IDEMPOTENTS IN COMPLEX BANACH ALGEBRAS

By

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ABSTRACT

Let A and Q be elements of a complex Banach algebra in $\mathcal{B}$ with identity I. Define $sp_Q(A)$, the spectrum of A relative to Q, as those complex numbers such that $A - zI - \bar{z}Q$ has no inverse in $\mathcal{B}$. It is proved that if J is any idempotent in $\mathcal{B}$ which commutes with A, then there exists a Q in $\mathcal{B}$ such that the spectrum of A relative to Q is disconnected and where we may integrate the Q-resolvent, $(A-zI-\bar{z}Q)^{-1}$, around a suitable contour and obtain the idempotent J. This result is also extended to finite resolutions of the identity.

1. Introduction.

The concept of the spectrum of A relative to Q, where A and Q commute and are elements in a complex Banach algebra was developed in [1]. One result from that paper was the following.

Theorem. Let C be a simple closed rectifiable curve which lies in the Q-resolvent set of A. Let

$$J = -P^{-1} \int_C (A-zI-\bar{z}Q)^{-1}(dz+Qd\bar{z}),$$

then J is an idempotent which commutes with A and Q and moreover $J = 0$ if and only if the interior of C belongs to the Q-resolvent set of A and $J = I$ if and only if the Q-spectrum of A lies entirely interior to C.

The question arises as to whether all idempotents which commute with A can be obtained using this more general concept of the spectrum. It is well known that in the case of the usual spectrum, there exist elements A with connected spectrum and where many nontrivial idempotents commute with A, but where the functional calculus of the usual spectrum cannot retrieve these idempotents using integration of the resolvent.

In this paper we prove that for every idempotent J which commutes with A there exists an element Q which commutes with A such that J can be retrieved as in the above theorem. This result is satisfying in that it demonstrates that the Q-spectra are disconnected, at least in the same abundance as there are nontrivial idempotents which commute with A.

2. Results.

Theorem 1. Let $\mathcal{B}$ be a complex Banach algebra with identity and let $A \in \mathcal{B}$ with $0 \notin \operatorname{sp} A$. Suppose J is any nontrivial idempotent in $\mathcal{B}$ which commutes with A. Then there exists $Q \in \mathcal{B}$ which commutes with A and J such that $\operatorname{sp} Q \cap T = \phi$ and

$$J = -P^{-1} \int_C (A - zI - \bar{z}Q)^{-1}(dz + Qd\bar{z})$$

for which a circle C about the origin which lies in $\operatorname{res}_Q(A)$.

Proof. Let us consider Q of the form

$Q = qJ$ for some fixed $q \in ¢$.

Then we set

$S(\lambda) = A - \lambda I - \bar{\lambda}Q = A - \lambda I - \bar{\lambda}qJ$ for $\lambda \in ¢$.

Let $\mathcal{B}^*$ be a maximal commutative subalgebra of $\mathcal{B}$ containing A and Q. Let $\Phi_{\mathcal{B}^*}$ denote the set of all algebra homomorphisms of $\mathcal{B}^*$ onto $¢$.

Then for each $\sigma \in \Phi_{\mathcal{B}^*}$ we have since $J^2 = J$ that

$\sigma(J) = 0$ or $\sigma(J) = 1$.

Now $\sigma\{S(\lambda)\} = \sigma(A) - \lambda - \bar{\lambda}q\,\sigma(J)$ and therefore $\lambda \in \operatorname{sp}_Q(A)$ if and only if $\sigma\{S(\lambda)\} = 0$ which means

$$\lambda = \frac{\sigma(A) - q\,\overline{\sigma(A)}\,\sigma(J)}{1 - |q|^2|\sigma(J)|^2} .$$

So we know by Corollary 1 [1] and the above that

$$\mathrm{sp}_Q(A) = \left\{ \lambda = \frac{\sigma(A) - q\,\overline{\sigma(A)}\,\sigma(J)}{1 - |q|^2 |\sigma(J)|^2} \;\middle|\; \sigma \in \Phi_{\mathcal{B}^*} \right\}$$

and that $\mathrm{sp}_Q(A)$ is a nonempty compact subset of $\mathbb{C}$. Note that since $\mathrm{sp}\ A = \left\{ \sigma(A) \mid \sigma \in \Phi_{\mathcal{B}^*} \right\}$ (see Rickart [2], Theorem 3.1.6) and $0 \notin \mathrm{sp}\ A$ we have $\mathrm{sp}_Q(A)$ is bounded away from 0 in $\mathbb{C}$.

We now break $\mathrm{sp}_Q(A)$ into two sets

$$S_0 = \left\{ \lambda = \frac{\sigma(A) - q\,\overline{\sigma(A)}\,\sigma(J)}{1 - |q|^2 |\sigma(J)|^2} \;\middle|\; \sigma(J) = 0 \text{ and } \sigma \in \Phi_{\mathcal{B}^*} \right\}$$

$$= \left\{ \lambda = \sigma(A) \mid \sigma(J) = 0 \text{ and } \sigma \in \Phi_{\mathcal{B}^*} \right\}$$

and

$$S_1 = \left\{ \lambda = \frac{\sigma(A) - q\,\overline{\sigma(A)}\,\sigma(J)}{1 - |q|^2 |\sigma(J)|^2} \;\middle|\; \sigma(J) = 1 \text{ and } \sigma \in \Phi_{\mathcal{B}^*} \right\}$$

$$= \left\{ \lambda = \frac{\sigma(A) - q\,\overline{\sigma(A)}}{1 - |q|^2} \;\middle|\; \sigma(J) = 1 \text{ and } \sigma \in \Phi_{\mathcal{B}^*} \right\}$$

Since J is a nontrivial idempotent in $\mathcal{B}$ it follows that S_0 and S_1 are both nonempty subsets of $\mathrm{sp}_Q(A)$, also since $\sigma(J) = 0$ or $\sigma(J) = 1$ must be the case we have that

$$\mathrm{sp}_Q(A) = S_0 \cup S_1 .$$

Now since $\sigma(A)$ is bounded away from 0 in $\mathbb{C}$ and $S_0 \subset \sigma(A)$ it is clear that if we choose q large enough and a positive real number with $q > 1$, then

$$S_1 = \left\{ \lambda = \frac{\sigma(A) - q\,\overline{\sigma(A)}}{1 - |q|^2} \;\middle|\; \sigma(J) = 1 \text{ and } \sigma \in \Phi_{\mathcal{B}^*} \right\}$$

can be made so close to zero in ¢ that

$$S_1 \cap S_0 = \emptyset$$

and such that there exists a circle C about 0 in ¢ such that $S_1 \subset C^0$ and $S_0 \subset (C \cup C^0)'$.

Having chosen such a $q > 1$, fixed, we will now prove that if we parametrize C as a simple closed rectifiable curve in a counterclockwise direction then

$$J = -P^{-1} \int_C R(z)\,d\phi(z) \equiv -P^{-1} \int_C (A-zI-\bar{z}Q)^{-1}(dz+Qd\bar{z}).$$

By Theorem 12 [1] we know that

$$K = -P^{-1} \int_C R(z)\,d\phi(z)$$

is a nontrivial idempotent in $\mathcal{B}$ which commutes with A and Q. Therefore if we write

$$K = -P^{-1} \int_C R(z)\,d\phi(z) = -P^{-1} \int_C (A - zI - \bar{z}qJ)^{-1}(dz + qJd\bar{z})$$

we have two cases to consider:

<u>Case 1</u>: Suppose $\sigma \in \Phi_{\mathcal{B}^*}$ and $\sigma(J) = 0$ then we have

$\sigma(K) = -\sigma(P^{-1}) \int_C (\sigma(A) - z)^{-1}\,dz = 0$ since $\sigma(A) \in (C \cup C^0)'$ by construction and our choice of q.

<u>Case 2</u>: Suppose $\sigma \in \Phi_{\mathcal{B}^*}$ and $\sigma(J) = 1$ then we have

$\sigma(K) = -\sigma(P^{-1}) \int_C [\sigma(A) - z - \bar{z}q]^{-1}(dz + q\,d\bar{z})$. If we let $\phi(z) = z + \bar{z}q$ and note that $\sigma(A) - z - \bar{z}q = 0$ only at $h_0 = \dfrac{\sigma(A) - q\,\overline{\sigma(A)}}{1 - |q|^2} \in C^0$ we have since $q > 1$ $\phi(h_0) = \dfrac{\sigma(A) - q\,\overline{\sigma(A)}}{1 - q^2} + \dfrac{\overline{\sigma(A)} - q\,\sigma(A)}{1 - q^2}\,q = \sigma(A)$ so we can write $\int_C [\sigma(A) - z - \bar{z}q]^{-1}(dz + q\,d\bar{z}) = \int_C \dfrac{d\phi(z)}{\phi(h_0) - \phi(z)}$ which by Theorem 7 [1] setting $f(z) = -1$ is equal to $-P = -\int_{|z|=1} [\phi(z)]^{-1}\,d\phi(z)$ which by Lemma 1 [1] is equal to $2\pi i$ since $q > 1$. Therefore $\sigma(K) = -\sigma(P^{-1})(2\pi i)$.

We recall that

$$P = \int_{|z|=1} (zI + \bar{z}Q)^{-1}(I\,dz + Q\,d\bar{z}) \quad \text{so}$$

$$\sigma(P) = \int_{|z|=1} [z + \bar{z}q\,\sigma(J)]^{-1}(I\,dz + q\,\sigma(J)\,d\bar{z}) \quad (\text{recall } \sigma(J) = 1)$$

$$= \int_{|z|=1} (z + \bar{z}q)^{-1}(dz + q\,d\bar{z}) = -2\pi i \quad \text{since } q > 1, \text{ again using}$$

Lemma 1 [1]. Therefore in Case 2 we have

$$\sigma(K) = -\frac{1}{\sigma(P)}(2\pi i) = 1.$$

By Cases 1 and 2 we have $\sigma(K) = 1$ when $\sigma(J) = 1$ and

$\sigma(K) = 0$ when $\sigma(J) = 0$.

Therefore $\sigma(J-K) = 0$ for all $\sigma \in \Phi_{\mathcal{B}^*}$ which implies $sp(J-K) = 0$ and $\lim_{n\to\infty} \|(J-k)^n\|^{\frac{1}{n}} = 0$. Now $(J-K)^2 = J - 2JK - K$ and $(J-K)^3 = J - K$ so by induction $(J-K)^{2n+1} = J - K$ for all positive integers n. But we have $\lim_{n\to\infty} \|(J-K)^{2n+1}\|^{2n+1} = 0$, which implies $\lim_{n\to\infty} \|J-K\|^{\frac{1}{2n+1}} = 0$ which means $\|J-K\| = 0$. Therefore $J = K$ and this completes the proof. □

Theorem 2. Let $\mathcal{B}$ be a complex Banach algebra with identity and let $A \in \mathcal{B}$ with $0 \in sp\ A$. Suppose J is a nontrivial idempotent in $\mathcal{B}$ which commutes with A. Then there exists a scalar β and a $Q \in \mathcal{B}$ which commutes with A and J such that $sp\ Q \cap T = \emptyset$ and

$$J = -P^{-1} \int_C (A - \beta I - zI - \bar{z}Q)^{-1}(dz+Qd\bar{z}) \quad \text{where}$$

C is a circle about the origin which lies in $res_Q(A - \beta I)$.

Proof. Choose a scalar β such that $0 \notin sp(A - \beta I)$ and then apply Theorem 1 to $A - \beta I$. □

A slight improvement on the above results is obtained in the following theorem.

Theorem 3. Let $\mathcal{B}$ be a complex Banach algebra with identity and let $A \in \mathcal{B}$ with $0 \notin sp\ A$.

Suppose $J_1 + J_2 + \dots + J_n = I$ is a finite resolution of the identity in $\mathcal{B}$, that is each J_i is a nontrivial idempotent and $J_iJ_j = 0$ for all $i \neq j$.

Further suppose that each J_i commutes with A, then there exists a $Q \in \mathcal{B}$ and a sequence of concentrix circles $C_1, C_2, \ldots, C_n$ about 0 such that

$$J_n = -P^{-1} \int_{C_n} (A - zI - \bar{z}Q)^{-1}(dz + Qd\bar{z})$$

$$J_{n-1} = -P^{-1} \int_{C_{n-1}} (A - zI - \bar{z}Q)^{-1}(dz + Qd\bar{z}) - J_n$$

$$\vdots$$

$$J_1 = -P^{-1} \int_{C_1} (A - zI - \bar{z}Q)^{-1}(dz + Qd\bar{z}) - J_n - J_{n-1} - \cdots - J_2.$$

Proof. Let $Q = q_1J_1 + q_2J_2 + \cdots + q_nJ_n$ for fixed constants $q_1, q_2, \ldots, q_n$. Then $S(\lambda) = A - \lambda I - \bar{\lambda}Q = A - \lambda I - \bar{\lambda}(q_1J_1 + \cdots + q_nJ_n)$. Suppose we take $\sigma_k \in \Phi_{\mathcal{B}^*}$ such that $\sigma_k(J_k) = 1$, $\sigma(J_i) = 0$ for $i \neq k$. Then $\sigma_k\{S(\lambda)\} = \sigma_k(A) - \lambda - \bar{\lambda}q_k\ \sigma_k(J_k) = 0$ if and only if

$$(*) \qquad \lambda = \frac{\sigma_k(A) - q_k\ \overline{\sigma_k(A)}}{1 - |q_k|^2}\ .$$

Such a σ_k exists since there exists $\sigma \in \Phi_{\mathcal{B}^*}$ with $\sigma(J_k) \neq 0$ and since J_k is idempotent this implies $\sigma(J_k) = 1$, however $1 = \sigma(I) = \sigma(J_1 + \cdots + J_n) = \sigma(J_1) + \cdots + \sigma(J_n)$ and since $\sigma(J_i) = 0$ or 1 for each i this implies $\sigma(J_k) = 1$ and $\sigma(J_i) = 0$ for $i \neq k$. Also note that for any $\sigma \in \Phi_{\mathcal{B}^*}$, σ is 1 at one and only one J_i.

Recall that

$$\left\{\sigma(A)\ \middle|\ \sigma \in \Phi_{\mathcal{B}^*}\right\} = \text{sp } A \quad \text{and}$$

since $0 \notin \text{sp } A$ and sp A is compact we have sp A is bounded away from 0.

We now look at a mapping of sets indexed by 1, 2, ..., n where

$$F_k : \text{sp } A \to ¢ \quad \text{defined by}$$

$$F_k\left(\sigma(A)\right) = \frac{\sigma(A) - q_k \overline{\sigma(A)}}{1 - |q_k|^2} .$$

We would like to be able to choose $q_1, q_2, \ldots, q_n$ constants so that $F_1(\text{sp } A), F_2(\text{sp } A), \ldots, F_n(\text{sp } A)$ lie in disjoint annuli centered at 0.

Suppose sp A is contained in the annulus about 0 with boundaries $z_1(\theta) = r_1 e^{i\theta}$ and $z_2(\theta) = r_2 e^{i\theta}$ with $0 < r_1 < r_2$ and $0 \le \theta < 2\pi$.

An analysis of F_k shows that the image of such an annulus under the map $F_k : ¢ \to ¢$ is an elliptical annulus about 0 whose boundaries are the images of the boundaries of the original annulus. Therefore

$$F_k\left(r_1 e^{i\theta}\right) = \frac{r_1 e^{i\theta} - q_k r_1 e^{i\theta}}{1 - |q_k|^2} = \frac{r_1}{1 - |q_k|^2} \left(e^{i\theta} - q_k e^{-i\theta}\right) \quad \text{and}$$

$$\left|F_k r_1(e^{i\theta})\right| = \frac{|r_1|}{|1 - |q_k|^2|} \left|e^{i\theta} - q_k e^{-i\theta}\right| \ge \frac{|r_2|}{|1 - |q_k|^2|} \left|1 - |q_k|\right|$$

$$= \frac{r_1}{1 + |q_k|} .$$

Also

$$\left|F_k\left(r_2 e^{i\theta}\right)\right| \le \frac{|r_2|}{|1 - |q_k|^2|} \left|1 + |q_k|\right| = \frac{r_2}{q_k - 1} \quad \text{for } q_k > 1.$$

We choose $q_1 = 2$ so we have $F_1(\text{sp } A)$ lies inside the annulus between

$\frac{r_1}{3} e^{i\theta}$ and $r_2 e^{i\theta}$ by the above analysis.

Now we choose $q_2 > q_1$ where q_2 is the smallest positive integer such that

$$\frac{r_2}{q_2 - 1} < \frac{r_1}{3} .$$

This will guarantee that $F_2(\text{sp } A)$ lies in an annulus which is closer to zero than the annulus which contains $F_1(\text{sp } A)$.

Continuing in this manner we choose $q_3 > q_2$ where q_3 is the smallest positive integer such that

$$\frac{r_2}{q_3 - 1} < \frac{r_1}{q_2 + 1} .$$

We continue this process n times to obtain the desired n annuli about 0. We name these annuli $A_1, A_2, \ldots, A_n$. Now we choose our circles with radii $S_1, S_2, \ldots, S_n$ where $r_2 < S_1$ is the radius of the circle C_1 about 0

$\frac{r_2}{q_2 - 1} < S_2 < \frac{r_1}{q_1 + 1}$ is the radius of the circle C_2 about 0

$\vdots$

$\frac{r_2}{q_n - 1} < S_n < \frac{r_1}{q_{n-1} + 1}$ is the radius of the circle C_n about 0.

By construction each of the annuli $A_1, A_2, \ldots, A_n$ contain elements of $\text{sp}_Q(A)$ and each element of $\text{sp}_Q(A)$ is in one of the annuli.

Therefore by Theorem 12 [1] we have that

$$J_n = -P^{-1} \int_{C_n} (A - zI - \bar{z}Q)^{-1}(dz + Qd\bar{z})$$

$$J_{n-1} = -P^{-1} \int_{C_{n+1}} (A - zI - \bar{z}Q)^{-1}(dz + Q\,d\bar{z}) - J_n$$

$$\vdots$$

$$J_1 = -P^{-1} \int_{C_1} (A - zI - \bar{z}Q)^{-1}(dz + Q\,d\bar{z}) - J_n - J_{n-1} - \cdots - J_2. \qquad \square$$

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