

INDEPENDENT DOMINATION CRITICAL GRAPHS

by

Suqin Ao


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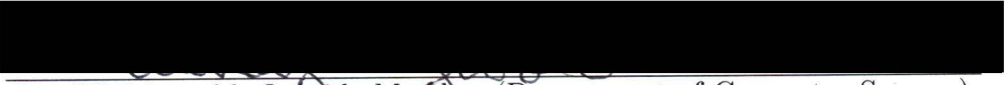
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We accept this thesis as conforming
to the required standard


Dr G MacGillivray, Supervisor (Department of Mathematics and Statistics)


Dr E J Cockayne, Department Member (Department of Mathematics
and Statistics)


Dr W Myrvold, Outside Member (Department of Computer Science)


Dr C Mynhardt, External Examiner (Department of Mathematics,
Applied Mathematics and Astronomy, University of South Africa)

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University of Victoria

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Supervisor Dr G MacGillivray

Abstract

The domination number of a graph G is the minimum size of any dominating set of G . The independent domination number of G is the minimum size of any independent dominating set of G .

We study several different definitions of graphs which are critical with respect to the domination number (respectively, independent domination number), and give examples of infinite families of graphs of each type. Comparing these families leads to four principal types of critical graphs, that is, graphs for which the deletion of any vertex causes the domination number (respectively, the independent domination number) to decrease, and graphs for which adding an edge causes the domination number (respectively, the independent domination number) to decrease. Since graphs for which such changes cause the domination number to decrease have been studied in the literature, we concentrate on those for which such changes cause the independent domination number to decrease. In each case we develop the theory of independent domination critical graphs which is analogous to that, already in the literature, for domination critical graphs. We extend this theory, and that for domination critical graphs, by giving a complete classification of the vertex domination critical graphs with the maximum possible diameter. We also explore Hamiltonian properties of edge independent domination critical graphs by proving that every 2-connected edge independent domination critical graph with independent domination number three has a Hamilton cycle. We then develop a complete classification of edge independent domination critical graphs with a cut-vertex. This leads to the result that any connected edge independent domination critical graph with more than six vertices has a Hamilton path.

Examiners:



Dr G MacGillivray, Supervisor (Department of Mathematics and Statistics)



Dr E J Cockayne, Department Member (Department of Mathematics and Statistics)



Dr W Myrvold, Outside Member (Department of Computer Science)



Dr C Mynhardt, External Examiner (Department of Mathematics, Applied Mathematics and Astronomy, University of South Africa)

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Chapter 1

Introduction

The domination number and independent domination number of graphs are two of the most studied graph parameters with over 200 papers having appeared in the literature (see [5], [11] and [12])

Frequently some insight into a particular graph parameter can be gained by studying graphs which are “critical” in the sense that an alteration to the graph yields a change in the value of the parameter. Since a subgraph of a graph G can, in general, have smaller or larger (independent) domination number than does G , there are several possible definitions of “(independent) domination critical” graphs. These are explored in Section 2.1.

Several of these types of domination critical graphs have already been explored in the literature [4, 7, 8, 14, 13, 16]. To date no one has undertaken a study of independent domination critical graphs and a comparison of the different notions of criticality. This thesis is an attempt to do just that.

In Section 1.1 we list some definitions and notations. An outline of the

thesis is given in Section 1.2

1.1 Preliminaries

In this section, we list some definitions and notations used in this thesis. For a more detailed description of basic graph theory, the reader is referred to [3]

A *graph* is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is a finite (possibly empty) set of *vertices* and $E(G)$ is a set of unordered pairs of distinct vertices. The elements of $E(G)$ are called *edges*. If $\{u, v\} \in E(G)$, we denote $\{u, v\}$ by uv .

The number of vertices of G is denoted by $p(G)$ and the number of edges of G is denoted by $q(G)$. When no confusion can result, we write p and q for $p(G)$ and $q(G)$, respectively.

Two graphs G_1 and G_2 are (*vertex*) *disjoint* if $V(G_1) \cap V(G_2) = \emptyset$.

Let G and H be two disjoint graphs. The *union* of G and H , denoted $G \cup H$, has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. We denote by nG the union of n vertex disjoint copies of G . The *join* of G and H , denoted $G + H$, is formed from $G \cup H$ by adding all possible edges with one end in $V(G)$ and the other in $V(H)$.

A complete bipartite graph $K_{1,m}$ is called a *star*.

A graph G is a *complete r -partite graph* if $V(G)$ can be partitioned into r

sets such that two vertices are joined if and only if they are in different sets, if all of these sets have size k , then G is denoted by $T_{r,rk}$. (Note $T_{r,rk}$ is the complement of rK_k .) The graph $T_{3,12}$ is shown in Figure 2.1.

The *open neighbourhood* of a vertex v in a graph $G = (V, E)$ is $N(v) = \{u \in V \mid uv \in E\}$, and its *closed neighbourhood* is $N[v] = N(v) \cup \{v\}$.

A vertex of degree 0 is called an *isolated vertex*, and a vertex of degree 1 is called an *end vertex*.

The *distance* between two vertices u, v , denoted by $d(u, v)$, is the *smallest* length of those paths connecting u and v . The *diameter* of a graph G , denoted by $\text{diam}(G)$, is $\max_{u,v \in V(G)} d(u, v)$.

A set $I \subseteq V(G)$ is an *independent set* of G , if for any $u, v \in I$, $uv \notin E(G)$, and the *independence number* of G , denoted by $\beta(G)$, is the largest size of such a set.

A set $D \subseteq V(G)$ is a *dominating set* of G , if for any vertex v not in D , v is adjacent to some vertex of D , and the *domination number* of G , $\gamma(G)$, is the minimum size of such a set.

An *independent dominating set* is a dominating set which is also independent, and the *independent domination number* of G , denoted by $\iota(G)$, is the minimum size of such a set. Clearly $\gamma(G) \leq \iota(G)$.

Let H be any subgraph of G and $D \subseteq V(G)$. If for any vertex $v \in V(H) - D$, v is adjacent to some vertex of D , then we say D *dominates* H .

or D dominates $V(H)$.

1.2 Outline

In this section, we give a brief outline of the thesis. The purpose of this thesis is to present a study of independent domination critical graphs. Our main results are discussed in Chapter 2, 3 and 4.

In Chapter Two we examine the seven possible definitions of criticality with respect to the domination number and the independent domination number. Infinite families of each type of critical graph are described. Comparing these families leads to the four principal types of critical graphs we treat in the remainder of the thesis, and to the conclusion that no two of these four types of criticality are the same. We also establish some basic properties of independent domination critical graphs. In the end of this chapter, we give a counterexample to disprove one of Sumner's conjectures ([13]).

Brigham, Chinn and Dutton [4], Fulman [7], and Hanson and MacGillivray [8] studied graphs in which the deletion of a vertex causes the domination number to decrease. We begin Chapter Three by briefly surveying these results, and extend the theory by giving a complete classification of the vertex domination critical graphs with the maximum possible diameter. We then turn our attention to the analogous concept for the independent domination number, and develop a similar theory. The methods developed in this chapter

yield further constructions of vertex independent domination critical graphs.

Graphs which are domination critical with respect to adding an edge have received some attention in the literature [13, 14, 15, 6, 7]. We begin Chapter Four by briefly surveying some of these results. We then turn our attention to the analogous concept for the independent domination number, and develop a similar theory. In the final section of Chapter Four, Hamiltonian properties of edge independent domination critical graphs are examined. In particular, we prove that every 2-connected edge independent domination critical graph with independent domination number three has a Hamilton cycle. This is done by developing an independent domination closure operation similar to the one in Hanson [9]. (The same question for edge domination critical graphs, due to Sumner [13] and Wojcicka [18], remains open.) We then develop a complete classification of edge independent domination critical graphs with a cut-vertex. (Such a classification for edge domination critical graphs has been initiated by Sumner [13] and Wojcicka [18], but is not complete.) This leads to the result that any connected edge independent domination critical graph with more than six vertices has a Hamilton path.

Finally, in Chapter Five, we list some suggestions for the future research on independent domination critical graphs.

Chapter 2

Domination Critical Graphs

In this chapter, we compare and contrast some different definitions of graphs which are critical with respect to the domination number or independent domination number. Section 2.1 gives the definitions of the seven different types of criticality we explore, and examples of infinite families of graphs of each of these seven types. Some properties and constructions of critical graphs are given in Section 2.2. In Section 2.3, we show that no two of these types of criticality are the same. Finally, we give a counterexample to disprove one of Sumner's conjectures [13].

2.1 Definitions and Examples

It is the purpose of this section to give definitions and some examples of various domination critical families.

We denote by $\mathcal{D}(G)$ and $\mathcal{I}(G)$ the set of all dominating sets of G and the

set of all independent dominating sets of G , respectively, that is

$$\mathcal{D}(G) = \{D : D \text{ is a dominating set of } G\},$$

$$\mathcal{I}(G) = \{I : I \text{ is an independent dominating set of } G\}.$$

Lemma 2.1 ([2]) *An independent set I of G is maximal if and only if I is an independent dominating set of G*

We now define the seven possible types of critical graphs. Graphs of the first four types are considered throughout this thesis, those of the last three types are considered only briefly in this section.

1. A γ -critical graph is a graph belonging to $\mathcal{G}_\gamma = \{G \mid \forall v \in V(G), \gamma(G - v) < \gamma(G)\}$,
2. An ι -critical graph is a graph belonging to $\mathcal{G}_\iota = \{G \mid \forall v \in V(G), \iota(G - v) < \iota(G)\}$,
3. An edge- γ -critical graph is a graph belonging to $\mathcal{G}_\gamma^+ = \{G \mid \forall e \notin E(G), \gamma(G + e) < \gamma(G)\}$,
4. An edge- ι -critical graph is a graph belonging to $\mathcal{G}_\iota^+ = \{G \mid \forall e \notin E(G), \iota(G + e) < \iota(G)\}$,
5. An e - γ -critical graph is a graph belonging to $\mathcal{G}_{\bar{\gamma}} = \{G \mid \forall e \in E(G), \gamma(G - e) > \gamma(G)\}$,

6. An e - i -critical graph is a graph belonging to $\mathcal{G}_i^- = \{G \mid \forall e \in E(G), \iota(G - e) > \iota(G)\}$;
7. An e' - i -critical graph is a graph belonging to $\mathcal{G}_i'^- = \{G \mid \forall e \in E(G), \iota(G - e) < \iota(G)\}$

The eighth possible definition, an e' - γ -critical graph, being a graph for which $\forall e \in E(G), \gamma(G - e) < \gamma(G)$, makes no sense as for any graph G and any edge $e \in E(G), \gamma(G - e) \geq \gamma(G)$. Further, by considering a vertex not in a minimum (independent) dominating set, or \overline{K}_n if there is no such vertex, it can be seen that $\gamma(G - v) > \gamma(G)$ ($\iota(G - v) > \iota(G)$) for all vertices $v \in V(G)$ is also impossible.

For a graph G with $\iota(G) = n$, if $G \in \mathcal{G}_i$ then we call G n - i -critical, similarly, if $G \in \mathcal{G}_i^+$ then call G n -edge- i -critical, and so on.

First, let us look at the families $\mathcal{G}_{\overline{\gamma}}$ and \mathcal{G}_i^- . Walikar and Acharya study e - γ -critical graphs in [16]. They show that the only e - γ -critical graphs are those for which each component is isomorphic to a star. On the other hand, Sumner and Blich characterized 2-edge- γ -critical graphs in [13]. They proved that a graph G is 2-edge- γ -critical if and only if \overline{G} is a union of stars.

Since a graph G is e - γ -critical if and only if each component of G is so, we have the following result.

Theorem 2.2 [4, 16] *A graph G is e - γ -critical if and only if \overline{G} is 2-edge- γ -critical.*

We have the same result for the e - ι -critical family \mathcal{G}_ι^- .

Theorem 2.3 *A graph G is e - ι -critical if and only if it is a union of stars, (and thus if and only if \overline{G} is 2-edge- γ -critical.) That is, $\mathcal{G}_\iota^- = \mathcal{G}_{\overline{\gamma}}^-$.*

Proof: Obviously, we only need to prove the necessity. Let G be an e - ι -critical graph, and let I be any minimum independent dominating set of G . We first prove that $V(G) - I$ is independent. Suppose not and let $e = uv \in E(G)$ such that $u, v \in V(G) - I$. Then, since I is an independent dominating set of $G - e$, $\iota(G - e) \leq |I| = \iota(G)$, a contradiction. Therefore, for any $e = uv \in E(G)$, we must have $u \in I$, say, and $v \in V(G) - I$. Next, we show that for any $v \in V(G) - I$, $d(v) = 1$. Suppose not and let $u, w \in N(v)$. Then from the above, we have $u, w \in I$. But now, since I is an independent dominating set of $G - uv$ (because $wv \in E(G - uv)$, I still dominates v), $\iota(G - uv) \leq |I| = \iota(G)$, a contradiction. Therefore, G is a union of stars. ■

In this thesis we study mainly four types of critical graphs: γ -critical, ι -critical, edge- γ -critical, and edge- ι -critical graphs. After making the following easy observation about e' - ι -critical graphs, we concentrate on describing some infinite families of graphs of each of these types.

Consider $K_{m,n}$, where $3 \leq m \leq n$. Then $\iota(K_{m,n}) = m$, and $\gamma(K_{m,n}) = \iota(K_{m,n} - e) = 2$ for every $e \in E(K_{m,n})$. Thus $K_{m,n}$ is e' - ι -critical. This can be generalized as follows.

Observation Let G be a graph with $\gamma(G) < \iota(G)$. If for every $uv \in E(G)$,

there exists a minimum dominating set D such that $u, v \in D$ and $G[D] - uv$ is independent, then G is e' - i -critical with $i(G - uv) \leq \gamma(G) < i(G)$.

Let G be a graph and $x \in V(G)$. We denote by H_x the subgraph induced by $V(G) - N[x]$.

Lemma 2.4 *For $n \geq 2$, if for every $x \in V(G)$, H_x is $(n - 1)$ - i -critical, then G is both n - i -critical and n -edge- i -critical*

Proof: Let $x \in V(G)$ and D_x be a minimum independent dominating set of H_x . Then $|D_x| = n - 1$ and $D = D_x \cup \{x\}$ is an independent dominating set of G . So, $i(G) \leq |D| = n$. If $i(G) < n$, then let I be a minimum independent dominating set of G . We have $|I| \leq n - 1$. Let $y \in I$. Then $I_y = I - \{y\}$ is an independent dominating set of H_y , and $|I_y| \leq n - 2$, a contradiction. Therefore, $i(G) = n$.

Now, we show that G is n - i -critical. For any $v \in V(G)$, since $n \geq 2$, there exists $x \in V(G)$ such that $v \in (V(G) - N[x])$. Since H_x is $(n - 1)$ - i -critical, $i(H_x - v) < n - 1$ and thus $i(G - v) < n$.

To see that G is n -edge- i -critical, let $uv \notin E(G)$. Then $u \in V(H_v)$. Since H_v is $(n - 1)$ - i -critical, there exists a minimum independent dominating set I_u of $H_v - u$ with $|I_u| < n - 1$. Thus $I = I_u \cup \{v\}$ is an independent dominating set of $G - u$. So, $i(G + uv) \leq |I| < n$. ■

Now, we describe various families of domination critical graphs.

First, for \mathcal{G}_i and \mathcal{G}_γ , we note that K_1 is the only 1- i -critical graph and also the only 1- γ -critical graph and graph G is 2- i -critical and/or 2- γ -critical if and only if $G \cong T_{m,2m}$ (defined as in Chapter 1) (see [4]).

For integers $r \geq 1$ and $n \geq 2$, let $m = n(2r + 1) - 2r$. The graph $H_{2r,m}$ is defined as follows:

The vertex set is

$$V(H_{2r,m}) = \{v_1, v_2, \dots, v_m\},$$

and the edge set is

$$E(H_{2r,m}) = \{v_i v_j : 0 < |i - j \pmod{m}| \leq r\}.$$

Figure 2.1 shows $H_{4,11}$.

Proposition 2.5 *For integers $r \geq 1$ and $n \geq 2$, let $m = n(2r + 1) - 2r$. Then the graph $H_{2r,m}$ is both n - γ -critical and n - i -critical.*

Proof First, since the set $I = \{v_1, v_{1+(2r+1)}, v_{1+2(2r+1)}, \dots, v_{1+(n-2)(2r+1)}, v_{1+(n-1)(2r+1)-r}\}$ is an independent dominating set of $H_{2r,m}$, $\gamma(H_{2r,m}) \leq i(H_{2r,m}) \leq |I| = n$. Next, since $H_{2r,m}$ is $2r$ -regular, and $|V(H_{2r,m})| = m$, we have $\gamma(H_{2r,m})(2r + 1) \geq m$, that is, $i(H_{2r,m}) \geq \gamma(H_{2r,m}) \geq m/(2r + 1)$. But $m = n(2r + 1) - 2r$ implies that $\gamma(H_{2r,m}) \geq n$. Therefore, we have $i(H_{2r,m}) = \gamma(H_{2r,m}) = n$.

Finally, $H_{2r,m}$ is i -critical (and thus γ -critical) since for any $v_j \in V(H_{2r,m})$, $I_j = \{v_{j+(r+1)}, v_{j+(r+1)+(2r+1)}, \dots, v_{j+(r+1)+(n-2)(2r+1)}\}$, where the additions

are modulo m , is an independent dominating set of $H_{2r,m} - v_j$ and $|I_j| = n - 1$.

■

For $t \geq 3$ and $m = 4t - 3$, the graph G_t is defined as follows:

The vertex set is

$$V(G_t) = \{v_1, v_2, \dots, v_m\},$$

and the edge set is

$$E(G_t) = \{v_i v_j : |i - j \pmod{m}| = 1, 3, \dots, 2t - 3\}$$

Figure 2.1 shows G_3 .

Proposition 2.6 *For $t \geq 3$, the graph G_t is 3- γ -critical and t - ι -critical.*

Proof. Note that G_t is vertex-transitive. We claim that $|N[x] \cap N[y]| \geq 2$ for all $x, y \in V(G_t)$. This is clear if x and y are adjacent. If $xy \notin E(G_t)$, then, without loss of generality, $x = v_1$ and $y = v_{2k+1}$ for some k ($1 \leq k \leq t - 1$). But then v_{2k} and v_{2k-2} are adjacent to both x and y . This proves the claim. Thus $|N[x] \cup N[y]| \leq 2(t - 1) + 2(t - 1) = 4t - 4 < 4t - 3$. So, no two vertices dominate all vertices of G_t . On the other hand, $\{v_1, v_2\}$ dominates $G_t - v_{2t}$. Therefore, $\gamma(G_t) = 3$ and G_t is 3- γ -critical. Now, since $I = \{v_1, v_3, v_5, \dots, v_{2t-1}\}$ is an independent dominating set of G_t , we have $\iota(G_t) \leq |I| = t$. On the other hand, for any $x \in V(G_t)$, consider H_x (the subgraph induced by $V(G) - N[x]$). Then H_x is easily seen to be isomorphic to the bipartite graph with vertex set $\{x_1, x_2, \dots, x_{t-1}, y_1, y_2, \dots, y_{t-1}\}$ in which

for $i = 1, 2, \dots, t-1$, x_i is adjacent to $y_{t-1}, y_{t-2}, \dots, y_{t-i}$. Since this graph has independent domination number $t-1$, we have $\iota(G_t) \geq t$. Therefore, $\iota(G_t) = t$.

To see that G_t is ι -critical, for any $v_k \in V(G_t)$, since $I_k = \{v_{k-2}, v_{k+2}, v_{k+4}, \dots, v_{k+(2t-4)}\}$ (where the additions are modulo m) is an independent dominating set of $G_t - v_k$, $\iota(G_t - v_k) \leq |I_k| = t-1$ ■

For $n \geq 3$, the graph Q_n is defined as follows.

The vertex set is

$$V(Q_n) = \{u_0, u_1, \dots, u_{n-1}, v_0, v_1, \dots, v_{n-1}, w_0, w_1, \dots, w_{n-1}\},$$

and the edge set is

$$E(Q_n) = \{u_i u_{i-1}, u_i v_{i-1}, u_i v_i, u_i w_i, v_i w_{i-1}, v_i w_i \mid i = 1, \dots, n\}$$

(where the additions are modulo n). The graph Q_3 is shown in Figure 2.1.

Proposition 2.7 *For $n \geq 3$, the graph Q_n is both n - ι -critical and n - γ -critical.*

Proof Since the independent set $I = \{v_0, v_1, \dots, v_{n-1}\}$ dominates Q_n , $\gamma(Q_n) \leq \iota(Q_n) \leq |I| = n$. On the other hand, for any $0 \leq i \neq j \leq n-1$, no one of $\{u_i, v_i, w_i\}$ can dominate all $\{u_j, v_j, w_j\}$, and thus, $\iota(Q_n) \geq \gamma(Q_n) \geq n$. Therefore, $\iota(Q_n) = \gamma(Q_n) = n$.

To see Q_n is critical, note that for any $0 \leq k, j, l \leq n-1$, $I_k = I - v_k$ dominates $Q_n - v_k$. $I_j = I - \{v_{j-1}, v_j\} + w_{j-1}$ dominates $Q_n - u_j$, and

$I_l = I - \{v_{l+1}, v_l\} + u_{l+1}$ dominates $Q_n - w_l$, and therefore, $\iota(Q_n - v) < \iota(Q_n)$ and $\gamma(Q_n - v) < \gamma(Q_n)$ for any $v \in V(Q_n)$ ■

Proposition 2 8 For $m \geq 2$ and $n \geq 3$, the graph $T_{m,mn}$ is n - ι -critical but not γ -critical

Proof. In this proof, let

$$V(T_{m,mn}) = \{v_{11}, v_{12}, \dots, v_{1n}, v_{21}, v_{22}, \dots, v_{2n}, \dots, v_{m1}, v_{m2}, \dots, v_{mn}\},$$

and

$$E(T_{m,mn}) = \{v_{ij}v_{kl} : i \neq k, 1 \leq i, k \leq m, 1 \leq j, l \leq n\}$$

Since any independent set $I_k = \{v_{k1}, v_{k2}, \dots, v_{kn}\}$ ($1 \leq k \leq m$) dominates $T_{m,mn}$ and no $n - 1$ independent vertices can dominate $T_{m,mn}$, $\iota(T_{m,mn}) = n$. For any $v_{kj} \in V(T_{m,mn})$, since $I_k - v_{kj}$ dominates $T_{m,mn} - v_{kj}$, $\iota(T_{m,mn} - v_{kj}) < \iota(T_{m,mn})$

On the other hand, since any two vertices v_{ij}, v_{kl} ($i \neq k$) dominate $T_{m,mn}$ and no one vertex can dominate $T_{m,mn}$, $\gamma(T_{m,mn}) = 2$. Since $n \geq 3$, for any $v_{kl} \in V(T_{m,mn})$, $\gamma(T_{m,mn} - v_{kl}) = 2$. Therefore, $T_{m,mn}$ is not γ -critical. ■

For $n \geq 1$, the graph G_{nn} is defined as follows:

The vertex set is

$$V(G_{nn}) = \{v_{ij} : 1 \leq i, j \leq n\},$$

and the edge set is

$$E(G_{nn}) = \{v_{ij}v_{kl} : i \neq k, j \neq l, 1 \leq i, j, k, l \leq n\}$$

Figure 2.1 shows G_{33} . (Observe that G_{nn} is the complement of the Cartesian product of K_n with itself.)

Proposition 2.9 *The graph \overline{G}_{nn} is both n - i -critical and n - γ -critical.*

Proof When $n = 1$, $\overline{G}_{11} = K_1$, which is both 1- i -critical and 1- γ -critical.

When $n = 2$, $\overline{G}_{22} = K_{2,2}$, which is both 2- i -critical and 2- γ -critical.

Suppose $\overline{G}_{(n-1)(n-1)}$ is both $(n-1)$ - i -critical and $(n-1)$ - γ -critical. Consider \overline{G}_{nn} , note that for any $x \in V(\overline{G}_{nn})$, $H_x \cong \overline{G}_{(n-1)(n-1)}$ (where H_x is the subgraph induced by $V(\overline{G}_{nn}) - N[x]$). By Lemma 2.4, \overline{G}_{nn} is n - i -critical. Since $\gamma(\overline{G}_{(n-1)(n-1)}) = n-1$, $\gamma(\overline{G}_{nn}) = n$, and therefore \overline{G}_{nn} is n - γ -critical.

■

Proposition 2.10 *For $n \geq 4$, the graph G_{nn} is n - i -critical but not γ -critical. The graph G_{33} is both 3- i -critical and 3- γ -critical.*

Proof Note that for $n \geq 4$ and for any $x \in V(G_{nn})$, the subgraph induced by $V(G_{nn}) - N[x]$ is $T_{2,2(n-1)} = K_{n-1, n-1}$ which is $(n-1)$ - i -critical but not γ -critical by Proposition 2.8. Also note that $G_{33} \cong \overline{G}_{33}$. ■

For $n \geq 3$, the graph O_n is defined as follows:

The vertex set is

$$V(O_n) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_n\},$$

and the edge set is

$$E(O_n) = \{v_i u_{i-1}, v_i u_i, v_i w_{i-1}, v_i w_i \mid i = 1, \dots, n\},$$

where the additions are modulo n . Alternatively, O_n is obtained from two $2n$ -cycles $u_1 u_2 \dots u_{2n}$ and $w_1 w_2 \dots w_{2n}$ by identifying u_i and w_i for each $i \in \{1, 3, \dots, 2n-1\}$. Figure 2.1 shows O_3 .

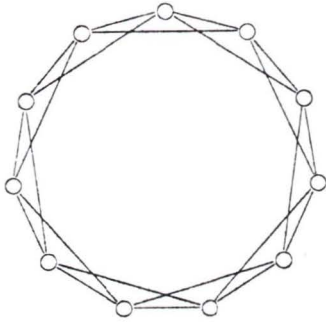
Proposition 2.11 *For $n \geq 3$, the graph O_n is both n - ι -critical and n - γ -critical.*

Proof: Since the independent set $I = \{v_1, v_2, \dots, v_n\}$ dominates O_n , $\gamma(O_n) \leq \iota(O_n) \leq |I| = n$. On the other hand, for any $1 \leq i \neq j \leq n$, no one of $\{u_i, v_i, w_i\}$ can dominate all $\{u_j, v_j, w_j\}$, and so, $\iota(O_n) \geq \gamma(O_n) \geq n$. Therefore, $\iota(O_n) = \gamma(O_n) = n$.

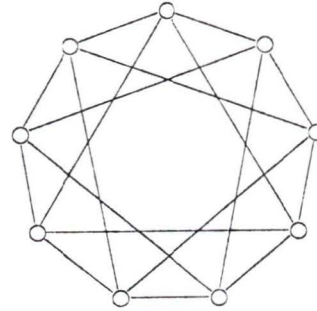
Since for any $1 \leq k, j, l \leq n$, $I_k = I - v_k$ dominates $O_n - v_k$, $I_j = I - \{v_{j+1}, v_j\} + w_j$ dominates $O_n - u_j$, and $I_l = I - \{v_{l+1}, v_l\} + u_l$ dominates $O_n - w_l$, for any $v \in V(O_n)$, $\iota(O_n - v) < \iota(O_n)$ and $\gamma(O_n - v) < \gamma(O_n)$. ■

Now, for \mathcal{G}_ι^+ and \mathcal{G}_γ^+ , we note that the only 1-edge- ι -critical, and the only 1-edge- γ -critical, graphs are K_n ($n \geq 1$), and a graph G is 2-edge- ι -critical and/or 2-edge- γ -critical if and only if \overline{G} is a union of stars (see [13]). So, $T_{m,2m}$ ($m \geq 1$) is a family of 2-edge- ι -critical and 2-edge- γ -critical graphs.

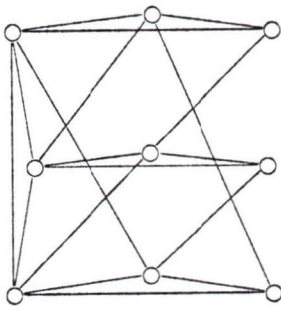
For $n \geq 2$ and $p \geq n$, a family of graphs $Q_{n,p}$ is constructed from $\overline{K}_n \cup K_p$ as follows: Let $p = a_1 + a_2 + \dots + a_n$ be a partition of the integer p , with



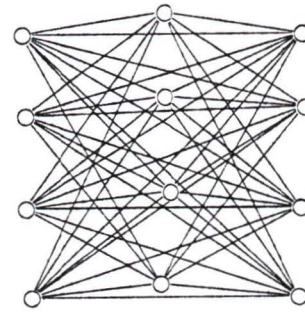
$H_{4,11}$



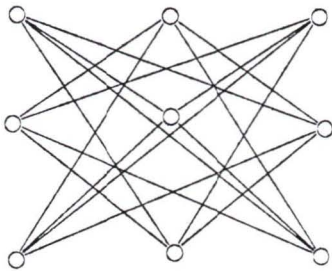
G_3



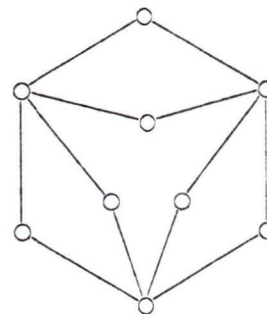
Q_3



$T_{3,12}$

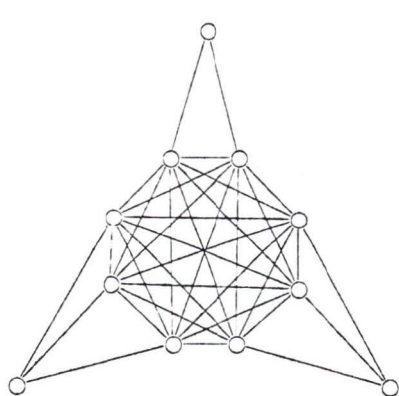


G_{33}

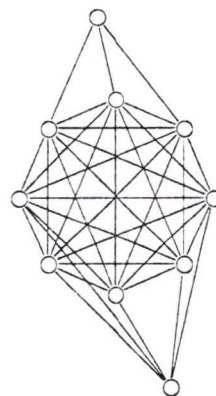


O_3

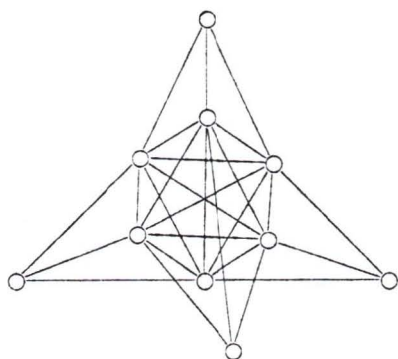
Figure 2.1 Critical Graphs



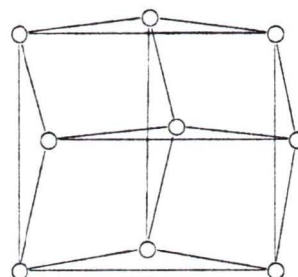
$Q_{3,8}$



$Q_{2,8}$



$P_{4,3}$



\bar{G}_{33}

Figure 2.2 Edge-Critical Graphs

$a_j \geq 1$ for $1 \leq j \leq n$. For K_p , let $V(K_p) = A_1 \cup A_2 \cup \dots \cup A_n$ with $|A_i| = a_i$, $i = 1, \dots, n$. Let $V(\overline{K}_n) = \{v_1, v_2, \dots, v_n\}$. For $j = 1, 2, \dots, n$, join v_j to every vertex of A_j . Figure 2.2 shows $Q_{3,8}$ and $Q_{2,8}$ corresponding to the partitions $8 = 2 + 3 + 3$ and $8 = 3 + 5$, respectively.

Proposition 2.12 *For $p \geq n \geq 3$, each graph $Q_{n,p}$ is both n -edge- ι -critical and n -edge- γ -critical.*

Proof. Since $I = V(\overline{K}_n) = \{v_1, v_2, \dots, v_n\}$ is an independent dominating set of $Q_{n,p}$, $\gamma(Q_{n,p}) \leq \iota(Q_{n,p}) \leq |I| = n$. On the other hand, for any $1 \leq i \neq j \leq n$, no vertex of $A_i \cup \{v_i\}$ can dominate v_j , and so, $\iota(Q_{n,p}) \geq \gamma(Q_{n,p}) \geq n$. Therefore, $\iota(Q_{n,p}) = \gamma(Q_{n,p}) = n$.

Now, consider any $uv \notin E(Q_{n,p})$. If $u, v \in V(\overline{K}_n)$, then $u = v_i, v = v_j$ for some $i \neq j$. Let $k \neq i, j$, and $z \in A_k$. Then $S = I - \{v_j, v_k\} \cup \{z\}$ is an independent dominating set of $Q_{n,p} - v_j$ (or $Q_{n,p} + uv$) and thus $\iota(Q_{n,p} + uv) < n$.

If $u \in V(\overline{K}_n), v \in V(K_p)$, then $u = v_j$ for some j and $v \in A_k$ for some $k \neq j$. Since $I - \{v_j, v_k\} \cup \{v\}$ is an independent dominating set of $Q_{n,p} - v_j$, $\iota(Q_{n,p} + uv) < n$. ■

For $m \geq n \geq 3$, let $p = \binom{m}{m-n+1}$. The graph $P_{m,n}$ is constructed from $\overline{K}_m \cup K_p$ as follows. Let $V(K_p) = \{v_1, \dots, v_p\}$ and let A_1, \dots, A_p be the $(m-n+1)$ -subsets of \overline{K}_m . Join v_j to every vertex of A_j , for each $j = 1, \dots, p$.

The graph $P_{4,3}$ is shown in Figure 2 2

Proposition 2.13 *For $m \geq n \geq 3$, the graph $P_{m,n}$ is n -edge- i -critical*

Proof: Since $I = \{v_1\} \cup (V(\overline{K}_m) - A_1)$ is an independent dominating set of $P_{m,n}$, $i(P_{m,n}) \leq |I| = n$. Also, note that $i(P_{m,n}) \geq n$ since K_p is complete and $m \geq n$.

Now, consider any $uv \notin E(P_{m,n})$. If $u, v \in V(\overline{K}_m)$, then there exists A_k such that $u, v \notin A_k$. Thus, $I_k = \{v_k\} \cup (V(\overline{K}_m) - A_k) - \{v\}$ is an independent dominating set of $P_{m,n} - v$.

If $u \in V(K_p)$, then $u = v_j$ for some j and $v \notin A_j$. In this case, $I_j = \{v_j\} \cup (V(\overline{K}_m) - A_j) - \{v\}$ is an independent dominating set of $P_{m,n} - v$. ■

Let $m \geq n \geq 3$. If G is both $(n-1)$ -edge- i -critical and $(n-1)$ - i -critical, then from $Q_{n,p}$ (defined as above) and G , by joining every vertex of G to every vertex v_j of \overline{K}_n in $Q_{n,p}$ we obtain a new graph $M_{n,p,G}$. Also, from $P_{m,n}$ (defined as above) and G , we can get a new graph $N_{n,m,G}$ by joining every vertex of G to every vertex of \overline{K}_m in $P_{m,n}$.

Proposition 2.14 *For $m \geq n \geq 3$, any graph $M_{n,p,G}$ or $N_{n,m,G}$ is n -edge- i -critical*

Proof: We show that $M_{n,p,G}$ is n -edge- i -critical. Since every $v_j \in V(\overline{K}_n) \subseteq V(Q_{n,p})$ is adjacent to every vertex of G , $I = V(\overline{K}_n) = \{v_1, v_2, \dots, v_n\}$ is an independent dominating set of $M_{n,p,G}$, and hence $i(M_{n,p,G}) \leq |I| = n$. On

the other hand, consider any independent dominating set D of $M_{n,p,G}$. If $D \cap G \neq \emptyset$ then since every vertex of G is adjacent to every $v_j \in V(\overline{K}_n) \subseteq V(Q_{n,p})$, $D \cap G$ must dominate G , and since $D \cap G$ cannot dominate any vertex of K_p in $Q_{n,p}$ and $\iota(G) = n - 1$, we must have $|D| \geq n$. If $D \cap G = \emptyset$, then since D must dominate $Q_{n,p}$ and $\iota(Q_{n,p}) = n$ we also have $|D| \geq n$. Thus $\iota(M_{n,p,G}) \geq n$, and so, we proved $\iota(M_{n,p,G}) = n$.

Now, consider any $uv \notin E(M_{n,p,G})$. If $u, v \in V(G)$, then since G is $(n-1)$ -edge- ι -critical, there exists an independent dominating set W of $G + uv$ with size $n - 2$ such that either $u \in W$ or $v \in W$. Let $z \in V(K_p)$, then $I = W \cup \{z\}$ is an independent dominating set of $M_{n,p,G} + uv$ and thus $\iota(M_{n,p,G} + uv) \leq |I| = n - 1$.

If $u, v \in V(Q_{n,p})$, by Proposition 2.12, and the definition of $M_{n,p,G}$, $\iota(M_{n,p,G} + uv) = \iota(Q_{n,p} + uv) < n$.

If $u \in V(G)$ and $v \in V(Q_{n,p})$, then $v \in V(K_p)$. Since G is $(n-1)$ - ι -critical, there exists an independent dominating set D_u of $G - u$ with $|D_u| < n - 1$. Thus, $I = D_u \cup \{v\}$ is an independent dominating set of $M_n - u$ and hence $\iota(M_{n,p,G} + uv) \leq |I| < n$.

Similarly, we can prove that $N_{n,m,G}$ is also n -edge- ι -critical. ■

Proposition 2.15 *The graph \overline{G}_{nn} is both n -edge- ι -critical and n -edge- γ -critical.*

Proof: For any $x \in V(\overline{G}_{nn})$, we have $H_x = \overline{G}_{(n-1)(n-1)}$, which is $(n-1)$ - ι -

critical by Proposition 2.9. Therefore, by Lemma 2.4, \overline{G}_{nn} is n -edge- ι -critical. Since $\gamma(\overline{G}_{nn}) = n$, \overline{G}_{nn} is n -edge- γ -critical. ■

Proposition 2.16 *For $m \geq 2$ and $n \geq 3$, the graph $T_{m,mn}$ is n -edge- ι -critical but not edge- γ -critical.*

Proof. Since for any $x \in V(T_{m,mn})$, $H_x = \overline{K}_{n-1}$ which is $(n-1)$ - ι -critical, $T_{m,mn}$ is n -edge- ι -critical by Lemma 2.4. For any $uv \notin E(T_{m,mn})$, since $\gamma(T_{m,mn} + uv) = 2 = \gamma(T_{m,mn})$ when $n \geq 3$, $T_{m,mn}$ is not edge- γ -critical. ■

Proposition 2.17 *For $n \geq 4$, the graph G_{nn} is n -edge- ι -critical but not edge- γ -critical, while G_{33} is both 3-edge- ι -critical and 3-edge- γ -critical.*

Proof. For any $x \in V(G_{nn})$, $H_x = K_{n-1, n-1} = T_{2, 2(n-1)}$ which is $(n-1)$ - ι -critical by Proposition 2.8. Therefore, by Lemma 2.4, G_{nn} is n -edge- ι -critical when $n \geq 3$. Now, for $n \geq 4$ and any $uv \notin E(G_{nn})$, $\gamma(G_{nn} + uv) = 3 = \gamma(G_{nn})$, so G_{nn} is not edge- γ -critical. When $n = 3$, the result follows from Proposition 2.15 since $G_{33} \cong \overline{G}_{33}$. ■

Some other examples of edge- γ -critical graphs can be found in [13] and [14].

2.2 Properties and Constructions of Independent Domination Critical Graphs

In this section, we discuss some properties and constructions of independent domination critical graphs. The constructions yield further families of critical

graphs. We begin with some properties of independent dominating sets in critical graphs.

Lemma 2.18 *Let G be an edge- ι -critical graph and $u, v \in V(G)$. If $uv \notin E(G)$, then there exists a minimum independent dominating set D of G such that $u, v \in D$.*

Proof: Since $\iota(G + uv) < \iota(G)$, there exists an independent dominating set W of $G + uv$ with $|W| = \iota(G) - 1$ such that either u or v but not both in W (otherwise, W would be an independent dominating set of G , a contradiction). In other words, there exists an independent set S of size $\iota(G) - 2$ such that either $S \cup \{v\} = W$ dominates $G - u$ or $S \cup \{u\} = W$ dominates $G - v$. Thus $D = S \cup \{u, v\}$ is a minimum independent dominating set of G . ■

The converse of Lemma 2.18 is not true. For example, every pair of nonadjacent vertices of C_7 are in some minimum independent dominating set, but C_7 is not edge- ι -critical.

Lemma 2.18 motivates the following notation. Let $G \in \mathcal{G}_\iota^+$, $uv \notin E(G)$, and S be an independent set of size $\iota(G) - 2$ such that either $S \cup \{v\}$ is independent and dominates $G - u$ or $S \cup \{u\}$ is independent and dominates $G - v$ (the existence of S is established in the preceding proof). If $S \cup \{v\}$ dominates $G - u$ we write $[v, S] \rightarrow u$, and if $S \cup \{u\}$ dominates $G - v$ we write $[u, S] \rightarrow v$.

Corollary 2.19 *If G is edge- ι -critical, then every vertex belongs to some minimum independent dominating set.*

Proof For any vertex $u \in V(G)$, if u is adjacent to all other vertices of G , then $\iota(G) = 1$ and $\{u\}$ dominates G . Otherwise, there exists $v \in V(G)$ such that $uv \notin E(G)$, and hence by Lemma 2.18, there exists a minimum independent dominating set D of G such that $u \in D$. ■

Lemma 2.20 *A graph G is ι -critical if and only if for every $v \in V(G)$,*

$$\iota(G - v) = \iota(G) - 1.$$

Proof If G is ι -critical, then for every $v \in V(G)$, $\iota(G - v) < \iota(G)$, that is, $\iota(G - v) \leq \iota(G) - 1$. But for every $v \in V(G)$, we have $\iota(G - v) \geq \iota(G) - 1$. Therefore, $\iota(G - v) = \iota(G) - 1$.

On the other hand, if for every $v \in V(G)$, $\iota(G - v) = \iota(G) - 1$, then $\iota(G - v) < \iota(G)$, and hence G is ι -critical. ■

Lemma 2.21 *If G is ι -critical, then every vertex $v \in V(G)$ belongs to some minimum independent dominating set.*

Proof For any $v \in V(G)$, since G is ι -critical, $\iota(G - v) = \iota(G) - 1$. Let D_v be a minimum independent dominating set of $G - v$. Since there is no vertex of D_v adjacent to v (otherwise, D_v dominates G , a contradiction), $D = D_v \cup \{v\}$ is independent. Also, since $|D| = \iota(G)$, $D = D_v \cup \{v\}$ is a minimum independent dominating set of G . ■

The converse of Lemma 2.21 is not true. For example, every vertex of C_6 is in some minimum independent dominating set, but C_6 is not ι -critical.

We now describe some more constructions of domination critical graphs.

Let G_1, G_2, \dots, G_m be disjoint graphs. The *join* of G_1, G_2, \dots, G_m is the graph $G_1 + G_2 + \dots + G_m$ obtained from the graph $G_1 \cup G_2 \cup \dots \cup G_m$ by adding additional edges joining each vertex of G_i with each vertex of G_j whenever $i \neq j$.

Note that if $G_j = \overline{K}_n$ for $j = 1, 2, \dots, m$, then $G_1 + G_2 + \dots + G_m \cong T_{m, mn}$.

Proposition 2.22 *Let G_1, G_2, \dots, G_m be n - ι -critical (respectively, n -edge- ι -critical) graphs with $n \geq 2$. Then $G_1 + G_2 + \dots + G_m$ is also n - ι -critical (respectively, n -edge- ι -critical).*

Proof: Let $G = G_1 + G_2 + \dots + G_m$. Since $\iota(G_k) = n$ for $k = 1, \dots, m$ by the definition of join, $\iota(G) = n$.

We show first that if G_1, G_2, \dots, G_m are n - ι -critical then G is also n - ι -critical. For any $v \in V(G)$, there exists k such that $v \in V(G_k)$. Since G_k is ι -critical, $\iota(G_k - v) < n$. But $n \geq 2$ and the definition of join imply that $\iota(G - v) = \iota(G_k - v)$. Therefore, G is n - ι -critical.

We now show that if G_1, G_2, \dots, G_m are n -edge- ι -critical then G is also n -edge- ι -critical. For any $uv \notin E(G)$, by the definition of join, there exists k such that $u, v \in V(G_k)$ and $uv \notin E(G_k)$. Since G_k is edge- ι -critical, $\iota(G_k + uv) < n$. Since $\iota(G + uv) = \iota(G_k + uv)$, G is n -edge- ι -critical. ■

Let G_1, G_2 and H be disjoint graphs such that for $j = 1, 2$, G_j has a labelled subgraph $H_j \cong H$. We define $G = G_1(H_1) \circ G_2(H_2)$ to be the graph obtained from $G_1 \cup G_2$ by identifying corresponding vertices of H_1 and H_2 and adding edges joining every vertex of $G_1 - H_1$ with every vertex of $G_2 - H_2$. For $m \geq 3$, let G_1, G_2, \dots, G_m and H be disjoint graphs such that for $j = 1, 2, \dots, m$, G_j has a labelled subgraph $H_j \cong H$. We define $G_1(H_1) \circ G_2(H_2) \circ \dots \circ G_m(H_m)$ to be $(G_1(H_1) \circ \dots \circ G_{m-1}(H_{m-1})) \circ G_m(H_m)$.

Note that when $V(H) = \emptyset$, $G_1(H_1) \circ G_2(H_2) \circ \dots \circ G_m(H_m) = G_1 + G_2 + \dots + G_m$. The following can thus be viewed as a generalization of Proposition 2.22.

Proposition 2.23 *Let G_1, G_2, \dots, G_m and H be disjoint graphs such that for $j = 1, 2, \dots, m$, G_j has a labelled subgraph $H_j \cong H$. If G_1, G_2, \dots, G_m are n - i -critical (respectively, n -edge- i -critical) and $\beta(H) \leq n - 2$, then $G = G_1(H_1) \circ G_2(H_2) \circ \dots \circ G_m(H_m)$ is also n - i -critical (respectively, n -edge- i -critical).*

Proof: Since for $j = 1, 2, \dots, m$, $i(G_j) = n$, $i(G) \geq n$ by the construction of G . Now, let D_1 be a minimum independent dominating set of G_1 . Then since $\beta(H) \leq n - 2 < |D_1| - 1$, there exists $x \in D_1$ such that $x \in V(G_1 - H_1)$, and thus, x dominates all $G_j - H_j$ for $j = 2, 3, \dots, m$. Hence, D_1 is an independent dominating set of G , and so, $i(G) \leq |D_1| = n$. Therefore, we have $i(G) = n$.

We show first that if G_1, G_2, \dots, G_m are n - i -critical then G is also n - i -critical. For any $v \in V(G)$, there exists k ($1 \leq k \leq m$) such that $v \in V(G_k)$. Since G_k is n - i -critical, there exists an independent dominating set D_v of $G_k - v$ with $|D_v| = n - 1$. Since $|D_v| > \beta(H_k)$, there exists some $z \in D_v$ such that $z \in V(G_k - H_k)$ and thus D_v is an independent dominating set of $G - v$. Therefore, $\iota(G - v) \leq |D_v| = n - 1$ and hence G is n - i -critical.

We now show that if G_1, G_2, \dots, G_m are n -edge- i -critical then G is also n -edge- i -critical. For any $uv \notin E(G)$, there exists j ($1 \leq j \leq m$) such that $u, v \in V(G_j)$ but $uv \notin E(G_j)$. Since G_j is n -edge- i -critical, there exists an independent set S of size $n - 2$ such that $D = S \cup \{u\}$ is independent and dominates $G_j - v$ or $S \cup \{v\}$ is independent and dominates $G_j - u$. Without loss of generality, assume the former. Now, $\beta(H_j) \leq n - 2$ implies that there exists some $w \in D$ such that $w \in V(G_j - H_j)$. Thus D is an independent dominating set of $G - v$. That is, $\iota(G + uv) \leq |D| = n - 1$. Hence G is n -edge- i -critical. ■

With respect to the above proof, note that if $\beta(H) \geq n - 1$, then it might be the case that (using the notation of Proposition 2.23) $D \subseteq V(H)$ dominates G_j but can not dominate G_k for some $k \neq j$. For example, let $G_1 = G_2 = K_{3,3}$ and $H = \overline{K}_2$. Then $K_{3,3}$ is 3- i -critical and $\beta(H) = 2$. But $G = K_{3,3}(\overline{K}_2) \circ K_{3,3}(\overline{K}_2)$ is not 3- i -critical because if $v \in V(G_1)$ is the vertex non-adjacent to all vertices of H , then $D = H$ and it cannot dominate G_2 .

Also, recall that graph $Q_{3,3}$ is constructed from $\overline{K}_3 \cup K_3$, let $V(\overline{K}_3) = \{v_1, v_2, v_3\}$ and $V(K_3) = \{u_1, u_2, u_3\}$. Now, if we take $G_1 = K_{3,3}$, $G_2 = Q_{3,3}$, $H_1 = \overline{K}_2$, and $H_2 = \{u_1, v_2\}$, then G_1 and G_2 are 3-edge- ι -critical, but $G = K_{3,3}(H_1) \circ Q_{3,3}(H_2)$ is not 3- ι -critical since $u_1v_3 \notin E(G)$, the vertices u_1 and v_2 must dominate all vertices of G except v_3 , and so $D = H_2$ cannot dominate G_1 .

Let G and H be disjoint graphs and $g \in V(G)$. We define $G(g, H)$ to be the graph obtained from G by replacing the vertex g with a copy of H , that is, put a copy of H in place of g and connect each vertex of H to every neighbour of g . Formally, $G(g, H)$ is the graph having vertex set

$$V(G(g, H)) = (V(G) - \{g\}) \cup V(H),$$

and edge set

$$E(G(g, H)) = E(G - g) \cup E(H) \cup \{hv \mid h \in V(H), v \in N(g)\}.$$

Proposition 2 24 *Let G be an n -edge- ι -critical graph with $g \in V(G)$ such that for each $u \in V(G) - N[g]$, there exists an independent set $S \subseteq V(G) - N[g]$ of size $n - 2$ such that $[g, S] \rightarrow u$. Then $G(g, K_m)$ is also n -edge- ι -critical.*

Proof: First, since K_m is complete and in $G(g, K_m)$, every vertex of K_m is adjacent to every neighbour of g , it is easy to see $\iota(G(g, K_m)) = n$

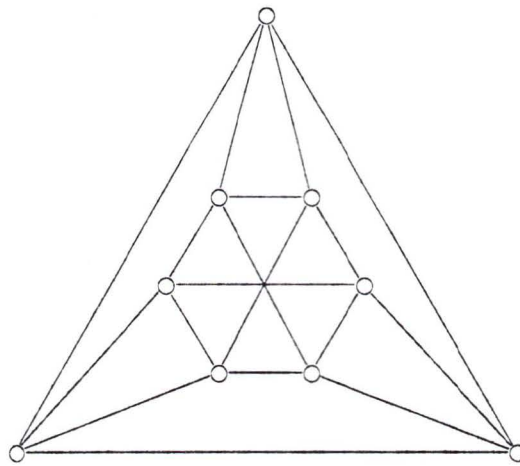


Figure 2.3 The graph A γ -critical, not i -critical

here

Let G and H be disjoint graphs, and let $u \in V(G)$ and $v \in V(H)$. The *coalescence of G and H with respect to u and v* is the graph $G \cdot_{uv} H$ defined to have vertex set

$$V(G \cdot_{uv} H) = (V(G) - \{u\}) \cup (V(H) - \{v\}) \cup \{w\},$$

where $w \notin V(G) \cup V(H)$, and edge set

$$E(G \cdot_{uv} H) = E(G - u) \cup E(H - v) \cup \{wx \mid ux \in E(G) \text{ or } vx \in E(H)\}$$

We call w the *vertex of identification* of G and H , and we consider $V(G)$ and $V(H)$ as subsets of $V(G \cdot_{uv} H)$ and regard w as an element of both of them. Informally, $G \cdot_{uv} H$ is the graph obtained from $G \cup H$ by identifying u and v . If the context is clear, or if the vertices u and v are not important, we write $G \cdot H$ instead of $G \cdot_{uv} H$.

We define $G_1 \cdot G_2 \cdot \dots \cdot G_n$ to be $(G_1 \cdot G_2 \cdot \dots \cdot G_{n-1}) \cdot G_n$.

In Chapter 3, we have the results (see Theorem 3.11 and 3.31) that $G = G_1 \cdot G_2 \cdot \dots \cdot G_n$ is ι -critical (respectively, γ -critical) if and only if all of G_1, G_2, \dots, G_n are ι -critical (respectively, γ -critical). Furthermore, $\iota(G) = \sum_{k=1}^n \iota(G_k) - (n - 1)$ (respectively, $\gamma(G) = \sum_{k=1}^n \gamma(G_k) - (n - 1)$).

Proposition 2.25 *For any integers k, l with $k > l \geq 2$, there exists a graph L with $\gamma(L) = l$ and $\iota(L) = k$ such that $L \in \mathcal{G}_\iota$ but $L \notin \mathcal{G}_\gamma$.*

Proof: For any $k > l \geq 2$, let $n = k - l + 2$, and W_1, W_2, \dots, W_{l-2} be disjoint copies of C_4 . Define $L = K_{n,n} \cdot W_1 \cdot W_2 \cdot \dots \cdot W_{l-2}$, (if $l = 2$, then $L = K_{n,n}$).

Since $K_{n,n}$ is n - ι -critical and C_4 is 2 - ι -critical, L is ι -critical by Theorem 3.31, and furthermore,

$$\iota(L) = n + \left(\sum_{j=1}^{l-2} 2 \right) - (l-2) = k$$

Since C_4 is 2 - γ -critical and $K_{n,n}$, with $\gamma(K_{n,n}) = 2$, is not γ -critical (note that $n = k - l + 2 > 2$), L is not γ -critical by Theorem 3.11. Further,

$$\gamma(L) = 2 + \left(\sum_{j=1}^{l-2} 2 \right) - (l-2) = l. \blacksquare$$

Proposition 2.26 *For any integers m, n with $n \geq m \geq 5$, there exists a graph Z with $\gamma(Z) = m$ and $\iota(Z) = n$, such that $Z \in \mathcal{G}_\gamma$ but $Z \notin \mathcal{G}_\iota$.*

Proof. For any $n \geq m \geq 5$, let $t = n - m + 3$ and W_1, W_2, \dots, W_{m-5} be disjoint copies of C_4 . Let A be the graph in Figure 2.3 and G_t be the one in Proposition 2.6. Define $Z = A \cdot W_1 \cdot W_2 \cdot \dots \cdot W_{m-5} \cdot G_t$, (if $m = 5$, then let $Z = A \cdot G_t$). Since C_4 is 2 - γ -critical and both A and G_t are 3 - γ -critical, by Theorem 3.11, Z is γ -critical and

$$\gamma(Z) = 3 + \left(\sum_{j=1}^{m-5} 2 \right) + 3 - (m-4) = m$$

Since C_4 is 2 - ι -critical and G_t is t - ι -critical but A is not ι -critical, Z is not ι -critical. Further, since $\iota(A) = 3$,

$$\iota(Z) = 3 + \left(\sum_{j=1}^{m-5} 2 \right) + t - (m-4) = n. \blacksquare$$

Note: If $n = m = 3$, then let $Z = A$. If $n = m = 4$, then let $Z = A \cdot C_4$.

From Section 2.1, the graphs $K_n, T_{n,2n}, Q_{n,p}$ ($n \geq 3$), and \overline{G}_{nn} are both edge- γ -critical and edge- ι -critical. Also, graph A in Figure 2.3 is both edge- γ -critical and edge- ι -critical. However, $T_{m,mn}$ ($n \geq 3$) and G_{nn} ($n \geq 4$) are edge- ι -critical but not edge- γ -critical, and Q_3 is edge- γ -critical but not edge- ι -critical.

Also, the graph \overline{G}_{nn} is ι -critical, γ -critical, edge- ι -critical, and edge- γ -critical. The graph O_n ($n \geq 3$) is ι -critical and γ -critical but neither edge- ι -critical nor edge- γ -critical.

On the other hand, all graphs obtained by subdividing one edge of the complete graph K_n with $n \geq 4$ are edge- ι -critical and edge- γ -critical but neither ι -critical nor γ -critical.

Brigham, Chinn and Dutton [4] showed that every γ -critical graph can be extended to one which is both γ -critical and edge- γ -critical by successively adding edges which do not decrease the domination number, until no such edge remain. This is also true for ι -critical graphs. That is, every ι -critical graph can be extended to a graph which is both ι -critical and edge- ι -critical by successively adding edges which do not decrease the independent domination number, until no such edges remain. We have the following theorem.

Theorem 2.27 *If G is ι -critical, then there exists a graph H with $V(H) = V(G)$ such that G is a subgraph of H , $\iota(G) = \iota(H)$ and H is both ι -critical and edge- ι -critical.*

Proof. Let H be a graph obtained from G by successively adding edges which do not decrease $\iota(G)$, until no such edges remain. Then $V(H) = V(G)$, G is a subgraph of H , and $\iota(G) = \iota(H)$.

To see H is ι -critical, note that for any $v \in V(H)$, $\iota(H - v) \leq \iota(G - v) < \iota(G) = \iota(H)$, since G is a subgraph of H and G is ι -critical,

The graph H is edge- ι -critical by construction. ■

Attempts to characterize γ -critical graphs of domination number greater than two have been unsuccessful. The following theorem shows that it is not possible to do so in terms of forbidden subgraphs.

Theorem 2 28 ([4]) *For any graph G there is a γ -critical graph H such that G is an induced subgraph of H .*

It is also not possible to characterize ι -critical graphs in terms of forbidden subgraphs. The following theorem is similar to Theorem 2 28.

Theorem 2 29 *For any graph G , there is an ι -critical graph H such that G is an induced subgraph of H .*

Proof. Assume $\iota(G) \geq 3$. (If not, we can add two isolated vertices to G so that $\iota(G) \geq 3$.)

Let v_1, v_2, \dots, v_n be the vertices of G , then $n \geq 3$. Define a new graph H by adding $2n$ new vertices $w_1, w_2, \dots, w_n, x_1, x_2, \dots, x_n$ to G so that

$$V(H) = \{v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_n, x_1, x_2, \dots, x_n\},$$

and

$$E(H) = E(G) \cup \{w_i x_j, w_i v_j, x_i v_j \mid 1 \leq i, j \leq n, i \neq j\}$$

Clearly, G is an induced subgraph of H . Since $\{w_j, x_j, v_j\}$ is an independent dominating set of H , $\iota(H) \leq 3$. On the other hand, since both $\{w_1, w_2, \dots, w_n\}$ and $\{x_1, x_2, \dots, x_n\}$ are independent (with $n \geq 3$) and $\iota(G) \geq 3$, we must have $\iota(H) \geq 3$ by the construction of H . Therefore, $\iota(H) = 3$. Now, for any $v \in V(H)$, there exists $1 \leq k \leq n$ such that $v \in \{w_k, x_k, v_k\}$ and $D_v = \{w_k, x_k, v_k\} - \{v\}$ is an independent dominating set of $H - v$. Therefore, H is ι -critical. ■

Sumner and Blitch [13] conjectured that if G is k -edge- γ -critical, then $\iota(G) = \gamma(G)$. We show that this conjecture is false by constructing a 4-edge- γ -critical graph B with $\iota(B) = 5$.

Let B_1 be the graph with

$$V(B_1) = \{x_{ij} \mid i = 1, 2, 1 \leq j \leq 4\}$$

and

$$E(B_1) = \{x_{1j} x_{2k} \mid 1 \leq j, k \leq 4\}.$$

Then $B_1 \cong K_{4,4}$. Suppose $V(K_{12}) = \{u_k, v_k, w_k \mid 1 \leq k \leq 4\}$, and let $B_2 = K_{12} - \{u_k w_k, v_k w_k \mid 1 \leq k \leq 4\}$. Then $B_2 \cong K_{12} - 4K_{1,2}$. The graph B is obtained from $B_1 \cup B_2$ by adding the edges $\{x_{ij} u_j \mid 1 \leq j \leq 4, i = 1, 2\}$.

The graph B is shown in Figure 2.4

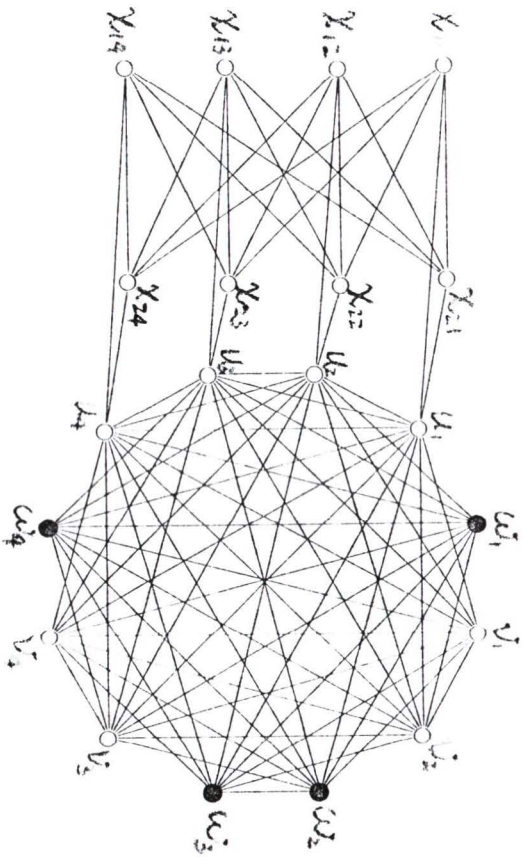
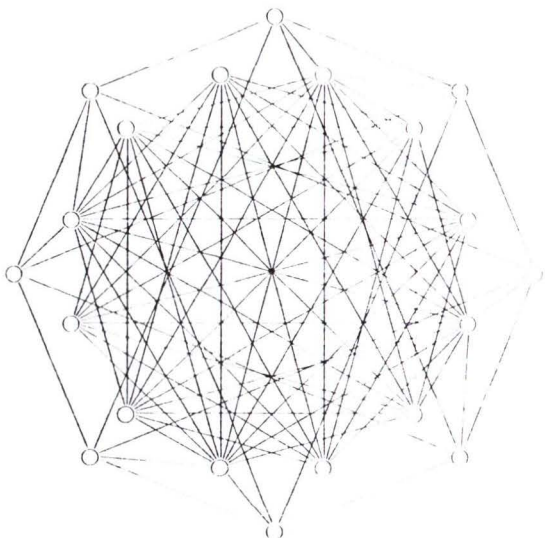


Figure 2.4 The graph B edge-critical with $\tilde{\nu}(B) \neq \nu(B)$

We first prove that $i(B) = 5$. It is easy to see that any independent dominating set I of B must contain one of w_1, w_2, w_3, w_4 . Without loss of generality, suppose $w_1 \in I$. Then, since v_1 must be dominated, either $u_1 \in I$ or $v_1 \in I$. The former case leads to $I = \{w_1, u_1, x_{12}, x_{13}, x_{14}\}$ or $I = \{w_1, u_1, x_{22}, x_{23}, x_{24}\}$, and the latter case leads to $I = \{w_1, v_1, x_{11}, x_{12}, x_{13}, x_{14}\}$ or $I = \{w_1, v_1, x_{21}, x_{22}, x_{23}, x_{24}\}$. Thus $i(B) = 5$.

Next, we prove that $\gamma(B) = 4$. Since $\{u_1, u_2, u_3, u_4\}$ dominates B , we have $\gamma(B) \leq |D| = 4$. We show no three vertices dominate B . Let D be a 3-subset of $V(G)$. If $D \cap V(B_2) = \emptyset$, then w_1 is not dominated, hence $D \cap V(B_2) \neq \emptyset$. Further, $|D \cap V(B_2)| \geq 2$, since no vertex of B_2 dominates $\{v_1, v_2, v_3, v_4, w_1, w_2, w_3, w_4\}$. In fact $|D \cap V(B_2)| = 2$ because no three vertices of B_2 dominate B_1 . Without loss of generality assume $D \cap V(B_1) = \{x_{11}\}$. Then the two vertices in $D \cap V(B_2)$ must dominate $\{x_{12}, x_{13}, x_{14}\}$. But any two vertices of B_2 dominate at most two of these vertices. Thus D is not a dominating set for B , and $\gamma(B) = 4$.

Finally, we prove that B is 4-edge- γ -critical. Up to symmetry there are eight possible edges which can be added to B . In each case we list a dominating set with size three of the graph $B + e$:

- (1) $\{x_{11}, u_3, u_4\}$ dominates $B + x_{11}x_{12}$
- (2) $\{x_{11}, x_{21}, u_1\}$ dominates $B + u_1w_1$
- (3) $\{x_{11}, x_{21}, v_1\}$ dominates $B + v_1w_1$
- (4) $\{x_{11}, x_{21}, w_1\}$ dominates $B + x_{11}v_1$
- (5) $\{x_{11}, u_3, v_1\}$ dominates $B + x_{11}w_1$
- (6) $\{x_{14}, u_3, u_2\}$ dominates $B + x_{11}u_2$
- (7) $\{x_{11}, x_{22}, w_2\}$ dominates $B + x_{11}v_2$
- (8) $\{x_{11}, x_{22}, v_2\}$ dominates $B + x_{11}w_2$ ■

Chapter 3

Vertex Domination Critical Graphs

It is the purpose of this chapter to study vertex domination critical graphs. First, in Section 3.1, we survey the previous work on γ -critical graphs and then establish a classification of all connected γ -critical graphs with the largest possible diameter. In Section 3.2, we develop an analogous study of ι -critical graphs. We discuss some bounds on the number of vertices of these graphs, give a construction technique which can produce more classes of ι -critical graphs, and classify all connected ι -critical graphs with the largest possible diameter.

3.1 γ -Critical Graphs

Brigham, Chinn and Dutton [4] discussed vertex domination critical graphs in the following sense. G is γ -critical if for any $v \in V(G)$, $\gamma(G - v) < \gamma(G)$. They presented some properties of γ -critical graphs and a method

of constructing them. If it is important to emphasize that the domination number of a γ -critical graph G is n , then we refer to G as an n - γ -critical graph

Lemma 3.1 ([4]) *The only 1- γ -critical graph is K_1 , the only 2- γ -critical graphs are $K_{2n} - F$ (or equivalently, $T_{m,2m}$), where $m \geq 1$ and F is a perfect matching.*

Lemma 3.2 ([4]) *For any graph G , $\forall v \in V(G)$, $\gamma(G - v) \geq \gamma(G) - 1$.*

Note that, by considering $K_{1,n}$ for example, there is no non-trivial upper bound on $\gamma(G - v)$

We denote by G_v the graph $G - v$ and by D_v some minimum dominating set of G_v .

Lemma 3.3 *If there exist vertices $u, v \in V(G)$ such that $N[v] \subseteq N[u]$, then G is not γ -critical*

Proof Pick a minimum dominating set D_u for G_u . Then some $x \in D_u$ dominates v . Since $x \in N[v] \subseteq N[u]$, D_u also dominates u , and thus D_u is a dominating set for G . This implies that G is not γ -critical ■

Corollary 3.4 ([4]) *If G has a vertex v such that $d(v) \geq 1$ and $G[N(v)]$ is complete, then G is not γ -critical*

The following theorems give bounds on p in terms of Δ , γ and q

Theorem 3.5 ([17]) $p \leq (\Delta + 1)\gamma$

Theorem 3.6 ([4]) *If there is a vertex $v \in V(G)$ such that $\gamma(G-v) < \gamma(G)$, then $p \leq (\Delta + 1)(\gamma - 1) + 1$*

If the equality holds in Theorem 3.6, then in any minimum dominating set of $G - v$, each vertex must dominate $\Delta + 1$ vertices. Fulman [7] proved the following result.

Theorem 3.7 ([7]) *If G is n - γ -critical and $p = (\Delta + 1)(n - 1) + 1$, then G is regular*

Theorem 3.8 ([4]) *If G is n - γ -critical, then $p \leq (2q + 3n - \Delta)/3$*

It is clear that a graph G is γ -critical if and only if every component of G is γ -critical. A similar statement holds for the blocks of G (see Theorem 3.11). Recall the definition of coalescence which was given in Chapter 2.

Lemma 3.9 ([4]) *Let H and G be nontrivial graphs. For any coalescence $H \cdot G$,*

$$\gamma(H) + \gamma(G) - 1 \leq \gamma(H \cdot G) \leq \gamma(H) + \gamma(G)$$

Furthermore, if both H and G are γ -critical, or if $H \cdot G$ is γ -critical, then

$$\gamma(H \cdot G) = \gamma(H) + \gamma(G) - 1$$

Theorem 3.10 ([4]) *The graph $H \cdot G$ is γ -critical if and only if both H and G are γ -critical.*

Theorem 3.11 ([4]) *A graph G is γ -critical if and only if each block of G is γ -critical. Further, if G is γ -critical with blocks G_1, \dots, G_n then*

$$\gamma(G) = \sum_{i=1}^n \gamma(G_i) - (n - 1).$$

Brigham et al [4] conjectured that for any n - γ -critical graph G , the diameter of G is not greater than $2(n - 1)$. This is proved by Hanson and MacGillivray [8].

Theorem 3.12 ([8]) *If G is connected and n - γ -critical, then $\text{diam}(G) \leq 2(n - 1)$ for $n \geq 2$.*

We give an alternate proof of this theorem using Lemma 3.13 below.

For a graph G , if $\text{diam}(G) = d$, then there exist two vertices a and b such that $d(a, b) = d$. For $j = 0, 1, \dots, d$, let $A_j = \{x \mid d(x, a) = j\}$. So, $A_0 = \{a\}$, $b \in A_d$, $V(G) = \bigcup_{j=0}^d A_j$, and $A_k \cap A_l = \emptyset$ for any $k \neq l$. Also, $uv \in E(G)$ if and only if $u, v \in A_{j-1} \cup A_j$ for some $1 \leq j \leq d$. For $k = 0, 1, \dots, d$, let $N_k = \bigcup_{j=0}^k A_j$, and $\overline{N}_k = V(G) - N_k$. As before, for any $x \in V(G)$, denote by D_x a minimum dominating set of $G - x$. We use these notations in the following lemma.

Lemma 3.13 *Let G be a 2-connected n - γ -critical graph with $\text{diam}(G) = d \geq 5$. Then for $4 \leq h \leq d - 1$ and all $x \in N_{d-h+1}$, $|D_x \cap \overline{N}_{d-h}| \geq \lfloor \frac{h}{2} \rfloor$.*

Proof. By induction on h .

Suppose $h = 4$. If the statement is false, then there exists $x \in N_{d-3}$ such that

$$|D_x \cap \overline{N}_{d-4}| \leq \left\lfloor \frac{4}{2} \right\rfloor - 1 = 1$$

Since $D_x \cap \overline{N}_{d-4}$ must dominate $A_{d-2} \cup A_{d-1} \cup A_d$, we have $D_x \cap \overline{N}_{d-4} = \{u\} \subseteq A_{d-1}$. But, since G is 2-connected, $|A_{d-1}| \geq 2$, and this implies that there exists $v \in A_{d-1}$ such that $N[v] \subseteq N[u]$. Hence, by Lemma 3.3, G is not γ -critical, a contradiction.

Assume the statement is true whenever $4 \leq h \leq i - 1$. Suppose the statement is false when $h = i$, that is, there exists $x \in N_{d-i+1}$ such that

$$|D_x \cap \overline{N}_{d-i}| \leq \left\lfloor \frac{i}{2} \right\rfloor - 1$$

It follows that there exists $l \leq i - 1$ such that $D_x \cap (A_{d-l} \cup A_{d-l+1}) = \emptyset$. Let k be the largest integer less than or equal to $i - 1$ such that

$$D_x \cap (A_{d-k} \cup A_{d-k+1}) = \emptyset$$

By the choice of k ,

$$m = |D_x \cap \overline{N}_{d-k+1}| \leq \left\lfloor \frac{i}{2} \right\rfloor - 1 - \frac{i - k - 1}{2}$$

Let $s \in A_{d-k+1}$ and consider D_s . Since $D_s \cap \overline{N}_{d-k+1}$ does not dominate any vertex in N_{d-k} , the set $D_s \cap N_{d-k+1}$ must dominate every vertex of N_{d-k} .

Suppose $|D_s \cap \overline{N}_{d-k+1}| \geq m$. Then $|D_s \cap N_{d-k+1}| \leq n - 1 - m$. Since $D_x \cap \overline{N}_{d-k+1}$ dominates \overline{N}_{d-k} , we have that $(D_s \cap N_{d-k+1}) \cup (D_x \cap \overline{N}_{d-k+1})$

dominates G . But,

$$\begin{aligned} |(D_s \cap N_{d-k+1}) \cup (D_x \cap \overline{N}_{d-k+1})| &= |D_s \cap N_{d-k+1}| + |D_x \cap \overline{N}_{d-k+1}| \\ &\leq (n-1-m) + m = n-1, \end{aligned}$$

a contradiction. Therefore

$$|D_s \cap \overline{N}_{d-k+1}| \leq m-1.$$

Since $s \in N_{d-k+2}$ and $k-1 \leq i-2$, the induction hypothesis implies

$$\left\lfloor \frac{k-1}{2} \right\rfloor \leq |D_s \cap \overline{N}_{d-k+1}| \leq m-1.$$

Therefore

$$\left\lfloor \frac{k-1}{2} \right\rfloor \leq \left\lfloor \frac{i}{2} \right\rfloor - 1 - \frac{i-k-1}{2} - 1,$$

which is impossible. This completes the proof. ■

Proof of Theorem 3.12: If G is 2-connected, then take $h = d-1$ in Lemma 3.13. We have for each $x \in N_2$, $|D_x \cap \overline{N}_1| \geq \lfloor \frac{d-1}{2} \rfloor$. Choose $x \notin A_0$. Then since $A_0 - \{x\} \neq \emptyset$, we have $D_x \cap N_1 \neq \emptyset$, and thus, $|D_x| \geq \lfloor \frac{d-1}{2} \rfloor + 1$. Therefore,

$$\gamma(G) = n = |D_x| + 1 \geq \left\lfloor \frac{d-1}{2} \right\rfloor + 2,$$

that is,

$$n \geq \frac{d-2}{2} + 2 = \frac{d+2}{2}$$

It follows that $d \leq 2n-2$.

If G is not 2-connected, then we prove the result by induction on n . From Lemma 3.1, the statement is immediate if $n = 2$.

Suppose for all $3 \leq m \leq n - 1$, the statement is true for m - γ -critical graphs. Let G be an n - γ -critical graph and v be a cut vertex of G . Then $G = H_1 \cdot H_2$ with v as the vertex of identification. By Theorem 3.11, both H_1 and H_2 are γ -critical with $\gamma(H_1) = k$, say, and $\gamma(H_2) = n - k + 1$. By the induction hypothesis, $\text{diam}(H_1) \leq 2k - 2$ and $\text{diam}(H_2) \leq 2(n - k + 1) - 2 = 2n - 2k$. Therefore, $\text{diam}(G) \leq \text{diam}(H_1) + \text{diam}(H_2) \leq (2k - 2) + (2n - 2k) = 2n - 2$. This completes the proof. ■

Lemma 3.14 *For $n \geq 3$, if G is connected and n - γ -critical with $\text{diam}(G) = 2(n - 1)$, then G is not a block.*

Proof: Let $n = 3$, and G be a connected 3- γ -critical graph with $\text{diam}(G) = d = 4$. Let a and b be vertices such that $d(a, b) = d$, and define A_0, A_1, \dots, A_d as above. Suppose G is a block. Then $|A_j| \geq 2$ for $j = 1, \dots, d - 1$. Now, pick $t \in A_2$ and consider D_t (a minimum dominating set of $G - t$). Since G is 3- γ -critical, $|D_t| = 2$ and $|D_t \cap N[a]| = |D_t \cap N[b]| = 1$. Since removing t left $N[a]$ and $N[b]$ unchanged, by Lemma 3.3, $D_t = \{a, b\}$. (If $a \notin D_t$, then we have $N[a] \subseteq N[x]$ for some $x \in D_t \cap A_1$ since x dominates $N[a]$. But this contradicts Lemma 3.3, and so, we must have $a \in D_t$. Similarly, $b \in D_t$.) But then D_t cannot dominate $A_2 - \{t\} \neq \emptyset$ (since $|A_2| \geq 2$), a contradiction. Therefore, G cannot be a block.

For $n \geq 4$, let G be a connected n - γ -critical graph with diameter $d = 2(n-1)$. Again let a and b be vertices such that $d(a, b) = d$, define A_0, A_1, \dots, A_d as above. Suppose G is a block. Then $|A_j| \geq 2$ for $j = 1, \dots, d-1$. Let $T = A_2 \cup A_3$. Pick any $t \in T$ and consider D_t , a minimum dominating set of $G-t$. Since G is n - γ -critical, $|D_t| = n-1$.

By Lemma 3.13, $|D_t \cap \overline{N}_4| \geq \lfloor \frac{d-4}{2} \rfloor = n-3$, and therefore, $|D_t \cap N_4| \leq 2$. Since $|D_t \cap N_4| \neq 1$ (otherwise, some vertex in $(N_4 - \{t\})$ cannot be dominated), we must have $|D_t \cap N_4| = 2$. Now, if $D_t \cap A_2 \neq \emptyset$, then $D_t \cap (A_3 \cup A_4) = \emptyset$ (otherwise, vertex a cannot be dominated). But then $|D_t \cap \overline{N}_2| = n-3 < n-2 = \frac{(d-2)}{2}$, contradicting Lemma 3.13. Therefore, $D_t \cap A_2 = \emptyset$. Suppose $D_t \cap A_4 \neq \emptyset$. Since removing t leaves $N[a]$ unchanged, Lemma 3.3 implies that $a \in D_t$ (as before, if $a \notin D_t$, then $N[a] \subseteq N[x]$ for some $x \in D_t \cap A_1$, contradicting Lemma 3.3), and therefore, no vertex in $A_2 - \{t\}$ can be dominated (such a vertex exists because $|A_2| \geq 2$). Therefore, $D_t \cap A_4 = \emptyset$ and we must have $|D_t \cap A_3| = 1$. That is, for any $t \in T$ we must have a $t' \in D_t \cap A_3$ such that D_t is a dominating set of $G-t$ and t' dominates $T - \{t\}$. Furthermore, if $t_1 \neq t_2$, we must have $t'_1 \neq t'_2$ since $t'_1 t_2 \in E(G)$ (because t'_1 dominates $T - \{t_1\}$) and $t'_2 t_2 \notin E(G)$ (as $t'_2 \in D_{t_2} \subseteq V(G) - N[t_2]$). In another words, we have a one-to-one mapping from $T = A_2 \cup A_3$ to A_3 . Since $|A_2| \geq 2$, this is impossible by the pigeonhole principle. Thus, G cannot be a block.

Furthermore, from the above proof, we come to the conclusion that $|A_2| = 1$, and therefore, by induction we can get $|A_{2k}| = 1$ for all $k = 1, 2, \dots, n-2$.

■

We now show how to construct all n - γ -critical graphs with diameter $2n-2$. Let G_1, G_2, \dots, G_{n-1} be 2 - γ -critical graphs. We define a coalescence of these graphs, denoted by $G_1 \diamond G_2 \diamond \dots \diamond G_{n-1}$, inductively as follows

(i) $G_1 \diamond G_2$ is any coalescence of G_1 and G_2

(ii) For $k \geq 2$, $G_1 \diamond G_2 \diamond \dots \diamond G_k \diamond G_{k+1} = (G_1 \diamond G_2 \diamond \dots \diamond G_k)_{uv} G_{k+1}$, where u is the unique vertex of G_k which is non-adjacent to the vertex of identification of $(G_1 \diamond G_2 \diamond \dots \diamond G_{k-1})$ and G_k (such a unique vertex u exists by Lemma 3.1), and v is any vertex of G_{k+1} .

By Theorem 3.11, $G = G_1 \diamond G_2 \diamond \dots \diamond G_{n-1}$ is n - γ -critical. Moreover,

$$\text{diam}(G) = \sum_{j=1}^{n-1} \text{diam}(G_j) = 2(n-1)$$

Theorem 3.15 For $n \geq 3$, a graph G is n - γ -critical with $\text{diam}(G) = 2(n-1)$ if and only if $G = G_1 \diamond G_2 \diamond \dots \diamond G_{n-1}$, where G_j is 2 - γ -critical for $j = 1, \dots, n-1$.

Proof. We only need to prove the necessity. Let G be an n - γ -critical graph with $\text{diam}(G) = 2n-2$. Let a and b be vertices such that $d(a, b) = 2n-2$, and define $A_0, A_1, \dots, A_{2n-2}$ as before. Then by Lemma 3.14, $|A_{2k}| = 1$ for all $k = 1, 2, \dots, n-2$ (see the end of the proof of Lemma 3.14). For

$j = 1, 2, \dots, n - 1$, let $G_j = G[A_{2j-2} \cup A_{2j-1} \cup A_{2j}]$. Then we have

$$G = G_1 \diamond G_2 \diamond \dots \diamond G_{n-1},$$

and by Theorem 3.11, G_j is $2\text{-}\gamma$ -critical for $j = 1, 2, \dots, n - 1$. ■

3.2 ι -Critical Graphs

In this section we study vertex independent domination critical graphs in the following sense: G is ι -critical if for any $v \in V(G)$, $\iota(G - v) < \iota(G)$. If it is important to emphasize that the independent domination number of a ι -critical graph G is n , then we refer to G as an n - ι -critical graph. As in Section 3.1, we will denote by G_v the graph $G - v$ and I_v (or D_v) some minimum independent dominating set of G_v . We have some analogous results to those in Section 3.1.

Lemma 3.16 *The only 1- ι -critical graph is K_1 , and the only 2- ι -critical graphs are $T_{m,2m}$ ($m \geq 1$).*

Proposition 3.17 *For a graph G , $\iota(G - v) \geq \iota(G) - 1$ for any $v \in V(G)$.*

Again, by considering $K_{1,n}$ for example, there is no non-trivial upper bound on $\iota(G - v)$.

Lemma 3.18 *If there exist distinct vertices $u, v \in V(G)$ such that $N[v] \subseteq N[u]$, then G is not ι -critical.*

Proof. Any minimum independent dominating set of $G - u$ includes a vertex of $N[v]$, but $N[v] \subseteq N[u]$, and so it must also be an independent dominating set of G . Hence $\iota(G - u) \geq \iota(G)$, and so G is not ι -critical ■

Corollary 3.19 *If G has a vertex v with $d(v) \geq 1$ such that $G[N[v]]$ is complete, then G is not ι -critical.*

Proof. Note that for every $u \in N[v]$, $N[v] \subseteq N[u]$. ■

Corollary 3.20 *If G is ι -critical, then G has no vertices of degree 1.*

Proof. Note that if $d(x) = 1$, then $G[N[x]] \cong K_2$ is complete. ■

Corollary 3.21 *If G is connected and ι -critical, then $\delta(G) \geq 2$.*

Now, similarly to [4], we have the following upper bounds on p in terms of q , Δ , and ι .

Theorem 3.22 $p \leq (\Delta + 1)\iota$

Proof. Let D be a minimum independent dominating set. Then $|D| + |D|\Delta \geq p$, since Δ is the maximum degree of G . That is,

$$p \leq \iota + \iota\Delta = \iota(\Delta + 1). \quad \blacksquare$$

Theorem 3.23 *If there is a vertex $u \in V(G)$ such that $\iota(G - u) < \iota(G)$, then $p \leq (\Delta + 1)(\iota - 1) + 1$.*

Proof Let u be a vertex such that $\iota(G-u) < \iota(G)$, and let D_u be a minimum independent dominating set of $G-u$. Then, by Lemma 3.17, $|D_u| = \iota(G) - 1$. Now, by Theorem 3.22, we have

$$|V(G-u)| \leq (\Delta + 1)|D_u| = (\Delta + 1)(\iota(G) - 1),$$

that is, $p \leq (\Delta + 1)(\iota(G) - 1) + 1$. ■

Similar to Theorem 3.7, we have the following result

Theorem 3.24 *If G is n - ι -critical and $p = (\Delta + 1)(n - 1) + 1$, then G is regular*

Proof First, we prove that for any $v \in V(G)$ and any minimum independent dominating set D_v of $G-v$, all vertices in D_v have degree Δ . Since G is n - ι -critical, $|D_v| = n - 1$, and D_v dominates all $p - 1 = (\Delta + 1)(n - 1)$ vertices of $G-v$. Since each element of D_v dominates at most $\Delta + 1$ vertices, this implies that each element of D_v dominates exactly $(\Delta + 1)$ vertices, and therefore has degree Δ . Moreover, $N[x] \cap N[y] = \emptyset$, for all $x, y \in D_v$ and $x \neq y$. That is, x and y cannot have any common neighbours.

Next, we prove that for any $u \in V(G)$, there exists $v \in V(G)$ such that $u \in D_v$. (So, from the above, $d(u) = \Delta$, and thus G is regular.) Let $u \in V(G)$ and choose $v \in D_u$. Note that $D_v \cap N[v] = \emptyset$, for otherwise D_v dominates G , which is impossible. Since $N[x] \cap N[y] = \emptyset$ for any $x, y \in D_v$, each $w \in V(G - v)$ is dominated precisely once by D_v . Thus, we have

$|D_v \cap N[w]| = 1$ for any $w \in (D_u - \{v\})$. Therefore, for any $w \in (D_u - \{v\})$, there is one and only one corresponding vertex in $D_v \cap V(G_u)$, and vice versa. In other words, we have a 1-1 correspondence between $D_u - \{v\}$ and $D_v \cap V(G_u)$. Thus, $|D_v \cap V(G_u)| = |D_u - \{v\}| = n - 2$. But $|D_v| = n - 1$, so we must have $u \in D_v$. ■

Theorem 3 25 *If G is n - ι -critical, then $p \leq (2q + 3n - \Delta)/3$.*

Proof. First assume G has no isolated vertices. Then by Corollary 3 21, $\delta(G) \geq 2$. Let u be a vertex such that $d(u) = \Delta$, and let $D = D_u \cup \{u\}$ be a minimum independent dominating set of G . Each vertex in $N(u)$ is adjacent to at least two vertices in $D - u$ and some vertex in D_u . Each of the remaining vertices in $V - D$ is adjacent to a vertex in D , and is incident with at least one more edge since $\delta(G) \geq 2$. Thus, there are at least $\lceil (p - n - \Delta)/2 \rceil$ more edges in G . Therefore, $q \geq 2\Delta + (p - n - \Delta) + \lceil (p - n - \Delta)/2 \rceil$, that is, $p \leq (2q + 3n - \Delta)/3$.

Next, suppose G has m isolated vertices. Then

$$p - m \leq [2q + 3(n - m) - \Delta]/3 = (2q + 3n - \Delta)/3 - m,$$

that is, $p \leq (2q + 3n - \Delta)/3$. ■

The bounds of the above theorems are the best possible in the sense that equality holds for some graphs. For C_6 , we have $(\Delta + 1)\iota = 6 = p$. For the infinite class of ι -critical graphs $H_{2r,m}$ defined before (Section 2 1), the

equality in Theorem 3.23 holds, and the equality of Theorem 3.25 holds for the infinite class of ι -critical cycles C_{3k+1} .

Obviously, a graph G is ι -critical if and only if every component of G is ι -critical. We will prove that a similar statement holds for the blocks of G . First, we have the following results about the coalescence of graphs G and H .

Theorem 3.26 *If G and H are disjoint nontrivial graphs, then for any coalescence $G \cdot H$,*

$$\iota(G) + \iota(H) - 1 \leq \iota(G \cdot H) \leq \min\{\iota(G) + \beta(H), \iota(H) + \beta(G)\},$$

where $\beta(G)$ is the independence number of G .

Proof Let D_G and D_H be minimum independent dominating sets of G and H , respectively, and let v be the vertex of identification of G and H . In order to prove the upper bound, there are four cases to consider:

Case 1: $v \in D_G \cap D_H$. Then $D_G \cup D_H$ is a minimum independent dominating set of $G \cdot H$ and therefore

$$\iota(G \cdot H) = |D_G| + |D_H| - 1 = \iota(G) + \iota(H) - 1.$$

Case 2: $v \notin D_G \cup D_H$. That is, $v \notin D_G$ and $v \notin D_H$. Then $D_G \cup D_H$ is an independent dominating set of $G \cdot H$ and therefore

$$\iota(G \cdot H) \leq |D_G| + |D_H| = \iota(G) + \iota(H).$$

Case 3: $v \in D_G$ but $v \notin D_H$ Suppose I_v is a maximal independent set of H containing v . Then, by Lemma 2.1, I_v is an independent dominating set of H . So, $D_G \cup I_v$ is an independent dominating set of $G \cdot H$ and thus

$$\iota(G \cdot H) \leq |D_G| + |I_v| - 1 \leq \iota(G) + \beta(H) - 1 < \iota(G) + \beta(H).$$

On the other hand, let I'_v be a maximal independent set of G not containing v . Then $I = I'_v \cup D_H$ is an independent dominating set of $G \cdot H$ and $v \notin I$. Therefore, we have

$$\iota(G \cdot H) \leq |I| \leq |D_H| + |I'_v| \leq \iota(H) + \beta(G)$$

Case 4: $v \in D_H$ but $v \notin D_G$ Similarly to Case 3, we get

$$\iota(G \cdot H) < \iota(H) + \beta(G),$$

and

$$\iota(G \cdot H) \leq \iota(G) + \beta(H).$$

Therefore, in any case, we have

$$\iota(G \cdot H) \leq \min\{\iota(G) + \beta(H), \iota(H) + \beta(G)\},$$

since $\iota(G) \leq \beta(G)$ and $\iota(H) \leq \beta(H)$.

Now, we prove that $\iota(G \cdot H) \geq \iota(G) + \iota(H) - 1$. Again, let v be the vertex of identification of G and H . If there are D_G and D_H (minimum independent dominating sets of G and H , respectively) such that $v \in D_G \cap D_H$, then by the case 1 above, we have $\iota(G \cdot H) = \iota(G) + \iota(H) - 1$. Otherwise, one of G

and H , say H , is such that for every minimum independent dominating set D_H of H , $v \notin D_H$. Then

$$\iota(G \cdot H) \geq \iota(G \cup (H - v)) \geq \iota(G) + (\iota(H) - 1),$$

and we are done. ■

Note that in the proof above (see Case 3 and Case 4), if $\iota(G) + \beta(H) = \iota(H) + \beta(G)$, then we have $\iota(G \cdot H) \leq \iota(G) + \beta(H) - 1$ (or $\iota(H) + \beta(G) - 1$). But the bounds of Theorem 3.26 are the best possible in the sense of the following proposition.

Proposition 3.27 *For any integers $1 \leq k \leq r$, $1 \leq l \leq s$, and $r - k \neq s - l$, let $-1 \leq t \leq \min\{s - l, r - k\}$. There exist graphs G and H with $\iota(G) = k$, $\beta(G) = r$, $\iota(H) = l$, $\beta(H) = s$ and a coalescence $G \cdot H$ such that $\iota(G \cdot H) = k + l + t$. (In other words, for some given set of independent domination numbers (k and l) and independence numbers (r and s), we can find graphs G and H with $\iota(G) = k$, $\beta(G) = r$, $\iota(H) = l$, $\beta(H) = s$ and a coalescence $G \cdot H$ so that $\iota(G \cdot H)$ assumes any given value $k + l + t$ in the interval $[k + l - 1, \min\{k + s, l + r\}]$.)*

Proof. If $t = -1$, let $G = K_{k,r}$ and $H = K_{l,s}$. Obviously, $\iota(G) = k$, $\beta(G) = r$, $\iota(H) = l$ and $\beta(H) = s$. Let u be a vertex in the partition set of size k in G and v be a vertex in the partition set of size l in H . Then $W = G \cdot_{uv} H$ satisfies $\iota(W) = k + l - 1$.

Suppose $t \geq 0$. Since $r - k \neq s - l$, without loss of generality, assume $s - l < r - k$. Then $t = s - l$. Let

$$G = \overline{K}_{k-2} \cup K_{1,r-k-t} \cup K_{1,t+2},$$

and

$$H = \overline{K}_{l-1} \cup K_{1,t+1}.$$

(If $k = 1$, then let $G = K_{1,r}$, if $k = 2$, then $G = K_{1,r-k-t} \cup K_{1,t+2}$, and if $l = 1$, then let $H = K_{1,s}$.) It is easy to see that $\iota(G) = k$, $\beta(G) = r$, $\iota(H) = l$, and $\beta(H) = s$. Let $u \in V(G)$ be a vertex of degree 1 in $K_{1,t+2}$ ($K_{1,r}$ if $k = 1$) and $v \in V(H)$ be a vertex of maximum degree in $K_{1,t+1}$ ($K_{1,s}$ if $l = 1$). Then the graph $W = G \underset{uv}{\cdot} H$ satisfies $\iota(W) = k + l + t$. ■

Theorem 3 28 *If G and H are ι -critical, then $\iota(G \cdot H) = \iota(G) + \iota(H) - 1$.*

Proof Suppose both G and H are ι -critical, and let v be the vertex of identification of G and H . Then by Lemma 3 17,

$$\iota(G \cdot H) - 1 \leq \iota(G \cdot H - v) = \iota(G - v) + \iota(H - v),$$

and by Lemma 2 20 we get $\iota(G \cdot H) \leq \iota(G) + \iota(H) - 1$. ■

Theorem 3 29 *If $G \cdot H$ is ι -critical, then $\iota(G \cdot H) = \iota(G) + \iota(H) - 1$.*

Proof: Suppose $G \cdot H$ is ι -critical and v is the vertex of identification of G and H . By Lemma 2 21, v is in a minimum independent dominating set of $G \cdot H$. Let $D = I_v \cup \{v\}$ be such a minimum independent dominating set.

Now, let $I_G = I_v \cap V(G)$ and $I_H = I_v \cap V(H)$. Then $D_G = I_G \cup \{v\}$ and $D_H = I_H \cup \{v\}$ are minimum independent dominating sets of G and H , respectively. (Otherwise, if A is an independent dominating set of G and $|A| < |D_G|$, then $A \cup I_H$ would be an independent dominating set of $G \cdot H$ and $|A \cup I_H| = |A| + |I_H| < |D_G| + |I_H| = \iota(G \cdot H)$, a contradiction.)

Therefore, we have $|D_G| = |I_G| + 1 = \iota(G)$ and $|D_H| = |I_H| + 1 = \iota(H)$. Hence

$$\begin{aligned} \iota(G \cdot H) &= |D| = |I_v| + 1 = |I_G \cup I_H| + 1 = |I_G| + |I_H| + 1 \\ &= (\iota(G) - 1) + (\iota(H) - 1) + 1 = \iota(G) + \iota(H) - 1 \quad \blacksquare \end{aligned}$$

The following theorem yields another construction for ι -critical graphs.

Theorem 3.30 *The graph $G \cdot H$ is ι -critical if and only if both G and H are ι -critical.*

Proof: Let v be the vertex of identification of G and H .

Suppose both G and H are ι -critical, and let $u \in V(G \cdot H)$.

If $u = v$, then

$$\begin{aligned} \iota(G \cdot H - v) &= \iota(G - v) + \iota(H - v) = (\iota(G) - 1) + (\iota(H) - 1) \\ &= (\iota(G) + \iota(H) - 1) - 1 = \iota(G \cdot H) - 1. \end{aligned}$$

If $u \neq v$, without loss of generality, suppose $u \in V(G) - \{v\}$. Let I_u and I_v be minimum independent dominating sets of $G - u$ and $H - v$, respectively.

Then since $I_v \cap N_H[v] = \emptyset$, $I_u \cup I_v$ is an independent dominating set of $(G \cdot H - u)$ and

$$\begin{aligned} \iota(G \cdot H - u) &\leq |I_u \cup I_v| = |I_u| + |I_v| \\ &= (\iota(G) - 1) + (\iota(H) - 1) = \iota(G \cdot H) - 1. \end{aligned}$$

But, $\iota(G \cdot H - u) \geq \iota(G \cdot H) - 1$, so we get $\iota(G \cdot H - u) = \iota(G \cdot H) - 1$. Therefore, by Lemma 2.20, $G \cdot H$ is ι -critical.

Now, suppose $G \cdot H$ is ι -critical but G is not. Then there is $u \in V(G)$ such that $\iota(G - u) \geq \iota(G)$

If $u = v$, then

$$\begin{aligned} \iota(G \cdot H - v) &= \iota(G - v) + \iota(H - v) \\ &\geq \iota(G) + \iota(H) - 1 = \iota(G \cdot H), \end{aligned}$$

a contradiction, since $G \cdot H$ is ι -critical

If $u \neq v$, then without loss of generality $u \in V(G) - \{v\}$. In this case we have

$$\begin{aligned} \iota(G \cdot H - u) &= \iota((G - u) \cdot H) \geq \iota(G - u) + \iota(H) - 1 \\ &\geq \iota(G) + \iota(H) - 1 = \iota(G \cdot H), \end{aligned}$$

a contradiction. Therefore, G is ι -critical. Similarly, H is also ι -critical. ■

Corollary 3.31 *If G has blocks G_1, G_2, \dots, G_n , then G is ι -critical if and only if each G_k ($k = 1, \dots, n$) is ι -critical. Further*

$$\iota(G) = \sum_{k=1}^n \iota(G_k) - (n - 1)$$

Proof. By induction on n , the statement being immediate if $n = 1$.

Suppose the statement is true for graphs with $m \leq n$ blocks. Let G have blocks $G_1, G_2, \dots, G_n, G_{n+1}$ indexed so that G_{n+1} contains only one cut vertex of G . Then $G = H \cdot G_{n+1}$, where H is the subgraph composed of blocks G_1, G_2, \dots, G_n . The results now follow from Theorems 3.28, 3.30 and the induction hypothesis. ■

Recall that a graph G is 2 - γ -critical if and only if G is 2 - ι -critical. The only such graphs are $T_{m,2m}$ which have diameter $2 = 2(\iota(T_{m,2m}) - 1)$. We have the following Lemma.

Lemma 3.32 *If G is n - ι -critical and $\text{diam}(G) = 2(n - 1)$, then $\gamma(G) = \iota(G) = n$ and thus G is n - γ -critical.*

Proof. Suppose $\gamma(G) < \iota(G)$. Then by Theorem 3.12, $\text{diam}(G) \leq 2(\gamma(G) - 1) < 2(n - 1)$, a contradiction. ■

With the above Lemma and Theorem 3.15, we can classify all n - ι -critical graphs with diameter $2(n - 1)$.

Corollary 3.33 *A graph G is n - ι -critical with $\text{diam}(G) = 2(n - 1)$ ($n \geq 3$) if and only if $G = G_1 \diamond G_2 \diamond \dots \diamond G_{n-1}$, where G_j is 2 - ι -critical for $j = 1, 2, \dots, n - 1$. ■*

Chapter 4

Edge Domination Critical Graphs

In this chapter we focus on the study of edge domination critical graphs. We survey previous work on edge- γ -critical graphs in Section 4.1. Then, in Section 4.2, we discuss some properties of edge- i -critical graphs and show that some properties in [13] and [14] for edge- γ -critical graphs are also true for edge- i -critical graphs. After that, we concentrate on Hamiltonian properties of edge- i -critical graphs.

4.1 Edge- γ -critical Graphs

Sumner and Blitch [13, 14] discussed edge- γ -critical graphs in the following sense. G is *edge- γ -critical* if for any $uv \notin E(G)$, $\gamma(G+uv) < \gamma(G)$. Obviously, for any graph G , $\gamma(G+uv) \geq \gamma(G) - 1$ for all $uv \notin E(G)$. Thus, G is *k -edge- γ -critical* if $\gamma(G) = k$ and for each edge $uv \notin E(G)$, $\gamma(G+uv) = k - 1$. In [13], 2-edge- γ -critical graphs are characterized and k -edge- γ -critical graphs

with $k \geq 3$ are studied. They mostly discussed some properties of connected 3-edge- γ -critical graphs.

Theorem 4.1 ([13]) *G is 2-edge- γ -critical if and only if \overline{G} is a union of stars.*

Theorem 4.2 ([13]) *Every 3-edge- γ -critical graph contains K_3 .*

Theorem 4.3 ([13]) *If G is a connected 3-edge- γ -critical graph of even order, then G contains a 1-factor.*

Theorem 4.4 ([13]) *The diameter of a 3-edge- γ -critical graph is at most three.*

For any graph G and integer $k \geq 0$, we define $S_k = \{v \mid v \in V(G), d(v) \leq k\}$ and $d_k = |S_k|$.

Theorem 4.5 ([13]) *Let G be a connected 3-edge- γ -critical graph. Then for $k \geq 1$, $d_k \leq 3k$. Further, if $|V(G)|$ is large compared to k , then $d_k \leq k + 1$.*

Theorem 4.6 ([13]) *If G is a k -edge- γ -critical graph with $k \geq 1$, then for every vertex $v \in V(G)$, $\gamma(G - v) \leq k$.*

4.2 Edge- ι -critical Graphs

In this section we study edge- ι -critical graphs in the following sense: G is *edge- ι -critical* if for any $uv \notin E(G)$, $\iota(G + uv) < \iota(G)$. We establish, for edge- ι -critical graphs, results analogous to those in Section 4.1.

Proposition 4.7 *There exist n -edge- i -critical graphs with independent sets of arbitrarily large size.*

Proof For any integer $m \geq n \geq 3$, graph $P_{m,n}$ defined in Section 2.1 is an n -edge- i -critical graph with an independent set of size m . ■

In general, removing a vertex from a graph can increase the independent domination number of the graph dramatically. For example, $i(K_{1,n}) = 1$ while if v is a vertex of maximum degree of $K_{1,n}$, then $i(K_{1,n} - v) = n$. But for k -edge- i -critical graphs, we have the following theorem.

We say a set S precisely dominates a subgraph H of a graph G if S dominates H and no vertex of $G - H$ is adjacent to a vertex in S .

Theorem 4.8 *If G is a k -edge- i -critical graph with $k \geq 1$, then for every vertex $v \in V(G)$, $i(G - v) \leq k$.*

Proof Since the case $k = 1$ is trivial, we assume $k \geq 2$. Let $v \in V(G)$. If there exist $x, y \in N(v)$ with $xy \notin E(G)$, then without loss of generality, there exists a set S with $|S| = k - 2$ such that $S \cup \{x\}$ is independent and precisely dominates $G - y$. So, we have $v \notin S$. Thus $S \cup \{x, y\}$ is an independent dominating set of $G - v$, and hence $i(G - v) \leq k$.

Now, assume that $N(v)$ is complete. Let $w \in V(G) - N[v]$. Since $vw \notin E(G)$, there exists a set $S \subseteq V(G) - N[v]$ with $|S| = k - 2$ such that $S \cup \{v\}$ is independent and precisely dominates $G - w$ or $S \cup \{w\}$ is independent and precisely dominates $G - v$. In the latter case we are done. So, suppose $S \cup \{v\}$

precisely dominates $G - w$. If every $x \in N(v)$ is adjacent to some vertex $u \in S \cup \{w\}$, then $S \cup \{w\}$ dominates $G - v$, and therefore $\iota(G - v) = k - 1$. On the other hand, if there exists some $x \in N(v)$ such that $\{x\} \cup S \cup \{w\}$ is independent, then since $N(v)$ is complete, $S \cup \{x, w\}$ dominates $G - v$, and so $\iota(G - v) \leq k$. ■

In what follows we give a complete characterization of disconnected 3-edge- ι -critical graphs, and some structural properties of connected 3-edge- ι -critical graphs. The latter results are similar to those in [13]. First, we have the following characterization of 2-edge- ι -critical graphs.

Theorem 4.9 *A graph G is 2-edge- ι -critical if and only if \overline{G} is a union of stars.*

Proof. We show that G is 2-edge- ι -critical if and only if G is 2-edge- γ -critical. The result then follows from Theorem 4.1. Clearly, if G is 2-edge- ι -critical, then G is 2-edge- γ -critical. Suppose G is 2-edge- γ -critical. It is enough to show $\iota(G) = 2$. Let u and v be non-adjacent vertices of G . Since G has domination number two and $G + uv$ has domination number one, we have without loss of generality that $N[u] = V(G) - v$. Thus $\{u, v\}$ is an independent dominating set for G . ■

Let G be a 3-edge- ι -critical graph. If $uv \notin E(G)$, then $\iota(G + uv) = 2$ implies that there exists a vertex x such that either $\{u, x\}$ dominates $G - v$, or else $\{v, x\}$ dominates $G - u$, and $x \notin N(u) \cup N(v)$. In the former case we

write $[u, x] \rightarrow v$, and in the latter case we write $[v, x] \rightarrow u$

Proposition 4.10 *If G is 3-edge- ι -critical, then G has at most 3 components. Furthermore, (1) if G has exactly 3 components, then $G = K_n \cup 2K_1$ for some $n \geq 1$, (2) if G has exactly 2 components, then either $G = K_1 \cup H$, where H is 2-edge- ι -critical, or $G = K_n \cup T_{m,2m}$, where $n \geq 1$ and $m \geq 2$.*

Proof If G is 3-edge- ι -critical, then it is obvious that G can have at most three components. Suppose G has exactly three components C_1 , C_2 , and C_3 . Then, since G is 3-edge- ι -critical, for $j = 1, 2, 3$, $\iota(C_j) = 1$, and C_j is 1-edge- ι -critical, and so $C_j = K_{n_j}$, where $n_j \geq 1$. Now, let $u \in V(C_1)$ and $v \in V(C_2)$. Since $uv \notin E(G)$, there exists x such that either $[u, x] \rightarrow v$ or $[v, x] \rightarrow u$, and we must have $x \in V(C_3)$. Therefore, either $n_2 = 1$ or $n_1 = 1$. Without loss of generality, suppose $n_2 = 1$. Similarly, from $ux \notin E(G)$ we get either $n_1 = 1$ or $n_3 = 1$, and the result follows.

Now, suppose G has exactly two components C_1 and C_2 . Then, since G is 3-edge- ι -critical, one of these, say C_1 , is 1-edge- ι -critical, and thus C_2 is 2-edge- ι -critical. Let $u \in V(C_1)$ and $v \in V(C_2)$. Since $uv \notin E(G)$ and $\iota(C_2) = 2$, there exists $x \in V(C_2)$ such that either $[u, x] \rightarrow v$ or $[v, x] \rightarrow u$. If $[v, x] \rightarrow u$, then $|V(C_1)| = 1$, $C_1 = K_1$, and $G = K_1 \cup H$, where H is 2-edge- ι -critical. If $[u, x] \rightarrow v$, then C_2 is 2- ι -critical (and 2-edge- ι -critical) since v is arbitrary. Thus, by Lemma 3.16, $C_2 = T_{m,2m}$ for some $m \geq 2$. Therefore, $G = K_n \cup T_{m,2m}$, where $n \geq 1$, $m \geq 2$. ■

We note that the above characterization still holds if 3-edge- i -critical is replaced by 3-edge- γ -critical, and 2-edge- i -critical is replaced by 2-edge- γ -critical.

In the following discussion, we consider 3-edge- i -critical graphs, unless stated otherwise. Most of these graphs will also be connected.

We define a relation \succ on $V(G)$ by $u \succ v$ if and only if $uv \notin E(G)$ and $\exists x$ such that $[u, x] \rightarrow v$. This relation defines a directed graph, which we denote by $D(\succ)$, on $V(G)$. It is possible that both $u \succ v$ and $v \succ u$ (for example, when $G = C_4$). If this does not happen, then $D(\succ)$ is an orientation of \overline{G} . In general, $D(\succ)$ is a subdigraph of the equivalent digraph of \overline{G} (see [3]).

Lemma 4.11 *Let G be a 3-edge- i -critical graph and S be an independent set of G with size $n \geq 4$. Then the vertices in S may be ordered as u_1, u_2, \dots, u_n in such a way that there exists a path $x_1x_2 \dots x_{n-1}$ in $G - S$ with $[u_i, x_i] \rightarrow u_{i+1}$ for $i = 1, 2, \dots, n - 1$.*

Proof. Since S is an independent set in G , it induces, in $D(\succ)$, a digraph with a spanning tournament. Since every such digraph has a directed Hamilton path, we may label the vertices in S as u_1, u_2, \dots, u_n so that this sequence is such a path. Hence for each $i = 1, 2, \dots, n - 1$, there exists x_i such that $[u_i, x_i] \rightarrow u_{i+1}$. Now, since $|S| = n \geq 4$, x_i dominates at least two vertices in S , thus $x_i \notin S$. Further, the vertices x_1, x_2, \dots, x_{n-1} are distinct: if $j \neq i, i + 1$, then $x_j u_{i+1} \in E(G)$ and $x_i u_{i+1} \notin E(G)$, so $x_j \neq x_i$. Suppose $1 \leq i \leq n - 2$. Then $x_i u_{i+2} \in E(G)$ and $x_{i+1} u_{i+2} \notin E(G)$, so $x_i \neq x_{i+1}$.

Finally, since for $i = 2, 3, \dots, n-1$, $x_{i-1}u_i \notin E(G)$ and $[u_i, x_i] \rightarrow u_{i+1}$, we have $x_{i-1}x_i \in E(G)$. Therefore, $x_1x_2 \dots x_{n-1}$ is the required path. ■

Note that the vertices $x_1, x_2, \dots, x_{n-1}, u_1, u_2, \dots, u_n$ are all distinct.

Lemma 4.12 *Let G be a 3-edge- i -critical graph with $\delta(G) \geq 1$. If S is an independent set of G with $|S| = n$, then there exists $u \in S$ with $d(u) \geq n-2$.*

Proof. If $n \leq 3$, since $\delta(G) \geq 1$, we have $d(u) \geq 1$ for every $u \in S$. So we assume $n \geq 4$ and let $S = \{u_1, \dots, u_n\}$ be ordered as in Lemma 4.11, and x_1, x_2, \dots, x_{n-1} be the associated path in $G - S$. Then $\{x_2, \dots, x_{n-1}\} \subseteq N(u_1)$, and hence $d(u_1) \geq n-2$. ■

We denote the size of a largest clique of G by $\omega(G)$.

Theorem 4.13 *If G is a connected 3-edge- i -critical graph with $n = |V(G)| \geq 9$, then $3 \leq \omega(G) \leq n-3$.*

Proof. Since the largest possible size for a clique in a connected 3-edge- i -critical graph is clearly $n-3$, we only need to prove that $\omega(G) \geq 3$. Suppose G does not contain a triangle. Then, since the Ramsey number $r(3, 4) = 9 \leq |V(G)|$, G must contain an independent set S of size at least four. Let $S = \{u_1, \dots, u_k\}$ be ordered as in Lemma 4.11, and let x_1, \dots, x_{k-1} be the associated path. Then $\{x_1, x_2, u_4\}$ is a triangle in G . ■

If $|V(G)| < 9$, then there are graphs which are edge- i -critical but contain no triangle. For example, graph $K_{3,3}$ is 3-edge- i -critical but contains no triangle.

Lemma 4.14 *If G is 3-edge- ι -critical, then no vertex of G has two neighbours of degree 1*

Proof. Suppose that $u, v \in V(G)$ are vertices of degree 1 and $N(u) = N(v) = \{z\}$. Since $\iota(G) = 3$, there exists w such that $wz \notin E(G)$. Now, from $uw \notin E(G)$, it follows that there exists x such that either $[u, x] \rightarrow w$ or $[w, x] \rightarrow u$. Because $xu \notin E(G)$, we have $x \neq z$ and therefore $xv \notin E(G)$. We also have $wv \notin E(G)$, so $v \notin N[u] \cup N[x]$ and $v \notin N[w] \cup N[x]$, a contradiction. ■

Theorem 4.15 *If G is 3-edge- ι -critical and $S \subseteq V(G)$ is a vertex cut, then $G - S$ has at most $|S| + 1$ components.*

Proof. First suppose that $S = \{v\}$ and that $G - v$ has three (or more) components C_1, C_2, C_3 . Then by Lemma 4.14, at most one of these can be trivial. Assume that $|V(C_1)| \geq 2$ and $|V(C_2)| \geq 2$. For $i = 1, 2$, let $x_i \in V(C_i)$ be adjacent to v . Since $x_1x_2 \notin E(G)$, with no loss of generality we may assume that $[x_1, x] \rightarrow x_2$ for some $x \in V(G)$. So, $x \neq v$ and hence $x \in V(C_3)$. But if $u \in V(C_2) - \{x_2\}$, u is not dominated by $\{x_1, x\}$. Therefore, $G - v$ has at most $|S| + 1 = 2$ components.

Now, assume that $|S| = n \geq 2$ and $G - S$ has components A_1, A_2, \dots, A_{n+2} , plus maybe some others. Let $u_i \in V(A_i)$ for $i = 1, 2, \dots, n + 2$. Then $I = \{u_1, \dots, u_{n+2}\}$ is independent in G with $|I| = n + 2 \geq 4$. Assume I is ordered as in Lemma 4.11, and let x_1, \dots, x_{n+1} be a path in $G - I$ with

$[u_i, x_i] \rightarrow u_{i+1}$ for $i = 1, 2, \dots, n+1$. Then, for $i = 1, 2, \dots, n+1$, since u_i does not dominate vertices in A_j for $j \neq i$ (it dominates only vertices in A_i and in S), x_i must belong to S . Since the vertices x_1, \dots, x_{n+1} are distinct, this contradicts $|S| = n$. ■

Recall the definitions of d_k and S_k from the previous section. We will show that a 3-edge- i -critical graph has at most $3k$ vertices of degree at most k , for any $k \geq 1$. (See Theorem 4.18.)

Lemma 4.16 *If G is a connected 3-edge- i -critical graph and for some $k \geq 2$, $d_k \geq 3k + 1$, then there do not exist $u, v, w \in S_k$ with $[u, v] \rightarrow w$.*

Proof. If $u, v, w \in S_k$ with $[u, v] \rightarrow w$, then $3k + 1 \leq |S_k| \leq d(u) + d(v) + 3 \leq 2k + 3$, and so $k \leq 2$. Therefore, the lemma holds if $k \geq 3$. If $k = 2$, then the existence of such u, v, w in S_k implies that G is a 7-cycle C_7 or the path P_7 , neither of which is 3-edge- i -critical. ■

Lemma 4.17 *If G is a connected 3-edge- i -critical graph with $d_k \geq 3k + 1$ for some $k \geq 2$, then $\beta(G[S_k]) \leq k + 1$.*

Proof. Suppose that $I = \{u_1, u_2, \dots, u_{k+2}\}$ is an independent subset of S_k , ordered as in Lemma 4.11. Then there exists a path $x_1x_2 \dots x_{k+1}$ in $G - I$ such that $[u_i, x_i] \rightarrow u_{i+1}$ for $i = 1, 2, \dots, k+1$. Moreover, by Lemma 4.16 $x_i \notin S_k$ for each i . Now, since $|S_k| = d_k \geq 3k + 1$ and $|I| = k + 2$, $S_k - I \neq \emptyset$. Let $v \in S_k - I$. Then for each $i = 1, 2, \dots, k+1$, v is adjacent to one of x_i or u_i which is impossible since $d(v) \leq k$. ■

Theorem 4.18 *If G is a connected 3-edge- ι -critical graph, then $d_k \leq 3k$ for $k \geq 1$.*

Proof: For $k = 1$, since G is connected, S_1 must be independent, and by Lemma 4.12, $d_1 \leq 3$.

Now, assume $k \geq 2$ and $d_k \geq 3k + 1$. Let H be the subdigraph of $D(\succ)$ induced by S_k . Then for all $w \in V(H)$, we have $d_H(w) \geq 2k$. Hence there exists a vertex $v \in V(H)$ with outdegree $d_H^+(v) \geq k$. Let $A = \{u_1, u_2, \dots, u_k\}$ be a k -subset of out-neighbours of v (v could have more out-neighbours). Thus for each $j = 1, 2, \dots, k$, there exists x_j such that $[v, x_j] \rightarrow u_j$ and, by Lemma 4.16, $x_j \notin S_k$. Now, since $|S_k| = d_k \geq 3k + 1$, $|A| = k$ and $d_G(v) \leq k$, there exists a subset B of k vertices in $(S_k - A) - \{v\}$ which are not adjacent to v in G . Then each element of B is adjacent to each x_j , for $j = 1, 2, \dots, k$. Since $B \subseteq S_k$, for all $b \in B$, $N_G(b) = \{x_1, \dots, x_k\}$. Thus $B \cup \{v, u_1\}$ is an independent set of size $k + 2$. But this is contrary to Lemma 4.17. ■

For general graphs if $\iota(G) = k$, then the diameter of G can be at most $3k - 1$ since every vertex can only dominate those vertices at distance at most two. (Note that the path P_{3k} has $\iota(P_{3k}) = k$ and diameter $3k - 1$.) The situation is even more restrictive for 3-edge- ι -critical graphs as is shown by the following theorem.

Theorem 4.19 *The diameter of a connected 3-edge- ι -critical graph is at most 3.*

Proof. Suppose the 3-edge- ι -critical graph G has diameter at least four. Let $a, b \in V(G)$ with $d(a, b) = \text{diam}(G) \geq 4$. Let $A = N(a)$, $B = N(b)$ and $C = V(G) - (N[a] \cup N[b])$. Then, for any $r \in A$ and any $s \in B$, we have $rs \notin E(G)$ (otherwise $d(a, b) \leq 3$).

First, we prove that one of A and B , say A , has the property that for every $r \in A$, there exists $r' \in A$ such that $rr' \notin E(G)$. Suppose, to the contrary, that there exist $r \in A$ and $s \in B$ such that $N[r] \supseteq A$ and $N[s] \supseteq B$. Then since $rs \notin E(G)$, there exists $x \notin N[r] \cup N[s]$ such that either $[r, x] \rightarrow s$ or $[s, x] \rightarrow r$. But this implies that either $x \in B$ (to dominate b) or $x \in A$ (to dominate a), a contradiction.

Next, we show that if A has the above property, then $G[B]$ is complete. For any $s \in B$, let r be any vertex in A , and let $r' \in A$ be a vertex such that $rr' \notin E(G)$. Since $rs \notin E(G)$, there exists x such that either $[r, x] \rightarrow s$ or $[s, x] \rightarrow r$. If $[r, x] \rightarrow s$, then x must be adjacent to both r' and b , which contradicts $d(a, b) \geq 4$. So, we can assume $[s, x] \rightarrow r$. Since $sa \notin E(G)$, $x \in N[a] - r$ and thus $N[s] \supseteq N[b]$. Since $s \in B$ is arbitrary, it follows that $G[B]$ is complete.

Finally, for every $u \in C$, since $bu \notin E(G)$, there exists $y \notin B \cup N(u)$ such that either $[b, y] \rightarrow u$ or $[u, y] \rightarrow b$. Now, $b, u \notin A$ implies that $y \in N[a]$. So, if $[b, y] \rightarrow u$, then (since no vertex in A dominates A) $y = a$ and $C = \{u\}$, and if $[u, y] \rightarrow b$, then $N[u] \supseteq B$. In either case, there exists $s \in B$ such that $su \in E(G)$ for every $u \in C$. Thus (since $G[B]$ is complete) $N[s] = C \cup N[b]$,

and hence $N[s] \cup N[a] = V(G)$, that is, $\iota(G) = 2$, a contradiction ■

4.3 Hamiltonian Properties

In this section Hamiltonian properties of edge- ι -critical graphs are examined. In particular, we prove that every 2-connected 3-edge- ι -critical graph has a Hamilton cycle. This is done by developing an independent domination closure operation similar to the one in Hanson [9]. We then develop a complete classification of 3-edge- ι -critical graphs with a cut-vertex, and this leads to the result that any connected 3-edge- ι -critical graph with more than six vertices has a Hamilton path.

Sumner ([13], [14]) conjectured that every connected 3-edge- γ -critical graph on more than 6 vertices has a Hamilton path, and this was proved by Wojcicka ([18]).

Sumner and Wojcicka (mentioned in [18]) also conjectured that every 2-connected 3-edge- γ -critical graph has a Hamilton cycle. Hanson ([9]) made some progress on this problem, but it is still open. His main tool was the following theorem.

Theorem 4 20 ([9]) *Suppose that G is 2-connected and 3-edge- γ -critical. If $N[u] \cup N[v] = V(G) - w$ for some pair of independent vertices $u, v \in V(G)$ with $d(w) \geq 3$, then $G + uv$ is Hamiltonian if and only if G is Hamiltonian.*

The following theorem is a development of Theorem 4 20.

Theorem 4.21 *Suppose that G is 2-connected and 3-edge- ι -critical. If $[u, v] \rightarrow w$ for some vertices u, v and w , then G is Hamiltonian if and only if $G + uv$ is Hamiltonian.*

Proof: First note that if G is Hamiltonian then $G + uv$ is obviously also Hamiltonian

Now, suppose for some $uv \notin E(G)$ and $[u, v] \rightarrow w$, $G + uv$ is Hamiltonian while G is not. Then $G + uv$ contains a Hamilton path $P = v_1v_2 \dots v_n$ from $u = v_1$ to $v = v_n$ where $n = |V(G)|$, and $N[v_1] \cup N[v_n] = V(G) - v_p$ with $v_p = w$. If $v_1v_i \in E(G)$, then $v_nv_{i-1} \notin E(G)$, otherwise $v_1v_2 \dots v_{i-1}v_nv_{n-1} \dots v_iv_1$ would be a Hamilton cycle of G . Let $M = \max\{i \mid v_1v_i \in E(G)\}$ and $m = \min\{j \mid v_jv_n \in E(G)\}$. There are three cases: $p > m \geq M$ (or $p < M \leq m$), $M < p < m$, and $m < p < M$.

Case 1: $p > m \geq M$ (here we must have $m = M$ or $M + 1$).

We consider two cases

Case 1.1: There exists $v_iv_j \in E(G)$ with $1 < i < M$ and $m < j < n$.

We first show $d(v_p) = 2$. In this case $v_nv_l \in E(G)$ for $l \in \{m, m + 1, \dots, n - 1\} - \{p\}$. If $v_nv_{j-1} \in E(G)$, then $v_1v_2 \dots v_iv_jv_{j+1} \dots v_nv_{j-1}v_{j-2} \dots v_{i+1}v_1$ would be a Hamilton cycle of G . Thus, $v_{j-1}v_n \notin E(G)$ and $j - 1 = p$, that is, $j = p + 1$. Now, since $v_kv_p \notin E(G)$ for $k = 1, 2, \dots, M - 1$ (otherwise, $v_1v_2 \dots v_kv_pv_{p+1} \dots v_nv_{p-1}v_{p-2} \dots v_{k+1}v_1$ would be a Hamilton cycle of G), $v_lv_p \notin E(G)$ for $l = M, M + 1, \dots, p - 2$ (otherwise, $v_1v_2 \dots v_lv_pv_{p+1}v_{p+2} \dots v_nv_{l+1}v_{l+2} \dots v_pv_lv_{l-1} \dots v_{l+1}v_1$ would be a Hamilton cycle of G), and $v_rv_p \notin$

$E(G)$ for $r = p+2, \dots, n$ (otherwise, $v_1v_2 \dots v_iv_{p+1}v_{p+2} \dots v_{r-1}v_nv_{n-1} \dots v_rv_p v_{p-1} \dots v_{i+1}v_1$ would be a Hamilton cycle of G), we have $d(v_p) = 2$

Since G is 3-edge- ι -critical, $v_iv_n \notin E(G)$ implies that there exists x such that either $[v_i, x] \rightarrow v_n$ or $[v_n, x] \rightarrow v_i$ and hence $x = v_p$. If $[v_n, v_p] \rightarrow v_i$, then $v_1 \in N[v_n] \cup N[v_p]$, which is a contradiction. If $[v_i, v_p] \rightarrow v_n$, then we have $N[v_i] \cup N[v_{p-1}] = V(G)$ (since $j = p+1$). If $\iota \neq p-2$, then $v_iv_{p-1} \notin E(G)$, and thus, $\iota(G) \leq 2$, a contradiction. If $\iota = p-2$, then $\iota = m-1, m = M$, and $p = m+1$. Since $v_{p+2} \notin N[v_i] \cup N[v_p]$, $v_{p+2} = v_n$, that is, $p+2 = n$. Therefore, $N[v_1] \cup N[v_{p+1}] = V(G)$ (since $m = M = p-1$), hence $\iota(G) \leq 2$, also a contradiction.

Case 1.2: $v_iv_j \notin E(G)$ for all $1 \leq i < M$ and $m < j \leq n$

We first show $d(v_p) = 2$. Since G is 2-connected, $m \neq M$, and so $m = M+1$. Since v_M and v_m are not cut vertices, there exist i ($1 \leq i < M$) and j ($m < j \leq n$) such that $v_iv_m \in E(G)$ and $v_Mv_j \in E(G)$. Therefore, $v_{j-1}v_n \notin E(G)$ (otherwise, $v_1v_2 \dots v_iv_mv_{m+1} \dots v_{j-1}v_nv_{n-1} \dots v_jv_Mv_{M-1} \dots v_{i+1}v_1$ would be a Hamilton cycle of G), and hence $j-1 = p$. Similar to Case 1.1, $v_kv_p \notin E(G)$ for $k = 1, \dots, M-1$ (otherwise, $v_1v_2 \dots v_kv_pv_{p+1} \dots v_nv_{p-1}v_{p-2} \dots v_{k+1}v_1$ would be a Hamilton cycle of G), $v_Mv_p \notin E(G)$ (otherwise, $v_{p-1}v_n \notin E(G)$ as mentioned above, a contradiction), $v_lv_p \notin E(G)$ for $l = m, \dots, p-2$ (otherwise, $v_1v_2 \dots v_iv_mv_{m+1} \dots v_lv_pv_{p-1} \dots v_{l+1}v_nv_{n-1} \dots v_{p+1}v_Mv_{M-1} \dots v_{i+1}v_1$ would be a Hamilton cycle of G), and $v_rv_p \notin E(G)$ for $r = p+2, \dots, n$ (otherwise, $v_1v_2 \dots v_iv_mv_{m+1} \dots v_pv_rv_{r+1} \dots v_nv_{r-1}v_{r-2} \dots v_{p+1}v_Mv_{M-1} \dots v_{i+1}v_1$

would be a Hamilton cycle of G). It follows that $d(v_p) = 2$.

Since G is 3-edge- ι -critical, $v_M v_n \notin E(G)$ implies that there exists x such that either $[v_M, x] \rightarrow v_n$ or $[v_n, x] \rightarrow v_M$ and hence $x = v_p$. If $[v_n, v_p] \rightarrow v_M$, then $v_1 \in N[v_n] \cup N[v_p]$, which is a contradiction. If $[v_M, v_p] \rightarrow v_n$, then we have $N[v_M] \cup N[v_{p-1}] = V(G)$. If $p - 1 \neq m$, then $v_M v_{p-1} \notin E(G)$, and thus $\iota(G) \leq 2$, a contradiction. Hence we may assume $p - 1 = m$, so $p = m + 1$. Since $v_{p+2} \notin N[v_M] \cup N[v_p]$, $v_{p+2} = v_n$ and $p + 2 = n$. Now, $v_i v_n \notin E(G)$ implies that $[v_i, v_p] \rightarrow v_n$ (since $v_1 \notin N[v_n] \cup N[v_p]$). Thus we have $N[v_i] \cup N[v_{p+1}] = V(G)$ (since $p - 1 = m$ and $v_i v_m \in E(G)$). Since $v_i v_{p+1} \notin E(G)$, we have $\iota(G) \leq 2$, also a contradiction.

Note that the case where $p < M \leq m$ is symmetrical to the case where $p > m \geq M$, so the proof of Case 1 is complete.

Case 2: $M < p < m$ (here we must have $m = M + 2$).

In this case we have $v_i v_j \notin E(G)$ for all $1 \leq i < M$ and $m < j \leq n$, otherwise, $v_1 v_2 \dots v_i v_j v_{j+1} \dots v_n v_{j-1} v_{j-2} \dots v_{i+1} v_1$ is a Hamilton cycle of G .

Again we show $d(v_p) = 2$. If $v_k v_p \in E(G)$ with $1 < k < M$, then since v_p, v_m are not cut vertices, either there exists j with $m < j < n$ such that $v_M v_j \in E(G)$ or there exists q with $m < q < n$ such that $v_p v_q \in E(G)$ and $v_M v_m \in E(G)$. In the former case, $v_1 v_2 \dots v_k v_p v_m \dots v_{j-1} v_n v_{n-1} \dots v_j v_M v_{M-1} \dots v_{k+1} v_1$ would be a Hamilton cycle of G , and in the latter case we have Hamilton cycle $v_1 v_2 \dots v_k v_p v_q v_{q+1} \dots v_n v_{q-1} v_{q-2} \dots v_m v_M v_{M-1} \dots v_{k+1} v_1$ in G . Therefore, $v_k v_p \notin E(G)$ for $k = 1, \dots, M - 1$. By symmetry, $v_p v_l \notin E(G)$ for

$l = m + 1, \dots, n$. Thus, $d(v_p) = 2$.

But now, since $v_1 v_n \notin E(G)$, there exists x such that either $[v_1, x] \rightarrow v_n$ or $[v_n, x] \rightarrow v_1$. Since in each case $x \notin N[v_1] \cup N[v_n]$, we have $x = v_p$. If $[v_1, v_p] \rightarrow v_n$, then since $v_q \notin N[v_1] \cup N[v_p]$ for all $m < q < n$, we have $m = n - 1$, and thus $d(v_n) = 1$. Similarly, if $[v_n, v_p] \rightarrow v_1$, then we have $d(v_1) = 1$. Each of these is a contradiction.

Case 3 $m < p < M$

In this case, we have v_1 adjacent to all of $\{v_2, v_3, \dots, v_{m-1}, v_{p+1}, v_{p+2}, \dots, v_M\}$ while v_n is adjacent to all of $\{v_m, v_{m+1}, \dots, v_{p-1}, v_{M+1}, v_{M+2}, \dots, v_{n-1}\}$.

Again we show $d(v_p) = 2$.

First, as before, $v_k v_p \notin E(G)$ for $k = 1, \dots, m - 2, M + 2, \dots, n$ (otherwise, either $v_1 v_2 \dots v_k v_p v_{p+1} \dots v_n v_{p-1} v_{p-2} \dots v_{k+1} v_1$ for $1 \leq k \leq m - 2$ or $v_1 v_2 \dots v_p v_k v_{k+1} \dots v_n v_{k-1} v_{k-2} \dots v_{p+1} v_1$ for $M + 2 \leq k \leq n$ would be a Hamilton cycle of G).

Also, $v_l v_p \notin E(G)$ for $l = m, \dots, p - 2, p + 2, \dots, M$ (otherwise, $v_1 v_2 \dots v_l v_p v_{p-1} \dots v_{l+1} v_n v_{n-1} \dots v_{p+1} v_1$ or $v_1 v_2 \dots v_{p-1} v_n v_{n-1} \dots v_l v_p v_{p+1} \dots v_{l-1} v_1$ would be a Hamilton cycle of G respectively), $v_p v_{m-1} \notin E(G)$ and $v_p v_{M+1} \notin E(G)$ (otherwise, $v_1 v_2 \dots v_{m-1} v_p v_{p-1} \dots v_m v_n v_{n-1} \dots v_{p+1} v_1$ or $v_1 v_2 \dots v_{p-1} v_n v_{n-1} \dots v_{M+1} v_p v_{p+1} \dots v_M v_1$ would be a Hamilton cycle of G , respectively). So, $d(v_p) = 2$.

Since G is 3-edge- v -critical, $v_1 v_n \notin E(G)$ and $d(v_p) = 2$ imply that either $[v_1, v_p] \rightarrow v_n$ or $[v_n, v_p] \rightarrow v_1$. Suppose $[v_1, v_p] \rightarrow v_n$. If $p - 1 \neq 2$, then

$v_1v_{p-1} \notin E(G)$ and $N[v_1] \cup N[v_{p-1}] = V(G)$, a contradiction. If $p-1 = m = 2$, then $p = m+1 = 3$. We also have $M = n-1$. Now, for each $p+1 < k < M$, $v_kv_n \notin E(G)$ implies that $[v_k, v_p] \rightarrow v_n$ (since $v_1 \notin N[v_n] \cup N[v_p]$). Thus $v_kv_M \in E(G)$ for all $p+1 < k < M$, but then we have $N[v_M] \cup N[v_p] = V(G)$, which is a contradiction since $M \neq p+1$ (otherwise if $M = p+1$, then $\iota(G) \leq 2$). Similarly, $[v_n, v_p] \rightarrow v_1$ also leads to a contradiction.

Since all of the cases lead to a contradiction, we have proved that G is Hamiltonian if $G + uv$ is Hamiltonian, and we are done. ■

If G is 3-edge- ι -critical, then we define the *domination closure* of G , denoted by $D^*(G)$, to be G together with all edges uv of \overline{G} where u, v are such that $(\text{in } G) [u, v] \rightarrow w$ for some vertex w .

Corollary 4.22 *If G is 2-connected and 3-edge- ι -critical, then $D^*(G)$ is Hamiltonian if and only if G is Hamiltonian.*

Proof. Clearly if G is Hamiltonian, then so is $D^*(G)$.

On the other hand, suppose $D^*(G)$ is Hamiltonian while G has no Hamiltonian cycle. Choose a minimal subset $\{e_1, e_2, \dots, e_k\} \subseteq E(D^*(G)) - E(G)$ such that $G + \{e_1, e_2, \dots, e_k\}$ is Hamiltonian and $G' = G + \{e_1, e_2, \dots, e_{k-1}\}$ is not Hamiltonian. Note that by Theorem 4.21, $k \geq 2$. Let $e_k = xy$. Then G' has a Hamiltonian path $P = v_1v_2 \dots v_n$, where $x = v_1$ and $y = v_n$. By minimality, each of e_1, e_2, \dots, e_{k-1} is on P .

The proof is now identical to that of Theorem 4.21. There we proved that

$d_G(v_p) = 2$ without using the criticality of G . The same argument show that $d_{G'}(v_p) = 2$. In fact, since G is 2-connected, both edges incident with v_p are edges of G . In the remainder of the proof a contradiction is obtained using the hypothesis that G is 3-edge- ι -critical. The contradiction is either that G is Hamiltonian or that $\iota(G) \leq 2$. The same arguments can be applied to G' . In the former case this is clear as e_1, e_2, \dots, e_k are all on P . In the latter case the only edges of P which are used in the arguments are edges of G . ■

We write $d^*(x)$ for the degree of vertex x in $D^*(G)$

Theorem 4 23 *If G is 2-connected and 3-edge- ι -critical, then G is Hamiltonian*

Proof Let w be any vertex of G . Define $A_w = \{x \in \overline{N}(w) : \exists y \in \overline{N}(w) \text{ s.t. } [x, y] \rightarrow w\}$, and $B_w = \overline{N}(w) - A_w$.

If $x \in B_w$, then there is $y \in \overline{N}(w)$ so that $[w, y] \rightarrow x$. Furthermore, $y \in B_w$, otherwise, there exists $z \in \overline{N}(w), z \neq y$ such that $[y, z] \rightarrow w$, but then $\{w, y\}$ does not dominate z , a contradiction. Therefore for each $x \in B_w$, there is $y \in B_w$ so that $[w, y] \rightarrow x$. Now suppose $x_1 \neq x_2 \in B_w$. Then there exist y_1 and y_2 in B_w so that $[w, y_1] \rightarrow x_1$ and $[w, y_2] \rightarrow x_2$. If $y_1 = y_2$, then we must have $x_1 = x_2$, a contradiction. Thus the mapping f on B_w defined by $f(x) = y$ if $[w, y] \rightarrow x$ is one-to-one and onto.

From the above argument and the definition of A_w , it follows that each vertex of $\overline{N}(w)$ is incident with an edge of $D^*(G)$ which is not an edge of G . We call such an edge a *new edge* with respect to w .

Suppose $[a, c] \rightarrow b$, so that ac is a new edge with respect to b . Then it is also a new edge with respect to each of a (since $c \in \overline{N}(a)$) and c (since $a \in \overline{N}(c)$). It is never a new edge with respect to any other vertex, since $\{a, c\} \in \overline{N}(w)$ if and only if $w \in \{a, b, c\}$.

Each vertex x is in $\overline{N}(w)$ for $|\overline{N}(x)| = |V(G)| - |N[x]| = |V(G)| - d(x) - 1$ vertices w . Each of these choices for w leads to a new edge incident with x , and each such new edge is counted exactly twice. (If xy is a new edge with respect to z , then the set $\{z, x, y\}$ is independent in G . Since $[x, y] \rightarrow w$, this is possible only if $w = y$ or $w = z$.) Thus,

$$d^*(x) \geq d(x) + |\overline{N}(x)|/2 = d(x) + |V(G)|/2 - d(x)/2 - 1/2 \geq |V(G)|/2$$

Hence, by Dirac's theorem, $D^*(G)$ is Hamiltonian. Therefore, by Corollary 4.22, G is also Hamiltonian. ■

Note: If G is 2-connected and 3-edge- ι -critical with $|V(G)|$ even, then G has a perfect matching (or 1-factor).

We now prove that every connected, 3-edge- ι -critical graph with more than six vertices has a Hamilton path. We first give a complete description of the 3-edge- ι -critical graphs with a cut vertex.

Lemma 4.24 *Let G be a connected, 3-edge- ι -critical graph. If v is a cut vertex of G , then $G - v$ has exactly two components C_1 and C_2 with $\iota(C_1) = 1$ and $\iota(C_2) = 2$.*

Proof: By Theorem 4.15, $G - v$ has exactly two components C_1 and C_2 .

First, if $\iota(C_1) \geq 3$ or $\iota(C_2) \geq 3$, let $x_i \in V(C_i)$ be adjacent to v . Then $\iota(G + x_1x_2) \geq 3$ (since v is adjacent to x_1 and x_2 , any minimum independent dominating set containing x_1 or x_2 does not contain v), a contradiction.

Now, suppose $\iota(C_1) = \iota(C_2) = 2$. Let $x_i, y_i \in V(C_i)$ such that $x_iy_i \notin E(G)$ for $i = 1, 2$. Since $x_1x_2 \notin E(G)$, there exists x such that $[x_1, x] \rightarrow x_2$ or $[x_2, x] \rightarrow x_1$. But $x_iy_i \notin E(G)$ implies that $x = v$. Now, since $y_1y_2 \notin E(G)$, there exists y such that either $[y_1, y] \rightarrow y_2$ or $[y_2, y] \rightarrow y_1$. Since $x_iy_i \notin E(G)$, $y = v$. But $vy_i \in E(G)$ (since $[x_1, v] \rightarrow x_2$), a contradiction.

Finally, if $\iota(C_1) = \iota(C_2) = 1$, let $x_i \in V(C_i)$ be a vertex adjacent to v for $i = 1, 2$. Since $x_1x_2 \notin E(G)$, there exists x such that either $[x_1, x] \rightarrow x_2$ or $[x_2, x] \rightarrow x_1$, and $x \neq v$. Without loss of generality, assume $x \in V(C_1)$. Then x_2 dominates every vertex of $C_2 \cup \{v\}$ and since $\iota(C_1) = 1$, we have $\iota(G) = 2$, a contradiction.

By the above, we have, say, $\iota(C_1) = 1$ and $\iota(C_2) = 2$. ■

Let $Q_{2,p}$ be the graph $Q_{n,p}$ defined in Section 2.1 with $n = 2$. We have the following complete description of the 3-edge- ι -critical graphs with a cut vertex.

Theorem 4.25 *Let G be a connected 3-edge- ι -critical graph. If v is a cut vertex of G , then $G - v$ has exactly two components C_1 and C_2 such that C_1 is complete, $\iota(C_2) = 2$ and*

1. if $|V(C_1)| \geq 2$, then $C_2 = T_{n,2n}$, for $n \geq 2$, and further, for every

$x \in V(C_1)$, $vx \in E(G)$, and for any pair u, u' of non-adjacent vertices of C_2 , $vu \in E(G)$ if and only if $vu' \in E(G)$,

or

2. if $|V(C_1)| = 1$, then $C_2 = S_1 + S_2 + \dots + S_m$, where $S_j = \overline{K}_2$ or $Q_{2,p}$, $1 \leq j \leq m$, and further, there exist non-adjacent vertices $u, u' \in \overline{N}[v]$ such that (i) $N[u] \cup N[u'] = V(C_2)$, and (ii) $\forall z \in (N(u) \cup N(u')) - (N(u) \cap N(u'))$, $uz \in E(G)$ and $N[z] \supseteq V(C_2) - \{u, u'\}$.

Proof. Since v is a cut vertex of G , by Lemma 4.24, $G - v$ has two components C_1 and C_2 with $\iota(C_1) = 1$ and $\iota(C_2) = 2$. There are two cases: $|V(C_1)| \geq 2$ and $|V(C_1)| = 1$.

Case 1: $|V(C_1)| \geq 2$

First, we prove that C_1 is complete. Suppose $x, y \in V(C_1)$ and $xy \notin E(G)$. Then there exists z such that (in G) either $[x, z] \rightarrow y$ or $[y, z] \rightarrow x$. Since $\iota(C_2) = 2$, $z = v$ and v dominates C_2 . Now, for any $u \in V(C_2)$, since $xu \notin E(G)$, there exists w such that either $[x, w] \rightarrow u$ or $[u, w] \rightarrow x$. Since $\iota(C_2) = 2$, $w \in V(C_2)$. But then y can not be dominated by $\{x, w, u\}$, a contradiction.

Next, we prove that $C_2 = \overline{K}_2 + \dots + \overline{K}_2 = T_{n,2n}$. Let $x \in V(C_1)$ be adjacent to v . For every $u \in V(C_2)$, since $xu \notin E(G)$, there exists u' such that either $[u, u'] \rightarrow x$ or $[x, u'] \rightarrow u$. Since C_1 is complete and $xv \in E(G)$, $u' \in V(C_2)$ and $[x, u'] \rightarrow u$. So, $N[u'] \supseteq V(C_2) - \{u\}$. Now, since $xu' \notin E(G)$,

there exists w such that $[x, w] \rightarrow u'$ or $[u', w] \rightarrow x$. As above, $[x, w] \rightarrow u'$ and $w \in C_2$, so $w = u$ and $N[u] \supseteq V(C_2) - \{u'\}$. Therefore, for every $u \in V(C_2)$, there exists $u' \in V(C_2)$ such that $N(u) = N(u') \supseteq V(C_2) - \{u, u'\}$, that is, $C_2 = \overline{K}_2 + \dots + \overline{K}_2 = T_{n, 2n}$.

Finally, we prove that for every $x \in V(C_1)$, $vx \in E(G)$, and that for any pair u, u' of non-adjacent vertices of C_2 , if $vu \notin E(G)$, then $vu' \notin E(G)$, and if $vu \in E(G)$, then $vu' \in E(G)$. Since there exists $u \in V(C_2)$ with $vu \notin E(G)$ (otherwise, $\iota(G) \leq 2$), there exists y such that either $[u, y] \rightarrow v$ or $[v, y] \rightarrow u$. If $[u, y] \rightarrow v$, then $y \in V(C_1)$, hence u dominates C_2 , a contradiction. Thus $[v, y] \rightarrow u$ and $y \in V(C_2)$. Since u' is the only vertex of C_2 not adjacent to u , $y = u'$ and $u'v \notin E(G)$. Thus v is adjacent to either both u and u' , or neither u nor u' . Further, as v dominates $V(C_1)$, $vx \in E(G)$ for every $x \in V(C_1)$.

Case 2 $|V(C_1)| = 1$

Let $C_1 = \{x\}$. Then, since G is connected, $vx \in E(G)$. We prove that $C_2 = S_1 + S_2 + \dots + S_m$, where $S_j = \overline{K}_2$ or $Q_{2,p}$, for $j = 1, \dots, m$.

Since $\iota(G) = 3$, there exists $u \in V(C_2)$ with $vu \notin E(G)$, and u' such that $[v, u'] \rightarrow u$ or $[u, u'] \rightarrow v$. But $x \notin N[u] \cup N[u']$ implies that $[v, u'] \rightarrow u$, $vu' \notin E(G)$ and $u' \in V(C_2)$. We prove that (i) $N[u] \cup N[u'] = V(C_2)$, and (ii) $\forall z \in (N(u) \cup N(u')) - (N(u) \cap N(u')) = A_1$, $uz \in E(G)$ and $N[z] \supseteq V(C_2) - \{u, u'\}$.

For (i), suppose there exists $w \in V(C_2) - N[u] \cup N[u']$. Then, from $wu \notin E(G)$, there exists y such that $[w, y] \rightarrow u$ or $[u, y] \rightarrow w$. Since x

must be dominated, $y = v$ or x . But $u' \notin N[w] \cup N[v] \cup N[x] \cup N[u]$, a contradiction.

For (ii), we first define S_1 . If $A_1 = \emptyset$, then $S_1 = G[\{u, u'\}] = \overline{K}_2$. Otherwise $A_1 \neq \emptyset$. Let $z \in A_1$, without loss of generality $z \in N(u)$ and $z \notin N(u')$. Suppose $uz \notin E(G)$. Then there exists y such that $[v, y] \rightarrow z$ or $[z, y] \rightarrow v$. Since such a vertex y must belong to $V(C_2)$, we have $[v, y] \rightarrow z$. On the other hand, from $vy \notin E(G)$, there exists w such that $[v, w] \rightarrow y$ or $[y, w] \rightarrow v$. Since w must be in $V(C_2)$ and $w \notin N[v] \cup N[y]$, $w = z$. So, we have $[v, z] \rightarrow y$. But $u' \notin N[v] \cup N[z]$, a contradiction. Therefore, $uz \in E(G)$. Now, from $zu' \notin E(G)$, we have t such that $[z, t] \rightarrow u'$ or $[u', t] \rightarrow z$. Since $uz \in E(G)$ and $u'u \notin E(G)$, $t = x$, and we have $[z, x] \rightarrow u'$ and $N[z] = (V(C_2) - \{u'\}) \cup \{v\}$. Similarly, if $z \in N(u')$ and $z \notin N(u)$, then $N[z] = (V(C_2) - \{u\}) \cup \{v\}$. So, $S_1 = G[A_1 \cup \{u, u'\}] = Q_{2,p}$.

Let $B_1 = N(u) \cap N(u')$. If $B_1 = \emptyset$, then $C_2 = S_1$. Otherwise $B_1 \neq \emptyset$. If for all $u_1 \in B_1$, $u_1v \in E(G)$, then similar to case 1, we can prove that $B_1 = \overline{K}_2 + \dots + \overline{K}_2$. Otherwise, there exists $u_1 \in B_1$ such that $u_1v \notin E(G)$. Then, we can similarly prove that there exists $u'_1 \in B_1$, such that $u_1u'_1 \notin E(G)$ and that: (i) $N[u_1] \cup N[u'_1] = V(C_2)$, and (ii) let $A_2 = (N(u_1) \cup N(u'_1)) - (N(u_1) \cap N(u'_1))$, if $A_2 = \emptyset$, then $S_2 = G[\{u_1, u'_1\}] = \{u_1, u'_1\} = \overline{K}_2$, otherwise for $\forall z \in A_2$, $uz \in E(G)$ and $N[z] \supseteq V(C_2) - \{u_1, u'_1\}$, and hence $S_2 = Q_{2,p}$.

In general, if $B_k = N(u_{k-1}) \cap N(u'_{k-1}) = \emptyset$, then $C_2 = S_1 + \dots + S_k$. Otherwise $B_k \neq \emptyset$. If for all $u_k \in B_k$, $u_kv \in E(G)$, then $B_k = \overline{K}_2 + \dots + \overline{K}_2$.

Otherwise, there exists $u_k \in B_k$ such that $u_k v \notin E(G)$, and thus there exists $u'_k \in B_k$ such that $u_k u'_k \notin E(G)$, and such that the following hold: (i) $N[u_k] \cup N[u'_k] = V(C_2)$. (ii) Let $A_{k+1} = (N(u_k) \cup N(u'_k)) - (N(u_k) \cap N(u'_k))$. If $A_{k+1} = \emptyset$, then $S_{k+1} = G[\{u_k, u'_k\}] = \{u_k, u'_k\} = \overline{K}_2$, otherwise for any $z \in A_{k+1}$, $uz \in E(G)$ and $N[z] \supseteq V(C_2) - \{u_k, u'_k\}$, and hence $S_{k+1} = G[A_{k+1} \cup \{u_k, u'_k\}] = Q_{2,p}$.

Since G is finite, we have $B_m = \emptyset$ for some m , and thus $C_2 = S_1 + \dots + S_m$, where $S_j = \overline{K}_2$ or $Q_{2,p}$ for all $1 \leq j \leq m$. The result now follows. ■

With this result we can prove that every connected, 3-edge- ι -critical graph with more than six vertices has a Hamilton path. First note that $Q_{2,p}$ has a Hamilton path, and by the definition of join, we have the following lemmas whose easy proofs are omitted.

Lemma 4.26 *If G has a Hamilton path, then so does $G + \overline{K}_2$ and $G + Q_{2,p}$.*

Lemma 4.27 *If $G = S_1 + S_2 + \dots + S_k$, where $S_j = \overline{K}_2$ or $Q_{2,p}$, for $j = 1, \dots, k$, then G has a Hamilton path, unless $G = \overline{K}_2$.*

Theorem 4.28 *If G is connected and 3-edge- ι -critical with $|V(G)| > 6$, then G has a Hamilton path.*

Proof. If G is 2-connected, then by Theorem 4.23, G is Hamiltonian. Thus assume G has a cut vertex v . By Theorem 4.25, $G - v$ has exactly two components C_1 and C_2 , such that C_1 is complete, $\iota(C_2) = 2$ and either (1) or (2) in the statement of Theorem 4.25 holds.

If (1) holds, then since $G[C_1 \cup \{v\}]$ is complete, it has a Hamilton path P that ends at v . Since $T_{n,2n}$ has a Hamilton path that starts at any vertex, $G[C_2 \cup \{v\}]$ has a Hamilton path Q that starts at v . Then the path PQ is a Hamilton path of G .

If (2) holds, then since G is connected and $\iota(G) = 3$, $C_2 \neq \overline{K}_2$. Thus, by Lemma 4.27, C_2 has a Hamilton path. Furthermore, $G[v \cup V(C_2)]$ has a Hamilton path Q that starts at v , except when $C_2 = Q_{2,4}$ (i.e. C_2 is a path with 4 vertices). Since $|V(G)| > 6$, this does not happen. Let $P = sv$. Then the path PQ is a Hamilton path of G . Therefore, if $|V(G)| > 6$ (thus, $C_2 \neq P_4$), then G has a Hamilton path. ■

For $k \geq 4$, the question of when a k -edge- ι -critical graph G contains a Hamilton cycle is still open.

On the other hand, for any $k \geq 4$, there are arbitrary large k -edge- ι -critical graphs with no Hamilton path. Let $m \geq 2k$, and $p = 1 + 1 + \dots + 1 + (m - 2k + 1)$ be a partition of $p = m - k$. Then graph $Q_{k,p}$ (defined in Section 2.1) is k -edge- ι -critical but has $k - 1 \geq 3$ vertices of degree 1, and hence no Hamilton path.

Chapter 5

Future Work

We conclude the thesis with some suggestions for future research on independent domination critical graphs.

- 1 Explore the relationships between the various critical families. In particular, characterize the graphs which are both ν -critical and edge- ν -critical, or edge- ν -critical and edge- γ -critical, etc.
- 2 For products other than those discussed in this thesis, determine when products of critical graphs are also critical.
- 3 Find a good bound on the diameter of a k -edge- ν -critical graph. We conjecture that the diameter of such a graph is at most $\lfloor \frac{3}{2}k - 1 \rfloor$ (we believe the same bound should hold for k -edge- γ -critical graphs).
- 4 Characterize the 3- ν -critical graphs, or more generally, the k - ν -critical graphs ($k \geq 3$).

- 5 Complete the characterization of 3-edge- λ -critical graphs by characterizing the connected 3-edge- λ -critical graphs. Better yet, characterize the k -edge- λ -critical graphs.
- 6 Determine whether every 2-connected k -edge- λ -critical graph with $k \geq 4$ is Hamiltonian.

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VITA

Surname AO Given Names SUQIN
Place of Birth P. R. CHINA Date of Birth 28/10/61

Educational Institutions Attended

University of Victoria	1992 to 1994
Hengyang Teachers College	1978 to 1982

Degree Awarded

B S	Hengyang Teachers College	1982
-----	---------------------------	------

Honours and Awards

Graduate Teaching Fellowship (U. Vic)	1992-1994
Graduate Studies Scholarship (U. Vic)	1993
People's Prize	1978-1982

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Author



(Signature)

SUQIN AO

(Name in Block Letter)

August 15, 1994

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