# STABILITY OF M-MATRIX PRODUCTS

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## ABSTRACT

We investigate various types of stability for powers and products of nonsingular M-matrices. Stability of the matrix powers is categorized according to the length of the longest simple circuit in the digraph of the matrix, while stability of the general products is categorized by the order of the matrices. Additional results are given regarding stability of the Hadamard product of M-matrices and for matrices whose digraph has a longest simple circuit of length two.

# 1. INTRODUCTION

In a previous paper [14] we derived sufficient conditions for products of M-matrices to have all principal minors positive; these conditions involve the directed graphs of the matrices. Here we use similar ideas to investigate various types of stability for powers and products of real, nonsingular M-matrices. We begin with some definitions and known results.

Matrix $A$ is (positive) stable if all its eigenvalues have positive real parts, or equivalently (by Lyapunov's theorem) there exists a positive definite matrix $P$ so that $AP + PA^t$ is positive definite. $A$ is diagonally (Volterra-Lotka) stable if this Lyapunov solution $P$ can be chosen as a diagonal matrix. Matrix $A$ is D-stable if $DA$ is stable for all positive diagonal matrices $D$; and $A$ is additively D-stable if $A + D$ is stable for all nonnegative diagonal $D$. Note that if $A$ is diagonally stable then it is both D-stable and additively D-stable, and also that if $P$ is a diagonal Lyapunov solution for $A$, then $P$ is a diagonal Lyapunov solution for $A^{-1}$ while $P^{-1}$ is a diagonal Lyapunov solution for $A^t$.

These types of stability are important in problems in economics and biology, and their characterizations are compared in [8]. For a $2 \times 2$ matrix the characterizations are easy, whereas for a $3 \times 3$ matrix the characterization of additive D-stability remains easy, but for D-stability [5, 10] and diagonal stability [8, 19] it becomes complicated. For matrices of these sizes D-stability implies additive D-stability [8], but in general this does not remain true for $n \geq 4$ [20]. For an $n \times n$ matrix diagonal stability has a characterization [1] which we use in our Example 3, and [19] has a computational procedure; several conditions which are either necessary or sufficient for D-stability are known [11]. For matrices with $m \leq 2$ (see section 2) diagonal stability (see [2]) and additive D-stability are straightforward to characterize, while D-stability has been characterized in [7, 3]. For normal matrices the types of stability defined above are all equivalent to stability, and for symmetric matrices these are also equivalent to positivity of all principal minors.

## 2. MATRIX POWERS

It is clear that $A^k$ is stable for all positive integers $k$ iff all eigenvalues of $A$ are real and positive. Thus if $A$ is symmetric and stable ($\equiv$ diagonally stable) then $A^k$ is also. For a normal matrix we have that $A^k$ is stable iff $A^k$ is diagonally stable [8].

For this work we take $A$ to be an $n \times n$ nonsingular M-matrix, thus $A$ is diagonally stable. We investigate conditions under which $A^k$ retains this stability. For a nonsymmetric M-matrix, it is not even true that $A^2$ remains stable, as shown by the following $5 \times 5$ matrix.

Example 1. $A = 5I_5 - 4P_5$,
where $P_n = [p_{ij}]$ is the permutation matrix with $p_{n,1} = p_{i,i+1} = 1$ for $i = 1,\ldots,n-1$; $p_{ij} = 0$ otherwise. Matrix $A^2$ has a pair of complex eigenvalues with negative real parts. Note that $A$ is a circulant and is normal.

To proceed further we restrict our class of M-matrices by looking at the directed graph of the matrix and denoting by $m$ the length of its longest simple circuit [14]. For example, irreducible tridiagonal matrices have $m = 2$, whereas the matrix given in Example 1 above has $m = 5$. With this restriction we have the following result.

THEOREM 1.

If $A$ is an irreducible M-matrix with $m = 2$, then $A^k$ is diagonally stable for all positive integers $k$.

Proof: Matrices in this class are diagonally symmetrizable [14, 17], that is there exists a positive diagonal matrix $D$ so that $R = DAD^{-1}$ is symmetric with positive eigenvalues. Taking powers, $R^k = DA^kD^{-1}$ is also symmetric with positive eigenvalues; thus $A$ and $A^k$ have all real positive eigenvalues. Now with $P = (D^{-1})^2$, which is also a positive diagonal matrix, form

$$A^k P + P(A^k)^t = D^{-1}R^k D(D^{-1})^2 + (D^{-1})^2 D R^k D^{-1}$$
$$= 2D^{-1}R^k D^{-1}$$

which is symmetric and has all principal minors positive, and thus is positive definite. □

Note that the diagonal Lyapunov solution $P$ can be chosen the same for all $k$; and this argument is also valid for any M-matrix which is diagonally symmetrizable.

We now show that certain powers of $A$ remain stable if the above condition on the directed graph is relaxed.

THEOREM 2.

If $A$ is an M-matrix with $m \geq 3$, then $A^k$ is stable for all positive integers $k$ which satisfy $k \leq m/(m-2)$.

Proof: For an M-matrix with eigenvalues $\{\lambda_i\}$ wedge conditions have been shown [15, 16 Th. 1], namely $0 \leq \arg \lambda_i \leq \pi/2 - \pi/m$. So $A^k$ is (positive) stable whenever $k(\pi/2-\pi/m) \leq \pi/2$, giving the result. □

Note that Theorem 2 has a nontrivial implication only for $m = 3$ or $4$ when $k$ is restricted to be integral, but that the conclusion remains valid for nonintegral $k$ (and thus arbitrary $m$) for the most natural definition of $A^k$.

When $m = 3$ the following example (from [14]) shows that it is not in general true that $A^3$ has nonnegative principal minors.

Example 2.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 6 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Thus $A^3$ is neither D-stable nor additively D-stable [8] (but is stable by Theorem 2 above).

In general when $m = n = 3$, $A^2$ can be shown using [5, 8] to be D-stable and additively D-stable, but not necessarily diagonally stable, as shown by Example 3.

Example 3.

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & -3 \\ -3 & 0 & 81 \end{bmatrix} .$$

With positive definite

$$C = \begin{bmatrix} 2000 & 196 & 15.301 \\ 196 & 933 & 27 \\ 15.301 & 27 & .99 \end{bmatrix} ,$$

we show $A^2$ is not diagonally stable by the characterization in [1 Th. 1]. The diagonal entries of $CA^2$ are $\{-.46, 0, -8.901\}$; as these are all nonpositive, $A^2$ is not diagonally stable. Alternatively this can be verified by using the characterization in [8 Th. 4] and showing numerically that the two quadratic inequalities have no common solution.

Note that in Example 3 the entries on the main diagonal of $A$ are not all equal. In fact for an M-matrix

$$B = \begin{bmatrix} 1 & -a & 0 \\ 0 & 1 & -b \\ -c & 0 & 1 \end{bmatrix} \quad \text{with} \quad a, b, c > 0 \quad \text{and} \quad abc < 1,$$

which has the same digraph as $A$ in Example 3, we can find a diagonal Lyapunov solution for $B^2$, for example $P = \text{diag.}(a, bc, z)$ with $ac^2 < z < c/b$.

For $m = 4$, $A^2$ can fail to be D-stable and additively D-stable; this is shown by the following example (again from [14]) as $A^2$ has a negative principal minor of size 2.

Example 4.

$$A = \begin{bmatrix} 17 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & -2 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} .$$

However when $n = 4$ but $m = 3$, we know $A^2$ has all principal minors positive [14], so this cannot be used to rule out the D-stability of $A^2$, a question which remains unresolved. Our results on matrix powers are summarized in Table 1.

| m \ k | 2 | 3 | 4 |
|---|---|---|---|
| 2 | diagonally stable | | |
| 3 | stable(2) | stable(1) | $5I_3-4P_3$ |
| 4 | stable(1) | $5I_4-4P_4$ | |
| 5 | $5I_5-4P_5$ | | |

Table 1. Summary of results on stability of $A^k$ for $A$ an irreducible $n \times n$ M-matrix with length of longest simple circuit $m$.

Matrices $5I_n - 4P_n$ are unstable, where $P_n$ is the permutation matrix defined in Example 1, and these show that the products are not necessarily stable.

(1) These matrix powers are stable but not necessarily diagonally stable, D-stable or additively D-stable; see Examples 2 and 4.

(2) For $n = 3$, $A^2$ is D-stable and additively D-stable, but not necessarily diagonally stable; see Example 3. For $n \geq 4$, D-stability and additive D-stability are unresolved.

## 3. MATRIX PRODUCTS

We now turn to M-matrix products, and take $A_i$ to be nonsingular $n \times n$ M-matrices. If $A_1$, $A_2$ are symmetric M-matrices, then it is well known that their product $A_1A_2$ is stable. We do not, in general, have D-stability for this product, as illustrated for $n = 4$ by Example 3 in [14]. This also shows that it is not in general true that the product of three symmetric M-matrices is stable. Stability of the product of two symmetric M-matrices and a positive diagonal matrix is of special interest as it is equivalent to stability of the product of two irreducible M-matrices with $m = 2$. Specifically, for $A_1$, $A_2$ both irreducible M-matrices with $m = 2$, the product $A_1A_2 = A_1D_1(D_1^{-1}A_2D_2)D_2^{-1}$, where $D_1$, $D_2$ are positive diagonal matrices with $D_1$ chosen to symmetrize $A_1$ and $D_2$ chosen to symmetrize $D_1^{-1}A_2$. Stability of this product for $n \geq 4$ and $m(A_1+A_2) \leq 3$ (including even the case when $A_1$ and $A_2$ have identical digraphs) remains unresolved.

Restricting our M-matrices to be permutation similar to tridiagonal matrices, we have the following stability result.

THEOREM 3.

If $A_i$ are tridiagonal M-matrices and $B_i = PA_iP^t$ for some permutation matrix $P$, then $\prod_{i=1}^{k} B_i$ is D-stable for all positive integers $k$.

Proof. For any positive diagonal $D$ and $A_i$ tridiagonal M-matrices, the product $D \prod_{i=1}^{k} A_i$ has a totally nonnegative inverse [9, 14 Lemma 2] and so has all real positive eigenvalues. The result holds for $\prod_{i=1}^{k} B_i$ as D-stability is retained under permutation similarity. □

Note that this result includes M-matrices whose digraphs are straight lines (with the same ordering of vertices) although a permutation similarity of a totally nonnegative matrix is not necessarily totally nonnegative.

When $n = 3$ with no restrictions on $m$, we can in fact use [5] and knowledge of the signs of symmetrically placed almost principal minors of M-matrices, to show that $A_1A_2$ is D-stable (and so also additively D-stable [8]). We already know from Example 3 that for matrices of this size with $m = 3$, it is not in general true that $A^2$ is diagonally stable. Moreover, for $3 \times 3$ symmetric, tridiagonal M-matrices $A_1$ and $A_2$ it is not in general true that the product is diagonally stable, as Example 5 shows.

Example 5.

$$A_1 = \begin{bmatrix} 1 & -.99 & 0 \\ -.99 & 1 & -.1 \\ 0 & -.1 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & -.1 & 0 \\ -.1 & 1 & -.99 \\ 0 & -.99 & 1 \end{bmatrix}.$$

Using the characterization in [8, 19] we see that the product $A_1A_2$ is not diagonally stable. This example also shows that a totally nonnegative matrix need not be diagonally stable (see [4]).

Looking at the product of three M-matrices when $n = 3$, we easily see that it need not be stable: take $AA(AD)$ with $A$ given by Example 2 above, and $D = \text{diag}(100, .01, .01)$; this product has two negative eigenvalues.

When $n = 4$, the product $A_1A_2$ need not even be stable, as the following example shows.

Example 6. (<u>cf</u>. Example 3 in [14])

$$A_1 = \begin{bmatrix} 6 & -1 & -2 & 0 \\ -1 & 6 & 0 & -2 \\ -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 60 & -1 & 0 & -200 \\ -10 & 6 & -200 & 0 \\ 0 & -2 & 100 & 0 \\ -20 & 0 & 0 & 100 \end{bmatrix}$$

has $A_1A_2$ unstable due to a pair of complex eigenvalues with negative real parts. Here $m(A_1) = m(A_2) = 2$, however $m(A_1+A_2) = 4$ as their digraphs are straight lines with different ordering of the vertices.

If we restrict matrices to have $m = 2$ and identical digraphs then Example 4 of [14] with $n = 4$ shows that a product of three matrices is in general not stable. We summarize our product results in Table 2; note that in contrast with Table 1 we use $n$ rather than $m$.

| n \ k | 2 | 3 | ≥4 |
|---|---|---|---|
| 2 | diagonally stable | | |
| 3 | stable(1) | not necessarily stable | |
| 4 | not necessarily stable | | |

Table 2. Summary of results on stability of M-matrix products $\prod_{i=1}^{k} A_i$, with $A_i$ $n \times n$ M-matrices.

(1) $A_1A_2$ is D-stable and additively D-stable, but not necessarily diagonally stable; see Example 5.

## 4. HADAMARD PRODUCTS

If $A_1, A_2$ are M-matrices then the Hadamard (entry-wise) product $A_1 \circ A_2^{-1}$ is also an M-matrix (Corollary 2 in [12]). We now consider the stability of M-matrix Hadamard products and their inverses.

THEOREM 4.

If $A_1, A_2$ are M-matrices then $A_1 \circ A_2$ is diagonally stable.

Proof: As $A_i (i=1,2)$ is an M-matrix, it has all diagonal entries positive, and there is a positive diagonal matrix $D_i$ so that $X_i = D_i^{-1} A_i D_i$ is row diagonally dominant. Now from $A_1 \circ A_2 = (D_1 X_1 D_1^{-1}) \circ (D_2 X_2 D_2^{-1}) = (D_1 D_2)(X_1 \circ X_2)(D_1 D_2)^{-1}$, $A_1 \circ A_2$ is diagonally similar to $X_1 \circ X_2$. But $X_1 \circ X_2$ is row diagonally dominant and has positive diagonal entries, and thus is diagonally stable. Hence $A_1 \circ A_2$ is also diagonally stable. □

Note that this theorem implies the other types of stability, and also that all principal minors of $A_1 \circ A_2$ are positive. From the Z-sign pattern of $A_i$, the matrix $A_1 \circ A_2$ is nonnegative, so its inverse is monotone and positive stable, and thus is an N-matrix [18]. However, this inverse Hadamard product does not in general retain the Z-sign pattern necessary for it to be an M-matrix (e.g. consider $(A_6 \circ A_6)^{-1}$ for $A_6$ in [18]). The above theorem can be extended to the Hadamard product of any number of M-matrices. When this number is odd, the inverse of the resulting Hadamard product is an M-matrix.

If we again impose the restriction that $A_i$ are M-matrices with $m = 2$, then the Kronecker product of any number of such matrices is stable, and is an N-matrix.

## 5. RELATED RESULTS

Both the set of M-matrices and the set of positive definite matrices are known to satisfy the (generalized) Hadamard-Fisher inequality [6, 12], namely

$$\det A[J\cap K] \det A[J\cup K] \le \det A[J] \det A[K] \tag{5.1}$$

where $J,K\subseteq\{1,2,\ldots,n\}$ are index sets, and $A[J]$ denotes the principal submatrix of $A$ containing rows and columns $J$. This inequality is inverse invariant for matrices with positive principal minors, and we can prove that it also holds for our restricted class of M-matrix powers.

THEOREM 5

If $A$ is an irreducible M-matrix with $m = 2$, then $A^k$ satisfies inequality (5.1) above for all positive integers $k$.

Proof: The diagonal symmetrization used in the proof of Theorem 1 shows that $\det A^k[J] = \det R^k[J]$. The positive definiteness of $R^k$ gives the inequality (5.1) for this matrix [6, 12], and thus it also holds for $A^k$. □

The same $3 \times 3$ matrix used above (Example 2) with $m = 3$, shows that this cannot be extended, for this matrix $A^2$ does not satisfy (5.1).

We now drop the restriction that $A$ is an M-matrix, and give results which again can be compared with those for symmetric matrices.

THEOREM 6

For a nonsingular matrix $A$ with $m = 2$:

(i) if $A$ has all principal minors positive, then $A$ is diagonally stable, and

(ii) if there is a nonsingular diagonal matrix $D$ such that $D^{-1}A^{-1}D \ge 0$ and $A$ is stable, then $A$ has all principal minors positive.

Proof: For both parts assume without loss of generality that A is in block upper triangular irreducible normal form.

(i) If $m = 2$ and A has all principal minors positive, it suffices to consider the case in which A is irreducible because the diagonal Lyapunov solution of each successive block may be multiplied by a sufficiently small positive number to produce a diagonal Lyapunov solution for A. For the irreducible case choose D so that BD is symmetric where B is the component-wise absolute value of A. This may straightforwardly be done as has been noted in [17]. Now $AD + DA^t$ is symmetric with positive principal minors, and D is the desired diagonal Lyapunov solution.

(ii) If $D^{-1}A^{-1}D \geq 0$ and A is stable, then $D^{-1}AD$ is an N-matrix. Since $m = 2$, it follows from [13] that each irreducible component of $D^{-1}AD$ is an M-matrix. Since M-matrices have positive principal minors, the assertion follows for A. Note that the assumption on $A^{-1}$ cannot, in general, be relaxed as the example $\begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}$ shows. □

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