HARVEST DECISIONS AND ASSET VALUATION FOR BIOLOGICAL RESOURCES EXHIBITING SIZE–DEPENDENT STOCHASTIC GROWTH

by

WILLIAM J. REED AND HARRY R. CLARKE

DM–440–IR

APRIL 1987
(Revised August 1988)

HARVEST DECISIONS AND ASSET VALUATION FOR BIOLOGICAL RESOURCES EXHIBITING SIZE–DEPENDENT STOCHASTIC GROWTH

William J. Reed* and Harry R. Clarke**

*Department of Mathematics
University of Victoria
Victoria, British Columbia
Canada V8W 2Y2

**Department of Economics
La Trobe University
Bundoora, Victoria
Australia 3083

August 1988

Summary

Optimal harvest rules and pre–harvest valuation procedures are derived for biological assets with stochastic size–dependent growth in a stochastic price environment. These issues are analyzed as continuous–time optimal stopping problems. Both the "single–rotation" and "ongoing–rotations" problem are addressed. The sensitivity of harvest and valuation rules is assessed. The results are compared with those derived from a "myopic look–ahead" procedure and with the corresponding findings for age–dependent growth.

Keywords: Tree problem, Wicksell rule, Faustmann rule, optimal stopping, diffusions, myopic–look–ahead rule.

1. Introduction.

In recent years a number of papers have applied stopping–rule methods to the analysis of asset management problems in a stochastic environment: see *e.g.* Malliaris and Brock [1983], Brock, Rothschild and Stiglitz [1988], Miller and Voltaire [1983], Brock and Rothschild [1986]. For the most part this literature derives harvest rules and determines asset values for a stylized "tree cutting" problem where revenues from harvesting are a stochastic process of the diffusion type. Recently Clarke and Reed [1987] have tried to make this analysis more useful for assessing practical resource issues by specifying separately the price of a biological asset and its size. Each of these attributes is then supposed to evolve as a diffusion process with the mean proportional growth rate in size depending on the age of the asset. This enables the derivation of harvest rules and biological asset valuation procedures, which depend on asset age, and which provide stochastic generalizations of the well–known Wicksell and Faustmann "tree cutting" rules for the "single rotation" and "ongoing rotation" asset management problems respectively. The analysis presupposes that biological growth processes are "age–dependent" and that the age of biological organisms can be determined in order to implement the resulting age–based harvest rules.

While this model specification may be appropriate for husbanded biological assets such as herds of livestock or intensively cultivated stands of trees, where environmental factors play no limiting role, it is less appropriate for assets of a more wild nature, such as untended forest stands or natural populations of fish, shellfish, wildlife etc. In such cases aggregate population growth is likely to be determined less by the age structure of the population than by its size or density in relation to environmental factors such as the availability of food or soil nutriments and sunlight. In fact this is the motivation for introducing "density–dependent" growth specifications in much of the "mathematical bioeconomics" literature: see *e.g.* Clark [1976, Chapt. 1]. Since for such resources, harvesting decisions are usually made at the aggregate population level, it would seem more appropriate to use a biological growth model exhibiting size–dependent rather than age–dependent growth. Furthermore even when harvest decisions are made at the level of individual organisms, it might well be the case that the age of an organism is not an observable quantity (at least prior to harvesting), but that size is. In such cases harvest rules depending on size rather than age would be required from the operational point of view.

The present paper thus develops harvest rules for biological assets subject to stochastic density– or size–dependent growth in a stochastic price environment. It also considers the problem of valuing such assets at times prior to harvesting. The results are compared with those for the analogous age–dependent growth model, both for the "single–rotation" and "ongoing–rotations" problems. Effects of changes in parameter values on the optimal harvest rule and asset value are also considered. Section 2 below discusses the model formulation and the optimization problem to be solved. Section 3 shows that the optimal harvest rule is a barrier rule on asset size while Section 4 discusses some sensitivity properties of this rule and the associated value of the asset. In Section 5 the differences between the optimal harvest rule and the "myopic look–ahead" rule are investigated and some numerical results presented. Section 6 discusses the "ongoing rotation" problem and compares its solution to that of the "single rotation" problem. Finally Section 7 summarizes the major conclusions and implications of the analysis.

2. Biological Asset Valuation Model.

Consider a biological asset, such as a tree, a stand of trees or a pond of fish, whose biological growth is subject to a degree of random variability, and for which the unit intrinsic value (*i.e.* price) also exhibits random variability. The valuation of the asset can be described in terms of the following variables:

P(t)	the unit price of the asset at time t
q(t)	= log P(t)
X(t)	the size of the asset (*e.g.* board feet of lumber, kilograms of fish biomass) at time t
y(t)	= log X(t)
R(t)	= P(t)X(t), the aggregate intrinsic value (revenue yielded by harvesting) at time t
V(X,P,t)	the asset's market value when its size is X and the price is P at time t
δ	positive, constant, instantaneous discount rate.

Suppose the evolution of price P(t) follows *geometric Brownian motion*, governed by the stochastic differential equation

(1) $$\frac{dP}{p} = \mu\, dt + \sigma_q dw_q$$

where $\{w_q(t)\}$ is a standard Wiener process ($\{dw_q(t)\}$ is white noise), μ is a parameter representing a constant upward or downward drift in prices and σ_q is a parameter representing the magnitude of random price effects. This stochastic differential equation can be re-written as

(2) $$dq = b\, dt + \sigma_q\, dw_q$$

where $b = \mu$ if the Stratonovich calculus is used and $b = \mu - \frac{1}{2}\sigma^2$ if the Itô calculus is preferred.[1] It should be noted from the outset, that *from an operational point of view* it makes no difference which calculus is employed since, in practice, parameters will have to be estimated from data. Thus if observations $\{P_t,\ t = 1,2,\cdots,n\}$ of price are made at regular intervals, then the mean and standard deviation of the sequence of numbers $[\log(P_{t+1}/P_t),\ t = 1,2,\cdots,n-1\}$ provide maximum likelihood estimates $\hat{b}$ and $\hat{\sigma}_q$ of the parameters b and σ_q in (2), *regardless of which stochastic calculus is employed.* Moreover, as will be proved later, the optimal harvesting rule and the market value function depend on b and σ_q. Replacing these parameters by their maximum likelihood estimates will provide the maximum likelihood estimates of the optimal harvesting rule and market value function, again regardless of which calculus is employed.

It is only for analytic purposes (specifically, comparative statics with respect to the uncertainty parameter σ_q^2) that the distinction becomes important. In consequence we shall ignore the dichotomy[2] in the sequel except in Section 4 on comparative statics. Thus we consider (2) as our starting point and b and σ_q as the basic parameters.

In specifying a stochastic model for biological growth one encounters a difficulty not encountered in deterministic modelling. For example in terms of y (= log(size)) one could have:

(3) $$dy = g(t)dt + \sigma_y \, dw_y \, ,$$

or

(4) $$dy = h(y)dt + \sigma_y \, dw_y \, .$$

In model (3) *proportional* growth in size depends upon the *current age* t, along with a stochastic component, whereas in model (4)it depends on *current size*, y, plus a stochastic component. We can refer to the former case as *pure age–dependent*[3] growth and the latter as *pure size–dependent* growth. Of course hybrids of (3) and (4) in which the deterministic component of growth is a function of both size and age are possible. In deterministic models (except those which include culling or thinning (see Clark [1976, page 263]), such a dichotomy does not arise since size and age are functionally related.

The question of which model is more appropriate is difficult to resolve. Possibly for individual organisms or for husbanded biological resources such as herds of livestock or intensively cultivated stands of trees where environmental factors play no limiting role, the pure age–dependent growth model (3) would be more appropriate. On the other hand for other biological resources of a more wild nature, such as untended forest stands or natural populations of fish, shellfish, etc., where growth is subject to density–dependent constraints, the pure size–dependent growth model might be more appropriate. In this paper we concentrate on this latter case. The age–dependent case is analyzed in Clarke and Reed [1987].

It should be noted that the model (4) is equivalent to

$$dX = X\,f(X)dt + \sigma_y X\,dw_y \tag{5}$$

where $f(X) = h(\log X)$ if the Stratonovich calculus is employed and $f(X) = h(\log X) + \frac{1}{2}\sigma_y^2$ if the Itô calculus is used. As with the specification of the price process the distinction is only of importance for comparative statics, and we therefore ignore it in the sequel (except for Section 4).

We shall assume that the function $h(y)$ (and therefore $f(X)$) is decreasing representing compensatory density–dependent growth. Such models are standard in the deterministic theory of fisheries management (*e.g.* Clark [1976]) and are widely used elsewhere in population biology. The stochastic model (4) has also been used extensively in population biology, especially in the logistic (*i.e.* $f(X)$ linear) form (see *e.g.* May [1974]). An important difference between the stochastic and deterministic models is that sample paths of the former may decrease due to random effects, while in the latter they do not. This attribute of the stochastic model may be thought of as reflecting random variability in mortality and fecundity rates.

Note our model hypothesizes *both* price and quantity to follow independent diffusions. The motivation, in terms of market conditions, for this to be true is the partial equilibrium view that each harvester is a price–taker. The general equilibrium task of simultaneously determining price and quantity *processes* is a difficult problem. We conjecture that a complete treatment of this topic would involve modelling the strategic behavior of asset managers with each manager trying to "plant" so that their "trees" mature when others are young. Everyone doing this will presumably force the system to a uniform distribution of harvest times. The simpler partial equilibrium view is adequate for our purpose.

Consider now the asset valuation problem. Assuming risk–neutrality and ignoring costs, the market value $V(X,P,t)$ of the asset at time t, when current size and price are X and P respectively, is the expected present value of revenues earned given that the harvest takes place at the time that maximizes the expected present value. Specificially

$$V(X,P,t) = \sup_{\tau \geq t} E\left\{e^{-\delta(\tau-t)}P(\tau)X(\tau) \mid P(t) = P,\ X(t) = X\right\}. \tag{6}$$

Since the stochastic processes for P and X are assumed stationary:

$$V(X,P,t) = V(X,P) = \sup_{\tau \geq 0} E\left\{e^{-\delta\tau}P(\tau)X(\tau) \mid P(0) = P,\ X(0) = X\right\}. \tag{7}$$

Using y and q as state variables the value is

$$V(X,P) = W(y,q) = \sup_{\tau \geq 0} E\left\{\exp(-\delta T+q(\tau)+y(\tau)) \mid q(0) = q,\ y(0) = y\right\}. \tag{8}$$

The determination of the market value of the asset thus requires the determination of the optimal stopping rule for the maximization of the expectation on the right hand side of (8).

3. The Optimal Harvesting Rule.

A *stopping rule* partitions the y–q (or X–P) space into two regions — a *stopping region*, and a *continuation region.* The process is stopped (asset harvested) when $\{y(t), q(t)\}$ leaves the continuation region for the first time. Using a stopping rule S, the corresponding value of the asset is

$$(9) \qquad W^S(y,q) = E\left\{\exp(-\delta\tau_S+q(\tau_S)+y(\tau_S)) \mid q(0) = q,\ y(0) = y\right\}$$

where τ_S is a random variable denoting the time at which the process is stopped (first hits boundary of stopping region). The expectation in (9) is taken with respect to τ_S, $q(\tau_S)$ and $y(\tau_S)$. It is easy to show, assuming zero correlation between disturbances, that on the continuation region, W^S satisfies the Hamilton–Jacobi–Bellman (H–J–B) equation[4]

$$(10) \qquad \delta W^S(y,q) = h(y)W^S_y + bW^S_q + \tfrac{1}{2}\sigma_y^2 W^S_{yy} + \tfrac{1}{2}\sigma_q^2 W^S_{qq}$$

(where subscripts denote partial derivatives) and that at the boundary of the stopping region

$$(11) \qquad W^S(y,q) = e^{y+q}$$

(*i.e.* that value equals intrinsic value on the stopping boundary). In addition, if the stopping rule S is optimal (*i.e.* if $W^S(y,q) \geq W^R(y,q)$ for all stopping rules R), W^S satisfies the so–called "*smooth–pasting*" conditions

$$(12) \qquad W^S_y(y,q) = \frac{\partial}{\partial y}\, e^{y+q} = e^{y+q}$$

(13) $$W^S_q(y,q) = \frac{\partial}{\partial q}\, e^{y+q} = e^{y+q}$$

at the boundary of the stopping region. It can be shown (see Krylov [1980, pp. 35–42]; Shiryayev [1978, pp. 157–162]) that conditions (10), (11), (12) and (13) along with the condition that $W^S(y,q) > e^{y+q}$ on the continuation region, determine an optimal stopping rule provided certain regularity conditions are satisfied, and appropriate initial conditions for W^S specified. Thus the optimal stopping problem can be solved as the *free–boundary problem* given by the partial differential equation (10) with appropriate initial conditions, and the free–boundary conditions (11), (12) and (13).[5]

We now show that there is a particularly simple solution to the above free–boundary problem; specifically that the optimal rule is a *barrier rule* of the form: stop whenever y exceeds some barrier value $\bar{y}$, *i.e.* harvest as soon as the asset size X reaches some target size $\bar{X}$.

Let $\bar{W}$ be the value function corresponding to the barrier $\bar{y}$. Then

(14) $$\bar{W}(y,q) = E\left\{\exp(-\delta T_{y,\bar{y}} + q(T_{y,\bar{y}}) + \bar{y}) \mid q(0) = q,\ y(0) = y\right\}$$

where $T_{y,\bar{y}}$ is the first passage time for the $\{y(t)\}$ process to reach $\bar{y}$ from an initial state y. Conditioning on $T_{y,\bar{y}}$, and using the solution to (1) for q (see Clarke and Reed [1987]), we have

(15) $$\bar{W}(y,q) = e^{q+\bar{y}}\, E(e^{-BT_{y,\bar{y}}})$$

where $B = \delta - b - \frac{1}{2}\sigma_q^2$, and the expectation is taken with respect to $T_{y,\bar{y}}$. We write

(16) $$M(y,\bar{y};B) = E(e^{-BT_{y,\bar{y}}}).$$

Regarded as a function of B, M is the moment generating function (m.g.f.) or Laplace transform of the first–passage–time random variable $T_{y,\bar{y}}$. By conditioning on the state an infinitesimal time dt after the initial time one can show[6] that M satisfies the equation

(17) $$\tfrac{1}{2}\sigma_y^2 M_{yy} + h(y)M_y - BM = 0$$

on $(-\infty, \bar{y})$. Using (17) one can verify directly that the value function $\bar{W}$ satisfies the H–J–B equation (10) on $(-\infty, \bar{y})$. Furthermore, since $M(\bar{y},\bar{y};B) = 1$, $\bar{W}$ satisfies the continuity condition (11) at $y = \bar{y}$ and also the second of the smooth–pasting conditions (13) at $y = \bar{y}$. If the other smooth–pasting condition is to hold we require

$e^{q+\bar{y}} M_y(\bar{y},\bar{y};B) = e^{q+\bar{y}}$, *i.e.*

(18) $$M_y(\bar{y},\bar{y};B) = 1.$$

This condition determines the magnitude of the optimal size barrier $\bar{y}$. Note that we still need to show that $\bar{W}(y,q) > e^{y+q}$ on $(-\infty, \bar{y})$ before we can conclude that the barrier rule at $\bar{y}$ is optimal.

In order to solve (18) we shall express M in terms of a new function. For the interval $[y, \bar{y}]$ consider a partition

$$y = y_0 < y_1 < \cdots < y_n = \bar{y}\ .$$

The random variable $T_{y,\bar{y}} = T_{y_0,y_1} + T_{y_1,y_2} + \cdots + T_{y_{n-1},y_n}$

where $T_{y_i,y_{i+1}}$ is the first–passage time from y_i to y_{i+1} $(i = 0,\cdots,n-1)$. Since the random variables on the r.h.s. of (19) are independent we can write

$$M(y,\bar{y};B) = E\left[\exp\left[-B \sum_{i=0}^{n-1} T_{y_i,y_{i+1}}\right]\right] \tag{20}$$

$$= \prod_{i=0}^{n-1} E\left[\exp\left[-BT_{y_i,y_{i+1}}\right]\right]$$

$$= \prod_{i=0}^{n-1} M(y_{i-1},y_i;B)$$

and

$$\log M(y,\bar{y};B) = \sum_{i=0}^{n-1} \log M(y_{i-1},y_i;B). \tag{21}$$

We now let $n \longrightarrow \infty$ in such a way that the mesh of the partition of $[y, \bar{y}] \longrightarrow 0$. Then the sum on the r.h.s. on (21) tends to an integral. Specifically

$$\log M(y,\bar{y};B) = -\int_y^{\bar{y}} \varphi(s)ds \tag{22}$$

where

$$\varphi(y) = - \lim_{\Delta \to 0} [\log(M(y,y+\Delta;B))/\Delta] , \tag{23}$$

i.e.

(24) $$M(y,\bar{y};B) = \exp\left\{-\int_{y}^{\bar{y}} \varphi(s)ds\right\}.$$

The function $\varphi(y)$ represents the negative of the derivative (with respect to y) of the cumulant generating function of the first passage time $T_{y,\bar{y}}$. From (24) it follows directly that

(25) $$\varphi(y) = M_y/M.$$

From this it follows that (17) can be re–expressed as

(26) $$\varphi'(y) = -\varphi^2(y) - \frac{2}{\sigma_y^2}[h(y)\varphi(y)-B]$$

which is a Ricatti equation.[7]

The condition (18) for the optimal barrier $\bar{y}$ can be expressed in terms of φ as

(27) $$\varphi(\bar{y}) = 1.$$

Thus $\bar{y}$ can be determined by integrating (26) in a forward sense until (27) is met.[8] To perform this, of course, we require an initial condition for φ. One approach is to consider absorbing or reflecting barriers for the X (and therefore y) processes. This is unnecessary in our case because the state $X = 0$ $(y = -\infty)$ is a *natural boundary* (Karlin and Taylor [1981, p. 235]; Gihman and Skorohod [1972, p. 165]), which the process $\{X(t)\}$ (or $\{y(t)\}$) can neither reach, nor escape from, in finite mean time. This implies that the m.g.f. $M(y,\bar{y};B) \to 0$ as $y \to -\infty$. However $\varphi(y)$ tends to a finite limit as $y \to -\infty$. It is shown in Appendix 1, that if $f(X)$ (in (5)) is assumed decreasing, with $f(0) < \infty$, (*i.e.* if $h(y)$ is decreasing with $h(-\infty) < \infty$)), then

(28) $$\varphi(y) \to C(f(0)) \text{ as } y \to -\infty$$

where

(29) $$C(z) = \frac{1}{\sigma_y^2}\left[\sqrt{z^2 + 2B\sigma_y^2} - z\right].$$

To summarize: we have now shown that the value function corresponding to the barrier rule, stop as soon as $y = \bar{y}$, (where $\bar{y}$ is determined by solving the Riccati equation (26) forwards from $\varphi(-\infty) = C(f(0))$ until the condition $\varphi(y) = 1$ is met), satisfies the H–J–B equation (10), the continuity condition (11) and the smooth–pasting conditions (12) and (13). Since the regularity conditions for the intrinsic value function e^{y+q} are also met, it only remains to show that $\bar{W}(y,q) > e^{y+q}$ on $(0, \bar{y})$ in order to conclude that this size–barrier rule is optimal. From (15) and (22) we have

(30) $$\bar{W}(y,q) = e^{q+\bar{y}} \exp\left\{-\int_y^{\bar{y}} \varphi(s)ds\right\}$$

$$= e^{q+y} \exp\left\{\int_y^{\bar{y}} [1 - \varphi(s)]ds\right\}$$

and thus that $\bar{W}(y,q) > e^{q+y}$ for $y < \bar{y}$, since for such a y, $\varphi(y) < 1$. We thus conclude that the barrier rule, stop as soon as $y = \bar{y}$ is optimal.

Complete determination of the optimal size barrier $\bar{y}$ requires numerical solution of the Riccati equation. To accomplish this it is convenient to switch back from the y–scale to the X–scale and let

$$\psi(X) = \varphi(y) = \varphi(\log X). \tag{31}$$

Then,

$$\varphi'(y) = X\psi'(X) \tag{32}$$

and (26) can be expressed as

$$\psi'(X) = \frac{1}{X}\left[-\psi^2(X) - \frac{2}{\sigma_y^2} f(X)\psi(X) + \frac{2B}{\sigma_y^2}\right] \tag{33}$$

with initial condition

$$\psi(0) = C(f(0)). \tag{34}$$

The optimal size barrier $\bar{X}$ is obtained by integrating (33) forwards until $\psi(X) = 1$. Numerical results are given in Section 5 by studying the behavior of solutions to the Riccati equation (26).

4. Comparative Statics.

It has been shown that the *optimal cutting rule depends only on the size of the asset and not on price.* Also it can be seen from (30) that *the market value of the asset is directly proportional to current price.* At first sight these results seem counter–intuitive. However it can be seen to be a consequence of specifying the price process as geometric Brownian motion (for which the expected proportional growth rate is unchanging) and of ignoring harvesting costs. As Brock *et al.* [1988] point out there is no optimal stopping rule for a geometric Brownian motion process, by itself (*i.e.* with no fixed stopping costs). Thus it is perhaps not surprising that, in a 2–dimensional situation in which one of the variables follows geometric Brownian motion, that that variable should not enter into the optimal stopping rule. We speculate that such a simple solution would not prevail for other price processes, nor would it prevail if costs were included. However since geometric Brownian motion appears to be the natural specification for the price process and since in many cases harvest costs will be small relative to revenue earned, the model appears to be a very plausible one.

The economic parameters δ, b and σ_q affect the optimal cutting size and value function only through $B = \delta - b - \frac{1}{2}\sigma_q^2$, which can be thought of as a discount rate adjusted for expected capital gains *i.e.* for expected growth in price. To investigate the effects of changes in B we firstly observe that $\varphi(y)$ is an increasing function provided h(y) (or f(X)) is decreasing. This follows from the fact that the first passage time, $T_{y,y+\Delta}$ will increase probabilistically as y increases and the mean drift h(y) decreases. Thus $\log[E\{\exp(-BT_{y,y+\Delta}\}]$ is decreasing. The fact that $\varphi(y)$ increases follows directly from its definition (23).

An explicit expression for the market value function is given in (30). In terms of the price and size vaiables P and X we have from (30) that

$$\log[V(X,P)/XP] = \int_{y}^{\bar{y}} [1-\varphi(s)]ds. \tag{35}$$

The shaded area in Figure 1(a) shows this for a given value of $y = \log X$. The optimal cutting size $\bar{y} = \log \bar{X}$ is also shown. Now it follows from (10) that $\partial M/\partial B < 0$ and thus from the definition of φ, (23), that $\partial\varphi(y)/\partial B > 0$ for every y. Thus an increase in B results in a uniform upward shift in $\varphi(y)$, and in consequence a decrease in the optimal barrier $\bar{y}$, and a decrease in $\log[V(X,P)/XP]$ (see Figure 1(b)). Thus *both the optimal cutting barrier* $\bar{X}$ *and the asset value* $V(X,P)$, *increase with an increase in the mean drift in price* μ *and decrease with an increase in the discount rate* δ.

A similar result holds with respect to any changes in the expected biological growth rate, *i.e. for any increase in the mean biological growth rate function*, $f(X)$, *both the optimal size barrier* $\bar{X}$ *and the asset value* $V(X,P)$ *increase.*

To see this consider a change from $f_{OLD}(X)$ to $f_{NEW}(X) = f_{OLD}(X)[1+\eta(X)]$ with $\eta(X) \geq 0$ for every X. There will be a corresponding increase in the function $h(y)$. Furthermore the trajectory for $\varphi(y)$ corresponding to $h_{NEW}(y)$ will start at the same place or lower from that for $h_{OLD}(y)$, and can never subsequently cross it.

We consider now the comparative statics with respect to the uncertainty parameters σ_q^2 and σ_y^2. Whether or not the parameter B depends on σ_q^2, depends on whether the geometric Brownian motion (1) is interpreted in the Itô or Stratonovich sense. If the former prevails then $B = \delta - \mu$ whereas if the latter holds then $B = \delta - \mu - \frac{1}{2}\sigma^2$. Thus using the Itô calculus yields the result that the optimal size barrier $\bar{X}$ and asset value $V(X,P)$ are independent of changes in price uncertainty, while using the Stratonovich calculus yields the different result that both quantities increase with increased price uncertainty. Thus to resolve the question of the effects of price uncertainty, one must resolve the modelling question as to which of the Itô or Stratonovich forms of geometric

Brownian motion best models the price of the commodity. Such a question is not easy to resolve and we make no attempt to do so here. We refer readers however to a discussion of this issue in Clarke and Reed [1987].

A similar difficulty arises with respect to the effects of the biological growth uncertainty parameter σ_y^2. With the Stratonovich interpretation for which h(y) is independent of σ_y^2 one can show (see Appendix 2) that both the optimal size barrier and asset value increase with increases in σ_y^2. However under the Itô interpretation the effects are ambiguous.[9]

The fact that the consequences of changes in uncertainty are contingent upon the form of stochastic calculus employed, is in many ways unsatisfactory. It is a difficulty already encountered in other areas where stochastic models are employed (see Mortensen [1969] and Turelli [1977] for a discussion of the problem in physics and biology respectively), and is ultimately a modelling problem. We remind readers however that from an operational point of view the choice of stochastic calculus is irrelevant since maximum likelihood estimates of the optimal size barrier and value function do not depend on the form of stochastic calculus employed.

5. <u>The Myopic–Look–Ahead Rule</u>.

A plausible harvest rule is the *myopic–look–ahead* (MLA) rule discussed by Ross [1970, p. 188]. The continuation region for this rule is the set of states for which the expected discounted intrinsic value of the asset an infinitesimal time ahead of the current time, exceeds the current intrinsic value. For continuous stochastic processes the MLA stopping boundary is the set of states for which these values are equal. Denoting the stopping time for this rule by τ we get

(36)
$$R(\tau) = E_{\tau}\left[e^{-\delta d\tau} R(\tau+d\tau)\right]$$

where the τ subscript in the expectation operator indicates that it is taken conditional on the values of the state variables at time τ. For the model of this paper this is:

(37)
$$\exp(y(\tau)+q(\tau)) = E_{\tau}[\exp(-\delta d\tau+y(\tau)+dy+q(\tau)+dq)].$$

Expanding the r.h.s. and using (2), (4) and the fact that $E(dw^2) = dt$ for a standard Wiener process, gives the following MLA stopping boundary:

(38)
$$h(y) = \delta - b - \tfrac{1}{2}\sigma_q^2 - \tfrac{1}{2}\sigma_y^2 = B - \tfrac{1}{2}\sigma_y^2 .$$

This is a barrier rule: stop whenever size y (or X) reaches a barrier size y_M (or X_M) given by the solution to (38) (or the solution to $f(X) = B - \tfrac{1}{2}\sigma_y^2$).

It is easily shown (see Clarke and Reed [1987]) using Itô's lemma that at a stopping time τ_M determined by this rule

(39)
$$E_{\tau_M}\{dR\} = \delta R(\tau_M)d\tau_M.$$

Thus, in this case, the MLA rule is a stochastic analogue of the Wicksellian "tree cutting" rule — harvest when the *expected* proportional rate of increase in the "instrinsic" value equals the discount rate.

In the case of age–dependent growth (Clarke and Reed [1987]) the MLA rule is a barrier rule on age, and is the optimal harvest rule. In the case of size–dependent growth the MLA rule, while it has the same form as the optimal stopping rule (a barrier rule on size), is in general *not* optimal, *i.e.* $y_M \neq \bar{y}$.

For the MLA rule the value function is (*cf.* (15) and (30))

(40) $$W^M(y,q) = e^{q+y_M}\, E\left[\exp\left[-BT_{y,y_M}\right]\right]$$

(41) $$= e^{q+y} \exp\left\{\int_y^{y_M} [1-\varphi(s)]ds\right\}.$$

Thus the degree of suboptimality (efficiency) of the MLA rule can be expressed as

(42) $$e \triangleq \frac{W^M(y,q)}{W(y,q)} = \exp\left\{-\int_{y_M}^{\bar{y}} [1-\varphi(s)]ds\right\}.$$

The fact that the r.h.s. of the above does not depend on y or q implies that *use of the sub–optimal* MLA *rule causes the same fixed proportional reduction*[10] *in value regardless of the current price and size.*

Precise determination of the efficiency, e, of the MLA rule, requires numerical solution of the Riccati equation and numerical integration to evaluate (42). This has been performed using NAG library (Anonymous [1984]) FORTRAN subroutines for two logistic growth models of the form $f(X) = \alpha - \beta X$. The results are reported in Table 1. For both cases parameter values

$$\delta = 0.03 \qquad b = 0 \qquad \sigma_q^2 = .01$$

were used. The logistic parameters used were

Case 1: $\alpha = .05$, $\beta = .01$ ("slow growth")

Case 2: $\alpha = .10$, $\beta = .02$ ("fast growth").

For both cases the asymptotic stationary distribution for the size of the asset has expected value 5 units. The coefficient of variation of this steady–state distribution is $(10\sigma_y^2)^{\frac{1}{2}}$ for "slow growth" and $(5\sigma_y^2)^{\frac{1}{2}}$ for "fast growth". Thus the respective values of the coefficient of variation are 0.10 and .0707 when $\sigma_y^2 = .001$, and .3126 and .2236 when $\sigma_y^2 = .01$. This is probably a realistic range of values for σ_y^2. To check the validity of the time scale, the passage time with no growth variation $(\sigma_y^2 = 0)$ from 25% of asymptotic size to 75% of asymptotic size was evaluated. For "slow growth" it is 43.9 years and for "fast growth" 21.97 years, which seems a realistic time scale for forestry problems.

It can be seen from Table 1 that the MLA harvest size X_M is somewhat less than the optimal size $\bar{X}$ (as indeed it must be — see footnote (10)), but only significantly so in the case where growth variance, σ_y^2 is high. More importantly the efficiency of the MLA rule is very high. Indeed in all cases the reduction in value through using the sub–optimal MLA rule is less than two percent, and in most cases less than one tenth of one percent. Thus for practical purposes, at least in forestry applications, it would seem that use of the MLA rule is as good as the optimal rule. This is useful information since the MLA rule is straightforward to compute.

Finally we may ask why the MLA rule is optimal in the case of age–dependent growth, but not optimal in the case of size–dependent growth. The answer lies in the fact that the age–barrier stopping set in the former case is *closed* while the size–barrier stopping set in the latter case is not closed. We use "closed" in the sense of Ross [1970, p. 188] to indicate that once the process has entered the stopping set it can never again leave it. Ross [1970, Theorem 9.31, p. 188] states that MLA stopping rules for which the stopping set is closed are optimal.

To see why the MLA rule cannot be optimal in the size–dependent growth case, consider a small perturbation in the optimal size barrier $\bar{y}$. The first–order condition for a maximum requires that the smooth–pasting condition (12) must hold. The effect on the value function W evaluated at the barrier $\bar{y}$, when the barrier is increase to $\bar{y} + d\bar{y}$, will depend not only on the mean growth function h at $y = \bar{y}$, but also on its behaviour *globally* on $(-\infty, \bar{y})$. This is because there is a non–zero probability that the process will drift away from the perturbed barrier. It may assume all values in the interval $(-\infty, \bar{y})$ before reaching the perturbed barrier, $y + d\bar{y}$. Thus the smooth–pasting condition (14) and the optimal barrier $\bar{y}$ will depend on $h(y)$ globally. The MLA barrier depends only on the *local* behaviour of h at $y = y_m$, and so cannot in general be optimal.

It should be noted that the above reasoning depends critically on the fact that size can decrease as well as increase. This is a consequence of modelling biological growth as a diffusion. For a discrete–time model, exhibiting monotonic growth with probability one, an MLA rule is indeed optimal. However as has been pointed out by Brock *et al.* [1988] the use of discrete–time models can lead to problems of "overshooting", which can cause ambiguity in the results.

6. The Ongoing Rotations Problem.

It has long been recognized in the forestry literature that the problem of determining the optimal cutting age for a tree or stand of trees involves the consideration of future forest 'rotations'. Once a stand is cut the land becomes available for new forest growth; the longer the cutting of the stand is delayed, the longer one must wait for revenues from future harvests.

In a deterministic analysis, when ongoing rotations are considered, the Wicksell formula $R'(T)/R(T) = \delta$ for the optimal cutting age is modified to the well–known Faustmann formula (Faustmann [1849], Samuelson [1976]) $R'(T)/R(T) = \delta/(1-e^{-\delta T})$. Since $\delta > 0$ the result is a lowering of the optimal cutting age. In the forestry literature the net present value of the revenues earned from all future harvests from a newly established site, using the optimal rotation (or cutting) age, is known as the *land* *expectation value*, and for a deterministic model is given by

$$R(T)/(1-e^{-\delta T})$$

where T is the optimal rotation age determined by solving the Faustmann equation. For the stochastic model of this paper, the land expectation value will depend on the initial price P_0. Specificially it is

$$\text{(43)} \qquad L(P_0) \triangleq \sup E\left\{ \sum_{i=1}^{\infty} e^{-\delta T_i} P(T_i)\hat{X}(T_i) \mid P(0) = P_0, X(0) = X_0 \right\}$$

where the supremum is taken over all sequences of cutting times $\{T_i\}_{i=1,2,\cdots}$; $\hat{X}_i = X(T_i - T_{i-1})$ is the size of a stand at absolute time T_i, and X_0 is the initial size of a newly planted stand.[11] It is shown in Clarke and Reed [1987] that the land expectation value is proportional to the initial price P_0 *i.e.*

$$\text{(44)} \qquad L(P_0) = \theta P_0$$

where θ is a constant of proportionality. It follows from (43) and (44) that

$$\text{(45)} \qquad L(P_0) = \theta P_0 = E\left\{ e^{-\delta\tau^*} P(\tau^*)\left[X(\tau^*) + \theta\right] \;\middle|\; P(0) = P_0 \right\}$$

where τ^* a random variable denoting the time (the stopping time) at which the process first enters the optimal stopping set.

To determine the optimal harvest rule we consider *for a fixed value of* θ the problem of determining a harvest rule to maximize

$$E\left\{e^{-\delta\tau}P(\tau)[X(\tau)+\theta] \mid P(0) = P_0\right\}$$

i.e.

(46)
$$E\left\{e^{-\delta\tau}e^{q(\tau)}\left[e^{y(\tau)}+\theta\right] \,\middle|\, q(0) = q_0\right\}.$$

Denote the value function for this rule by $V^\theta(X,P)$ (or by $W^\theta(y,q)$). To determine θ we have to solve (numerically) for θ the equation

(47)
$$\theta P_0 = V^\theta(X_0,P_0) ,$$

since, with the correct value of θ, both sides represent the land expectation value (*i.e.* equation (45)).

In order to be able to carry this out we need to be able to solve, for any value of θ, the single–rotation optimal harvest problem with intrinsic value function

(48)
$$R(X,P) = (X+\theta)P = e^q(e^y+\theta).$$

For a stopping rule which is a barrier rule on size (*i.e.* stop when $y = \tilde{y}$), the value function is (*cf.* (15) and (30))

(49)
$$W^\theta(y,q) = e^q(e^{\tilde{y}}+\theta)\, M(y,\tilde{y};B)$$

$$= e^q(e^{\tilde{y}}+\theta)\, \exp\left\{-\int_y^{\tilde{y}} \varphi(s)ds\right\}$$

where φ as before is the solution to the Riccati equation (26) with initial condition (28). As in Section 3, it can be shown that this value function satisfies the H–J–B equation (10), the continuity condition at the boundary (*cf.* (11)),

$$W^{\theta}(\tilde{y},q) = R = e^{q}(e^{\tilde{y}}+\theta) \tag{50}$$

and the smooth–pasting condition (*cf.* (15))

$$W^{\theta}(\tilde{y},q) = R_q = e^{q}(e^{\tilde{y}}+\theta). \tag{51}$$

In order to satisfy the other smooth–pasting condition (*cf.* (14))

$$W^{\theta}(\tilde{y},q) = R_y = e^{q} \cdot e^{\tilde{y}} \tag{52}$$

we require

$$\varphi(\tilde{y})e^{q}(e^{\tilde{y}}+\theta) = e^{q} \cdot e^{\tilde{y}} \tag{53}$$

i.e.

$$\varphi(\tilde{y}) = e^{\tilde{y}}/(e^{\tilde{y}}+\theta). \tag{54}$$

For a given value of θ the Riccati equation (26) with initial condition (28) can be integrated in a forward sense until the condition (54) is met. The barrier rule, stop at $y = \tilde{y}$, will be optimal.[12] For numerical solution it is easier to switch back to the

X–variable (transformation (31)) and solve (33) with initial condition (34) until the condition

(55) $$\psi(X) = X/(X+\theta)$$

is met.

Before presenting numerical results we observe that the r.h.s. of (55) is less than 1. Thus since φ is increasing the condition (54) will be met for a smaller value of y than the condition $\varphi(y) = 1$, which characterizes the optimal boundary for the single rotation. Thus *the optimal cutting size in the case of ongoing rotations is smaller than that in the single rotation case.* This is a stochastic generalization of the deterministic result that the Faustmann cutting size is smaller than the Wicksellian one.

In the case of age–dependent growth (Clarke and Reed [1987]) the MLA rule involves cutting initially slower growing trees at a younger age than faster growing ones; if a stand of trees grows slowly initially, then it pays the resource owner to abandon that rotation and replace the stand with a new one with greater growth potential. In the case of size–dependent growth this no longer happens. In fact *a slow growing stand is harvested optimally at a later age than a fast growing one.* This is illustrated in Figure 2. In this case it pays to cut initially fast growing stands at an early age so that they can be replaced by smaller stands with greater growth potential. Slow growing stands on the other hand, since size and not age is the determinant of growth, still maintain high growth potential.

We turn now to numerical results. Using (50) and (32), equation (47) which determines θ can be expressed

(56) $$\theta = (\tilde{X}_\theta + \theta)\ \exp\left\{-\int_{X_0}^{\tilde{X}_\theta} \frac{\psi(S)}{S}\, dS\right\}$$

where $\tilde{X}_\theta$ is the value of X for which (55) holds. Equation (56) was solved numerically using the Secant method (see *e.g.* Press *et al.* [1986, p. 248]). Determination of $\tilde{X}_\theta$, for a given θ, involved integrating the Riccati equation (33). Only Case 1 (the "slow growth" case) discussed in Section 5 was considered using values of the growth variance $\sigma_y^2 = .01$, .001 and 0. Other parameter values were as given in Section 5. The initial size of a newly planted stand of trees was set at $X_0 = 0.1$ (= 2% of mean asymptotic size). The results are shown in Table 2. As in the single rotation case, both the optimal size barrier $\tilde{X}$ and the MLA size barrier $\tilde{X}_M$ increase with increasing growth variance. The difference between them is slightly less than in the single rotations problem. The suboptimality of the MLA rule is even less in the ongoing–rotations case than in the single–rotations case, and is negligible. Finally, the last column of Table 2 indicates how the land expectation value increases with the growth variance.[13]

7. Conclusions and Final Remarks.

This paper derives harvest rules for biological assets displaying size–dependent growth under conditions of price and growth uncertainty. The analysis has been necessarily somewhat involved and it is useful to now summarize the major conclusions in a non–technical way so that the important issues raised for resource managers can be clearly understood.

Over a single rotation a barrier rule on size is shown to be optimal. The size of this barrier can be computed numerically by solving a Riccati equation from a given initial condition until a predetermined endpoint condition is satisfied. A rather surprising fact is that the optimal harvesting rule does not depend on price. This is a consequence of specifying the price process as geometric Brownian motion and of ignoring costs. While using geometric Brownian motion to model the price process seems natural and is indeed customary in many areas of economics the procedure of ignoring harvesting costs may be more questionable. However if costs are small relative to the total revenue it seems reasonable to assume that the barrier rule on size provides a good approximation to the optimal harvesting rule.

The optimal harvest rule derived for the case of size–dependent growth differs in an important way from that for age–dependent growth. With age–dependent growth the optimal rule is a barrier rule on age (Clarke and Reed, [1987]), and as such is an "open loop" rule — *i.e.* the timing of the harvest can be determined in advance and is independent of the particular sample paths followed by the price and size processes. For size–dependent growth, in contrast, the optimal harvest rule is a "closed–loop" or feedback rule — at each point in time the decision as to whether to harvest or not is made on the basis of the current size (although not unit price) of the asset. As such the timing of the harvest depends on the particular sample path followed by the size process.

A further distinction between the two specifications follows from the above fact. In the case of age–dependent growth, with an open–loop harvest rule, the effects of uncertainty can be handled through the use of an "uncertainty adjusted" discount rate *i.e.* one can solve the timing and valuation problems in the stochastic context, by using deterministic (Wicksell) methods, but with the discount rate adjusted downward to account for price and growth uncertainty. This is not the case in general when growth is size–dependent.

Another distinction arises from the fact that the "myopic–look–ahead" approach (the MLA rule) provides the optimal solution in the case of age–dependent growth, but does not do so for size–dependent growth. However the numerical examples given in the paper suggest that the MLA approach would provide a good approximation to the optimal procedure in many practical resource management problems involving size–dependent growth. This is useful to know, since the MLA procedure is much simpler computationally. Furthermore it suggests that in situations involving both size and age dependency in the growth process, for which computation of the optimal rule would be extremely difficult, a good approximate solution might be obtained through use of the MLA procedure. The same implication applies for cases in which harvest costs are significant.

The comparative static properties of the optimal size barrier and the market value of an asset exhibiting size–dependent growth, can be determined analytically. Both the size barrier and the market value increase with increases in the mean growth rate of prices or size. However effects of changes in price or growth uncertainty depend on how the stochastic differential equation models for price and biological growth are interpreted. These results are qualitatively consistent with those for age–dependent growth (Clarke and Reed, [1987]).

With "ongoing rotations" a barrier on size again provides the optimal harvest rule and this can also be computed numerically by solving a Riccati equation. This barrier is smaller than that for the "single rotation" case, a result which generalizes the well–known deterministic comparison between cutting rules of the Faustmann and Wicksell types respectively. This result is also consistent with the corresponding age–dependent finding of Clarke and Reed [1987]. In this latter analysis however solutions for the case of "ongoing rotations", when both prices and growth are stochastic, could only be determined approximately by using an MLA rule as a heuristic approximation to the optimal harvest rule. This heuristic prescribed a harvest rule dependent on both age and size. The rule indicated that initially slow–growing assets should be harvested early to make way for assets with better growth potential in subsequent rotations. The result is reversed with a size–dependent growth specification since, in this case, initially slow–growing assets will be harvested at the same fixed optimal size as faster growing assets, and hence will be harvested later than them.

The general upshot of the present analysis and the work of Clarke and Reed [1987] is that, for practical resource management decision–making in a stochastic environment, the choice between age and size–dependent growth specifications has an important impact on the form of optimal management rules. While in reality probably both age and size play a role in determining proportional growth rates for biological assets, the empirical difficulties associated with model specification will likely preclude the use, in practice, of models which reflect this fact. The age–dependent and size–dependent growth specifications can be thought of as the two extremes of a continuum. The choice between them in a particular modelling exercise will depend in part on biological knowledge about the growth process, but also and perhaps to a larger extent, on operational considerations concerning the observability of the two variables. If, for example, size is costly or impossible to estimate, but age is readily known, then the age–dependent specification might be chosen. On the other hand, if age is difficult to determine, but size is readily known then a size–dependent specification would appear more attractive.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge the comments of Professor W.A. Brock and the contributions of Professors R. Lamberson, R. McKelvey and P. van den Driessche voiced in discussions of this work. Support for the first author was provided by National Sciences and Engineering Research Council of Canada (Grant no. A–7252).

APPENDIX 1

An Initial Condition for the Riccati Equation

It is assumed that the mean growth rate function $f(X)$ in (6) is decreasing with $f(0) < \infty$. This corresponds to compensatory growth (see *e.g.* Clark [1976, p. 11]) and implies that $X = 0$ is a natural boundary. In terms of $h(y) = f(e^y)$, it implies that h is decreasing with $h(-\infty) = f(0) < \infty$.

Consider now some finite value y_a of y. For $y < y_a$,

(A1.1) $$f(0) = h(-\infty) > h(y) > h(y_a) = \mu_a \,,$$

say; and in consequence

(A1.2) $$E\left[\exp\left[-BT^{a}_{y,y+\Delta}\right]\right] > M(y,y+\Delta;B) > \left[\exp - \left[-BT^{0}_{y,y+\Delta}\right]\right]$$

where $T^{a}_{y,y+\Delta}$ and $T^{0}_{y,y+\Delta}$ are first passage times for processes with *constant* mean drift μ_a and $f(0)$ respectively.

From the well–known results (see *e.g.* Ross [1983, p. 203]) on Brownian motion with constant drift we have

$$E\left[\exp\left[-BT^{a}_{y,y+\Delta}\right]\right] = \exp(-C(\mu_a)\Delta)$$

and

$$E\left[\exp\left[-BT^{0}_{y,y+\Delta}\right]\right] = \exp(-C(f(0))\Delta)$$

where the function C is defined in (31). Thus we have from (A2.2)

$$-C(\mu_a) > \frac{\log[M(y\,,\,y+\Delta;B)]}{\Delta} > -C(f(0)),$$

which from (25) implies

$$C(\mu_a) \le \varphi(y) \le C(f(0)),$$

for $y < y_a$. Now as $y_a \to -\infty$, $\mu_a \to f(0)$ and therefore $\varphi(y) \to C(f(0))$.

APPENDIX 2

Effects of Changes in Growth Variance σ_y^2

To determine the optimal cutting barrier $\bar{y}$ one has to integrate the Riccati equation (26), *i.e.*

$$\text{(A2.1)} \qquad \sigma_y^2\,\varphi'(y) + \sigma_y^2\,\varphi^2(y) + 2h(y)\varphi(y) - 2B = 0$$

from an initial condition $\varphi(-\infty) = \varphi_0$, where φ_0 is the positive solution to

$$\text{(A2.2)} \qquad \sigma_y^2\,\varphi_0^2 + 2f(0)\varphi_0 - 2B = 0,$$

until the value, $\bar{y}$ of y for which

$$\text{(A2.3)} \qquad \varphi(\bar{y}) = 1$$

is reached. The value of the asset can be determined from (35).

Consider now the effects on φ of changes in σ_y^2 under the assumption that $h(y)$ does not depend on σ_y^2. Let $\gamma(y) = \partial\varphi(y)/\partial\sigma_y^2$. It follows (assuming continuity of the derivatives) that $\gamma'(y) = \partial\varphi'(y)/\partial\sigma_y^2$. Thus differentiating (A2.1) and A2.2) w.r.t. σ_y^2 gives

$$\text{(A2.4)} \qquad \sigma_y^2[\gamma'(y)+2\varphi(y)\gamma(y)] + \varphi'(y) + \varphi^2(y) + 2h(y)\gamma(y) = 0$$

and

$$\text{(A2.5)} \qquad 2\sigma_y^2\,\varphi_0\gamma_0 + \varphi_0^2 + 2f(0)\gamma(0) = 0.$$

The evolution of φ and γ is governed by the pair of differential equations (A2.1) and (A2.4) with initial conditions (A2.2) and (A2.5). From (A2.5) we have

$$\gamma_0 = \frac{\varphi_0^2}{2(\sigma_y^2 \varphi_0 + f(0))}$$

which is less than zero (from (A2.2)). We claim now that $\gamma(y) < 0$ for $y < \bar{y}$. Suppose the contrary. Then at some value of y, $\gamma(y) = 0$ with $\gamma' = -(\varphi'(y)+\varphi^2(y))/\sigma_y^2 < 0$, which is a contradiction. Thus we conclude, for $y < \bar{y}$,

$$\frac{\partial}{\partial\sigma_y^2}\varphi(y) = \gamma(y) < 0.$$

It follows from the arguments used in Section 4 that the optimal size barrier $\bar{X}$ and the value function $V(X,P)$ both increase with increases in σ_y^2, provided that $h(y)$ does not depend on σ_y^2, *i.e.* when the growth equation (5) is interpreted in the Stratonovich sense.

FOOTNOTES

(1) See *e.g.* Karlin and Taylor [1981, Chapt. 15] for definitions of Stratonovich and Itô stochastic integrals, and their relationship. Economists have tended to prefer the Itô specification on *a priori* grounds. While such arguments may be valid in such fields as finance where markets can be closely approximated by models which hypothesise continuous, frictionless trading, we feel they may have less validity in resource–economic contexts. Phenomenological arguments suggest the Stratonovich specification is superior in this latter area. For a discussion of the modelling implications of the two specifications for the tree–cutting problem see Clarke and Reed [1987].

(2) It should be noted that dichotomy in choice of stochastic model is not a consequence of using continuous time. A similar situation arises in discrete time where one can choose between models:

$$P_{t+1} - P_t = \mu P_t + P_t \epsilon_t \quad (\text{with } E(\epsilon_t) = 0)$$

or

$$\log P_{t+1} - \log P_t = \mu + \nu_t \quad (\text{with } E(\nu_t) = 0)$$

or, indeed, many other models. The choice of models can be thought of as one of choosing the 'natural' scale for measuring prices.

(3) Since in the model (3) for age–dependent growth it is *proportional* growth that depends on age, the *absolute* growth depends on both size and age.

(4) If the y–process and the q–process were assumed to be correlated there would be an additional term $\sigma_{yq}W^S_{yq}$ on the r.h.s. of (12), with σ_{yq} representing the covariance between the two white noise processes $\sigma_y dw_y$ and $\sigma_q dw_q$. However there appears to be no reason to suppose that pre–harvest price and size should be correlated.

(5) For examples of optimal stopping problems solved as free–boundary problems, see Van Moerbeke [1974].

(6) By conditioning on the state an infinitesimal time dt after the initial time, one can write $M(y,\bar{y};B) = E\{e^{-Bdt}M(y+dy,\bar{y};B)\} + o(dt)$, where the expectation is taken with respect to dy. Expanding the r.h.s gives $(1-Bdt)\{M+M_yE(dy)+\frac{1}{2}M_{yy}E(dy^2)\} + o(dt)$ which when equated to the l.h.s. gives, in the limit as $dt \to 0$, the equation (19).

(7) Note that (25) defines the well–known Riccati transformation (see *e.g.* Birkhoff and Rota [1962, p. 30]) for reducing a linear second–order differential equation to a first–order Riccati equation. Note also that this transformation was used by Malliaris and Brock [1982, p. 200] for solving the one–dimensional tree problem with constant drift and variance.

(8) The condition (27) for the optimal barrier $\bar{y}$, can also be derived by setting the derivative of (17) w.r.t. $\bar{y}$ equal to zero. However this would involve assuming that the optimal rule is a barrier rule on y.

(9) Indeed setting $\sigma_q^2 = 0$ yields a special case of the one–dimensional model of Brock, Rothschild and Stiglitz [1983] who have shown that *local* changes in variance can result in both increases and decreases in asset value depending upon the local concavity or convexity of the value function. In the current model changes in σ_y^2

globally change the variance function, and it seems likely that the overall effects of such a change would depend in a complicated way upon the nature of the expected growth function $f(X)$.

(10) A theorem of Miroschnichenko [1975] states, among other things, that the continuation region for the MLA rule is a subset of the continuation region for the optimal stopping rule. Thus $y_M \leq \bar{y}$. Since $\varphi(y) < 1$ for $y < \bar{y}$ it follows that the exponent in (44) is non–positive, and thus $e \leq 1$, as indeed must be the case. For a heuristic proof of the relevant part of Miroschnichenko's theorem see Malliaris and Brock [1982, p. 196].

(11) Here, as throughout the paper, we ignore costs. In addition to costs of harvesting and maintenance, costs of re–establishment are ignored in the ongoing–rotations problem.

(12) Miller and Voltaire [1983] consider the ongoing–rotations problem for the model of Brock, Rothschild and Stiglitz [1983] in which "intrinsic revenue" is the state variable. They prove that a barrier rule on this variable is optimal.

(13) For a qualitative idea of how the price variance σ_q^2 affects land expectation value and harvest rules see Clarke and Reed [1987], where a numerical example with deterministic growth but stochastic price is presented.

Figure 1 (a) shows the function $\varphi(y)$, the optimal harvesting barrier $\bar{y} = \log \bar{X}$ and the scaled value function $\log\{V(X,P)/XP\}$ (shaded area). The broken line and the cross–hatched area in **Figure 1(b)** show the new values of φ and $\log\{V/XP\}$ corresponding to an upward shift in the parameter $B = \delta - b - \frac{1}{2}\sigma_q^2$.

Figure 2. The application of the optimal stopping rule in the ongoing–rotations problem. Two sample paths are shown: $X(t,\omega_1)$ corresponds to an initially fast–growing tree and $X(t,\omega_2)$ to an initially slower growing tree. Note that the faster–growing tree is harvested sooner.

Growth variance σ_y^2	Optimal size barrier $\bar{X}$	MLA rule size barrier X_M	Percentage deviation of MLA barrier $100(\bar{X}-X_M)/\bar{X}$ (%)	Efficiency of MLA rule 100e (%)
	Case 1. $\alpha = .05$	$\beta = .01$	**("slow growth")**	
.01	3.465	3.0	13.41	99.14
.001	2.599	2.55	1.89	99.98
.0001	2.510	2.505	0.19	100.00
0.0	2.5	2.5	0.0	100.00
	Case 2. $\alpha = .10$	$\beta = .02$	**("fast growth")**	
.01	4.566	4.0	12.39	98.35
.001	3.846	3.775	1.84	99.95
.0001	3.756	3.753	0.20	99.99
0.0	3.75	3.75	0.0	100.00

Table 1. Comparisons of the myopic look–ahead (MLA) rule with the optimal harvesting rule. The mean growth rate function is of the logistic form, $f(X) = \alpha - \beta X$. The discount rate is $\delta = .03$, and the parameters of the price process are $b = 0$ and $\sigma_q^2 = .01$.

Growth variance σ_y^2	Optimal size barrier $\tilde{X}$	MLA rule size barrier $\tilde{X}_M$	Efficiency of MLA rule (%)	Land expectation value parameter $\theta = V(X_0,P)/P$
.01 (single rotation)	2.814 (3.465)	2.487 (3.0)	99.63 99.14)	.5119
.001 (single rotation)	2.045 2.599)	2.022 2.55)	99.99 (99.98)	.4327
0 (single roation)	1.959 (2.5)	1.959 (2.5)	100.0 (100.0)	.4242

Table 2. Optimal harvest rule and MLA harvest rule for the ongoing rotations problem. The mean growth rate function is of the logistic form $f(X) = \alpha - \beta X$ with parameter values $\alpha = .05$, $\beta = .01$ (Case 1: "slow growth"). Newly planted stands are assumed to have size $X_0 = 0.1$ (= 1/50 of asymptotic mean size). The discount rate is $\delta = .03$ and the parameters of the price process are $b = 0$ and $\sigma_q^2 = .01$. Values for the single rotation problem are given in parentheses.

REFERENCES

Anonymous, *Fortran Library Manual Mark II*, Vol. I (Oxford, Numerical Algorithms Group 1984).

Brock, W.A. and M. Rothschild, "Comparative Statics for Multidimensional Optimal Stopping Problems". In H. Sonnenschein (ed.) *Models of Economic Dynamics*, Springer–Verlag Lecture Notes in Economics and Mathematical Systems, no. 264 (New York, 1984).

Brock, W.A., M. Rothschild and J.E. Stiglitz, "Stochastic Capital Theory." In G. Feiwal (ed.) *Essays in Honour of Joan Robinson*, Cambridge: Cambridge University Press, 1988.

Birkhoff, B. and G.C. Rota, *Ordinary Differential Equations* (Waltham, Mass.: Blaisdell Publishing Co., 1962).

Clarke, H.R. and W.J. Reed, "The Tree–Cutting Problem in a Stochastic Environment: The Case of Age–Dependent Growth." Research Paper No. DM–438–IR, Department of Mathematics, University of Victoria (April, 1987).

Clark, C.W., *Mathematical Bioeconomics: The Optimal Management of Renewable Resources* (New York: Wiley–Interscience, 1976).

Faustmann, M., "On the Determination of the Value which Forest Land and Immature Stands Possess for Forestry," 1849. Reprinted in M. Gane (ed.) *Martin Faustmann and the Evolution of Discounted Cash Flow* (in English) Oxford Institute Paper No. 42, (1968).

Gihman, I.I. and A.V. Skorohod, *Stochastic Differential Equations* (New York: Springer–Verlag, 1972).

Karlin, S. and H.M. Taylor, *A Second Course in Stochastic Processes*, 1st edition (New York: Academic Press, 1981).

Krylov, N.V., *Controlled Diffusion Processes* (New York: Springer–Verlag, 1980).

May, R.M., "Stability in Randomly Fluctuating Versus Deterministic Environments," *American Naturalist*, **107** (1973), 621–650.

Malliaris, A.B. AND W.A. Brock, *Stochastic Methods in Economics and Finance* (Amsterdam: North–Holland, 1982).

McDonald, R. and D. Siegel, "The Value of Waiting to Invest," *Quarterly Journal of Economics*, **101** (1986), 707–728.

Miller, R.A. and K. Voltaire, "A Stochastic Analysis of the Tree Paradigm," *Journal of Economic Dynamics and Control*, **6** (1983), 371–386.

Miroschnichenko, T.P., "Optimal Stopping of the Integral of a Wiener Process," *Theory of Probability and Its Applications*, **20** (1975), 387–391.

Mortenson, R.E., "Mathematical Properties of Modelling Stochastic Nonlinear Dynamic Systems," *Journal of Statistical Physics* (1969), 271–296.

Pindyck, R.S., "The Optimal Production of an Exhaustible Resource when Price is Exogenous and Stochastic," *Scandinavian Journal of Economics*, **83** (1981), 279–288.

Press, W.H., B.P. Flannery, S.A. Turkolsky and W.T. Vetterling, *Numerical Recipes: The Art of Scientific Computing* (Cambridge: Cambridge University Press, 1986).

Ross, S.M., *Applied Probability Models with Optimization Applications* (San Francisco: Holden–Day, 1970).

Ross, S.M., *Introduction to Stochastic Dynamic Programming* (New York: Academic Press, 1983).

Samuelson, P.A., "Economics of Forestry in an Evolving Society," *Economic Inquiry*, **14** (1976), 446–492.

Samuelson, P.A., "Rational Theory of Warrant Pricing," *Industrial Management Review*, **6** (1965), 13–31.

SETHI, S.P. AND J.P. LEHOCZKY, "A Comparison of the Itô and Stratonovich Formulations of Problems in Finance," *Journal of Economic Dynamics and Control*, **3** (1981), 343–356.

SHIRYAYEV, A.N., *Optimal Stopping Rules* (New York: Springer, 1978).

TURELLI, M., "Random Environments and Stochastic Calculus," *Theoretical Population Biology*, **12** (1977), 140–178.

VAN MOERBEKE, P., "Optimal Stopping and Free Boundary Problems," *Rocky Mountain Journal of Mathematics*, **4** (1974), 539–577.

Figure 1(a)

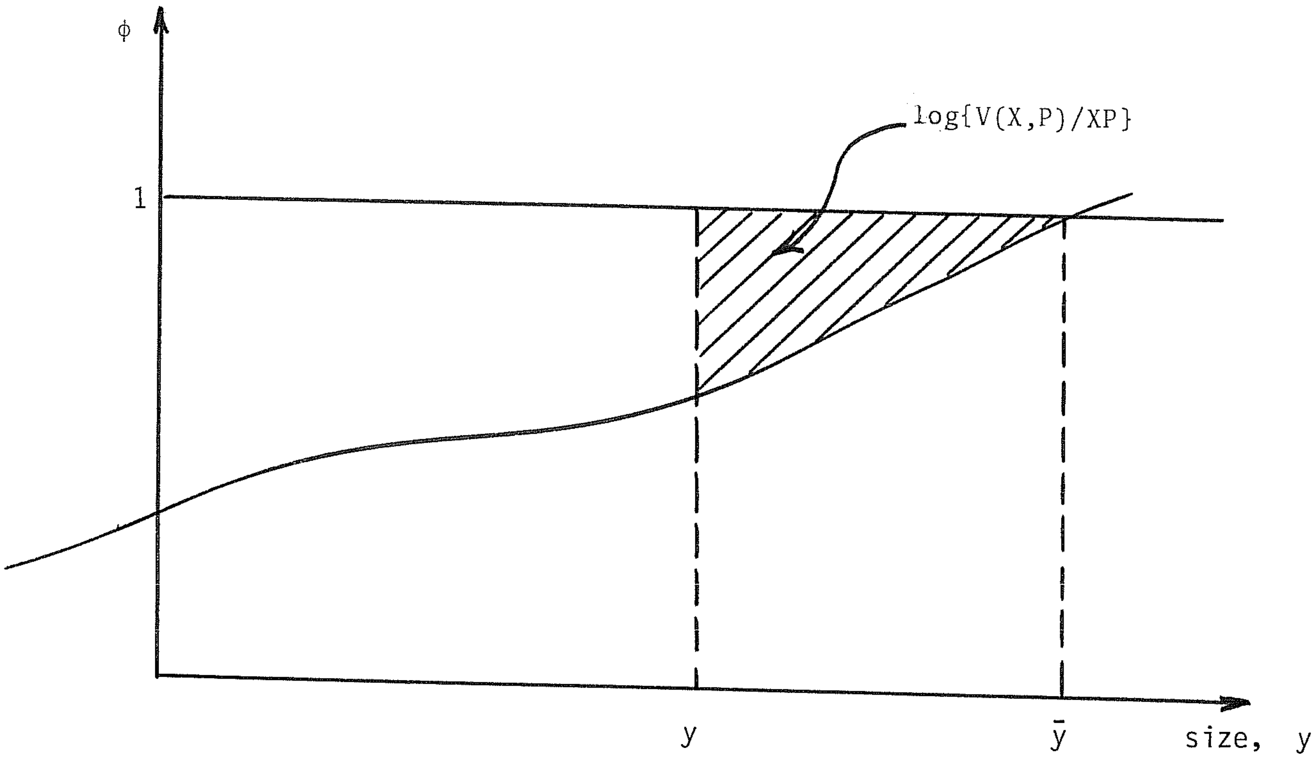


Figure 1(b)

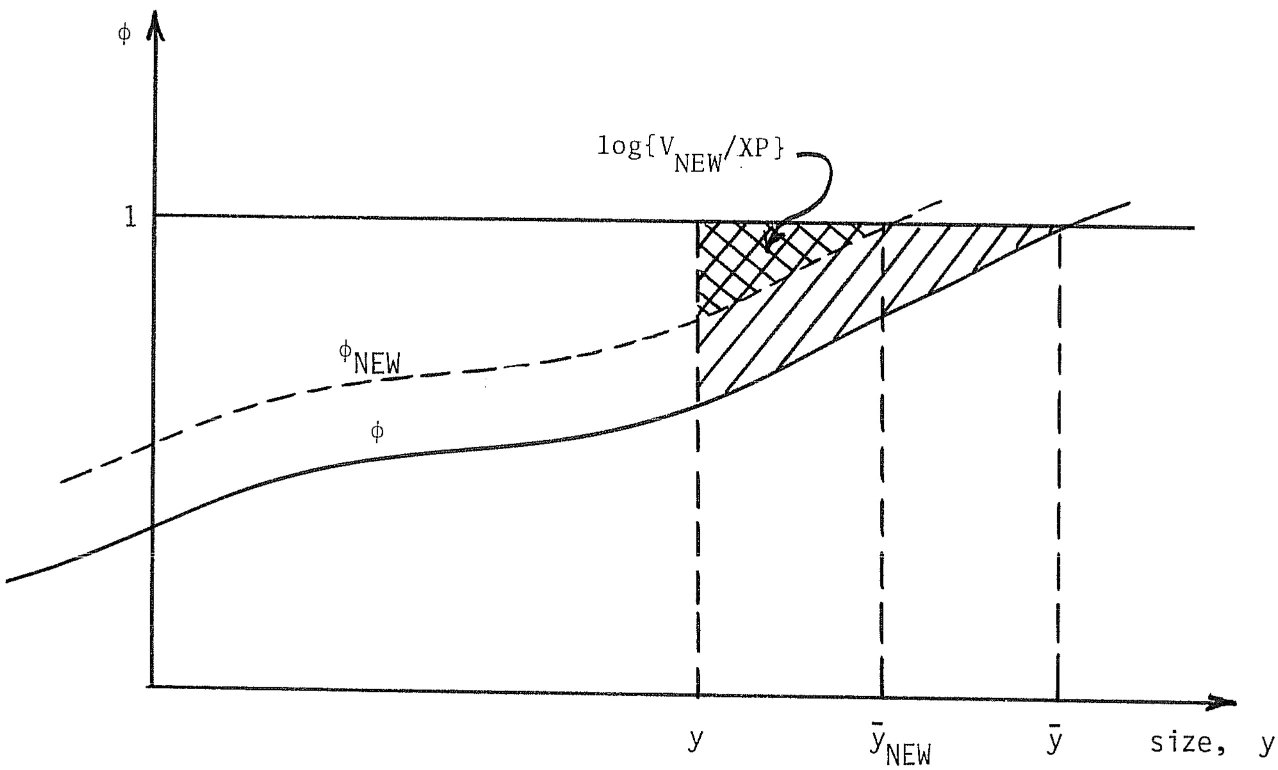
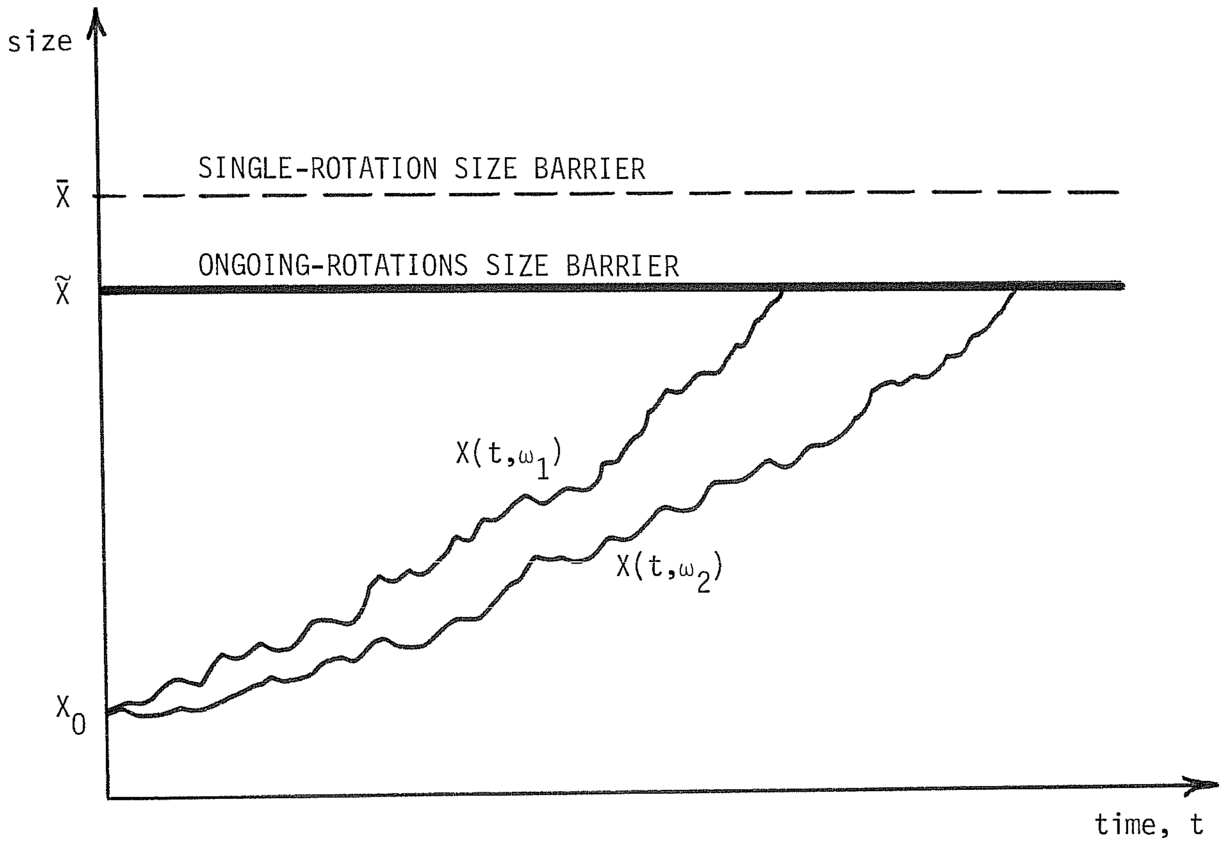


Figure 2


size
SINGLE-ROTATION SIZE BARRIER
$\bar{X}$
ONGOING-ROTATIONS SIZE BARRIER
$\tilde{X}$
$X(t,\omega_1)$
$X(t,\omega_2)$
X_0
time, t