General Fractional-Order Anomalous Diffusion with Non-Singular Power-Law Kernel

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GENERAL FRACTIONAL-ORDER ANOMALOUS DIFFUSION WITH NON-SINGULAR POWER-LAW KERNEL

by

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In this paper, we investigate general fractional derivatives with a non-singular power-law kernel. The anomalous diffusion models with non-singular power-law kernel are discussed in detail. The results are efficient for modelling the anomalous behaviors within the frameworks of the Riemann-Liouville and Liouville-Caputo general fractional derivatives.

Key words: *general fractional derivative with non-singular power-law kernel, Riemann-Liouville general fractional derivative, anomalous diffusion, Liouville-Caputo general fractional derivative*

Introduction

The Riemann-Liouville and Liouville-Caputo fractional derivatives (FD) are known to have important roles in engineering applications, such as (for example) in heat transfer, viscoelasticity, and others, see [1-4] and the references cited therein. The theory of the Riemann-Liouville and Liouville-Caputo FD is used to model the anomalous diffusion behaviors. For example, the anomalous diffusion in the rotating flow was observed in [5]. The anomalous diffusion involving the stochastic pathway was discussed in [6]. The anomalous diffusion in the disordered (complex) media was reported in [7]. The anomalous diffusion with the external forces was presented in [8]. The anomalous diffusion related to the thermal equilibrium was considered in [9]. The anomalous diffusion in the sub diffusive case was proposed in [10].

Recently, the Riemann-Liouville and Liouville-Caputo general fractional derivatives (GFD) with non-singular Mittag-Leffler function kernels were introduced in [11] and their applications in the rheological models were discussed in [12]. More recently, the Riemann-Liouville and Liouville-Caputo GFD with non-singular power-law kernel were presented in [13].

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The Liouville-Weyl and Liouville-Caputo GFD with non-singular power-law kernel were proposed to model the anomalous diffusion problems in [14].

In the spirit of the previous ideas, the chief aim of this paper is to model general fractional anomalous diffusion problems with non-singular power-law kernel.

Preliminary

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_-$, and $\mathbb{N}$ be the sets of the real numbers, positive real numbers, negative real numbers, and positive integer numbers, respectively.

In order to introduce the derivations of the Riemann-Liouville and Liouville-Caputo GFD with non-singular power-law kernel, we start with the n-fold integral in the form, see [1]:

$$\overbrace{\int_0^x \mathrm{d}t \cdots \int_0^x \Theta(t)\mathrm{d}t}^{n-\text{times}} = \frac{1}{\Gamma(1+n)}\int_0^x (x-t)^n \Theta(t)\mathrm{d}t \tag{1}$$

where $n \in \mathbb{N}$, $x \in \mathbb{R}$, and $\Theta(t)$ is a real function.

From eq. (1) we have, see [1, 4]:

$$\overbrace{\int_0^x \mathrm{d}t \cdots \int_0^x \Theta(t)\mathrm{d}t}^{n-\text{times}} = \int_0^x \Lambda(x-t)\Theta(t)\mathrm{d}t \tag{2}$$

where the kernel is represented in the form, see [1, 4]:

$$\Lambda(x-t) = \frac{(x-t)^n}{\Gamma(1+n)} \tag{3}$$

Replacing n by α in eq. (2), where $\alpha \in \mathbb{R}$, we obtain:

$$\Phi(x) = \int_0^x \frac{(x-t)^\alpha}{\Gamma(1+\alpha)}\Theta(t)\mathrm{d}t \tag{4}$$

When $\alpha = -\beta \in \mathbb{R}_-$, from eq. (4) we obtain the generalized Abel integral equation in the form [1]:

$$\Phi(x) = \frac{1}{\Gamma(1-\beta)}\int_0^x \frac{\Theta(t)}{(x-t)^\beta}\mathrm{d}t \tag{5}$$

The left-handed Riemann-Liouville FD of the function $\Theta(t)$ of order β is defined as, see [1-4]:

$${}_{a}^{\mathrm{RL}}\mathbb{D}_x^{(\beta)}\Theta(x) = \frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \frac{\Theta(t)}{(x-t)^\beta}\mathrm{d}t \tag{6}$$

where $a, \beta \in \mathbb{R}$ and $0 < \beta < 1$.

The right-handed Riemann-Liouville FD of the function $\Theta(t)$ of order β is defined as, see [1-4]:

$${}_{x}^{\mathrm{RL}}\mathbb{D}_b^{(\beta)}\Theta(x) = \frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b \frac{\Theta(t)}{(t-x)^\beta}\mathrm{d}t \tag{7}$$

where $b, \beta \in \mathbb{R}$ and $0 < \beta < 1$.

The left-handed Riemann-Liouville FD of the function $\Theta(t)$ of order β is defined as, see [4]:

$$ {}_{a}^{\mathrm{RL}}\mathbb{D}_{x}^{(\beta)}\Theta(x) = \frac{1}{\Gamma(m-\beta)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{m}\int_{a}^{x}\frac{\Theta(t)}{(x-t)^{\beta-m+1}}\mathrm{d}t \tag{8} $$

where $a, \beta \in \mathbb{R}$ and $m-1 < \beta < m$.

The right-handed Riemann-Liouville FD of the function $\Theta(t)$ of order β is defined as, see [4]:

$$ {}_{x}^{\mathrm{RL}}\mathbb{D}_{b}^{(\beta)}\Theta(x) = \frac{1}{\Gamma(m-\beta)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{m}\int_{x}^{b}\frac{\Theta(t)}{(t-x)^{\beta-m+1}}\mathrm{d}t \tag{9} $$

where $b, \beta \in \mathbb{R}$ and $m-1 < \beta < m$.

The left-handed Liouville-Caputo FD of the function $\Theta(t)$ of order β is defined as, see [2-4]:

$$ {}_{a}^{\mathrm{LC}}\mathbb{D}_{x}^{(\beta)}\Theta(x) = \frac{1}{\Gamma(1-\beta)}\int_{a}^{x}\frac{1}{(x-t)^{\beta}}\frac{\mathrm{d}\Theta(t)}{\mathrm{d}x}\mathrm{d}t \tag{10} $$

where $a, \beta \in \mathbb{R}$ and $0 < \beta < 1$.

The right-handed Liouville-Caputo FD of the function $\Theta(t)$ of order β is defined as, see [2-4]:

$$ {}_{x}^{\mathrm{LC}}\mathbb{D}_{b}^{(\beta)}\Theta(x) = \frac{1}{\Gamma(1-\beta)}\int_{x}^{b}\frac{1}{(t-x)^{\beta}}\frac{\mathrm{d}\Theta(t)}{\mathrm{d}x}\mathrm{d}t \tag{11} $$

where $b, \beta \in \mathbb{R}$ and $0 < \beta < 1$.

The left-handed Liouville-Caputo FD of the function $\Theta(t)$ of order β is defined as, see [2-4]:

$$ {}_{a}^{\mathrm{LC}}\mathbb{D}_{x}^{(\beta)}\Theta(x) = \frac{1}{\Gamma(m-\beta)}\int_{a}^{x}\frac{1}{(x-t)^{\beta-m+1}}\frac{\mathrm{d}^{m}\Theta(t)}{\mathrm{d}x^{m}}\mathrm{d}t \tag{12} $$

where $a, \beta \in \mathbb{R}$ and $m-1 < \beta < m$.

The right-handed Liouville-Caputo FD of the function $\Theta(t)$ of order β is defined as, see [2-4]:

$$ {}_{x}^{\mathrm{LC}}\mathbb{D}_{b}^{(\beta)}\Theta(x) = \frac{1}{\Gamma(m-\beta)}\int_{x}^{b}\frac{1}{(t-x)^{\beta-m+1}}\frac{\mathrm{d}^{m}\Theta(t)}{\mathrm{d}x^{m}}\mathrm{d}t \tag{13} $$

where $b, \beta \in \mathbb{R}$ and $m-1 < \beta < m$.

For $0 < \beta < 1$, one has [1, 4]:

$$ {}_{a}^{\mathrm{RL}}\mathbb{D}_{x}^{(\beta)}\Theta(x) = \frac{1}{\Gamma(1-\beta)}\frac{\Theta(a)}{(x-a)^{\beta}} + {}_{a}^{\mathrm{LC}}\mathbb{D}_{x}^{(\beta)}\Theta(x) \tag{14} $$

$$ {}^{\mathrm{RL}}_{\ x}\mathbb{D}_b^{(\beta)}\Theta(x)=\frac{1}{\Gamma(1-\beta)}\frac{\Theta(b)}{(b-x)^{\beta}}-{}^{\mathrm{LC}}_{\ x}\mathbb{D}_b^{(\beta)}\Theta(x) \tag{15}$$

Remark 1. For more details of the Riemann-Liouville and Liouville-Caputo FD, readers refer to see [1-10].

General fractional-order derivatives

When $\alpha=-\beta\in\mathbb{R}_+$, from eq. (4) we obtain:

$$ \Phi(x)=\frac{1}{\Gamma(1+\beta)}\int_0^x (x-t)^{\beta}\,\Theta(t)\,\mathrm{d}t \tag{16}$$

The left-handed Riemann-Liouville GFD of the function $\Theta(t)$ of order β is defined as [13]:

$$ {}^{\mathrm{GRL}}_{\ \ a}\mathbb{D}_x^{(\beta)}\Theta(x)=\frac{1}{\Gamma(1+\beta)}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x (x-t)^{\beta}\,\Theta(t)\,\mathrm{d}t \tag{17}$$

where $a,\beta\in\mathbb{R}$ and $0<\beta<1$.

The right-handed Riemann-Liouville GFD of the function $\Theta(t)$ of order β is defined:

$$ {}^{\mathrm{GRL}}_{\ \ x}\mathbb{D}_b^{(\beta)}\Theta(x)=\frac{1}{\Gamma(1+\beta)}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b (t-x)^{\beta}\,\Theta(t)\,\mathrm{d}t \tag{18}$$

where $b,\beta\in\mathbb{R}$ and $0<\beta<1$.

The left-handed Riemann-Liouville GFD of the function $\Theta(t)$ of order β is defined as [13]:

$$ {}^{\mathrm{GRL}}_{\ \ a}\mathbb{D}_x^{(\beta)}\Theta(x)=\frac{1}{\Gamma(m+\beta)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{m}\int_a^x (x-t)^{\beta+m-1}\,\Theta(t)\,\mathrm{d}t \tag{19}$$

where $a,\beta\in\mathbb{R}$ and $m-1<\beta<m$.

The right-handed Riemann-Liouville GFD of the function $\Theta(t)$ of order β is defined:

$$ {}^{\mathrm{GRL}}_{\ \ x}\mathbb{D}_b^{(\beta)}\Theta(x)=\frac{1}{\Gamma(m+\beta)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{m}\int_x^b (t-x)^{\beta+m-1}\,\Theta(t)\,\mathrm{d}t \tag{20}$$

where $b,\beta\in\mathbb{R}$ and $m-1<\beta<m$.

The left-handed Liouville-Caputo GFD of the function $\Theta(t)$ of order β is defined as, see [13]:

$$ {}^{\mathrm{GLC}}_{\ \ a}\mathbb{D}_x^{(\beta)}\Theta(x)=\frac{1}{\Gamma(1+\beta)}\int_a^x (x-t)^{\beta}\,\frac{\mathrm{d}\Theta(t)}{\mathrm{d}x}\,\mathrm{d}t \tag{21}$$

where $a,\beta\in\mathbb{R}$ and $0<\beta<1$.

The right-handed Liouville-Caputo GFD of the function $\Theta(t)$ of order β is defined:

$$ {}^{\mathrm{GLC}}_{\ \ x}\mathbb{D}_b^{(\beta)}\Theta(x)=\frac{1}{\Gamma(1+\beta)}\int_x^b (t-x)^{\beta}\,\frac{\mathrm{d}\Theta(t)}{\mathrm{d}x}\,\mathrm{d}t \tag{22}$$

where $b,\beta\in\mathbb{R}$ and $0<\beta<1$.

The left-handed Liouville-Caputo GFD of the function $\Theta(t)$ of order β is defined as, see [13]:

$$ {}^{\mathrm{GLC}}_{\ a}\mathbb{D}_x^{(\beta)}\Theta(x)=\frac{1}{\Gamma(m+\beta)}\int_a^x (x-t)^{\beta+m-1}\frac{\mathrm{d}^m\Theta(t)}{\mathrm{d}x^m}\mathrm{d}t \tag{23}$$

where $a,\beta\in\mathbb{R}$ and $m-1<\beta<m$.

The right-handed Liouville-Caputo GFD of the function $\Theta(t)$ of order β is defined as:

$$ {}^{\mathrm{GLC}}_{\ x}\mathbb{D}_b^{(\beta)}\Theta(x)=\frac{1}{\Gamma(m+\beta)}\int_x^b (t-x)^{\beta+m-1}\frac{\mathrm{d}^m\Theta(t)}{\mathrm{d}x^m}\mathrm{d}t \tag{24}$$

where $b,\beta\in\mathbb{R}$ and $m-1<\beta<m$.

For $0<\beta<1$, we obtain:

$$ {}^{\mathrm{GRL}}_{\ a}\mathbb{D}_x^{(\beta)}\Theta(x)=\frac{(x-a)^{\beta}\,\Theta(a)}{\Gamma(1-\beta)}+{}^{\mathrm{GLC}}_{\ a}\mathbb{D}_x^{(\beta)}\Theta(x) \tag{25}$$

$$ {}^{\mathrm{GRL}}_{\ x}\mathbb{D}_b^{(\beta)}\Theta(x)=\frac{(b-x)^{\beta}\,\Theta(b)}{\Gamma(1-\beta)}-{}^{\mathrm{GLC}}_{\ x}\mathbb{D}_b^{(\beta)}\Theta(x) \tag{26}$$

Remark 2. For more details of the definitions of the left-handed Riemann-Liouville and Liouville-Caputo GFD with non-singular power-law and Mittag-Leffler-function kernels, readers refer to [11-18].

The Laplace transforms of eqs. (18) and (21) are as follows [13]:

$$\mathbb{L}\left[\left({}^{\mathrm{GRL}}_{\ 0}\mathbb{D}_x^{(\beta)}\Theta\right)(x)\right]=\frac{1}{s^{\beta}}\Theta(s) \tag{27}$$

$$\mathbb{L}\left[\left({}^{\mathrm{GLC}}_{\ 0}\mathbb{D}_x^{(\beta)}\Theta\right)(x)\right]=\frac{1}{s^{1+\beta}}\left[s\Theta(s)-\Theta(0)\right] \tag{28}$$

where the Laplace transform is defined by [1, 4]:

$$\mathbb{L}\left[\Phi(x)\right]=\Phi(s):=\int_0^{\infty}e^{-sx}\Phi(x)\mathrm{d}x \tag{29}$$

Remark 3. For more details of the definitions of the left-handed Riemann-Liouville and Liouville-Caputo FD and GFD, readers refer to [1-22].

New results

Let us consider the following expressions of the GFD with non-singular power-law kernel:

$$ {}^{\mathrm{GRL}}_{\ a}\mathbb{D}_x^{(i\beta)}\Theta(x)=\frac{1}{\Gamma(1+i\beta)}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x (x-t)^{i\beta}\,\Theta(t)\mathrm{d}t \tag{30}$$

$$ {}^{\mathrm{GRL}}_{\ x}\mathbb{D}_b^{(i\beta)}\Theta(x)=\frac{1}{\Gamma(1+i\beta)}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b (t-x)^{i\beta}\,\Theta(t)\mathrm{d}t \tag{31}$$

$$ {}_{a}^{\mathrm{GLC}}\mathbb{D}_{x}^{(i\beta)}\Theta(x)=\frac{1}{\Gamma(1+i\beta)}\int_{a}^{x}(x-t)^{i\beta}\frac{\mathrm{d}\Theta(t)}{\mathrm{d}x}\mathrm{d}t \tag{32} $$

$$ {}_{x}^{\mathrm{GLC}}\mathbb{D}_{b}^{(i\beta)}\Theta(x)=\frac{1}{\Gamma(1+i\beta)}\int_{x}^{b}(t-x)^{i\beta}\frac{\mathrm{d}\Theta(t)}{\mathrm{d}x}\mathrm{d}t \tag{33} $$

For $0<\beta<1$, we have the following GFD with non-singular power-law kernel:

$$ {}_{a}^{\mathrm{GRL}}\mathbb{D}_{x}^{(\beta)}\Theta(x)=\frac{\mathrm{d}}{\mathrm{d}x}\int_{a}^{x}\left[\sum_{i=0}^{\infty}\frac{1}{\Gamma(1+i\beta)}(x-t)^{i\beta}\right]\Theta(t)\mathrm{d}t=\frac{\mathrm{d}}{\mathrm{d}x}\int_{a}^{x}E_{\beta}\left[(x-t)^{\beta}\right]\Theta(t)\mathrm{d}t \tag{34} $$

$$ {}_{a}^{\mathrm{GRL}}\mathbb{D}_{x}^{(\beta)}\Theta(x)=\frac{\mathrm{d}}{\mathrm{d}x}\int_{a}^{x}\left[\sum_{i=0}^{\infty}\frac{(-1)^{i}}{\Gamma(1+i\beta)}(x-t)^{i\beta}\right]\Theta(t)\mathrm{d}t=\frac{\mathrm{d}}{\mathrm{d}x}\int_{a}^{x}E_{\beta}\left[-(x-t)^{\beta}\right]\Theta(t)\mathrm{d}t \tag{35} $$

$$ {}_{x}^{\mathrm{GRL}}\mathbb{D}_{b}^{(\beta)}\Theta(x)=\frac{\mathrm{d}}{\mathrm{d}x}\int_{a}^{x}\left[\sum_{i=0}^{\infty}\frac{1}{\Gamma(1+i\beta)}(t-x)^{i\beta}\right]\Theta(t)\mathrm{d}t=\frac{\mathrm{d}}{\mathrm{d}x}\int_{x}^{b}E_{\beta}\left[(t-x)^{\beta}\right]\Theta(t)\mathrm{d}t \tag{36} $$

$$ {}_{x}^{\mathrm{GRL}}\mathbb{D}_{b}^{(\beta)}\Theta(x)=\frac{\mathrm{d}}{\mathrm{d}x}\int_{x}^{b}\left[\sum_{i=0}^{\infty}\frac{(-1)^{i}}{\Gamma(1+i\beta)}(t-x)^{i\beta}\right]\Theta(t)\mathrm{d}t=\frac{\mathrm{d}}{\mathrm{d}x}\int_{x}^{b}E_{\beta}\left[-(t-x)^{\beta}\right]\Theta(t)\mathrm{d}t \tag{37} $$

$$ {}_{a}^{\mathrm{GLC}}\mathbb{D}_{x}^{(\beta)}\Theta(x)=\int_{a}^{x}\left[\sum_{i=0}^{\infty}\frac{1}{\Gamma(1+i\beta)}(x-t)^{i\beta}\right]\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t}\mathrm{d}t=\int_{a}^{x}E_{\beta}\left[(x-t)^{\beta}\right]\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t}\mathrm{d}t \tag{38} $$

$$ {}_{a}^{\mathrm{GLC}}\mathbb{D}_{x}^{(\beta)}\Theta(x)=\int_{a}^{x}\left[\sum_{i=0}^{\infty}\frac{(-1)^{i}}{\Gamma(1+i\beta)}(x-t)^{i\beta}\right]\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t}\mathrm{d}t=\int_{a}^{x}E_{\beta}\left[-(x-t)^{\beta}\right]\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t}\mathrm{d}t \tag{39} $$

$$ {}_{x}^{\mathrm{GLC}}\mathbb{D}_{b}^{(\beta)}\Theta(x)=\int_{x}^{b}\left[\sum_{i=0}^{\infty}\frac{1}{\Gamma(1+i\beta)}(t-x)^{i\beta}\right]\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t}\mathrm{d}t=\int_{x}^{b}E_{\beta}\left[(t-x)^{\beta}\right]\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t}\mathrm{d}t \tag{40} $$

$$ {}_{x}^{\mathrm{GLC}}\mathbb{D}_{b}^{(\beta)}\Theta(x)=\int_{x}^{b}\left[\sum_{i=0}^{\infty}\frac{(-1)^{i}}{\Gamma(1+i\beta)}(t-x)^{i\beta}\right]\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t}\mathrm{d}t=\int_{x}^{b}E_{\beta}\left[-(t-x)^{\beta}\right]\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t}\mathrm{d}t \tag{41} $$

where the Mittag-Leffler function is defined as in [1]:

$$ E_{\beta}\left[(x-t)^{\beta}\right]=\sum_{i=0}^{\infty}\frac{(x-t)^{i\beta}}{\Gamma(1+i\beta)} $$

In a similar way, from eqs. (17), (18), (21), and (22) we find for $0<\beta<1$ that:

$$ {}_{a}^{\mathrm{GRL}}\mathbb{D}_{x}^{(\beta)}\Theta(x)=\frac{\mathrm{d}}{\mathrm{d}x}\int_{a}^{x}\left[\sum_{i=0}^{\infty}\frac{1}{\Gamma(1-i\beta)}\frac{1}{(x-t)^{i\beta}}\right]\Theta(t)\mathrm{d}t=\frac{\mathrm{d}}{\mathrm{d}x}\int_{a}^{x}\Xi_{\beta}\left[(x-t)^{\beta}\right]\Theta(t)\mathrm{d}t \tag{42} $$

$$ {}_{a}^{\mathrm{GRL}}\mathbb{D}_{x}^{(\beta)}\Theta(x)=\frac{\mathrm{d}}{\mathrm{d}x}\int_{a}^{x}\left[\sum_{i=0}^{\infty}\frac{(-1)^{i}}{\Gamma(1-i\beta)}\frac{1}{(x-t)^{i\beta}}\right]\Theta(t)\mathrm{d}t=\frac{\mathrm{d}}{\mathrm{d}x}\int_{a}^{x}\Xi_{\beta}\left[-(x-t)^{\beta}\right]\Theta(t)\mathrm{d}t \tag{43} $$

$$ {}^{\mathrm{GRL}}_{\;x}\mathbb{D}_b^{(\beta)}\Theta(x)=\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b\left[\sum_{i=0}^{\infty}\frac{1}{\Gamma(1-i\beta)}\frac{1}{(t-x)^{i\beta}}\right]\Theta(t)\,\mathrm{d}t=\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b\Xi_\beta\left[(t-x)^\beta\right]\Theta(t)\,\mathrm{d}t \tag{44}$$

$$ {}^{\mathrm{GRL}}_{\;x}\mathbb{D}_b^{(\beta)}\Theta(x)=\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b\left[\sum_{i=0}^{\infty}\frac{(-1)^i}{\Gamma(1+i\beta)}\frac{1}{(t-x)^{i\beta}}\right]\Theta(t)\,\mathrm{d}t=\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b\Xi_\beta\left[-(t-x)^\beta\right]\Theta(t)\,\mathrm{d}t \tag{45}$$

$$ {}^{\mathrm{GLC}}_{\;a}\mathbb{D}_x^{(\beta)}\Theta(x)=\int_a^x\left[\sum_{i=0}^{\infty}\frac{1}{\Gamma(1-i\beta)}\frac{1}{(x-t)^{i\beta}}\right]\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t}\mathrm{d}t=\int_a^x\Xi_\beta\left[(x-t)^\beta\right]\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t}\mathrm{d}t \tag{46}$$

$$ {}^{\mathrm{GLC}}_{\;a}\mathbb{D}_x^{(\beta)}\Theta(x)=\int_a^x\left[\sum_{i=0}^{\infty}\frac{(-1)^i}{\Gamma(1-i\beta)}\frac{1}{(x-t)^{i\beta}}\right]\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t}\mathrm{d}t=\int_a^x\Xi_\beta\left[-(x-t)^\beta\right]\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t}\mathrm{d}t \tag{47}$$

$$ {}^{\mathrm{GLC}}_{\;x}\mathbb{D}_b^{(\beta)}\Theta(x)=\int_x^b\left[\sum_{i=0}^{\infty}\frac{1}{\Gamma(1+i\beta)}\frac{1}{(t-x)^{i\beta}}\right]\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t}\mathrm{d}t=\int_x^b\Xi_\beta\left[(t-x)^\beta\right]\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t}\mathrm{d}t \tag{48}$$

$$ {}^{\mathrm{GLC}}_{\;x}\mathbb{D}_b^{(\beta)}\Theta(x)=\int_x^b\left[\sum_{i=0}^{\infty}\frac{(-1)^i}{\Gamma(1-i\beta)}\frac{1}{(t-x)^{i\beta}}\right]\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t}\mathrm{d}t=\int_x^b\Xi_\beta\left[-(t-x)^\beta\right]\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t}\mathrm{d}t \tag{49}$$

where

$$\Xi_\beta\left[(x-t)^\beta\right]=\sum_{i=0}^{\infty}\frac{1}{\Gamma(1+i\beta)}\frac{1}{(x-t)^{i\beta}}$$

Modelling the general fractional anomalous diffusion with non-singular power-law kernel

We now consider the Riemann-Liouville general fractional time anomalous diffusion with non-singular power-law kernel:

$$ {}^{\mathrm{GRL}}_{\;0}\mathbb{D}_\tau^{(\beta)}\Theta(x,\tau)=\kappa\frac{\partial^2\Theta(x,\tau)}{\partial x^2} \tag{50}$$

subject to the initial condition

$$\Theta(x,0)=g(x) \tag{51}$$

where κ is the diffusion coefficient and the Riemann-Liouville general fractional partial derivative of the function $\Theta(x,\tau)$ of order β with respect to the time variable, τ, is defined by:

$$ {}^{\mathrm{GRL}}_{\;0}\mathbb{D}_\tau^{(\beta)}\Theta(x,\tau)=\frac{1}{\Gamma(1+\beta)}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau(\tau-t)^\beta\,\Theta(x,t)\,\mathrm{d}t \tag{52}$$

Let us consider the Liouville-Caputo general fractional time anomalous diffusion with non-singular power-law kernel:

$$ {}^{\mathrm{GLC}}_{\;0}\mathbb{D}_\tau^{(\beta)}\Theta(x,\tau)=\kappa\frac{\partial^2\Theta(x,\tau)}{\partial x^2} \tag{53}$$

subject to the initial condition

$$\Theta(x,0)=\delta(x) \tag{54}$$

where $\delta(x)$ is the Dirac delta function [4] and the Liouville-Caputo general fractional partial derivative of the function $\Theta(x,\tau)$ of order β with respect to the time variable, τ, is defined by:

$${}^{\mathrm{GLC}}_{\ 0}\mathbb{D}^{(\beta)}_{\tau}\Theta(x,\tau)=\frac{1}{\Gamma(1+\beta)}\int_0^\tau(\tau-t)^\beta\frac{\mathrm{d}\Theta(x,t)}{\mathrm{d}t}\mathrm{d}t \tag{55}$$

Conclusion

The present study addressed the derivations of the Riemann-Liouville and Liouville-Caputo GFD with non-singular power-law kernel. The relationship between the GFD with non-singular power-law and Mittag-Leffler function kernels were discussed. The Riemann-Liouville and Liouville-Caputo general fractional time anomalous diffusion models with non-singular power-law kernel were obtained. The models are successfully adopted to model the anomalous behaviors of the complex phenomena.

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Nomenclature

t – time co-ordinate, [s]
x – space co-ordinate, [m]

Greek symbols

β – fractional order, [–]
κ – diffusion coefficient, [m^2s^{-1}]

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