Some Applications of a Differential Subordination

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ABSTRACT. A number of interesting criteria were given by earlier workers for a normalized analytic function to be in the familiar class $\mathcal{S}^*$ of starlike functions. The main object of the present paper is to extend and improve each of these earlier results. An application associated with an integral operator $\mathcal{F}_c$ $(c > -1)$ is also considered.

KEY WORDS AND PHRASES: Differential subordination, analytic functions, starlike functions, integral operator, Gauss hypergeometric function, Digamma function.

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1. INTRODUCTION

Let $\mathcal{A}(n)$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \qquad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also let $\mathcal{S}^*$ be the class of starlike functions in $\mathcal{U}$, defined by (*cf.*, *e.g.*, [2] and [11])

$$\mathcal{S}^* := \left\{ f(z) \in \mathcal{A}(1) : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathcal{U}) \right\}. \tag{1.2}$$

For analytic functions $g(z)$ and $h(z)$ with $g(0) = h(0)$, $g(z)$ is said to be subordinate to $h(z)$ if there exists an analytic function $w(z)$ such that $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathcal{U})$, and $g(z) = h(w(z))$. We denote this subordination by $g(z) \prec h(z)$.

For a function $f(z)$ belonging to the class $\mathcal{A}(1)$, Bernardi [1] defined the integral operator $\mathcal{F}_c$ as follows:

$$(\mathcal{F}_c f)(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt \tag{1.3}$$

$$(c > -1;\ z \in \mathcal{U}).$$

We note that $\mathcal{F}_c f \in \mathcal{A}(n)$ if $f \in \mathcal{A}(n)$. In particular, the operator $\mathcal{F}_1$ was studied earlier by Libera [3] (see also Owa and Srivastava [8, p. 126 *et seq.*]).

R. Singh and S. Singh [10] proved that, if $f(z) \in \mathcal{A}(1)$ and

$$\Re\{f'(z) + zf''(z)\} > -\frac{1}{4} \qquad (z \in \mathcal{U}), \tag{1.4}$$

then $f(z) \in \mathcal{S}^*$.

Recently, Yi and Ding [12] improved the above-mentioned result of R. Singh and S. Singh [10] by showing that, if $f(z) \in \mathcal{A}(1)$ and

$$\Re\{f'(z) + zf''(z)\} > 1 - \frac{3}{4(1-\log 2)^2 + 2} \approx -0.263 \qquad (z \in \mathcal{U}), \tag{1.5}$$

then $f(z) \in \mathcal{S}^*$.

Furthermore, Nunokawa and Thomas [6] proved that, if $f(z) \in \mathcal{A}(1)$ and

$$\Re\{f'(z)\} > -0.0175\cdots \qquad (z \in \mathcal{U}), \tag{1.6}$$

then $\mathcal{F}_1 f \in \mathcal{S}^*$.

In this paper we shall extend and improve each of these earlier results in [6] and [12] and also consider an interesting application associated with the integral operator $\mathcal{F}_c$.

2. PRELIMINARY RESULTS

The following results will be required in our investigation.

LEMMA 1 (Yi and Ding [12, Lemma 1]). *Suppose that the function* $\phi : \mathbb{C}^2 \times \mathcal{U} \to \mathbb{C}$ *satisfies the condition* $\Re\{\phi(ix, y; z)\} \leq \delta$ *for all real* x *and* $y \leq -\frac{1}{2}(1+x^2)$ *and all* $z \in \mathcal{U}$. *If* $p(z) = 1 + p_1 z + p_2 z + \cdots$ *is analytic in* $\mathcal{U}$ *and*

$$\Re\{\phi(p(z), zp'(z); z)\} > \delta \qquad (z \in \mathcal{U}), \tag{2.1}$$

then $\Re\{p(z)\} > 0$ *in* $\mathcal{U}$.

LEMMA 2 (Owa and Nunokawa [7, Theorem 1]). *Let* $p(z)$ *be analytic in* $\mathcal{U}$ *with*

$$p(0) = 1, \; p'(0) = \cdots = p^{(n-1)}(0) = 0.$$

If $p(z)$ *satisfies the inequality:*

$$\Re\{p(z) + \alpha z p'(z)\} > \beta \qquad (z \in \mathcal{U}), \tag{2.2}$$

then

$$\Re\{p(z)\} > \beta + (1-\beta)\left\{2\int_0^1 \frac{d\rho}{1+\rho^{n\,\Re(\alpha)}} - 1\right\} \qquad (z \in \mathcal{U}), \tag{2.3}$$

where $\alpha \neq 0$, $\Re(\alpha) \geq 0$, *and* $\beta < 1$.

LEMMA 3 (Owa and Nunokawa [7, Example 1]). *Let* $\alpha > 0$ *and* $\beta < 1$. *If* $f(z) \in \mathcal{A}(n)$ *satisfies the inequality:*

$$\Re\{f'(z) + \alpha z f''(z)\} > \beta \qquad (z \in \mathcal{U}), \tag{2.4}$$

then

$$\Re\{f'(z)\} > \beta + (1-\beta)\{2\delta(n,\alpha) - 1\} \qquad (z \in \mathcal{U}), \tag{2.5}$$

where

$$\delta(n,\alpha) = \int_0^1 \frac{d\rho}{1+\rho^{n\alpha}}. \tag{2.6}$$

Incidentally, the value of $\delta(n,\alpha)$ in (2.6) can be expressed as the Gauss hypergeometric function

$${}_2F_1\left(1, \frac{1}{n\alpha}; 1 + \frac{1}{n\alpha}; -1\right), \tag{2.7}$$

which may also be rewritten in terms of the difference of two Digamma (or ψ-) functions:

$$\frac{1}{2n\alpha}\left[\psi\left(\frac{1+n\alpha}{2n\alpha}\right) - \psi\left(\frac{1}{2n\alpha}\right)\right] \qquad \left(\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}\right). \tag{2.8}$$

We note also that the inequality (2.4) is equivalent to the subordination given by

$$f'(z) + \alpha z f''(z) \prec \frac{1 + (1 - 2\beta)z}{1 - z}. \tag{2.9}$$

3. MAIN RESULTS

The following Theorem is a generalization of the main result of Yi and Ding [12].

THEOREM. *Let $\delta(n, \alpha)$ be as defined in Lemma 3 and let $\theta = 0.911621907$, $\alpha \geq 0.17418$, and*

$$\alpha - \frac{(1-\alpha)^2}{3\alpha} \tan^2\theta < \frac{2\delta(n,\alpha) - 1}{\{1 - \delta(n,\alpha)\}\{2\delta(n,1) - 1\}}. \tag{3.1}$$

If $f \in \mathcal{A}(n)$ satisfies the inequality:

$$\Re\{f'(z) + \alpha z f''(z)\}$$
$$> 1 - \frac{\frac{2}{\alpha} + \left(1 - \frac{(1-\alpha)^2}{3\alpha^2}\tan^2\theta\right)}{\frac{2}{\alpha} + 4\{1 - \delta(n,1)\}\{1 - \delta(n,\alpha)\}\left(1 - \frac{(1-\alpha)^2}{3\alpha^2}\tan^2\theta\right)} \quad (z \in \mathcal{U}), \tag{3.2}$$

then $f(z) \in \mathcal{S}^*$.

PROOF. Making use of Lemma 3 and the inequality (3.2), we obtain

$$\Re\{f'(z)\} > \beta + (1-\beta)\{2\delta(n,\alpha) - 1\}$$
$$= 2\{\delta(n,\alpha) - 1\}\left[\frac{\frac{2}{\alpha} + \left(1 - \frac{(1-\alpha)^2}{3\alpha^2}\tan^2\theta\right)}{\frac{2}{\alpha} + 4\{1 - \delta(n,1)\}\{1 - \delta(n,\alpha)\}\left(1 - \frac{(1-\alpha)^2}{3\alpha^2}\tan^2\theta\right)}\right] + 1$$
$$=: \gamma \qquad (z \in \mathcal{U}), \tag{3.3}$$

where

$$\beta = 1 - \frac{\frac{2}{\alpha} + \left(1 - \frac{(1-\alpha)^2}{3\alpha^2}\tan^2\theta\right)}{\frac{2}{\alpha} + 4\{1 - \delta(n,1)\}\{1 - \delta(n,\alpha)\}\left(1 - \frac{(1-\alpha)^2}{3\alpha^2}\tan^2\theta\right)}. \tag{3.4}$$

Since $\alpha \geq 0.17418$ and

$$\frac{1}{2} < \delta(n,\alpha) < 1 \qquad (\alpha > 0;\ n \in \mathbb{N}), \tag{3.5}$$

we have

$$\frac{2}{\alpha} + 4\{1 - \delta(n,1)\}\{1 - \delta(n,\alpha)\}\left(1 - \frac{(1-\alpha)^2}{3\alpha^2}\tan^2\theta\right) > 0.$$

Hence, by (3.1), we find from (3.3) that

$$0 < \gamma < 1. \tag{3.6}$$

If we put $p(z) = z^{-1}f(z)$, then

$$\Re\{f'(z)\} = \Re\{p(z) + zp'(z)\} > \gamma \qquad (z \in \mathcal{U}), \tag{3.7}$$

which, in view of Lemma 2, implies that

$$\Re\left\{\frac{f(z)}{z}\right\} > \gamma + (1-\gamma)\{2\delta(n,1) - 1\} \qquad (z \in \mathcal{U}). \tag{3.8}$$

By using (3.5) and (3.6), we get

$$\Re\left\{\frac{f(z)}{z}\right\} > 0 \qquad (z \in \mathcal{U}). \tag{3.9}$$

Next we let

$$q(z) = \frac{zf'(z)}{f(z)} \quad \text{and} \quad \lambda(z) = \frac{f(z)}{z}.$$

Then

$$\Re\{\lambda(z)\} > \gamma + (1-\gamma)\{2\delta(n,1) - 1\} \qquad (z \in \mathcal{U}) \tag{3.10}$$

and

$$\begin{aligned} f'(z) + \alpha z f''(z) &= \lambda(z)[\alpha z q'(z) + (1-\alpha)q(z) + \alpha\{q(z)\}^2] \\ &= \phi(q(z), zq'(z); z), \end{aligned} \tag{3.11}$$

where $\phi(u, v; z) = \lambda(z)[\alpha u^2 + (1-\alpha)u + \alpha v]$.

By setting $\lambda(z) = a + bi$, we have

$$\begin{aligned} \Re\{\phi(ix, y; z)\} &\leq -\frac{1}{2}\left\{3\alpha\, a x^2 + 2b(1-\alpha)\, x + \alpha\, a\right\} \\ &\leq -\frac{a}{2}\left\{\alpha - \frac{1}{3\alpha}(1-\alpha)^2\left(\frac{b}{a}\right)^2\right\} \end{aligned} \tag{3.12}$$

for all real x and $y \leq -\frac{1}{2}(1+x^2)$. Since $\Re\{f'(z)\} > 0 \quad (z \in \mathcal{U})$ implies $\lambda(z) \prec L(z) := -1 - \frac{2}{z}\log(1-z)$, we have $\lambda(\mathcal{U}) \subset L(\mathcal{U})$, where (see [9])

$$L(\mathcal{U}) \subset \{\omega : \Re(\omega) > 2\log 2 - 1\} \cap \{\omega : |\Im(\omega)| < \pi\} \cap \{\omega : |\arg(\omega)| < \theta = 0.911621907\}.$$

By using (3.8) and (3.12), we obtain

$$\begin{aligned}\Re\{\phi(ix,y;z)\} &\leq -\frac{a}{2}\left\{\alpha - \frac{(1-\alpha)^2}{3\alpha}\tan^2\theta\right\} \\ &\leq \beta \qquad (z \in \mathcal{U}).\end{aligned} \tag{3.13}$$

Hence, by Lemma 1, we get

$$\Re\{q(z)\} = \Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \qquad (z \in \mathcal{U}). \tag{3.14}$$

This evidently completes the proof of the Theorem.

COROLLARY 1. *Let* $\theta = 0.911621907$, $\alpha \geq 0.17418$, *and*

$$\alpha - \frac{(1-\alpha)^2}{3\alpha}\tan^2\theta < \frac{2\delta(1,\alpha)-1}{\{1-\delta(1,\alpha)\}(2\log 2 - 1)}. \tag{3.15}$$

If $f \in \mathcal{A}(1)$ *satisfies the inequality:*

$$\begin{aligned}&\Re\{f'(z) + \alpha z f''(z)\} \\ &> 1 - \frac{\frac{2}{\alpha} + \left(1 - \frac{(1-\alpha)^2}{3\alpha^2}\tan^2\theta\right)}{\frac{2}{\alpha} + 4(1-\log 2)\{1-\delta(1,\alpha)\}\left(1 - \frac{(1-\alpha)^2}{3\alpha^2}\tan^2\theta\right)} \qquad (z \in \mathcal{U}),\end{aligned} \tag{3.16}$$

then $f(z) \in \mathcal{S}^*$.

REMARK 1. For $\alpha = 1$, Corollary 1 would immediately yield the main result of Yi and Ding [12, p. 614, Theorem].

REMARK 2. A result of Ponnusamy [9, Theorem 4] can be obtained by taking $\beta = 0$ in the proof of our Theorem.

It is not difficult to apply the definition (1.3) in order to show that

$$f'(z) = (\mathcal{F}_c f)'(z) + \frac{1}{c+1}\, z(\mathcal{F}_c f)''(z). \tag{3.17}$$

Thus, by the Theorem, we arrive at the following application:

COROLLARY 2. *Let* $\theta = 0.911621907$, $-1 < c \leq 4.741187$, *and*

$$\frac{1}{c+1} - \frac{c^2}{3(c+1)}\tan^2\theta < \frac{2\delta\left(n, \frac{1}{c+1}\right) - 1}{\left\{1 - \delta\left(1, \frac{1}{c+1}\right)\right\}\{2\delta(n,1) - 1\}}. \tag{3.18}$$

If $f \in \mathcal{A}(n)$ *satisfies the inequality:*

$$\Re\{f'(z)\} > 1 - \frac{2(c+1) + \left(1 - \frac{1}{3}\,c^2 \tan^2\theta\right)}{2(c+1) + 4\{1 - \delta(n,1)\}\left\{1 - \delta\left(n, \frac{1}{c+1}\right)\right\}\left(1 - \frac{1}{3}\,c^2 \tan^2\theta\right)} \quad (z \in \mathcal{U}), \tag{3.19}$$

then $\mathcal{F}_c f \in \mathcal{S}^*$, *where* $\mathcal{F}_c$ *is defined by* (1.3).

By setting $c = n = 1$ in Corollary 2, we obtain Corollary 3 below, which shows that the constant -0.0175 in the inequality (1.6) of Nunokawa and Thomas [6] can be reduced further.

COROLLARY 3. *Let* $\theta = 0.911621907$. *If* $f \in \mathcal{A}(1)$ *satisfies the inequality:*

$$\Re\{f'(z)\} > 1 - \frac{5 - \frac{1}{3}\tan^2\theta}{4 + 8(1-\log 2)^2\left(1 - \frac{1}{3}\tan^2\theta\right)} \approx -0.025311\cdots \qquad (z \in \mathcal{U}), \tag{3.20}$$

then $\mathcal{F}_1 f \in \mathcal{S}^*$.

PROOF. Since

$$\frac{1}{2} - \frac{1}{6}\tan^2\theta = 0.222356 \ (\theta = 0.911621907) \quad \text{and} \quad \frac{3 - 4\log 2}{(2\log 2 - 1)^2} = 1.523967\cdots, \tag{3.21}$$

the proof of Corollary 3 is completed by setting $c = n = 1$ in Corollary 2.

REMARK 3. Several non-sharp results, obtained by various other authors (*cf.*, *e.g.*, [9]), would correspond to the further special cases of Corollary 2 when $c = 0$ and $c = 1$.

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REFERENCES

[1] BERNARDI, S.D., Convex and starlike univalent functions, *Trans. Amer. Math. Soc.* **135**(1969), 429–446.

[2] DUREN, P.L., *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften **259**, Springer-Verlag, New York, Berlin, Heidelberg, and Tokyo, 1983.

[3] LIBERA, R.J., Some classes of regular univalent functions, *Proc. Amer. Math. Soc.* **16**(1965), 755–758.

[4] MILLER, S.S. and MOCANU, P.T., Second order differential inequalities in the complex plane, *J. Math. Anal. Appl.* **65**(1978), 289–305.

[5] NUNOKAWA, M., On starlikeness of Libera transformation, *Complex Variables Theory Appl.* **17**(1991), 79–83.

[6] NUNOKAWA, M. and THOMAS, D.K., On the Bernardi integral operator, in *Current Topics in Analytic Function Theory* (H.M. Srivastava and S. Owa, Editors), World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992, 212–219.

[7] OWA, S. and NUNOKAWA, M., Applications of a subordination theorem, *J. Math. Anal. Appl.* **188**(1994), 219–226.

[8] OWA, S. and SRIVASTAVA, H.M., Some applications of the generalized Libera integral operator, *Proc. Japan Acad. Ser. A Math. Sci.* **62**(1986), 125–128.

[9] PONNUSAMY, S., Differential subordination and starlike functions, *Complex Variables Theory Appl.* **19**(1992), 185–194.

[10] SINGH, R. and SINGH, S., Convolution properties of a class of starlike functions, *Proc. Amer. Math. Soc.* **106**(1989), 145–152.

[11] SRIVASTAVA, H.M. and OWA, S. (Editors), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992.

[12] YI, L. and DING, S., A new criterion for starlike functions, *Internat. J. Math. and Math. Sci.* **19**(1996), 613–614.