A Class of Meromorphically Univalent Functions with Fixed Coefficients

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Abstract

The main object of the present paper is to investigate various interesting properties and characteristics of a certain class $\mathcal{M}_\lambda(p, \alpha, \beta)$ of meromorphically univalent functions with fixed coefficients, which is related rather closely to the class of meromorphically univalent functions studied recently by Altintaş *et al.* [1].

1. Introduction and Definitions

Let $\mathcal{M}(p)$ denote the class of functions $f(z)$ of the form:

$$f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n \qquad (a_n \geqq 0;\ p \in \mathbb{N} := \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are analytic *and* univalent in the punctured unit disk

$$\mathcal{U}^* := \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\} = \mathcal{U} \setminus \{0\},$$

and which have a simple pole at the origin ($z = 0$) with residue 1 there. Altintaş *et al.* [1] introduced and studied systematically a class $\mathcal{M}(p, \alpha, \beta)$ of functions $f(z) \in \mathcal{M}(p)$, which also satisfy the inequality:

$$\Re\left\{z\, f(z) - \alpha\, z^2\, f'(z)\right\} > \beta \tag{1.2}$$

$$(z \in \mathcal{U}^*; \quad \alpha > 1; \quad 0 \leqq \beta < 1).$$

As a matter of fact, various classes of meromorphically univalent functions have been investigated rather extensively by many authors (*cf.*, *e.g.*, Aouf [2], Chen *et al.* [3], Clunie [4], Libera [5], Mogra *et al.* [6], and Uralegaddi [9]; see also Srivastava and Owa [8]).

We begin by recalling the following result involving the class $\mathcal{M}(p, \alpha, \beta)$, which we shall need in our present investigation.

Theorem A (Altintaş *et al.* [1, p. 76, Theorem 1]). *Let a function $f(z)$ defined by* (1.1) *be in the class $\mathcal{M}(p)$. Then $f(z)$ belongs to the class $\mathcal{M}(p, \alpha, \beta)$ if and only if*

$$\sum_{n=p}^{\infty} (n\alpha - 1)\, a_n \leqq 1 + \alpha - \beta \qquad (\alpha > 1; \quad 0 \leqq \beta < 1). \tag{1.3}$$

The result is sharp for the function $f(z)$ *given by*

$$f(z) = \frac{1}{z} + \frac{1+\alpha-\beta}{p\alpha-1}\, z^p \qquad (p \in \mathbb{N}).$$

In view of Theorem A, for a function $f(z)$ defined by (1.1) and in the class $\mathcal{M}(p,\alpha,\beta)$, we readily have

$$a_p \leq \frac{1+\alpha-\beta}{p\alpha-1} \qquad (p \in \mathbb{N}). \tag{1.4}$$

Thus we may take

$$a_p := \frac{\lambda(1+\alpha-\beta)}{p\alpha-1} \qquad (p \in \mathbb{N}; \quad 0 \leq \lambda \leq 1) \tag{1.5}$$

and let $\mathcal{M}_\lambda(p,\alpha,\beta)$ denote the subclass of $\mathcal{M}(p,\alpha,\beta)$ consisting of functions $f(z)$ of the form:

$$f(z) = \frac{1}{z} + \frac{\lambda(1+\alpha-\beta)}{p\alpha-1}\, z^p + \sum_{n=p+1}^{\infty} a_n\, z^n \tag{1.6}$$

$$(a_n \geq 0; \quad p \in \mathbb{N}; \quad \alpha > 1; \quad 0 \leq \beta < 1; \quad 0 \leq \lambda \leq 1).$$

In Section 2 of this paper we shall first deduce (from Theorem A) some coefficient inequalities which would provide a necessary and sufficient condition for a function $f(z)$ to be in the class $\mathcal{M}_\lambda(p,\alpha,\beta)$. Then in Section 3 we shall show that the class $\mathcal{M}_\lambda(p,\alpha,\beta)$ is closed under *arithmetic mean* and under *convex linear combinations*. Finally, in Section 4 we shall obtain the radii of meromorphic convexity and meromorphic starlikeness of order γ $(0 \leq \gamma < 1)$ for functions belonging to the class $\mathcal{M}_\lambda(p,\alpha,\beta)$.

We remark in passing that the various results presented in this paper are sharp and that the techniques employed for their derivation are essentially analogous to those used earlier in simpler situations (*cf.*, *e.g.*, Silverman and Silvia [7]).

2. A Theorem Involving Coefficient Inequalities

Making use of the definition (1.5) in Theorem A, it is fairly straightforward to deduce

Theorem 1. *Let the function* $f(z)$ *defined by* (1.6) *be in the class* $\mathcal{M}(p)$. *Then* $f(z)$ *is in the class* $\mathcal{M}_\lambda(p,\alpha,\beta)$ *if and only if*

$$\sum_{n=p+1}^{\infty} (n\alpha - 1)\, a_n \leq (1-\lambda)(1+\alpha-\beta) \tag{2.1}$$

$$(\alpha > 1; \quad 0 \leq \beta < 1; \quad 0 \leq \lambda \leq 1).$$

The result is sharp for the functions $f(z)$ *given by*

$$f(z)=\frac{1}{z}+\frac{\lambda(1+\alpha-\beta)}{p\alpha-1}\,z^p+\frac{(1-\lambda)(1+\alpha-\beta)}{n\alpha-1}\,z^n \tag{2.2}$$
$$(n=p+1,p+2,p+3,\cdots).$$

The following result is a rather immediate consequence of Theorem 1:

Corollary. *If a function* $f(z)$ *defined by* (1.6) *is in the class* $\mathcal{M}_\lambda(p,\alpha,\beta)$, *then*

$$a_n\leq\frac{(1-\lambda)(1+\alpha-\beta)}{n\alpha-1} \tag{2.3}$$
$$(n=p+1,p+2,p+3,\cdots).$$

The result is sharp for the functions $f(z)$ *given by* (2.2).

3. A Set of Closure Theorems

Theorem 2. *Let*

$$f_j(z)=\frac{1}{z}+\frac{\lambda(1+\alpha-\beta)}{p\alpha-1}\,z^p+\sum_{n=p+1}^{\infty}a_{n,j}\,z^n \tag{3.1}$$
$$(a_{n,j}\geq 0;\quad j=1,\cdots,m).$$

If $f_j(z)\in\mathcal{M}_\lambda(p,\alpha,\beta)$ $(j=1,\cdots,m)$, *then the function* $g(z)$ *given by*

$$g(z):=\frac{1}{z}+\frac{\lambda(1+\alpha-\beta)}{p\alpha-1}\,z^p+\sum_{n=p+1}^{\infty}b_n\,z^n \tag{3.2}$$

is also in the class $\mathcal{M}_\lambda(p,\alpha,\beta)$, *where*

$$b_n:=\frac{1}{m}\sum_{j=1}^{m}a_{n,j}\,. \tag{3.3}$$

Proof. In view of the hypothesis of Theorem 2, it follows from Theorem 1 that

$$\sum_{n=p+1}^{\infty}(n\alpha-1)\,a_{n,j}\leq(1-\lambda)(1+\alpha-\beta)\qquad(j=1,\cdots,m). \tag{3.4}$$

Therefore, making use of the definition (3.3), we have

$$\begin{aligned}\sum_{n=p+1}^{\infty} (n\alpha-1)\, b_n & \\ &= \sum_{n=p+1}^{\infty} (n\alpha-1)\left(\frac{1}{m}\sum_{j=1}^{m} a_{n,j}\right) \\ &= \frac{1}{m}\sum_{j=1}^{m}\sum_{n=p+1}^{\infty} (n\alpha-1)\, a_{n,j} \\ &\leq (1-\lambda)(1+\alpha-\beta),\end{aligned} \tag{3.5}$$

which, again by virtue of Theorem 1, proves Theorem 2.

Theorem 3. *Let*

$$f_p(z) = \frac{1}{z} + \frac{\lambda(1+\alpha-\beta)}{p\alpha-1}\, z^p \tag{3.6}$$

and

$$f_n(z) = \frac{1}{z} + \frac{\lambda(1+\alpha-\beta)}{p\alpha-1}\, z^p + \frac{(1-\lambda)(1+\alpha-\beta)}{n\alpha-1}\, z^n \tag{3.7}$$

$$(n = p+1, p+2, p+3, \cdots).$$

Then the function $f(z)$ is in the class $\mathcal{M}_\lambda(p,\alpha,\beta)$ if and only if it can be expressed in the form:

$$f(z) = \sum_{n=p}^{\infty} \mu_n\, f_n(z) \tag{3.8}$$

$$\left(\mu_n \geq 0; \quad \sum_{n=p}^{\infty} \mu_n = 1\right).$$

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{n=p}^{\infty} \mu_n\, f_n(z) \\ &= \frac{1}{z} + \frac{\lambda(1+\alpha-\beta)}{p\alpha-1}\, z^p + \sum_{n=p+1}^{\infty} \frac{(1-\lambda)(1+\alpha-\beta)}{n\alpha-1}\, \mu_n\, z^n,\end{aligned} \tag{3.9}$$

where the coefficients μ_n $(n=p+1,p+2,p+3,\cdots)$ are given as in (3.8). Then, since

$$\begin{aligned}\sum_{n=p+1}^{\infty}(n\alpha-1)\,\frac{(1-\lambda)(1+\alpha-\beta)}{n\alpha-1}\,\mu_n &= (1-\lambda)(1+\alpha-\beta)\sum_{n=p+1}^{\infty}\mu_n \\ &= (1-\lambda)(1+\alpha-\beta)(1-\mu_p) \\ &\leq (1-\lambda)(1+\alpha-\beta),\end{aligned} \tag{3.10}$$

we conclude that $f(z)\in\mathcal{M}_\lambda(p,\alpha,\beta)$, by virtue of Theorem 1.

Conversely, let us assume that the function $f(z)$ defined by (1.6) is in the class

$$\mathcal{M}_\lambda(p,\alpha,\beta).$$

Then, in view of the inequality (2.3), we set

$$\mu_n=\frac{n\alpha-1}{(1-\lambda)(1+\alpha-\beta)}\,a_n \qquad (n=p+1,p+2,p+3,\cdots) \tag{3.11}$$

and

$$\mu_p=1-\sum_{n=p+1}^{\infty}\mu_n, \tag{3.12}$$

and we thus arrive at (3.8).

This evidently completes the proof of Theorem 3.

4. Radii of Meromorphic Convexity and Meromorphic Starlikeness

Theorem 4. *Let the function* $f(z)$ *defined by* (1.6) *be in the class* $\mathcal{M}_\lambda(p,\alpha,\beta)$. *Then* $f(z)$ *is meromorphically convex of order* γ $(0\leq\gamma<1)$ *in*

$$0<|z|<r=r_\lambda(p,\alpha,\beta),$$

where $r_\lambda(p,\alpha,\beta)$ *is the largest value of* r *for which*

$$\frac{\lambda p(p-\gamma+2)(1+\alpha-\beta)}{p\alpha-1}\,r^{p+1}+\frac{n(n-\gamma+2)(1-\lambda)(1+\alpha-\beta)}{n\alpha-1}\,r^{n+1}\leq 1-\gamma \tag{4.1}$$

$$(n=p+1,p+2,p+3,\cdots;\quad 0\leq\gamma<1).$$

The result is sharp for the function $f_n(z)$ given by

$$f_n(z) = \frac{1}{z} + \frac{\lambda(1+\alpha-\beta)}{p\alpha-1}\, z^p + \frac{(1-\lambda)(1+\alpha-\beta)}{n\alpha-1}\, z^n \tag{4.2}$$

for some n $(n = p+1, p+2, p+3, \cdots)$.

Proof. It is sufficient to show that

$$\left|\frac{z\,f''(z)}{f'(z)} + 2\right| \leqq 1-\gamma \tag{4.3}$$

$$(0 < |z| < r_\lambda(p,\alpha,\beta); \quad 0 \leqq \gamma < 1),$$

where $r_\lambda(p,\alpha,\beta)$ is the largest value of r for which the inequality (4.1) holds true.

Observe that

$$\begin{aligned} &\left|\frac{z\,f''(z) + 2\,f'(z)}{f'(z)}\right| \\ &= \left|\frac{\lambda p(p+1)(1+\alpha-\beta)(p\alpha-1)^{-1}\, z^{p+1} + \sum\limits_{n=p+1}^{\infty} n(n+1)\, a_n\, z^{n+1}}{-1 + \lambda p(1+\alpha-\beta)(p\alpha-1)^{-1}\, z^{p+1} + \sum\limits_{n=p+1}^{\infty} n\, a_n\, z^{n+1}}\right| \\ &\leqq \frac{\lambda p(p+1)(1+\alpha-\beta)(p\alpha-1)^{-1}\, |z|^{p+1} + \sum\limits_{n=p+1}^{\infty} n(n+1)\, a_n |z|^{n+1}}{1 - \lambda p(1+\alpha-\beta)(p\alpha-1)^{-1}\, |z|^{p+1} - \sum\limits_{n=p+1}^{\infty} n\, a_n |z|^{n+1}} \\ &\leqq 1-\gamma \qquad (0 < |z| < r; \quad 0 \leqq \gamma < 1), \end{aligned}$$

if and only if

$$\frac{\lambda p(p-\gamma+2)(1+\alpha-\beta)}{p\alpha-1}\, r^{p+1} + \sum_{n=p+1}^{\infty} n(n-\gamma+2)\, a_n\, r^{n+1} \leqq 1-\gamma. \tag{4.4}$$

Since $f(z) \in \mathcal{M}_\lambda(p,\alpha,\beta)$, we may apply the inequality (2.1) and set

$$a_n = \frac{(1-\lambda)(1+\alpha-\beta)}{n\alpha-1}\, \kappa_n \tag{4.5}$$

$$\left(\kappa_n \geqq 0; \quad n = p+1, p+2, p+3, \cdots; \quad \sum_{n=p+1}^{\infty} \kappa_n \leqq 1\right).$$

Now, for each fixed r, we choose the positive integer $n_0 = n_0(r)$ for which

$$\frac{n_0(n_0-\gamma+2)}{n_0\alpha-1}\, r^{n_0+1}$$

is maximal. Then

$$\sum_{n=p+1}^{\infty} n(n-\gamma+2)\, a_n\, r^{n+1} \leq \frac{n_0(n_0-\gamma+2)(1-\lambda)(1+\alpha-\beta)}{n_0\alpha-1}\, r^{n_0+1}. \tag{4.6}$$

Hence $f(z) \in \mathcal{M}_\lambda(p,\alpha,\beta)$ is meromorphically convex of order γ in

$$0 < |z| < r_\lambda(p,\alpha,\beta),$$

provided that

$$\frac{\lambda p(p-\gamma+2)(1+\alpha-\beta)}{p\alpha-1}\, r^{p+1} + \frac{n_0(n_0-\gamma+2)(1-\lambda)(1+\alpha-\beta)}{n_0\alpha-1}\, r^{n_0+1} \leq 1-\gamma. \tag{4.7}$$

We find the value $r_0 = r_{\lambda,0}(p,\alpha,\beta)$ and the corresponding integer $n_0(r_0)$ so that

$$\frac{\lambda p(p-\gamma+2)(1+\alpha-\beta)}{p\alpha-1}\, r_0^{p+1} + \frac{n_0(n_0-\gamma+2)(1-\lambda)(1+\alpha-\beta)}{n_0\alpha-1}\, r_0^{n_0+1} = 1-\gamma. \tag{4.8}$$

Then this value r_0 is the radius of meromorphic convexity of order γ for functions $f(z)$ belonging to the class $\mathcal{M}_\lambda(p,\alpha,\beta)$.

In a similar manner, we can prove our last result (Theorem 5 below) providing the radius of meromorphic starlikeness of order γ $(0 \leq \gamma < 1)$ for functions in the class $\mathcal{M}_\lambda(p,\alpha,\beta)$.

Theorem 5. *Let the function* $f(z)$ *defined by* (1.6) *be in the class* $\mathcal{M}_\lambda(p,\alpha,\beta)$. *Then* $f(z)$ *is meromorphically starlike of order* γ $(0 \leq \gamma < 1)$ *in*

$$0 < |z| < R = R_\lambda(p,\alpha,\beta),$$

where $R_\lambda(p,\alpha,\beta)$ *is the largest value of* R *for which*

$$\frac{\lambda(p-\gamma+2)(1+\alpha-\beta)}{p\alpha-1}\, R^{p+1} + \frac{(1-\lambda)(n-\gamma+2)(1+\alpha-\beta)}{n\alpha-1}\, R^{n+1} \leq 1-\gamma \tag{4.9}$$

$$(n = p+1, p+2, p+3, \cdots; \quad 0 \leq \gamma < 1).$$

The result is sharp for the function $f_n(z)$ *given by* (4.2) *for some* n

$$(n = p+1, p+2, p+3, \cdots).$$

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