ESTIMATING THE HISTORIC PROBABILITY OF STAND-REPLACEMENT FIRE USING THE AGE-CLASS DISTRIBUTION OF UNDISTURBED FOREST

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Abstract

A simple probabilistic model for the destruction of forest stands by fire is used to determine a likelihood function for the problem of estimating the per-annum probability of stand-replacement fire using data in the form of areas in various age classes of undisturbed old-growth forest. This enables the estimation of either an age-independent fire probability or of the parameters of a parametrically defined age-dependent probability. The model includes contagion effects (the probability of a stand burning being increased by other stands burning). Maximum likelihood is used to obtain estimates of the age-independent per annum probability of stand-replacement fire (or the parameters of an age-dependent model), along with a contagion parameter. Confidence intervals for the fire probability are also obtained.

Keywords: Stand-replacement fire; even-aged stands; maximum likelihood; contagion; Polya's Urn Scheme; Dirichlet distribution.

1. INTRODUCTION

Much research over the past decade has demonstrated that the long-run timber supply and the flow of economic benefits from a forest can be seriously reduced by the presence of the risk of fire. This conclusion has been established using analytic methods at the stand level (Martell, 1980; Reed & Errico, 1985) and both simulation (Van Wagner, 1983) and optimization methods (Reed & Errico, 1986) at the forest level. This research provides a strong justification for forest fire protection. However by itself it does not provide any indication as to what level of fire protection is appropriate. In order to address this question one needs to assess quantitatively the benefits of various levels of fire protection and then perform a cost-benefit analysis. The costs of protection are easily assessed; the benefits less readily so, for they depend on the fire probability, still present in a protected forest, and on the fire probability which would prevail in the absence of protection. The former probability is fairly easily estimated from recent data on areas burned in protected forests, with the given level of protection; the latter probability is much harder to assess. In some cases, in remote areas where people-caused fires are very rare, one could use as an estimate of the no-protection fire probability, the estimated fire probability in historic times. In areas where people-caused fires are more prevalent this method would not be appropriate, since the fire probability which would prevail in the absence of protection would likely be considerably higher than the historic one. Nevertheless, even in such regions determination of the historic fire probability is of scientific interest. It is the purpose of this paper to address the question of estimating the historic probability of destructive fires. Many of the forests for which the method of estimation is appropriate exist in remote northern or mountainous regions. For such forests the method should be useful as a component of the cost-benefit analysis of fire protection.

Various methods have been used to estimate the incidence of fire in times before the systematic reporting and recording of fires. These include the use of fire scars (*e.g.* Heinselman, 1973) and examination of lake bed sediments for the incidence of charcoal layers (*e.g.* Swain, 1973). These methods all rely on the correct dating of fires which occurred a considerable time in the past. This is often a difficult and costly procedure. Another approach, first pursued by Van Wagner (1978), does not require a record of individual fires, but rather relies on the current age-class distribution of forest which has not been disturbed by logging. Van Wagner writes "It is generally acknowledged that many North American forest types, especially boreal ones, are mainly dependent in the natural state on periodic fire for their continued existence." Forests of this type, in which parts are periodically destroyed by fire and replaced by the growth of new stands, will comprise even-aged stands, each of which will have originated following a destructive fire.[1] Van Wagner notes that for such forests "the present age-class structure is the key to its past fire history."

The central idea of Van Wagner was, loosely put, that if there is a constant probability of a destructive fire in any one stand, then the age-class distribution over the whole forest should, in the long run, closely follow an exponential distribution (or if the probability of fire depends on stand age it should follow some other distribution, such as the Weibull). Thus if one plots on semi-log paper, the frequency distribution by age against age, for a large forest area, one should obtain approximately a straight line whose (negative) slope reflects the per-annum probability of destructive fire.

Van Wagner's paper was written primarily to demonstrate how a fairly simple mathematical model of the probability of the occurrence of stand-replacement fire could be used to explain existing age-class distributions of forest undisturbed by harvesting. An estimate of the fire cycle (the reciprocal of the per annum probability of stand-replacement fire) followed from a plot (of logarithm of area frequency *vs.* age), whose main purpose was to validate the predictive power of the mathematical model. Since estimation was not the main goal it is not surprising that the estimation procedure was not grounded in an established paradigm of statistical inference. For example the estimation procedure provides no way of assessing the precision of the estimate *i.e.* it does not provide the standard error of the estimate, nor a confidence or likelihood interval for the fire cycle.

It is the purpose of this paper to develop the approach of Van Wagner using the method of maximum likelihood and to make explicit the assumptions involved in its application. The starting point will be data in the form of the age-class distribution, by area, of a forest comprising many even-aged stands each of which originated following a stand replacement fire. The aim is to estimate the per annum probability of stand destruction for such stands. The reciprocal of this estimated probability will provide an estimate of the average time elapsed between destructive fires on a given site (*i.e.* the period of *the fire cycle*).

2. THEORY

We shall consider a large forest area which has never been harvested and which comprises a number of even-aged stands all of which can be assumed to have originated following a destructive "stand-replacement" fire.[2] We shall assume that the whole forest area is homogeneous with respect to the probability of destructive fire. For such a forest area we shall suppose that data in the form of areas in various age classes are available. An example is given in Table 1 which shows areas (in hectares) in age-classes 41-60, 61-80, 81-100, 101-120 and 121-140 years for montane spruce in the biogeoclimactic zone of the Kootenay Timber Supply Area (T.S.A.) in southeastern British Columbia.

To begin we shall consider only the case in which all age-classes are of equal width (say T years), although later the case in which the oldest age class is of a different width or is open-ended, will be discussed. The age class containing stands of ages between $(r\text{-}1)T$ and rT years will be labelled as age-class r; and the youngest and oldest age classes will be age classes a and $a + k$ respectively. Thus for the age classes in Table 1 we have $a = 3$, $k = 4$ and $T = 20$ (the index r running from 3 to 7). Denote the areas (in hectares) of even-aged stands in age-classes $a,a+1,\cdots,a+k$ by $x_a,x_{a+1},\cdots,x_{a+k}$. Using these data we wish to estimate the per-annum probability of stand-replacement fire using the method of maximum likelihood. In order to do this we need a plausible stochastic model for the way in which such data were generated. To this end we make the following assumptions.

(A) On any particular forest site the even-aged stand growing there originated following a destructive fire on the site.[3]

(B) For all sites in the forest, in times before the intervention of Europeans in the area, the probability of a stand-replacement fire *depended only on the age of the stand* growing there, and not on the location of the site, nor explicitly on time. Thus we are assuming that the historic probability of stand-replacement fire, while it might be *age-dependent*, was otherwise homogeneous with respect to time and space.

(C) Since the intervention of settlers of European origin, the probability of destructive fire has been the same for all age-classes and is homogeneous over space, *even though it may have changed over time* since the first intervention.

While these assumptions are probably not strictly met, in most cases they may serve as a first approximation and they do permit the development of a tractable statistical model.

Assumption (B) allows the probability of a destructive fire to be characterized by what is known in the statistical and engineering literature as a *hazard-rate function* (see *e.g.* Kalbfleisch, 1985a, p. 221)

(1)
$$h(t) = \lim_{\Delta \to 0} \left\{ \begin{matrix} P \text{ (stand destroyed between ages} \\ t \text{ and } t+\Delta \mid \text{'alive' at age } t)/\Delta \end{matrix} \right\}$$

In order to avoid confusion with the usual forestry usage of the term fire hazard, we shall eschew use of the word "hazard" and refer to $h(t)$ as the *age-specific per annum probability of a stand-replacement fire.*[4] The probability that a given stand survives to age x is given by the *survivor function,* $S(x)$, which is related to $h(t)$ by

(2)
$$S(x) = \exp\left\{-\int_0^x h(t)\,dt\right\}$$

(see *e.g.* Kalbfleisch, *ibid.*).

We shall assume that the per annum probability of stand-replacement fire is either *age-independent*, $h(t) \equiv \lambda$, or that it is age-specific following a simple parametric form, such as that of the *Weibull model* (*e.g.* Kalbfleisch, 1985a, p. 223):

(3)
$$h(t) = \upsilon \beta t^{\beta - 1}.$$

With $\beta > 1$, the stand-replacement fire probability increases with age, and with $\beta < 1$ it decreases with age. The case $\beta = 1$ reduces to the age-independent case above. We shall estimate the parameters λ, or υ and β, using the method of maximum likelihood.

To determine the likelihood function (the probability of observing the given data) we note firstly that (from assumptions (A) and (B)) that up until the time of European intervention fires on a given site would occur in a *renewal process* (see *e.g.* Ross, 1983, Chapt. 3). In the age-independent case the renewal process would be a Poisson process.

From the theory of renewal processes it follows (see *e.g.* Ross, *ibid.*) that after many cycles of destructive fire-re-establishment-destructive fire, etc. the probability distribution of the age of the stand at any fixed time is characterized by the equilibrium probability density function (p.d.f.) $f_e(x)$ given by

(4) $$f_e(x) = \frac{1}{\mu} S(x)$$

where μ is the expected time between stand-replacement fires (*i.e.* the *fire cycle*). Of course μ also represents the expected lifetime of a stand and is determined by

(5) $$\mu = \int_0^\infty x h(x) S(x) \, dx,$$

since $h(x)S(x)$ is the p.d.f. of the time between stand-replacement fires (renewals) (see *e.g.*, Ross, *ibid.*). It follows that the probability that, at any fixed time (before European intervention), a stand would be in age-class r is

(6) $$\psi_r = \int_{(r-1)T}^{rT} f_e(x) \, dx = \frac{1}{\mu} \int_{(r-1)T}^{rT} S(x) \, dx.$$

We now consider the period since intervention by people of European origin over the last 100-150 years, which has undoubtedly affected the probability of stand-replacement fire. On the one hand the possibility of man-caused (deliberate or accidental) fires will have increased this probability, but at the same time efforts at protecting against fire and limiting the spread of fires will have influenced the probability in the other direction. To model this let us suppose that intervention first occurred τ years ago and that the fire probability in subsequent years has been *age-independent*,[5] with the per annum probability of stand-replacement fire for stands of all ages, y years ago given by $\sigma(-y)$, $0 \leq y \leq \tau$. With this assumption the probability that a given site *at the present time* contains "old-growth" forest whose age belongs to the interval $[t, t+dt]$ is thus

(7)
$$[f_e(t-\tau)dt] \times \left[\exp\left(-\int_{-\tau}^{0} \sigma(z)\,dz\right)\right].$$

The exponential term represents the probability of survival over the last τ years. The probability that a particular site, at the present time, contains "old-growth" forest in age-class r is obtained by integrating (7) over the class interval and is given by

(8)
$$\begin{aligned}\psi_r' &= \int_{(r-1)T}^{rT} fe(t-\tau)\ \exp\left[-\int_{-\tau}^{0} \sigma(z)\,dz\right] dt \\ &= \frac{1}{\mu}\ \exp\left[-\int_{-\tau}^{0} \sigma(z)\,dz\right] \int_{(r-1)T}^{rT} S(t-\tau)\,dt.\end{aligned}$$

Now we are considering only forest which is "old-growth" *i.e.* of age at least τ years, (*i.e.* $(a\text{-}1)T \geq \tau$). Given that a site at present contains "old-growth" forest in one of the age-classes $a, a+1, \cdots, a+k$ the *conditional probability* that it contains trees in age class r is

(9)
$$\theta_r = \psi_r' / \sum_{r=a}^{a+k} \psi_r'$$

which, using (8), reduces to

(10)
$$\theta_r = \frac{\int_{(r-q)T}^{rT} S(t-\tau)\,dt}{\int_{(a-1)T}^{(a+k)T} S(t-\tau)\,dt} = \frac{\int_{(r-1)T-\tau}^{rT-\tau} S(z)\,dz}{\int_{(a-1)T-\tau}^{(a+k)T-\tau} S(z)\,dz}$$

Observe that this conditional probability *does not depend* on the per annum probability of destruction $\upsilon(-y)$ prevailing since intervention, but it does depend on the time τ at which such intervention occurred. However in the important special case of an *age-independent* fire probability, $h(t) \equiv \lambda$, the dependence on τ also vanishes, since in this case $S(t) = e^{-\lambda t}$, and (10) reduces to

(11)
$$\theta_r = \frac{\exp[-\lambda(r-a)T] - \exp[-\lambda(r-a+1)T]}{1 - \exp[-\lambda(k+1)T]}$$

i.e.

(12)
$$\theta_r = \frac{q^{r-a}(1-q)}{1-q^{k+1}}$$

where $q = e^{-\lambda T}$, (N.B. q represents the probability of a stand surviving from the beginning to end of any given age class).

If the per annum probability of stand-replacement fire $h(t)$ exhibits only slight age dependence then, since $S(t)$ will deviate only slightly from an exponential form, the dependence of the conditional probability θ_r on τ should only be slight. Thus a lack of accuracy in specifying the date of the first intervention should not cause much error in θ_r. Also in the age-independent case if the time since the last destructive fire on the site is known only up to an additive constant, (which could be the case if the current even-aged stand succeeded a pioneer species on the burned over site - see footnotes 2 and 3) it would have no effect on the conditional probabilities θ_r. Thus if the per annum probability of stand-replacement fire shows little dependence on stand age, the distinction between stand age and time since last destructive fire is not important (see footnote 3).

If one could reasonably assume that each site burned or did not burn, at any time, independently of what happened at other sites, then one could write down the likelihood function as a multinomial probability of observing $x_n, x_{a+1}, \cdots, x_{a+k}$ hectares in classes $a, a+1, \cdots, a+k$ with probabilities $\theta_a, \theta_{a+1}, \cdots, \theta_{a+k}$. Unfortunately this is not a reasonable assumption. The fact of a fire on one site will increase the probability of fires on adjacent sites. Thus there is a *contagion effect* - the probability of a given area being in a particular age class will increase the probability of other areas being in the same age class. Various models for contagious effects have been proposed and used in biological and other applications. However most involve univariate distributions (*e.g.* negative binomial, Neyman type A etc.) for sample frequencies and are not appropriate for the problem here which involves a multivariate distribution of areas in the various age classes. Another widely used model of contagion is that of Pólya (see *e.g.* Feller, 1968, Chapt. V.2), which in its limiting form gives rise to a *multivariate Dirichlet distribution* for the distribution of the proportions falling in various categories.[6] We shall use this multivariate p.d.f. for the *proportional areas* $y_r = x_r / \sum_{r=a}^{a+k} x_r$ in the various age classes to generate the likelihood function rather than the multinomial probability mass function for the *absolute* areas $x_a, \cdots, x_{a+k},$ discussed above. Specifically for this contagious model the likelihood is

$$L = \frac{\Gamma(\rho^{-1})}{\prod_{r=1}^{a+k} \Gamma(\theta_r \rho^{-1})} y_a^{[(\theta_a/\rho)-1]} y_{a+1}^{[(\theta_{a+1}/\rho)-1]} \cdots y_{a+k}^{[(\theta_{a+k}/\rho)-1]} \qquad (13)$$

where the θ_r $(r = a, \cdots, a+k)$ are as given in (10) (or (12)); and ρ is a *contagion parameter*, large value of ρ representing a high degree of contagion.[7] Note that with this model the proportion of the observed undisturbed "old-growth" forest falling in age-class r is a random variable with *expected value* θ_r.[8]

If we assume that the per annum probability of destructive fire is age-independent ($h(t) \equiv \lambda$) the likelihood (13) will involve two parameters λ and ρ. From the point of view of numerically maximizing the log-likelihood it is more convenient to consider the parameters

$$q = e^{-\lambda T}; \qquad p = \ln\left\{\rho \frac{(1-q^{k+1})}{1-q}\right\}. \qquad (14)$$

In this case the log-likelihood[9] is:

$$\begin{aligned} \ell(p,q) = \text{loggamma}\left[\frac{1-q^{k+1}}{1-q} e^{-p}\right] \\ - \sum_{r=a}^{a+k} \text{loggamma}\left[q^{r-a} e^{-p}\right] \\ + \sum_{r=a}^{a+k} \left[q^{r-q} e^{-p} - 1\right] \ln(y_r) \end{aligned} \qquad (15)$$

which must be maximized with respect to p and q. The case in which the per annum probability of stand-replacement fire is age-dependent, following a Weibull form is discussed in Section 4.

Numerical methods are required to maximize the log-likelihood (15) with respect to the parameters p and q. The invariance property of maximum likelihood (ML) estimates (see *e.g.* Kalbfleisch, 1985b) enables the direct computation of the ML estimates of λ and ρ from the ML estimates $\hat{q}$ and $\hat{p}$ via (14). Details are given in the example in Section 3.

3. A NUMERICAL EXAMPLE

Table 1 gives the areas of even-aged stands in age-classes of equal width for the Montane Spruce biogeoclimactic zone of southeastern British Columbia. Although much of this forest was established since the first incursions of Europeans to the area, it can in essence be considered as "old growth" since the area has never been harvested and it is believed that fire conditions prevailing up to forty years ago are essentially the same as those much earlier since up to that time there was no fire protection, and very little human encroachment on the forest.[10]

If we assume that the per annum probability of stand-replacement fire is age independent, the log-likelihood function (15) must be maximized with respect to the parameters p and q. The only constraint is that q be non-negative. Since derivatives of (15) cannot be computed analytically, an optimization routine which does not require such derivatives (such as the Nelder-Mead Simplex method) (Nelder & Mead, 1965) must be used. In fact the *"FindMinimum"* routine of *Mathematica* (Wolfram, 1991) was used.[11]

The ML estimates of q and p obtained from the maximization procedure were

$$\hat{q} = 0.67661, \qquad \hat{p} = -3.5032 \tag{16}$$

with the maximal value of ℓ being 7.8777. Figure 1 provides a contour plot of the log-likelihood (15) with contours at levels (moving outwards) 2.30, 3.00 and 4.60 units below the maximum. They correspond to 10%, 5% and 1% *likelihood regions* (Kalbfleisch, 1985b) or to approximate 90%, 95% and 99% *confidence regions*. Figure 2 shows the *profile* (or concentrated) *log-likelihood*

$$\hat{\ell}(q) = \max_{p} \ell(q,p) \tag{17}$$

The horizontal line is at $1.92 = \frac{1}{2}\chi^2_{2;05}$ below the maximum, and the interval $[q_L, q_u]$ provides an approximate 95% confidence interval for q.[12] It is [0.563, 0.820].

Using the invariance properties of ML estimates, one can obtain the ML estimate of the per annum probability of stand-replacement fire. It is

$$\hat{\lambda} = -\frac{1}{20}\ln(\hat{q}) = .01953 \tag{18}$$

while that of the contagion parameter ρ is

$$\hat{\rho} = \frac{1-\hat{q}}{1-\hat{q}^5}\exp(\hat{p}) = .01134. \tag{19}$$

An approximate 95% confidence interval for the per annum probability of destructive fire is 0.010 - 0.028. The corresponding ML estimate of the expected fire cycle (expected time between destructive fires) is 51.2 years (with an approximate 95% confidence interval of 35-101 years).

For the sake of comparison, the graphical method of Van Wagner (1978) was used on these data. Figure 3 shows a plot of log (proportional area) *vs*. age. Also included is the least-squares regression line, which has a slope of 0.0230. It can be seen that the points lie fairly close to the line (which is what one would expect if the per-annum probability of stand-replacement fire were age-independent). The slope, which gives a rough estimate of the fire probability, is compatible with the ML estimate of 0.0195 obtained here.

Unfortunately there are no estimates of the historic fire probability, obtained by other means, available for this region,[10] with which a comparison could be made.

4. AGE-DEPENDENCE IN THE PROBABILITY OF STAND-REPLACEMENT

One can use the method outlined in Section 3 without the assumption of an age-independent stand-replacement probability. However one needs to specify a simple parametric form for the age-dependence. Such a parametric form with a long pedigree in the literature on statistical survival analysis is the *Weibull model*, which assumes that the per-annum probability of destructive fire for stands of age t is

(19) $$h(t) = \upsilon \beta t^{\beta-1}.$$

This model has also been widely used in forest fire studies (see *e.g.* Rowe *et al.*, 1975, Johnson & Rowe, 1977 and Johnson and Van Wagner, 1985). For the Weibull model the survivor function is

(20) $$S(x) = \exp(-\upsilon x^{\beta}).$$

Substituting this into the expression (10) for the conditional probabilities θ_r results in expressions involving the *incomplete gamma function*

(21) $$\Gamma(x;\alpha) = \int_0^x t^{\alpha-1}e^{-t}dt.$$

Specifically

(22) $$\theta_r = \frac{\Gamma(\upsilon[rT-\tau]^{\beta};\beta^{-1}) - \Gamma(\upsilon[(r-1)T-\tau]^{\beta};\beta^{-1})}{\Gamma(\upsilon[(a+k)T-\tau]^{\beta};\beta^{-1}) - \Gamma(\upsilon[(a-1)T-\tau]^{\beta};\beta^{-1})}$$

Upon making the substitution

(23) $$m = \log\{\rho[\Gamma(\upsilon[(a+k)T-\tau]^{\beta};\beta^{-1}) - \Gamma(\upsilon[(a-1)T-\tau]^{\beta};\beta^{-1})]\}$$

the log-likelihood (13) can be expressed as:

(24) $$\begin{aligned} \ell = {} & \text{loggamma}\left[e^{-m}\{\Gamma(\upsilon[(a+k)T-\tau]^{\beta};\beta^{-1}) - \Gamma(\upsilon[(a-1)T-\tau]^{\beta};\beta^{-1})\}\right] \\ & - \sum_{r=a}^{k} \text{loggamma}\left[e^{-m}\{\Gamma(\upsilon[rT-\tau)]^{\beta};\beta^{-1}) - \Gamma(\upsilon[(r-1)T-\tau]^{\beta};\beta^{-1})\}\right] \\ & + \sum_{r=a}^{a+k}\left[e^{-m}\{\Gamma(\upsilon[rT-\tau]^{\beta};\beta^{-1}) - \Gamma(\upsilon[(r-1)T-\tau]^{\beta};\beta^{-1})\} - 1\right]\ln(y_r). \end{aligned}$$

For given values of τ, m, υ and β this function can be evaluated numerically. Rather than try to maximize the function numerically over τ, m, υ and β simultaneously, the procedure followed was to set τ (the time since European intervention) at a fixed value, and maximize the log-likelihood over the three parameters m, υ and β. The parameter τ was then varied and the process repeated. With τ set at 40 years the maximum likelihood estimates of the parameters were:

(25) $$\hat{\upsilon} = 0.1052; \quad \hat{\beta} = 1.0023; \quad \hat{m} = -4.6324$$

and the maximized log-likelihood was

(26) $$\hat{\ell} = 7.8792.$$

The ML estimate of $\hat{\beta}$ is very close to 1, indicating very little age dependence. When τ was varied the ML estimates changed hardly at all - as indeed one would expect, since in this case the dependence of the conditional probabilities θ_r in (10), on τ would be very slight.

Using (23) the ML estimate of the contagion parameter ρ can be obtained. It is $\hat{\rho} = 0.01128$, which is very close to the value obtained with the age-independent model. The ML estimate of the age-dependent per annum probability of stand-replacement fire is

$$0.1956\, t^{.0023}.$$

This ranges from 0.01970 at age $t = 20$ to 0.01978 when $t = 140$. Thus there appears to be no evidence of age dependence in the probability of destructive fire.[13]

5. AGE-CLASSES OF UNEQUAL WIDTH

In many cases forest inventory may have older aged forest lumped into one or two age classes. For example, in the British Columbia Forest Service inventory stands of age up to 140 years are usually grouped into 20 years age-classes while older stands are put into categories 141-240 years and older than 240 years. Adjusting the method for such data is conceptually not difficult - it simply involves changing the limits of integration in the expression (10).

If the k youngest age-classes are of width T years (ranging from $(a\text{-}1)$ to $(a+k\text{-}1)T$) while the $(k+1)$-st age class is of width αT years and the oldest age class includes stands of age *greater than* $(a+k+\alpha-1)T$, then in the case of an age-independent destruction probability, the conditional probabilities θ_r in (9) can be expressed as:

(27) $$\theta_r = q^{r-a}(1-q) \qquad r = a, a+1, \cdots, a+k-1$$

(28) $$\theta_{a+k} = q^k(1-q^\alpha)$$

and

(29) $$\theta_{a+k+1} = q^{k+\alpha}.$$

These expressions can be used directly in the log-likelihood (15). In the case when there is no forest in the oldest age class (no forest of age greater than $(a+k+\alpha-1)T$ years) the corresponding conditional probabilities are:

(30) $$\theta_r = \frac{q^{r-1}(1-q)}{1-q^{\alpha+k}} \qquad r = a, a+1, \cdots, a+k-1.$$

and

(31) $$\theta_{a+k} = \frac{q^k(1-q^\alpha)}{1-q^{\alpha+k}}.$$

In the case of age-dependent stand-replacement probabilities the conditional probabilities can be obtained by making similar changes to the indices, in the expression (22). Specifically when there is forest in the oldest age class (as described above), for $r = a, \cdots, a+k-1$ the numerator in (22) is unchanged, but the denominator is replaced by

(32) $$\Gamma(\beta^{-1}) - \Gamma(\upsilon[(a-1)T-\tau]^\beta; \beta^{-1})$$

where the first term is the ordinary (complete) gamma function. This denominator is the same for the two oldest age-classes while the numerators are for θ_{a+k}

(33) $$\Gamma\left(\upsilon[(a+k-1+\alpha)T-\tau]^{\beta};\,\beta^{-1}\right)-\Gamma\left(\upsilon[(a+k-1)T-\tau]^{\beta};\,\beta^{-1}\right)$$

and for θ_{a+k+1}

(34) $$\Gamma(\beta^{-1})-\Gamma\left(\upsilon[(a+k-1+\alpha)T-\tau]^{\beta};\,\beta^{-1}\right).$$

When there are no trees in the oldest age-class the denominator in (22) becomes

$$\Gamma\left[\upsilon[(a+k-1+\alpha)T-\tau]^{\beta};\,\beta^{-1}\right]-\Gamma\left(\upsilon[(a-1)T-\tau]^{\beta};\,\beta^{-1}\right)$$

for all age-classes, while the numerators for θ_r, $r = a,\cdots,a+k-1$ are unchanged, and that for θ_{a+k} is as given in (33).

6. CONCLUDING REMARKS

This paper has considered the estimation of the per-annum probability of stand-replacement fire, using as data the age-class distribution of undisturbed old-growth forest. The reciprocal of this estimated probability provides an estimate of the expected time between stand-replacement fires on a given site *i.e.* of the fire cycle.

The method is only appropriate for estimating the probability of fires which are completely destructive and result in a new even-aged stand developing on the site. It will not work for fires which only burn some of the trees on a given site - such fires would lead to uneven-aged stands.

The data required for the method are the areas of undisturbed forest in various non-overlapping age-classes. The more, and the narrower, the age-classes the better the situation from the estimation point of view. However an investigator usually has little choice in this matter, having to accept the forest inventory data available. The main point here though is that it would not normally pay an investigator to pool age-classes into larger ones.[14] This would only result in a loss of information and thus less precision in the estimation procedure.

The fact that inventory data may include age-classes of unequal width, with possibly the top age-class being open-ended poses no difficulty for the method. It simply results in somewhat more complicated formulae in the components of the log-likelihood function. These difficulties are addressed in Section 5. Similarly the specification of an age-dependent probability of stand-replacement fire poses no conceptual difficulty for the method. There may however be a practical difficulty when using an age-dependent model. The difficulty arises from the fact that specifying age-dependency (using say a Weibull model) results in the inclusion of two parameters more than in the age-independent case (one extra parameter for $h(t)$, and the extra parameter τ reflecting the time elapsed since the first intervention in the forest by people of European origin. Our approach has been to fix τ using historical knowledge and to maximize the log-likelihood over the other parameters. The robustness of the results to the specified value of τ can then be checked by repeating the procedure with different values of τ. In spite of this simplification, using an age-dependent model will still result in a likelihood function with at least three parameters. If there are only a few age-classes in the inventory of undisturbed "old-growth" forest one faces the possible difficulty of an over-parameterized model. This can result in considerable numerical difficulties, such as the existence of multiple local maxima for the log-likelihood or failure in the convergence of the numerical maximization scheme.

In most cases, if this difficulty arises, it is probably better to eschew the use of an age-dependent model, using instead the simple two-parameter age-independent model. It is only in cases where the per-annum probability of stand-replacement fire varies considerably on the range of ages starting at the age of the youngest trees in the inventory at the time of the first European intervention $((a-1)T-\tau)$ and up, that an age-specific model would be important. This can be seen from the expression (10) since over ages where the fire probability is constant the survivor function $S(z)$ will behave like an exponential function, and the effects of age-dependence at younger ages will cancel out on the division of the integrals in the r.h.s. of (10). Fitting an age-independent model in cases where there really is serious age-dependency, should give a reasonable estimate of the *average* stand-replacement fire probability.

The spread of fires and their overall extent has not been modelled explicitly in the method described. However the contagion model used, should reflect this aspect of forest fires. While the Pólya model is described in terms of drawing balls from urns, it does capture the essence of the contagion effect present in forest fires, *viz.* that given that one site is in a particular age-class, r, say (*i.e.* that is experienced a stand-replacement fire at a time about rT years in the past), the probability of other sites being in that age class (*i.e.* of having experienced a stand-replacement fire at the same time) is increased. The degree of contagion is reflected by the parameter ρ. If there is a high degree of contagion, *i.e.* if fires spread easily, this should be reflected in a large value of ρ and *vice-versa*.

One aspect of stand-replacement fires is that they leave little physical evidence of their having occurred. For example if no trees survive the fire there will be no fire scars to provide a record of it. While in some cases soil core samples might provide some evidence of a fire having occurred, precise dating of the fire could still be difficult. However if one can reasonably assume that in a given region even-aged stands originate only following a completely destructive fire then the dating of such a fire simply involves determining the age of the stand currently growing on the site. Thus an inventory of undisturbed forest, which includes age information, provides a complete record of stand-replacement fires. The method described in this article is simply an application of maximum likelihood techniques to obtain estimates of the per annum probability of stand-replacement fires, using such data.

Advantages of the method are that it does not require difficult and costly collection of data; that no sampling plan is required for the collection of data and that essentially a complete record of the latest fire on every site is used in the estimation scheme. Disadvantages are that the method can only be used for areas populated by even-aged stands which originated following a fire; that it cannot estimate the probability of occurrence of partially destructive fires, (which, say, only burn the understorey) and that it cannot be used in areas where logging has seriously disturbed the forest (unless of course a complete record of the ages and areas of stands harvested is available, for in this case the pre-logging age-class distribution could be re-constructed).

In spite of these limitations the method provides an inexpensive and simple way of estimating historic fire probabilities, and furthermore one which may be applicable in situations where other methods, which rely on dating fires at sampled points, cannot easily be applied.

FOOTNOTES

1. Weaver (1974) describes a number of such forest types in Western North America including true fir and spruce at high elevations.

2. We are concerned here only with fires which completely destroy some area of a forest which will subsequently be occupied by a new even-aged stand. The current stand on a given site does not necessarily have to comprise trees of a pioneer species. It could be the case that the current even-aged stand succeeded another pioneer species, *e.g.* an even-aged stand of Engelman Spruce (*Picea engelmani*) might succeed a pioneering stand of quaking aspen (*Populus tremuloides*).

3. Although we include the situation outlined in Footnote 2 in which the current even-aged stand succeeded an even-age stand of another pioneer species. In this case the "age" of the stand should strictly be the time since the stand-replacement fire which will be the age of the current stand plus the time elapsed between the destructive fire and the establishment of the current stand. However it will be seen later that in many cases this distinction is not important. In such cases it is the *differences* in stand ages that are important, not their actual ages.

4. Although it is not strictly the per-annum probability of destruction, it is (provided time is measured in years), very close to it. The actual probability of destruction in the year following the t-th anniversary of establishment is $1 - \exp\left\{-\int_t^{t+1} h(s)\,ds\right\}$. For example for an age-independent "hazard-rate function" $h(t) \equiv .01$ per annum, the actual per-annum probability of destruction using this model is $1 - e^{-0.01} = 0.00995$. The complication arises because $h(t)$ is not a probability *per se*, but rather a probability rate.

The advantage of characterizing the risk of destruction in terms of the "hazard rate" $h(t)$ is that it allows for the use of "continuous mathematics" *i.e.* of the differential and integral calculus.

5. This assumption is made in order to have a tractable estimation problem. A "post-intervention" fire probability which is both age-dependent and changing over time in an unknown way would not permit computation of the likelihood. Since the range of ages of surviving undisturbed "old-growth" forest is often not large, the assumption of a constant fire destruction probability over these ages is a reasonable one.

6. The multivariate Pólya urn model (see *e.g.* Johnson & Kotz, 1977, p. 194) considers an urn initially containing $b_0, b_1, \cdots, b_k$ balls of colours $0, 1, \cdots k$. Balls are drawn successively at random from the urn. If a ball of colour j is drawn, c additional balls of that colour are placed in the urn. This of course increases the probability of drawing a ball of the same colour on subsequent drawings - a contagious effect. The multivariate distribution of the *proportions* $y_0, y_1, \cdots, y_k$, of balls of colours $0, 1, \cdots, k$, approaches, as the number of drawings approaches infinity, a *Dirichlet distribution* with (multivariate) probability density

$$f(y_0, y_1, \cdots, y_k) = \frac{\Gamma\left(\sum_{r=1}^{k} b_r/c\right)}{\prod_{r=1}^{k} \Gamma\left(\frac{b_r}{c}\right)} y_0^{(b_0/c-1)} y_1^{(b_1/c-1)}, \cdots, y_k^{(b_k/c-1)}$$

on the simplex $y_0 + y_1 + \cdots + y_k = 1$; $0 \le y_r \le 1$, $r = 0, 1, 2, \cdots, k$, (see *e.g.* Johnson and Kotz, 1977, p. 378), where $\Gamma(\alpha)$ is the usual gamma function

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-5} dt.$$

The use of this model appears to have originated with a paper of Pólya in 1923 and has subsequently been used as a prototype for many models involving contagion or after effects (see Feller, 1968, Chapter V.2 for references).

7. In terms of the urn model presented in Footnote 5, the parameter ρ corresponds to $c/\Sigma b_r$, *i.e.* the number of additional balls added at each drawing, expressed as a fraction of the initial endowment of balls; the parameters θ_r correspond to the proportions of balls initially of colour r *i.e.* the probabilities that the first ball drawn is of colour r $(r = a,\cdots,a+k)$.

8. In fact the marginal distribution of y_r is a beta distribution with parameters θ_r/ρ and $(1-\theta_r)/\rho$. The limiting form of the joint distribution of $y_a,\cdots,y_{a+k}$ as the contagion parameter tends to zero is a degenerate (atomistic) distribution with all its mass at the point $\theta_a,\theta_{a+1},\cdots,\theta_{a+k}$, which is exactly the result one would obtain for the *proportions* in the multinomial model (no contagion) as the total number of units (*e.g.* sites of unit area) passed to infinity.

9. In expression (15) the function loggamma(x) is $\ln(\Gamma(x))$. When x is large $\Gamma(x)$ can be a very large number. Direct computation of $\Gamma(x)$ can cause overflow problems even if $\ln(\Gamma(x))$ is a machine-size number. To avoid overflow problems it is better to use a "log-gamma" function (such as LogGamma in *Mathematica* (Wolfram, 1991)).

10. D. Errico, B.C. Forest Service, *personal communication.*

11. Regrettably, and somewhat surprisingly, details of the routine employed by *Mathematica* are not provided with the *Mathematica* documentation.

12. This confidence interval is based on the theory of likelihood ratio test - see *e.g.* Kalbfleisch (1985b); also see Footnote 13.

13. A formal test of this can be constructed using the likelihood ratio test methodology (see *e.g.* Kalbfleisch, 1985b) although it is hardly needed in this case. To test the hypothesis $H_o: \beta = 0$ *vs.* $H_a: \beta \neq 0$ for the Weibull model (19), one would compare twice the difference between the maximized log-likelihoods (24) and (15) with a χ_1^2 distribution. Here the difference between the maximized log-likelihoods is only in the second decimal place, so clearly there is no evidence of age-dependence.

14. An exception to this could be the case when some age-classes are empty. A zero value for any of the proportions y_r in (15) will cause the log-likelihood to have a value of minus infinity. Pooling age-classes is perhaps the easiest way to deal with this difficulty. However a better alternative is to consider the conditional probabilities θ_r (in (9) and (10)) only for age-classes r which are non-empty. Thus the summation in the denominator of (9) would include only non-empty age-classes.

Table 1

Age-class distribution of undisturbed old-growth forest in the biogeoclimactic zone of the Kootenay T.S.A. characterized by montane spruce

Age Class	3	4	5	6	7
Age Range (years)	41-60	61-80	81-100	101-120	121-140
Area (ha)	83545	49542	51685	32473	10317

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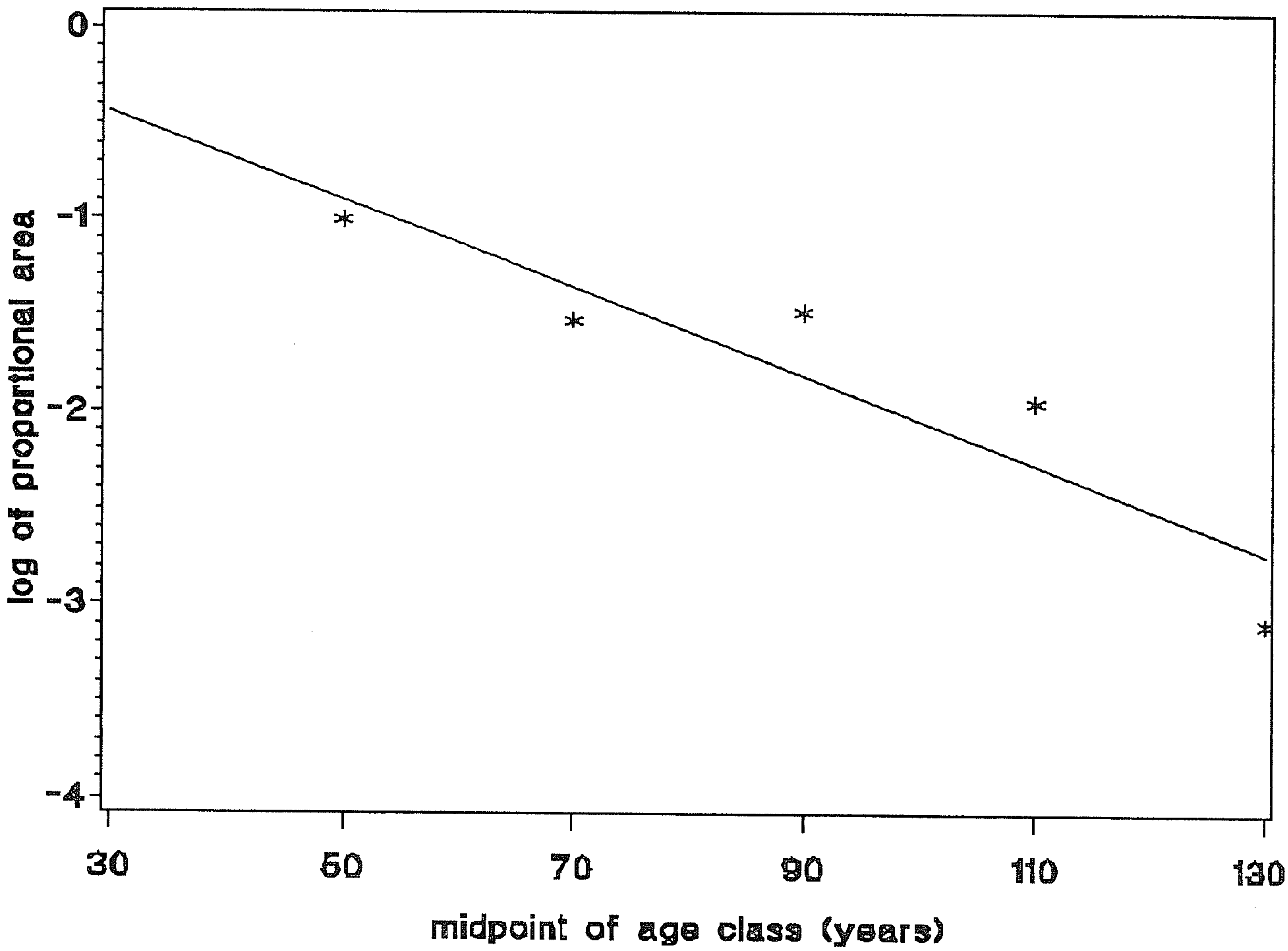
FIGURE TITLES

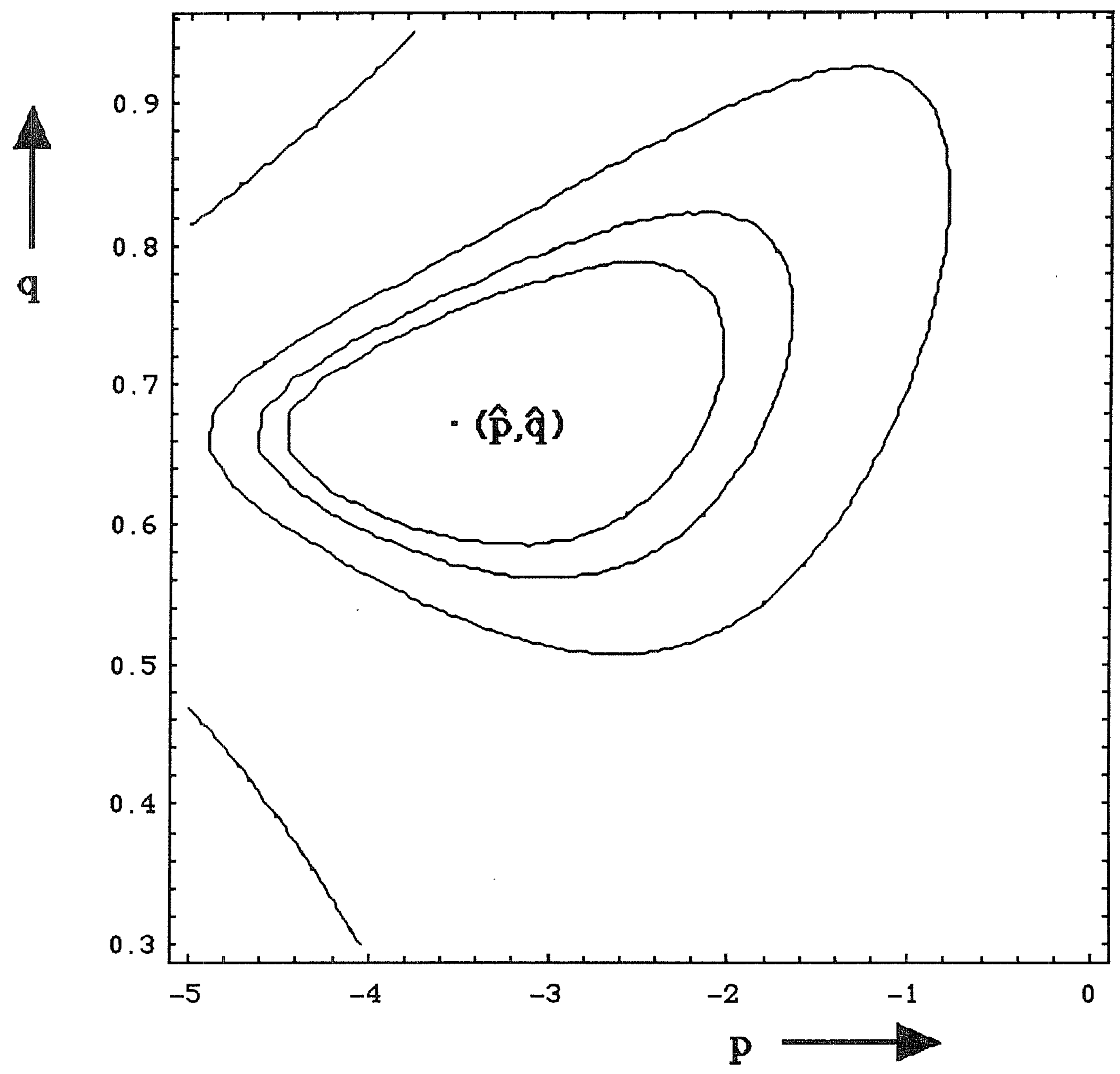
Fig. 1. Contour plot of the log-likelihood (15) for the data in Table 1. The maximum of the log-likelihood is at the point $(\hat{p}, \hat{q})$.

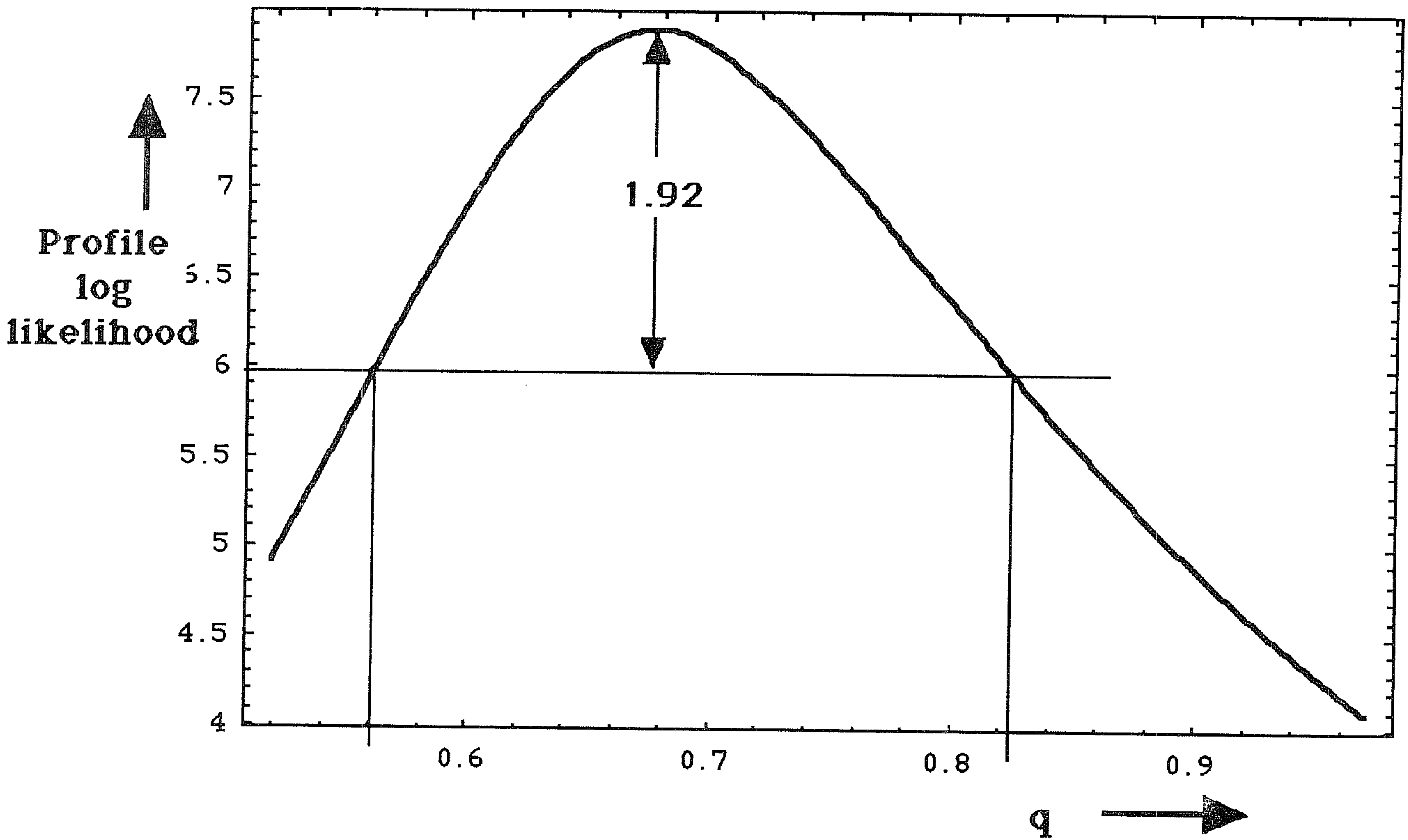
Fig. 2. The profile (or concentrated) log-likelihood (17) for the data in Table 1. An approximate 95% confidence interval for q (and hence for the per-annum probability of destructive fire, λ) is determined from the intersection of the horizontal line 1.92 units below the maximum.

Fig. 3. The logarithm of the proportional areas in various age classes plotted against the mid-point of the age class. The line shown is the least-squares regression line (of log (area) on age).



0
-1
-2
-3
-4
log of proportional area
30
50
70
90
110
130
midpoint of age class (years)


0.9
0.8
0.7
0.6
0.5
0.4
0.3
q
(p̂,q̂)
-5
-4
-3
-2
-1
0
p


Profile
log
likelihood
7.5
7
6.5
6
5.5
5
4.5
4
1.92
0.6
0.7
0.8
0.9
q