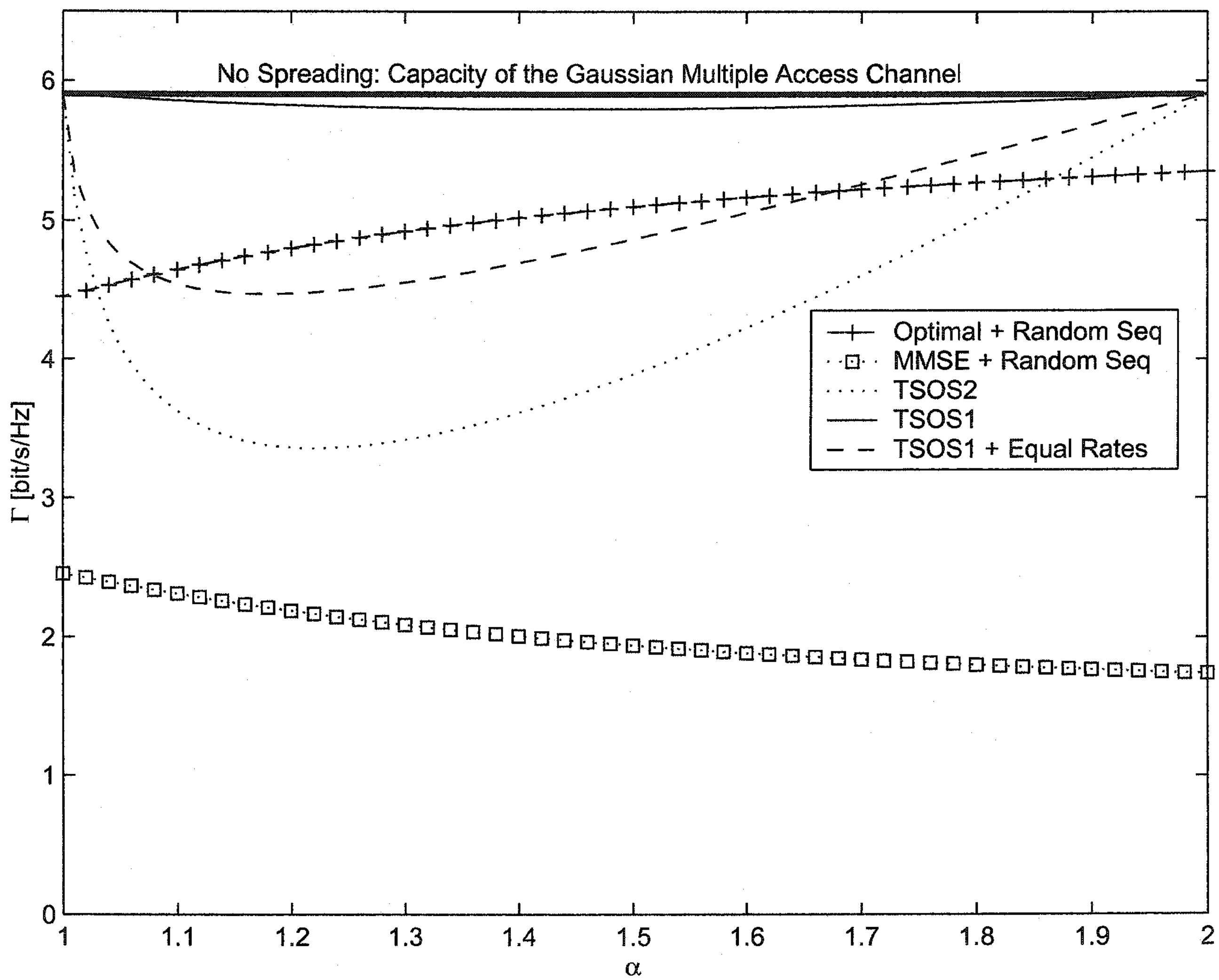

Figure 5.1. *Comparison of spectral efficiencies of the optimal and MMSE detector for long random sequences with feedback detector for TSOS and two orders of interference cancellation (TSOS1, TSOS2) and the TSOS1 scheme with equal rates of all users as a function of the load of the system α for $\bar{E}_b/N_0 = 10dB$.*

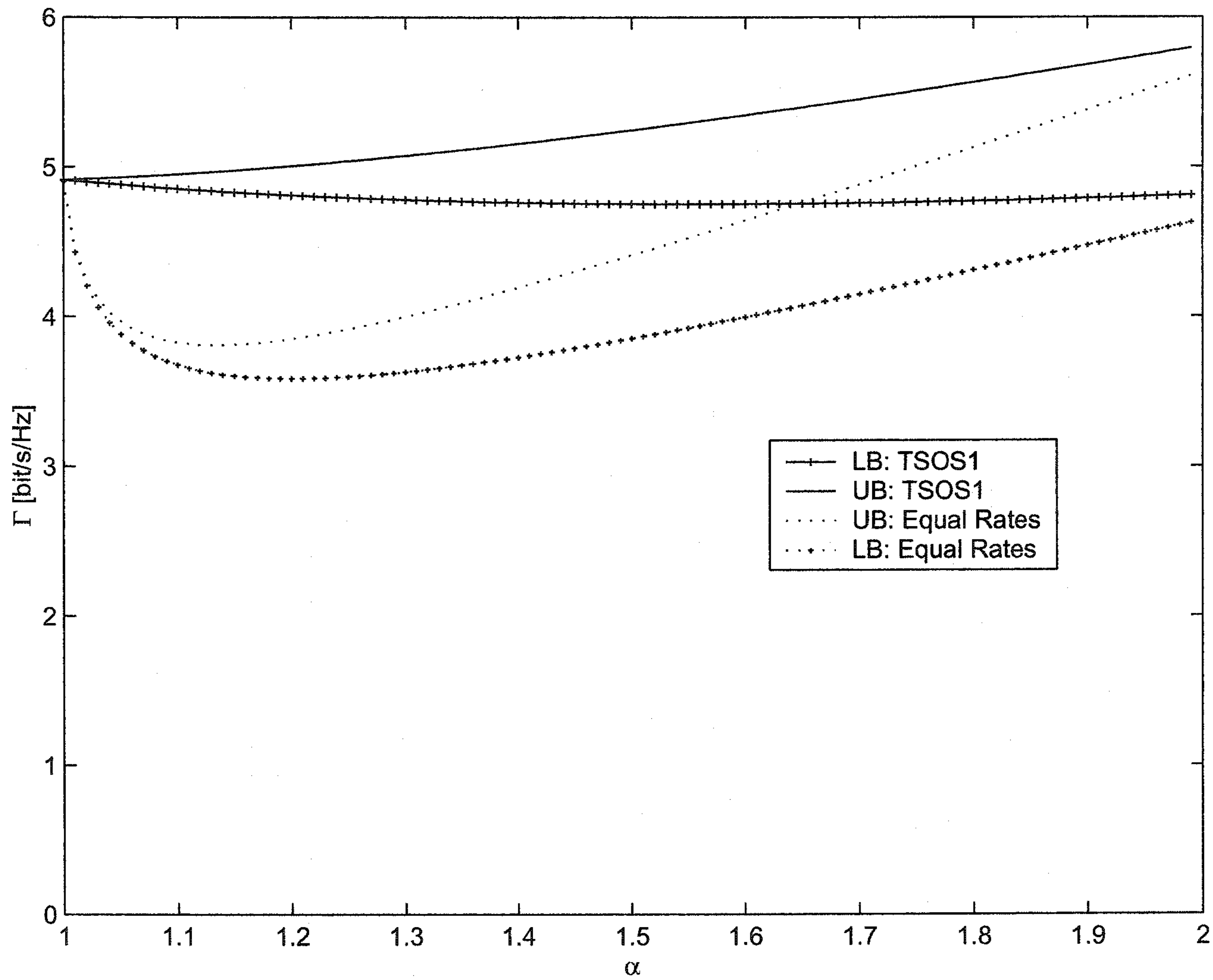

Figure 5.2. *Upper and lower bounds on spectral efficiency of TSOS1 scheme with decision feedback detector in Rayleigh fading for equal powers and equal average rates (P_1 and P_2 chosen according to (5.3)) as a function of load of the system α for $\bar{E}_b/N_0 = 10dB$.*

Chapter 6

Imperfect Channel State Information

Spectral efficiency presents an ultimate limit on the performance of a certain communication system. For the CDMA systems, spectral efficiency has been derived for almost all known multiuser receivers in case of synchronous reception, fading and non-fading environment as well as single and multi-cell cellular networks. The most pervasive model employed in all these analyses is the large system random signature model. However, in the case of fading environments or in the case of imperfect power control we cannot assume the perfect channel state information and the inevitable channel estimation error limits the performance of these systems. Our main result in this presentation is the derivation of the lower and upper bounds on the spectral efficiency of the optimum multiuser receiver. We are particularly interested in the lower bound on spectral efficiency since it is attainable by the optimum receiver with metric modified to minimize the influence of channel estimation error. It is shown that the optimum receiver is very vulnerable to channel estimation errors even when the distribution of estimation errors is known.

6.1 System Model

Received signal in a K-user symbol and chip synchronous DS-CDMA model with processing gain N is given with equation (1.1) of Chapter 1. In order to simplify the extension of the previous results on the information-theoretic issues of fading channels [26], the discrete-time vector model is derived from (1.4) for the analysis of the spectral efficiency of the optimal receiver. We assume also the perfect estimation of the phase of the received modulated signal is performed, making the coherent reception feasible. This implies that fading gain coefficients can be regarded as real numbers since only fading amplitude is being estimated.

We assume that transmitted powers of all users are equal to P, however different users can undergo different fading channels and have different received powers. The amplitude channel gain of user k due to the flat fading channel is denoted with $a_k = \sqrt{g_k}$. Furthermore, to describe the effect of imperfectly estimated channel gains we break the measured channel gain $a_k = \bar{a}_k + \tilde{a}_k$ into $\bar{a}_k$, which is the measured channel gain (or attenuation) of user k, and $\tilde{a}_k$ which corresponds to the estimation error of that channel gain with average over all estimation trials $\mathrm{E}[\tilde{a}_k] = 0$ and variance $\mathrm{var}[\tilde{a}_k] = \Omega_k \bar{a}_k^2$. These gains we arrange in diagonal matrices $\tilde{\mathbf{A}} = \mathrm{diag}[\tilde{a}_1, \ldots, \tilde{a}_K]$ and $\bar{\mathbf{A}} = \mathrm{diag}[\bar{a}_1, \ldots, \bar{a}_K]$ and we also let $\mathbf{\Omega} = \mathrm{diag}[\Omega_1, \ldots, \Omega_2]$. We assume that $\bar{a}_k$ and $\tilde{a}_k$ are statistically independent and therefore averaged over estimation errors $\mathrm{E}[a_k^2] = \mathrm{E}[(\bar{a}_k + \tilde{a}_k)^2] = \mathrm{E}[\bar{a}_k^2](\Omega_k + 1)$. For the sake of analysis of flat fading channels, we assume that measured power channel gain $\bar{a}_k^2$ of user k has a known probability distribution $f_k(g)$ where g is the realization of the channel gain. For simplicity, we assume throughout the Chapter that all users have the same probability distribution, i.e. $f_k(g) = f(g)$. The user fading processes are assumed to be ergodic which allows us to calculate the spectral efficiency by averaging the spectral efficiency of a receiver for specific realization of fading over all possible realizations of fading.

Several conclusions on spectral efficiency of optimal detector are derived for the random sequences model where the number of users and processing gain increase to infinity while maintaining their ratio $\alpha = \frac{K}{N}$ fixed. These assumptions, although theoretical in nature, provide insight into the operation of receivers for practical CDMA systems where the number of users is finite and signature sequences are pseudorandomly chosen. The material in this Chapter follows up on the material on the spectral efficiency of the CDFR with random sequences presented in Chapter 4.

6.2 Spectral Efficiency in Channels with ICSI

In this section we derive the spectral efficiency of the optimal detector in cases when received user powers are estimated with error.

First the upper and lower bounds on spectral efficiency (the sum capacity) of the optimum detector will be found by employing the results of [26]. It is shown in [26] that the mutual informations, which define the capacity region (Chapter 1 and [12]) provide more details on the definition of the capacity region of a discrete time vector channel, are lower bounded by

$$I(\{X_i\}_{i\in J};\mathbf{Y} \mid \{X_i\}_{i\in J^c}) \geq \frac{1}{2}\log_2\left|\left(\sum_{k=1}^{K}\mathrm{cov}[\tilde{a}_k\mathbf{s}_kX_k]+\mathrm{cov}[\mathbf{N}]\right)^{-1}\left(\sum_{k\in J}\mathrm{cov}[\bar{a}_k\mathbf{s}_kX_k]\right)+\mathbf{I}\right| \tag{6.1}$$

and upper bounded by

$$I(\{X_i\}_{i\in J};\mathbf{Y} \mid \{X_i\}_{i\in J^c}) \leq \frac{1}{2}\log_2\left|\mathrm{cov}[\mathbf{N}]^{-1}\left(\sum_{k\in J}\mathrm{cov}[(\bar{a}_k+\tilde{a}_k)\mathbf{s}_kx_k]\right)+\mathbf{I}\right| \tag{6.2}$$

where $X_i, i = 1,\dots,K$ and $Y_i, i = 1,\dots,N$ are inputs and outputs of a certain channel. The expressions in (6.1) and (6.2) are generalized expressions of Section III B in [26] for K users. J is a certain subset of users (i.e. $J \subseteq \{1,\dots,K\}$) and J^c is its complement $J^c = \{1,\dots,K\}/J$. To evaluate the spectral efficiency of the optimal detector we concentrate only on the analysis of the sum rate of all users in the system. We denote the upper bound on the sum rate capacity $\bar{C}_{sum}$ and the lower bound $\underline{C}_{sum}$. These bounds on the capacity can be derived from (6.1) and (6.2) for $J = \{1,\dots,K\}$.

It has been shown in [20] and [27] that the optimal receiver which uses the imperfectly estimated channel gains but perfectly estimated phases can attain the lower bound on sum capacity predicted in [26] if the decision metric is modified in order to account for the influence estimation error. The next Proposition gives the lower bound on sum capacity of the optimal detector.

Proposition 7 *The spectral efficiency of the optimal detector in channels with ICSI is lower bounded by*

$$\underline{C}_{sum}(\alpha, SNR, \boldsymbol{\Omega}, \boldsymbol{S}, \bar{\boldsymbol{A}}) = \frac{1}{2}\log_2|(\boldsymbol{S}^T\bar{\boldsymbol{A}}^2\boldsymbol{\Omega}\boldsymbol{S}\, SNR+\boldsymbol{I})^{-1}(\mathbf{S}^{\mathrm{T}}\bar{\boldsymbol{A}}^2(\boldsymbol{\Omega}+\boldsymbol{I})\boldsymbol{S}\, SNR+\boldsymbol{I})|. \tag{6.3}$$

For equal relative estimation errors $\boldsymbol{\Omega} = \Omega\,\mathbf{I}$ the previous equation simplifies to

$$\underline{C}_{sum}(\alpha, SNR, \Omega, \boldsymbol{S}, \bar{\boldsymbol{A}}) = C^{opt}(\alpha, SNR(1+\Omega), \boldsymbol{S}, \bar{\boldsymbol{A}}) - C^{opt}(\alpha, SNR\Omega, \boldsymbol{S}, \bar{\boldsymbol{A}}) \tag{6.4}$$

where $C^{opt}(\alpha, SNR, \mathbf{S})$ is spectral efficiency of the optimal detector in ergodic flat fading channels with perfect channel state information and $SNR = P/\sigma^2$.

Proof: We start from equation (6.1) by substituting $J = \{1, \ldots, K\}$ and calculating the covariance matrices for fixed actual channel gains $\tilde{\mathbf{A}}$. In this case we have

$$\underline{C}_{sum} = \frac{1}{2}\log_2 |(\mathbf{S}^T \operatorname{cov}[\tilde{\mathbf{A}}^2]\mathbf{S}\,\mathrm{P} + \sigma^2\mathbf{I})^{-1}(\mathbf{S}^T\bar{\mathbf{A}}^2\mathbf{S}\,P) + \mathbf{I}|. \tag{6.5}$$

Rearranging the terms in the previous equation and using that $\operatorname{cov}[\tilde{\mathbf{A}}^2] = \bar{\mathbf{A}}^2\mathbf{\Omega}$ we can easily arrive at (6.3).

To derive the equation (6.4) we denote by $\sigma_i(\mathbf{S}^T\bar{\mathbf{A}})$ the singular values of the matrix $\mathbf{S}^T\bar{\mathbf{A}}$. Now using the fact that $\mathbf{\Omega} = \Omega\,\mathbf{I}$ and the known fact that eigenvalues of a rational matrix function are the eigenvalues of the initial matrix transformed by the same rational function we can get

$$\bar{C}_{sum}(\alpha, \mathrm{SNR}, \Omega, \mathbf{S}, \bar{\mathbf{A}}) = \frac{1}{2}\sum_{i=1}^{K}\log_2\frac{1+\sigma_i(\mathbf{S}^T\bar{\mathbf{A}})^2(\Omega+1)\mathrm{SNR}}{1+\sigma_i(\mathbf{S}^T\bar{\mathbf{A}})^2\Omega\mathrm{SNR}}. \tag{6.6}$$

Now, using the fact that $C^{opt}(\alpha, \mathrm{SNR}, \mathbf{S}, \bar{\mathbf{A}}) = \frac{1}{2}\sum_{i=1}^{K}\log_2(1+\sigma_i(\mathbf{S}^T\bar{\mathbf{A}})^2\mathrm{SNR})$, the equation (6.4) readily follows. □

Note that although the equation (6.4) applies to any choice of signature sequences $\mathbf{S}$ and the distribution of channel gains $\bar{\mathbf{A}}$ it is easy to calculate $C^{opt}(\alpha, \mathrm{SNR}, \mathbf{S}, \bar{\mathbf{A}})$ for long randomly chosen signature sequences with fixed load α in flat fading channels as presented in Theorem IV.1 in [49]. The lower bound on sum capacity of optimal receiver in flat Rayleigh fading and no fading channels for several values of parameter Ω are presented in Figure 6.1. The dependence of the spectral efficiency on the system load for $E_b/N_0 = 10dB$ is given in this figure. The case $\Omega = 0$ corresponds to the perfect channel state information while $\Omega = 1$ corresponds to the complete absence of knowledge about the channel gains. The no fading case is included only for reference since equal user channel fading gains would normally be perfectly known in these circumstances. In the setting analyzed here, it is assumed that all fading gains are separately estimated with the same error variance.

It should be pointed out here that the assumption that relative estimation errors of channel gains of all users are equal ($\mathbf{\Omega} = \Omega\,\mathbf{I}$) is employed in Proposition 1 for convenience in simplifying the general equation of (6.3). Comparing to the large random sequences model and linear MMSE estimator of channel gains discussed in [48] in Theorem 2. it is obvious that this assumption is accurate for non-fading channels as well as for the channel with complete absence of knowledge about fading gains. This assumption is approximately valid also for LMMSE estimators in fading channels whose channel gains are not too spread out.

The upper bound on spectral efficiency (sum capacity) is derived easily using the same arguments for large random matrices as in [38] and (6.2)

$$\bar{C}_{sum}(\alpha, \mathrm{SNR}, \Omega, \mathbf{S}, \bar{\mathbf{A}}) = \frac{1}{2}\log_2 |\mathbf{S}\bar{\mathbf{A}}^2(\mathbf{\Omega} + I)\mathbf{S}^{\mathrm{T}}\mathrm{SNR} + \mathbf{I}| \tag{6.7}$$

which is equal to $C^{opt}(\alpha, \mathrm{SNR}(1+\Omega), \mathbf{S}, \mathbf{A})$ for $\mathbf{\Omega} = \Omega\,\mathbf{I}$. However, this upper bound is of theoretical interest only, since it can be attained only if channel gain estimation error is, somehow, perfectly known at the receiver.

6.3 Conclusions

Starting from the general results on the capacity loss in case of imperfect channel state information at the receiver, upper and lower bounds on the capacity loss of the optimal receiver in flat fading channels are derived. These results are then specialized for the case of equal relative estimation errors of all users. The spectral efficiency of the optimal detector for the large random sequence model is evaluated numerically for different estimation errors. It has been shown that the spectral efficiency of this detector is very vulnerable to channel gain estimation error.

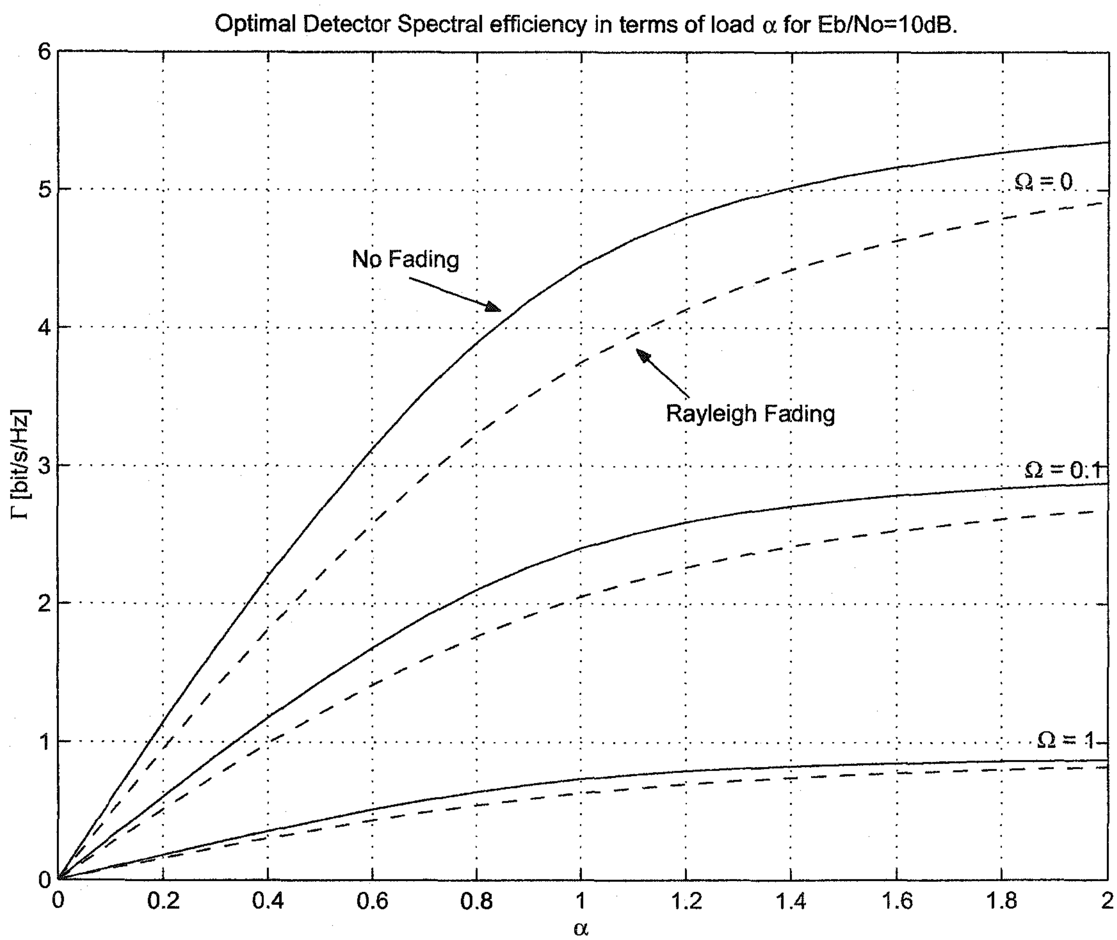

Figure 6.1. *Influence of the estimation error on the lower bound on spectral efficiency of the optimal detector for large random sequence model and Rayleigh fading for* $E_b/N_0 = 10dB$.

Chapter 7

Summary and Suggestions for Future Work

Chapter 1 presented an introduction that defined the scope of the thesis. This chapter also presented a brief survey of some known information-theoretic results on wireless multiuser channels that were the starting point for the results presented in this thesis.

In Chapter 2, optimal spreading sequence allocation in flat fading channels was discussed. A novel approach that uses results from asymptotic order statistic was used to analyze the sum capacity of the CDMA system in the limit of large number of users and processing gain. The analysis was carried out for the power control and no-power control case. These results make possible an exact comparison between the performance of the sum capacity of random sequence allocation and the sum capacity of optimal sequence allocation in the large system model. Furthermore, based on the optimal sequence allocation for no power control case, sequence allocation that divides all user spreading sequences in orthogonal subspaces was proposed. This allocation simplifies the complexity of the joint detector/decoder at the receiver.

Chapter 3 focused on the analysis of the whole capacity region of a multiple-access channel, extending the results on sum capacity of a multiple-access channel. A simple characterization of the vertices of the synchronous CDMA Gaussian multiple-access capacity region through Cholesky Decomposition of a certain matrix was presented. This Cholesky characterization was later used in order to formulate and simplify the numerical algorithm that finds the optimal sequence allocation that maximizes the symmetric capacity of the multiple access channel. This chapter also presented a reinterpretation of previously derived upper bound on symmetric capacity with optimally allocated spreading sequences. Extensive simulations showed that this upper bound was, almost always, attainable with the numerical procedure proposed in this Chapter. The asymptotic order statistic method, discussed first in Chapter 2, was used to discuss the capacity profile of the random sequence large system model for increasing and decreasing ordering of powers in the optimal decision feedback receiver.

Extending the analysis of the large system random sequence allocation model, Chapter 4 discussed the sum capacity of the Conventional Decision Feedback Receiver. Two basic models for this receiver were discussed: with and without power ordering at the receiver. A numerical procedure for the calculation of the optimal power control law was proposed for both cases and simulation results were given. It was also shown that the spectral efficiency increased without bounds with the increasing system load. The power control law which equalizes the capacities of all users was also evaluated and symmetric capacity results compared with sum capacity results.

Chapter 5 discussed a simple sequence allocation that divided users into two groups of orthogonal sequences. This separation simplified the receiver operation significantly. It was shown that the sum capacity of such a scheme was very high, close to the maximal possible. The case of power control which equalized the capacities in both groups was also discussed. Upper and lower bounds on the sum capacity of such an allocation scheme in flat fading channels were also discussed.

The case of the imperfect channel state information available at the receiver was discussed in Chapter 6. The sum capacity loss for the case of optimal detector/decoder was discussed and a simple expression for the sum capacity in case when estimation errors were proportional to the fading gain was derived.

7.1 Future Work

The material in this thesis opens several research alleys for future work. Some of the ideas that may be used in future research endeavors are presented below and are sorted according to the chapters of this thesis.

The optimal power/sequence allocation in Chapter 2 was derived using the constraint on the total radiated power of all users. It may be interesting to see how some other constraints, like constraints on average individual powers of users, affect the optimal power/sequence allocation. Another issue that stems from the material of this chapter is the usage of the asymptotic approach in the analysis of some diversity systems where selection or hybrid selection/maximal ratio combining receivers is used for large numbers of diversity branches (for example Ultra Wide Band Communications).

Several future research issues may originate from Chapter 3. During the preparation of the thesis, the author was not able to provide the formal proof that validates the choice of ordering and the convergence of the numerical algorithm that is used in Chapter 3 to obtain spreading sequences that maximize the symmetric capacity. Also, extensions of the case where different QoS requirements are assigned to different users along with the optimal sequence allocation can also be addressed. These present open challenging problems. The simplification of the algorithm that finds optimal sequences with power control that maximize symmetric capacity through the usage of Cholesky updating algorithm [70] is currently under investigation.

The issue of imperfect CSI can be addressed further in considerable detail since there are not many results on this topic, especially for multiuser receivers. It may be interesting to see how the imperfect channel state information at the receiver affects the sum capacity of other multiuser receivers that are analyzed in [49] in the case of perfect CSI. The case of imperfect CSI at the transmitter is a situation that has a strong practical appeal and it would be interesting to see how this estimation error at the receiver affects the optimal power/sequence allocation of a multiuser CDMA channel.

Bibliography

[1] C. E. Shannon, "A mathematical theory of communication," *Bell Syst. Tech. J.*, pp. 623 – 656, July – Oct. 1948.

[2] A. Glavieux C. Berrou and P. Thitimajshima, "Near Shannon limit error-correcting coding and decoding: Turbo codes," *in Proc. 1993 IEEE Int. Conf. Communications*, pp. 1064 – 1070, Geneva, Switzerland, 1993.

[3] S. Verdú, "Capacity region of Gaussian CDMA channels: The symbol synchronous case," *in Proc. 24th Annu. Allerton Conference Communication, Control, and Computing*, pp. 1025 – 1034, Oct. 1986.

[4] M. Varanasi and T. Guess, "Optimum decision feedback multiuser equalization with successive decoding achieves the total capacity of the Gaussian multiple access channel," *in Proc. Asilomar Conf. Signal, Systems and Computing*, pp. 1405 – 1409, Nov. 1997.

[5] M. K. Simon and M-S. Alouini, *Digital Communication over Fading Channels: A Unified Approach to Performance Analysis*, John Wiley & Sons, New York, 2000.

[6] R. Ahlswede, "Multi-way communication channels," *in Proc. 2nd Int. Symp. Information Theory*, pp. 103 – 135, 1971.

[7] H. Liao, *Multiple access channels, Ph.D Thesis*, Univ. Hawaii, 1972.

[8] H. Liao, "A coding theorem for multiple-access communications," *Proc. Int. Symp. Info. Theory*, 1972.

[9] A. D. Wyner, "Recent results in the Shannon theory," *IEEE Trans. on Inform. Theory*, pp. 2 – 9, Jan. 1974.

[10] T. Cover, "Some advances in broadcast channels," *Adv. Commun. Syst.*, pp. 229 – 260, vol. 4 1975.

[11] R. G. Gallager, "A perspective on multiaccess channel," *IEEE Trans. on Inform. Theory*, pp. 124 – 142, March 1985.

[12] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, John Wiley & Sons, New York, 1991.

[13] S. Verdú, "Multiple-access channels with and without frame-synchronism," *IEEE Trans. on Inform. Theory*, pp. 605 – 619, May 1989.

[14] R. S. Cheng and S. Verdú, "Gaussian multiple-access channels with intersymbols interference: Capacity region and multiuser water-filling," *IEEE Trans. on Inform. Theory*, pp. 773 – 785, May 1993.

[15] S. Verdú, "The capacity region of the symbol-asynchronous Gaussian multiple-access," *IEEE Trans. on Inform. Theory*, pp. 733 – 751, July 1989.

[16] S. Shamai and A. D. Wyner, "Information theoretic considerations for symmetrix cellular, multiple-access fading channels - parts I, II," *IEEE Trans. on Inform. Theory*, pp. 1877 – 1911, Nov. 1997.

[17] A. A. Alsugair and R. S. Cheng, "Symmetric capacity and signal design for L-out-of-K symbol-synchronous CDMA Gaussian channels," *IEEE Trans. on Inform. Theory*, pp. 1072 – 1082, July 1995.

[18] M. Rupf and J. L. Massey, "Optimum sequence multisets for synchronous code-division multiple-access channels," *IEEE Trans. on Inform. Theory*, pp. 1261 – 1266, July 1994.

[19] P. Viswanath and V. Anantharam, "Optimal sequences and sum capacity of synchronous CDMA systems," *IEEE Trans. on Inform. Theory*, pp. 1984 – 1992, Sept. 1999.

[20] E. Biglieri, J. Proakis and S. Shamai (Shitz), "Fading channels: Information-theoretic and communications aspects," *IEEE Trans. on Inform. Theory*, pp. 2619 – 2692, Oct. 1998.

[21] S. Shamai (Shitz) L. H. Ozarow and A. D. Wyner, "Information theoretic considerations for cellular mobile radio," *IEEE Trans. Veh. Technol.*, pp. 359 – 378, May 1994.

[22] A. J. Goldsmith and P. P. Varaiya, "Capacity of fading channels with channel side information," *IEEE Trans. on Inform. Theory*, pp. 1986 – 1992, Nov. 1997.

[23] R. Knopp and P. A. Humblet, "Information capacity and power control in single cell multiuser communications," *IEEE Trans. on Inform. Theory*, pp. 1384 – 1400, July 2000.

[24] A. J. Goldsmith and Soon-Ghee Chua, "Variable-rate variable-power MQAM for fading channels," *IEEE Trans. on Comm.*, pp. 1218 – 1230, Oct. 1997.

[25] S. T. Chung and A. J. Goldsmith, "Degrees of freedom in adaptive modulation: a unified view," *IEEE Trans. on Comm.*, pp. 1561 – 1571, Sept. 2001.

[26] M. Medard, "The effect upon channel capacity in wireless communications of perfect and imperfect knowledge of the channel," *IEEE Trans. on Inform. Theory*, pp. 933 – 946, May 2000.

[27] A. Lapidoth and S. Shamai (Shitz), "Fading channels: How perfect need "perfect

side information " be?," *IEEE Trans. on Inform. Theory*, pp. 1118 – 1134, May 2002.

[28] A. Lapidoth and P. Narayan, "Reliable communication under channel uncertainity," *IEEE Trans. on Inform. Theory*, pp. 2148 – 2177, Oct. 1998.

[29] S. Hanly and D. N. C. Tse, "Multiaccess fading channels - part II: Delay limited capacities," *IEEE Trans. on Inform. Theory*, pp. 2816 – 2831, Nov. 1998.

[30] G. Taricco G. Caire and E. Biglieri, "Optimal power control for the fading channel," *IEEE Trans. on Inform. Theory*, pp. 1468 – 1489, July 1999.

[31] S. Hanly and D. N. C. Tse, "Resource pooling and effective bandwidths in CDMA systems with multiuser receivers and spatial diversity," *IEEE Trans. on Inform. Theory*, pp. 1328 – 1351, May 2001.

[32] P. Viswanath, V. Anantharam and D. Tse, "Optimal sequences, power control and capacity of synchronous CDMA systems with linear MMSE multiuser receivers," *IEEE Trans. on Inform. Theory*, pp. 1968 – 1983, Sept. 1999.

[33] M. K. Varanasi and T. Guess, "Bandwidth-efficient multiple access (BEMA): A new strategy based on signal design for multiple receivers under quality-of-service constraints," *IEEE Trans. on Commun.*, pp. 844 – 854, May 2001.

[34] T. Guess, "Joint signal decision and power control for CDMA with decision-feedback receivers and asymmetric QoS constraints," *in Proc. 2001 Conf. on Information Sciences and Systems*, pp. 549 – 553, March 2001.

[35] U. E. Teletar, "Capacity of multi-antenna Gaussian channels," *European Trans. on Telecomm.*, pp. 585 – 596, Nov.-Dec. 1999.

[36] G. J. Foschini and M. J.Gans, "On limits of wireless communication in a fading environment when using multiple antennas," *Wireless Personal Communications*, pp. 311 – 335, Mar. 1998.

[37] E. Biglieri, G. Caire and G. Taricco, "Limiting performance of block-fading channels with multiple antennas," *IEEE Trans. on Inform. Theory*, pp. 1273 – 1289, May 2001.

[38] S. Verdú and S. Shamai (Shitz), "Spectral efficiency of CDMA with random spreading," *IEEE Trans. on Inform. Theory*, pp. 622 – 640, March 1999.

[39] D. N. C. Tse and S. V. Hanly, "Linear multiuser receivers: Effective interference, effective bandwidth and user capacity," *IEEE Trans. on Inform. Theory*, pp. 641 – 657, March 1999.

[40] A. J. Grant and P. D. Alexander, "Random sequence multisets for synchronous code-division multiple-access channels," *IEEE Trans. on Inform. Theory*, pp. 2832 – 2836, Nov. 1998.

[41] Z. D. Bai and Y. Q. Yin, "Limit of the smallest eigenvalues of a large dimensional sample covariance matrix," *Ann. Probab.*, pp. 1275 – 1294, June 1993.

[42] J. W. Silverstein and Z. D. Bai, "On the empirical distribution of eigenvalues of a class of large-dimensional random matrices," *J. of Multivariate Anal.*, pp. 175 – 192, Feb. 1995.

[43] J. W. Silverstein, "Strong convergence of the empirical distribution of eigenvalues of large dimensional random matrices," *J. of Multivariate Anal.*, pp. 331–330, Feb. 1995.

[44] R. Müller and P. Schramm, "Spectral efficiency of CDMA systems with linear mmse interference suppression," *IEEE Trans. on Commun.*, pp. 722 – 731, May 1999.

[45] R. Müller, "Multiuser receivers for randomly spread signals: Fundamental limits with and without decision-feedback," *IEEE Trans. on Inform. Theory*, pp. 268 – 283, Jan. 2001.

[46] S. Shamai (Shitz), S. Verdú and B. M. Zaidel, "Spectral efficiency of randomly spread DS-CDMA in a multi-cell model," *in Proc. of Thirty-seventh Allerton Conf. on Communications, Control and Computing*, pp. 22 – 24, Sep. 1999.

[47] Kiran and D. N. C. Tse, "Effective bandwidths and effective interference for linear multiuser receivers in asynchronous systems," *IEEE Trans. on Inform. Theory*, pp. 1426 – 1447, July 2000.

[48] J. Evans and D. N. C. Tse, "Large system performance of linear multiuser receivers in multipath fading channels," *IEEE Trans. on Inform. Theory*, pp. 2059 – 2078, Sep. 2000.

[49] S. Shamai (Shitz) and S. Verdú, "The impact of frequency-flat fading on the spectral efficiency of CDMA," *IEEE Trans. on Inform. Theory*, pp. 1302 – 1327, May 2001.

[50] T. Guess M. Varanasi, "Optimum decision feedback multiuser equalization with successive decoding achieves the total capacity of the Gaussian multiple-access channel," *in Proc. of Asilomar Conf.*, Nov. 1997.

[51] D. N. C. Tse and O. Zeitouni, "Performance of linear multiuser receivers in random environments," *IEEE Trans. on Inform. Theory*, pp. 171 – 188, Jan. 2000.

[52] G. Caire E. Biglieri and G. Taricco, "CDMA system design through asymptotic analysis," *IEEE Trans. on Comm.*, pp. 1882 – 1895, Nov. 2000.

[53] A. M. Tulino and S. Verdú, "Asymptotic analysis of improved linear receivers for BPSK-CDMA subject to fading," *IEEE J. on Sel. Areas in Comm.*, pp. 1544 – 1555, Aug. 2001.

[54] R. Müller and S. Verdú, "Design and analysis of low-complexity interference mitigation on vector channels," *IEEE J. on Sel. Areas in Comm.*, pp. 1544 – 1555, Aug. 2001.

[55] T. Tanaka, "Statistical mechanics of CDMA multiuser detection," *Europhys. Lett.*, pp. 540 – 546, 2001.

[56] T. Tanaka, "Average-case analysis of multiuser detectors," *Proc. IEEE Int. Symp. Info. Theory*, p. 287, June 2001.

[57] R. Müller, "On channel capacity, uncoded error probability, ML-detection and spin glasses," *Proc. Workshop on Concepts in Inform. Theory*, pp. 79 – 81, June 2002.

[58] H. S. Witsenhausen, "A determinant maximization problem occurring in the theory of data communication," *SIAM J. Appl. Math*, pp. 515 – 522, Nov. 1975.

[59] E. Castillo, *Extreme Value Theory in Engineering*, Academic Press Inc., San Diego, 1998.

[60] S. Ulukus and R. D. Yates, "Iterative construction of optimum signature sequence sets in synchronous CDMA systems," *IEEE Trans. on Inform. Theory*, pp. 1989 – 1998, July 2001.

[61] C. Rose, "CDMA codeword optimization: Interference avoidance and convergence via class warfare," *IEEE Trans. on Inform. Theory*, pp. 2368 – 2382, Sept. 2001.

[62] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York, 1979.

[63] H. A.David, *Order Statistics*, John Wiley & Sons, New York, 1981.

[64] V. K. Bhargava D. Djonin, "Asymptotic analysis of the conventional decision feedback receiver in fading channels," *in Proc. ICC Conf.*, pp. 1368 – 1372, Nov. 2002.

[65] P.Billingsley, *Probability and Measure*, John Wiley & Sons, New York, 1986.

[66] J. Yang and S. Roy, "Joint transmitter-receiver optimization for multi-input multi-output systems with decision feedback," *IEEE Trans. on Inform. Theory*, pp. 1334 – 1347, Nov. 1994.

[67] D. Gerakoulis and S. Ghassemzadeh, "Extended orthogonal code designs with applications in CDMA," *in Proc. of IEEE Sixth International Symposium on Spread Spectrum Techniques and Applications*, pp. 657 – 661, June 2000.

[68] A. Ephremides and B. Hajek, "Information theory and communication networks: An unconsummated union," *IEEE Trans. on Inform. Theory*, pp. 2619 – 2692, Oct. 1998.

[69] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1999.

[70] G. H. Golub and C. F. Van Loan, *Matrix Computations*, The Johns Hopkins University Press, 1989.

[71] W. Yu, W. Rhee, S. Boyd and J. M. Cioffi, "Iterative water-filling for Gaussian vector multiple access channels," *in Proc. of the ISIT Symposium*, p. 322, June 2001.

[72] *Matlab Support Material*, www.mathworks.com.

[73] A. J. Viterbi, "Very low rate convolution codes for maximum theoretical performance of spread-spectrum multiple-access channels," *IEEE J. Sel. Areas Commun.*, pp. 631 – 649, May 1990.

[74] Y. C. Yoon, R. Kohno and H. Imai, "Combination of an adaptive antenna array and a canceller of interference for direct-sequence spread-spectrum multiple-access system," *IEEE J. Sel. Areas Commun.*, pp. 675 – 682, May 1990.

[75] A. L. C. Hui and K. B. Letaief, "Successive interference cancellation for multiuser asynchronous DS/CDMA detectors in multipath fading links," *IEEE Trans. Commun.*, pp. 384 – 391, Mar. 1998.

[76] P. Patel and J. Holtzmann, "Analysis of simple successive interference cancellation scheme in a DS/CDMA system," *IEEE J. Sel. Areas Commun.*, pp. 796 – 807, June 1994.

[77] H. Sari, F. Vanheverbeke and M. Moenclaey, "Multiple access using two sets of orthogonal signal waveforms," *IEEE Commun. Lett.*, pp. 4–6, Jan. 2000.

[78] R.Müller, *Power and Bandwidth Efficiency of multiuser systems with random spreading*, Univ. Erlangen-Nürnberg, Nürnberg, Erlangen-Nürnberg, Nürnberg, 1999.

[79] A. Antoniou and W.-S. Lu, *Optimization: Methods, Algorithms, and Applications*, Kluwer Academic, to be published, 2002.

[80] D. Djonin and V. K. Bhargava, "On the spreading sequence allocation for oversaturated CDMA systems under performance and complexity considerations," *to be published in IEEE Trans. on Wireless Communications*, April 2003.

[81] T. Guess and M. K. Varanasi, "Error exponents for maximum-likelihood and successive decoders for the Gaussian CDMA channel," *IEEE Trans. on Inform. Theory*, pp. 1683–1690, July 2000.

[82] H. Sari, F. Vanheverbeke and M. Moenclaey, "DS/CDMA with two sets of orthogonal spreading sequences and iterative detection," *IEEE Commun. Lett.*, pp. 289 – 291, Sept. 2000.

[83] D. V. Djonin and V. K. Bhargava, "Low complexity receivers for over-saturated cdma systems," *in Proc. of Globecom 2001*, pp. 846 – 850, Nov. 2001.

[84] T. W. Anderson, *An Introduction to Multivariate Statistical Analysis*, John Wiley & Sons, 1958.

[85] V. A. Marcenko and L. A. Pastur, "Distribution of eigenvalues of some sets of random matrices," *Math. USSR-Sb*, pp. 457 – 483, 1967.

[86] C. G. Khatri, "On certain distribution problems based on positive definite quadratic functions in normal vectors," *Ann. Math. Stat.*, pp. 468–470, Apr. 1966.

Appendix A

Random Matrices

This appendix reviews several important results on the behavior of the eigenvalue distributions of random matrices. It complements the random sequence model which was used in Chapters 3,4 and 6 in order to analyze the sum capacity and the symmetric capacity of a CDMA multiple-access channel. The analysis of random sequences falls under the purview of multivariate statistical analysis (see for example an introductory text in [84]).

We begin with the definition of a random matrix

Definition 8 *A random matrix $\boldsymbol{S}$ is a matrix $\boldsymbol{S} = \{s_{ij}\}, i = 1, \ldots, n_1,\ j = 1, \ldots, n_2$ of random variables $s_{11}, \ldots, s_{n_1 n_2}$.*

Now we review two results on eigenvalue distributions of random matrices without proof which are used to derive spectral efficiencies and effective interferences of various receivers in [38], [49] and [39].

Next Proposition due to Bai and Yin [41] gives the asymptotic distribution of eigenvalues of a large dimensional random correlation matrix.

Proposition 8 *Let $\boldsymbol{S}$ be a $K \times N$ dimensional matrix consisting of independent identically distributed binary random variables. Let $K/N \to \alpha \in (0, \infty)$ as $K \to \infty$. Than, the percentage of the K eigenvalues of the correlation matrix $\boldsymbol{R} = \boldsymbol{S}\,\boldsymbol{S}^T$ that lie below x converges to the cumulative probability density function*

$$F_\alpha(x) = [1 - \alpha^{-1}]^+ \delta(x) + \frac{\sqrt{[x-a]^+[b-x]^+}}{2\pi\alpha x} \tag{A.1}$$

where $[x]^+ = \max\{0, x\}$, and $a = (1-\sqrt{\alpha})^2, b = (1+\sqrt{\alpha})^2$.

The following important result [42] [1] for is an extension of [85] and was used extensively in the analysis of the effective interference and effective bandwidth given in [39].

In order to review this proposition we will need the definition of the Stieltjes transform of a distribution function

$$m_F(z) = \int \frac{1}{\lambda - z} dF(z). \tag{A.2}$$

Now we can present the statement of this Proposition in terms of Stieltjes transform.

Proposition 9 *Let*

i) $\boldsymbol{S}_K = \{S_{ij}^K\}$ *be an* $N \times K$ *dimensional matrix of independent (across* i, j*) random variables* S_{ij}^K.

ii) $K/N \to \alpha \in (0, \infty)$ *as* $K \to \infty$.

iii) $\boldsymbol{T}_K = diag\{t_1, \ldots, t_K\}$, $t_i \in \mathcal{R}$, *and the empirical distribution function of* $\{t_1, \ldots, t_K\}$ *converges almost surely in distribution to a probability distribution function* H *as* $K \to \infty$.

iv) $\boldsymbol{B}_K = \boldsymbol{I}_N + \boldsymbol{S}_K \boldsymbol{T}_K \boldsymbol{S}_K^T$ *where* $\boldsymbol{A}_K$

v) $\boldsymbol{T}_K$ *and* $\boldsymbol{S}_K$ *are independent*

Then almost surely the empirical distribution function of the eigenvalues of $\boldsymbol{B}_K$ *converges to, as* $K \to \infty$ *to a nonrandom cumulative distribution function* F *whose Stieltjes transform* $m_F(x)$ *satisfies for* $z \in \boldsymbol{C}^+$

$$m_F(z) = \frac{1}{-z + \alpha \int \frac{y dF(y)}{1 + y m_F(z)}} \tag{A.3}$$

The notion of the *large random sequence model* is often encountered in this thesis, and the following remark formalizes this model.

Remark 5 *By large random sequence model in this thesis we assume that all conditions of the previous proposition are satisfied. The signature sequences correspond to matrix* $\boldsymbol{S}$, *while random channel gains/powers correspond to matrix* $\boldsymbol{T}$ *of the previous proposition.*

Among many other results on the asymptotical properties of random matrix, we emphasize [43] which deals with the asymptotic properties of a matrix $\mathbf{S}\ \mathbf{S}^T\mathbf{G}$ where $\mathbf{S}$

[1] The exposition of this Proposition is slightly simplified compared to the original result.

is a random matrix while $\mathbf{G}$ is a matrix with known empirical eigenvalue distribution function.

For completeness of this short survey we mention that some results on eigenvalue distributions of some non-asymptotic random matrices is also known and can be found in [86].