

THE EXPONENT AND CIRCUMDIAMETER OF
PRIMITIVE DIRECTED GRAPHS

by

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*We accept this dissertation as conforming
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Abstract

The exponent of a primitive directed graph (digraph) D , denoted $\gamma(D)$, is the smallest m such that for each ordered pair of vertices (u, v) , there exists a $u \rightsquigarrow v$ walk of length m . A walk touches another if they have a vertex in common. If $\lambda(D)$ is the set of all cycle lengths, then the circumdiameter of D , denoted $d^c(\lambda(D))$, is the maximum over all ordered pairs of vertices (u, v) of the length of a shortest $u \rightsquigarrow v$ walk that touches cycles of all lengths. It is well known that $\gamma(D) \leq \phi(\lambda(D)) + d^c(\lambda(D))$, in which $\phi(\lambda(D))$ is the Frobenius-Schur index. The main results of this thesis include several new sufficient conditions and families of digraphs for which equality holds in the above upper bound. A primitive digraph D on n vertices has large exponent if $\gamma(D) \geq \lfloor \frac{(n-1)^2+1}{2} \rfloor + 2$. Additional sufficient conditions for equality in the above upper bound for $\gamma(D)$ and a new upper bound for $d^c(\lambda(D))$ are given for digraphs with large exponent. Expressions for the exponent, diameter and circumdiameter of some primitive digraphs on n vertices with large exponent and circumference n or $n - 1$ are also given.

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Chapter 1

Introduction

This chapter includes basic definitions and notation, results from the literature, and other preliminary results that are required in further chapters. In Section 1.6, primitive digraphs with large exponent are defined and many of their properties are discussed.

1.1 Basic Definitions and Notation

The following concepts are primarily from Chapter 3 of *Combinatorial Matrix Theory* by R. A. Brualdi and H. J. Ryser [2].

A *directed graph (digraph)* D on n vertices is a finite set of vertices $V(D) = \{1, 2, \dots, n\}$ together with a set of *arcs* $E(D) = \{i \rightarrow j \mid i, j \in V(D)\}$. Note that i may equal j , and an arc $i \rightarrow i$ is called a loop. A *subdigraph* of D is a digraph H in which $V(H) \subseteq V(D)$ and $E(H) \subseteq E(D)$.

Two digraphs with the same number of vertices are *isomorphic* if one can be obtained from the other by renumbering the vertices. Theorems, lemmas and definitions in this thesis pertaining to digraphs are considered to be

true up to isomorphism and/or the reversal of every arc. The Wielandt digraph on $n \geq 3$ vertices is the digraph W_n , with $E(W_n) = \{1 \rightarrow n, n \rightarrow (n-1), \dots, 2 \rightarrow 1, 1 \rightarrow (n-1)\}$. It is depicted in Figure 1.1 for $n \geq 5$.

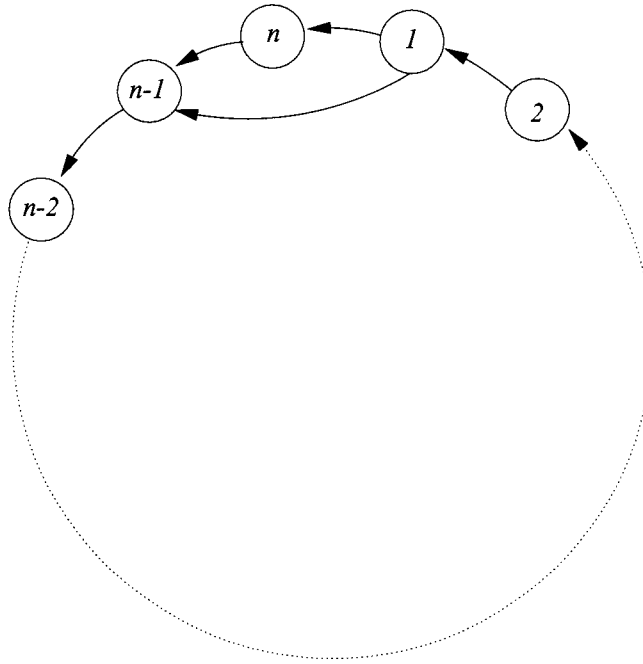


Figure 1.1: W_n

A $u \rightsquigarrow v$ walk W of length $|W| \geq 1$ in a digraph D (for u, v not necessarily distinct) is defined to be a sequence $u = u_1, u_2, \dots, u_{|W|+1} = v$ of vertices in D such that for $i = 1, 2, \dots, |W|$, $u_i \rightarrow u_{i+1}$ is an arc of D . To emphasize the adjacency between consecutive vertices in the sequence, the notation $u_1 \rightarrow u_2 \rightarrow \dots \rightarrow u_{|W|+1}$ is used for a $u \rightsquigarrow v$ walk. A *path* P is a walk with all vertices distinct. One walk *intersects* another if they have at least one vertex in common.

A *cycle* is a walk with distinct vertices except that the beginning and ending vertex are the same. A j -*cycle* is a cycle with length j . A Hamilton cycle in a digraph on n vertices is a cycle of length n . The set of all cycle lengths in the digraph D is denoted by $\lambda(D)$. The *circumference* of a digraph D is

the length of a longest cycle, and the *girth* is the length of a shortest cycle. For example, $\lambda(W_n) = \{n - 1, n\}$, implying that the circumference of W_n is n and its girth is $n - 1$.

A digraph D is *strongly connected* if there is a $u \rightsquigarrow v$ path for each ordered pair of vertices (u, v) . The *distance* $\delta[u, v]$ from vertex u to vertex v in a strongly connected digraph D is the length of a shortest $u \rightsquigarrow v$ path, with $\delta[u, u]$ defined to be zero. The *diameter* $\delta(D)$ of a strongly connected digraph D on n vertices is $\max\{\delta[u, v] \mid u, v \in V(D)\}$, which is less than or equal to $n - 1$. For example, W_n is strongly connected and $\delta(W_n) = \delta[n, 1] = n - 1$.

A digraph D is *primitive* if for some fixed positive integer m , there exists a $u \rightsquigarrow v$ walk of length m for each ordered pair of vertices (u, v) . The *exponent* of a primitive digraph D , denoted by $\gamma(D)$, is the smallest such m . It is well known (see, e.g., [21, p. 49]) that a digraph D is primitive if and only if it is strongly connected and $\gcd \lambda(D) = 1$. For example (see [22]), W_n is primitive and $\gamma(W_n) = (n - 1)^2 + 1$. The number $(n - 1)^2 + 1$ is denoted by ω_n .

Let D be a digraph on n vertices. The *adjacency matrix* $A(D) \equiv [a_{ij}]$ is the $n \times n$ matrix in which $a_{ij} = 1$ if there exists an arc $i \rightarrow j$ in D and $a_{ij} = 0$ otherwise. The adjacency matrix $A(D)$ is *irreducible* if and only if D is strongly connected, and $A(D)$ is defined to be a *primitive matrix* if and only if D is a primitive digraph. A special boolean arithmetic in which $0 + 0 = 0$, $1 + 0 = 1$ and $1 + 1 = 1$ is used for computing powers of adjacency matrices, implying that the exponent of a primitive matrix $A(D)$ is the smallest positive integer m such that $(A(D))^m$ is equal to the ‘all ones’ matrix. It is easily shown that the exponent of a digraph D is equal to the exponent of $A(D)$. A strongly connected digraph D is *minimally strong* if the digraph obtained by removing any one arc is not strongly connected. The adjacency matrix of a minimally strong digraph is nearly reducible.

The following notation is adapted from Heap and Lynn [5] and Lewin and Vitek [10]. For a strongly connected digraph D , let $d^c(\{c_1, c_2, \dots, c_s\})[u, v]$

denote the length of a shortest $u \rightsquigarrow v$ walk that intersects cycles of lengths c_1, c_2, \dots, c_s in D . Let $d^c(\{c_1, c_2, \dots, c_s\})$ be $\max\{d^c(\{c_1, c_2, \dots, c_s\})[u, v]\}$, noting that here u may equal v . If u is on cycles of lengths c_1, c_2, \dots, c_s , then $d^c(\{c_1, c_2, \dots, c_s\})[u, u]$ is defined to be zero. The *circumdiameter* of a strongly connected digraph D is $d^c(\lambda(D))$. Note that $d^c(\lambda(D)) \geq d^c(\{c_1, c_2, \dots, c_s\}) \geq \delta(D)$ for any subset of cycle lengths $\{c_1, c_2, \dots, c_s\}$ in D . For example, the circumdiameter of W_n is equal to $d^c(\lambda(W_n))[n, n]$, which is equal to n . The definitions of the concepts involved lead immediately to the following result.

Lemma 1.1.1 *Let D be a strongly connected digraph. If D' is a strongly connected subdigraph of D with $V(D') = V(D)$, then $\delta(D') \geq \delta(D)$. If, in addition, D is primitive and $\lambda(D') = \lambda(D)$, then $\gamma(D') \geq \gamma(D)$ and $d^c(\lambda(D')) \geq d^c(\lambda(D))$.*

1.2 Applications of Primitive Digraphs

Two applications of primitive digraphs are now considered.

1.2.1 Leslie Matrices

The following concepts are taken from [3, Chapter 2]. Age is a continuous variable, but can be broken up into a discrete set of age classes numbered 1 to n . Suppose that time is broken up into discrete intervals that are the same as these age classes. A population at time t can be represented by a column vector $u(t)$ with i^{th} entry equal to the number of individuals in age class i , with post reproductive age classes being ignored.

A *Leslie matrix* is a square nonnegative matrix $L = [l_{ij}]$ such that $u(t+1) = Lu(t)$. The only entries of L that can be positive are in the subdiagonal and the first row. For $2 \leq i \leq n$, $l_{i,i-1}$ is positive and represents the probability

that an individual survives age class $i - 1$, moving it into age class i . For $1 \leq i \leq n$, l_{1i} represents the per capita fertility of age class i , where l_{1n} is positive. The following matrix L is an example of a Leslie matrix with constant entries for a population with five age classes:

$$L = \begin{bmatrix} 0 & 0 & 0 & 0.8 & 0.5 \\ 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 \end{bmatrix}$$

This Leslie matrix $L = [l_{ij}]$ gives the strongly connected life cycle digraph with an arc $j \rightarrow i$ if and only if $l_{ij} > 0$ as in Figure 1.2. A sufficient condition for primitivity of the life cycle digraph is the existence of two adjacent age classes with positive fertility; thus most life cycle digraphs are primitive. Note that if there are exactly two adjacent reproductive age classes, then the life cycle digraph is isomorphic to W_n . The strong ergodic theorem [3, p. 86] shows that if the life cycle digraph is primitive, then the population tends towards a stable age distribution vector.

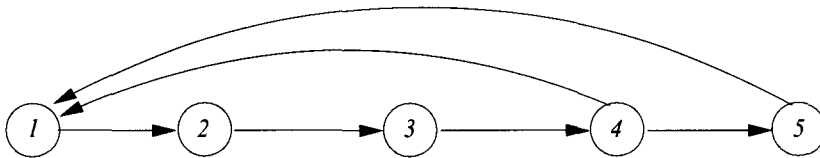


Figure 1.2: Life Cycle Digraph

1.2.2 Markov Chains

The following concepts are taken from [14, Chapter 5]. A *Markov chain* is a sequence of trials, each of which results in exactly one of the outcomes (states) in $\{1, 2, \dots, n\}$. It is characterized by a nonnegative *transition matrix* $P = [p_{ij}]$, where p_{ij} is the probability of transitioning from state i at the end of one trial to state j at the end of the next trial. The transition matrix P is *row stochastic*, meaning that $p_{ij} \geq 0$ and the entries in any row sum

to one. The i^{th} entry in the *probability (row) vector* $v(t)$ represents the probability that trial t will have outcome i . Thus $v(t) = v(t-1)P$, implying that a Markov chain has no memory of past outcomes; the outcome of trial t depends only on the outcome of trial $t-1$. This leads to the conclusion that $v(t) = v(0)P^t$, where $v(0)$ is the initial probability vector.

For example, consider the following weather pattern of a fictional region, where outcome 1 is sun, outcome 2 is cloud, and outcome 3 is rain. If it is sunny one day, then the probability of clouds the following day is $\frac{4}{5}$ and the probability of rain is $\frac{1}{5}$. If it is cloudy one day, then the probability of sun the following day is $\frac{3}{10}$, the probability of clouds is $\frac{1}{10}$ and the probability of rain is $\frac{3}{5}$. If it is rainy one day, then the probability of sun the following day is $\frac{3}{10}$, the probability of clouds is $\frac{1}{2}$ and the probability of rain is $\frac{1}{5}$. The transition matrix for this Markov chain is

$$P = \begin{bmatrix} 0 & \frac{4}{5} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{10} & \frac{3}{5} \\ \frac{3}{10} & \frac{1}{2} & \frac{1}{5} \end{bmatrix},$$

which has a primitive transition digraph. If it is sunny on the first day of the study, then the probability that it will be sunny, cloudy or rainy 30 days later is found by computing

$$v(0)P^{30} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.2308 & 0.4066 & 0.3626 \\ 0.2308 & 0.4066 & 0.3626 \\ 0.2308 & 0.4066 & 0.3626 \end{bmatrix} = \begin{bmatrix} 0.2308 \\ 0.4066 \\ 0.3626 \end{bmatrix}.$$

Note that the probability that it will be sunny 30 days later is 0.2308. The transition matrix P gives a transition digraph that has an arc $i \rightarrow j$ if and only if $p_{ij} > 0$ as in Figure 1.3.

In general, a strongly connected transition digraph corresponds to a family of Markov chains each with the property that it is possible to reach any trial outcome from any other trial outcome. A Markov chain corresponding to a primitive transition digraph tends to a stationary probability vector over the long term.

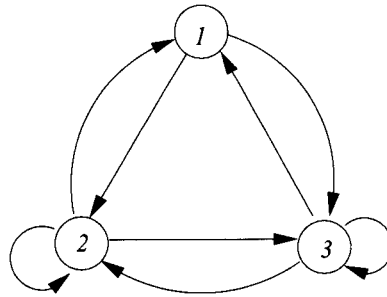


Figure 1.3: Transition Digraph

1.3 Fundamental Bounds for the Exponent

In 1950, Wielandt stated the following upper bound for the exponent of a primitive digraph [22]; his proof appears in [16]. Moreover, Wielandt concluded that any digraph for which equality holds in the following upper bound is isomorphic to W_n .

Theorem 1.3.1 [22] *If D is a primitive digraph on n vertices, then $\gamma(D) \leq \omega_n$.*

Another fundamental upper bound for the exponent of a primitive digraph was published in 1963 by Dulmage and Mendelsohn [4].

Theorem 1.3.2 [4, Theorem 1] *If D is a primitive digraph on n vertices with girth j , then $\gamma(D) \leq n + j(n - 2)$.*

All digraphs with exponent exactly equal to $n + j(n - 2)$ are characterized in [17]. Note that the Dulmage Mendelsohn bound of Theorem 1.3.2 is equivalent to the Wielandt bound of Theorem 1.3.1 for primitive digraphs with maximum girth $j = n - 1$. A possible improvement on the Dulmage-Mendelsohn bound was given in 1998 by Liu [11].

Theorem 1.3.3 [11, Theorem 2.2] *If D is a primitive digraph on n vertices with girth j and J is the set of vertices on j -cycles, then $\gamma(D) \leq n - |J| + (n - 1)j$.*

The above bound improves the Dulmage-Mendelsohn bound when $|J| > j$, and corresponds to the upper bound of [6, Corollary 1] when $j = 1$. The following two results give better upper bounds than Theorems 1.3.1 and 1.3.2, respectively, on the exponent of minimally strong primitive digraphs.

Theorem 1.3.4 [1, Theorem 4.2] *If D is a minimally strong primitive digraph on n vertices, then $6 \leq \gamma(D) \leq n^2 - 4n + 6$.*

Theorem 1.3.5 [15, Theorem 4.1] *If D is a minimally strong primitive digraph on n vertices with girth j , then $\gamma(D) \leq n + j(n - 3)$.*

A list of bounds for the exponent of various classes of primitive digraphs is given in [12, p. 112].

1.4 Diameter Bounds for the Exponent

If D is a digraph with adjacency matrix $A(D)$ and s is a positive integer, then D^s is the digraph with adjacency matrix $(A(D))^s$. Note by [18, Theorem 3.1] that $\delta(D^s) \leq \delta(D)$. The following upper bound is due to Liu.

Theorem 1.4.1 [11, Lemma 2.1] *If D is a primitive digraph on n vertices with girth j and J is the set of all vertices on j -cycles, then $\gamma(D) \leq \min\{n - |J|, \delta(D)\} + \delta(D^j)j$.*

The following bound of Shen in terms of diameter and girth is an improvement on the Dulmage-Mendelsohn bound (Theorem 1.3.2) for any primitive digraph

D on n vertices with $\delta(D) \leq n - 2$, and it is equivalent to the Dulmage-Mendelsohn bound if $\delta(D) = n - 1$.

Theorem 1.4.2 [20, Theorem 2] *If D is a primitive digraph with girth j , then $\gamma(D) \leq \delta(D) + 1 + j(\delta(D) - 1)$.*

If D is a primitive digraph on n vertices with girth $j = 1$ and $\delta(D) \leq n - 2$, then the Liu upper bound of Theorem 1.4.1 is better than that of Holladay and Varga [6, Corollary 1], since by [18, Theorem 3.1] $\delta(D^j) \leq \delta(D)$. The Liu bound of Theorem 1.4.1 is less than the Dulmage-Mendelsohn bound of Theorem 1.3.2 and the Liu bound of Theorem 1.3.3 if $\delta(D) \leq n - 2$, and less than the Shen bound of Theorem 1.4.2 if $\delta(D^j) < \delta(D)$. Note that for minimally strong primitive digraphs, the upper bound of Theorem 1.4.1 is less than that of Theorem 1.3.5 if $\delta(D) \leq n - 3$, and at least as good when $\delta(D) = n - 2$.

Neufeld [13, Section 5] describes a family of primitive digraphs \mathcal{F}_δ for $\delta \geq 2$. If $D \in \mathcal{F}_\delta$, then the vertex set of D is $V_0 \cup V_1 \cup \dots \cup V_\delta$, in which the V_i are pairwise disjoint and nonempty and V_0 consists of a single distinguished vertex. For $0 \leq i \leq \delta$, $u \rightarrow v$ is an arc of D for each $u \in V_i$ and $v \in V_{i+1}$, in which addition is taken modulo $\delta + 1$. The remaining arcs in D may be any set of arcs from V_δ to V_1 satisfying the condition that for each $u \in V_\delta$, there exists an arc $u \rightarrow v$ for some $v \in V_1$, and for each $v \in V_1$, there exists an arc $u \rightarrow v$ for some $u \in V_\delta$. The structure of these digraphs \mathcal{F}_δ is depicted in Figure 1.4 for $\delta \geq 6$; see [13, Figure 1]. From [13], each $D \in \mathcal{F}_\delta$ has diameter δ , and at least one cycle of each length δ and $\delta + 1$. The distinguished vertex is on a cycle of length $\delta + 1$ but on no cycle of length δ , and the length of every cycle in D is a nonnegative linear combination of δ and $\delta + 1$.

An upper bound for the exponent in terms of diameter is now given.

Theorem 1.4.3 [13, Theorem 4.1] [19, Main Theorem] *If D is a primitive digraph, then $\gamma(D) \leq \delta(D)^2 + 1$.*

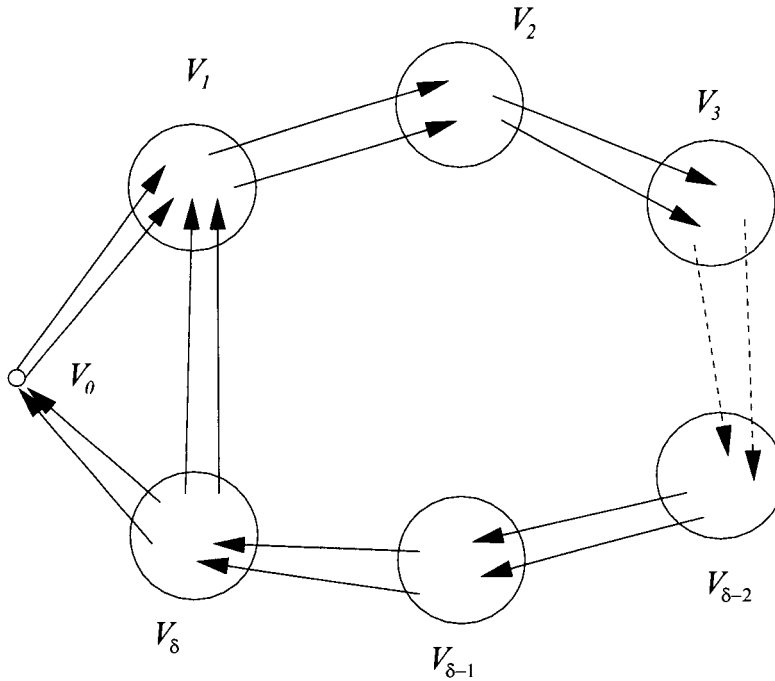


Figure 1.4: Structure of Neufeld Digraphs \mathcal{F}_δ

Neufeld also characterizes all digraphs that attain the upper bound for the exponent in Theorem 1.4.3.

Theorem 1.4.4 [13, Theorem 5.1] *Suppose that D is a primitive digraph with diameter δ . Then $\gamma(D) = \delta^2 + 1$ if and only if $D \in \mathcal{F}_\delta$.*

1.5 A Circumdiameter Bound

If $S = \{p_1, p_2, \dots, p_s\}$ is a set of relatively prime positive integers, then the *Frobenius-Schur index* of S , denoted by $\phi(S)$, is defined to be the least integer N such that each integer $n \geq N$ can be expressed in the form $a_1 p_1 + a_2 p_2 + \dots + a_s p_s$ for some nonnegative integers a_1, a_2, \dots, a_s . The following lemma and its proof, which give the explicit formula for $\phi(\{k, j\})$,

can be found in Brualdi and Ryser [2].

Lemma 1.5.1 [2, Lemma 3.5.5] *If k and j are two relatively prime positive integers, then $\phi(\{k, j\}) = (k - 1)(j - 1)$.*

The following upper bound for the exponent in terms of the Frobenius-Schur index and the circumdiameter is used extensively in subsequent chapters.

Theorem 1.5.2 [4, Theorem 3] [5, Theorem 4.1] *If D is a primitive digraph, then $\gamma(D) \leq \phi(\lambda(D)) + d^c(\lambda(D))$.*

Let D be a primitive digraph with $\lambda(D) = \{c_1, c_2, \dots, c_s\}$. An ordered pair of vertices (u, v) in D has the *unique path property* [4] if whenever there is a $u \rightsquigarrow v$ walk W with $|W| \geq d^c(\lambda(D))[u, v]$, then there exist nonnegative integers a_1, a_2, \dots, a_s so that $|W| = d^c(\lambda(D))[u, v] + a_1 c_1 + a_2 c_2 + \dots + a_s c_s$. Here u may be equal to v , in which case we say that u has the *unique path property*. Note that if all $u \rightsquigarrow v$ paths have the same length, then the ordered pair (u, v) has the unique path property. A related lower bound for the exponent is now given, which leads to a useful corollary identifying equality in Theorem 1.5.2.

Theorem 1.5.3 [4, Theorem 4] *If D is a primitive digraph that has an ordered pair of vertices (u, v) with the unique path property, then $\gamma(D) \geq \phi(\lambda(D)) + d^c(\lambda(D))[u, v]$.*

Corollary 1.5.4 [4, Corollary 2] *If D is a primitive digraph that has an ordered pair of vertices (u, v) with the unique path property and $d^c(\lambda(D)) = d^c(\lambda(D))[u, v]$, then $\gamma(D) = \phi(\lambda(D)) + d^c(\lambda(D))$.*

Note that W_n has cycle lengths n and $n - 1$, thus $\phi(\lambda(W_n)) = (n - 1)(n - 2)$. Vertex n in W_n is on exactly one cycle, implying that vertex n has the unique

path property. Since $d^c(\lambda(W_n)) = d^c(\lambda(W_n))[n, n] = n$, it follows from Corollary 1.5.4 that $\gamma(W_n) = \phi(\lambda(W_n)) + d^c(\lambda(W_n))$. Thus the exponent of W_n is equal to $(n - 1)(n - 2) + n$, which is equal to $(n - 1)^2 + 1 = \omega_n$.

1.6 Primitive Digraphs with Large Exponent

A primitive digraph D on $n \geq 3$ vertices has *large exponent* [8] if

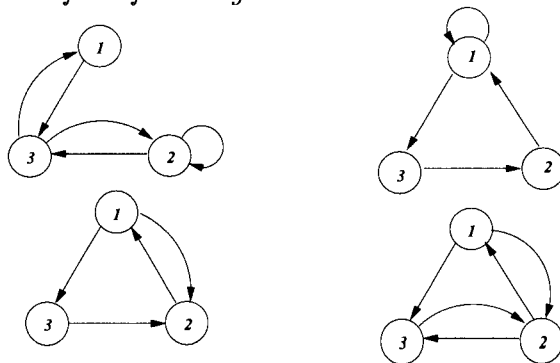
$$\gamma(D) \geq \lfloor \frac{\omega_n}{2} \rfloor + 2 = \lfloor \frac{(n - 1)^2 + 1}{2} \rfloor + 2. \quad (1.1)$$

Note that since $\omega_n \geq \lfloor \frac{\omega_n}{2} \rfloor + 2$ for $n \geq 3$, it follows that W_n has large exponent for $n \geq 3$. The following result of Lewin and Vitek [10] restricts the cycle structure of digraphs with large exponent.

Theorem 1.6.1 [10, Theorem 3.1] *A primitive digraph with large exponent has cycles of exactly two different lengths.*

Throughout this thesis, these two cycle lengths will be referred to as k and j , where k and j are relatively prime, and without loss of generality $k > j$.

Example 1.6.2 *All digraphs on $n = 3$ vertices with large exponent are isomorphic to one of the following :*



In [9], Kirkland, Olesky, and van den Driessche give an explicit relationship between k , j and n for primitive digraphs with large exponent. Note that the hypothesis $n \geq 4$ was omitted from [8, Theorem 1] and [9, Theorem 1.2]. The first digraph of Example 1.6.2 has cycles of lengths $k = 2$ and $j = 1$, exponent $4 = \lfloor \frac{\omega_3}{2} \rfloor + 2$, a loop at vertex 1, two vertex disjoint cycles and $(k - 2)j \leq \lfloor \frac{\omega_3}{2} \rfloor - 1$, therefore the hypothesis $n \geq 4$ is required in the following results.

Theorem 1.6.3 [9, Theorem 1.2] *Let $n \geq 4$, and let k and j be relatively prime with $n \geq k > j$. There exists a primitive digraph D on n vertices with $\lambda(D) = \{k, j\}$ and large exponent if and only if $(k - 2)j \geq \lfloor \frac{\omega_n}{2} \rfloor + 2 - n$.*

Corollary 1.6.4 *A primitive digraph on $n \geq 4$ vertices with large exponent has no loops.*

Proof: Suppose D is a primitive digraph with large exponent and $\lambda(D) = \{k, 1\}$. Thus $k - 2 \geq \lfloor \frac{\omega_n}{2} \rfloor + 2 - n$ by Theorem 1.6.3, implying that $k > n$ if $n \geq 4$, which is a contradiction. Thus D has no loops if $n \geq 4$. \square

The following lemma, which is also proved in [8, proof of Theorem 1], gives a lower bound on j for digraphs on n vertices with large exponent and shows that $k + j > n$.

Lemma 1.6.5 *If D is a primitive digraph on $n \geq 4$ vertices with large exponent and $\lambda(D) = \{k, j\}$, then $j \geq \lceil \frac{n-1}{2} \rceil$. Furthermore, if $j = \frac{n-1}{2}$, then $k = n$; if $j = \frac{n}{2}$, then $k = n - 1$; and if $j = \frac{n+1}{2}$, then $k \geq n - 2$.*

Proof: Suppose that $n = 2m + 1$ is odd. By [8, Corollary 1.1], $j \geq \lceil \frac{\gamma(D) - n}{n-2} \rceil \geq \lceil \frac{2m^2 - 2m + 1}{2m-1} \rceil = m$, thus $j \geq \lceil \frac{n-1}{2} \rceil$. If $j = \frac{n-1}{2}$, then by Theorem 1.6.3, $(k - 2)j \geq \lfloor \frac{\omega_n}{2} \rfloor + 2 - n$, implying that $\frac{n-1}{2}(k - 2) \geq \frac{(n-1)^2}{2} + 2 - n$, from which it follows that $k - 2 \geq n - 3 + \frac{2}{n-1}$. Thus $k \geq n$, giving $k = n$. Suppose that $n = 2m$ is even. From [8, Corollary 1.1],

$j \geq \lceil \frac{\gamma(D)-n}{n-2} \rceil \geq \lceil \frac{2m^2-4m+3}{2m-2} \rceil = m = \frac{n}{2}$, thus $j \geq \lceil \frac{n-1}{2} \rceil$. If $j = \frac{n}{2}$, then by Theorem 1.6.3, $\frac{n}{2}(k-2) \geq \frac{(n-1)^2+1}{2} + 2 - n$, from which it follows that $k-2 \geq n-4 + \frac{6}{n}$, thus $k \geq n-1$. Since k and j are relatively prime, $k = n-1$. If $n = 2m+1$ is odd and $j = \frac{n+1}{2}$, then by Theorem 1.6.3, $(k-2)j \geq \lfloor \frac{\omega_n}{2} \rfloor + 2 - n$, implying that $\frac{n+1}{2}(k-2) \geq \frac{(n-1)^2}{2} + 2 - n$, from which it follows that $k-2 \geq n-5 + \frac{10}{n+1}$. Thus $k \geq n-2$. \square

For fixed n such that $3 \leq n \leq 40$, Figure 1.5 displays all possible circumferences k for a primitive digraph on n vertices with large exponent. These circumferences can be obtained using Example 1.6.2 and Theorem 1.6.3 with $j = k-1$. The proof of Theorem 1.6.3 that is given in [9] defines a family of primitive digraphs with large exponent for each possible pair n, k with $n \geq 4$ in Figure 1.5.

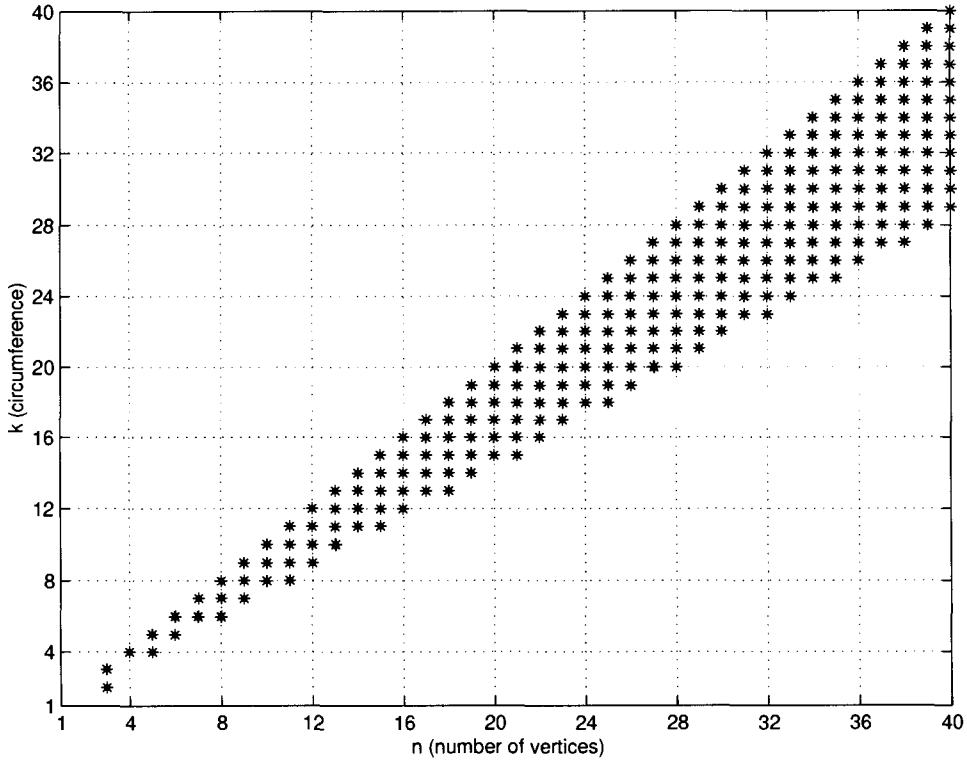


Figure 1.5: Possible n, k Values for Primitive Digraphs with Large Exponent

The following lemma and its proof are taken from [8, proof of Theorem 1].

Lemma 1.6.6 *In a primitive digraph on $n \geq 4$ vertices with large exponent, no two cycles are vertex disjoint.*

Proof: Let D be a digraph on $n \geq 4$ vertices with large exponent and $\lambda(D) = \{k, j\}$. If $j > \frac{n}{2}$, then a pair of vertex disjoint cycles would result in the contradiction $|V(D)| \geq 2j > n$.

Suppose that $n = 2m$ is even and $j \leq m$. From Lemma 1.6.5, $j = \frac{n}{2}$ and $k = n - 1$. If there are two vertex disjoint cycles, then they are j -cycles and every vertex is on a j -cycle with at most one vertex not on a k -cycle. Thus $d^c(\lambda(D)) \leq n$, and by Theorem 1.5.2, $\gamma(D) \leq (k - 1)(j - 1) + n < \lfloor \frac{(n-1)^2+1}{2} \rfloor + 2$, contradicting the fact that D has large exponent.

Suppose that $n = 2m + 1$ is odd and $j \leq m$. From Lemma 1.6.5, $j = \frac{n-1}{2}$ and $k = n$. If the digraph has two vertex disjoint cycles, then they are j -cycles and every vertex is on a k -cycle with at most one vertex not on a j -cycle. Thus $d^c(\lambda(D)) = n$, and by Theorem 1.5.2, $\gamma(D) \leq (k - 1)(j - 1) + n < \lfloor \frac{(n-1)^2+1}{2} \rfloor + 2$, again giving a contradiction. \square

Lemma 1.6.7 *If D is a primitive digraph on $n \geq 4$ vertices with circumference k that has a doubly directed k -cycle, then D does not have large exponent.*

Proof: Let D be a primitive digraph on $n \geq 4$ vertices with large exponent and $\lambda(D) = \{k, j\}$. Suppose that D has a doubly directed k -cycle. By Theorem 1.6.3, $k \neq 2$, thus $j = 2$, implying that $n \leq 5$ and $k = n$. However, by [9, Lemma 2.1] D has exactly one n -cycle. Thus if D is a primitive digraph with circumference k that has a doubly directed k -cycle, then D does not have large exponent. \square

Corollary 1.6.8 *If D is a primitive digraph on $n \geq 4$ vertices that has a symmetric adjacency matrix, then D does not have large exponent.*

The following theorem gives a useful lower bound on the diameter of a primitive digraph with large exponent.

Theorem 1.6.9 [10, Theorem 3.2] *If D is a primitive digraph with cycles of exactly two lengths k and j with $k > j$, and if the adjacency matrix of D is not symmetric, then there is an edge $u \rightarrow v$ in D such that $\delta[v, u] = k - 1$.*

If D is a primitive digraph on $n \geq 4$ vertices with large exponent and cycles of exactly two lengths k and j with $k > j$, then the results given in Lemma 1.6.5 imply that $k + j > n$. By Corollary 1.6.8, Theorem 1.6.9 and [10, Lemma 3.2]

$$k - 1 \leq \delta(D) \leq d^c(\lambda(D)) \leq n + k - j - 1. \quad (1.2)$$

For D with these properties, using Lemma 1.5.1, the circumdiameter bound for the exponent (Theorem 1.5.2) can be restated as

$$\gamma(D) \leq (k - 1)(j - 1) + d^c(\lambda(D)). \quad (1.3)$$

Since by Corollary 1.6.8, $A(D)$ is not symmetric, [10, Theorem 4.1] gives

$$(k - 1)j \leq \gamma(D) \leq n + (k - 2)j. \quad (1.4)$$

The upper bound in (1.4) can be obtained by substituting the maximum possible circumdiameter from (1.2) into (1.3). If the additional substitution $k = n$ is made, the resulting upper bound is the Dulmage and Mendelsohn bound of Theorem 1.3.2.

The next corollary follows from Theorem 1.4.1.

Corollary 1.6.10 *If D is a primitive digraph on $n \geq 4$ vertices with large exponent and girth j , then $\gamma(D) \leq n - |J| + \delta(D)j$, where J is the set of all vertices on j -cycles.*

Proof: Let $\lambda(D) = \{j, k\}$. By Theorem 1.4.1, $\gamma(D) \leq \min\{n - |J|, \delta(D)\} + \delta(D^j)j$. Since D has large exponent, it follows by Lemma 1.6.5 that $k + j > n$. By Corollary 1.6.8, Theorem 1.6.9 and [10, Lemma 3.2] it follows that $k - 1 \leq \delta(D)$. Thus $n - |J| \leq n - j \leq k - 1 \leq \delta(D)$, implying that $\gamma(D) \leq n - |J| + \delta(D^j)j$. By [18, Theorem 3.1] $\delta(D^j) \leq \delta(D)$, implying that $\gamma(D) \leq n - |J| + \delta(D)j$. \square

1.7 Thesis Outline

The remainder of this thesis focusses on properties of primitive digraphs, emphasizing those with cycles of exactly two lengths; these include digraphs with large exponent. In Chapter 2, a family of digraphs on n vertices and cycles of exactly two lengths k and $k - 1$ is defined that contains a digraph with each diameter between $k - 1$ and $n - 1$. A new upper bound for the circumdiameter of a primitive digraph with cycles of exactly two lengths k and j with $k + j > n$ is given in Chapter 3 (in Theorem 3.1.1) and some sufficient conditions for which $\gamma(D) = \phi(\lambda(D)) + d^c(\lambda(D))$, i.e., equality holds in Theorem 1.5.2, are established. In Chapter 4, three families of digraphs are given for which equality holds in Theorem 1.5.2. Two families of digraphs with large exponent for which equality does not hold in Theorem 1.5.2 are given in Sections 3.2 and 4.4. In Chapter 5, expressions are given for the diameter, circumdiameter and exponent of all primitive digraphs on n vertices with circumference n and large exponent and some primitive digraphs with circumference $n - 1$ and large exponent.

Appendix A contains digraphs that support the first statement in Theorem 3.2.6, proving that equality holds in Theorem 1.5.2 for all primitive digraphs with large exponent on $3 \leq n \leq 8$ vertices. Appendix B contains digraphs that show this equality fails for primitive digraphs with large exponent on $n = 9$ vertices. Finally, Appendix C describes the software used to construct the lists of digraphs in Appendices A and B.

Chapter 2

Diameters of Primitive Digraphs with Two Cycle Lengths

In this chapter, the set of possible diameters of primitive digraphs D on $n \geq 3$ vertices with $\lambda(D) = \{k, j\}$, $k > j$ is considered. Clearly, $\delta(D) \leq n-1$, and if $A(D)$ is not symmetric, then $\delta(D) \geq k-1$; see Theorem 1.6.9. For fixed n and k , the following theorem shows that a primitive digraph D can be constructed with $\lambda(D) = \{k, k-1\}$ and any diameter such that $k-1 \leq \delta(D) \leq n-1$. Its exponent is given in the subsequent lemma, and conditions under which it has large exponent are stated.

Vertex m replicates vertex l [9] in a digraph D with no loops if and only if

$$\{i \mid l \rightarrow i \in E(D)\} = \{i \mid m \rightarrow i \in E(D)\}$$

and

$$\{i \mid i \rightarrow l \in E(D)\} = \{i \mid i \rightarrow m \in E(D)\}.$$

Note that in $A(D)$, rows (and columns) m and l are identical. Replicated vertices are represented in figures by an ellipse containing a list of vertex numbers.

Theorem 2.1 *Let n and k be fixed so that $n \geq k \geq 3$. For any d such that $k - 1 \leq d \leq n - 1$, there exists a primitive digraph $D(n, k, d)$ on n vertices with diameter d and cycles of exactly two lengths, k and $k - 1$.*

Proof: Fix n and k so that $n \geq k \geq 3$. Consider the Wielandt digraph W_k with vertices $\{1, 2, \dots, k\}$, cycle $1 \rightarrow k \rightarrow (k - 1) \rightarrow \dots \rightarrow 2 \rightarrow 1$ and the arc $1 \rightarrow (k - 1)$. If $d \geq k$ then add the vertices $\{k + 1, k + 2, \dots, d + 1\}$ and the arcs $1 \rightarrow (d + 1)$ and $(k + 1) \rightarrow (2k - d - 1)$. If $d > k$, then add the arcs $(d + 1) \rightarrow d \rightarrow (d - 1) \rightarrow \dots \rightarrow (k + 2) \rightarrow (k + 1)$. If $d + 2 \leq n$ then add vertices $\{d + 2, d + 3, \dots, n\}$, and arcs so that each of these vertices replicates vertex 1. The resulting digraph is denoted $D(n, k, d)$; see Figure 2.1.

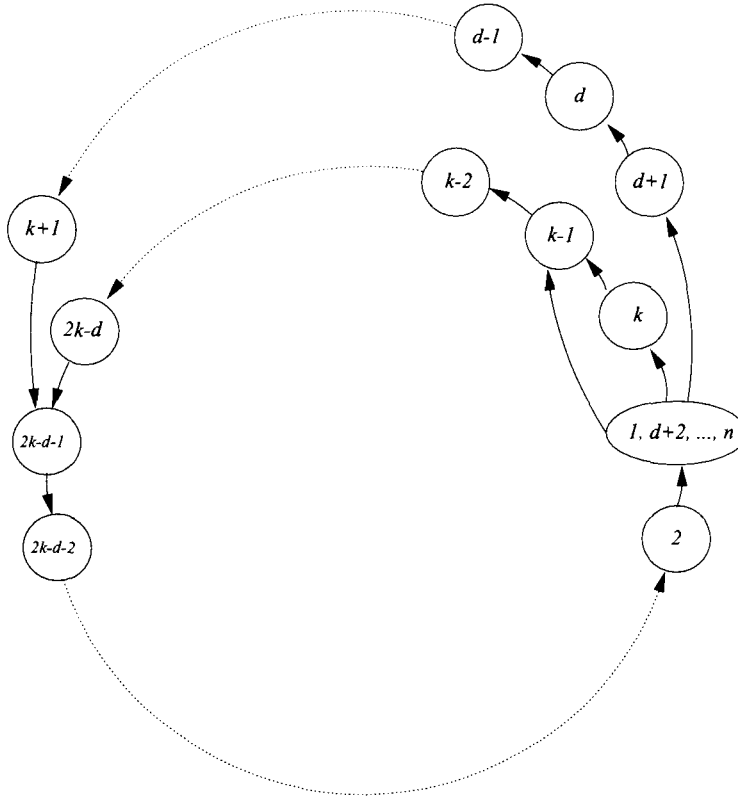


Figure 2.1: Digraph $D(n, k, d)$ if $n - 2 \geq d \geq k$

$D(n, k, d)$ always contains the two cycles $1 \rightarrow k \rightarrow (k-1) \rightarrow \cdots \rightarrow 2 \rightarrow 1$ with length k and $1 \rightarrow (k-1) \rightarrow (k-2) \rightarrow \cdots \rightarrow 2 \rightarrow 1$ with length $k-1$. If $d \geq k$, then $D(n, k, d)$ also contains the k -cycle $1 \rightarrow (d+1) \rightarrow d \rightarrow (d-1) \rightarrow \cdots \rightarrow (k+1) \rightarrow (2k-d-1) \rightarrow (2k-d-2) \rightarrow \cdots \rightarrow 2 \rightarrow 1$. Furthermore, if $d+2 \leq n$, then $D(n, k, d)$ contains additional cycles in which each of the vertices $d+2, d+3, \dots, n$ replaces vertex 1 in the above cycles, since vertices $d+2, d+3, \dots, n$ replicate vertex 1. Thus the only two cycle lengths of $D(n, k, d)$ are k and $k-1$, implying that $D(n, k, d)$ is primitive.

To prove that $\delta(D(n, k, d)) = d$, it is sufficient to show that $\delta[u, v] \leq d$ for any $u, v \in V(D(n, k, d))$ and exhibit a pair of vertices u, v for which $\delta[u, v] = d$. If u and v are vertices in $\{1, 2, \dots, k\}$, then $\delta[u, v] \leq k-1 \leq d$. If $k \leq d$ and u and v are vertices in $\{k+1, k+2, \dots, d+1\}$, then $\delta[u, v] \leq k-1 \leq d$. If u is a vertex in $\{1, 2, \dots, k\}$ and $k+1 \leq v \leq d+1$, then $\delta[u, v] \leq \delta[k, k+1] = \delta[k, 1] + \delta[1, k+1] = k-1 + [(d+1) - (k+1) + 1] = d$. Also, $\delta[v, u] \leq \delta[d+1, 2k-d] = \delta[d+1, k+1] + 1 + \delta[2k-d-1, 2k-d] \leq (d+1) - (k+1) + 1 + k-1 = d$.

If u is a vertex in $\{d+2, d+3, \dots, n\}$ and v is in $\{1, 2, \dots, d+1\}$, then $\delta[u, v] = \delta[1, v] \leq d$ from above, and $\delta[v, u] = \delta[v, 1] \leq d$ from above. If u and v are vertices in $\{d+2, d+3, \dots, n\}$, then $\delta[u, v] = k-1$. Thus $\delta[u, v] \leq d$ for all u, v and $\delta[k, k+1] = d$, implying that $\delta(D(n, k, d)) = d$. \square

Lemma 2.2 *If $d = k-1$, then the circumdiameter of the digraph $D(n, k, d)$ of Theorem 2.1 is k and its exponent is $(k-1)^2 + 1$. If $d \geq k$, then the circumdiameter of $D(n, k, d)$ is d and its exponent is $(k-1)(k-2) + d$.*

Proof: If $d = k-1$, then $D(n, k, d)$ is just the Wielandt digraph W_k with the addition of arcs so that vertices in $\{k+1, k+2, \dots, n\}$ replicate vertex 1. Thus $d^c(\lambda(D(n, k, d))) = d^c(\lambda(W_k)) = k$, and $\gamma(D(n, k, d)) = \gamma(W_k) = (k-1)^2 + 1$.

Suppose that $d \geq k$. If u and v are distinct vertices in $\{1, 2, \dots, k\}$, then

they share a k -cycle. Thus $d^c(\lambda(D(n, k, d)))[u, v] \leq k - 1 < d$. If $u \in \{1, 2, \dots, k - 1\}$, then $d^c(\lambda(D(n, k, d)))[u, u] = 0$. Vertex k is on a k -cycle only, thus $d^c(\lambda(D(n, k, d)))[k, k] = k \leq d$.

If $k + 1 \leq u \leq d + 1$, then $d^c(\lambda(D(n, k, d)))[u, k] \leq k \leq d$ since any circumpath from u to k must pass through vertex 2, which is on cycles of both lengths. Similarly $d^c(\lambda(D(n, k, d)))[k, u] \leq \delta[k, k + 1] = d$. If $v \in \{1, 2, \dots, k - 1\}$ and $d \geq k + 1$, then $d^c(\lambda(D(n, k, d)))[u, v] \leq \delta[d + 1, 2k - d] = d - 1$ and $d^c(\lambda(D(n, k, d)))[v, u] \leq \delta[k - 1, k + 1] = \delta[k, k + 1] - 1 = d - 1$. If $v \in \{1, 2, \dots, k - 1\}$ and $d = k$, then $d^c(\lambda(D(n, k, d)))[u, v] \leq d^c(\lambda(D(n, k, d)))[k + 1, 1] = k = d$.

If $d \geq k$ and u and v are vertices in $\{k + 1, k + 2, \dots, d + 1\}$ with $u \leq v$, then $d^c(\lambda(D(n, k, d)))[u, v] \leq k \leq d$ since any circumpath from u to v must pass through vertex 2, which is on cycles of both lengths. These vertices are on k -cycles only, implying that any circumpath from v to u must include at least one k -cycle in order to intersect cycles of both lengths. Thus $d^c(\lambda(D(n, k, d)))[v, u] \leq \delta[d + 1, k + 1] + k = (d + 1) - (k + 1) + k = d$.

If $n \geq d + 2$, then each of the vertices in $\{d + 2, d + 3, \dots, n\}$ replicates vertex 1, and the above cases imply that if u and/or v is in $\{d + 2, d + 3, \dots, n\}$, then $d^c(\lambda(D(n, k, d)))[u, v] \leq d$. Thus $d^c(\lambda(D(n, k, d))) \leq d$ and since $d^c(\lambda(D(n, k, d))) \geq d$, it follows that $d^c(\lambda(D(n, k, d))) = d$.

If $d \geq k$, then by (1.3), $\gamma(D(n, k, d)) \leq (k - 1)(k - 2) + d^c(\lambda(D(n, k, d))) = (k - 1)(k - 2) + d$. To precisely determine the exponent, note that any path from vertex k to vertex $k + 1$ is of the form $k \rightarrow (k - 1) \rightarrow \dots \rightarrow 2 \rightarrow a \rightarrow (d + 1) \rightarrow d \rightarrow \dots \rightarrow (k + 1)$, where a is a vertex in $\{1, d + 2, d + 3, \dots, n\}$. These paths all have length d (see the proof that $\delta(D(n, k, d)) = d$). This implies that any walk from vertex k to vertex $k + 1$ has length $a_1 k + a_2(k - 1) + d$ for some nonnegative integers a_1, a_2 , implying that the ordered pair $(k, k + 1)$ has the unique path property. Thus by Corollary 1.5.4, $\gamma(D(n, k, d)) = (k - 1)(k - 2) + d$ when $d \geq k$. \square

because then $a_1k + a_2(k-1) = (k-1)(k-2) - 1$ which is a contradiction.

Note that if $d = k-1$, then $\gamma(D(n, k, d)) = [\delta(D(n, k, d))]^2 + 1$, and $D(n, k, d)$ belongs to the family of digraphs \mathcal{F}_d described in Section 1.4. The following is an immediate consequence of the definition of large exponent.

Corollary 2.3 *If $d = k-1$, then $D(n, k, d)$ has large exponent if and only if $(k-1)^2 + 1 \geq \lfloor \frac{\omega_n}{2} \rfloor + 2$. If $d \geq k$, then $D(n, k, d)$ has large exponent if and only if $(k-1)(k-2) + d \geq \lfloor \frac{\omega_n}{2} \rfloor + 2$.*

For the digraph defined in Theorem 2.1, it is now shown that equality holds in the result of Theorem 1.5.2.

Corollary 2.4 $\gamma(D(n, k, d)) = \phi(\lambda(D(n, k, d))) + d^c(\lambda(D(n, k, d)))$.

Proof: By Lemma 1.5.1, $\phi(\lambda(D(n, k, d))) = (k-1)(k-2)$. By Lemma 2.2, if $d = k-1$, then $\phi(\lambda(D(n, k, d))) + d^c(\lambda(D(n, k, d))) = (k-1)(k-2) + k = (k-1)^2 + 1 = \gamma(D(n, k, d))$; whereas if $d \geq k$, then $\phi(\lambda(D(n, k, d))) + d^c(\lambda(D(n, k, d))) = (k-1)(k-2) + d = \gamma(D(n, k, d))$. \square

In the remainder of this section, the set of possible diameters of primitive digraphs D on $n \geq 3$ vertices with large exponent is considered. In the following result, a construction is given to show the existence of such digraphs with diameter δ , where δ is restricted by the inequality of Theorem 1.4.3.

Theorem 2.5 *There exists a primitive digraph on n vertices with diameter δ and large exponent if and only if $\delta^2 + 1 \geq \lfloor \frac{\omega_n}{2} \rfloor + 2$.*

Proof: Let D be a primitive digraph on n vertices with diameter δ such that $\delta^2 + 1 < \lfloor \frac{\omega_n}{2} \rfloor + 2$. Since $\gamma(D) \leq \delta^2 + 1$ by Theorem 1.4.3, D does

not have large exponent. Thus a primitive digraph D on n vertices with diameter δ and large exponent exists only if $\delta^2 + 1 \geq \lfloor \frac{\omega_n}{2} \rfloor + 2$.

Suppose that $\delta^2 + 1 \geq \lfloor \frac{\omega_n}{2} \rfloor + 2$. Consider the digraph D on n vertices with a cycle $1 \rightarrow 2 \rightarrow \dots \rightarrow (\delta+1) \rightarrow 1$, arc $(\delta+1) \rightarrow 2$, and arcs so that vertices $\delta+2, \delta+3, \dots, n$ replicate vertex 2. Thus $D \in \mathcal{F}_\delta$ (see Section 1.4), implying that D is primitive with diameter δ , and $\gamma(D) = \delta^2 + 1$ by Theorem 1.4.4. Since $\delta^2 + 1 \geq \lfloor \frac{\omega_n}{2} \rfloor + 2$, D has large exponent. Thus if $\delta^2 + 1 \geq \lfloor \frac{\omega_n}{2} \rfloor + 2$, then a primitive digraph on n vertices exists with diameter δ and large exponent. \square

If D is a primitive digraph on $n \geq 3$ vertices with $\lambda(D) = \{k, j\}$, $k > j$ and $A(D)$ is not symmetric, then from the above discussion $k-1 \leq \delta(D) \leq n-1$. Recall that if D on $n \geq 4$ vertices has large exponent, then $A(D)$ is not symmetric (Corollary 1.6.8) and $(k-2)j \geq \lfloor \frac{\omega_n}{2} \rfloor + 2 - n$ (Theorem 1.6.3). For a fixed n and k , the following example illustrates that there may not exist a digraph D on n vertices with large exponent, circumference k and each of the possible diameters δ such that $k-1 \leq \delta \leq n-1$.

Let D be a digraph with $n = 8$ vertices and cycle lengths $k = 6$ and $j = 5$. Since $(k-2)j = 20 > 19 = \lfloor \frac{\omega_n}{2} \rfloor + 2 - n$, there exist such digraphs D with large exponent by Theorem 1.6.3. Since $A(D)$ is not symmetric, it follows that $5 = k-1 \leq \delta(D) \leq n-1 = 7$. However, $\delta(D)$ cannot be equal to 5 (by Theorem 1.4.3) since this would imply that $\gamma(D) \leq \delta(D)^2 + 1 = 26 < 27 = \lfloor \frac{\omega_n}{2} \rfloor + 2$. Thus there does not exist such a digraph D with large exponent and $\delta(D) = 5$.

For fixed n such that $3 \leq n \leq 40$, Figure 2.2 displays all possible diameters δ for a primitive digraph with large exponent. These results were obtained using Theorem 2.5. Note that the pairs (n, δ) are a subset of the pairs $(n, k-1)$ of Figure 1.5. The example of the previous paragraph with $n = 8$ illustrates that they are a proper subset.

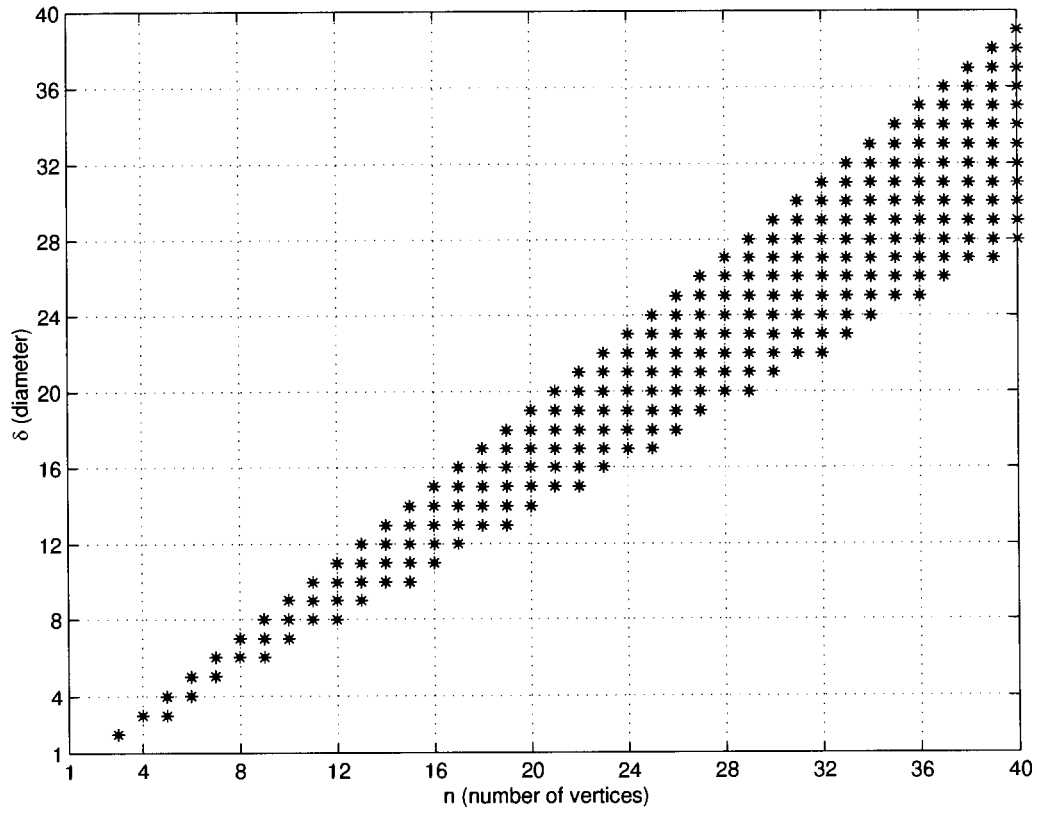


Figure 2.2: Possible n, δ Values for Primitive Digraphs with Large Exponent

Chapter 3

Exponent and Circumdiameter

3.1 A New Bound for the Circumdiameter

A possible improvement on the upper bound of (1.2) follows for primitive digraphs on $n \geq 4$ vertices with cycles of exactly two cycle lengths k and j in which $k + j > n$. Its proof is modelled after the proof of [10, Lemma 3.2].

Theorem 3.1.1 *Let D be a primitive digraph on n vertices with cycles of exactly two lengths k and $j < k$, let K be the set of all vertices on k -cycles and J the set of all vertices on j -cycles. If $k + j > n$, then*

$$d^c(\lambda(D)) \leq \max\{k, \delta(D), n + k - |J| - 1, n + j - |K| - 1\}.$$

Proof: Since $k + j > n$, it follows that each j -cycle intersects every k -cycle and each k -cycle intersects every j -cycle.

Let u and v be vertices in D . Suppose $u \in J$. Then $d^c(\lambda(D))[u, u] \leq j$, and thus $d^c(\lambda(D))[u, u] \leq n + k - |J| - 1$ since $|J| \leq n$ and each j -cycle intersects

every k -cycle. If $\delta[u, v] \geq n - |K|$, then every $u \rightsquigarrow v$ walk intersects a k -cycle, implying that $d^c(\lambda(D))[u, v] = \delta[u, v] \leq \delta(D)$. If $\delta[u, v] < n - |K|$, then there exists a $u \rightsquigarrow v$ walk that starts with a j -cycle and intersects a k -cycle, implying that $d^c(\lambda(D))[u, v] \leq j + \delta[u, v] \leq j + n - |K| - 1$. Thus $d^c(\lambda(D))[u, v] \leq \max\{\delta(D), j + n - |K| - 1\}$.

Suppose $u \notin J$. Thus $u \in K$ and $d^c(\lambda(D))[u, u] = k$ since D is strongly connected and each k -cycle intersects every j -cycle. If $\delta[u, v] \geq n - |J|$, then every $u \rightsquigarrow v$ walk intersects a j -cycle, implying that $d^c(\lambda(D))[u, v] = \delta[u, v] \leq \delta(D)$. If $\delta[u, v] < n - |J|$, then there exists a $u \rightsquigarrow v$ walk that starts with a k -cycle and intersects a j -cycle, implying that $d^c(\lambda(D))[u, v] \leq k + \delta[u, v] \leq k + n - |J| - 1$. Thus $d^c(\lambda(D))[u, v] \leq \max\{k, \delta(D), k + n - |J| - 1\}$. \square

The above upper bound on the circumdiameter implies an upper bound on the exponent.

Corollary 3.1.2 *Let D be a primitive digraph on n vertices with cycles of exactly two lengths k and $j < k$, let K be the set of all vertices on k -cycles and J the set of all vertices on j -cycles. If $k + j > n$ and $m = \max\{k, \delta(D), n + k - |J| - 1, n + j - |K| - 1\}$, then $\gamma(D) \leq (k - 1)(j - 1) + m$.*

It is easily shown that the upper bound of Corollary 3.1.2 is always at least as good as that of [17, Lemma 2.1]. The following corollary draws a natural conclusion for a primitive digraph with large exponent.

Corollary 3.1.3 *Let D be a primitive digraph on n vertices with cycles of lengths k and $j < k$, let K be the set of all vertices on k -cycles and J the set of all vertices on j -cycles. If D has large exponent, then $d^c(\lambda(D)) \leq \max\{k, \delta(D), n + k - |J| - 1, n + j - |K| - 1\}$.*

The following three examples show that the maximum in Corollary 3.1.3 (Theorem 3.1.1) can be attained by any one of the four given values and can

be strictly less than the previously known upper bound of (1.2).

Example 3.1

Let D be the primitive digraph on $n = 14$ vertices depicted in Figure 3.1. Then D has cycles of lengths $j = 9$ and $k = 11$, and the diameter of D is 11 (e.g., $\delta[13, 6] = 11$). If J is the set of all vertices on j -cycles, then $|J| = 14$. If K is the set of all vertices on k -cycles, then $|K| = 12$. The digraph D has large exponent since $\gamma(D) = 91 > \lfloor \frac{\omega_n}{2} \rfloor + 2 = 87$. By (1.2), $d^c(\lambda(D)) \leq n + k - j - 1 = 15$. Since $k = 11$, $\delta(D) = 11$, $n + k - |J| - 1 = 10$, and $n + j - |K| - 1 = 10$, it follows by Theorem 3.1.1 that $d^c(\lambda(D)) \leq \delta(D) = k = 11$. Note that $\delta(D) = k < n + k - j - 1$ in this case and $d^c(\lambda(D)) = d^c(\lambda(D))[13, 6] = 11$.

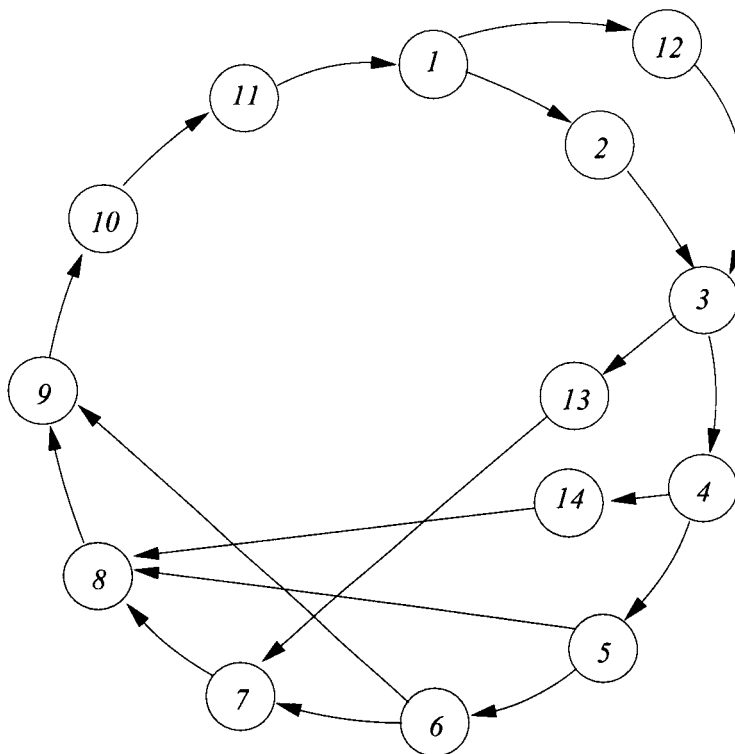


Figure 3.1: Circumdiameter = $k = \delta(D)$

Example 3.2

Let D be the primitive digraph on $n = 12$ vertices depicted in Figure 3.2. Then D has cycles of lengths $j = 8$ and $k = 9$, and the diameter of D is 9 (e.g., $\delta[10, 11] = 9$). If J is the set of all vertices on j -cycles, then $|J| = 10$. If K is the set of all vertices on k -cycles, then $|K| = 12$. The digraph D has large exponent since $\gamma(D) = 66 > \lfloor \frac{\omega n}{2} \rfloor + 2 = 63$. By (1.2), $d^c(\lambda(D)) \leq n + k - j - 1 = 12$. Since $k = 9$, $\delta(D) = 9$, $n + k - |J| - 1 = 10$, and $n + j - |K| - 1 = 7$, by Theorem 3.1.1, $d^c(\lambda(D)) \leq n + k - |J| - 1 = 10$. Note that $n + k - |J| - 1 < n + k - j - 1$ in this case, and $d^c(\lambda(D)) = d^c(\lambda(D))[10, 3] = 10$.

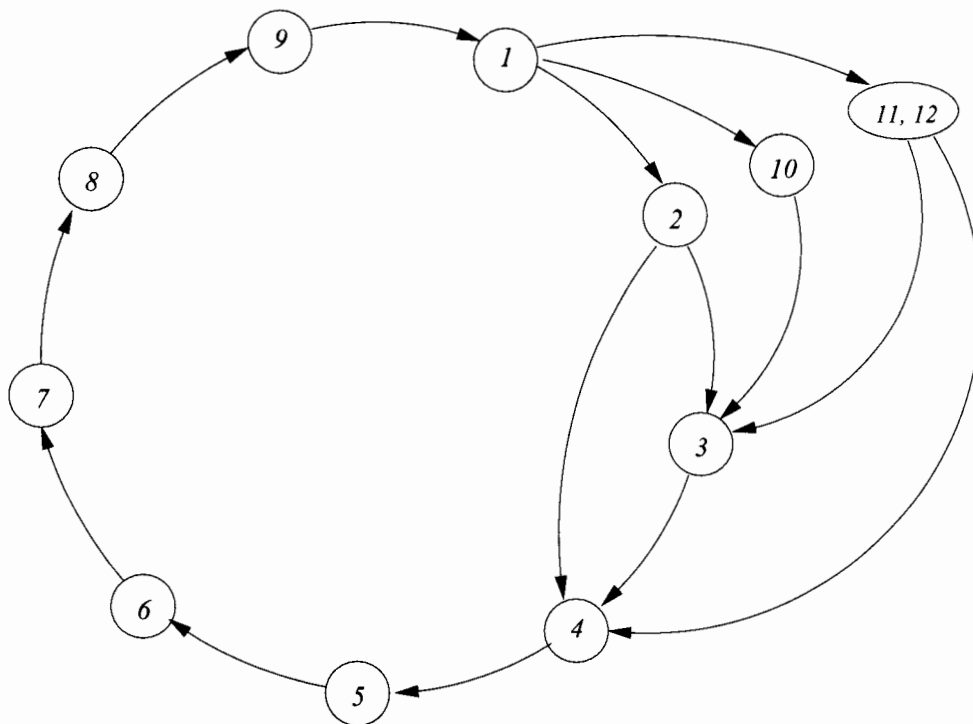


Figure 3.2: Circumdiameter = $n + k - |J| - 1 < n + k - j - 1$

Example 3.3

Let D be the primitive digraph on $n = 15$ vertices depicted in Figure 3.3. Then D has cycles of lengths $j = 10$ and $k = 11$, and the diameter of D is 11 (e.g., $\delta[2, 15] = 11$). If J is the set of all vertices on j -cycles, then $|J| = 15$. If K is the set of all vertices on k -cycles, then $|K| = 12$. The digraph D has large exponent since $\gamma(D) = 102 > \lfloor \frac{\omega n}{2} \rfloor + 2 = 100$. By (1.2), $d^c(\lambda(D)) \leq n + k - j - 1 = 15$. Since $k = 11$, $\delta(D) = 11$, $n + k - |J| - 1 = 10$, and $n + j - |K| - 1 = 12$, by Theorem 3.1.1, $d^c(\lambda(D)) \leq n + j - |K| - 1 = 12$. Note that $n + j - |K| - 1 < n + k - j - 1$ in this case and $d^c(\lambda(D)) = d^c(\lambda(D))[1, 15] = 12$.

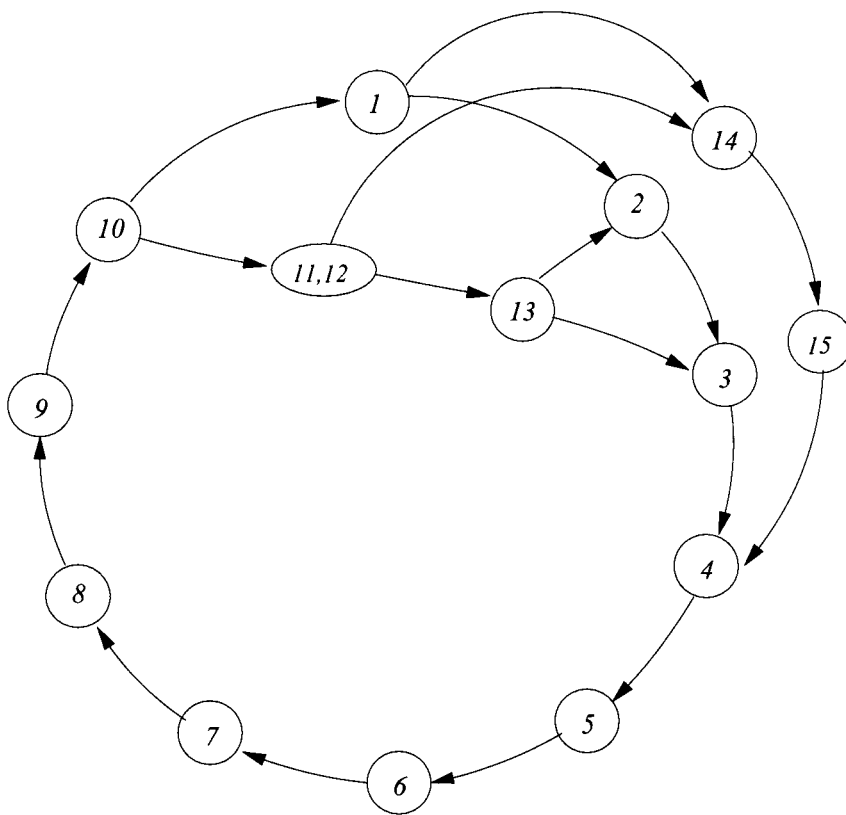


Figure 3.3: Circumdiameter = $n + j - |K| - 1 < n + k - j - 1$

3.2 Equality in an Upper Bound for the Exponent

The aim of this section is to present some conditions under which equality is attained in the exponent bound of Theorem 1.5.2. For primitive digraphs on n vertices with large exponent, it is shown that equality holds for all $3 \leq n \leq 8$ and strict inequality is demonstrated using a family with $n \geq 9$. Recall from Section 1.5 that equality is attained under the conditions of Corollary 1.5.4.

Theorem 3.2.1 *If D is a primitive digraph on $n \geq 3$ vertices with cycles of exactly two lengths k and j with $k + j > n$ and a vertex u such that $d^c(\lambda(D)) = d^c(\lambda(D))[u, u]$, then $\gamma(D) = (k - 1)(j - 1) + d^c(\lambda(D))$.*

Proof: Suppose u is on cycles of both lengths. If $x \neq u$ is a vertex in D , then $d^c(\lambda(D))[u, x] \geq 1 > 0 = d^c(\lambda(D))[u, u]$, giving a contradiction to $d^c(\lambda(D)) = d^c(\lambda(D))[u, u]$. Thus u is on cycles of only one length. If u is only on k -cycles, then $d^c(\lambda(D)) = k$ since each k -cycle intersects every j -cycle and each j -cycle intersects every k -cycle. Furthermore any $u \rightsquigarrow u$ walk of length $\geq k$ must include a_1 k -cycles and a_2 j -cycles for some positive integer a_1 and nonnegative integer a_2 . This implies that u has the unique path property. Similarly, if u is only on j -cycles, then $d^c(\lambda(D)) = j$, implying that u has the unique path property. Thus $\gamma(D) = (k - 1)(j - 1) + d^c(\lambda(D))$ by Corollary 1.5.4. \square

Any primitive digraph D on $n \geq 4$ vertices that has large exponent and cycles of lengths k and j satisfies $k + j > n$ by Lemma 1.6.5, implying by Theorem 3.2.1 that if such a D has a vertex u with $d^c(\lambda(D)) = d^c(\lambda(D))[u, u]$, then $\gamma(D) = (k - 1)(j - 1) + d^c(\lambda(D))$.

Theorem 3.2.2 *If D is a primitive digraph on $n \geq 4$ vertices with large exponent and cycles of lengths k and j with $d^c(\lambda(D)) = k - 1$, then $\gamma(D) = (k - 1)(j - 1) + d^c(\lambda(D))$.*

Proof: By Theorem 1.5.2, $\gamma(D) \leq (k-1)(j-1) + (k-1)$, which equals $(k-1)j$. Since $A(D)$ is not symmetric by Corollary 1.6.8, it follows that $\gamma(D) \geq (k-1)j$ by [10, Corollary 3.1]. Thus $\gamma(D) = (k-1)j = (k-1)(j-1) + d^c(\lambda(D))$. \square

All primitive digraphs on $n \geq 4$ vertices with large exponent and circumference n and all primitive digraphs on $n \geq 6$ vertices with large exponent and circumference $n-1$ are characterized in [9], and elaborated on in Chapter 5.

Theorem 3.2.3 *If D is a primitive digraph on $n \geq 4$ vertices with large exponent and cycles of lengths $k \geq n-1$ and j , then*

$$\gamma(D) = (k-1)(j-1) + d^c(\lambda(D)).$$

Proof: Let u and v be vertices in D such that $d^c(\lambda(D)) = d^c(\lambda(D))[u, v]$. Note that the cases $u = v$ and $d^c(\lambda(D)) = k-1$ are covered in the previous two theorems.

Case 1) $u \neq v$, $k = n$ and $d^c(\lambda(D)) \geq k$.

Suppose that some $u \rightsquigarrow v$ path intersects a j -cycle. Since u and v are each on a k -cycle, $d^c(\lambda(D)) \leq n-1 = k-1$, contradicting the above assumption. Thus we may assume that no $u \rightsquigarrow v$ path intersects a j -cycle.

By [9, Lemma 2.1], D has only one k -cycle, therefore there is exactly one $u \rightsquigarrow v$ path. Thus $d^c(\lambda(D)) = \delta[u, v] + k$ since D has no vertex disjoint cycles by Lemma 1.6.6. Any $u \rightsquigarrow v$ walk of length $\geq \delta[u, v] + k$ includes a path of length $\delta[u, v]$, a_1 k -cycles and a_2 j -cycles for some positive integer a_1 and nonnegative integer a_2 . Thus the ordered pair (u, v) has the unique path property, implying that $\gamma(D) = (k-1)(j-1) + d^c(\lambda(D))$ by Corollary 1.5.4.

Case 2) $u \neq v$, $k = n-1$ and $d^c(\lambda(D)) \geq k$.

Subcase i) At least one of u or v is on a j -cycle.

Since at least one of u or v is also on a k -cycle, $d^c(\lambda(D)) \leq n - 1 = k$, implying that $d^c(\lambda(D)) = k = n - 1$. Thus all $u \rightsquigarrow v$ paths have length $n - 1$ and any $u \rightsquigarrow v$ walk has length of the form $(n - 1) + a_1k + a_2j$ for some nonnegative integers a_1 and a_2 , implying that the ordered pair (u, v) has the unique path property. Therefore $\gamma(D) = (k - 1)(j - 1) + d^c(\lambda(D))$ by Corollary 1.5.4.

Subcase ii) Both u and v are on k -cycles only.

By Theorem 1.6.3, $n \geq 5$. If $n = 5$ and $k = 4$, then $j = 3$. If there exists a $u \rightsquigarrow v$ path of length at least 2, then it must intersect cycles of both lengths, implying that $d^c(\lambda(D)) \leq n - 1 = 4$. Since $d^c(\lambda(D)) \geq k = 4$, it follows that $d^c(\lambda(D)) = 4$. Therefore, by Theorem 1.5.2, $\gamma(D) \leq (4 - 1)(3 - 1) + 4 = 10$. Since $\gamma(D) \geq \lfloor \frac{5+5}{2} \rfloor + 2 = 10$, it follows that $\gamma(D) = 10 = (k - 1)(j - 1) + d^c(\lambda(D))$. If there is an arc $u \rightarrow v$ and no $u \rightsquigarrow v$ path of length at least 2, then by Lemma 1.6.6, $d^c(\lambda(D)) = 1 + k = 5$. Every $u \rightsquigarrow v$ walk of length at least 5 involves the $u \rightarrow v$ arc, a_1 k -cycles and a_2 j -cycles for some positive integer a_1 and nonnegative integer a_2 , implying that the ordered pair (u, v) has the unique path property. Thus $\gamma(D) = (k - 1)(j - 1) + d^c(\lambda(D))$ by Corollary 1.5.4.

If $n \geq 6$, then by construction [9, Theorems 3.3 and 3.4], all $u \rightsquigarrow v$ paths have the same length, and $d^c(\lambda(D)) = \delta[u, v] + k$. Any $u \rightsquigarrow v$ walk of length $\geq \delta[u, v] + k$ includes a path of length $\delta[u, v]$, a_1 k -cycles and a_2 j -cycles for some positive integer a_1 and nonnegative integer a_2 . Thus the ordered pair (u, v) has the unique path property, implying that $\gamma(D) = (k - 1)(j - 1) + d^c(\lambda(D))$ by Corollary 1.5.4. \square

Lemma 3.2.4 *If u and v are distinct vertices in a primitive digraph D such that $d^c(\lambda(D)) = d^c(\lambda(D))[u, v]$ and $\delta[u, v] = n - 1$, then $\gamma(D) = \phi(\lambda(D)) + d^c(\lambda(D))$.*

Proof: Since $\delta[u, v] = n - 1$, it follows that there are no $u \rightsquigarrow v$ paths of any other length. Thus the ordered pair (u, v) has the unique path property,

implying that $\gamma(D) = \phi(\lambda(D)) + d^c(\lambda(D))$ by Corollary 1.5.4. \square

Lemma 3.2.5 *Suppose that D is a primitive digraph and $u, v \in V(D)$.*

If there is no $u \rightsquigarrow v$ walk of length $\phi(\lambda(D)) + d^c(\lambda(D)) - 1$, then

$\gamma(D) = \phi(\lambda(D)) + d^c(\lambda(D))$ and u, v are such that $d^c(\lambda(D))$ is equal to $d^c(\lambda(D))[u, v]$.

Proof: If there is no such $u \rightsquigarrow v$ walk, then by Theorem 1.5.2

$\gamma(D) = \phi(\lambda(D)) + d^c(\lambda(D))$. Let w, x be vertices in $V(D)$ such that $d^c(\lambda(D)) > d^c(\lambda(D))[w, x]$. There is a $w \rightsquigarrow x$ walk of every length at least $\phi(\lambda(D)) + d^c(\lambda(D))[w, x]$, including a $w \rightsquigarrow x$ walk of length $\phi(\lambda(D)) + d^c(\lambda(D)) - 1$. Therefore if u and v are vertices such that there is no $u \rightsquigarrow v$ walk of length $\phi(\lambda(D)) + d^c(\lambda(D)) - 1$, then $d^c(\lambda(D)) = d^c(\lambda(D))[u, v]$. \square

If D is a primitive digraph on $3 \leq n \leq 8$ vertices, then the following result shows that having large exponent is sufficient for equality in Theorem 1.5.2 and (1.3); however this is not sufficient if $n \geq 9$.

Theorem 3.2.6 *If D is a primitive digraph with large exponent on $3 \leq n \leq 8$ vertices, then $\gamma(D) = \phi(\lambda(D)) + d^c(\lambda(D))$. For every $n \geq 9$ there exists a primitive digraph $L(n)$ on n vertices with large exponent such that $\gamma(L(n)) = \phi(\lambda(L(n))) + d^c(\lambda(L(n))) - 1$.*

Proof: If D is a primitive digraph on 3 vertices with large exponent, then D is isomorphic to one of the digraphs of Example 1.6.2, implying that $\gamma(D) = \phi(\lambda(D)) + d^c(\lambda(D))$.

Let D be a primitive digraph on $4 \leq n \leq 8$ vertices with large exponent and $\lambda(D) = \{k, j\}$. If $n \leq 7$, then $k \geq n - 1$ by Theorem 1.6.3, implying that $\gamma(D) = \phi(\lambda(D)) + d^c(\lambda(D))$ by Theorem 3.2.3, which also covers the cases $n = 8, k \geq 7$. If $n = 8$, then $k \geq 6$ by Theorem 1.6.3. A complete list, up to isomorphism and the reversal of every arc, of all primitive digraphs on 8

vertices with large exponent and circumference 6 can be found in Appendix A, along with their circumdiameters and exponents. The above quoted theorems and Appendix A show that any primitive digraph D on $n \leq 8$ vertices with large exponent satisfies $\gamma(D) = \phi(\lambda(D)) + d^c(\lambda(D))$.

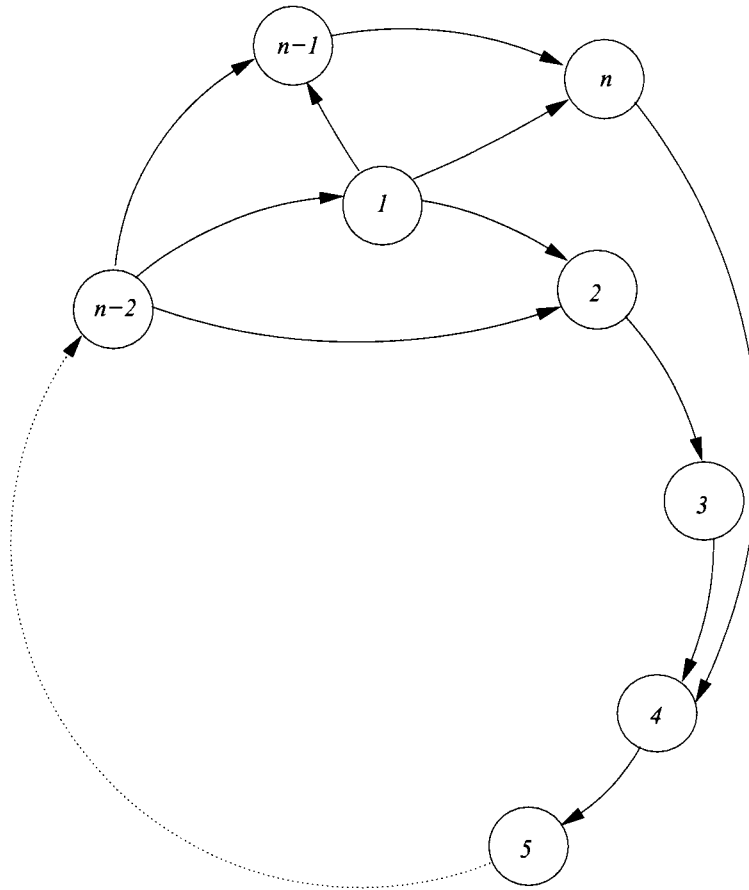


Figure 3.4: $L(n)$

Let $L(n)$ be the primitive digraph on $n \geq 9$ vertices with arcs $1 \rightarrow 2 \rightarrow \cdots \rightarrow (n-2) \rightarrow 1$, $(n-2) \rightarrow 2$, $(n-2) \rightarrow (n-1) \rightarrow n \rightarrow 4$, $1 \rightarrow (n-1)$ and $1 \rightarrow n$ as depicted in Figure 3.4. Since $L(n)$ has cycles of exactly two lengths $k = n-2$ and $j = n-3$, it follows by (1.4) that $\gamma(L(n)) \geq (n-3)^2$. This implies that $L(n)$ has large exponent. Every vertex is on cycles of

both lengths, implying that $d^c(\lambda(L(n)))[u, v] = \delta[u, v]$ for every ordered pair of vertices (u, v) . If u and v are on the same k -cycle or j -cycle, then $d^c(\lambda(L(n)))[u, v] = \delta[u, v] \leq n - 3$. However, if u and v are not on the same k -cycle or j -cycle, then there exists a $u \rightsquigarrow v$ path that is a subpath of either $2 \rightarrow 3 \rightarrow \cdots \rightarrow n$ or $(n - 1) \rightarrow n \rightarrow 4 \rightarrow 5 \rightarrow \cdots \rightarrow (n - 2) \rightarrow 2 \rightarrow 3$, both of which have length $n - 2$. Since $\delta[2, n] = \delta[n - 1, 3] = n - 2$, it follows that $d^c(\lambda(L(n))) = n - 2$, implying by Lemma 1.5.1 that $\phi(\lambda(L(n))) + d^c(\lambda(L(n))) = (n - 3)(n - 4) + (n - 2) = (n - 3)^2 + 1$.

By Lemma 3.2.5, for every u, v such that $d^c(\lambda(L(n))) \neq d^c(\lambda(L(n)))[u, v]$, there exists a $u \rightsquigarrow v$ walk of length $\phi(\lambda(L(n))) + d^c(\lambda(L(n))) - 1$. Recall from above that $d^c(\lambda(L(n))) = d^c(\lambda(L(n)))[u, v]$ only when (u, v) is $(n - 1, 3)$ or $(2, n)$, and note that there exists an $(n - 1) \rightsquigarrow 3$ path of length $(n - 1)$ through vertex 1 and a $2 \rightsquigarrow n$ path of length $n - 1$ through vertices 1 and $n - 1$. These paths, together with $n - 5$ cycles of length $n - 2$, form walks of length $n - 1 + (n - 5)(n - 2) = (n - 3)^2 = \phi(\lambda(L(n))) + d^c(\lambda(L(n))) - 1$, and thus it follows that there is a $u \rightsquigarrow v$ walk of length $\phi(\lambda(L(n))) + d^c(\lambda(L(n))) - 1$ for every $u, v \in V(L(n))$. Thus $\gamma(L(n)) \leq \phi(\lambda(L(n))) + d^c(\lambda(L(n))) - 1$.

Suppose there exists a $2 \rightsquigarrow 1$ walk of length $\phi(\lambda(L(n))) + d^c(\lambda(L(n))) - 2 = (n - 3)^2 - 1$. There is exactly one $2 \rightsquigarrow 1$ path which has length $n - 3$, implying that $(n - 3)^2 - 1 = (n - 3) + a_1(n - 2) + a_2(n - 3)$ for some nonnegative a_1 and a_2 . Therefore $(n - 3)(n - 4) - 1 = a_1(n - 2) + a_2(n - 3)$, giving a contradiction. Thus there is no $2 \rightsquigarrow 1$ walk of length $\phi(\lambda(L(n))) + d^c(\lambda(L(n))) - 2$, implying that $\gamma(L(n)) = \phi(\lambda(L(n))) + d^c(\lambda(L(n))) - 1$. \square

Chapter 4

Special Families of Digraphs

Let D be a primitive digraph on $n \geq 3$ vertices with cycles of lengths k and j where $n \geq k > j$. Recall from previous sections that

$\gamma(D) = \phi(\lambda(D)) + d^c(\lambda(D))$ if :

1. D has an ordered pair of vertices (u, v) that have the unique path property and $d^c(\lambda(D)) = d^c(\lambda(D))[u, v]$; see Corollary 1.5.4.
2. D has an ordered pair of vertices (u, v) such that $d^c(\lambda(D))$ is equal to $d^c(\lambda(D))[u, v]$ and $\delta[u, v] = n - 1$; see Lemma 3.2.4.
3. D has large exponent and there exists a vertex u such that $d^c(\lambda(D)) = d^c(\lambda(D))[u, u]$; see Theorem 3.2.1.
4. D on $n \geq 4$ vertices has large exponent and $d^c(\lambda(D)) = k - 1$; see Theorem 3.2.2.
5. D on $n \geq 4$ vertices has large exponent and $k \geq n - 1$; see Theorem 3.2.3.
6. D has large exponent and $3 \leq n \leq 8$; see Theorem 3.2.6.

7. D is isomorphic to the digraph $D(n, k, d)$ as defined in the proof of Theorem 2.1; see Corollary 2.4.

Recall that $\gamma(D) \neq \phi(\lambda(D)) + d^c(\lambda(D))$ if $n \geq 9$, D has large exponent and is isomorphic to $L(n)$ as defined in the proof of Theorem 3.2.6.

In this chapter, three families of primitive digraphs are presented in Sections 4.1, 4.2 and 4.3 for which equality is attained in Theorem 1.5.2. A family of primitive digraphs is introduced in Section 4.4 demonstrating that for minimally strong digraphs with large exponent, the inequality of Theorem 1.5.2 can be strict. Note that the family of primitive digraphs $L(n)$ is not minimally strong.

4.1 Neufeld Digraphs with Cycles of Two Lengths

Recall the family of primitive digraphs \mathcal{F}_δ with $\delta \geq 2$ defined in Section 1.4. A primitive digraph $D \in \mathcal{F}_\delta$ on n vertices has large exponent if and only if $\delta^2 + 1 \geq \lfloor \frac{\omega_n}{2} \rfloor + 2$.

Lemma 4.1.1 *A digraph D in \mathcal{F}_δ has cycles of exactly two lengths δ and $\delta + 1$ if and only if at least one of $V_1, V_2, \dots, V_\delta$ has exactly one vertex. In this case, $d^c(\lambda(D)) = \delta + 1$.*

Proof: Suppose $|V_i| = 1$ for some $1 \leq i \leq \delta$. Then every closed walk of length $\geq 2\delta$ must pass through this vertex more than once, implying that the only possible cycle lengths are δ and $\delta + 1$. Since D is primitive and the length of each cycle in D is a nonnegative linear combination of δ and $\delta + 1$ [13], $\lambda(D) = \{\delta, \delta + 1\}$.

If $|V_i| \geq 2$ for every $1 \leq i \leq \delta$, then cycles of lengths δ , $\delta + 1$ and 2δ can be constructed, i.e., D does not have cycles of exactly two lengths δ and $\delta + 1$. The contrapositive gives the desired result.

If J is the set of vertices on δ -cycles, and K is the set of vertices on $(\delta + 1)$ -cycles, then $|J| = n - 1$ and $|K| = n$. By Theorem 3.1.1, $d^c(\lambda(D)) \leq \max\{k, \delta(D), n+k-|J|-1, n+j-|K|-1\} = \max\{\delta+1, \delta, \delta-1, \delta+1\} = \delta+1$. Since $d^c(\lambda(D))[1, 1] = \delta + 1$, it follows that $d^c(\lambda(D)) = \delta + 1$. \square

Note that the exponent $\delta^2 + 1$ for a Neufeld digraph cannot be expressed in terms of n , since for any fixed value of δ many different values of n are possible. See Figure 2.2 for Neufeld digraphs with exactly two cycle lengths.

Proposition 4.1.2 *If $n = 4$ or $n \geq 6$, then there exists a primitive digraph $N_n \in \mathcal{F}_\delta$ on n vertices with cycles of exactly two lengths $\delta = \lfloor \frac{n+1}{2} \rfloor$ and $\delta + 1$ that does not have large exponent.*

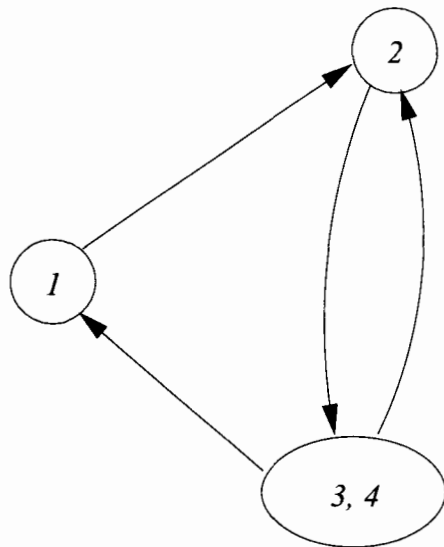
Proof: Let $V_0 = \{1\}$ and $V_1 = \{2\}$. If n is even, then define $V_2 = \{3, 4\}, V_3 = \{5, 6\}, \dots, V_\delta = \{n, n-1\}$. If $n \geq 7$ is odd, then define $V_2 = \{3\}, V_3 = \{4, 5\}, \dots, V_\delta = \{n-1, n\}$. Thus $V(N_n) = V_0 \cup V_1 \cup \dots \cup V_\delta$. Add arcs $u \rightarrow v$ for each $u \in V_i$ and each $v \in V_{i+1}$, where addition is taken modulo $\delta + 1$, for $0 \leq i \leq \delta$. Add the arcs $n \rightarrow 2$ and $(n-1) \rightarrow 2$. The resulting digraph is N_n .

N_n has diameter δ and cycles of exactly two lengths $\delta + 1$ and δ . Since δ and $\delta + 1$ are relatively prime, N_n is primitive. By construction $N_n \in \mathcal{F}_\delta$, implying that $\gamma(N_n) = \delta^2 + 1$ by Theorem 1.4.4. However, $\delta^2 + 1 = \lfloor \frac{n+1}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor + 1 < \lfloor \frac{\omega_n}{2} \rfloor + 2$ for $n = 4$ or $n \geq 6$, implying that N_n does not have large exponent. \square

Figure 4.1 illustrates the digraph N_4 .

Theorem 4.1.3 *If $D \in \mathcal{F}_\delta$ has cycles of exactly two lengths, then $\gamma(D) = \phi(\lambda(D)) + d^c(\lambda(D))$.*

Proof: Since $\lambda(D) = \{\delta, \delta + 1\}$, $\phi(\lambda(D)) = \delta(\delta - 1)$. By Lemma 4.1.1, $d^c(\lambda(D)) = \delta + 1$. By Theorem 1.4.4, $\gamma(D) = \delta^2 + 1$, which is equal to $\phi(\lambda(D)) + d^c(\lambda(D))$. \square

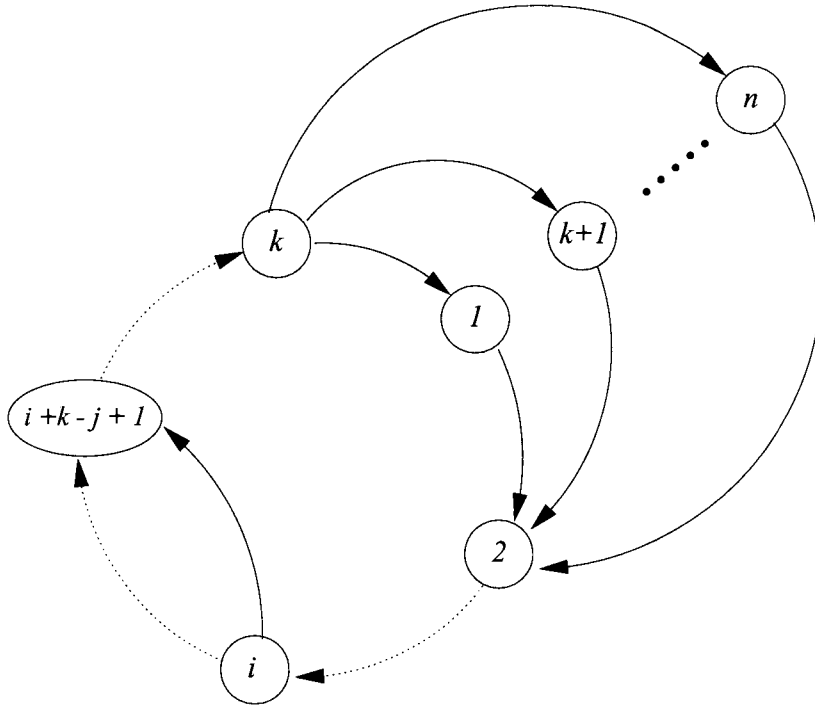
Figure 4.1: N_4

4.2 Ear Digraphs

For a fixed value of i such that $1 \leq i \leq k$ and relatively prime k and j such that $n - 1 \geq k > j \geq 2$, define $R(n, k, j, i)$ as the digraph on $n \geq 4$ vertices with the following arcs : (i) $1 \rightarrow 2 \rightarrow \cdots \rightarrow k \rightarrow 1$, (ii) $k \rightarrow m \rightarrow 2$ for all m such that $k + 1 \leq m \leq n$ and (iii) $i \rightarrow i + k - j + 1$ if $i \leq j - 1$ or $i \rightarrow i - j + 1$ if $i \geq j$. This digraph is depicted in Figure 4.2 for $i \geq 3$, $i \leq j - 2$ and $n \geq k + 2$. Since it has cycles of exactly two lengths k and j that are relatively prime, $R(n, k, j, i)$ is primitive. The family $R(n, k, j, i)$ has many digraphs in common with those defined in Section 4 of [10], for example $R(10, 8, 7, 8)$ is depicted in [10, p. 94].

The following three theorems give the diameter, circumdiameter and exponent of $R(n, k, j, i)$, and show that $\gamma(R(n, k, j, i)) = \phi(\lambda(R(n, k, j, i))) + d^c(\lambda(R(n, k, j, i)))$.

Theorem 4.2.1 *If $2 \leq i \leq j - 1$, then $R(n, k, j, i)$ has diameter $k - 1$,*

Figure 4.2: $R(n, k, j, i)$

otherwise it has diameter k .

Proof: Let $S_1 = \{2, 3, \dots, k\}$ and $S_2 = \{1, k+1, \dots, n\}$. Suppose that $2 \leq i \leq j-1$. If u and v are in S_1 , then $\delta[u, v] \leq \delta[i+1, i] = k-1$ since they share a k -cycle. If $u \in S_1$ and $v \in S_2$, then $\delta[u, v] \leq k-1$ and $\delta[v, u] \leq k-1$, since they share a k -cycle. If u and v are distinct vertices in S_2 , then $\delta[u, v] = j \leq k-1$, implying that $\delta(R(n, k, j, i)) = k-1$.

Suppose that $i = 1$ or $j \leq i \leq k$. If u and v are in S_1 , then $\delta[u, v] \leq k-1$ since they share a k -cycle. If $u \in S_1$ and $v \in S_2$, then $\delta[u, v] \leq k-1$ and $\delta[v, u] \leq k-1$, since they share a k -cycle. If u and v are in S_2 , then $\delta[u, v] \leq \delta[n, 1] = k$, implying that $\delta(R(n, k, j, i)) = k$. \square

Theorem 4.2.2 *If $i \neq 1$ and $i \neq j$, then the circumdiameter of $R(n, k, j, i)$*

is $2k - j - 1$ and its exponent is $\phi(\lambda(R(n, k, j, i))) + d^c(\lambda(R(n, k, j, i))) = k + j(k - 2)$.

Proof: Let J be the set of vertices on j -cycles, i.e., $J = \{i - j + 1, i - j + 2, \dots, i - 1, i\}$ if $i \geq j$ or $J = \{i + k - j + 1, i + k - j + 2, \dots, k, 1, 2, \dots, i - 1, i\} \cup \{k + 1, k + 2, \dots, n\}$ if $2 \leq i \leq j - 1$. Since all vertices are on k -cycles, if at least one of u and v is in J , then $d^c(\lambda(R(n, k, j, i)))[u, v] = \delta[u, v] \leq k - 1 \leq 2k - j - 1$. If u and v are not in J , then $d^c(\lambda(R(n, k, j, i)))[u, v] \leq d^c(\lambda(R(n, k, j, i)))[i + 1, i + k - j] = (i + k - j) - (i + 1) + k = 2k - j - 1$. This implies that $d^c(\lambda(R(n, k, j, i))) = 2k - j - 1$.

All $(i + 1) \rightsquigarrow (i + k - j)$ paths have the same length, implying that the ordered pair $(i + 1, i + k - j)$ has the unique path property. Thus by Corollary 1.5.4, $\gamma(R(n, k, j, i)) = \phi(\lambda(R(n, k, j, i))) + d^c(\lambda(R(n, k, j, i))) = (k - 1)(j - 1) + 2k - j - 1 = k + j(k - 2)$. \square

Theorem 4.2.3 *If $i = 1$ or $i = j$, then the circumdiameter of $R(n, k, j, i)$ is $2k - j$ and its exponent is $\phi(\lambda(R(n, k, j, i))) + d^c(\lambda(R(n, k, j, i))) = k + 1 + (k - 2)j$.*

Proof: Let J be the set of vertices on j -cycles, i.e., $J = \{1, k - j + 2, k - j + 3, \dots, k\}$ if $i = 1$ or $J = \{1, 2, \dots, j\}$ if $i = j$. Since all vertices are on k -cycles, if at least one of u and v is in J , then $d^c(\lambda(R(n, k, j, i)))[u, v] = \delta[u, v] \leq k \leq 2k - j - 1$. If $i = 1$ and u and v are not in J , then $d^c(\lambda(R(n, k, j, i)))[u, v] \leq d^c(\lambda(R(n, k, j, i)))[k + 1, k - j + 1] = (k - j) + k = 2k - j$. If $i = j$ and u and v are not in J , then $d^c(\lambda(R(n, k, j, i)))[u, v] \leq d^c(\lambda(R(n, k, j, i)))[j + 1, k + 1] = (k + 1) - (j + 1) + k = 2k - j$. Thus $d^c(\lambda(R(n, k, j, i))) = 2k - j$.

If $d^c(\lambda(R(n, k, j, i))) = d^c(\lambda(R(n, k, j, i)))[u, v]$, then all $u \rightsquigarrow v$ paths have the same length, implying that the ordered pair (u, v) has the unique path property. Thus by Corollary 1.5.4, $\gamma(R(n, k, j, i)) = \phi(\lambda(R(n, k, j, i))) + d^c(\lambda(R(n, k, j, i))) = (k - 1)(j - 1) + 2k - j = k + 1 + j(k - 2)$. \square

The following theorem gives necessary and sufficient conditions for $R(n, k, j, i)$ to be a Neufeld digraph as described in Section 1.4.

Theorem 4.2.4 $R(n, k, j, i)$ is in \mathcal{F}_δ if and only if $j = k - 1$ and $2 \leq i \leq k - 2$. Moreover, $\delta = j = k - 1$.

Proof: Recall the definition of \mathcal{F}_{k-1} from Section 1.4, and suppose that $2 \leq i \leq k - 2$. Define $V_0 = \{i + 1\}$, $V_1 = \{i + 2\}, \dots, V_{k-i-1} = \{k\}$, $V_{k-i} = \{1, k + 1, \dots, n\}$, $V_{k-i+1} = \{2\}, \dots, V_{k-1} = \{i\}$. Then $V(R(n, k, k - 1, i)) = V_0 \cup V_1 \cup \dots \cup V_{k-1}$, the V_m are pairwise disjoint and nonempty, and V_0 consists of a single distinguished vertex $i + 1$. For $0 \leq i \leq k - 1$, $u \rightarrow v$ is an arc of $R(n, k, k - 1, i)$ for each $u \in V_m$ and $v \in V_{m+1}$, in which addition is taken modulo k . There is exactly one arc from the single vertex in V_{k-1} to the single vertex in V_1 , implying that $R(n, k, j, i) \in \mathcal{F}_\delta$ if $j = k - 1 = \delta$ and $2 \leq i \leq k - 2$.

If $j \leq k - 2$, then $R(n, k, j, i)$ does not have cycle lengths δ and $\delta + 1$ for any δ , implying that $R(n, k, j, i)$ is not in \mathcal{F}_δ for any δ . Suppose that $j = k - 1$. If $i = 1$, then $V_0 = \{2\}$, $V_1 = \{3\}$ and $V_{k-1} = \{1, k + 1, \dots, n\}$, implying that $R(n, k, j, i)$ is not in \mathcal{F}_{k-1} since there is no arc $(k + 1) \rightarrow 3$. If $i = k - 1$, then $V_0 = \{k\}$, $V_1 = \{1, k + 1, \dots, n\}$ and $V_{k-1} = \{k - 1\}$, implying that $R(n, k, j, i)$ is not in \mathcal{F}_{k-1} since there is no arc $(k - 1) \rightarrow (k + 1)$. If $i = k$, then $V_0 = \{1, k + 1, \dots, n\}$, implying that $R(n, k, j, i)$ is not in \mathcal{F}_{k-1} . Thus $R(n, k, j, i)$ is in \mathcal{F}_δ only if $j = k - 1 = \delta$ and $2 \leq i \leq k - 2$. \square

The next corollary follows from Theorems 4.2.2 and 4.2.3 and the definition of large exponent.

Corollary 4.2.5 *If $i = 1$ or $i = j$, then $R(n, k, j, i)$ has large exponent if and only if $k + 1 + (k - 2)j \geq \lfloor \frac{\omega n}{2} \rfloor + 2$, otherwise $R(n, k, j, i)$ has large exponent if and only if $k + (k - 2)j \geq \lfloor \frac{\omega n}{2} \rfloor + 2$.*

4.3 Generalized Wielandt Digraphs

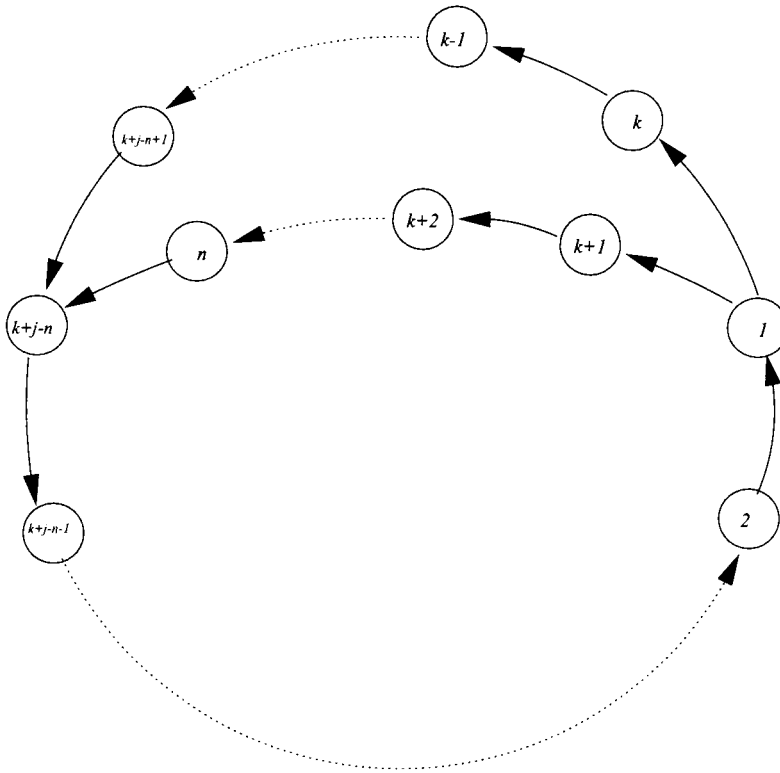
For $n \geq k > j \geq 2$, $k + j \geq n + 2$ and k and j relatively prime, let $\widetilde{W}(n, k, j)$ be the primitive digraph on $n \geq 3$ vertices containing the k -cycle $1 \rightarrow k \rightarrow (k - 1) \cdots \rightarrow 2 \rightarrow 1$. If $k \leq n - 1$, then the path $1 \rightarrow (k + 1) \rightarrow (k + 2) \rightarrow \cdots \rightarrow n \rightarrow (k + j - n)$ is included, whereas if $k = n$, then the arc $1 \rightarrow j$ is included. This digraph is depicted in Figure 4.3 for $k - 1 > k + j - n + 1$, $n > k + 2$ and $k + j - n - 1 > 2$. Note that $\widetilde{W}(n, k, j)$ has exactly two cycles, one of length k and one of length j . Also note that $\widetilde{W}(n, k, j)$ is minimally strong if and only if $k \leq n - 1$, and that $\widetilde{W}(n, n, n - 1)$ is the Wielandt digraph W_n (see Section 1.3). The primitive digraph D constructed in the proof of [9, Theorem 1.2] is the same as $\widetilde{W}(n, k, j)$ except that n , k and j do not necessarily satisfy $(k - 2)j \geq \lfloor \frac{cn}{2} \rfloor + 2 - n$ for $\widetilde{W}(n, k, j)$.

Theorem 4.3.1 *The diameter of $\widetilde{W}(n, k, j)$ is $n - 1$, its circumdiameter is $n + k - j - 1$ and its exponent is $\phi(\lambda(\widetilde{W}(n, k, j))) + d^c(\lambda(\widetilde{W}(n, k, j))) = n + (k - 2)j$.*

Proof: If $k \leq n - 1$, then there is exactly one $k \rightsquigarrow n$ path with length $n - 1$, implying that $\delta(\widetilde{W}(n, k, j)) = n - 1$. If $k = n$, then there is exactly one $n - 1 \rightsquigarrow n$ path of length $n - 1$, implying that $\delta(\widetilde{W}(n, k, j)) = n - 1$.

Since $d^c(\lambda(\widetilde{W}(n, k, j)))[k, k + j - n + 1] = k - (k + j - n + 1) + k = n + k - j - 1$ and $k + j > n$, it follows from [10, Lemma 3.2] that $d^c(\widetilde{W}(n, k, j)) = n + k - j - 1$.

Every $k \rightsquigarrow (k + j - n + 1)$ walk intersecting cycles of both lengths involves a walk of length $d^c(\lambda(\widetilde{W}(n, k, j)))[k, k + j - n + 1]$ and a nonnegative combination of k -cycles and j -cycles, implying that the ordered pair $(k, k + j - n + 1)$ has the unique path property. Therefore by Corollary 1.5.4, $\gamma(\widetilde{W}(n, k, j)) = \phi(\lambda(\widetilde{W}(n, k, j))) + d^c(\lambda(\widetilde{W}(n, k, j))) = n + (k - 2)j$ by Lemma 1.5.1. \square

Figure 4.3: $\widetilde{W}(n, k, j)$

For $k \leq n-1$, the digraphs $\widetilde{W}(n, k, j)$ are depicted in [15, Figure 6], and their circumdiameter and exponent are given in the proof of [15, Theorem 4.9]. Members of this family achieve the upper bounds given for the exponent in Theorem 1.5.2 and [10, Theorem 4.1], and the Wielandt bound of Theorem 1.3.1, as demonstrated in the following corollary.

Corollary 4.3.2 *The circumdiameter of $\widetilde{W}(n, n, j)$ is $2n - j - 1$. When n and j are relatively prime, the exponent of $\widetilde{W}(n, n, j)$ is $n + (n - 2)j$. The circumdiameter of $\widetilde{W}(n, n, n - 1)$ is n and its exponent is $(n - 1)^2 + 1 = \omega_n$.*

4.4 A Special Class of Minimally Strong Digraphs

Recall the definition of a *minimally strong* digraph from Section 1.1. In this section it is shown that for each $n \geq 22$ there exists a minimally strong primitive digraph D on n vertices with large exponent for which $\gamma(D) \leq \phi(\lambda(D)) + d^c(\lambda(D)) - 1$. It is unknown whether or not this is the smallest such n .

Theorem 4.4.1 *For any $n \geq 22$, there exists a minimally strong primitive digraph $M(n)$ on n vertices with large exponent such that*

$$\gamma(M(n)) = \phi(\lambda(M(n))) + d^c(\lambda(M(n))) - 1.$$

Proof: Suppose that $n \geq 22$ and $M(n)$ is the digraph on n vertices consisting of the $(n - 6)$ -cycle $1 \rightarrow 2 \rightarrow \dots \rightarrow (n - 6) \rightarrow 1$ and the arcs $(n - 6) \rightarrow (n - 5) \rightarrow 3$, $(n - 6) \rightarrow (n - 4) \rightarrow (n - 3) \rightarrow n \rightarrow 5$, $1 \rightarrow (n - 2) \rightarrow (n - 3)$ and $1 \rightarrow (n - 1) \rightarrow n$ depicted in Figure 4.4. Thus $\lambda(M(n)) = \{n - 6, n - 7\}$.

The only vertices with indegree more than one are 3, 5, $n - 3$ and n . Each of the arcs that terminates at any of these vertices originates at a vertex with outdegree one, implying that removing any of these arcs leaves a digraph that is not strongly connected. The only vertices with outdegree more than one are 1 and $n - 6$. Each of the arcs that originates from any of these vertices terminates at a vertex with indegree one, implying that removing any of these arcs would result in a digraph that is not strongly connected. All other arcs originate at a vertex of outdegree one and terminate at a vertex of indegree one, implying that removing any of these arcs would result in a digraph that is not strongly connected. Thus $M(n)$ is minimally strong with cycles of exactly two relatively prime lengths, also implying that $M(n)$ is primitive.

By construction, the k -cycles are $1 \rightarrow 2 \rightarrow \dots \rightarrow (n - 6) \rightarrow 1$ and $1 \rightarrow (n - 2) \rightarrow (n - 3) \rightarrow n \rightarrow 5 \rightarrow 6 \rightarrow \dots \rightarrow (n - 6) \rightarrow 1$, and the j -cycles are

$3 \rightarrow 4 \rightarrow \cdots \rightarrow (n-5) \rightarrow 3, 5 \rightarrow 6 \rightarrow \cdots \rightarrow (n-6) \rightarrow (n-4) \rightarrow (n-3) \rightarrow n \rightarrow 5$ and $1 \rightarrow (n-1) \rightarrow n \rightarrow 5 \rightarrow 6 \rightarrow \cdots \rightarrow (n-6) \rightarrow 1$. Thus the set of all vertices on k -cycles is $V(M(n)) \setminus \{n-5, n-4, n-1\}$ and the set of all vertices on j -cycles is $V(M(n)) \setminus \{2, n-2\}$. Since no two adjacent vertices in $M(n)$ are on cycles of one length only, $d^c(\lambda(M(n)))[u, v] = \delta[u, v]$ for all distinct u and v in $V(M(n))$. Note that $d^c(\lambda(M(n)))[u, u] \leq n-6$ for all $u \in V(M(n))$.

Let $S = \{(2, n), (n-5, n), (n-4, 4), (n-2, 4)\}$. It will now be shown that if $(u, v) \in S$, then $d^c(\lambda(M(n))) = d^c(\lambda(M(n)))[u, v]$, and if $(u, v) \notin S$, then $d^c(\lambda(M(n)))[u, v] < d^c(\lambda(M(n)))$. Consider the paths :

$2 \rightarrow 3 \rightarrow \cdots \rightarrow (n-6) \rightarrow 1 \rightarrow (n-1) \rightarrow n,$
 $(n-5) \rightarrow 3 \rightarrow 4 \rightarrow \cdots \rightarrow (n-6) \rightarrow 1 \rightarrow (n-1) \rightarrow n,$
 $(n-4) \rightarrow (n-3) \rightarrow n \rightarrow 5 \rightarrow 6 \rightarrow \cdots \rightarrow (n-6) \rightarrow (n-5) \rightarrow 3 \rightarrow 4$ and
 $(n-2) \rightarrow (n-3) \rightarrow n \rightarrow 5 \rightarrow 6 \rightarrow \cdots \rightarrow (n-6) \rightarrow (n-5) \rightarrow 3 \rightarrow 4.$

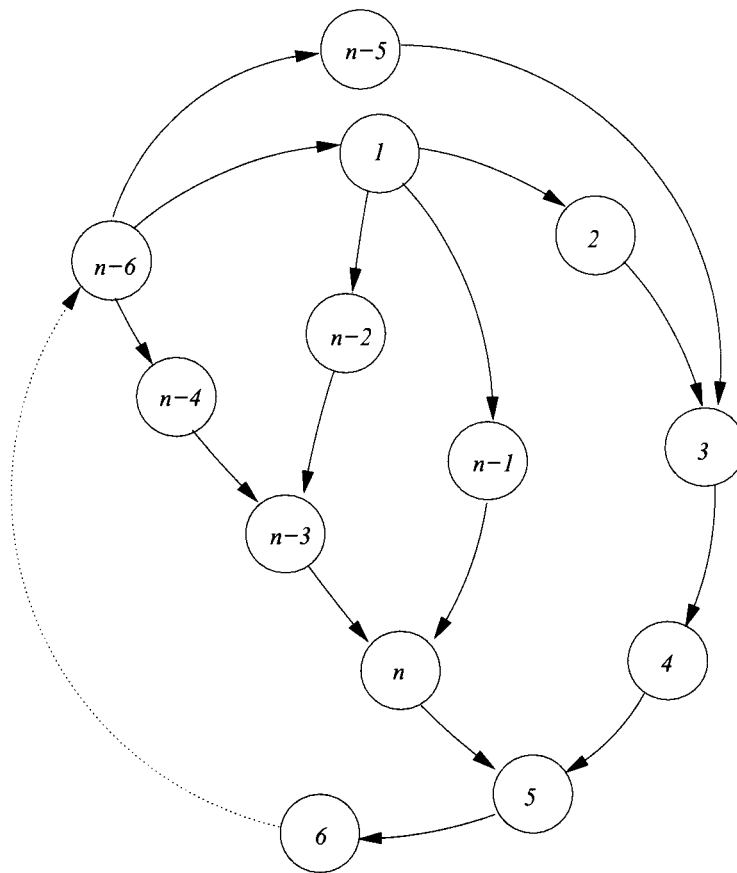
Each of these paths is a path of shortest length from the first to the last vertex, each intersects cycles of both lengths and has length $n-5$. Suppose that $u \in V(D)$. Note that $\delta[u, n-4] \leq \delta[1, n-4] = n-7$, $\delta[n-4, u] \leq \delta[n-4, 4] = n-5$ with equality for only $(n-4) \rightsquigarrow 4$, $\delta[u, n-3] \leq \delta[2, n-3] = n-6$, $\delta[n-3, u] \leq \delta[n-3, 4] = n-6$, $\delta[u, n-5] \leq \delta[1, n-5] = n-7$, $\delta[n-5, u] \leq \delta[n-5, n] = n-5$ with equality for only $(n-5) \rightarrow n$, $\delta[u, n-2] \leq \delta[2, n-2] = n-6$, $\delta[n-2, u] \leq \delta[n-2, 4] = n-5$ with equality for only $(n-2) \rightsquigarrow 4$, $\delta[n-1, u] \leq \delta[n-1, 4] = n-6$ and $\delta[n, 2] = n-8$.

If $(u, v) \notin S$, then either : (i) u and v share a k -cycle or a j -cycle, (ii) there is a $u \rightsquigarrow v$ path, along one of the aforementioned four paths, that intersects cycles of both lengths, or (iii) $\delta[u, v] < n-5$ by the previous paragraph. Any one of these conditions implies that $d^c(\lambda(M(n)))[u, v] < n-5$. If $(u, v) \in S$, then $d^c(\lambda(M(n)))[u, v] = n-5$, implying that $d^c(\lambda(M(n))) = n-5$ and $(u, v) \in S$ are the only pairs of vertices such that $d^c(\lambda(M(n)))[u, v] = d^c(\lambda(M(n))) = n-5$.

If $(u, v) \notin S$, then by Lemma 3.2.5 there exists a $u \rightsquigarrow v$ walk of length $\phi(\lambda(M(n))) + d^c(\lambda(M(n))) - 1$. If $(u, v) \in S$, then, in addition to the

above paths of length $n - 5$, there exists a $u \rightsquigarrow v$ path of length $n - 4$ that intersects cycles of both lengths. Thus there exists a $u \rightsquigarrow v$ walk of length $(n - 9)(n - 6) + 0(n - 7) + n - 4 = \phi(\lambda(M(n))) + d^c(\lambda(M(n))) - 1$. Thus $\gamma(M(n)) \leq \phi(\lambda(M(n))) + d^c(\lambda(M(n))) - 1$.

Suppose that there is a $2 \rightsquigarrow 2$ walk of length $\phi(\lambda(M(n))) + d^c(\lambda(M(n))) - 2 = (n - 7)(n - 8) + (n - 5) - 2$. Since vertex 2 is on $(n - 6)$ -cycles only, it follows that $a_1(n - 6) + a_2(n - 7) + n - 6 = (n - 7)(n - 8) + (n - 5) - 2$, implying that $a_1(n - 6) + a_2(n - 7) = (n - 7)(n - 8) - 1$, giving a contradiction. Thus $\gamma(M(n)) \geq \phi(\lambda(M(n))) + d^c(\lambda(M(n))) - 1$, implying that $\gamma(M(n)) = \phi(\lambda(M(n))) + d^c(\lambda(M(n))) - 1$, which is at least $\lfloor \frac{\omega_n}{2} \rfloor + 2$ for $n \geq 22$. \square

Figure 4.4: $M(n)$

Chapter 5

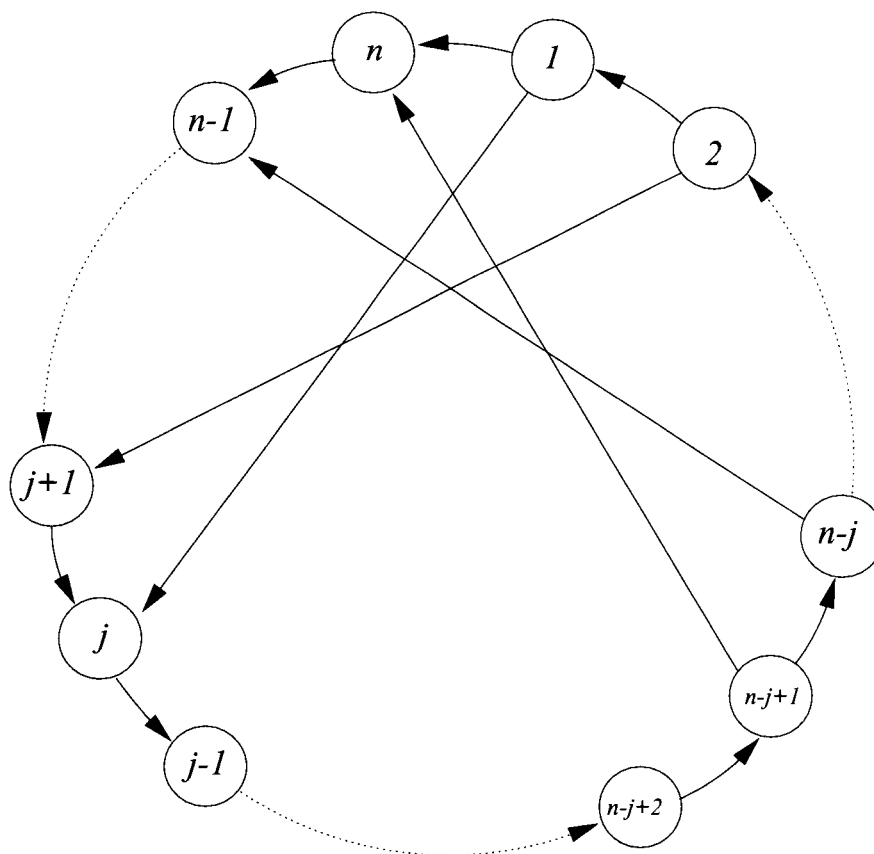
Large Exponent and Circumference n or $n - 1$

5.1 Circumference n

Theorem 5.1.1 gives expressions for the diameter, circumdiameter, and exponent of digraphs with large exponent and circumference n . The characterizations of these digraphs given in [9, Theorems 2.2 and 2.3] are used throughout this section.

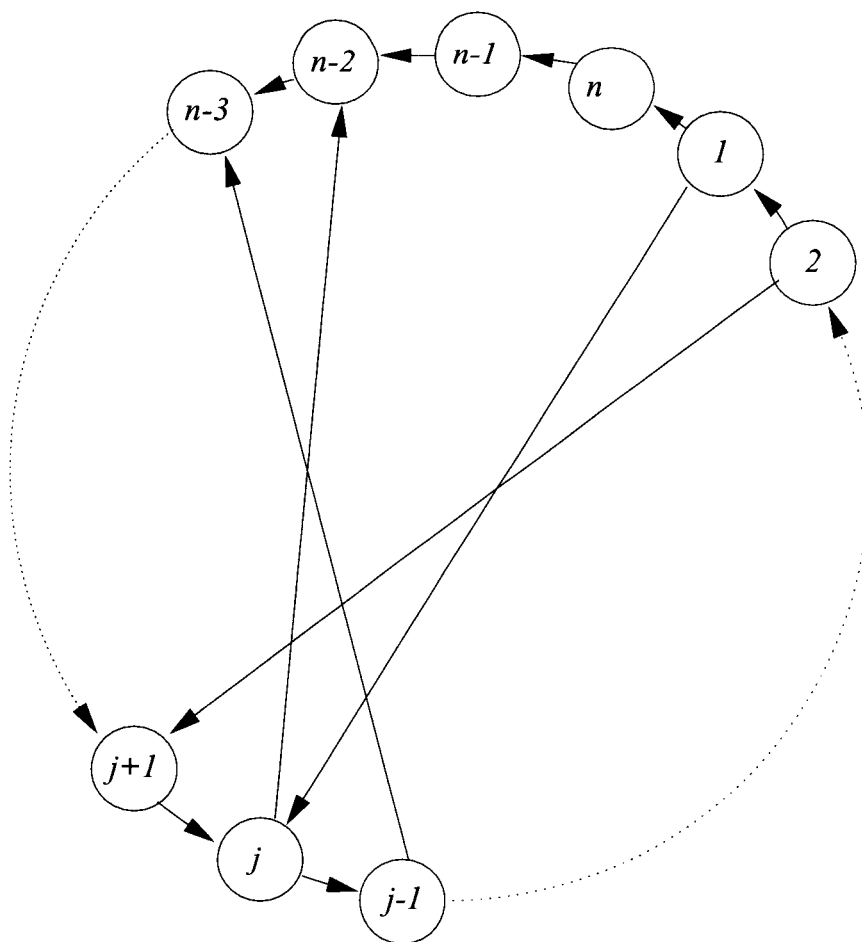
For fixed $n \geq 3$ and fixed j with $n > j \geq \frac{n+1}{2}$, let $\tilde{D}(n, j)$ be the primitive digraph on n vertices with large exponent constructed by taking the Hamilton cycle $1 \rightarrow n \rightarrow (n-1) \rightarrow \cdots \rightarrow 2 \rightarrow 1$, and adding in the arcs $i \rightarrow (i+j-1)$ for $1 \leq i \leq n-j+1$. This digraph is defined in [9, Theorem 2.2], and is depicted in Figure 5.1 for $n \geq 10$. Note that [9, Theorem 2.2] allows $j = \frac{n}{2}$. However, a digraph with cycles of lengths n and $\frac{n}{2}$ only is not primitive. Thus all primitive digraphs described in [9, Theorem 2.2] are included in the definition of $\tilde{D}(n, j)$.

For $n \geq 3$ odd and $j = \frac{n-1}{2}$, let $\hat{D}(n)$ be as $\tilde{D}(n, j)$ except that

Figure 5.1: $\tilde{D}(n, j)$

$1 \leq i \leq \frac{n-1}{2} = j$. This digraph is defined in [9, Theorem 2.3], and is depicted in Figure 5.2 for $n \geq 9$.

Theorem 5.1.1 *If D is any primitive digraph on $n \geq 3$ vertices with large exponent, cycles of length n and j , and J is the set of vertices on j -cycles, then $\delta(D) = n - 1$, $d^c(\lambda(D)) = 2n - |J| - 1$ and $\gamma(D) = (n - 1)(j - 1) + d^c(\lambda(D)) = (n - 1)(j + 1) - |J| + 1$.*

Figure 5.2: $\hat{D}(n)$

Proof: D must be a subdigraph of $\tilde{D}(n, j)$ or $\hat{D}(n)$ by [9, Theorems 2.2 and 2.3]. The only path from vertex n to vertex 1 in $\tilde{D}(n, j)$ or $\hat{D}(n)$ is $n \rightarrow (n-1) \rightarrow \dots \rightarrow 2 \rightarrow 1$, which has length $n-1$. Thus the diameters of both $\tilde{D}(n, j)$ and $\hat{D}(n)$ are $n-1$, implying by Lemma 1.1.1 that $\delta(D) = n-1$.

If $|J| = n$, then all vertices are on cycles of both lengths, implying that $d^c(\lambda(D)) = \delta(D) = n-1 = 2n - |J| - 1$.

If $|J| = n-1$ and K is the set of all vertices on n -cycles, then Theorem

3.1.1 gives $d^c(\lambda(D)) \leq \max\{n, n-1, 2n-|J|-1, j-1\} = n = 2n-|J|-1$. If $u \in V(D) \setminus J$, then $d^c(\lambda(D))[u, u] = n$, implying that $d^c(\lambda(D)) = d^c(\lambda(D))[u, u] = n = 2n-|J|-1$.

Suppose that $|J| \leq n-2$. If u and v are distinct vertices in J , then $d^c(\lambda(D))[u, v] = \delta[u, v] \leq n-1 < 2n-|J|-1$, and $d^c(\lambda(D))[u, u] = 0 < 2n-|J|-1$. If $u, v \in V(D) \setminus J$ (not necessarily distinct), then $d^c(\lambda(D))[u, v] \leq d^c(\lambda(D))[\hat{u}, \hat{v}]$, in which \hat{u} and \hat{v} are such that there exists a $\hat{u} \rightsquigarrow \hat{v}$ path of length $n-|J|-1$ on vertices in $V(D) \setminus J$. As D contains at least one j -cycle, $d^c(\lambda(D))[\hat{u}, \hat{v}] = (n-|J|-1) + n = 2n-|J|-1$.

Thus $d^c(\lambda(D)) = 2n-|J|-1$, implying by Theorem 3.2.3 that $\gamma(D) = (n-1)(j-1) + d^c(\lambda(D)) = (n-1)(j+1) - |J| + 1$. \square

Corollary 5.1.2 *The circumdiameter of $\tilde{D}(n, j)$ is $n-1$ and its exponent is $(n-1)(j-1) + d^c(\lambda(\tilde{D}(n, j))) = (n-1)j \geq \lfloor \frac{\omega_n}{2} \rfloor + 2$. The circumdiameter of $\hat{D}(n)$ is $n+1$ and its exponent is $(\frac{n-1}{2}-1)(n-1) + d^c(\lambda(\hat{D}(n))) = \lfloor \frac{\omega_n}{2} \rfloor + 2$.*

Proof: All vertices of $\tilde{D}(n, j)$ are on j -cycles, implying that $d^c(\lambda(\tilde{D}(n, j))) = n-1$ and (by Theorem 5.1.1) $\gamma(\tilde{D}(n, j)) = (n-1)(j-1) + d^c(\lambda(\tilde{D}(n, j))) = (n-1)j$, which is greater than or equal to $\lfloor \frac{\omega_n}{2} \rfloor + 2$ since $j \geq \frac{n+1}{2}$.

All but two vertices of $\hat{D}(n)$ are on $(\frac{n-1}{2})$ -cycles, implying that $d^c(\lambda(\hat{D}(n))) = n+1$ and $\gamma(\hat{D}(n)) = (\frac{n-1}{2}-1)(n-1) + d^c(\lambda(\hat{D}(n))) = \lfloor \frac{\omega_n}{2} \rfloor + 2$ by Theorem 5.1.1. \square

A proper subdigraph of $\tilde{D}(n, j)$ is depicted in Figure 5.3 (see Example 1.6.2) for the smallest possible values of n and j , i.e., $n = 3$ and $j = 2$. It has diameter $n - 1 = 2$. There are two vertices on 2-cycles (i.e., $|J| = 2$), implying that its circumdiameter is 3. Its exponent is $5 > \lfloor \frac{\omega_3}{2} \rfloor + 2 = 4$.

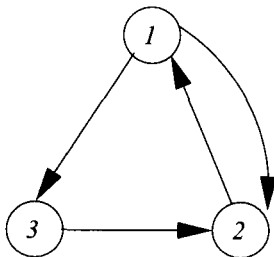


Figure 5.3: A Proper Subdigraph of $\tilde{D}(3, 2)$

The digraph $\hat{D}(n)$ is depicted in Figure 5.4 (see Example 1.6.2) for the smallest possible value of n , i.e., $n = 3$. It has diameter $n - 1 = 2$. There is exactly one vertex with a loop (i.e., $|J| = 1$), implying that its circumdiameter is 4. Its exponent is $4 = \lfloor \frac{\omega_3}{2} \rfloor + 2$.

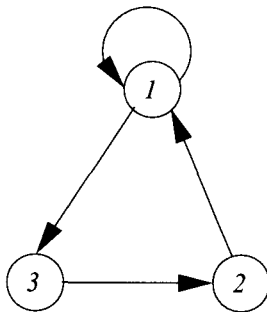


Figure 5.4: $\hat{D}(3)$

5.2 Circumference $n - 1$ and Diameter $n - 1$

The characterizations of the class of primitive digraphs with large exponent, circumference $n - 1$ and diameter $n - 1$ that are defined in [9, Theorems 3.3 and 3.4] are used in this section to derive their circumdiameters and exponents. The diameters of their strongly connected subdigraphs are also determined.

For fixed $n \geq 6$ and fixed j with $n - 2 \geq j > \frac{n}{2}$, let $D_1(n, j)$ be the primitive digraph on n vertices with large exponent constructed by taking the $(n - 1)$ -cycle $1 \rightarrow (n - 1) \rightarrow (n - 2) \rightarrow \cdots \rightarrow 2 \rightarrow 1$, arcs $i \rightarrow (i + j - 1)$ for $1 \leq i \leq n - j$ and arcs $1 \rightarrow n$, $n \rightarrow (n - 2)$, $n \rightarrow (j - 1)$. This digraph is defined in [9, Theorem 3.3 (b)], and depicted in Figure 5.5 for $n \geq 11$.

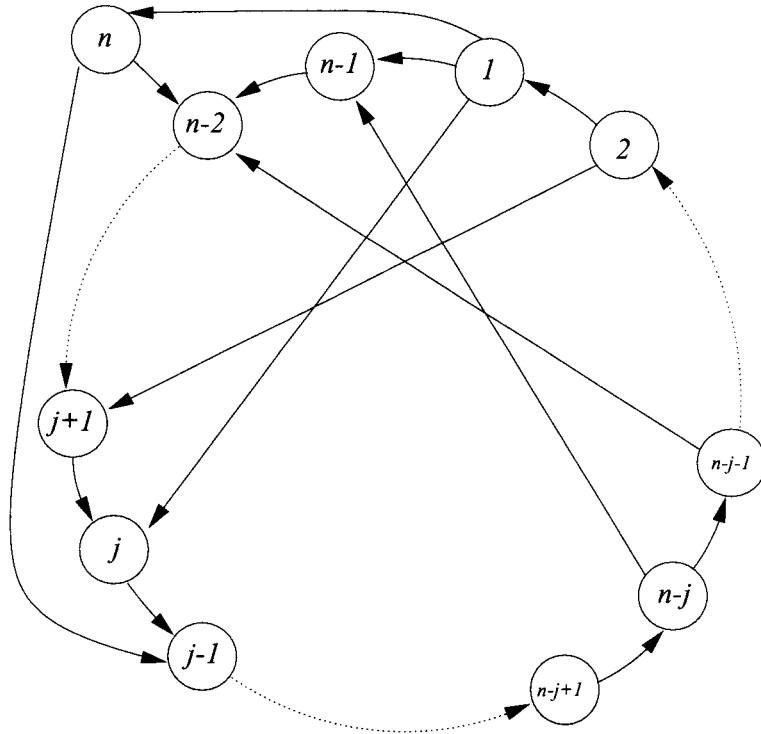


Figure 5.5: $D_1(n, j)$

Lemma 5.2.1 *The diameter of $D_1(n, j)$ is $n-1$, its circumdiameter is $n-1$, and its exponent is $(n-2)(j-1) + d^c(\lambda(D_1(n, j))) = (n-2)j + 1 \geq \lfloor \frac{\omega_n}{2} \rfloor + 2$.*

Proof: By [9, Theorem 3.3(b)], $D_1(n, j)$ is a digraph with large exponent and cycles of lengths $n-1$ and j only.

To prove that $\delta(D_1(n, j)) = n-1$, it suffices to demonstrate a single pair of vertices u, v for which $\delta[u, v] = n-1$. Since no $(n-1) \rightsquigarrow 1$ path contains a chord of an $(n-1)$ -cycle, $\delta[n-1, 1] = n-2$ because vertices $n-1$ and 1 are adjacent on an $(n-1)$ -cycle. The only arc into vertex n is $1 \rightarrow n$, and so vertex n does not lie on any $(n-1) \rightsquigarrow 1$ path. This implies that $\delta[n-1, n] = \delta[n-1, 1] + \delta[1, n]$, which is equal to $(n-2) + 1 = n-1$. Thus $\delta(D_1(n, j)) = n-1$. Every vertex in $D_1(n, j)$ is on an $(n-1)$ -cycle and on a j -cycle, implying that $d^c(\lambda(D_1(n, j))) = \delta(D_1(n, j)) = n-1$. By Theorem 3.2.3, $\gamma(D_1(n, j)) = (n-2)(j-1) + d^c(\lambda(D_1(n, j))) = (n-2)j + 1$. \square

For fixed even $n \geq 8$, $j = \frac{n}{2}$, and fixed $i_0 \in \{n-3, n-2, n-1\}$, let $D_2(n, i_0)$ be the primitive digraph on n vertices with large exponent constructed by taking the $(n-1)$ -cycle $1 \rightarrow (n-1) \rightarrow (n-2) \rightarrow \cdots \rightarrow 2 \rightarrow 1$, arcs $i \rightarrow (i+j-1)$ for $1 \leq i \leq \frac{n}{2} - 3$, and arcs so that n replicates i_0 . This digraph is defined in [9, Theorem 3.4 (a)], and depicted in Figure 5.6 for $n \geq 14$ and $i_0 = n-2$.

Lemma 5.2.2 *The diameter of $D_2(n, i_0)$ is $n-1$, its circumdiameter is $n+1$, and its exponent is $(n-2)(\frac{n}{2}-1) + d^c(\lambda(D_2(n, i_0))) = \lfloor \frac{\omega_n}{2} \rfloor + 2$.*

Proof: By [9, Theorem 3.4 (a)], $D_2(n, i_0)$ is a digraph with large exponent and cycles of lengths $n-1$ and $\frac{n}{2}$ only.

To prove that $\delta(D_2(n, i_0)) = n-1$, it suffices to find a pair of vertices u, v so that $\delta[u, v] = n-1$. Vertices $n-3$, $n-2$ and $n-1$ are not on $\frac{n}{2}$ -cycles, and since n replicates one of these vertices, it is not on an $\frac{n}{2}$ -cycle. If i_0 is the vertex that n replicates, then the only arcs into vertices n and i_0 are

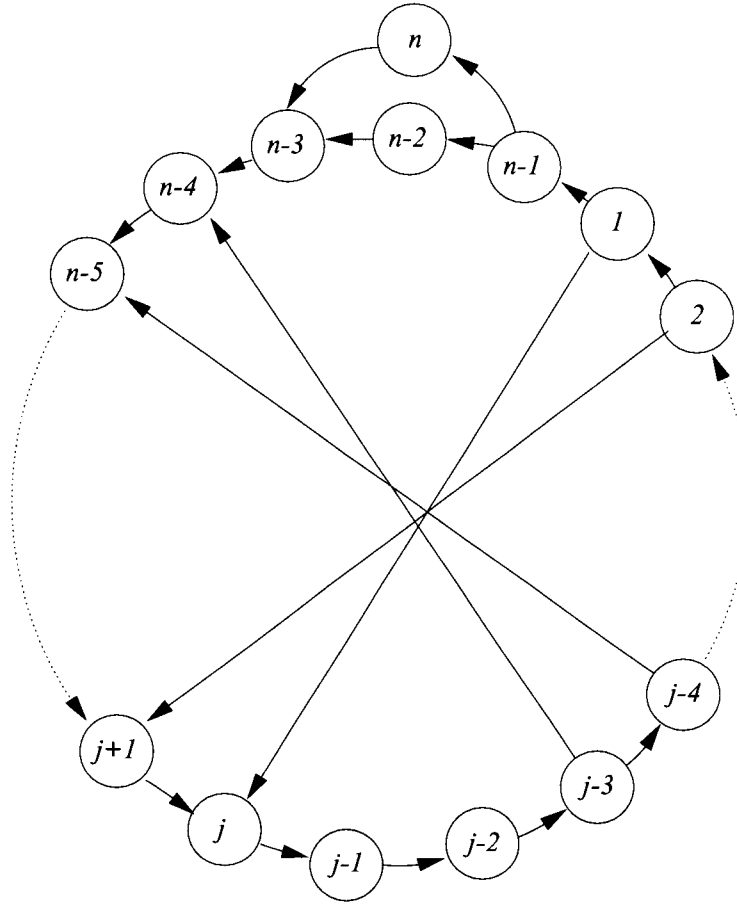


Figure 5.6: $D_2(n, i_0)$ with $i_0 = n - 2$

$(i_0 + 1) \rightarrow n$ and $(i_0 + 1) \rightarrow i_0$ (in which addition and subtraction on vertex subscripts are modulo $n - 1$). This implies that $\delta[i_0, i_0 + 1] = n - 2$, and thus $\delta[i_0, n] = \delta[i_0, i_0 + 1] + \delta[i_0 + 1, n]$, which is equal to $(n - 2) + 1 = n - 1$. Thus $\delta(D_2(n, i_0)) = n - 1$.

Let J be the set of vertices on $\frac{n}{2}$ -cycles, i.e., $J = \{1, 2, \dots, n - 4\}$. Let u and v be vertices in $V(D_2(n, i_0))$, not necessarily distinct. If at least one of u or v is in J , then all $u \rightsquigarrow v$ walks intersect cycles of both lengths, implying that $d^c(\lambda(D_2(n, i_0)))[u, v]$ is less than or equal to $n - 1$. If u and v are both

in $V(D_2(n, i_0)) \setminus J$ and there is a $u \rightsquigarrow v$ walk on vertices in $V(D_2(n, i_0)) \setminus J$, then $d^c(\lambda(D_2(n, i_0)))[v, u] \leq n - 1$ and $d^c(\lambda(D_2(n, i_0)))[u, v]$ is less than or equal to $d^c(\lambda(D_2(n, i_0)))[n - 1, n - 3]$, which is equal to $n + 1$. Therefore $d^c(\lambda(D_2(n, i_0))) = n + 1$.

By Theorem 3.2.3, $\gamma(D_2(n, i_0)) = (n - 2)(\frac{n}{2} - 1) + d^c(\lambda(D_2(n, i_0))) = \lfloor \frac{\omega_n}{2} \rfloor + 2$.
 \square

For fixed even $n \geq 6$, and $j = \frac{n}{2}$, let $D_3(n)$ be the primitive digraph on n vertices with large exponent constructed by taking the $(n - 1)$ -cycle $1 \rightarrow (n - 1) \rightarrow (n - 2) \rightarrow \dots \rightarrow 2 \rightarrow 1$, arcs $i \rightarrow (i + j - 1)$ for $1 \leq i \leq \frac{n}{2} - 3$, and arcs $1 \rightarrow n, n \rightarrow (n - 2), n \rightarrow (j - 1)$. This digraph is defined in [9, Theorem 3.4 (b)] and depicted in Figure 5.7 for $n \geq 14$.

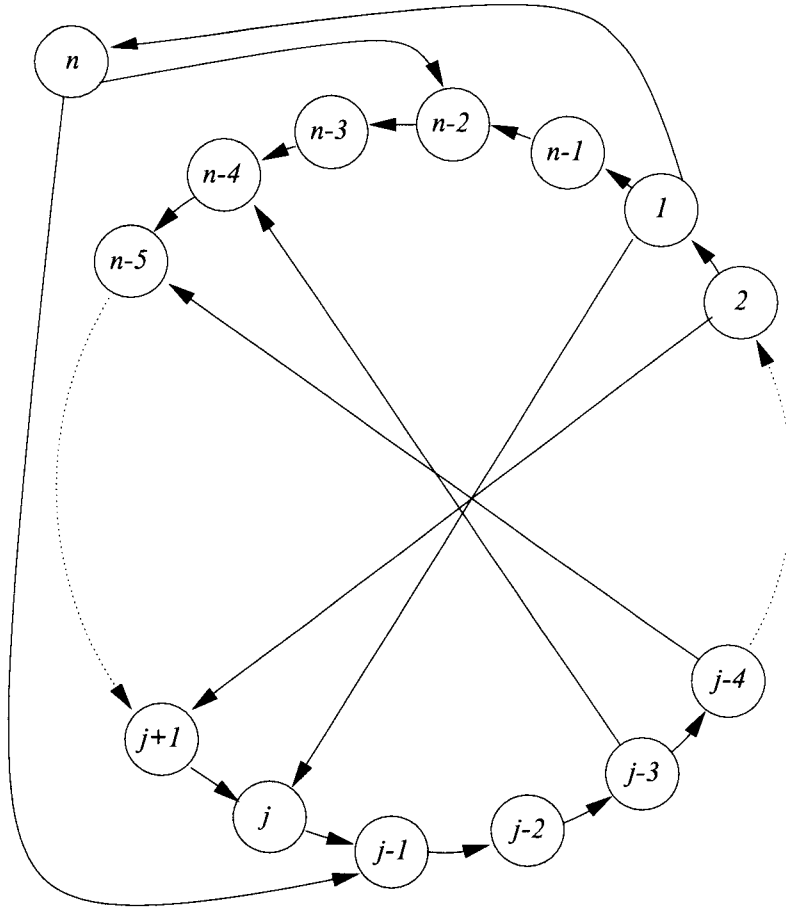
Lemma 5.2.3 *The diameter of $D_3(n)$ is $n - 1$, its circumdiameter is $n + 1$, and its exponent is $(n - 2)(\frac{n}{2} - 1) + d^c(\lambda(D_3(n))) = \lfloor \frac{\omega_n}{2} \rfloor + 2$.*

Proof: By [9, Theorem 3.4 (b)], $D_3(n)$ is a digraph with large exponent and cycles of lengths $n - 1$ and $\frac{n}{2}$ only.

To prove that $\delta(D_3(n)) = n - 1$, it suffices to find a pair of vertices u, v so that $\delta[u, v] = n - 1$. Vertex $n - 1$ is not on an $\frac{n}{2}$ -cycle. The only arc into vertex $n - 1$ is $1 \rightarrow n - 1$, therefore $\delta[n - 1, 1] = (n - 1) - 1 = n - 2$. The only arc into vertex n is $1 \rightarrow n$, and so vertex n does not lie on any $(n - 1) \rightsquigarrow 1$ path. This implies that $\delta[n - 1, n] = \delta[n - 1, 1] + 1$, which is equal to $(n - 2) + 1 = n - 1$. Thus $\delta(D_3(n)) = n - 1$.

If J is the set of vertices on j -cycles and K is the set of vertices on $(n - 1)$ -cycles, then $|J| = n - 3$ and $|K| = n$. By Theorem 3.1.1, $d^c(\lambda(D_3(n))) \leq \max\{n - 1, n - 1, n + 1, \frac{n}{2} - 1\} = n + 1$. Since $d^c(\lambda(D_3(n)))[n - 1, n - 3] = n + 1$, it follows that $d^c(\lambda(D_3(n))) = n + 1$.

By Theorem 3.2.3, $\gamma(D_3(n)) = (n - 2)(\frac{n}{2} - 1) + d^c(\lambda(D_3(n)))$, which is equal to $\lfloor \frac{\omega_n}{2} \rfloor + 2$. \square

Figure 5.7: $D_3(n)$

It follows from Lemma 1.1.1 and the results of this section that if D is a primitive subdigraph on n vertices of $D_1(n, j)$, $D_2(n, i_0)$ or $D_3(n)$ then $\delta(D) = n - 1$.

5.3 Circumference $n - 1$ and Diameter $n - 2$

The characterizations of the class of primitive digraphs with large exponent, circumference $n - 1$ and diameter $n - 2$ that are defined in [9, Theorems

3.3 and 3.4] are used in this section to derive their circumdiameters and exponents.

For fixed $n \geq 6$, fixed j with $n - 2 \geq j > \frac{n}{2}$, and fixed $i_0 \in \{1, 2, \dots, n - 1\}$, let $D_4(n, j, i_0)$ be the primitive digraph on n vertices with large exponent constructed by taking the $(n - 1)$ -cycle $1 \rightarrow (n - 1) \rightarrow (n - 2) \rightarrow \dots \rightarrow 2 \rightarrow 1$, arcs $i \rightarrow (i + j - 1)$ for $1 \leq i \leq n - j$, and arcs so that n replicates i_0 . This digraph is defined in [9, Theorem 3.3 (a)] and depicted in Figure 5.8 for $n \geq 11$ and $i_0 = 1$.

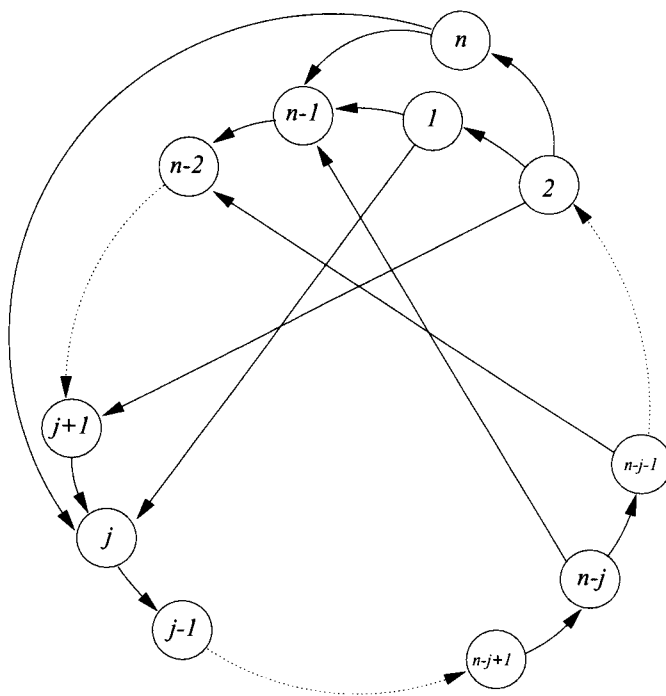


Figure 5.8: $D_4(n, j, i_0)$ with $i_0 = 1$

Lemma 5.3.1 *The diameter of $D_4(n, j, i_0)$ is $n - 2$, its circumdiameter is $n - 2$, and its exponent is $(n - 2)(j - 1) + d^c(\lambda(D_4(n, j, i_0))) = (n - 2)j \geq \lfloor \frac{\omega_n}{2} \rfloor + 2$.*

Proof: By [9, Theorem 3.3 (a)], $D_4(n, j, i_0)$ is a primitive digraph with large

exponent and cycles of lengths $n-1$ and j only. If H is the primitive digraph $D_4(n, j, i_0)$ without vertex n and all of its incoming / outgoing arcs, then it is isomorphic to $\tilde{D}(n-1, j)$, implying that $\delta(H) = \delta(\tilde{D}(n-1, j)) = (n-1) - 1 = n-2$. Thus $\delta[u, v] \leq n-2$ for $1 \leq u, v \leq n-1$ in $D_4(n, j, i_0)$. Vertex n shares an $(n-1)$ -cycle with all other vertices except i_0 , thus $\delta[u, n] \leq n-2$ and $\delta[n, u] \leq n-2$ for all $1 \leq u \leq n-1$ such that $u \neq i_0$. In addition, $\delta[n, i_0] = \delta[i_0, n] = j \leq n-2$. All paths from vertex $n-1$ to 1 have length exactly $n-2$, implying that $\delta(D_4(n, j, i_0)) = n-2$. Since every vertex in $D_4(n, j, i_0)$ is on cycles of both lengths, $d^c(\lambda(D_4(n, j, i_0))) = \delta(D_4(n, j, i_0)) = n-2$. By Theorem 3.2.3, $\gamma(D_4(n, j, i_0)) = (n-2)(j-1) + d^c(\lambda(D_4(n, j, i_0))) = (n-2)j$. \square

For fixed even $n \geq 8$, $j = \frac{n}{2}$, and fixed $i_0 \in \{1, 2, \dots, n-4\}$, let $D_5(n, i_0)$ be the primitive digraph on n vertices with large exponent constructed by taking the $(n-1)$ -cycle $1 \rightarrow (n-1) \rightarrow (n-2) \rightarrow \dots \rightarrow 2 \rightarrow 1$, arcs $i \rightarrow (i+j-1)$ for $1 \leq i \leq \frac{n}{2}-3$, and arcs so that n replicates i_0 . This digraph is defined in [9, Theorem 3.4 (a)] and depicted in Figure 5.9 for $n \geq 14$.

Lemma 5.3.2 *The diameter of $D_5(n, i_0)$ is $n-2$, its circumdiameter is $n+1$, and its exponent is $(\frac{n}{2}-1)(n-2) + d^c(\lambda(D_5(n, i_0))) = \lfloor \frac{\omega n}{2} \rfloor + 2$.*

Proof: By [9, Theorem 3.4 (a)], $D_5(n, i_0)$ is a primitive digraph with cycles of lengths $n-1$ and $\frac{n}{2}$ only. To prove that $\delta(D_5(n, i_0)) = n-2$, it is sufficient to show that $\delta[u, v] \leq n-2$ for all u, v in $D_5(n, i_0)$, and then find a u, v pair for which $\delta[u, v] = n-2$. Any u, v pair in $\{1, 2, \dots, n-1\}$ shares an $(n-1)$ -cycle, implying that $\delta[u, v] \leq n-2$ for $u, v \in \{1, 2, \dots, n-1\}$.

Since n replicates a fixed $i_0 \in \{1, 2, \dots, n-4\}$, $\delta[u, n] = \delta[u, i_0] \leq n-2$ and $\delta[n, u] = \delta[i_0, u] \leq n-2$ for $u \in \{1, 2, \dots, i_0-1, i_0+1, \dots, n-1\}$. Also $\delta[i_0, n] = \delta[n, i_0] = \frac{n}{2}$, since n replicates i_0 , which is on an $\frac{n}{2}$ -cycle. Since vertices $n-1$ and $n-2$ are not on $\frac{n}{2}$ -cycles, $\delta[n-2, n-1] = n-2$. Thus $\delta(D_5(n, i_0)) = n-2$.

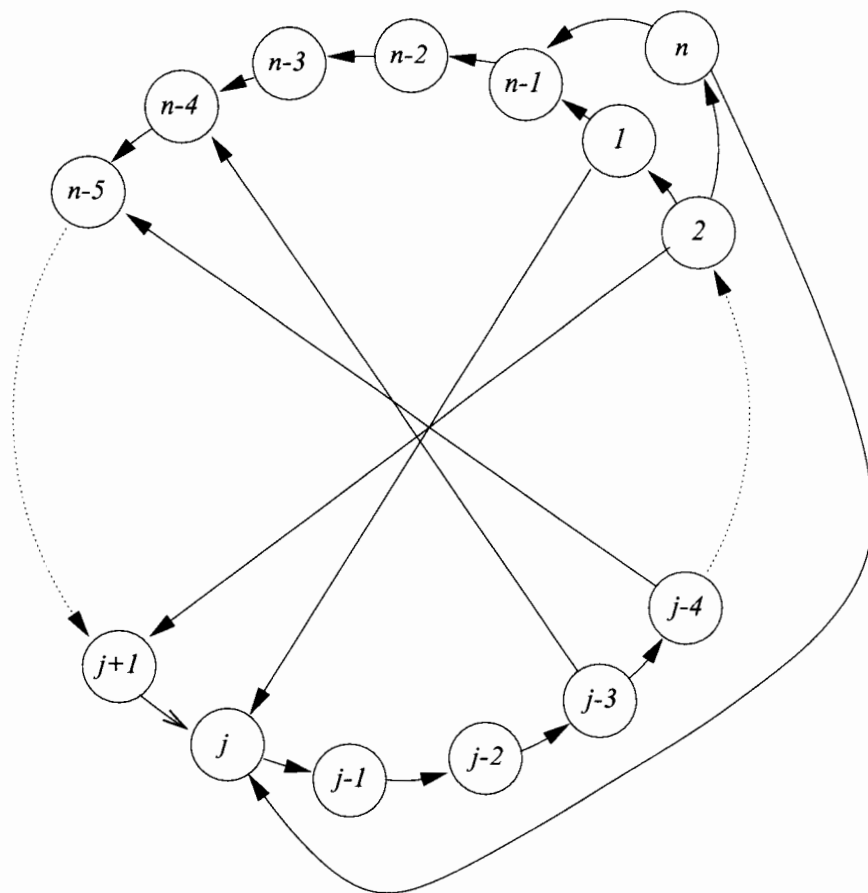


Figure 5.9: $D_5(n, i_0)$ with $i_0 = 1$

If J is the set of vertices on $\frac{n}{2}$ -cycles and K is the set of vertices on $(n-1)$ -cycles, then $|J| = n-3$ and $|K| = n$. By Theorem 3.1.1, $d^c(\lambda(D_5(n, i_0))) \leq \max\{n-1, n-2, n+1, \frac{n}{2}-1\} = n+1$. Since $d^c(\lambda(D_5(n, i_0)))[n-1, n-3] = n+1$, it follows that $d^c(\lambda(D_5(n, i_0))) = n+1$.

By Theorem 3.2.3, $\gamma(D_5(n, i_0)) = (n-2)(\frac{n}{2}-1) + d^c(\lambda(D_5(n, i_0))) = \lfloor \frac{\omega_n}{2} \rfloor + 2$.

□

Chapter 6

Conclusions and Questions for Future Research

Let D be a primitive digraph on $n \geq 3$ vertices with cycles of lengths k and j where $n \geq k > j$. Recall from previous sections that $\gamma(D) = \phi(\lambda(D)) + d^c(\lambda(D))$ if :

1. D has an ordered pair of vertices (u, v) that have the unique path property and $d^c(\lambda(D)) = d^c(\lambda(D))[u, v]$; see Corollary 1.5.4.
2. D has an ordered pair of vertices (u, v) such that $d^c(\lambda(D))$ is equal to $d^c(\lambda(D))[u, v]$ and $\delta[u, v] = n - 1$; see Lemma 3.2.4.
3. D has large exponent and there exists a vertex u such that $d^c(\lambda(D)) = d^c(\lambda(D))[u, u]$; see Theorem 3.2.1.
4. D on $n \geq 4$ vertices has large exponent and $d^c(\lambda(D)) = k - 1$; see Theorem 3.2.2.
5. D has large exponent and $k \geq n - 1$; see Theorem 3.2.3 for $n \geq 4$. If $n = 3$, then D is isomorphic to one of the digraphs of Example 1.6.2. If $k = n \geq 4$, then D is isomorphic to a primitive subdigraph of $\tilde{D}(n, j)$

or $\widehat{D}(n)$ as defined in Chapter 5. If $k = n - 1$ and $n \geq 6$, then D is isomorphic to a primitive subdigraph of $D_1(n, j)$, $D_2(n, i_0)$, $D_3(n)$, $D_4(n, j, i_0)$ or $D_5(n, i_0)$ as defined in Chapter 5. Note that $D_2(n, i_0)$ and $D_5(n, i_0)$ are defined only for $n \geq 8$.

6. D has large exponent and $3 \leq n \leq 8$; see Theorem 3.2.6.
7. D is isomorphic to the digraph $D(n, k, d)$ as defined in the proof of Theorem 2.1; see Corollary 2.4.
8. $D \in \mathcal{F}_\delta$ for $\delta \geq 2$ has cycles of exactly two lengths; see Theorem 4.1.3.
9. D is isomorphic to $R(n, k, j, i)$ as defined in Section 4.2; see Theorems 4.2.2 and 4.2.3.
10. D is isomorphic to $\widetilde{W}(n, k, j)$ as defined in Section 4.3; see Theorem 4.3.1.

Conditions 1 through 7 above are the same as those stated at the beginning of Chapter 4. Condition 5 above is also stated in terms of the specific digraphs as defined in Chapter 5. Conditions 7 through 10 give families of digraphs. These ten conditions beg the question, *what are necessary and sufficient conditions for a primitive digraph D to satisfy*

$$\gamma(D) = \phi(\lambda(D)) + d^c(\lambda(D))? \quad (6.1)$$

The first three conditions above follow from Corollary 1.5.4. If a digraph D satisfies (6.1), does it follow that there exists an ordered pair of vertices (u, v) with the unique path property such that $d^c(\lambda(D)) = d^c(\lambda(D))[u, v]$? In other words, is Corollary 1.5.4 "if and only if"?

The proofs of theorems in conditions 4, 5 and 6 rely heavily on the fact that these digraphs have large exponent. Recall that (6.1) is not satisfied for any digraph D on n vertices with large exponent that is isomorphic to $L(n)$ for $n \geq 9$ as defined in the proof of Theorem 3.2.6, or $M(n)$ for $n \geq 22$

as defined in the proof of Theorem 4.4.1. Thus the fact that a primitive digraph D has large exponent is not sufficient for it to satisfy (6.1).

The final four conditions above apply to specific families of digraphs. Any digraph D that is isomorphic to a member of these specific families, to $L(n)$ for $n \geq 9$ as defined in the proof of Theorem 3.2.6, or $M(n)$ for $n \geq 22$ as defined in the proof of Theorem 4.4.1, satisfies either (6.1) or $\gamma(D) = \phi(\lambda(D)) + d^c(\lambda(D)) - 1$. Are these the only two possibilities for primitive digraphs with large exponent?

Chapter 1 presents a number of bounds on the exponent of a primitive digraph. One of these is the Shen bound of Theorem 1.4.2. None of the primitive digraphs described in Chapter 5, i.e., $\tilde{D}(n, j)$, $\widehat{D}(n)$, $D_1(n, j)$, $D_2(n, i_0)$, $D_3(n)$, $D_4(n, j, i_0)$ or $D_5(n, i_0)$ has an exponent that is equal to this upper bound. Do any of their primitive subdigraphs?

Chapter 2 explores the relationship between the diameter and circumference of primitive digraphs with cycles of exactly two lengths by introducing a family of digraphs, $D(n, k, d)$, which has circumference k and diameter d . What is the probability that a primitive digraph in this family has large exponent? Is there another family of digraphs that always has large exponent, and allows any possible combination of n , k , and d ? Is it possible to characterize all digraphs with large exponent and diameter $n - 1$?

Chapter 3 shows that a primitive digraph D with large exponent and circumference n or $n - 1$ satisfies (6.1). The family $L(n)$ defined in the proof of Theorem 3.2.6 has large exponent, circumference $n - 2$ and does not satisfy (6.1). What are necessary and sufficient conditions for a primitive digraph with large exponent and circumference $n - 2$ to satisfy (6.1)?

Chapter 5 explores some properties of primitive digraphs on n vertices with large exponent, circumference n or $n - 1$ and girth j . The diameter, circumdiameter and exponent of $\tilde{D}(n, j)$, $\widehat{D}(n)$, $D_1(n, j)$, $D_2(n, i_0)$, $D_3(n)$, $D_4(n, j, i_0)$ and $D_5(n, i_0)$ are given. These properties are also given for

primitive subdigraphs of $\tilde{D}(n, j)$ and $\hat{D}(n)$. Finally, the diameter of a primitive subdigraph of $D_1(n, j)$, $D_2(n, i_0)$ or $D_3(n)$ is given. What are the circumdiameter and exponent for a primitive subdigraph of $D_1(n, j)$, $D_2(n, i_0)$ or $D_3(n)$? What are the diameter, circumdiameter and exponent of a primitive subdigraph of $D_4(n, j, i_0)$ or $D_5(n, i_0)$?

The software used to generate the lists of digraphs that appear in Appendices A and B is described in Appendix C. Another suggestion for future research is to improve the efficiency of this software. Currently it runs in exponential time.

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Appendix A

List of Digraphs with Large Exponent, $n = 8$, $k = 6$ and $j = 5$

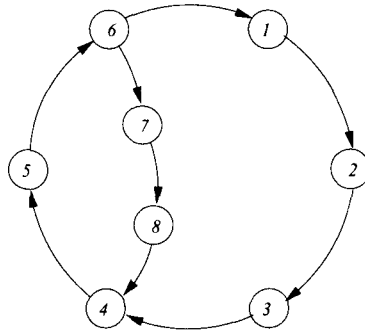
Let D be a primitive digraph on $n \leq 8$ vertices with large exponent and $\lambda(D) = \{k, j\}$. If $n \leq 7$, then $k \geq n - 1$ by Theorem 1.6.3, implying that $\gamma(D) = \phi(\lambda(D)) + d^c(\lambda(D))$ by Theorem 3.2.3, which also covers $n = 8$, $k \geq 7$. If $n = 8$, then by Theorem 1.6.3, $k = 6$, $j = 5$ is the only case not covered by an explicit result. This case is the focus of Appendix A.

By Theorem 1.4.3, $\delta(D)^2 + 1 \geq \lfloor \frac{\omega_8}{2} \rfloor + 2 = 27$, implying that $\delta(D) = 6$ or $\delta(D) = 7$. If $D \in \mathcal{F}_\delta$, then $\delta = 5$, giving a contradiction. Thus $D \notin \mathcal{F}_\delta$ for any δ .

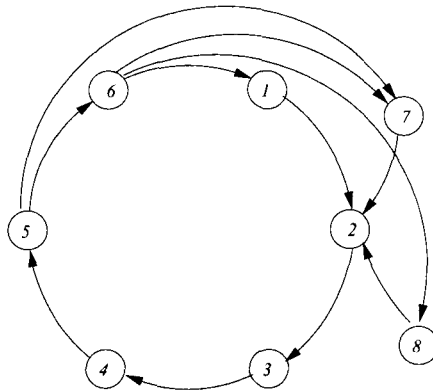
By (1.2) and Theorem 1.5.2, $7 \leq d^c(\lambda(D)) \leq 8$. If there exists a vertex $u \in D$ such that $d^c(\lambda(D)) = d^c(\lambda(D))[u, u]$, then $d^c(\lambda(D)) \leq 6$, giving a contradiction. Thus $d^c(\lambda(D)) \neq d^c(\lambda(D))[u, u]$ for any $u \in D$. Also $d^c(\lambda(D)) \geq 7 > 5 = k - 1$, therefore Theorems 3.2.1 and 3.2.2 do not apply.

The following digraph is isomorphic to the digraph $\widetilde{W}(8, 6, 5)$ defined in

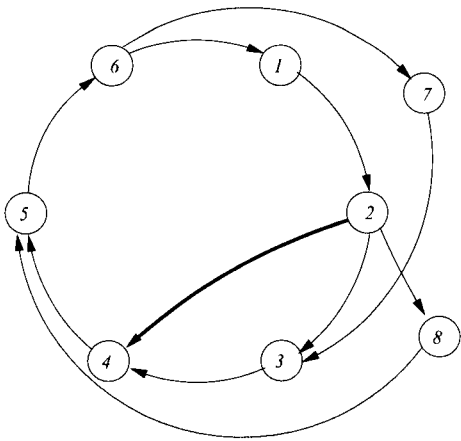
Section 4.3, implying that it has circumdiameter 8 and exponent 28 by Theorem 4.3.1.



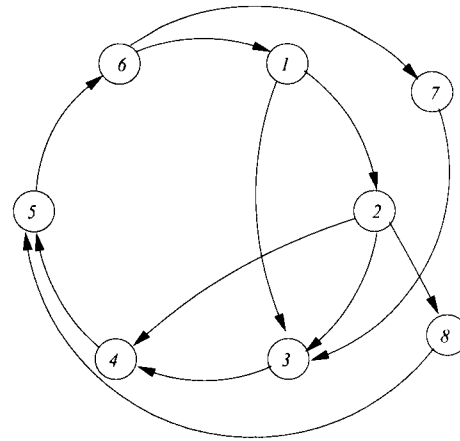
If $i \neq 1$ and $i \neq 5$, then the digraph $R(8, 6, 5, i)$, defined in Section 4.2, has circumdiameter 6. If $i = 1$ or $i = 5$, then $R(8, 6, 5, i)$ has circumdiameter 7. The following digraph is isomorphic to $R(8, 6, 5, 5)$. With a reversal of every arc it is isomorphic to $R(8, 6, 5, 1)$.



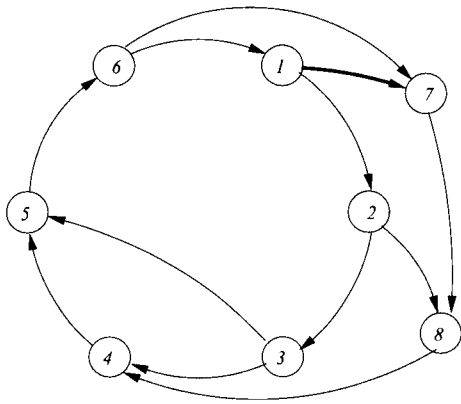
The following digraphs with exponent 27 each have a path of length $n - 1$, implying that they each have diameter 7. Beneath each is a pair of vertices in square brackets signifying the initial and terminal vertex of the path of length $n - 1$. This pair also gives the circumdiameter, 7, of these digraphs, implying that for each D , $\gamma(D) = \phi(\lambda(D)) + d^c(\lambda(D)) = 27$ by Lemma 3.2.4. A figure with a bold arc represents two possible digraphs, i.e., with or without the arc, that have the same circumdiameter, exponent, diameter, and a common pair of vertices giving the circumdiameter and diameter.



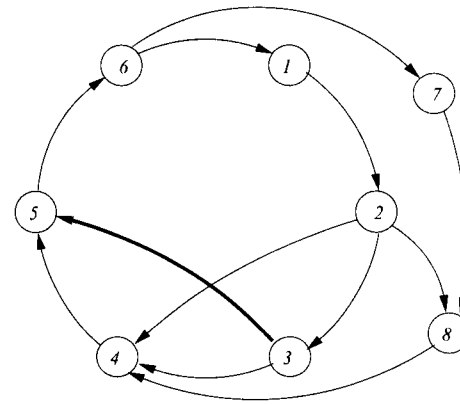
[7, 8]



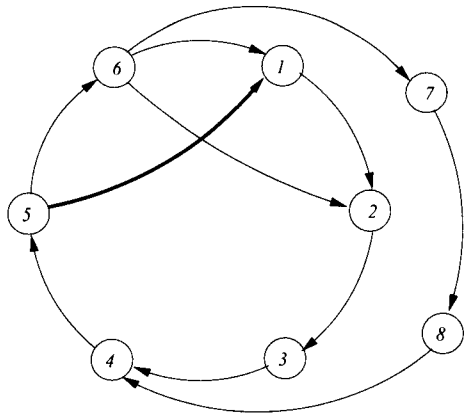
[7, 8]



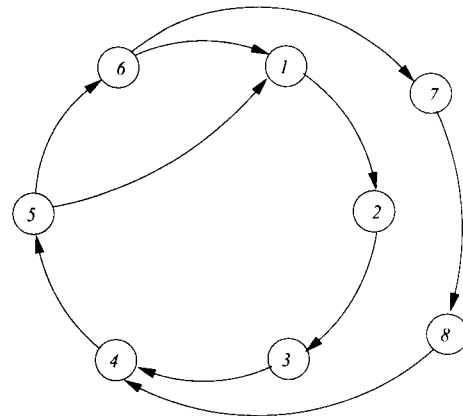
[7, 3]



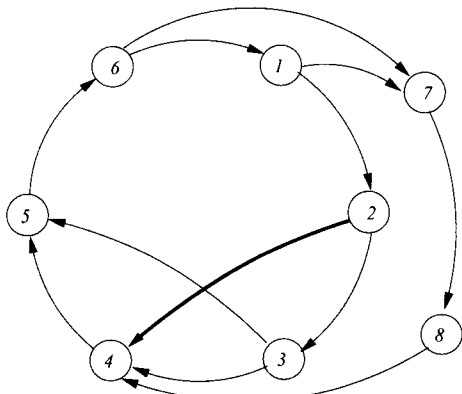
[7, 3]



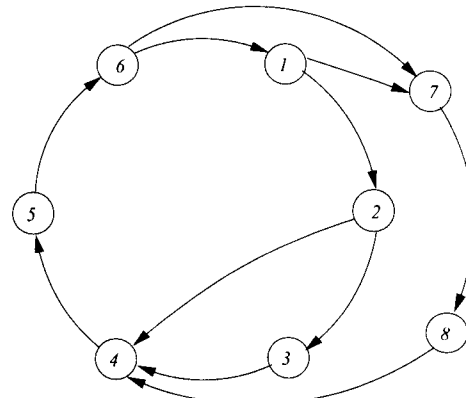
[1, 8]



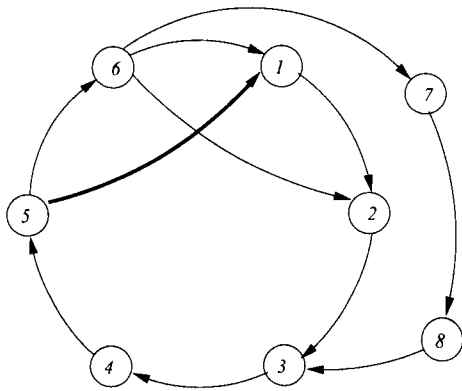
[1, 8]



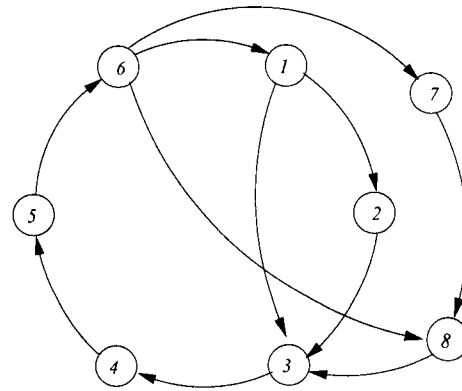
[7, 3]



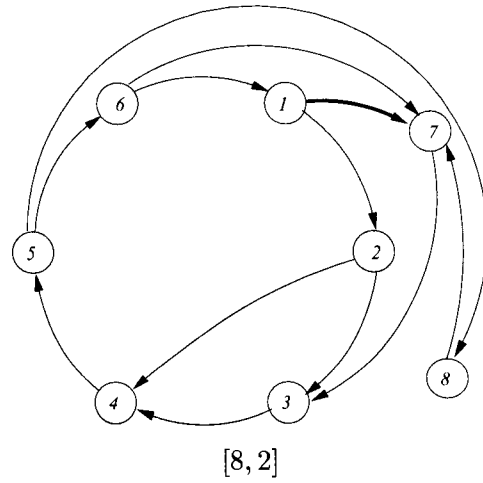
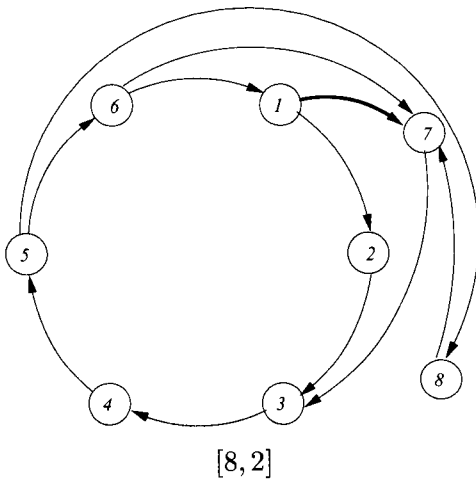
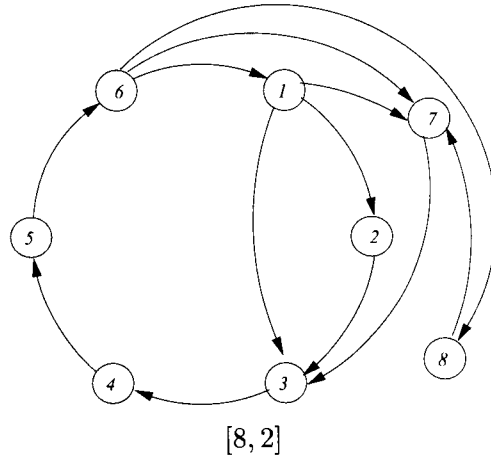
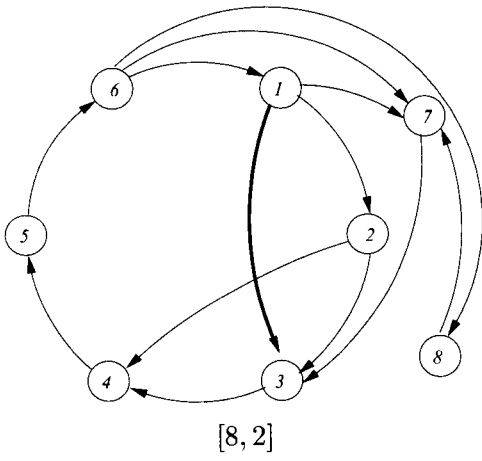
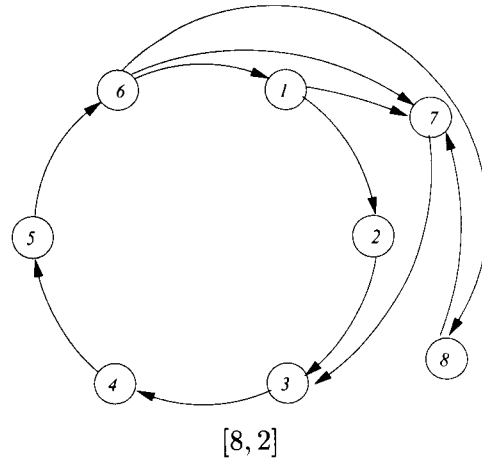
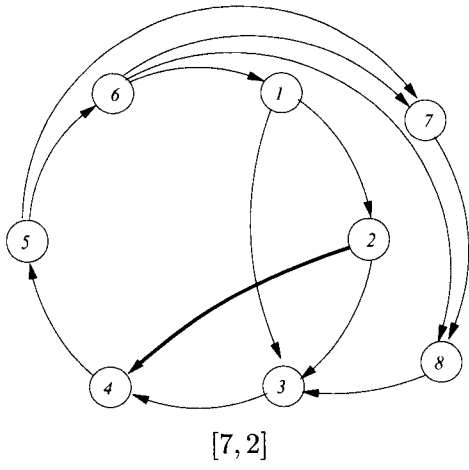
[7, 3]

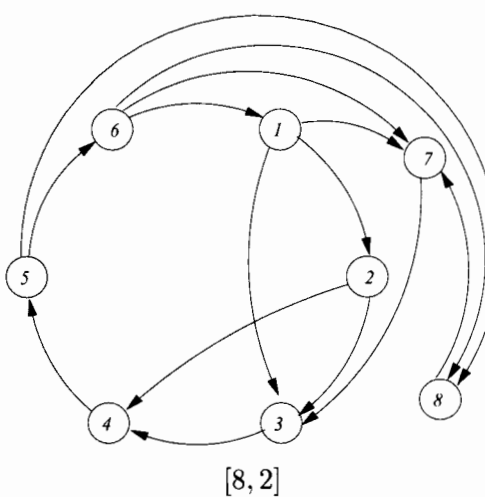
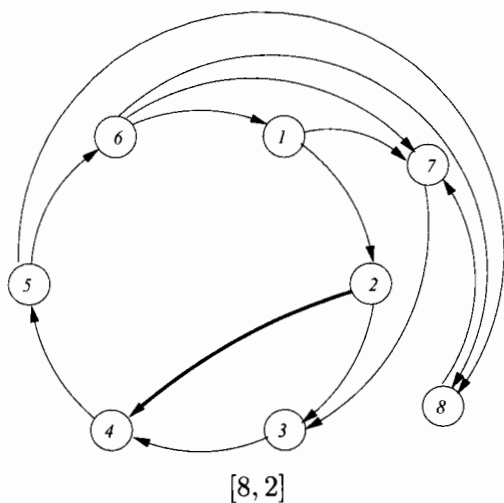
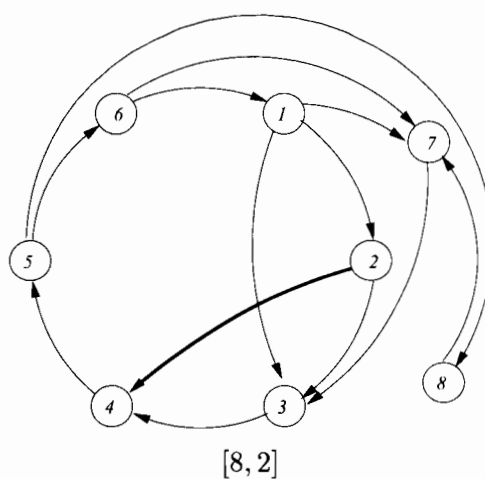
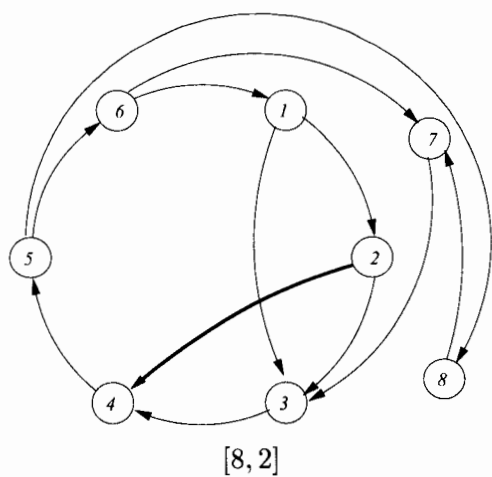


[1, 8]



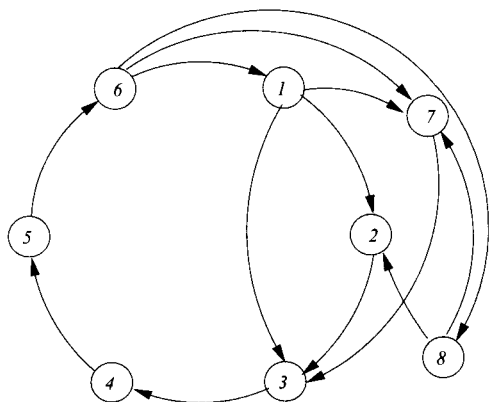
[7, 2]



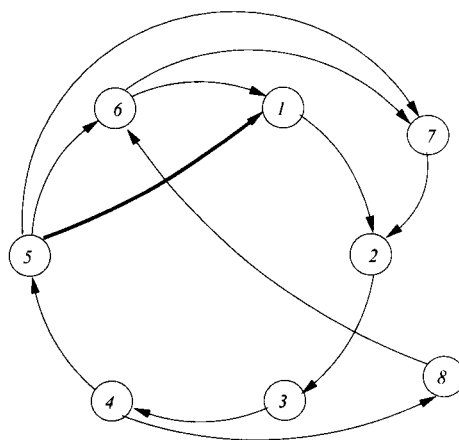


If $\lambda(D) = \{6, 5\}$, then without loss of generality, D has the arcs $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 1$ and $6 \rightarrow 7$. At most one of $1 \rightarrow 3$, $3 \rightarrow 5$ or $5 \rightarrow 1$ may be present. At most one of $2 \rightarrow 4$, $4 \rightarrow 6$ or $6 \rightarrow 2$ may be present. Some of the following arcs or their reverse (but not both) may be present; $2 \rightarrow 7$, $3 \rightarrow 7$, $1 \rightarrow 8$, $2 \rightarrow 8$, $3 \rightarrow 8$, $4 \rightarrow 8$, $5 \rightarrow 8$, $6 \rightarrow 8$, $7 \rightarrow 8$. All remaining

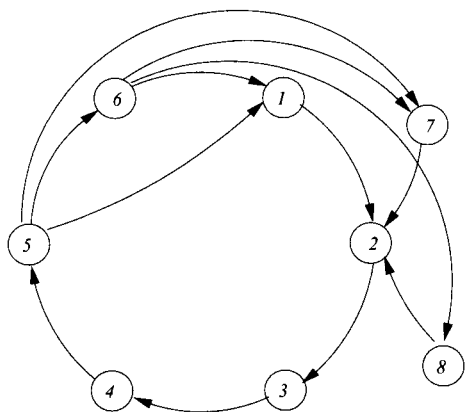
combinations of these arcs (not covered by previous digraphs) resulting in a primitive digraph with large exponent, up to isomorphism or the reversal of every arc, are listed here. They each have an ordered pair of vertices (u, v) with the unique path property such that $d^c(\lambda(D)) = d^c(\lambda(D))[u, v]$.



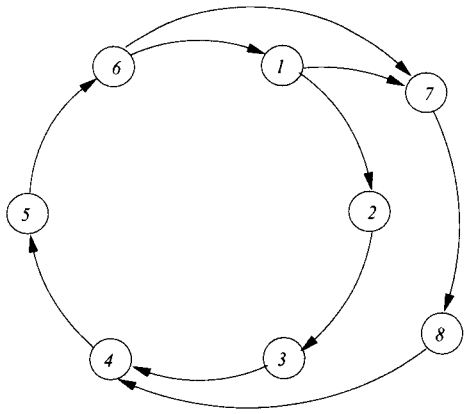
$$\delta(D) = 6, d^c(\lambda(D))[8, 2] = 7, \gamma(D) = 27$$



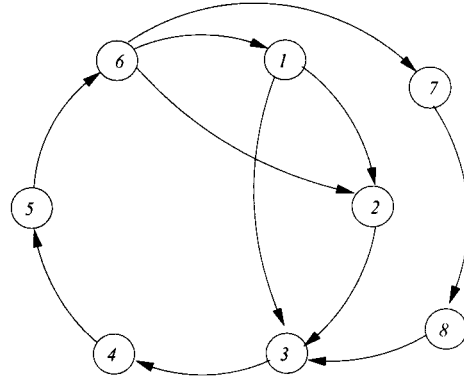
$$\delta(D) = 6, d^c(\lambda(D))[8, 1] = 8, \gamma(D) = 28$$



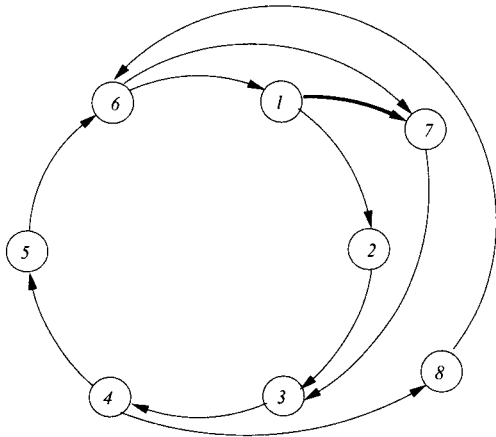
$$\delta(D) = 6, d^c(\lambda(D))[6, 8] = 7, \gamma(D) = 27$$



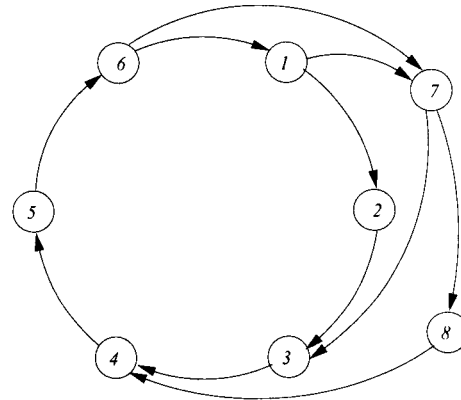
$\delta(D) = 7, d^c(\lambda(D))[1, 3] = 8, \gamma(D) = 28$



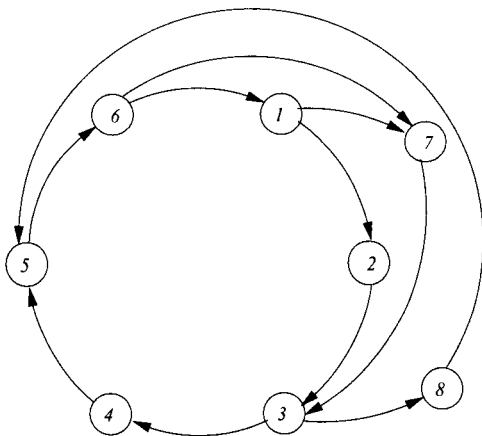
$\delta(D) = 6, d^c(\lambda(D))[7, 8] = 7, \gamma(D) = 27$



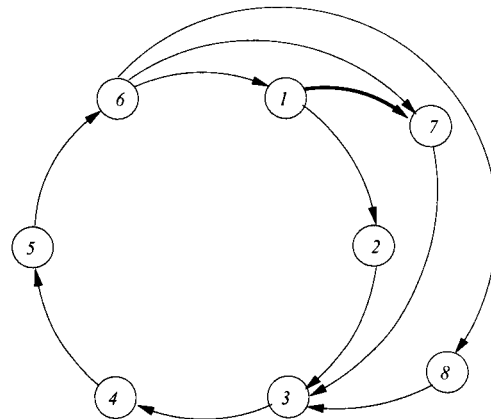
$\delta(D) = 6, d^c(\lambda(D))[1, 2] = 7, \gamma(D) = 27$



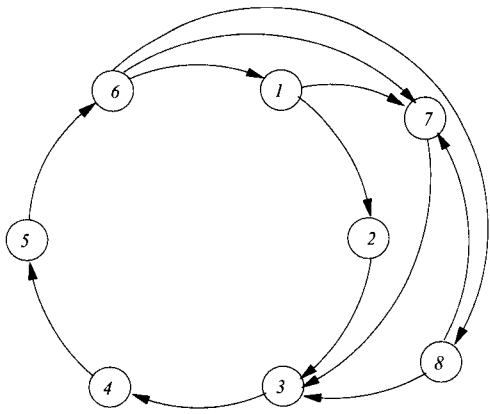
$\delta(D) = 6, d^c(\lambda(D))[1, 2] = 7, \gamma(D) = 27$



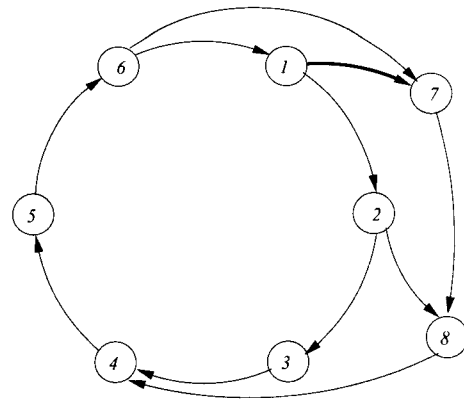
$\delta(D) = 6, d^c(\lambda(D))[1, 2] = 7, \gamma(D) = 27$



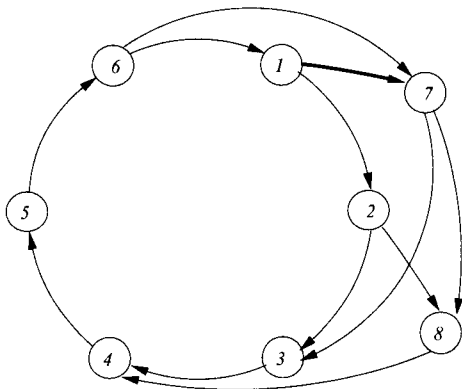
$\delta(D) = 6, d^c(\lambda(D))[1, 2] = 7, \gamma(D) = 27$



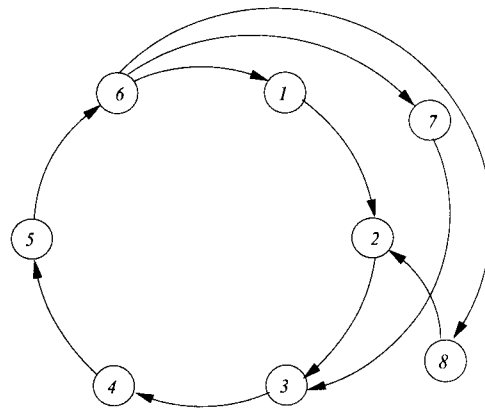
$\delta(D) = 6, d^c(\lambda(D))[1, 2] = 7, \gamma(D) = 27$



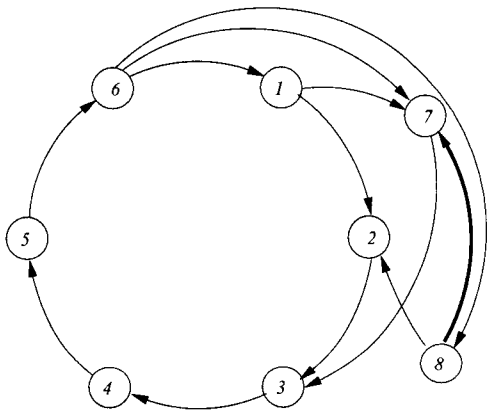
$\delta(D) = 7, d^c(\lambda(D))[1, 3] = 8, \gamma(D) = 28$



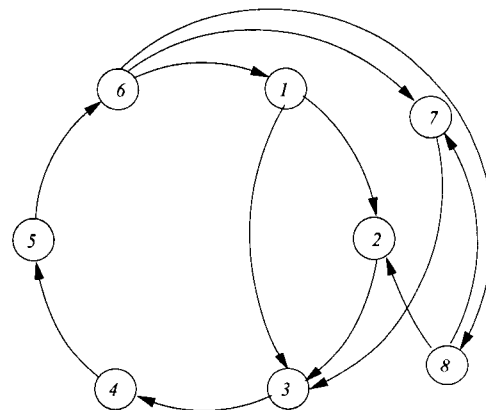
$\delta(D) = 6, d^c(\lambda(D))[1, 2] = 7, \gamma(D) = 27$



$\delta(D) = 6, d^c(\lambda(D))[1, 2] = 7, \gamma(D) = 27$



$\delta(D) = 6, d^c(\lambda(D))[1, 2] = 7, \gamma(D) = 27$



$\delta(D) = 6, d^c(\lambda(D))[8, 2] = 7, \gamma(D) = 27$

Appendix B

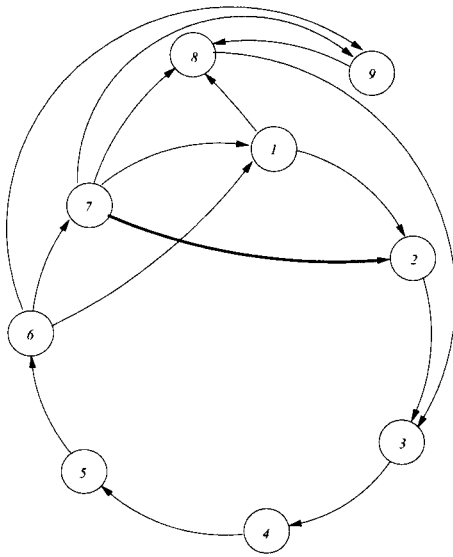
List of Digraphs D with Large Exponent, $n = 9$, $k = 7$ and $j = 6$ and $\gamma(D) \neq \phi(\lambda(D)) + d^c(\lambda(D))$

If D is a primitive digraph on $n = 9$ vertices with exponent $\geq \lfloor \frac{9n}{2} \rfloor + 2 = 34$, then by Theorem 1.6.3, D has circumference at least 7. By Theorem 3.2.3 if D has circumference at least 8, then $\gamma(D) = \phi(\lambda(D)) + d^c(\lambda(D))$. If D has cycles of lengths 7 and 5, then by (1.4), $\gamma(D) \leq n + j(k - 2) = 34$, implying that $\gamma(D) = 34 = \phi(\lambda(D)) + d^c(\lambda(D))$.

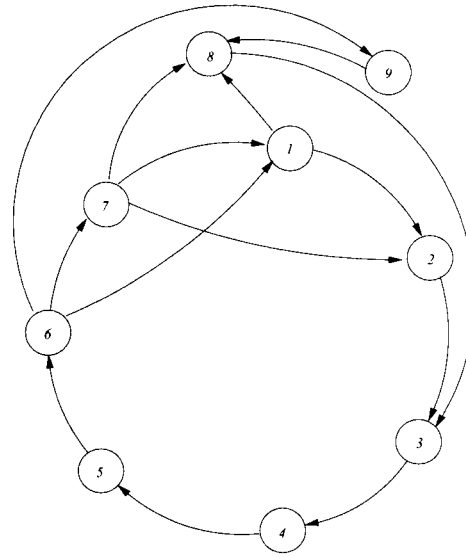
Suppose that D has cycles of length 7 and 6. Without loss of generality, D has the arcs $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 1$ and $1 \rightarrow 8$. At most one of $1 \rightarrow 3$, $3 \rightarrow 5$ or $5 \rightarrow 7$ may be present. At most one of $7 \rightarrow 2$, $2 \rightarrow 4$, $4 \rightarrow 6$, or $6 \rightarrow 1$ may be present. Some of the following arcs or their reverse (but not both) may be present : $1 \rightarrow 9$, $2 \rightarrow 9$, $3 \rightarrow 8$, $3 \rightarrow 9$, $4 \rightarrow 8$, $4 \rightarrow 9$, $5 \rightarrow 9$, $6 \rightarrow 9$, $7 \rightarrow 9$, $8 \rightarrow 9$. Some of the following arcs may be present : $2 \rightarrow 8$, $5 \rightarrow 8$, $6 \rightarrow 8$, $7 \rightarrow 8$. Testing all valid combinations of the

above arcs gives the following primitive digraphs D with exponent 36 and, circumdiameter 7, implying that $\gamma(D) = \phi(\lambda(D)) + d^c(\lambda(D)) - 1$.

Underneath each digraph D are pairs of vertices $[u, v]$ in square brackets that give the circumdiameter, and pairs of vertices (u, v) in round brackets such that there exists no $u \rightsquigarrow v$ walk of length 35. A figure with a bold arc represents two possible digraphs i.e., with or without the arc, that have the same circumdiameter, exponent, diameter, and a common pair of vertices giving the circumdiameter and diameter.

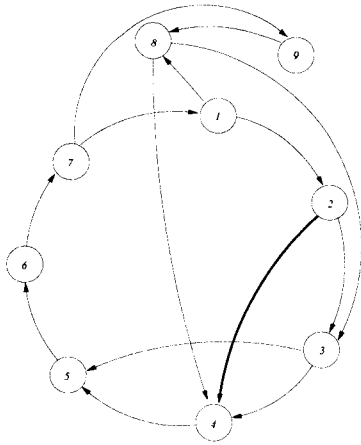


$[9, 2], (1, 7), (2, 1)$

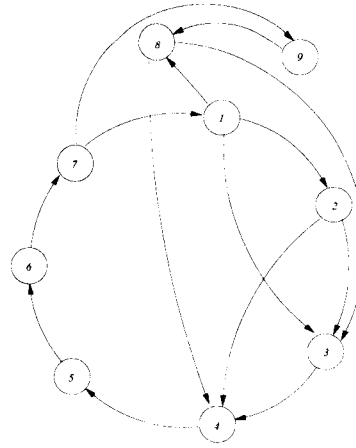


$[9, 2], (1, 7), (9, 7)$

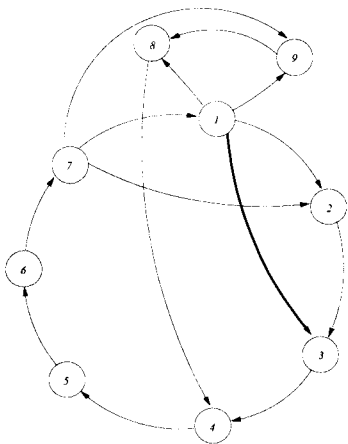
Note that the third digraph on the following page (without the bold arc) is isomorphic to $L(9)$ as defined in the proof of Theorem 3.2.6.



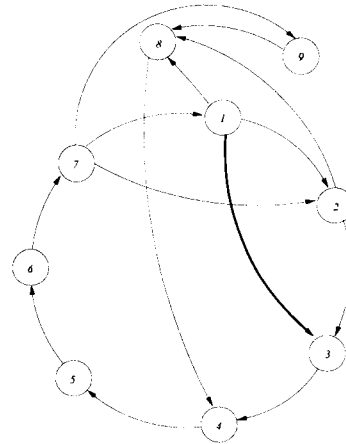
$[9, 2], (4, 3)$



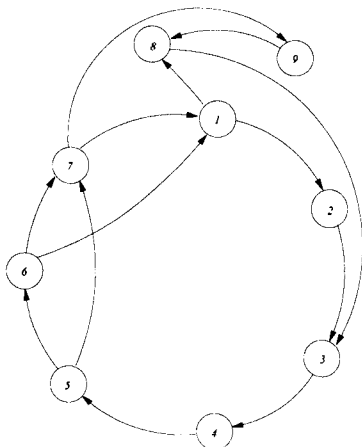
$[9, 2], (3, 2), (3, 8)$



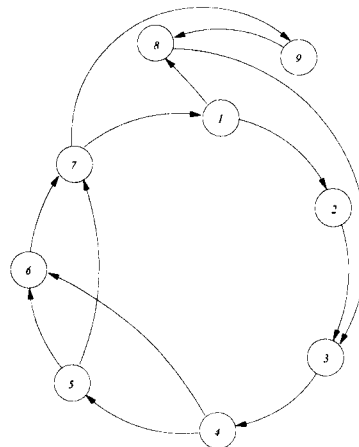
$[2, 8], [9, 3], (2, 1), (9, 1)$



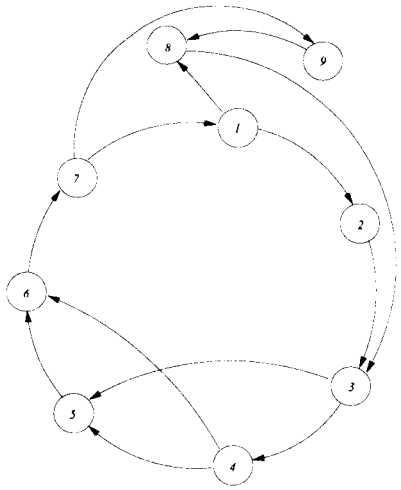
$[9, 3], (2, 1), (2, 9), (9, 1), (9, 9)$



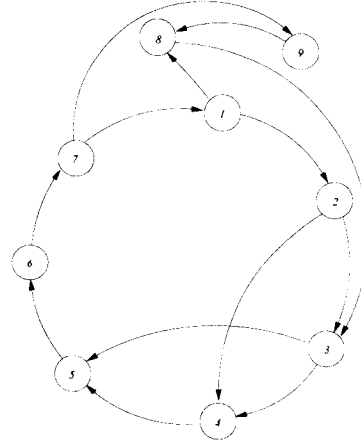
$[9, 2], (7, 6)$



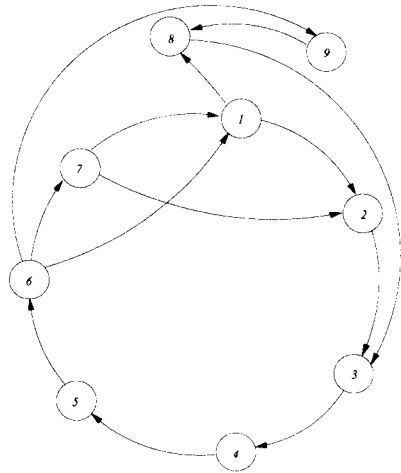
$[9, 2], (6, 5)$



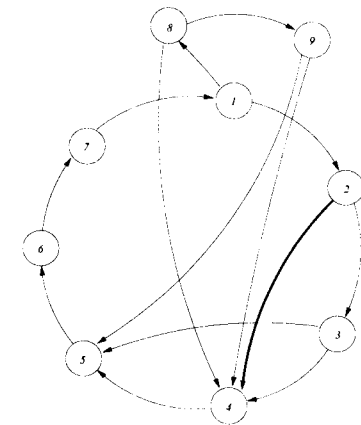
[9, 2], (9, 2)



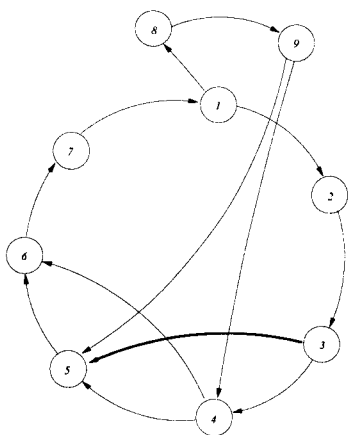
[9, 2], (9, 2)



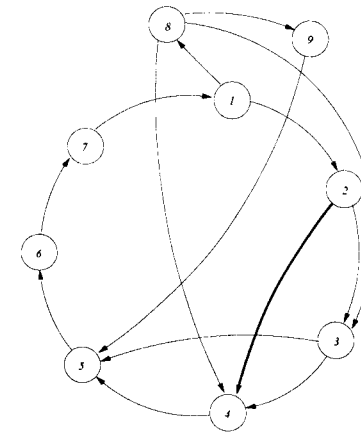
[9, 2], (1, 7), (1, 9), (9, 7), (9, 9)



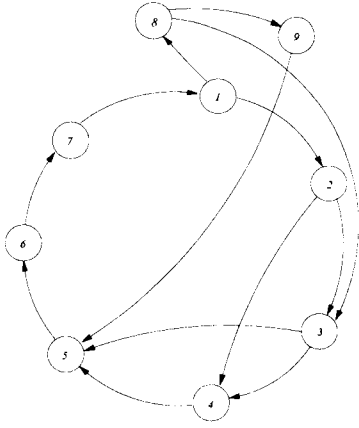
[2, 9], [8, 3], (4, 3), (4, 9)



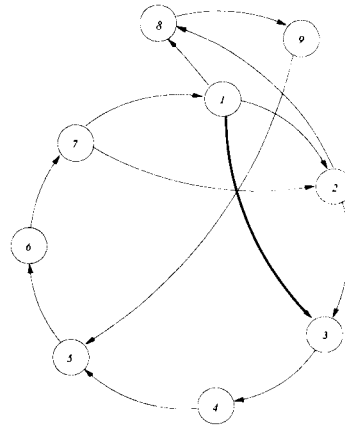
[2, 9], [8, 3], (5, 4)



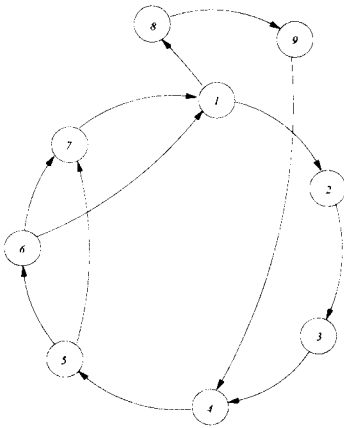
[2, 9], (4, 3), (4, 9), (9, 3), (9, 9)



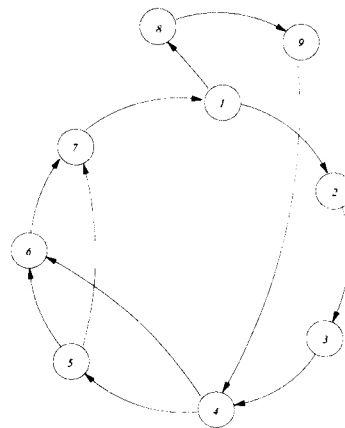
$[2, 9], (4, 3), (4, 9), (9, 3), (9, 9)$



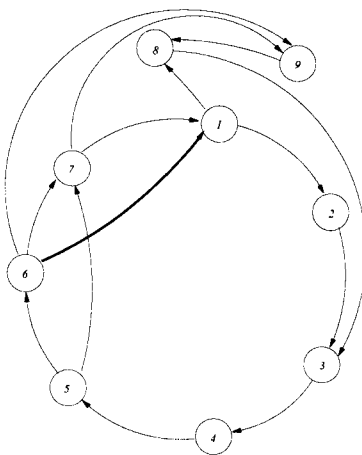
$[3, 9], [8, 4], (2, 1)$



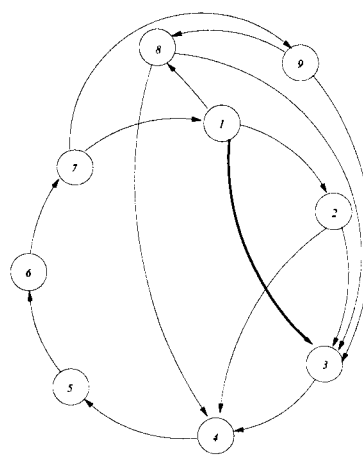
$[2, 9], [8, 3], (7, 6)$



$[2, 9], [8, 3], (6, 5)$



$[9, 2], (7, 6)$



$[9, 2], (3, 2), (3, 8)$

Appendix C

Software Used for Appendices A and B

All software was written using MATLAB Student version 6.0.

Appendix A gives a complete list up to isomorphism of primitive digraphs with large exponent on $n = 8$ vertices with cycles of lengths $k = 6$ and $j = 5$. Appendix B gives a complete list up to isomorphism of primitive digraphs D on $n = 9$ vertices with large exponent and cycles of lengths $k = 7$ and $j = 6$ that do not satisfy $\gamma(D) = \phi(\lambda(D)) + d^c(\lambda(D))$. The software used to construct both of these lists uses an adjacency matrix $A = A(D)$ to represent a digraph D .

A base matrix B with 1's in entries corresponding to a k -cycle $1 \rightarrow 2 \rightarrow \dots \rightarrow k \rightarrow 1$ and an arc terminating in $k+1$ is first constructed. It is assumed that the entries in B corresponding to the arcs $1 \rightarrow k, k \rightarrow k-1, \dots, 2 \rightarrow 1$ are all zero so that there are no 2-cycles in D . It is further assumed that the only chords allowed across the k -cycle are those that create a j -cycle; thus all entries in B corresponding to other chords across the k -cycle are set to zero. All of these entries are *fixed* in the base matrix B . However, there

are many entries in B that correspond to arcs that may or may not be in a primitive digraph D with large exponent and cycles of lengths k and j .

Given the fixed entries in B , two mutually exclusive lists are created that account for the remaining entries. One is a list of pairs of indices $L2 = [(u, v)_i]$ such that $u \rightarrow v$ can be present or not present in the corresponding digraph. Second, there is a list of pairs of indices $L3 = [(u, v)_i]$ such that exactly one of the arcs $u \rightarrow v$ or $v \rightarrow u$ or neither is allowed. The software now loops through testing all valid combinations of arcs in the corresponding digraph. The adjacency matrix $A = A(D)$ is first initialized to the base matrix B . Each combination of possible arcs is expressed as a base two number together with a base three number. If the i^{th} digit of the base two number is one, then the arc $u \rightarrow v$ corresponding to the i^{th} entry in $L2$ is assumed to be present in the combination, where i is varied through the length of the array $L2$. If the i^{th} digit of the base three number is 0 then neither $u \rightarrow v$ nor $v \rightarrow u$ is present, if it is 1 then only $u \rightarrow v$ is present, and if it is 2, then only $v \rightarrow u$ is present, corresponding to the i^{th} entry in $L3$, where i is varied through the length of the array $L3$.

When an instance of $A = A(D)$ is generated, its exponent is calculated by taking progressive powers of A until the first one is found with no zero entries, or until it is determined that A is not primitive. If A has large exponent, then further calculations are performed, otherwise the next valid combination of entries is generated. The distance matrix $M = [m_{uv}]$, in which $m_{uv} = \delta[u, v]$, is calculated using the efficient Floyd Warshall algorithm [7, p. 210]. This algorithm has an initialization step, and a calculation step for each vertex. Entries of M are initialized as follows : $m_{uu} = 0$ for all $u = 1, 2, \dots, n$; $m_{uv} = 1$ if there exists an arc $u \rightarrow v$ in D ; otherwise $m_{uv} = \infty$.

During the calculation step for each vertex w , the value of each m_{uv} is set to the minimum of m_{uv} and the sum $m_{uw} + m_{wv}$.

The circumdiameter of D is then calculated as follows. A list J of vertices on

j -cycles is constructed by computing A^j and noting which diagonal entries are nonzero. Similarly a list K of vertices on k -cycles is generated.

The circumdistance matrix $C = [c_{uv}]$ is initialized to the distance matrix M . If u and v are distinct vertices with at least one on a j -cycle and at least one on a k -cycle, then $c_{uv} = m_{uv}$. If neither u nor v is on a j -cycle, then c_{uv} is set to the minimum over all $w \in J$ of $m_{uw} + m_{wv}$. Similarly, if neither u nor v is on a k -cycle, then c_{uv} is set to the minimum over all $w \in K$ of $m_{uw} + m_{wv}$. For each vertex u , c_{uu} is set to zero if u is on cycles of both lengths, j if u is only on a j -cycle, or k if u is only on a k -cycle. The circumdiameter of D is then the maximum entry in the circumdistance matrix C .

The quantity $(k-1)(j-1) + d^c(\lambda(D))$ is compared with the exponent. For $n = 8$, $k = 6$ and $j = 5$, these two quantities were found to be always equal. For $n = 9$, $k = 7$ and $j = 6$, all exceptions up to isomorphism are listed in Appendix B.

Example C.1

The digraph in Figure C.1 (which is listed in Appendix B and has exponent 36) has bold arcs corresponding to fixed entries in the base matrix B , and additional arcs.

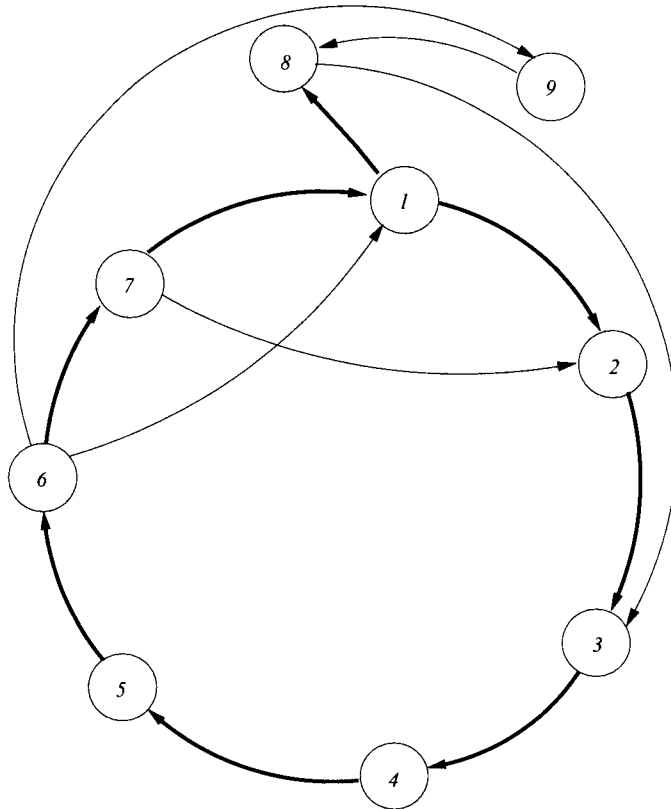


Figure C.1: Example Digraph D

The initialization step of the Floyd Warshall algorithm gives

updated M after calculation step for vertex $w = 8$ and distance matrix M after calculation step for vertex $w = 9$:

$$M = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 1 & 6 \\ 5 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 5 \\ 4 & 5 & 0 & 1 & 2 & 3 & 4 & 5 & 4 \\ 3 & 4 & 5 & 0 & 1 & 2 & 3 & 4 & 3 \\ 2 & 3 & 4 & 5 & 0 & 1 & 2 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 & 0 & 1 & 2 & 1 \\ 1 & 1 & 2 & 3 & 4 & 5 & 0 & 2 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 & 5 & 0 & 5 \\ 6 & 7 & 2 & 3 & 4 & 5 & 6 & 1 & 0 \end{bmatrix}$$

For this example, $K = \{1, 2, \dots, 8\}$ and $J = \{1, 2, \dots, 9\}$. The circumdistance matrix C follows. Note that only the 9,9 entry is different from that in the distance matrix.

$$C = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 1 & 6 \\ 5 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 5 \\ 4 & 5 & 0 & 1 & 2 & 3 & 4 & 5 & 4 \\ 3 & 4 & 5 & 0 & 1 & 2 & 3 & 4 & 3 \\ 2 & 3 & 4 & 5 & 0 & 1 & 2 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 & 0 & 1 & 2 & 1 \\ 1 & 1 & 2 & 3 & 4 & 5 & 0 & 2 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 & 5 & 0 & 5 \\ 6 & 7 & 2 & 3 & 4 & 5 & 6 & 1 & \mathbf{6} \end{bmatrix}$$

For this example, $(k-1)(j-1) + d^c(\lambda(D)) = (6)(5) + 7$, which is one greater than the exponent.