

Kernels and Quasi-kernels in Digraphs

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Abstract

Given a digraph $D = (V, A)$, a *quasi-kernel* of D is an independent set $Q \subseteq V$ such that for every vertex v not contained in Q , either there exists a vertex $u \in Q$ such that v dominates u , or there exists a vertex w such that v dominates w and w dominates some vertex $u \in Q$. A *sink* in a digraph $D = (V, A)$ is a vertex $v \in V$ that dominates no vertex of D . In this thesis we prove that if D is a semicomplete multipartite, quasi-transitive or locally semicomplete digraph that contains no sink, then D has two disjoint quasi-kernels.

For a digraph $D = (V, A)$ and a subset $X \subseteq V$, *pushing* the set X means that we reverse the orientation of each arc with exactly one endpoint in X . In this thesis, we show that it is NP-complete to decide whether an arbitrary digraph $D = (V, A)$ admits a subset $X \subseteq V$ such that after pushing X the resultant digraph contains no directed odd cycle. In addition we show that it is NP-hard to decide whether an arbitrary digraph $D = (V, A)$ admits a subset $X \subseteq V$ such that after pushing X the resultant digraph is kernel-perfect. Finally, we characterize, in terms of forbidden subdigraphs, multipartite tournaments $M = (V, A)$ that contain a subset $X \subseteq V$ for which pushing X results in a multipartite tournament that contains no odd cycle.

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Chapter 1

Introduction

A *kernel* of a digraph is an independent set K such that each vertex $v \notin K$ dominates some vertex of K . Kernels were first encountered in the work of von Neumann and Morgenstern [31] who showed that, in digraphs associated with certain combinatorial games, the existence of a kernel implies a winning strategy. Kernels of digraphs have also proven useful outside of game theory; notably, a property derived from the notion of a kernel played a key role in the search for a proof of the Strong Perfect Graph Conjecture, now the Strong Perfect Graph Theorem [8]. In particular, a graph G is perfect if and only if every orientation of G for which each complete subgraph is acyclic has a kernel. The interested reader will refer to [6, 7] for a detailed description.

It is important to note that not every digraph has a kernel, for example a directed odd cycle has no kernel. It turns out that it is NP-complete to decide if an arbitrary digraph has a kernel [9].

A digraph D is *kernel-perfect* if every induced subdigraph of D contains a kernel. There are a number of conditions that are sufficient for a digraph to be kernel-perfect; Richardson [29] proved that if a digraph contains no directed odd-cycle, then it is kernel-perfect. Kernel-perfection has been explored as a means to prove the *Line Graph Conjecture* [5, 13].

A *quasi-kernel* of a digraph is an independent set Q such that for each vertex $v \notin Q$, either v dominates a vertex of Q , or there exists a vertex u such that v dominates u and u dominates a vertex of Q . It follows from this definition that a kernel is a quasi-kernel. Although not every digraph has a kernel, Chvátal and Lovász proved that every digraph contains a quasi-kernel [10].

The study of quasi-kernels in digraphs began with the study of kings in digraphs, in particular, kings in tournaments. A *king* in a digraph D is a vertex v such that for every vertex u , different than v , either v dominates u or there exists a vertex w such that v dominates w and w dominates u . Observe that if v is a king in a digraph D , then v is a quasi-kernel in the

converse of D . It is, in general, not true that every digraph contains a king. Landau [25], however, proved that every tournament contains a king. If v is a vertex in a digraph D , and v dominates no vertex of D , then we call v a *sink* of D . Clearly, if the converse of a tournament T contains a sink v , then v is the unique king of T . Moon [27] showed that if the converse of a tournament T contains no sink, then T contains at least 3 kings.

A number of results on kings in tournaments have been generalized to statements about quasi-kernels in arbitrary digraphs. For example, Jacob and Meyniel [20], generalizing the earlier result of Moon, proved that every digraph containing no kernel has at least 3 quasi-kernels. Observe that if a digraph D contains a sink v , then v is necessarily contained in any quasi-kernel of D . Consequently, a digraph that contains a sink has no pair of disjoint quasi-kernels. Gutin *et al.* in the technical report [14], preceding [15], conjectured that if a digraph D contains no sink, then D has two disjoint quasi-kernels. In this thesis we prove the truth of the conjecture for a number of special classes of digraphs. Explicitly, we prove that if D is a semicomplete multipartite, quasi-transitive, or locally semicomplete digraph that contains no sink, then D has two disjoint quasi-kernels.

Given a digraph $D = (V, A)$ and a subset $X \subseteq V$, *pushing* the set X

means that we reverse the orientation of every arc with exactly one endpoint in X . The above operation, the *push operation* was first studied in the context of digraphs by Fisher and Ryan [12]. Typically, study of the push operation has focused on the decision problem of recognizing whether an arbitrary digraph D may be pushed to a digraph D' with a given property. A number of NP-completeness results involving the push operation have been obtained. For example, it is NP-complete to decide whether an arbitrary digraph can be pushed to a digraph that is Hamiltonian [21] or acyclic [19]. In contrast, deciding if a multipartite tournament can be made acyclic using the push operation is solvable in polynomial time. In this thesis we prove that the problem of deciding whether an arbitrary digraph $D = (V, A)$ admits a subset $X \subseteq V$ for which the digraph obtained by pushing X in D contains no odd cycle is NP-complete. In addition we show that the problem of deciding whether an arbitrary digraph $D = (V, A)$ admits a subset $X \subseteq V$ for which the digraph obtained by pushing X in D is kernel-perfect is NP-hard.

We characterize, in terms of forbidden subdigraphs, multipartite tournaments that can be pushed to contain no directed odd cycle, as well as those that can be made kernel-perfect using the push operation.

Chapter 2

Preliminaries

This chapter surveys the requisite graph theoretic background of the thesis. The chapter is comprised of four sections. In the first, we summarize the definitions, notation and terminology required in subsequent work. In the second, we describe the classes of digraphs that will be the object of our later study. The third section introduces the concepts of *kernels* and *quasi-kernels* of digraphs. In the fourth, and final, section we introduce the *push operation*.

Our notation is consistent with Bang-Jensen and Gutin [2].

2.1 Basic definitions and notation

A *digraph* D consists of a non-empty finite set $V(D)$ (or simply V) of elements called *vertices*, and a set $A(D)$ (or simply A) of ordered pairs of distinct vertices called *arcs*. The sets V and A are called, respectively, the *vertex set* and *arc set* of D . For an arc $uv \in A$ we call u its *tail* and v its *head*, and refer to both as *endpoints* of the arc uv . If $uv \in A$, then we say that u *dominates* v , and denote this relationship by $u \rightarrow v$. If $uv \notin A$, then we write $u \nrightarrow v$. For a pair of vertices $x, y \in V$, if at least one of xy or $yx \in A$, then we say that x and y are *adjacent*. In the body of this thesis we allow both uv and vu to be arcs in a digraph D .

Let $D = (V, A)$ and let $X, Y \subseteq V$. The notation (X, Y) will denote the set of arcs with tail in X and head in Y , e.g. $(X, Y) = \{xy \in A : x \in X, y \in Y\}$. $X \rightarrow Y$ means that every vertex in X dominates some vertex in Y , while $X \mapsto Y$ means that every vertex in X dominates every vertex in Y .

If D is a digraph, then the digraph H is a *subdigraph* of D if $V(H) \subseteq V(D)$ and $A(H) \subseteq A(D)$. If in addition every arc of $A(D)$ with both endpoints in $V(H)$ is contained in $A(H)$, then we call H an *induced subdigraph* of D , and say that H is *induced by* $V(H)$. For a digraph $D = (V, A)$ and a subset $X \subseteq V$ we denote by $D - X$ the subdigraph of D induced by the vertex set

$V - X$. If the subdigraph of D induced by X contains no arc, then we say that X is an *independent set*.

Given a digraph $D = (V, A)$ and a vertex $v \in V$, we define $N^+(v)$ to be the set of vertices dominated by v , and $N^-(v)$ to be the set of vertices that dominate v . We call the sets $N_D^+(v)$ and $N_D^-(v)$, the *out-neighbourhood* and the *in-neighbourhood* of v respectively. Given a vertex $v \in V$ we denote by $d_D^+(v)$ the quantity $|N_D^+(v)|$ and call $d_D^+(v)$ the *out-degree* of v . The *in-degree*, $d_D^-(v)$, of a vertex $v \in V$ is defined in an analogous manner.

We define $N_D^+(X)$ and $N_D^-(X)$ of a set $X \subseteq V$ to be the sets of vertices $v \notin X$ that are dominated by, respectively dominate, some vertex in X . The *closed out-neighbourhood*, $N_D^+[X]$, of a set X is defined to be $N_D^+(X) \cup X$. The *closed in-neighbourhood* is defined in a similar manner. The *second out* and *in-neighbourhoods* of a set $X \subseteq V$ are, respectively, $N^+(N^+(X)) - N^+[X]$ and $N^-(N^-(X)) - N^-[X]$.

When the digraph D in question is clear from the context we will abbreviate the above notation by omitting the subscript D , e.g. we write $N^+(v)$ rather than $N_D^+(v)$.

A *path* P , of length k , in a digraph D is a sequence $P = v_0 v_1 \dots v_k$ of distinct vertices of D such that $v_{i-1} \rightarrow v_i$ for each $i \in \{1, \dots, k-1\}$. We call

the path P a (v_1, v_k) -*path*. Note that the length of a path P is $|V(P)| - 1$. A *cycle* of a digraph D is a sequence $C = v_0 v_1 \dots v_{k-1} v_0$ of vertices such that $v_0, v_1, \dots, v_{k-1}$ are distinct and $v_i \rightarrow v_{i+1}$ (with indices taken modulo k). We define the *length* of a cycle to be the number of vertices that it contains. In particular a cycle is *even* if it contains an even number of vertices and *odd* otherwise.

Let u and v be vertices in a digraph D . The *distance* from u to v in D , denoted by $\text{dist}(u, v)$, is defined to be the minimum length of a (u, v) -path, if such a path exists, and $\text{dist}(u, v) = \infty$ if no (x, y) -path exists. By convention, we define $\text{dist}(v, v) = 0$. A digraph $D = (V, A)$ is said to be *strongly connected*, or simply *strong*, if $\text{dist}(u, v) < \infty$ for each pair of vertices $u, v \in V$. If D is not strong then we call the maximal strong induced subdigraphs of D the *strong components* of D . The distance from a set X to a set Y is as follows:

$$\text{dist}(X, Y) = \max\{\text{dist}(u, v) : u \in X, v \in Y\}$$

We will find useful the operation of digraph *composition*: Let D be a digraph with vertex set $\{v_1, \dots, v_n\}$, and let $D_1, \dots, D_n$ be digraphs that are pairwise vertex disjoint. The *composition* $D[D_1, \dots, D_n]$ is the digraph H on the vertex set $V(D_1) \cup \dots \cup V(D_n)$ whose arc set is formed by combining

the arc sets $A(D_i)$ for $i = 1, \dots, n$, with the set of arcs required so that if $v_i v_j \in A(D)$, then $D_i \mapsto D_j$ in $D[D_1, \dots, D_n]$.

For a digraph $D = (V, A)$ the *underlying graph* of D is the graph $G = (V, E)$ where $E = \{ \{u, v\} : uv \text{ or } vu \in A \}$. For a graph G , the *complement* of G is the graph $\overline{G}$ where $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{ \{u, v\} : \{u, v\} \notin E(G) \}$.

2.2 Classes of digraphs

In this section we introduce the classes of digraphs that will be studied later in this thesis: semicomplete multipartite digraphs, quasi-transitive digraphs and, finally, locally semicomplete digraphs. The semicomplete multipartite digraphs are the most fundamental class of digraphs that we will encounter and, accordingly, we begin with these.

2.2.1 Semicomplete multipartite digraphs

A digraph D is said to be *multipartite* if there exists a partition $V_1, \dots, V_p$, of V , such that each V_i is an independent set. The sets V_i are termed *partite sets*. Multipartite digraphs for which $p = 2$ are known as *bipartite* digraphs.

Note that, in general, we must further restrict the class of multipartite digraphs in order to obtain structures of interest as, trivially, every digraph is a multipartite digraph since we may take individual vertices to be the partite sets.

A *multipartite tournament* is a multipartite digraph $M = (V_1, \dots, V_p; A)$ in which for arbitrary $u \in V_i$, and $v \in V_j$, where $i \neq j$, exactly one of uv , vu is an element of A . If we allow both uv and vu to be elements of A we obtain the class of semicomplete multipartite digraphs: A *semicomplete multipartite digraph*, is a multipartite digraph $M = (V_1, \dots, V_p; A)$ in which for arbitrary $u \in V_i$, and $v \in V_j$, where $i \neq j$, either $uv \in A$, $vu \in A$, or both $uv, vu \in A$. For an example of a multipartite tournament and a semicomplete multipartite digraph see Figure 2.1.

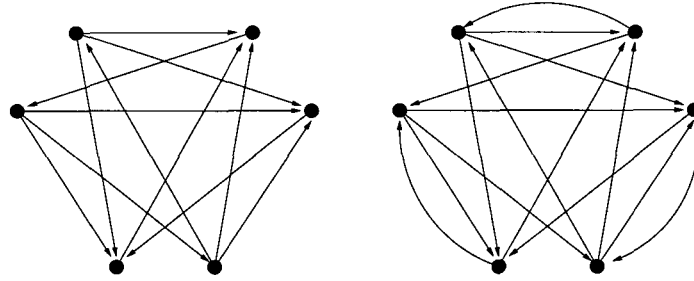


Figure 2.1: A 3-partite tournament, and a 3-partite semicomplete multipartite digraph.

There are a number of interesting subclasses of semicomplete multipartite

digraphs. One that we will find of particular interest is the class of semicomplete digraphs. A digraph $D = (V, A)$ is *semicomplete* if for every pair u, v of distinct vertices of V , at least one of the arcs uv, vu is present in A .

From further restrictions we obtain the most highly structured class of digraphs: tournaments.

A digraph $D = (V, A)$ is a *tournament* if for every pair of distinct vertices $u, v \in V$, exactly one of the arcs uv, vu is present in A .

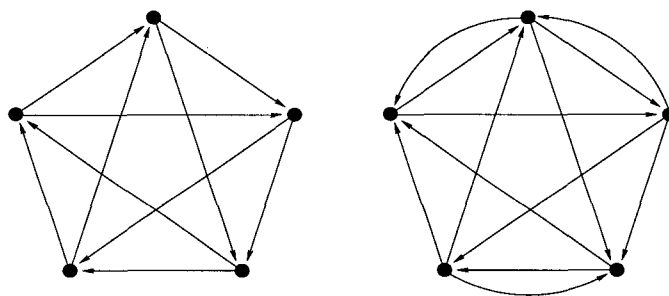


Figure 2.2: A tournament, and a semicomplete digraph.

2.2.2 Quasi-transitive digraphs

In the preceding subsection we obtained fundamental classes of digraphs by requiring the existence of arcs between maximal independent sets. One can obtain classes with equally interesting structure through restrictions based on distance between vertices. Quasi-transitive and transitive digraphs are two such classes.

A digraph $D = (V, A)$ is *quasi-transitive* if for every pair of vertices $u, v \in V$ for which there exists a path of length 2 from u to v , at least one of the arcs uv, vu is an element of A .

A digraph $D = (V, A)$ is called *transitive* if for every pair of vertices $u, v \in V$ for which there exists a path of length 2 from u to v , the arc uv is an element of A .

In our later work, we will require the following characterization of quasi-transitive digraphs:

Theorem 2.1 [3] *Let D be a quasi-transitive digraph.*

1. *If D is non-strong, then there exist a transitive oriented graph T with vertex set $V = \{u_1, u_2, \dots, u_t\}$ and strong quasi-transitive digraphs $H_1, H_2, \dots, H_t$ such that $D = T[H_1, H_2, \dots, H_t]$, where H_i is substituted for u_i , $i = 1, 2, \dots, t$.*
2. *If D is strong, then there exists a strong semicomplete digraph S with vertex set $V = \{v_1, \dots, v_s\}$ and quasi-transitive digraphs $Q_1, \dots, Q_s$ such that Q_i is either a vertex or is non-strong and $D = S[Q_1, \dots, Q_s]$, where Q_i is substituted for v_i , for $i = 1, 2, \dots, s$.*

□

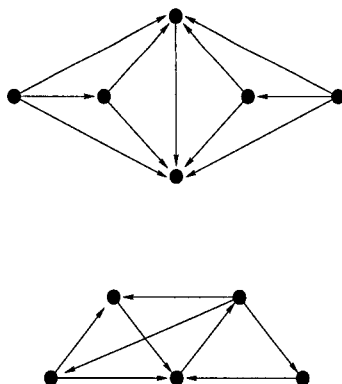


Figure 2.3: A transitive digraph (top), and a quasi-transitive digraph.

2.2.3 Locally semicomplete digraphs

In the previous two subsections we used global criteria as a means of classification. It is also possible to obtain interesting classifications based on local criteria; by imposing structure on the neighbourhoods of individual vertices for instance.

A digraph D is *locally semicomplete* if for each $v \in V$ both $N^+(v)$ and $N^-(v)$ induce semicomplete digraphs. If, in addition, for each $v \in V$ both $N^+(v)$ and $N^-(v)$ induce tournaments, we say that D is a *local tournament* or a *locally tournament* digraph. Figure 2.4 gives an example of a locally semicomplete digraph.

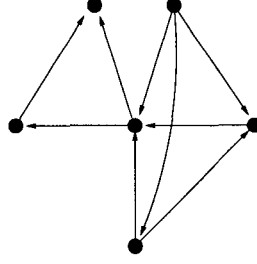


Figure 2.4: A locally semicomplete digraph.

2.3 Kernels and quasi-kernels

A *kernel* of a digraph $D = (V, A)$ is an independent set $K \subseteq V$ such that for each $v \in V - K$, there exists a vertex $u \in K$ for which $v \rightarrow u$. A digraph D is said to be *kernel-perfect* if every induced subdigraph of D contains a kernel.

It is an easy exercise to show that every acyclic digraph contains a unique kernel. In fact, if D is an acyclic digraph, then one may efficiently determine the kernel K of D by letting K be the set of sinks of D and recursively adding the sinks of $D - N^-[K]$ to K . Since every induced subdigraph of an acyclic digraph is acyclic, it follows that every acyclic digraph is kernel-perfect. It is also true that every bipartite digraph is kernel-perfect.

Proposition 2.2 *Every bipartite digraph is kernel-perfect.*

Proof: Since every induced subdigraph of a bipartite digraph is bipartite

it is enough to show that every bipartite digraph contains a kernel. Suppose, to the contrary, that $D = (V_1, V_2; A)$ is a vertex minimal counterexample. If D contains no sink, then clearly V_1 and V_2 are both kernels of D . So assume that D contains a sink v . By our assumption, $D - N^-[v]$ contains a kernel K . Hence, $K \cup \{v\}$ is a kernel of D , a contradiction. $\square$

A non-trivial sufficient condition for a digraph to be kernel-perfect is the following result of Richardson:

Theorem 2.3 [29] *Every digraph containing no odd cycle is kernel-perfect.*

$\square$

A *quasi-kernel* of a digraph $D = (V, A)$ is an independent set $Q \subseteq V$ such that for each $v \in V - Q$, there exists a vertex $u \in Q$ for which $\text{dist}(v, u) \leq 2$.

There are a number of results on quasi-kernels of digraphs that will be fundamental in our later work. It was first proved by Chvátal and Lovász in [10] that every digraph contains a quasi-kernel. We present a proof of this fundamental result, which is due to Thomassé, cf. [4].

Theorem 2.4 [10] *Every digraph contains a quasi-kernel.*

Proof: Let $D = (V, A)$ be a digraph and let $\prec$ be an arbitrary total order of V . Let $D' = (V, \{xy \in A : x \prec y\})$ and $D'' = (V, \{xy \in A : y \prec x\})$. Since both D' and D'' are acyclic, D' and D'' are kernel perfect. Let K' be a kernel of D' , and let K'' be the kernel of the subdigraph of D'' induced by K' . Since $\text{dist}(v, K') \leq 1$ for each $v \in V$ and $\text{dist}(K', K'') \leq 1$, it follows that $\text{dist}(v, K'') \leq 2$ for each $v \in V$. By construction, K'' is an independent set and hence is a quasi-kernel of D . $\square$

Theorem 2.4 can be extended as follows:

Theorem 2.5 *Let $D = (V, A)$ be a digraph and let $v \in V$. Then there exists a quasi-kernel Q of D such that $v \in N^-[Q]$.*

Proof: Let us suppose that $S \subset V$ is the set of vertices for which the claim does not hold. Let Q be a quasi-kernel of the subdigraph of D induced by S . Consider a quasi-kernel Q' of $V - N^-[Q]$. Let $X = Q - (Q \cap N^-(Q'))$. We claim that $Q' \cup X$ is a quasi-kernel of D .

Select an arbitrary $v \in V$. If $v \notin N^-[Q]$, then $v \in N^{-2}[Q']$. Thus, $\text{dist}(v, Q' \cup X) \leq 2$. If $v \in N^-[Q]$, then either $v \in N^-[X]$ or $v \in N^-[Q - X]$. Since $Q - X \rightarrow Q'$ it follows that $\text{dist}(v, Q' \cup X) \leq 2$. Hence, $Q' \cup X$ is a quasi-kernel of D . Since $\text{dist}(v, Q' \cup X) \leq 1$ for each $v \in Q$, we have obtained

a contradiction to our definition of S . Therefore, $S = \emptyset$. $\square$

The lack of a kernel in a digraph forces the existence of a number of quasi-kernels. Jacob and Meyniel [20] proved the following:

Theorem 2.6 [20] *Every digraph with no kernel has at least three quasi-kernels.* $\square$

It was conjectured by Gutin *et al.* [14] that if D is a digraph that has no sink, then D contains a pair of disjoint quasi-kernels. This conjecture, however, is false. The following counterexample was given in [15]:

Erdős, in [11], proved the existence of tournaments T possessing the property that for every pair of vertices $u, v \in V$, there exists $w \in V$ such that $u \rightarrow w$ and $v \rightarrow w$. Let T be such a tournament. Let D be the digraph obtained from T by adding, for each $v \in V(T)$, a new vertex v' and the arc $v'v$. By construction, D contains no sink. Note that any quasi-kernel of D must contain exactly one vertex of T . Let us suppose that Q_1 and Q_2 are a pair of quasi-kernels of D that contain the vertices u and v of T respectively. Since there exists $w \in V$ such that $u \rightarrow w$ and $v \rightarrow w$, it follows that $\text{dist}(w, Q_1) = 2$ and $\text{dist}(w, Q_2) = 2$. Therefore, $w' \in Q_1 \cap Q_2$. Hence, D contains no pair of disjoint quasi-kernels.

Although not true in general, we will show that the above conjecture holds for a number of special classes of digraphs.

Quasi-kernels, indirectly and not in name, were first encountered in the work of Vaughan [30] and Landau [25] in the context of special vertices of tournaments now called *kings*. A *king* in a tournament T is a vertex t such that $\text{dist}(t, v) \leq 2$ for all $v \in T$. Kings of digraphs have been studied extensively, for an introduction see [28] and §2.10 of [2]. Moon [27] proved the following:

Proposition 2.7 *If T is a tournament containing no sink, then T has at least three kings.*

□

Observe that Theorem 2.6 is a direct generalization of Proposition 2.7.

2.4 The push operation

Let $D = (V, A)$ be a digraph and let $X \subseteq V$. We define D^X to be the digraph obtained from D by reversing the orientation of each arc with exactly one endpoint in X . We say that the vertices of X are *pushed*, and that D^X is the result of *pushing* X in D . In the context of digraphs, the push operation was first studied by Fisher and Ryan [12]. Research on the push operation

has typically involved the classification of digraphs $D = (V, A)$ that contain a set of vertices $X \subseteq V$ for which D^X has a desired property. Recent work has concentrated on pushing vertices so that the resulting digraph is acyclic or Hamiltonian. The interested reader will refer to the papers: [18, 19, 21, 22, 23, 24, 26]. Below we state several trivial properties of the push operation.

Let D be a digraph and let $X, Y \subseteq V(D)$, then $(D^X)^Y = (D^Y)^X = D^{X \Delta Y}$, where $X \Delta Y$ is the symmetric difference of the sets X and Y . It follows from the previous fact that $(D^X)^X = D$. Since $D^\emptyset = D^V = D$, we have $D^X = (D^X)^V = D^{V-X}$.

In addition we will find it useful to define an equivalence relation on the set of all digraphs as follows: For two digraphs D and D' we say that D is related to D' , denoted by $D \equiv D'$, if and only if $D^X \cong D'$ for some $X \subseteq V(D)$. It is easy to verify that the relation $\equiv$ is indeed an equivalence relation. We denote the equivalence class containing a digraph D by $[D]$.

Chapter 3

Disjoint quasi-kernels in semicomplete multipartite digraphs

In this chapter we prove that every semicomplete multipartite digraph that contains no sink has a pair of disjoint quasi-kernels. Initially, we will find it useful to construct from a semicomplete multipartite digraph M , a digraph that preserves, with respect to distance, certain structural properties of M .

Given $M = (V_1, V_2, \dots, V_m; A)$, let $M^* = (\{v_1, v_2, \dots, v_m\}; A^*)$ where

$v_i v_j \in A^*$ if and only if $\text{dist}(V_i, V_j) = 1$, that is, for each $u \in V_i$ there exists $v \in V_j$ such that $uv \in A$. We refer to M^* as the digraph *associated with* M . Figure 3.1 shows a semicomplete multipartite digraph M and its associated digraph M^* . Note that either $\text{dist}(v_i, v_j) = 1$ or $\text{dist}(v_j, v_i) = 1$ (or both). Thus, M^* is a semicomplete digraph. Since any independent set in a semicomplete digraph has cardinality one, any kernel or quasi-kernel of M^* consists of a single vertex.

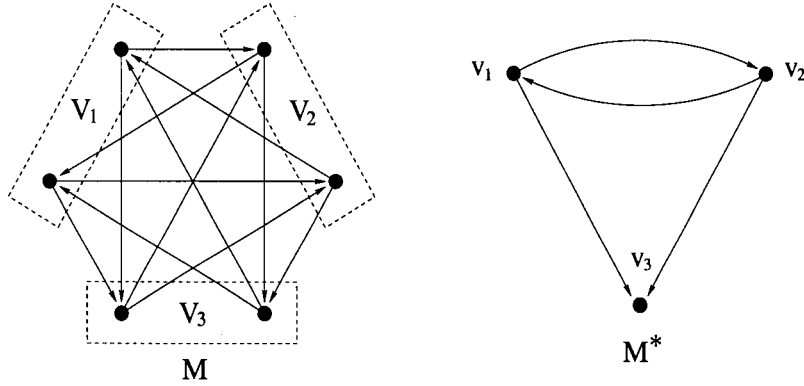


Figure 3.1: A semicomplete multipartite digraph M and its associated M^* .

Proposition 3.1 *Let $M = (V_1, V_2, \dots, V_m; A)$ be a semicomplete multipartite digraph and $M^* = (\{v_1, v_2, \dots, v_m\}; A^*)$ the digraph associated with M . For $j = 1, 2, \dots, m$, the partite set V_j is a kernel of M if and only if $\{v_j\}$ is a kernel of M^* .*

Proof: The partite set V_j is a kernel of M if and only if $\text{dist}_M(v, V_j) = 1$

for all $v \notin V_j$. This is equivalent to the statement that $v_i v_j \in A^*$ for all $i \neq j$ by the construction of M^* , that is, $\{v_j\}$ is a kernel of M^* . $\square$

Proposition 3.2 *Let $M = (V_1, V_2, \dots, V_m; A)$ be a semicomplete multipartite digraph and $M^* = (\{v_1, v_2, \dots, v_m\}; A^*)$ the digraph associated with M . For $j = 1, 2, \dots, m$, if $\{v_j\}$ is a quasi-kernel of M^* , then V_j is a quasi-kernel of M .*

Proof: Suppose $\{v_j\}$ is a quasi-kernel of M^* . Then $\text{dist}(v_i, v_j) \leq 2$ for all $i \in \{1, 2, \dots, m\}$. Let $v \in V_k$ where $k \neq j$. Since $\text{dist}_{M^*}(v_k, v_j) \leq 2$, it follows from the definition of M^* that $\text{dist}_M(v, V_j) \leq 2$. Hence, V_j is a quasi-kernel of M . $\square$

The converse of Proposition 3.2 does not hold. For instance, for the pair M and M^* in Figure 3.1, the partite set V_1 is a quasi-kernel of M , while $\{v_1\}$ is not a quasi-kernel of M^* .

Propositions 3.1 and 3.2 allow us to use M^* to quickly obtain a lower bound on the number of disjoint quasi-kernels of M .

Theorem 3.3 *Let $M = (V_1, V_2, \dots, V_m; A)$ be a semicomplete multipartite*

digraph and suppose that V_j is a kernel of M . If $I = \{i : \forall u \in V_j, \exists v \in V_i \text{ such that } uv \in A\}$, then there are at least $|I| + 1$ partite sets that are quasi-kernels of M .

Proof: By Proposition 3.1, $\{v_j\}$ is a kernel of the associated digraph M^* . It follows from the definition of I and the construction of M^* that $|I| = |N^+(v_j)|$. Since $\{v_j\}$ is a kernel of M^* , each $v \in N^+(v_j)$ is a quasi-kernel of M^* and hence, M^* has $|I| + 1$ quasi-kernels. By Proposition 3.2 there are at least $|I| + 1$ partite sets that are quasi-kernels of M . $\square$

Theorem 3.4 *If $M = (V_1, V_2, \dots, V_m; A)$ is a semicomplete multipartite digraph that has no kernel, then there are at least three partite sets that are quasi-kernels of M .*

Proof: Since M has no kernel, Proposition 3.1 implies that the associated digraph M^* has no kernel. Proposition 2.7 implies that M^* has three quasi-kernels, say $\{v_1\}, \{v_2\}, \{v_3\}$. By Proposition 3.2, the partite sets V_1, V_2, V_3 are quasi-kernels of M . $\square$

If M^* contains no sink, then M^* has at least two quasi-kernels and hence,

by Proposition 3.2, M has at least two disjoint quasi-kernels. However, if M^* contains a sink, then we are unable to infer from the structure of M^* that there exist disjoint quasi-kernels of M .

We note that if Q is a quasi-kernel of $M = (V_1, V_2, \dots, V_m; A)$, then Q is a subset of some partite set V_j since Q is an independent set. It may be the case that Q is properly contained in V_j , however, this implies that V_j itself is a quasi-kernel. Consequently, given $M = (V_1, V_2, \dots, V_m; A)$ some partite set V_j is a quasi-kernel of M .

Theorem 3.5 *Every semicomplete multipartite digraph M with no sink must contain two disjoint quasi-kernels.*

Proof: Denote $M = (V_1, V_2, \dots, V_m; A)$. Assume without loss of generality that V_m is a quasi-kernel of M . If V_j is a quasi-kernel for some $j \in \{1, \dots, m-1\}$, then we are done. So assume that V_j is not a quasi-kernel of M for $j \neq m$. Let $H = M - V_m$. We claim that H has no kernel.

Suppose to the contrary that V_k is a kernel of H for some $k \in \{1, 2, \dots, m-1\}$. Then $\text{dist}(V_i, V_k) \leq 1$ for all $i = 1, 2, \dots, m-1$. Since M contains no sink, $\text{dist}(V_m, V(H)) = 1$ and hence $\text{dist}(V_m, V_k) \leq 2$. Thus, V_k is a quasi-kernel of M that is disjoint from V_m , a contradiction. Hence, H has no kernel.

Let V_n be a quasi-kernel of H and X_n be the set of vertices $v \in V_m$ for which $\text{dist}(v, V_n) > 2$. Since V_n is not a quasi-kernel of M , $X \neq \emptyset$. We claim that X_n is a quasi-kernel of M .

First, note that X_n is an independent set and that $N^-[V_n] \cap V(H) \mapsto X_n$. Let $x \in V(M) - X_n$. The selection of X_n implies that $\text{dist}(x, V_n) \leq 2$. Since $N^-[V_n] \cap V(H) \mapsto X_n$ we have $\text{dist}(x, X_n) \leq 2$. Hence, X_n is a quasi-kernel of M .

Without loss of generality, let $\mathbb{Q} = \{V_1, V_2, \dots, V_r\}$ be the set of partite sets that are quasi-kernels of H . Since H has no kernel, Theorem 3.4 implies that $r \geq 3$. As above, we select for each $V_i \in \mathbb{Q}$, the subset $X_i \subseteq V_m$ for which $\text{dist}(X_i, V_i) > 2$. Since each $V_i \in \mathbb{Q}$ is not a quasi-kernel of M , the corresponding subset X_i of V_m is non-empty for $i = 1, 2, \dots, r$. Observe that $\bigcup_{i=1}^r V_i$ consists of all vertices contained in quasi-kernels of M . We have now $\bigcap_{i=1}^r N^{-2}(V_i) = \emptyset$, as otherwise, no quasi-kernel Q satisfies $v \in N^-[Q]$ for any $v \in \bigcap_{i=1}^r N^{-2}(V_i)$, contradicting the claim of Theorem 2.5. Note that $N^+(X_i) \subset N^{-2}(V_i)$ for $i = 1, \dots, r$. Thus, $\bigcap_{i=1}^r X_i = \emptyset$. Let $I \subseteq \{1, 2, \dots, r\}$. We claim that if $Y = \bigcap_{i \in I} X_i$ is non-empty, then Y is a quasi-kernel of M .

Observe that $N_H^-[V_i] \mapsto Y$ for each $i \in I$. Consider an arbitrary vertex $v \in V(M - Y)$. The definition of Y implies that there exists an index $j \in I$

for which $\text{dist}(v, V_j) \leq 2$. Since $N_H^-[V_j] \mapsto Y$, it follows that $\text{dist}(v, Y) \leq 2$.

If there exist indices i and j such that $X_i \cap X_j = \emptyset$, then X_i and X_j are disjoint quasi-kernels of M . If not, then we select $S \subset \{1, 2, \dots, r\}$ for which $Y = \bigcap_{i \in S} X_i$ is non-empty. Assume that the cardinality of S is maximal with respect to this property, and note that the containment of S is proper. Thus, there exists an index $j \in \{1, 2, \dots, r\}$ for which $Y \cap X_j = \emptyset$. Our claims now imply that Y and X_j are disjoint quasi-kernels of M . $\square$

The claim of Theorem 3.5 is sharp in that there exist semicomplete multipartite digraphs that have exactly two disjoint quasi-kernels. For instance a 4-cycle $C = abcd$ is a semicomplete multipartite digraph with $\{a, c\}$ and $\{b, d\}$ being the only two quasi-kernels of C . An infinite family of semicomplete multipartite digraphs with exactly one pair of disjoint quasi-kernels can be constructed from C and an arbitrary semicomplete multipartite digraph M by letting $M \mapsto C$. As above, the only two disjoint quasi-kernels of any digraph in this family are $\{a, c\}$ and $\{b, d\}$.

Chapter 4

Disjoint quasi-kernels in quasi-transitive digraphs

In the previous section we proved that every semicomplete multipartite digraph with no sink contains two disjoint quasi-kernels. In this section we prove an analogous result for the class of quasi-transitive digraphs. Our proof will mirror the following characterization of quasi-transitive digraphs.

Theorem 4.1 [3] *Let D be a quasi-transitive digraph.*

1. *If D is strong, then there exists a strong semicomplete digraph S with vertex set $V = \{v_1, v_2, \dots, v_s\}$ and there are quasi-transitive digraphs $H_1, H_2, \dots, H_s$, where H_i is either a vertex or non-strong, such that*

$D = S[H_1, H_2, \dots, H_s]$, where for $i = 1, 2, \dots, t$, H_i is substituted for u_i .

2. If D is non-strong, then there exists a transitive oriented graph T with vertex set $V = \{u_1, u_2, \dots, u_t\}$ and strong quasi-transitive digraphs $H_1, H_2, \dots, H_t$ such that $D = T[H_1, H_2, \dots, H_t]$, where for $i = 1, 2, \dots, t$, H_i is substituted for u_i .

We begin with the case where D is a strong quasi-transitive digraph.

First, we have the following lemma.

Lemma 4.2 *If I is an independent set in a strong quasi-transitive digraph $D = (V, A)$, where $|V| \geq 2$, then there exists a vertex $v \in V$ such that $I \mapsto v$.*

Proof: Since D is strong, case (1) of Theorem 4.1 implies that $D = S[H_1, H_2, \dots, H_s]$ where S is a strong semicomplete digraph and each H_i is a non-strong quasi-transitive digraph or a single vertex. Hence, if I is an independent set in D , then $I \subseteq V(H_j)$ for some $j \in \{1, 2, \dots, s\}$. Since S is strong, S contains no sink. Therefore, $v_j \in V(S)$ dominates some vertex $v_l \in V(S)$. It now follows that $I \mapsto v$ for any vertex $v \in V(H_l)$. $\square$

Proposition 4.3 *Let D be a strong quasi-transitive digraph. If D has no sink and Q is a quasi-kernel of D , then there exists a quasi-kernel disjoint from Q .*

Proof: Since D has no sink, D has at least two vertices. By Lemma 4.2, there exists some vertex $v \in V$ such that $Q \mapsto v$. Theorem 2.5 implies that there exists a quasi-kernel Q' of D such that $v \in N^-[Q']$. If $v \in Q'$, then $Q' \cap Q = \emptyset$ as otherwise neither Q nor Q' are independent sets. If $v \in N^-(Q')$ and there exists a vertex x such that $x \in Q \cap Q'$, then x is adjacent to every out-neighbour of v in Q' , contradicting the fact that Q' is an independent set. Hence, $Q \cap Q' = \emptyset$ □

We now turn to the non-strong case.

Proposition 4.4 *Every non-strong quasi-transitive digraph D with no sink contains two disjoint quasi-kernels.*

Proof: According to Theorem 4.1 there exists a transitive oriented graph $T = (V, A)$, with $V = \{u_1, u_2, \dots, u_t\}$ and strong quasi-transitive digraphs $H_1, H_2, \dots, H_t$, such that $D = T[H_1, H_2, \dots, H_t]$ where H_i is substituted for u_i for $i = 1, 2, \dots, t$. Without loss of generality, let $u_1, u_2, \dots, u_k$ be the

sinks of T , and $H_1, H_2, \dots, H_k$ be the corresponding strong quasi-transitive digraphs. Since D contains no sink, each H_i has no sink. By Proposition 4.3, each H_i contains two disjoint quasi-kernels, say $Q_{i,1}$ and $Q_{i,2}$. We claim that $Q_1 = \bigcup_{i=1}^k Q_{i,1}$ and $Q_2 = \bigcup_{i=1}^k Q_{i,2}$ are disjoint quasi-kernels of D .

As $u_1, u_2, \dots, u_k$ are sinks of T , for $i, j \in \{1, 2, \dots, k\}$ we have $(H_i, H_j) = \emptyset$. Thus, Q_1 and Q_2 are independent sets. Since $Q_{i,1} \cap Q_{i,2} = \emptyset$ for $i = 1, 2, \dots, k$, it follows that $Q_1 \cap Q_2 = \emptyset$. It remains to show that if $v \in V(D)$, then $\text{dist}(v, Q_j) \leq 2$ for $j = 1, 2$.

Without loss of generality, consider Q_1 . Let $v \in V(D) - Q_1$ and suppose that $v \in V(H_j)$. If $j \in \{1, 2, \dots, k\}$, then, since $Q_{j,1} \subseteq V(H_j)$, we have $\text{dist}(v, Q_{j,1}) \leq 2$. Hence, $\text{dist}(v, Q_1) \leq 2$ since $Q_{j,1} \subseteq Q_1$. If $j \in \{k+1, k+2, \dots, t\}$, then the structure of D implies that $H_j \mapsto H_l$ for some $l \in \{1, 2, \dots, k\}$. Hence, $H_j \mapsto Q_{l,1}$. Since $Q_{l,1} \subseteq Q_1$, we have $\text{dist}(v, Q_1) = 1$. Thus, Q_1 is a quasi-kernel of D . Therefore, Q_1 and Q_2 are disjoint quasi-kernels of D . □

From Propositions 4.3 and 4.4 we immediately obtain the following:

Theorem 4.5 *Every quasi-transitive digraph with no sink contains two disjoint quasi-kernels.* □

Chapter 5

Disjoint quasi-kernels in locally semicomplete digraphs

In this chapter we prove that every locally semicomplete digraph that contains no sink has a pair of disjoint quasi-kernels. Our treatment of this problem will be based upon a classification of locally semicomplete digraphs taken from the papers by Bang-Jensen, Guo, Gutin, and Volkmann [1], and Huang [17]. Before stating this classification we need the following definitions:

A *round local tournament* is an oriented graph D for which there exists a labeling $v_1, v_2, \dots, v_n$ of its vertices such that for each $i = 1, 2, \dots, n$

the following are satisfied: $N^+(v_i) = \{v_{i+1}, v_{i+2}, \dots, v_{i+d^+(v_i)}\}$ and $N^-(v_i) = \{v_{i-d^-(v_i)}, \dots, v_{i-1}\}$ with subscripts taken modulo n . The labeling $v_1, v_2, \dots, v_n$ is referred to as a *round labeling* of D .

A digraph D is *round decomposable* if there exists a round local tournament R , on $n \geq 2$ vertices, such that $D = R[D_1, D_2, \dots, D_n]$ where each D_i is a strong semicomplete digraph. We call $R[D_1, D_2, \dots, D_n]$ a *round decomposition* of D .

The following theorem is summarized from [1] and [17].

Theorem 5.1 *Let D be a connected locally semicomplete digraph. Then exactly one of the following possibilities holds:*

1. *D is round decomposable, ie. $D = R[D_1, D_2, \dots, D_n]$, where R is a round local tournament on $n \geq 2$ vertices and D_i is a strong semicomplete digraph for $i = 1, 2, \dots, n$.*
2. *D is a locally semicomplete digraph that is not round decomposable and $\overline{U(D)}$ is bipartite.*

□

Our proof that every locally semicomplete digraph with no sink contains two disjoint quasi-kernels will be divided into two cases according to Theorem 5.1 above: when D is round decomposable and respectively when D is

not round decomposable. Initially, we prove the existence of disjoint quasi-kernels in round local tournaments. First, we have the following lemma.

Lemma 5.2 *If $R = (V, A)$ is a round local tournament, then $R - N^-(v)$ contains a kernel for each $v \in V$.*

Proof: Let $v_1, v_2, \dots, v_n$ be a round labeling of R . If R is acyclic, then we are done. So suppose that R contains a cycle $C = v_{c_1} \dots v_{c_r}$. Without loss of generality, consider v_1 . Since $v_1, \dots, v_n$ is a round labeling of R , there exists an index c_i such that $c_i < 1 < c_{i+1}$ modulo n . Thus, $v_{c_i} \rightarrow v_1$. Hence, $N^-(v_1) \cap V(C) \neq \emptyset$ for any cycle C in R . It follows that $R - N^-(v)$ is acyclic for any $v \in V$ and hence has a kernel. $\square$

Theorem 5.3 *If $R = (V, A)$ is a round local tournament that contains no sink, then R has two disjoint quasi-kernels.*

Proof: Let $v_1, v_2, \dots, v_n$ be a round labeling of R . Since R has no sink it must contain a cycle. Suppose that $C = v_{c_1} v_{c_2} \dots v_{c_r}$ is a shortest cycle of R . Note that the set E of vertices of C with even indices form a quasi-kernel of D and that $N^+(E) \cap V(C)$ is a quasi-kernel disjoint from E . Let $X_i = N^+(v_{c_i}) - N^-(v_{c_{i+2}})$, with subscripts taken modulo r . If $X_j = \emptyset$ for each

$j \in \{1, 2, \dots, r\}$, then it is easy to see that a pair of disjoint quasi-kernels of C is a pair of disjoint quasi-kernels of D . Hence, we assume that there exists i for which $X_i \neq \emptyset$ and, without loss of generality, that $X_1 \neq \emptyset$. Note that since R is a round local tournament, the out-neighbourhood of any vertex induces a transitive tournament and hence has a kernel. We claim that if $\{v_k\}$ is the kernel of X_1 , $q = \max\{j : v_{c_j} \nrightarrow v_k\}$, and $l = \max\{i : c_{q-2i} > c_2\}$, then $Q = \{v_k, v_{c_q}, v_{c_{q-2}}, \dots, v_{c_{q-2l}}\}$ is a quasi-kernel of R . See Figure 5.1.

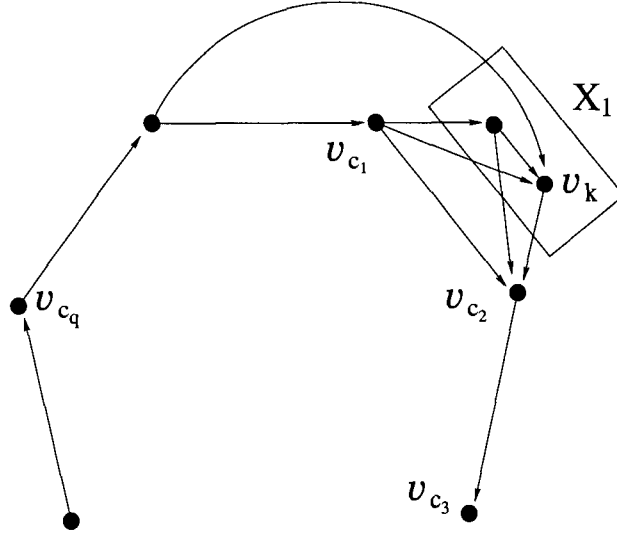


Figure 5.1: A schematic of C .

First, observe that $c_1 \leq k < c_2$. Since C is a minimal cycle and R is a round digraph, $S = \{v_{c_q}, v_{c_{q-2}}, \dots, v_{c_{q-2l}}\}$ is an independent set. It follows from the selection of v_k that $(S, v_k) = \emptyset$ and $(v_k, S) = \emptyset$. Hence, Q is an independent set. It remains to show that $\text{dist}(v_i, Q) \leq 2$ for $i = 1, \dots, n$.

Select an arbitrary $v_i \in V$. If $i > c_q$, then by selection of q , either $v_i \rightarrow v_k$ or $v_i \rightarrow v_{q+1} \rightarrow v_k$. If $c_q \geq i \geq c_{q-2l}$, then it is easy to see that $\text{dist}(v_i, Q) \leq 2$. If $c_{q-2l} > i > k$, then we consider the following cases: $c_{q-2l} = c_3$ and $c_{q-2l} = c_4$. In the first case, the selection of v_k implies that $v_i \rightarrow v_{c_3}$. In the second case, either $v_i \rightarrow v_{c_3}$ or $v_i \rightarrow v_{c_4}$ and thus $\text{dist}(v_i, Q) \leq 2$. Hence, $\text{dist}(v_i, Q) \leq 2$ for $i = 1, 2, \dots, n$.

For each $v \in Q$ let X contain the out-neighbour of v on C , that is, let $X = \{N^+(v) \cap C : v \in Q\}$. By Lemma 5.2, $V - N^-(X)$ contains a kernel K . Observe that each vertex of $N^-(X)$ has an arc to $V - N^-(X)$. Since $Q \subseteq N^-(X)$, it follows that K is a quasi-kernel of R disjoint from Q . $\square$

Before we move on to the final result of this chapter we need the following lemmas.

Lemma 5.4 [15] *Let x be a vertex in a digraph D . If x is a non-sink, then D has a quasi-kernel not including x .* $\square$

Lemma 5.5 *If $D = (V, A)$ is locally semicomplete but not round decomposable, and has no sink, then D contains two disjoint quasi-kernels.*

Proof: Since D is not round decomposable, Theorem 5.1 implies that $\overline{U(D)}$ is bipartite. Hence, $U(D)$ consists of two complete graphs G_1 and G_2 with

some number of arcs between them. Since any quasi-kernel of D can contain at most one vertex from each of $V(G_1)$ and $V(G_2)$, every quasi-kernel of D contains at most two vertices. If D has a quasi-kernel Q consisting of a single vertex, then by Lemma 5.4, D has two disjoint quasi-kernels. So assume that each quasi-kernel of D contains exactly two vertices.

Let $Q = \{u, v\}$ be a quasi-kernel of D . Assume without loss of generality that $u \in V(G_1)$ and that $v \in V(G_2)$. Let $v_1, v_2, \dots, v_n$ be a labeling of V for which $v = v_r$ and $N^+(v) = \{v_{r+1}, v_{r+2}, \dots, v_n\}$. Let $D' = (V, A')$ where $A' = \{v_i v_j \in A : i < j\}$ and let $D'' = (V, A'')$ where $A'' = \{v_i v_j \in A : j < i\}$. By construction, both D' and D'' are acyclic. Let K be the kernel of D' . Since v_n is a sink of D' and $(v =)v_r \rightarrow v_n$, it follows that $v_n \in K$ and $v \notin K$. Let K' be the kernel of the subdigraph of D'' induced by K . Then, for $v_i \in V$, we have $\text{dist}(v_i, K) \leq 1$ and $\text{dist}(K, K') \leq 1$. Thus, $\text{dist}(v_i, K') \leq 2$ for $i = 1, 2, \dots, n$. By construction, K' is an independent set. Therefore, K' is a quasi-kernel of D .

If $u \notin K'$, then K' is a quasi-kernel of D that is disjoint from Q . So suppose that $K' = \{u, x\}$. Note that $x \in V(G_2)$. We repeat the above argument with u substituted for v to obtain a quasi-kernel $K'' = \{v, y\}$ such that $y \in V(G_1)$. Hence, K' and K'' are disjoint quasi-kernels of D . $\square$

We may now prove the final result of this chapter.

Theorem 5.6 *Every locally semicomplete digraph D with no sink contains two disjoint quasi-kernels.*

Proof: As stated in Theorem 5.1 every locally semicomplete digraph falls into one of two categories. By Lemma 5.5, we need only consider the case when D is round decomposable.

Suppose that D is round decomposable with a unique round decomposition given by $D = R[D_1, D_2, \dots, D_n]$, where $R = (\{v_1, v_2, \dots, v_n\}, A)$ is a round local tournament on $n \geq 2$ vertices and D_i is a strong semicomplete digraph for $i = 1, 2, \dots, n$. By Theorem 5.3, R contains two disjoint quasi-kernels, say Q and Q' . For $i = 1, 2, \dots, n$ let Q_i be a quasi-kernel of D_i . Note that $Q_i \cap Q_j = \emptyset$ for $i, j \in \{1, 2, \dots, n\}$. We claim that $X = \bigcup_{v_i \in Q} Q_i$ and $Y = \bigcup_{v_i \in Q'} Q_i$ are disjoint quasi-kernels of D .

Since Q and Q' are disjoint quasi-kernels of R , the structure of D implies that $X \cap Y = \emptyset$. Since Q and Q' are independent sets, the structure of D implies that X and Y are independent sets. It remains to show that $\text{dist}(u, X) \leq 2$ and $\text{dist}(u, Y) \leq 2$ for each $u \in V(D)$.

Without loss of generality consider X . Select an arbitrary vertex $u \in V(D)$. Suppose that $u \in V(D_j)$. Since Q is a quasi-kernel of R , there exists a vertex $v_k \in Q$ for which $\text{dist}_R(v_j, v_k) \leq 2$. Hence, $\text{dist}_D(u, Q_k) \leq 2$. Since $Q_k \subseteq X$, we have $\text{dist}_D(u, X) \leq 2$.

Therefore, X and Y are disjoint quasi-kernels of D . $\square$

Chapter 6

Kernel-perfection through the push operation

In this chapter we study kernel-perfection through use of the push operation. Previously, though not in this context, the acyclic subclass of kernel-perfect digraphs has been studied with respect to the push operation. A digraph $D = (V, A)$ is said to be *acyclically pushable* if there exists $X \subseteq V$ for which D^X is acyclic. Given a digraph $D = (V, A)$, the *Acyclically Pushable Problem* (APP) asks whether or not there exists $X \subseteq V$ for which D^X is acyclic. In [19, 21] it is proved that APP is NP-complete for arbitrary digraphs; this result will be of particular relevance to our work.

Given a digraph $D = (V, A)$, if there exists $X \subseteq V$ for which D^X does not contain an odd cycle, then we say that D is *R-pushable*. Similarly, if there exists $X \subseteq V$ for which D^X is kernel-perfect, then we say that D is *K-pushable*. It follows from Richardson's Theorem (Theorem 2.3) that every R-pushable digraph is K-pushable.

In this chapter we investigate the complexity of the problems of deciding whether an arbitrary digraph is R-pushable, and respectively K-pushable. In particular, we show that the former decision problem is NP-complete. In general, it is difficult to verify that a given digraph is kernel-perfect, hence we are unable to show that the latter problem is contained in NP. However, we show the latter problem to be NP-hard.

6.1 R-pushable digraphs

Given a digraph $D = (V, A)$ the *R-Pushable Problem* (RPP) asks whether or not there exists $X \subseteq V$ for which D^X contains no odd cycle. In this section we prove that RPP is NP-complete for arbitrary digraphs. Our proof involves the digraph G defined in the following lemma.

Lemma 6.1 *Let $G = (V, A)$ where $V = \{a, b, c, d, w\}$ and*

$$A = \{ ab, bc, cd, da, wa, wb, wc, wd \}$$

(see Figure 6.1). If $S = \{ a, b, c, d \}$, then for $X \subseteq V$, G^X contains no odd cycle if and only if $X \cap S = S$ or $X \cap S = \emptyset$.

□

It is easy to verify Lemma 6.1.

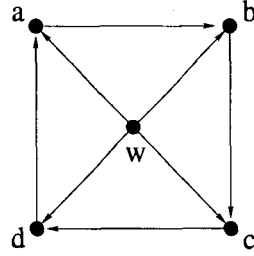


Figure 6.1: The digraph G .

Note that for the digraph G of Lemma 6.1, if G^X has no odd cycle, then either $d^+(w) = 0$ or $d^-(w) = 0$. In either case, w is contained in no cycle of G^X . We therefore make the assumption that if G^X does not contain an odd cycle, then $X = V$ or $X = \emptyset$.

Theorem 6.2 *It is NP-complete to decide whether an arbitrary digraph is R -pushable.*

Proof: The problem is clearly in NP. We show how to reduce from APP.

From a given instance $D = (V, A)$ of APP we construct an instance H of

RPP as follows. For each vertex $v \in V(D)$, let H contain a new copy G_v of the digraph G in Figure 6.1. For each arc $uv \in A(D)$, let $G_u \mapsto G_v$ in H . We claim that D is acyclically pushable if and only if H is R-pushable.

Suppose that D^X is acyclic for some $X \subseteq V(D)$. Let $Y = \bigcup_{v \in X} V(G_v)$ and let C be a cycle of H^Y . Since D^X is acyclic, it follows from the definition of Y and the construction of H that $V(C) \subseteq V(G_v)$ for some $v \in V(D)$. By Lemma 6.1, C has even length. Hence, H^Y contains no odd cycle.

Conversely, let us suppose that there exists $Y \subseteq V(H)$ for which H^Y contains no odd cycle. By Lemma 6.1 and the comment following, we assume that $Y = \bigcup_{v \in X} V(G_v)$ for some subset $X \subseteq V(D)$. It follows from the construction of H that D^X contains no odd cycle. If D^X contains an even cycle $C = v_1 v_2 \cdots v_{2k} v_1$, then $a_1 b_1 a_2 \cdots a_{2k} a_1$ is an odd cycle in H^Y , a contradiction. Hence, D^X is acyclic. $\square$

6.2 K-pushable digraphs

Given a digraph $D = (V, A)$ the *K-Pushable Problem* (KPP) asks whether or not there exists $X \subseteq V$ for which D^X is kernel-perfect. In this section we

prove that KPP is NP-hard for arbitrary digraphs. We require the digraph G defined in the following lemma.

Lemma 6.3 *Let $G = (V, A)$ where $V = \{a, b, c, d, e, w\}$ and,*

$$A = \{ab, bc, cd, de, ea, eb, wa, wb, wc, wd, we\}$$

see Figure 6.2. If $S = \{a, b, c, d, e\}$, then for $X \subseteq V$, G^X is kernel-perfect if and only if $X \cap S = S$ or $X \cap S = \emptyset$. $\square$

Consider the digraph G defined in Lemma 6.3. Note that if G^X is kernel-perfect, then either $d^+(w) = 0$ or $d^-(w) = 0$. Since in either case there can be no cycle containing w , we make the assumption that if G^X is kernel-perfect, then $X = V$ or $X = \emptyset$.

The following lemma simplifies our proof.

Lemma 6.4 *Let $D = (V, A)$ be a digraph and let $S_1, S_2, \dots, S_r$ be the strong components of D . Then D is kernel-perfect if and only if S_i is kernel-perfect for $i = 1, \dots, r$.*

Proof: It follows from the definition that if D is kernel-perfect, then S_i is kernel-perfect for $i = 1, \dots, r$. We prove the converse by induction on r . The base case, $r = 1$, follows from the assumption that S_1 is kernel-perfect.

Suppose that the statement is true for all $r \leq k$. Let D be a digraph with exactly $k + 1$ strong components, $S_1, S_2, \dots, S_{k+1}$, each of which is kernel-perfect. Without loss of generality, assume that $S_1, S_2, \dots, S_{k+1}$ is an acyclic ordering of the strong components of D for which $(S_i, S_j) = \emptyset$ if $i > j$. Let H be an induced subdigraph of D . If $V(H)$ intersects at most k of the vertex sets $V(S_i)$, then by the induction hypothesis H is kernel-perfect. Thus, H contains a kernel and we are done. So assume that $V(H) \cap V(S_i) \neq \emptyset$ for $i = 1, 2, \dots, k+1$. By the induction hypothesis, $H - (V(H) \cap V(S_1))$ is kernel-perfect and hence has a kernel K . Since S_1 is kernel-perfect, the subdigraph of D induced by $(V(S_1) \cap V(H)) - N^-(K)$ has a kernel K_1 . We claim that $K \cup K_1$ is a kernel of H .

Our assumptions on the ordering $S_1, S_2, \dots, S_{k+1}$ imply that $(K, K_1) = \emptyset$. By selection of K_1 we have $(K_1, K) = \emptyset$. Hence, $K \cup K_1$ is an independent set. Select an arbitrary vertex $v \in V(H)$. The definition of K_1 implies that if $v \notin N^-[K]$, then $v \in N^-[K_1]$. Hence, $\text{dist}(v, K \cup K_1) \leq 1$. Therefore, $K \cup K_1$ is a kernel of H . This completes the proof. $\square$

We now prove our main result.

Theorem 6.5 *It is NP-hard to decide whether an arbitrary digraph is K-*

pushable.

Proof: We show how to reduce from APP. From an instance $D = (V, A)$ of APP we construct an instance H of KPP as follows. For each vertex $v \in V$ let H contain a new copy G_v of the digraph G pictured in Figure 6.2.

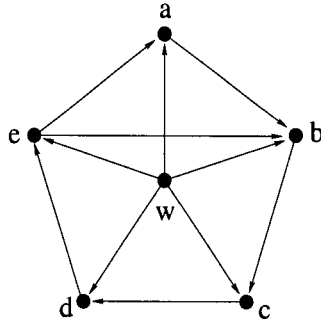


Figure 6.2: The digraph G .

For each arc $uv \in A(D)$ we let $\{a_u a_v, a_u c_v, d_u a_v\} \subseteq A(H)$ (see Figure 6.3).

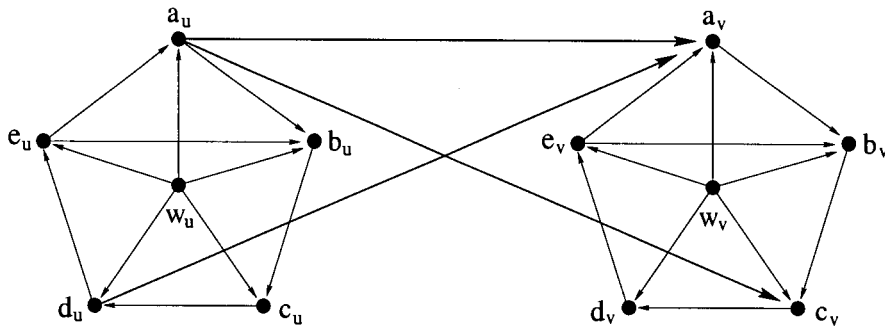


Figure 6.3: Components G_u and G_v for $uv \in A(D)$.

We claim that D is acyclically pushable if and only if H is K-pushable.

Suppose that there exists a subset $Y \subseteq V(H)$ for which H^Y is kernel-perfect. By Lemma 6.3 and the comment following, we assume that $Y = \bigcup_{v \in S} V(G_v)$ for some $S \subseteq V(D)$. If D^S is acyclic, then we are done. So assume that C is a cycle of D^S with minimal length. If the length of C is odd, then it follows from the construction of H that the subdigraph of H^Y induced by $\{a_v : v \in V(C)\}$ is an odd cycle of H , a contradiction. Hence, D^S contains no odd cycle. Thus, C has even length. Now, if $u \in V(C)$ then the subdigraph of H^Y induced by the vertex set $\{a_v : v \in V(C - u)\} \cup \{c_u, d_u\}$ is an odd cycle, a contradiction. Hence, D^X is acyclic.

Conversely, suppose that there exists $X \subseteq V(D)$ for which D^X is acyclic. It follows from the construction of H and Lemma 6.4 that if $Y = \bigcup_{v \in X} V(G_v)$, then H^Y is kernel-perfect.

Therefore, D is acyclically pushable if and only if H is K-pushable. $\square$

Chapter 7

A characterization of R-pushable multipartite tournaments

In this chapter we characterize, in terms of forbidden subdigraphs, R-pushable multipartite tournaments. Throughout the chapter we will confine our discussion to oriented graphs, that is, we will speak only of digraphs with no cycle of length two. Observe that a multipartite tournament contains no odd cycle if and only if it contains no 3-cycle. Thus, a multipartite tournament M is R-pushable if and only if M^X contains no 3-cycle for some

$X \subseteq V(M)$. This fact will be used implicitly throughout the chapter. We begin by establishing a few properties of R-pushable digraphs.

Lemma 7.1 *A digraph D is R-pushable if and only if every subdigraph of D is R-pushable.*

Proof: If every subdigraph of D is R-pushable, then it is clear that D is R-pushable since D is a subdigraph of itself.

Conversely, suppose that D^X contains no odd cycle for some $X \subseteq V(D)$. If H is a subdigraph of D , then $H^{X \cap V(H)}$ is a subdigraph of D^X and hence contains no odd cycle. □

Recall that $[D]$ denotes the equivalence class containing the digraph D under the push operation.

Lemma 7.2 *Let D be a digraph and let $H \in [D]$. Then D is R-pushable if and only if H is R-pushable.*

Proof: Suppose that D is R-pushable. Let D^X contain no odd cycle. Since $H \in [D]$, we have $H = D^Y$ for some $Y \subseteq V$. It follows that $H^{X \Delta Y}$ contains no odd cycle.

Conversely, suppose that there exists a subset $Y \subseteq V$ for which H^Y contains no odd cycle. Since $H = D^X$ for some $X \subseteq V$, it follows that $H^{Y \Delta X}$ contains no odd cycle. $\square$

Lemma 7.3 *None of the digraphs B_1, B_2, B_3, B_4 in Figure 7.1 are R -pushable.*

$\square$

The proof of Lemma 7.3 is a tedious case analysis; it is omitted.

Lemmas 7.2 and 7.3 immediately imply the following:

Lemma 7.4 *No R -pushable digraph contains any digraph $B \in \bigcup_{i=1}^4 [B_i]$ as a subdigraph.* $\square$

Lemma 7.5 *Let $D = (V, A)$ be a digraph on n vertices. If $d(v) = n - 1$ for some vertex $v \in V$, then there exists $X \subseteq V$ for which D^X contains no cycle of length three if and only if D contains no $B \in [B_1]$.*

Proof: Since each $B \in [B_1]$ contains a cycle of length three, necessity is clear. To prove sufficiency, suppose that D contains no $B \in [B_1]$. Consider $D^{N^-(v)}$. Assume that $D^{N^-(v)}$ contains a 3-cycle $C = abca$, as otherwise we are done. It is clear that v dominates C in $D^{N^-(v)}$. Thus, as $D^{N^-(v)}$ contains

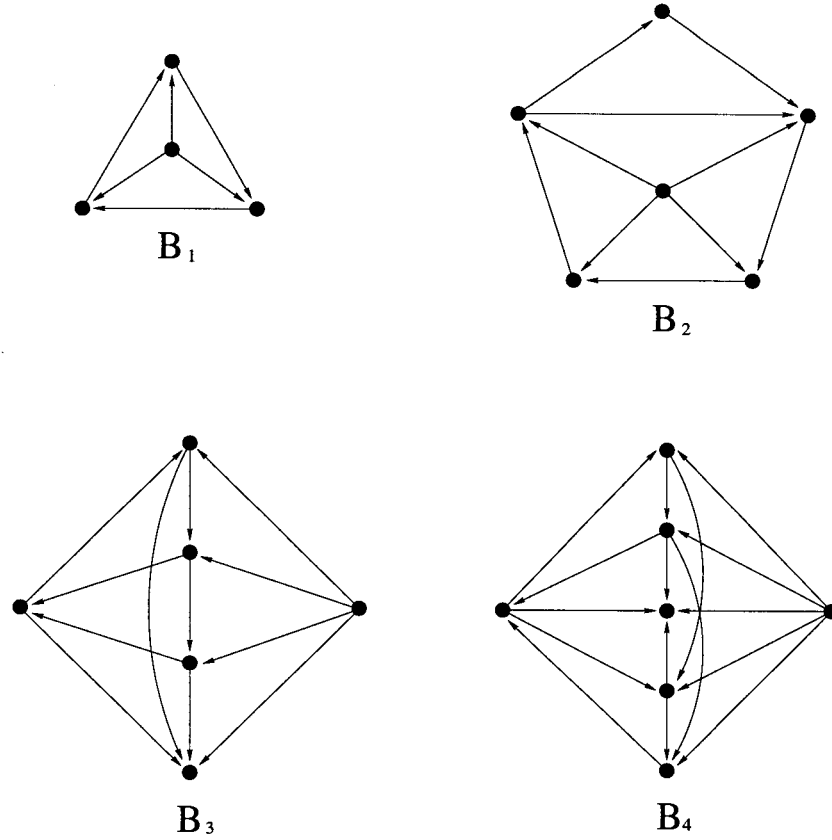


Figure 7.1: Forbidden subdigraphs.

some $B \in [B_1]$, D contains some $B \in [B_1]$.

□

It is an easy exercise to show that if D is strong, then D contains no odd cycle if and only if D is bipartite [16]. We use this fact in the following:

Theorem 7.6 *If D is a digraph then D contains no odd cycle if and only if each strong component of D is bipartite.*

Proof: Suppose that D contains no odd cycle. Thus, if S is a strong component of D , then S contains no odd cycle. Therefore, if S is a strong component of D , then S is bipartite.

Conversely, suppose that the strong components of D are bipartite. Thus, no strong component of D contains an odd cycle. Observe that each cycle of D intersects exactly one strong component of D . Hence, D contains no odd cycle. $\square$

We have the following lemmas.

Lemma 7.7 *Let $D = (V, A)$ be a strong bipartite tournament. For any partition $X \cup Y$ of V there exists a 4-cycle C of D that contains at least one vertex from X and one vertex of Y .*

Proof: Let $C = v_1v_2 \dots v_{2k}v_1$ be a shortest cycle of D whose vertex set intersects both X and Y . If C is a 4-cycle, then we are done. So assume that $|V(C)| \geq 6$. Without loss of generality, let $v_1 \in X$ and $v_2 \in Y$. Since D is bipartite, $\{v_1, v_3, \dots, v_{2k-1}\}$ is an independent set. Similarly, $\{v_2, v_4, \dots, v_{2k}\}$ is an independent set. Hence, v_1 and v_4 are adjacent. If $v_4 \rightarrow v_1$, then $v_1v_2v_3v_4v_1$ is a 4-cycle of D that has the desired property. Thus, $v_1 \rightarrow v_4$. If $v_4 \in Y$, then $v_1v_4 \dots v_{2k}v_1$ is a cycle shorter than C whose vertex set intersects

both X and Y , a contradiction. Hence, $v_4 \in X$. Analogously, $v_{2k-1} \rightarrow v_2$ and $v_{2k-1} \in Y$. But now $v_1v_4v_5 \dots v_{2k}v_1$ is a cycle shorter than C whose vertex set intersects both X and Y , a contradiction. Therefore, C is a 4-cycle. $\square$

Lemma 7.8 *Let $M = (V_1, V_2, V_3; A)$ be a strong multipartite tournament with $V_3 = \{x\}$. If the subdigraph of M induced by $V_1 \cup V_2$ is a strong bipartite tournament, then there exists a 4-cycle C of $M - \{x\}$, and a vertex $v \in V(C)$ for which $x \rightarrow v$ and $u \rightarrow x$ for some neighbour u of v on C .*

Proof: Let $X = N^+(x) \cap (V_1 \cup V_2)$ and $Y = N^-(x) \cap (V_1 \cup V_2)$. Since $X \cup Y$ partitions $V_1 \cup V_2$, the result follows immediately from Lemma 7.7. $\square$

Let $D = (V, A)$ be a digraph and $X \subseteq V$. Hereafter, denoted by $D\langle X \rangle$ will be the subdigraph of D induced by X . For a fixed acyclic ordering $S_1, S_2, \dots, S_n$ of the strong components of D , define $\sigma(v) = i$ for all $v \in S_i$, $i = 1, 2, \dots, n$.

Lemma 7.9 *Let $D = (V, A)$ be a digraph and let $\{v_1, v_2, \dots, v_6, x, y\} \subseteq V$. Suppose that $S_1, S_2, \dots, S_n$ is an acyclic ordering of the strong components of D . Let $\{v_1v_2, v_3v_4, v_5v_6\} \subseteq A$. If $x \mapsto \{v_1, v_4, v_5\}$, $\{v_2, v_3, v_6\} \mapsto x$,*

$y \mapsto \{v_1, \dots, v_6\}$ and

$$\sigma(v_1) < \sigma(v_2), \sigma(v_3), \sigma(v_4), \sigma(v_5), \sigma(v_6)$$

$$\sigma(v_2), \sigma(v_3) < \sigma(v_4), \sigma(v_5), \sigma(v_6)$$

$$\sigma(v_4), \sigma(v_5) < \sigma(v_6)$$

then D contains some $B \in \bigcup_{i=1}^3 [B_i]$.

This situation is represented in Figure 7.2. If an arc is not explicitly given in the figure, then it may or may not exist. Each arc that is not drawn but does exist, excluding those between pairs of vertices within a dashed box (these may be of an arbitrary direction), the tail of each arc must have a lower index than the head.

Proof: We prove our assertion by case analysis of the possible relationship between the vertex v_2 and the vertex v_3 . That is, we examine the following possibilities respectively: $v_2 \rightarrow v_3$, $v_3 \rightarrow v_2$, and $\{v_2, v_3\}$ is an independent set. We assume that in the acyclic ordering of the strong components of D , $(S_i, S_j) = \emptyset$ for $i > j$.

Suppose that $v_2 \rightarrow v_3$. If $v_1 \rightarrow v_4$, then $D\langle\{v_1, v_2, v_3, v_4, x, y\}\rangle$ contains an element of $[B_3]$. So assume that $v_1 \nrightarrow v_4$. Since $\sigma(v_1) < \sigma(v_4)$ it follows

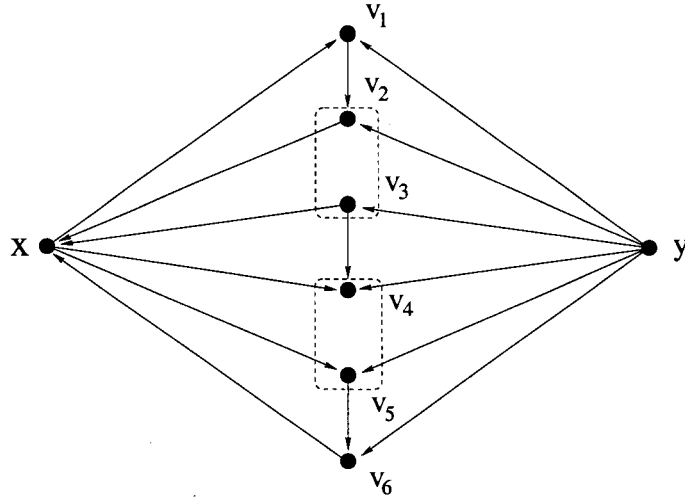


Figure 7.2: The situation of Lemma 7.9.

that $\{v_1, v_4\}$ is an independent set. As $\sigma(v_1) < \sigma(v_3)$ and $v_3 \rightarrow v_4$, we have $v_1 \rightarrow v_3$. If $v_3 \rightarrow v_5$, then $D(\{v_1, v_3, v_5, v_6, x, y\})$ contains a digraph in $[B_2]$. So assume that $v_3 \nrightarrow v_5$. Since $\sigma(v_3) < \sigma(v_5)$, it follows that $\{v_3, v_5\}$ is an independent set. As $v_5 \rightarrow v_6$, we have $v_3 \rightarrow v_6$. We deal with the following possibilities respectively: v_4 and v_5 are adjacent, and $\{v_4, v_5\}$ is an independent set.

If v_4 and v_5 are adjacent, then $D(\{v_3, v_4, v_5, v_6, x, y\})$ contains a digraph in $[B_3]$.

If $\{v_4, v_5\}$ is an independent set, then since $v_3 \rightarrow v_4$, we have $v_3 \rightarrow v_5$. Hence, $D(\{v_3, v_5, v_6, x\}) = B_1$.

Therefore, if $v_2 \rightarrow v_3$, then D contains a digraph $B \in \bigcup_{i=1}^3 [B_i]$.

Suppose that $v_3 \rightarrow v_2$. If $v_1 \rightarrow v_4$, then $D\langle\{v_1, v_2, v_3, v_4, x, y\}\rangle$ contains a digraph in $[B_3]$. So assume that $v_1 \nrightarrow v_4$. As $\sigma(v_1) < \sigma(v_4)$, we have that $\{v_1, v_4\}$ is an independent set. Since $\sigma(v_1) < \sigma(v_3)$ we have $v_1 \rightarrow v_3$. If $v_3 \rightarrow v_5$, then $D\langle\{v_1, v_3, v_5, v_6, x, y\}\rangle$ contains a digraph in $[B_2]$. So assume that $v_3 \nrightarrow v_5$. Now, $\sigma(v_3) < \sigma(v_5)$ implies that $\{v_3, v_5\}$ is an independent set. Since $v_5 \rightarrow v_6$, it follows that $v_3 \rightarrow v_6$. We examine the following cases respectively: v_4 and v_5 are adjacent, and $\{v_4, v_5\}$ is an independent set.

If v_4 and v_5 are adjacent, then $D\langle\{v_3, v_4, v_5, v_6, x, y\}\rangle$ contains a digraph in $[B_3]$.

If $\{v_4, v_5\}$ is an independent set, then since $v_3 \rightarrow v_4$, we have $v_3 \rightarrow v_5$. Hence, $D\langle\{v_3, v_5, v_6, x\}\rangle = B_1$.

Therefore, if $v_3 \rightarrow v_2$, then D contains a digraph $B \in \bigcup_{i=1}^3 [B_i]$.

Suppose that $\{v_2, v_3\}$ is an independent set. Since $v_1 \rightarrow v_2$ and $\sigma(v_1) < \sigma(v_3)$ it follows that $v_1 \rightarrow v_3$. Similarly, $v_2 \rightarrow v_4$. If $v_1 \rightarrow v_4$, then $D\langle\{v_1, v_2, v_4, x\}\rangle = B_1$. So assume that $v_1 \nrightarrow v_4$. Since $\sigma(v_1) < \sigma(v_4)$, it follows that $\{v_1, v_4\}$ is an independent set. There are the following possibilities: $v_4 \rightarrow v_5$ or $v_5 \rightarrow v_4$, and $\{v_4, v_5\}$ is an independent set.

If $v_4 \rightarrow v_5$, then $D\langle\{v_1, v_2, v_4, v_5, x, y\}\rangle$ contains a digraph in $[B_2]$.

If $v_5 \rightarrow v_4$, then since $\{v_1, v_4\}$ is an independent set and $\sigma(v_1) < \sigma(v_5)$, we have $v_1 \rightarrow v_5$. Hence, $D\langle\{v_1, v_3, v_4, v_5, x, y\}\rangle$ contains a digraph in $[B_3]$.

If $\{v_4, v_5\}$ is an independent set, then since $\{v_1, v_4\}$ is an independent set, it follows that $\{v_1, v_4, v_5\}$ is an independent set. Since $v_1 \rightarrow v_2$ and $\sigma(v_2) < \sigma(v_5)$, it is the case that $v_2 \rightarrow v_5$. Thus, $D\langle\{v_1, v_2, v_5, v_6, x, y\}\rangle$ contains a digraph in $[B_2]$.

Therefore, if $\{v_2, v_3\}$ is an independent set, then D contains a digraph $B \in \bigcup_{i=1}^3 [B_i]$. This completes the proof. $\square$

We note that in addition the conclusion of Lemma 7.9 holds in the following cases: $b = c$ and $d \neq e$, $b = c$ and $d \neq e$, $b \neq c$ and $d = e$. As we have examined in the proof a situation identical to that which would result from any of the three stated cases being true, these cases have been omitted for clarity.

We are now ready to prove the main result of this chapter. The following notation and terminology will simplify our discussion:

Let D be digraph and let $x \in V(D)$. We refer to a 3-cycle $xyzx$ as an x -triangle. A transitive triple of the form $(\{x, y, z\}, \{yx, xz, yz\})$ will be referred to as an x -anti-triangle and will be denoted by yxz . Note that for

each such transitive triple T , there is a unique vertex $x \in V(T)$ for which T is an x -anti-triangle: the vertex that satisfies $d^+(x) = d^-(x) = 1$. For a multipartite tournament M and a vertex $v \in V(M)$, we denote by V_v the partite set of M that contains v .

Theorem 7.10 *If M is a multipartite tournament, then M is R-pushable if and only if M contains no $B \in \bigcup_{i=1}^4 [B_i]$.*

Proof: Observe that necessity follows from Lemma 7.4 so we need only prove sufficiency. To obtain a contradiction, suppose that M is a non-R-pushable multipartite tournament and that M contains no $B \in \bigcup_{i=1}^4 [B_i]$. Assume that M is vertex minimal and let $x \in V(M)$. By hypothesis, $M - \{x\}$ is R-pushable. So assume that $M - \{x\}$ contains no odd cycle. By Theorem 7.6, the strong components of $M - \{x\}$ are bipartite tournaments (possibly consisting of a single vertex). Note that every odd cycle of M passes through x .

Let $S_1, S_2, \dots, S_n$ be an acyclic ordering of the strong components of $M - \{x\}$. Since M contains no $B \in \bigcup_{i=1}^4 [B_i]$ and is not R-pushable, Lemma 7.5 implies that each partite set of M contains at least 2 vertices. Let $y \in V_x - \{x\}$ and let S_r be the strong component of $M - \{x\}$ that contains y . Clearly, the digraph obtained from $M - \{x\}$ by pushing $\bigcup_{i=1}^{r-1} V(S_i)$ contains

no odd cycle. Assume that $M - \{x\}$ is this digraph. Observe that there exists a vertex $y \in V_x - \{x\}$ contained in an initial strong component of $M - \{x\}$.

We make the following claims:

Claim 1: Let $P = v_1v_2 \dots v_k$ be a path in $M - \{x\}$ for which x is adjacent to v_i , for $i = 1, 2, \dots, k$. If either $x \rightarrow v_1$ and $v_2 \rightarrow x$, or $v_1 \rightarrow x$ and $x \rightarrow v_2$, then there exists a vertex y in M such that $y \mapsto V(P)$.

Proof of Claim 1: Let $S_1 = (W_1, W_2; A)$ be the initial strong component of $M - \{x\}$. Without loss of generality, our earlier assumption implies that $W_1 \subset V_x$. If $V(P) \cap W_2 = \emptyset$, then our claim is clear. So suppose that $V(P) \cap W_2 \neq \emptyset$. Since the arguments are symmetric we assume that $x \rightarrow v_1$ and $v_2 \rightarrow x$. Hence, $\sigma(v_1) \neq \sigma(v_2)$, as otherwise Lemma 7.8 implies that M contains some digraph $B \in [B_2]$. Since $v_1 \rightarrow v_2$, we have $V(P) \cap W_2 = \{v_1\}$. Thus, there exists $y \in W_1$ such that $y \rightarrow v_1$ as S_1 is strong. Since $\sigma(y) < \sigma(v_i)$ for $i \neq 1$, we have $y \mapsto V(P)$. $\square$

In the following, we frequently appeal to the existence of the vertex y of Claim 1. For ease of exposition, we will often do so implicitly and denote this vertex by y in each such case. Since M contains no $B \in [B_2]$, Claim 1 and Lemma 7.8 imply that if $x \rightarrow u$ and $v \rightarrow x$ for some pair of adjacent

vertices u and v , then $\sigma(u) \neq \sigma(v)$.

Claim 2: There exists an index $k \in \{1, 2, \dots, n\}$ such that one of the following holds:

- (1) $\bigcup_{i=1}^k V(S_i)$ contains at least one vertex from each x -triangle ,
and no pair of vertices from any x -anti-triangle .
- (2) $\bigcup_{i=k}^n V(S_i)$ contains at least one vertex from each x -triangle ,
and no pair of vertices from any x -anti-triangle .

Proof of Claim 2: Suppose that the claim does not hold. Then, M must contain at least two x -triangles or at least two x -anti-triangles. We assume that M contains two x -triangles, as otherwise we may push x to arrive at this case.

Suppose that the vertex dominated by x on every x -triangle lies in the same strong component. Let this strong component have index k . If there exists no x -anti-triangle axb with $\sigma(b) \leq k$, then k satisfies part (1) of the claim. So assume that axb is an x -anti-triangle for which $\sigma(b) \leq k$. Let m be the minimal index of a strong component containing the vertex dom-

inating x on an x -triangle. Let $xcdx$ be an x -triangle with $\sigma(d) = m$. If there exists no x -anti-triangle exf , with $\sigma(e) \geq m$, then the index m satisfies part (2) of the claim. So assume that exf is an x -anti-triangle for which $\sigma(e) \geq m$. Now, after relabeling the vertices a, b, c, d, e, f by $v_1, v_2, v_3, v_4, v_5, v_6$ respectively, it is easy to see that if y is the vertex of Claim 1, then $M^{\{x\}}(\{v_1, v_2, \dots, v_6, x, y\})$ satisfies the hypothesis of Lemma 7.9. Thus, $M^{\{x\}}$ contains some $B \in \bigcup_{i=1}^3 [B_i]$. Therefore, M contains some $B \in \bigcup_{i=1}^3 [B_i]$, a contradiction.

Hence, we assume that abx and $xcdx$ are two x -triangles such that $\sigma(a) < \sigma(c)$. Assume that $\sigma(b)$ is the minimal index of a strong component that contains the vertex dominating x on an x -triangle. In addition, assume that $\sigma(c)$ is the maximal index of a strong component that contains the vertex dominating x on an x -anti-triangle. If there exists no x -anti-triangle exf with $\sigma(f) \leq \sigma(c)$, then $\sigma(c)$ satisfies part (1) of the claim. So assume that exf is an x -anti-triangle for which $\sigma(f) \leq \sigma(c)$. Similarly, if there exists no x -anti-triangle gxh with $\sigma(g) \geq \sigma(b)$, then $\sigma(b)$ satisfies part (2) of the claim. So assume that gxh is an x -anti-triangle for which $\sigma(g) \geq \sigma(b)$.

We have established the following relationships:

$$\sigma(e) < \sigma(f) \leq \sigma(c) < \sigma(d)$$

$$\sigma(a) < \sigma(b) \leq \sigma(g) < \sigma(h)$$

If $\sigma(e) \geq \sigma(b)$, then after relabeling the vertices a, b, e, f, c, d respectively as $v_1, v_2, v_3, v_4, v_5, v_6$, it is easy to see that $M\langle\{v_1, v_2, \dots, v_6, x, y\}\rangle$ satisfies the hypothesis of Lemma 7.9. Hence, M has a subdigraph $B \in \bigcup_{i=1}^3 [B_i]$, a contradiction. Therefore, $\sigma(e) < \sigma(b)$. Similarly, if $\sigma(h) \leq \sigma(c)$, then after relabeling the vertices a, b, g, h, c, d as $v_1, \dots, v_6$ respectively, it is easy to see that $M\langle\{v_1, v_2, \dots, v_6, x, y\}\rangle$ satisfies the hypothesis of Lemma 7.9. This implies that M contains some $B \in \bigcup_{i=1}^3 [B_i]$, a contradiction. Hence, $\sigma(h) > \sigma(c)$. It now follows that:

$$\sigma(e) < \sigma(b), \sigma(c), \sigma(d), \sigma(f), \sigma(g), \sigma(h)$$

and,

$$\sigma(h) > \sigma(a), \sigma(b), \sigma(c), \sigma(e), \sigma(f), \sigma(g)$$

This situation is pictured in Figure 7.3. A dashed arc in Figure 7.3 indicates that if the endpoints of the arc are adjacent, then the arc is in the direction shown.

If $e \rightarrow a$, then by relabeling e, a, b, g, h as v_1, v_2, v_4, v_5, v_6 respectively, it is clear that $M^{\{x\}}\langle\{v_1, v_2, v_4, v_5, v_6, x, y\}\rangle$ satisfies the hypothesis of Lemma 7.9

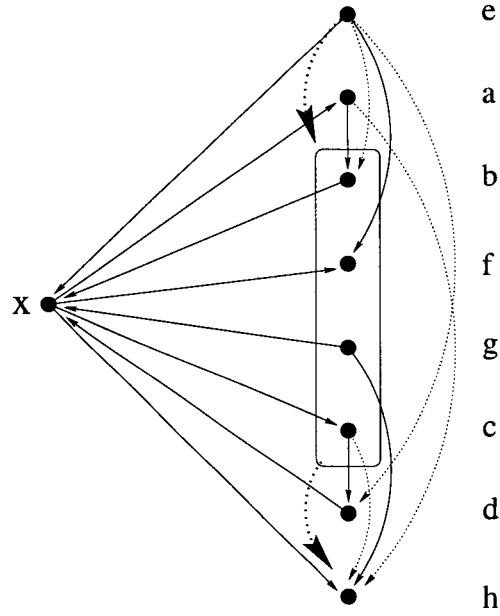


Figure 7.3: The situation in Claim 2.

in the case $v_2 = v_3$. This implies that M contains some $B \in \bigcup_{i=1}^3 [B_i]$, a contradiction.

If $a \rightarrow e$, then $\sigma(a) < \sigma(e)$. Therefore, after relabeling a, e, f, c, d as v_1, v_2, v_4, v_5, v_6 respectively, it is easy to see that $M(\{v_1, v_2, v_4, v_5, v_6, x, y\})$ satisfies the hypothesis of Lemma 7.9 in the case of $v_2 = v_3$. As above, this implies a contradiction.

Since $x \rightarrow a$ and $e \rightarrow x$, it follows that $a \neq e$. Therefore, $\{a, e\}$ is an independent set. A similar argument reveals that $\{d, h\}$ is also an independent set.

Suppose that $e \rightarrow d$. If $e \rightarrow c$, then $M(\{c, d, e, x\}) = B_1$, a contradiction. So assume that $e \nrightarrow c$. Since $\sigma(e) < \sigma(c)$, it follows that $e \neq c$. Therefore, $\{c, e\}$ is an independent set. Since $e \rightarrow f$, we have that f and c are adjacent. Now, $M(\{c, d, e, f, x, y\})$ contains an element of $[B_3]$, a contradiction. Hence, $e \nrightarrow d$. Therefore, $\{d, e\}$ is an independent set as $\sigma(e) < \sigma(d)$. Thus, $\{a, d, e, h\}$ is an independent set. Since $a \rightarrow b$ and $c \rightarrow d$, we have $e \rightarrow \{b, c\}$ and $\{b, c\} \rightarrow h$.

By selection of a and c , $\sigma(a) < \sigma(c)$. Since $\{a, e\}$ is an independent set and $e \rightarrow c$, it follows that $a \rightarrow c$. A similar argument reveals that $b \rightarrow d$.

We have established the arc set pictured in Figure 7.4.

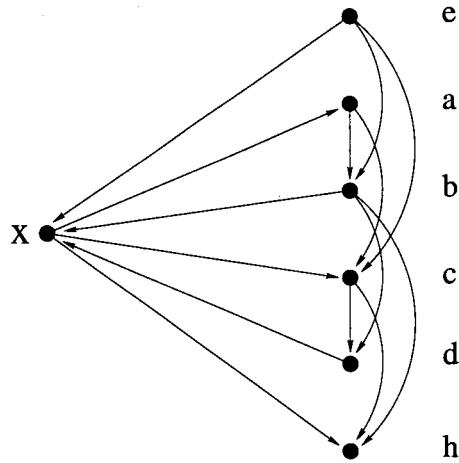


Figure 7.4: The established arc set of Claim 2.

Thus, $M(\{a, b, c, d, h, x, y\})$ contains the digraph B_4 , a contradiction.

This proves the claim. $\square$

Let k be an index that satisfies Claim 2. We suppose that k satisfies part (1) of Claim 2 as otherwise we may push x to reach this case. Consider, the digraph obtained from M by pushing $S = \bigcup_{i=1}^k V(S_i)$. Since S contains at least one vertex from each x -triangle, pushing S destroys all existing x -triangles. Since $S_1, S_2, \dots, S_n$ is an acyclic ordering of the strong components of $M - \{x\}$, pushing S creates no cycle in $M - \{x\}$. M is not R-pushable by supposition and hence M^S contains an odd cycle through x . Therefore, in M there exists an arc uv whose endpoints are dominated by x and such that $u \in S$ and $v \notin S$.

From the existence of uv we will derive a contradiction. As the existence of uv is a logical consequence of our assumption that M is not R-pushable, this will complete the proof.

By selection of k , there exists an x -triangle $xabx$ with $\sigma(a) < \sigma(v)$, and an x -anti-triangle cxu with $\sigma(d) > \sigma(u)$ and $\sigma(d) > \sigma(a)$.

Suppose that $a \rightarrow d$. If $b = c$, then $M(\{a, b, d, x\})$ is B_1 , a contradiction. If b and c are adjacent, then $M(\{a, b, c, d, x, y\})$ contains a digraph in $[B_3]$. Therefore, $\{c, b\}$ is an independent set. Since $a \rightarrow b$, a and c are adjacent.

If $a \rightarrow c$, then $M(\{a, c, d, x\})$ is B_1 , a contradiction. If $c \rightarrow a$, then $\sigma(c) < \sigma(a)$. Thus, a and c are two vertices of an x -anti-triangle contained in strong components with indices $\leq k$ contradicting our choice of k . Hence, $a \nrightarrow d$. Since $\sigma(a) < \sigma(d)$, it follows that $\{a, d\}$ is an independent set.

Since $\{a, d\}$ is an independent set and $c \rightarrow d$, it follows that a and c are adjacent. If $c \rightarrow a$, then $\sigma(c) < \sigma(a)$. Thus, as $\sigma(a) < k$, cxa is an x -anti-triangle for which $\sigma(a), \sigma(c) < k$. This contradicts our selection of k . Hence, $a \rightarrow c$.

Since $\{a, d\}$ is an independent set and $a \rightarrow b$, it follows that b and d are adjacent. If $d \rightarrow b$, then $M(\{a, b, c, d, x, y\})$ contains a digraph in $[B_2]$, a contradiction. Therefore, $b \rightarrow d$.

Suppose that a and v are adjacent. As $\sigma(a) < \sigma(v)$, we have $a \rightarrow v$. Since $\{a, d\}$ is an independent set, d and v are adjacent. However, if $d \rightarrow v$ or $v \rightarrow d$, then $M(\{a, c, d, v, x, y\})$ contains a digraph of $[B_2]$. Therefore, $\{a, v\}$ is an independent set. Hence, $\{a, v, d\}$ is an independent set.

Since $u \rightarrow v$ and $\sigma(u) < \sigma(d)$, it follows that $u \rightarrow d$. In addition we know that u and a are adjacent. If $u \rightarrow a$, then, by Claim 1, there exists a vertex $y \in V_x - \{x\}$ such that $y \rightarrow \{a, c, d\}$. Hence, $M(\{u, a, c, d, x, y\})$ contains a digraph in $[B_2]$, a contradiction. If $a \rightarrow u$, then, by Claim 1, there exists

a vertex $y \in V_x - \{x\}$ such that $y \rightarrow \{a, b, d\}$. Hence, $M\langle\{a, b, u, d, x, y\}\rangle$ contains an element of $[B_2]$, a contradiction.

Since each case leads to a contradiction, uv cannot exist. Therefore, M is R-pushable, a contradiction. $\square$

The following corollary to Theorem 7.10 shows that R-pushable multipartite tournaments are closely related to R-pushable digraphs in general.

Corollary 7.11 *If D is a digraph, then D is R-pushable if and only if D is a subdigraph of an R-pushable multipartite tournament.*

Proof: Observe that sufficiency follows from Lemma 7.1, so we need only show necessity. Suppose that D is R-pushable. Select $X \subseteq V$ for which D^X contain no odd cycle. Since D^X contains no odd cycle, the strong components of D are bipartite by Theorem 7.6. Let $S_1, S_2, \dots, S_n$ be an acyclic ordering of the strong components of D . Let M be the digraph obtained from D^X by adding arcs to D^X so that $S_i \mapsto S_j$ for $i < j$. Since $S_1, \dots, S_n$ is an acyclic ordering of the strong components of D^X , it follows that M contains no odd cycle. Hence, D is a subdigraph of the R-pushable multipartite tournament M . $\square$

We note that in the case of multipartite tournaments, the property of being kernel-perfect, and the property of containing no odd cycle are identical. Hence, we have the following:

Theorem 7.12 *If M is a multipartite tournament, then M is K -pushable if and only if M contains no $B \in \bigcup_{i=1}^4 [B_i]$. $\square$*

Bibliography

- [1] J. Bang-Jensen, Y. Guo, G. Gutin, and L. Volkmann, *A classification of locally semicomplete digraphs*, Discrete Math. 167/168 (1997), 101–114, 15th British Combinatorial Conference (Stirling, 1995).
- [2] J. Bang-Jensen and G. Gutin, **Digraphs: theory, algorithms, and applications.**, Springer-Verlag, 2001.
- [3] J. Bang-Jensen and J. Huang, *Quasi-transitive digraphs.*, J. Graph Theory 20 (1995), no. 2, 141–161.
- [4] J.A. Bondy, *Short proofs of classical theorems*, J. Graph Theory 44 (2003), 159–165.
- [5] O.V. Borodin, A.V. Kostochka, and D.R. Woodall, *On kernel-perfect orientations of line graphs*, Discrete Math. 191 (1998), 45–49.
- [6] E. Boros and V. Gurvich, *Perfect graphs are kernel solvable*, Discrete Math. 159 (1996), 35–55.
- [7] E. Boros and V. Gurvich, *Perfect graphs, kernels, and cores of cooperative games*, RRR 12, Rutgers Center for Operations Research, Rutgers University, 2003.
- [8] M. Chudnovsky, N. Robertson, P.D. Seymour, and R. Thomas, *The strong perfect graph theorem*, Retrieved March 7, 2005 from <http://www.math.gatech.edu/~thomas/spgc.ps.gz>.
- [9] V. Chvátal, *On the computational complexity of finding a kernel*, Report no. CRM-300, Centre de Recherches Mathématiques, Université de Montréal, 1973.

- [10] V. Chvátal and L. Lovász, *Every directed graph has a semi-kernel*, Lecture notes in mathematics 411 (1974), 175.
- [11] P. Erdős, *On a problem in graph theory*, Math. Gaz. 43 (1963), 220–223.
- [12] D.C. Fisher and J. Ryan, *Tournament games and positive tournaments*, J. Graph Theory 19 (1995), 217–236.
- [13] F. Galvin, *The list chromatic index of a bipartite multigraph*, J. Combin. Theory Ser. B 63 (1995), 153–158.
- [14] G. Gutin, K.M. Koh, E.G. Tay, and A. Yeo, *On the number of quasi-kernels in digraphs*, Report Series 01-7, University of Aarhus, 2001.
- [15] G. Gutin, K.M. Koh, E.G. Tay, and A. Yeo, *On the number of quasi-kernels in digraphs*, J. Graph Theory 46 (2004), 48–56.
- [16] F. Harary, R.Z. Norman, and D. Cartwright, **Structural models**, John Wiley and Sons, 1965.
- [17] J. Huang, *On the structure of local tournaments*, J. Combin. Theory Ser. B 63 (1995), no. 2, 200–221.
- [18] J. Huang, G. MacGillivray, and K.L.B. Wood, *Pushing the cycles out of multipartite tournaments*, Discrete Math. 231 (2001), 279–287.
- [19] J. Huang, G. MacGillivray, and A. Yeo, *Pushing vertices in digraphs without long induced cycles*, Discrete Applied Math. 121 (2002), no. 1-3, 181–192.
- [20] H. Jacob and H. Meyniel, *About quasi-kernels in a digraph*, Discrete Math. 154 (1996), no. 1-3, 279–280.
- [21] W.F. Klostermeyer, *Pushing vertices and orienting edges*, Ars Combin. 51 (1999), 65–76.
- [22] W.F. Klostermeyer, C. Myers, and L. Šoltés, *An analogue of Camion’s theorem in squares of cycles*, Congr. Numer. 132 (1998), 205–218.
- [23] W.F. Klostermeyer and L. Šoltés, *Hamiltonicity and reversing arcs in digraphs*, J. Graph Theory 28 (1998), 13–30.

- [24] W.F. Klostermeyer and L. Šoltés, *An extremal connectivity parameter of tournaments*, Discrete Math. 233 (2000), no. 1-3, 157–173.
- [25] H. G. Landau, *On dominance realtions and the structure of animal societies. III. the condition for a score structure*, Bull. Math. Biophys. 15 (1953), 143–148.
- [26] G. MacGillivray and K.L.B. Wood, *Re-orienting tournaments by pushing vertices*, Ars Combin. 57 (2000), 33–47.
- [27] J.W. Moon, *Solution to problem 463.*, Math. Mag. 35 (1962), 189.
- [28] K. B. Reid, *Tournaments: Scores, kings, generalizations and special topics*, Congr. Numer. 115 (1996), 171–211.
- [29] M. Richardson, *Solution of irreflexive relations*, Ann. Math. 58 (1953), 573–580.
- [30] H. E. Vaughan, *On well-ordered subsets and maximal elements of ordered sets*, Pacific J. Math. 2 (1952), 407–412.
- [31] J. von Neumann and O. Morgenstern, **Theory of games and economic behaviour**, Princeton University Press, 1944,1947,1953.