Exploring the Fluid Landscape: Three New Regimes of Relativistic Hydrodynamics

by

Juan Hernandez
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Supervisory Committee

Dr. Pavel Kovtun, Supervisor
(Department of Physics and Astronomy)

Dr. Adam Ritz, Departmental Member
(Department of Physics and Astronomy)

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ABSTRACT

In this work, we use the recently developed equilibrium generating functional and systematic derivative expansion approach to hydrodynamics to explore three new regimes of relativistic hydrodynamics. First, we derive the equations of motion and write the constitutive relations to first order in derivatives for relativistic fluids coupled to an external vector field. Next, for relativistic fluids in strong magnetic fields $B^\mu \sim O(1)$, we derive the equations of motion and present the constitutive relations to first order in derivatives. From the resulting system of equations, we find the hydrodynamic modes for these systems. We also find the constraints on the transport coefficients due to the entropy production argument and derive the corresponding Kubo formulas. Finally, we repeat the same analysis for relativistic fluids coupled to dynamical electromagnetic fields with $\langle B^\mu \rangle \sim O(1)$.

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Chapter 1

Introduction

Hydrodynamics is an effective low energy macroscopic description for many-body systems that are in local thermal equilibrium. The "classical" formulation of hydrodynamics has been known for over a century and consists of a set of conservation equations supplemented with constitutive relations relating the conserved currents to the hydrodynamic variables inherited from thermodynamics [1]. The fact that hydrodynamics is a relatively old subject by no means implies that it is yet fully understood, and formal studies of the hydrodynamic framework remain an active research area. In the formulation of non-relativistic hydrodynamics, the effects of thermal fluctuations were taken into account by adding stochastic currents and integrating over their fluctuations with a given weight function, in a way that is reminiscent of statistical/quantum field theory [2, 3]. Recent interest in relativistic hydrodynamics [4] has spiked due to it's recent connection with a variety of areas such gravitational dynamics via the AdS/CFT correspondence [5, 6, 7], and as an effective description of the quark-gluon plasma generated in heavy-ion collisions [8, 9] and for recently discovered strange metals [10, 11, 12, 13, 14]. With increased interest have come many advances in the formal studies of hydrodynamics. Some of these advancements include the systematic derivative expansion [15], the manifestation of chiral anomalies in the hydrodynamic framework [16], the derivation of the parity breaking terms in 2+1 dimensional hydrodynamics [17], the formulation of hydrostatics/thermodynamics from equilibrium partition functions [18, 19], studies of the convergence and resurgence properties of the hydrodynamic expansion [20, 21, 22, 23], elucidation of the role of the entropy current [24, 25], the classification of hydrodynamic coefficients [26], the search for a hydrodynamic generating functional for out of equilibrium systems [27, 28] and the emergence of supersymmetry in effective hydrodynamic actions [29, 30, 31, 32]. The purpose of this thesis is to employ some of these recent advancements to study relatively unexplored regimes of relativistic hydrodynamics.

1.1 Introduction to hydrodynamics

We now present a quick overview of the classical approach to hydrodynamics as presented in [1]. Traditionally, hydrodynamics was mainly used to describe liquids and gases at scales much larger than their microscopic constituents. Relativistic hydrodynamics has also widely been used to study astrophysical processes [33] for quite some time, and has recently expanded into descriptions of quark-gluon plasmas [8, 9] and strange metals [10, 11, 12, 13, 14].

Let us begin by reviewing the standard non-relativistic hydrodynamic framework. The mathematical description of normal non-relativistic fluids in 3+1 dimensions is contained in five functions which give the fluid's velocity $\mathbf{v}(t, \mathbf{x})$ and any two thermodynamic quantities, such as pressure $p(t, \mathbf{x})$ and mass density $\rho(t, \mathbf{x})$. An important assumption is the conservation of fluid mass, stated in integral form

$$\frac{\partial}{\partial t} \int_{V_0} \rho dV = -\oint_{\partial V_0} \rho \mathbf{v} \cdot d\mathbf{A} \,, \tag{1.1}$$

that is, the change in total fluid mass enclosed in an volume V_0 changes only by the fluid entering or leaving the volume through it's boundary ∂V_0. Using Green's theorem, the right hand side can be turned into a volume integral of the divergence of $\rho\mathbf{v}$, leading to the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \,, \tag{1.2}$$

where $\mathbf{j} = \rho\mathbf{v}$ is the mass flux density.

Now let's consider the force felt by the fluid inside V_0

$$\mathbf{F} = -\oint_{\partial V_0} p d\mathbf{A} = -\int_{V_0} \nabla p dV = \int_{V_0} \mathbf{f} dV \,, \tag{1.3}$$

where $\mathbf{f} = -\nabla p$ is the force density felt by the fluid. Equating this force density to the fluid acceleration $\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial x^i}{\partial t}\frac{\partial \mathbf{v}}{\partial x^i} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v}$ times its mass density ρ, we find Euler's equation

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+v^i\frac{\partial \mathbf{v}}{\partial x^i}\right)=-\nabla p\,. \tag{1.4}$$

Euler's equation can be interpreted as a momentum conservation equation by defining the momentum flux density tensor $\Pi_{ij}=p\delta_{ij}+\rho v_i v_j$, and using the continuity equation (1.2) to write

$$\frac{\partial}{\partial t}(\rho v_i)=-\frac{\partial \Pi_{ij}}{\partial x_j}\,. \tag{1.5}$$

The interpretation of the momentum flux density comes from integrating (1.5)

$$\frac{\partial}{\partial t}\int_{V_0}\rho v^i dV=-\oint_{\partial V_0}\Pi^{ij}dA_j\,, \tag{1.6}$$

so Π^{ij} gives the i-th component of momentum flux through the j-th unit area element.

For adiabatic flows, we demand a local version of the second law

$$\frac{d\tilde{s}}{dt}=\frac{\partial \tilde{s}}{\partial t}+v^i\frac{\partial \tilde{s}}{\partial x^i}=0\,, \tag{1.7}$$

where $\tilde{s}(p,\rho)$ is the entropy density per unit mass. Using the continuity equation (1.2), the adiabatic equation (1.7) can be written as an entropy continuity equation

$$\frac{\partial s}{\partial t}+\nabla\cdot(s\mathbf{v})=0\,, \tag{1.8}$$

where $s=\rho\tilde{s}$ is the entropy density per unit volume. Fluid flows that preserve entropy are called non-dissipative, or adiabatic.

For adiabatic fluids, the five equations (1.2), (1.5) and (1.8) together with an equation of state $s(p,\rho)$ relating the three thermodynamic functions p, ρ and s give a full set of equations to describe the fluid flow. Alternatively, the continuity equation, the Euler equation and the energy density per unit volume $\varepsilon=\rho\tilde{\varepsilon}$ can be used to turn the entropy conservation equation (1.8) into an energy conservation equation

$$\frac{\partial}{\partial t}\left(\varepsilon+\tfrac{1}{2}\rho\mathbf{v}^2\right)+\nabla\cdot\left((w+\tfrac{1}{2}\rho\mathbf{v}^2)\mathbf{v}\right)=\frac{\partial}{\partial t}\left(\varepsilon+\tfrac{1}{2}\rho\mathbf{v}^2\right)+\nabla\cdot\mathbf{j}_\epsilon=0\,, \tag{1.9}$$

where $w=\varepsilon+p$ is the enthalpy density per unit volume and $\mathbf{j}_\varepsilon=(w+\frac{1}{2}\rho\mathbf{v}^2)\mathbf{v}$ is the energy flux density per unit volume.

As a straightforward example we can consider an incompressible fluid $\rho=$ constant. The energy per unit mass $\tilde{\varepsilon}$ satisfies the thermodynamic relation $d\tilde{\varepsilon}=Td\tilde{s}-pd\tilde{V}=$

$Td\tilde{s} + p/\rho^2 d\rho$, where $\tilde{V} = 1/\rho$ is the volume per unit mass. The continuity equation leads to $\nabla\cdot\mathbf{v} = 0$, from which the energy conservation equation reads

$$\frac{\partial}{\partial t}\left(\varepsilon + \tfrac{1}{2}\rho\mathbf{v}^2\right) + (\mathbf{v}\cdot\nabla)\left(w + \tfrac{1}{2}\rho\mathbf{v}^2\right) = \frac{d}{dt}\left(w + \tfrac{1}{2}\rho\mathbf{v}^2\right) - \frac{\partial p}{\partial t} = 0\,. \tag{1.10}$$

In the presence of a gravitational field g, the fluid energy density is $\rho g h+$ internal energy density. The internal energy density depends on the fluid's temperature and is therefore constant for a fluid in a static flow. For a static flow $\frac{\partial p}{\partial t} = 0$ in a system where the fluid temperature is uniform, we find Bernoulli's equation

$$\tfrac{1}{2}\rho\mathbf{v}^2 + \rho g h + p = \text{constant}\,. \tag{1.11}$$

When the fluid is in a dissipative flow, we add the viscous stress tensor Π'_{ij} to the momentum flux density tensor [1]:

$$\Pi_{ij} = p\delta_{ij} + \rho v_i v_j - \Pi'_{ij} = (p - \zeta\nabla\cdot\mathbf{v})\,\delta_{ij} + \rho v_i v_j - \eta\sigma_{ij}\,, \tag{1.12}$$

where $\sigma_{ij} = \frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i} - \frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{v}$. The transport coefficients η and ζ are called viscosity coefficients and are functions of the thermodynamic variables $(T\,,\ \mu\,,\ \mathbf{v})$. The viscous stress tensor Π'_{ij} vanishes in adiabatic flows. With these modifications Euler's equation (1.4) turns into the Navier-Stokes equation

$$\rho\left(\frac{\partial\mathbf{v}}{\partial t} + v^i\frac{\partial\mathbf{v}}{\partial x^i}\right) = -\nabla p + \eta\frac{\partial^2\mathbf{v}}{\partial x^k\partial x^k} + \left(\zeta + \tfrac{1}{3}\eta\right)\nabla\left(\nabla\cdot\mathbf{v}\right)\,. \tag{1.13}$$

Similarly, we add a viscous and a heat vector to the energy flux density [1]

$$j^i_\varepsilon = (w + \tfrac{1}{2}\rho\mathbf{v}^2)v^i - \Pi'^{ij}v_j - \kappa\frac{\partial T}{\partial x_i}\,, \tag{1.14}$$

where T is the local temperature of the fluid and κ is another transport coefficient called the heat conductivity. The modified energy conservation equation is

$$\frac{\partial}{\partial t}\left(\varepsilon + \rho\mathbf{v}^2\right) + \frac{\partial}{\partial x^i}\left((w + \tfrac{1}{2}\rho\mathbf{v}^2)v^i - \Pi'^{ij}v_j - \kappa\frac{\partial T}{\partial x^i}\right) = 0\,. \tag{1.15}$$

Using the Navier-Stokes equation and the thermodynamic relation $d\varepsilon = Tds + \frac{w-sT}{\rho}d\rho$ the modified energy conservation can be written as the general equation for heat transfer

$$T\left(\frac{\partial s}{\partial t}+\frac{\partial}{\partial x^i}(sv^i)\right)=\Pi'_{ij}\frac{\partial v^i}{\partial x^j}+\frac{\partial}{\partial x^i}\left(\kappa\frac{\partial T}{\partial x^i}\right)\,. \tag{1.16}$$

This equation reduces to the entropy continuity equation (1.8) when there is no viscosity and no heat transfer. The fact that the continuity equation (1.2) doesn't receive modifications in non-adiabatic flows comes from the particular *choice* of out-of equilibrium definition of the fluid velocity $\mathbf{v}$. We implicitly picked $\mathbf{v}$ to be the velocity of the fluid particles carrying mass, which leaves the continuity equation intact. This is related to the concept of frame transformations, which will be explained in section 2.2.1.

Magnetohydrodynamics

The field of magnetohydrodynamics was born when Hannes Alfv́en proposed to link Maxwell's equations with those of hydrodynamics to study the dynamics of electrically conducting fluids in the presence of magnetic fields [34]. To have a classical hydrodynamic description of these fluids, the magnetic field must be small compared to the temperature, $B \ll T^2$. We now summarize how the hydrodynamic framework gets modified as presented in [35]. Maxwell's equations are

$$\nabla\cdot\mathbf{E}=\frac{\rho_c}{\varepsilon_0}\,,\quad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\,, \tag{1.17a}$$

$$\nabla\cdot\mathbf{B}=0\,,\quad \nabla\times\mathbf{B}=\mu_0\mathbf{j}_e+\varepsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}\,, \tag{1.17b}$$

where ρ_c is the charge density, $\mathbf{j}_e$ the electric current density, ε_0 the permittivity of free space and μ_0 the permeability of free space. In most cases the fluid is electrically neutral so $\rho_c=0$. Electric and magnetic polarization are also ignored. In the non-relativistic limit, the electric displacement term in Ampère's law (1.17b) can be neglected.

The Navier-Stokes equation (1.13) receive a new contribution due to the Lorentz force law

$$\mathbf{f}_e=\rho_c\mathbf{E}+\mathbf{j}_e\times\mathbf{B}\,. \tag{1.18}$$

The Coulomb force term is negligible in the electrically neutral fluid approximation ($\rho_c=0$). Using Ampère's law (1.17b) and the identity $\mathbf{B}\times(\nabla\times\mathbf{B})=\frac{1}{2}\nabla\mathbf{B}^2-$

$(\mathbf{B}\cdot\nabla)\,\mathbf{B}$, the Navier-Stokes equation (1.13) turns to

$$\rho\left(\frac{\partial\mathbf{v}}{\partial t}+v^i\frac{\partial\mathbf{v}}{\partial x^i}\right)=-\nabla\left(p+\frac{1}{2\mu_0}\mathbf{B}^2\right)-\frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\,\mathbf{B}+\eta\frac{\partial^2\mathbf{v}}{\partial x^k\partial x^k}+\left(\zeta+\tfrac{1}{3}\eta\right)\nabla\left(\nabla\cdot\mathbf{v}\right)\,. \tag{1.19}$$

This can be written in the form of the momentum conservation equation (1.5) with a modified momentum flux density tensor

$$\Pi_{ij}=\left(p+\tfrac{1}{2\mu_0}\mathbf{B}^2-\zeta\nabla\cdot\mathbf{v}\right)\delta_{ij}-\tfrac{1}{\mu_0}B_iB_j+\rho v_iv_j-\eta\sigma_{ij}\,. \tag{1.20}$$

The energy conservation equation (1.15) receives a modification due to the magnetic field energy and the Poynting vector $\mathbf{P}=\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}$ which carries the electromagnetic momentum density.

$$\frac{\partial}{\partial t}\left(\varepsilon+\rho\mathbf{v}^2+\tfrac{1}{2\mu_0}\mathbf{B}^2\right)+\frac{\partial}{\partial x^i}\left((w+\tfrac{1}{2}\rho\mathbf{v}^2)v^i+P^i-\Pi'^{ij}v_j-\kappa\frac{\partial T}{\partial x^i}\right)=0\,. \tag{1.21}$$

The continuity equation (1.2) doesn't receive modifications from the magnetic fields. A consequence of Maxwell's equations is the charge conservation equation

$$\frac{\partial\rho_c}{\partial t}+\nabla\cdot\mathbf{j}_e=0\,. \tag{1.22}$$

The equations of magnetohydrodynamics consist of the modified hydrodynamic equations (1.2) (1.19) and (1.21), Maxwell's equations (1.17) and the generalized form of Ohm's law

$$\mathbf{j}_e=\sigma\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\,. \tag{1.23}$$

Relativistic hydrodynamics

In his elegant 6 page paper [36] in 1940, Carl Eckart unified the frameworks of hydrodynamics and special relativity, giving a relativistic formulation of hydrodynamics. For relativistic fluids in flat space-time, the fluid velocity $\mathbf{v}$ is promoted to a covariant four-vector $u^\mu=\gamma(1,\mathbf{v}/c)$, where $\gamma=1/\sqrt{1-\mathbf{v}^2/c^2}$. The momentum flux density tensor Π_{ij}, the momentum density $\rho\mathbf{v}$ and the energy density $\varepsilon+\frac{1}{2}\rho\mathbf{v}^2$ defined in section 1.1 are replaced by their relativistic counterparts and form part of the covariant energy-momentum tensor $T^{\mu\nu}$. In the frame of reference where the fluid is at rest ($u^\mu=(1,\mathbf{0})$), we have $T^{\mu\nu}=\mathrm{diag}(\epsilon,p,p,p)$, where $\epsilon=nc^2+\varepsilon$ is the relativistic energy per unit volume, $n=\rho/\gamma$ is the relativistic mass per unit proper volume and

p is the relativistic pressure, which is the same as the non-relativistic pressure. We can therefore write

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + p\eta^{\mu\nu}\,. \tag{1.24}$$

Note that in the non-relativistic limit $\mathbf{v}^2 \ll c^2$ we get the corresponding non-relativistic versions of energy, momentum and momentum flux densities

$$T^{00} = \frac{\epsilon + p}{1 - \mathbf{v}^2/c^2} - p \approx \rho c^2 + \varepsilon + \tfrac{1}{2}\rho \mathbf{v}^2\,, \tag{1.25a}$$

$$cT^{0i} = \frac{(\epsilon + p)v^i}{1 - \mathbf{v}^2/c^2} \approx (\rho c^2 + \varepsilon + p + \tfrac{1}{2}\rho \mathbf{v}^2)v^i\,, \tag{1.25b}$$

$$T^{ij} = \frac{(\epsilon + p)v^i v^j}{c^2(1 - \mathbf{v}^2/c^2)} + p\delta^{ij} \approx \rho v^i v^j + p\delta^{ij}\,. \tag{1.25c}$$

The non-relativistic energy density $\varepsilon + \frac{1}{2}\rho\mathbf{v}^2$ in the energy conservation equation (1.9) corresponds to the non-relativistic limit of T^{00} minus the non-relativistic rest energy ρc^2. The non-relativistic momentum flux density tensor Π^{ij} used in the non-relativistic momentum conservation equation (1.5) agrees with the non-relativistic limit of T^{ij}. The momentum flux density ρv^i appearing in the momentum conservation equation (1.5) agrees with the non-relativistic limit of T^{0i}/c, while the non-relativistic energy flux $\left(w + \frac{1}{2}\rho\mathbf{v}^2\right) v^i$ appearing in the non-relativistic energy conservation equation (1.9) corresponds to the non-relativistic limit of cT^{0i} minus the non-relativistic rest energy density flux $\rho c^2 v^i$. Finally, two of the non-relativistic hydrodynamic equations (momentum conservation (1.5) and energy conservation (1.9)) appear as the non-relativistic limit of the energy-momentum conservation equation

$$\partial_\mu T^{\mu\nu} = 0\,, \tag{1.26}$$

keeping in mind that the relativistic derivatives are taken with respect to the coordinates (ct, x, y, z). The continuity equation (1.2) is found by imposing a fluid flux conservation equation

$$\partial_\mu (n u^\mu) = 0\,. \tag{1.27}$$

When out of equilibrium, the particular out of equilibrium choice of fluid variables which keeps the continuity equation (1.27) and the energy density T^{00} intact in the fluid rest frame is known as the Eckart *hydrodynamic* frame (not to be confused with inertial frames used relativity). See section 2.2.1 for an introduction to hydrodynamic frames. In this hydrodynamic frame, the current/fluid flux remains the same (nu^μ) while the energy-momentum tensor receives contributions to the pressure, momentum density and shears

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + (p - \zeta \partial_\lambda u^\lambda)\Delta^{\mu\nu} - \eta\sigma^{\mu\nu} + \kappa(u^\mu \Delta^{\nu\alpha}\partial_\alpha T + u^\nu \Delta^{\mu\alpha}\partial_\alpha T)\,, \tag{1.28}$$

where $\Delta^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$ and $\sigma^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}\eta_{\alpha\beta}\partial_\lambda u^\lambda)$ is the covariant tensor version of σ_{ij} introduced above the Navier-Stokes equation (1.13).

1.2 Outline

In this thesis, we rely on the philosophy presented in [18, 19] of hydrodynamics as a generalization of (local) equilibrium thermodynamics to systems out of equilibrium in order to study the hydrodynamic framework under three relatively unexplored regimes:

1. Relativistic hydrodynamics in the presence of an external vector field,
2. Relativistic hydrodynamics in the presence of strong magnetic fields,
3. Relativistic hydrodynamics for dynamical electromagnetic fields.

The structure of this work is as follows.

In chapter 2 we give a summary of the construction of the relativistic hydrodynamic framework from equilibrium partition functions. The central concepts to hydrodynamics (derivative expansion, transport coefficients, frame transformations, Kubo formulas, etc.) are first introduced here, and the general idea of how these pieces fit together is elucidated step by step using the "normal" hydrodynamic regime as an example. Section 2.1 sets up the equilibrium/thermodynamic terms in the derivative expansion. The non-equilibrium terms are added and constrained in section 2.2.

Chapter 3 focuses on the framework of anisotropic hydrodynamics in the presence of an external vector field. Several of the concepts introduced in chapter 2 will require some modifications but the overall structure remains the same. Some of these modifications are summarized in section 3.3. The chapter ends with the constitutive relations for anisotropic hydrodynamics to first order in derivatives. Constraints on the non-equilibrium dissipative transport coefficients and derivation of the Kubo formulas are left for later work.

Chapter 4 can be broadly broken down into two parts:

- Hydrodynamics in the presence of strong external magnetic fields. These magnetic fields are assumed small compared to the temperature to avoid the emergence of non-hydrodynamical degrees of freedom. The construction in sections 4.2 and 4.3 follows the structure of chapter 2 with the addition of an analysis of the eigenmodes of the resulting set of equations in section 4.3.5. The modifications required for parity-violating systems are studied in section 4.3.9.
- Hydrodynamics with dynamical electromagnetic fields. Section 4.4 sets up the new formalism, connects it with the covariant formulation of Maxwell's equations in matter and proceeds to a similar analysis of this formalism. Section 4.5 summarizes a recent construction of magnetohydrodynamics in a dual formulation [37] which uses the Hodge dual of the field strength as conserved current, and compares it with the results of section 4.4.

Chapter 2

Relativistic hydrodynamic framework

Hydrodynamics is an effective low energy macroscopic description for many-body systems that are in local thermal equilibrium [1]. The structure of the hydrodynamic equations will be sensitive to the symmetries of the microscopic system, but not to its precise details. The low energy information of the particular microscopic system will be captured by the transport coefficients in the constitutive relations, which can be expressed as small frequency and small wavelength limits of the correlation functions of conserved currents. These conserved currents will provide the relevant description for systems after coarse graining, since they are protected by the symmetries of the theory. On length scales much larger than the mean free path of the microscopic excitations or quasiparticles, the description in terms of these quasiparticles is not adequate. On the other hand, because they are conserved, the currents will still be there long after the quasiparticles have scattered or decayed. The relevant degrees of freedom in hydrodynamics are inherited from thermodynamics. In this chapter, we explain the generating functional and systematic derivative expansion approach to hydrodynamics and follow the construction of the hydrodynamic framework of normal fluids as a useful example.

2.1 Thermodynamics

Let us start with equilibrium thermodynamics. For a diffeomorphism and $U(1)$ gauge invariant system in equilibrium subject to an external non-dynamical gauge field A_μ

and an external non-dynamical metric $g_{\mu\nu}$, we write the logarithm of the partition function $W_s = -i \ln Z$ as

$$W_s[g, A] = \int d^{d+1}x \sqrt{-g}\, \mathcal{F} \,, \tag{2.1}$$

and we will call $\mathcal{F}$ the free energy density[1]. [Conventions: metric is mostly plus, $\epsilon^{0123}{=}1/\sqrt{-g}$.] For a system with short-range correlations in equilibrium and for external sources A and g which only vary on scales much longer than the correlation length, $\mathcal{F}$ is a local function of the external sources, and W_s is extensive in the thermodynamic limit. It is only when the external sources vary slowly that the system can be described by the hydrostatic framework. The density $\mathcal{F}$ may then be written as an expansion in derivatives of the external sources [19, 18]. The current J^μ and the energy-momentum tensor $T^{\mu\nu}$ are defined by varying W_s with respect to the external sources

$$\delta W_s = \int d^{d+1}x \sqrt{-g}\, \left(\tfrac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} + J^\mu \delta A_\mu \right) \,, \tag{2.2}$$

and automatically satisfy

$$\nabla_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda \,, \tag{2.3a}$$

$$\nabla_\mu J^\mu = 0 \,, \tag{2.3b}$$

due to gauge- and diffeomorphism-invariance of $W_s[g, A]$. See the derivation of the conservation equations in appendix A. The object $W_s[g, A]$ is the generating functional of static (zero frequency) correlation functions of $T^{\mu\nu}$ and J^μ in equilibrium. Of course, the conservation laws (2.3) are also true out of equilibrium, being a consequence of gauge- and diffeomorphism-invariance in the microscopic theory. The equations relating $T^{\mu\nu}$ and J^μ to the hydrodynamic variables are called the constitutive relations.

Being in equilibrium means that there exists a timelike Killing vector V such that the Lie derivative of the sources with respect to V vanishes. The equilibrium temperature T, velocity u^α and the chemical potential μ are functions of the Killing

[1] Note that we are working in the grand canonical ensemble, so this energy density is a local version of the grand canonical potential.

vector and the external sources [19, 18]

$$T=\frac{1}{\beta_0\sqrt{-V^2}}\,,\qquad u^\mu=\frac{V^\mu}{\sqrt{-V^2}}\,,\qquad \mu=\frac{V^\mu A_\mu+\Lambda_V}{\sqrt{-V^2}}\,. \tag{2.4}$$

Here β_0 is a constant setting the normalization of temperature, and Λ_V is a gauge parameter which ensures that μ is gauge-invariant [38]. Note that this gives a local thermodynamic theory, since the thermodynamic variables can be space dependent. The Killing vector field defines the notion of time, and the static sources are assumed to vary slowly in space, so that $O(\partial^{n+1})\ll O(\partial^n)$. The temperature sets the relevant dimension for the derivative expansion: $\partial T/T^2\,,\ \partial\mu/T^2\ll 1$. It is important that the fluid rest frame is aligned with the Killing vector field; otherwise the system is not in equilibrium but instead in a steady state. We will write the energy-momentum tensor using the decomposition with respect to the timelike velocity vector u^μ,

$$T^{\mu\nu}=\mathcal{E}u^\mu u^\nu+\mathcal{P}\Delta^{\mu\nu}+\mathcal{Q}^\mu u^\nu+\mathcal{Q}^\nu u^\mu+\mathcal{T}^{\mu\nu}\,, \tag{2.5}$$

where $\Delta^{\mu\nu}\equiv g^{\mu\nu}+u^\mu u^\nu$ is the transverse projector, $\mathcal{Q}^\mu$ is transverse to u_μ, and $\mathcal{T}^{\mu\nu}$ is transverse to u_μ, symmetric, and traceless. Explicitly, the coefficients are $\mathcal{E}\equiv u_\mu u_\nu T^{\mu\nu}$, $\mathcal{P}\equiv\frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu}$, $\mathcal{Q}_\mu\equiv-\Delta_{\mu\alpha}u_\beta T^{\alpha\beta}$ and $\mathcal{T}_{\mu\nu}\equiv\frac{1}{2}(\Delta_{\mu\alpha}\Delta_{\nu\beta}+\Delta_{\nu\alpha}\Delta_{\mu\beta}-\frac{2}{3}\Delta_{\mu\nu}\Delta_{\alpha\beta})T^{\alpha\beta}$. Similarly, we will write the current as

$$J^\mu=\mathcal{N}u^\mu+\mathcal{J}^\mu \tag{2.6}$$

where the charge density is $\mathcal{N}\equiv-u_\mu J^\mu$, and the spatial current is $\mathcal{J}_\mu\equiv\Delta_{\mu\lambda}J^\lambda$. The decompositions (2.5) and (2.6) are just identities, true for any symmetric $T^{\mu\nu}$ and any vector J^μ. This decomposition will remain true for systems out of equilibrium.

The derivative of the fluid velocity can be decomposed even out of equilibrium in 3+1 dimensions as

$$\nabla_\mu u_\nu=-u_\mu a_\nu-\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}u^\rho\Omega^\sigma+\tfrac{1}{2}\sigma_{\mu\nu}+\tfrac{1}{3}\Delta_{\mu\nu}\nabla{\cdot}u\,, \tag{2.7}$$

where $\Omega^\mu\equiv\epsilon^{\mu\nu\alpha\beta}u_\nu\nabla_\alpha u_\beta$ is the vorticity, $a^\mu\equiv u^\lambda\nabla_\lambda u^\mu$ is the acceleration and $\sigma^{\mu\nu}\equiv\Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\nabla_\alpha u_\beta+\nabla_\beta u_\alpha-\frac{2}{3}\Delta_{\alpha\beta}\nabla{\cdot}u\right)$ is the shear viscosity tensor. The decomposition (2.7) is an identity, true for any timelike unit vector u^μ. As we will see soon, $\sigma_{\mu\nu}$ and $\nabla{\cdot}u$ vanish in equilibrium. The electromagnetic field strength tensor

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ can also be decomposed in 3+1 dimensions as

$$F_{\mu\nu} = u_\mu E_\nu - u_\nu E_\mu - \epsilon_{\mu\nu\rho\sigma} u^\rho B^\sigma \,, \tag{2.8}$$

where $E_\mu \equiv F_{\mu\nu} u^\nu$ is the electric field, and $B^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}$ is the magnetic field, satisfying $u{\cdot}E = u{\cdot}B = 0$. The decomposition (2.8) is just an identity, true for any antisymmetric $F_{\mu\nu}$ and any timelike unit u^μ. Electric and magnetic fields are not independent, but are related by the "Bianchi identity" $\epsilon^{\mu\nu\alpha\beta}\nabla_\nu F_{\alpha\beta} = 0$, which in equilibrium becomes

$$\nabla{\cdot}B = B{\cdot}a - E{\cdot}\Omega \,, \tag{2.9a}$$

$$u_\mu \epsilon^{\mu\nu\rho\sigma} \nabla_\rho E_\sigma = u_\mu \epsilon^{\mu\nu\rho\sigma} E_\rho a_\sigma \,. \tag{2.9b}$$

Relations (2.9) are curved-space versions of the familiar flat-space equilibrium identities $\boldsymbol{\nabla}{\cdot}\mathbf{B} = 0$ and $\boldsymbol{\nabla}{\times}\mathbf{E} = 0$.

2.1.1 Equilibrium constraints

The condition of static external sources imposes constraints on certain thermodynamic parameters. The requirement that $\mathcal{L}_V g = 0$ gives one symmetric tensor equation. Decomposing this equation with respect to the fluid velocity u^μ in a similar way to (2.5) gives two scalar equations, one transverse vector equation and one transverse traceless equation

$$u^\lambda \partial_\lambda T = 0 \,, \;\; \nabla{\cdot}u = 0 \,, \;\; a^\mu + \Delta^{\mu\nu}\partial_\nu T/T = 0 \,, \;\; \sigma_{\mu\nu} = 0 \,. \tag{2.10}$$

Similarly, the vector equation coming from $\mathcal{L}_V A = 0$ can be decomposed similarly to (2.6) into

$$u^\lambda \partial_\lambda \mu = 0 \,, \quad E^\mu - T\Delta^{\mu\nu}\partial_\nu \frac{\mu}{T} = 0 \,. \tag{2.11}$$

These will restrict the number of terms that can appear in the derivative expansion of the free energy density $\mathcal{F}$ and in the equilibrium constitutive relations. The coefficients appearing in the equilibrium constitutive relations will be called the thermodynamic coefficients. These coefficients are fully fixed by the equilibrium generating functional W_s and, if the microscopic theory has chiral anomalies, the anomaly coefficients.

2.1.2 Derivative expansion

In order to write down the density $\mathcal{F}$ in the derivative expansion, we need to specify the derivative counting of the external sources A and g. The possible choices for these derivative countings are determined by the equilibrium conditions

$$a^\mu = -\Delta^{\mu\nu}\partial_\nu T/T, \quad E^\mu = T\Delta^{\mu\nu}\partial_\nu \frac{\mu}{T}\,. \tag{2.12}$$

Since finite temperature is required in order to have a classical description of the theory, we always require $a^\mu \sim O(\partial)$. The concept of chemical potential is not relevant when there are no free conserved charges in the theory, in which case we can have $E^\mu \sim O(1)$. This would amount to a hydrostatic description of an insulator. If we want to describe a theory with free charges, the chemical potential μ is again a relevant variable and we find $E \sim O(\partial)$ because of equation (2.12), reflecting the charge screening present in these theories. These correspond to hydrostatic descriptions of conductors. For normal fluids, no equation requires B^μ or Ω^μ to be small, and these may freely be taken at either $O(1)$ or $O(\partial)$. The framework of fluids in strong magnetic fields is explored in chapter 4. Non-relativistic fluids in magnetic fields and with finite vorticity have been simulated in [39, 40]. In superfluids, the covariant derivative of the Goldstone boson $\xi_\mu \equiv -\partial_\mu\psi + A_\mu$, is a new hydrodynamic variable known as the superfluid velocity [41]. In these fluids, there is an additional hydrodynamic equation $\partial_\mu\xi_\nu - \partial_\nu\xi_\mu = F_{\mu\nu}$, leading to $B^\mu\,,\ E^\mu \sim O(\partial)$. This electromagnetic screening is a consequence of the spontaneous U(1) symmetry breaking required to form superfluids, much like the Meissner effect in superconductors [42]. Recent work on relativistic superfluid hydrodynamics includes [43, 44, 45]. For the remainder of this chapter, we take the traditional derivative counting $E^\mu\,,\ B^\mu\,,\ \Omega^\mu \sim O(\partial)$.

We now proceed to expand the free energy density in a derivative expansion, starting with the $O(1)$ term

$$\mathcal{F} = p(T,\mu) + O(\partial)\,, \tag{2.13}$$

which corresponds to the equilibrium pressure[2]. The ideal constitutive relations are then found by varying the generating functional (2.1) with respect to the sources as

[2]The fact that the $O(1)$ part of the free energy density corresponds to the equilibrum pressure follows from taking the limit where the thermodynamic fields are constant everywhere. In the grand canonical ensemble, $-W_s$ is, up to a factor of T, the grand canonical potential, which is equal to $-pV$. See e.g. chapters 2 and 3 of [46].

in (2.2), which gives

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + p\Delta^{\mu\nu}\,, \quad J^\mu = nu^\mu\,. \tag{2.14}$$

The equilibrium charge density n and the equilibrium energy density ϵ are functions of temperature and chemical potential and are related to the equilibrium pressure by $dp = sdT + nd\mu$ and $\epsilon = sT + n\mu - p$. The equilibrium entropy density is important to find constraints on non-equilibrium transport coefficients (see section 2.2.3). Looking at the energy-momentum tensor and current in the frame where $u^\mu = (1, \mathbf{0})$, we get $T^{\mu\nu} = \mathrm{diag}(\epsilon, p, p, p)$ and $J^\mu = (n, \mathbf{0})$. One can proceed expanding the free energy density order by order in derivatives to define the thermodynamic terms at higher derivative orders

$$\mathcal{F} = p(T,\mu) + \sum_{m=1}^{k}\sum_{n=1}^{N_m} M_n^{(m)}(T,\mu)\ s_n^{(m)} + O(\partial^{k+1}), \tag{2.15}$$

where $s_n^{(m)}$ are $O(\partial^m)$ gauge and diffeomorphism invariant functions of $g_{\mu\nu}\,,\ A_\mu$ and V^μ that don't vanish in equilibrium, which we will refer to $O(\partial^m)$ equilibrium scalars. For every derivative order m, N_m is the number of independent gauge and diffeomorphism invariant terms that don't vanish in equilibrium. For normal fluids, there are three possible $O(\partial)$ scalars: $\nabla{\cdot}u\,,\ u^\lambda\partial_\lambda T\,,\ u^\lambda\partial_\lambda\mu$ all of which vanish in equilibrium. Thus, there are no $O(\partial)$ equilibrium scalars, and $N_1 = 0$. Examples of $O(\partial^2)$ equilibrium scalars are $g^{\mu\nu}\partial_\mu T\partial_\nu T$ and $g^{\mu\nu}\partial_\mu\partial_\nu T$. The $M_n^{(m)}$ are independent thermodynamic functions of temperature and chemical potential which will appear as thermodynamic coefficients in the equilibrium constitutive relations. For the derivative counting we have picked $(B^\mu\,,\ E^\mu\,,\ \Omega^\mu \sim O(\partial))$, varying the generating functional then gives the equilibrium constitutive relations to $O(\partial^k)$. As we will see in chapter 4, picking a different derivative counting can lead to terms of $O(\partial^m)$ in the derivative expansion of the free energy density (2.15) giving terms of $O(\partial^{m+1})$ in the equilibrium constitutive relations. As an example of $O(\partial)$ equilibrium constitutive relations, we turn to a 2+1 dimensional parity violating system with a $U(1)$ conserved current. This was first studied without the use of equilibrium generating functionals in [17] and later with the use of this formalism in [18, 19]. For this system there are two equilibrium first order scalars:

$$\Omega = -\epsilon^{\mu\nu\rho}u_\mu\partial_\nu u_\rho\,, \quad B = -\tfrac{1}{2}\epsilon^{\mu\nu\rho}u_\mu F_{\nu\rho}\,,$$

which are the two-dimensional vorticity and magnetic field, so the free energy density up to $O(\partial)$ is

$$\mathcal{F} = p(T,\mu) + M_\Omega(T,\mu)\,\Omega + M_B(T,\mu)\,B + O(\partial^2)\,. \tag{2.16}$$

Varying the generating functional with respect to the metric and the gauge field gives the first order equilibrium constitutive relations

$$\mathcal{E} = \epsilon + (TM_{\Omega,T} + \mu M_{\Omega,\mu} - 2M_\Omega)\,\Omega + (TM_{B,T} + \mu M_{B,\mu} - M_B)\,B\,, \tag{2.17a}$$

$$\mathcal{P} = p\,, \tag{2.17b}$$

$$\mathcal{N} = n + (M_{\Omega,\mu} - M_B)\,\Omega + M_{B,\mu}B\,, \tag{2.17c}$$

$$\mathcal{Q}^\mu = (M_{\Omega,\mu} - M_B)\,\epsilon^{\mu\nu\rho}u_\nu E_\rho + (TM_{\Omega,T} + \mu M_{\Omega,\mu} - 2M_\Omega)\,\epsilon^{\mu\nu\rho}u_\nu\partial_\rho T/T\,, \tag{2.17d}$$

$$\mathcal{J}^\mu = M_{\Omega,\mu}\epsilon^{\mu\nu\rho}u_\nu E_\rho + (TM_{B,T} + \mu M_{B,\mu} - M_B)\,\epsilon^{\mu\nu\rho}u_\nu\partial_\rho T/T\,. \tag{2.17e}$$

These were derived in [18, 19], and are related to the equilibrium expression found in [17] by a frame transformation. Frame transformations will be explained in Section 2.2.1.

2.2 Hydrodynamics

Hydrodynamics is an extension of thermodynamics to fluids that are out of equilibrium, keeping variations in space-time slow. Schematically, this means $\partial \ll 1/\ell_{MFP}$ where ℓ_{MFP} is the mean free path of the microscopic particles. The relevant dimension for the derivative expansion is set by the temperature. To have a well defined derivative expansion we require $\partial T/T^2\,,\ \partial\mu/T^2 \ll 1$. Convergence properties of the hydrodynamic expansion has been a subject of recent interest, studied in [20, 21, 22, 23]. The energy-momentum tensor $T^{\mu\nu}$ and conserved current J^μ come from the variation of an, in general, non-local generating functional $W = -i\ln Z$, similarly to (2.2). Then the conservation equations (2.3) will remain valid due to gauge and diffeomorphism invariance of the out-of equilibrium generating functional W. We will also call these the hydrodynamic equations. Being an extension of thermodynamics, it is not a surprise that the degrees of freedom in hydrodynamics are inherited from thermodynamics. That is, the hydrodynamic variables will be the temperature, chemical potential and fluid velocity, which are now promoted to be time dependent. These hydrodynamic variables don't have a unique definition out of equilibrium. This is because one can in principle add any non-equilibrium scalars or vectors to their equilibrium definition (2.4) and these different definition will agree in equilibrium. This ambiguity in the definition of the hydrodynamic variables leads to the concept of hydrodynamic frames and frame transformations.

2.2.1 Frame transformations

The ambiguity in the out of equilibrium definition of hydrodynamic variables is related to the fact that these are auxiliary variables useful to describe the hydrodynamic limit of the microscopic theories. They are not inherited from the microscopic theory, but rather from the equilibrium thermodynamic description of the theory. In other words, there are no "thermodynamic operators" in the microscopic theory whose expectation values give the local definitions of temperature, chemical potential and fluid velocity. In contrast, the hydrodynamic energy-momentum tensor and conserved current do arise as expectation values of the microscopic energy-momentum tensor and conserved current operators. Thus, when out of equilibrium, we can redefine the hydrodynamic variables $T \to T + \delta T$, $\mu \to \mu + \delta\mu$ and $u^\mu \to u^\mu + \delta u^\mu$ where δT, $\delta\mu$ and δu^μ are made of non-equilibrium terms (such as $\nabla{\cdot}u$, etc.) so that, in equilibrium, the new definitions coincide with the thermodynamic definition (2.4). Note that the normalization condition of u^μ imposes δu^μ to be transverse ($u^\mu \delta u_\mu = 0$). These redefinitions of the hydrodynamic variables are known as frame transformations. In principle, one can consider redefinitions by terms that don't vanish in equilibrium (like $\delta u^\mu = a^\mu$) but this will make the equilibrium constraints found in Section 2.1.1 true in equilibrium only up to $O(\partial^2)$. If we restrict frame transformations to non-equilibrium terms, the equilibrium constraints are exact.

These frame transformations will change the way the out of equilibrium constitutive relations will *look*, but since the energy-momentum and current have a microscopic description, these are independent of frame choice. We describe this by writing $\delta T^{\mu\nu} = 0$ and $\delta J^\mu = 0$. This is a shorthand way of saying $T^{\mu\nu}(T, \mu, u^\mu) = T^{\mu\nu}(T + \delta T, \mu + \delta\mu, u^\mu + \delta u^\mu)$ and similarly for J^μ. Linearising in the variable redefinitions and using the ideal constitutive relations, we find

$$\delta\mathcal{E} = 0\,, \quad \delta\mathcal{P} = 0\,, \quad \delta\mathcal{N} = 0\,, \tag{2.18a}$$

$$\delta\mathcal{Q}^\mu = -\left(\mathcal{E}+\mathcal{P}\right)\delta u^\mu\,, \quad \delta\mathcal{J}^\mu = -\mathcal{N}\delta u^\mu\,, \tag{2.18b}$$

$$\delta\mathcal{T}^{\mu\nu} = 0\,. \tag{2.18c}$$

Isolating the $O(1)$ parts of the out of equilibrium energy, pressure and charge densities we can define the $O(\partial)$ scalars in the contitutive relations $f_\mathcal{E}\,,\ f_\mathcal{P}\,,\ f_\mathcal{N}$ by $\mathcal{E} = \epsilon + f_\mathcal{E}$, $\mathcal{P} = p + f_\mathcal{P}$, $\mathcal{N} = n + f_\mathcal{N}$. The transformation of these $O(\partial)$ scalars can then be found from (2.18):

$$\delta f_\mathcal{E} = \left(\frac{\partial\epsilon}{\partial T}\right)_\mu \delta T + \left(\frac{\partial\epsilon}{\partial\mu}\right)_T \delta\mu\,, \tag{2.19a}$$

$$\delta f_\mathcal{P} = \left(\frac{\partial p}{\partial T}\right)_\mu \delta T + \left(\frac{\partial p}{\partial\mu}\right)_T \delta\mu\,, \tag{2.19b}$$

$$\delta f_\mathcal{N} = \left(\frac{\partial n}{\partial T}\right)_\mu \delta T + \left(\frac{\partial n}{\partial\mu}\right)_T \delta\mu\,. \tag{2.19c}$$

Since there are three first order scalars and two possible hydrodynamic scalar variations (i.e., δT and $\delta\mu$), we find one frame independent combination

$$f = f_\mathcal{P} - \left(\frac{\partial p}{\partial\epsilon}\right)_n f_\mathcal{E} - \left(\frac{\partial p}{\partial n}\right)_\epsilon f_\mathcal{N}. \tag{2.20}$$

Similarly, since we have one possible transverse hydrodynamic vector variation (δu^μ) we find one frame independent transverse vector

$$\ell^\mu = \mathcal{J}^\mu - \frac{n}{\epsilon+p}\mathcal{Q}^\mu\,, \tag{2.21}$$

and one frame independent transverse traceless tensor $\mathcal{T}^{\mu\nu}$. In order to fix the frame ambiguity when writing the constitutive relations, we must specify the frame choice in which they will be written. Two popular conventions are the Landau-Lifshitz frame [1] $f_\mathcal{E} = f_\mathcal{N} = \mathcal{Q}^\mu = 0$ and the Eckart frame [36] $f_\mathcal{E} = f_\mathcal{N} = \mathcal{J}^\mu = 0$. Given our preference of restricting to definitions of the hydrodynamic variables that coincide with (2.4) in equilibrium, we will be using the thermodynamic Landau-Lifshitz frame used in [47] where the equilibrium terms are left in the thermodynamic frame of [18, 19] while the non-equilibrium terms are in the Landau-Lifshitz frame

$$f_{\mathcal{E}} = \bar{f}_{\mathcal{E}}\,, \quad f_{\mathcal{P}} = \bar{f}_{\mathcal{P}} + f_{\text{non-eq.}}\,, \quad f_{\mathcal{N}} = \bar{f}_{\mathcal{N}}\,, \tag{2.22a}$$

$$\mathcal{Q}^{\mu} = \bar{\mathcal{Q}}^{\mu}\,, \quad \mathcal{J}^{\mu} = \bar{\mathcal{J}}^{\mu} + \ell^{\mu}_{\text{non-eq.}}\,, \tag{2.22b}$$

$$\mathcal{T}^{\mu\nu} = \bar{\mathcal{T}}^{\mu\nu} + \mathcal{T}^{\mu\nu}_{\text{non-eq.}}\,. \tag{2.22c}$$

Here the barred terms are the ones derived from the variation of the equilibrium generating functional and the "non-eq." subscript indicates the non-equilibrium terms of the frame invariants f , ℓ^{μ} and $\mathcal{T}^{\mu\nu}$. With this in mind, we can proceed to add the non-equilibrium terms in the derivative expansion of the constitutive relations.

2.2.2 Constitutive relations

For a system out of equilibrium, the terms that vanish in equilibrium can appear in the out of equilibrium constitutive relations. The $O(1)$ functions in front of them are called the non-equilibrium transport coefficients. Two familiar examples of non-equilibrium transport coefficients are shear viscosity and charge conductivity. The non-equilibrium transport coefficients may or may not be constrained by the requirement of local entropy production, allowing for a further classification of dissipative vs adiabatic non-equilibrium transport coefficients. All thermodynamic coefficients are adiabatic, while non-equilibrium transport coefficients can be either dissipative or adiabatic. A more detailed classification of transport coefficient is done in [26], in which dissipative vs adiabatic is taken as the first differentiator. Adiabatic transport coefficients are then further classified into seven classes, which include the hydrostatic (thermodynamic) transport coefficients and hydrodynamic (non-equilibrium adiabatic) transport coefficients discussed in this work.

For a parity-preserving theory with the traditional derivative counting, there are three non-equilibrium first order scalars $u^{\lambda}\partial_{\lambda}T$ $u^{\lambda}\partial_{\lambda}\mu$, and $\nabla{\cdot}u$. These are related by the ideal hydrodynamic equations $\nabla{\cdot}J = 0$, $u_{\mu}\nabla_{\nu}T^{\mu\nu} = E{\cdot}J$, so only one is independent up to $O(\partial^2)$. Similarly, there are two transverse non-equilibrium one derivative vectors $a^{\mu} + \Delta^{\mu\nu}\partial_{\nu}T/T$ and $E^{\mu} - T\Delta^{\mu\nu}\partial_{\nu}\frac{\mu}{T}$ which are related by one transverse vector equation $\Delta_{\mu\nu}\nabla_{\rho}T^{\rho\nu} = 0$, so only one is independent up to $O(\partial^2)$. Finally, there is one independent transverse traceless tensor $\sigma^{\mu\nu}$. Picking $\nabla{\cdot}u$ and $V^{\mu} = E^{\mu} - T\Delta^{\mu\nu}\partial_{\nu}\frac{\mu}{T}$ as the independent first order scalar and vector and using the thermodynamic Landau-Lifshitz frame (which coincides with the Landau-Lifshitz frame in this case) we get the out of equilibrium constitutive relations

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + (p - \zeta \nabla\cdot u)\, \Delta^{\mu\nu} - \eta\sigma^{\mu\nu}\,, \quad J^\mu = n u^\mu + \sigma V^\mu\,. \tag{2.23}$$

This defines the bulk viscosity ζ, shear viscosity η and charge conductivity σ, which are the non-equilibrium transport coefficients. The exact functional form $\zeta(T,\mu)\,,\ \eta(T,\mu)$ and $\sigma(T,\mu)$ depends on the details of the microscopic theory. These three non-equilibrium transport coefficients are dissipative, meaning they will satisfy some inequality constraints due to the requirement of the positivity of entropy production.

2.2.3 Entropy production

One method to find constraints on transport coefficients is to impose a local version of the second law of thermodynamics: the existence of a local entropy current with positive semi-definite divergence for every non-equilibrium configuration consistent with the hydrodynamic equations. It was shown in [24, 25] that the constraints on transport coefficients derived from the entropy current are the same as the equality constraints derived from the equilibrium generating functional, plus the inequality constraints on dissipative transport coefficients. We now review how these inequality constraints are found.

We take the entropy current to be

$$S^\mu = S^\mu_{\text{canon}} + S^\mu_{\text{eq.}}\,,$$

where the canonical part of the entropy current is

$$S^\mu_{\text{canon}} = \frac{1}{T}\left(p u^\mu - T^{\mu\nu}u_\nu - \mu J^\mu\right)\,, \tag{2.24}$$

and $S^\mu_{\text{eq.}}$ is found from the equilibrium partition function, as described in [24, 25]. The constraints on transport coefficients follow by demanding $\nabla_\mu S^\mu \geqslant 0$. Using hydrodynamic equations (2.3a),(2.3b), the divergence of the canonical entropy current is

$$\nabla_\mu S^\mu_{\text{canon}} = \nabla_\mu\left(\frac{p}{T}u^\mu\right) - T^{\mu\nu}\nabla_\mu \frac{u_\nu}{T} + J^\mu\left(\frac{E_\mu}{T} - \partial_\mu \frac{\mu}{T}\right)\,.$$

The $S^\mu_{\text{eq.}}$ part of the entropy current is explicitly built to cancel out the part of $\nabla_\mu S^\mu_{\text{canon}}$ that arises from the equilibrium terms in the constitutive relations, i.e. the terms in

$T^{\mu\nu}$ and J^μ derived from the equilibrium generating functional. For parity-preserving normal fluids to $O(\partial)$, $S^\mu_{\text{eq.}}$ is zero. We thus focus on the non-equilibrium terms, and write the thermodynamic frame constitutive relations (2.23) as $T^{\mu\nu} = T^{\mu\nu}_{\text{eq.}} + T^{\mu\nu}_{\text{non-eq.}}$ and $J^\mu = J^\mu_{\text{eq.}} + J^\mu_{\text{non-eq.}}$. The divergence of the entropy current is then

$$\begin{aligned}\nabla_\mu S^\mu &= \frac{1}{T} J^\mu_{\text{non-eq.}} \left(E_\mu - T\partial_\mu \frac{\mu}{T}\right) - T^{\mu\nu}_{\text{non-eq.}} \nabla_\mu \frac{u_\nu}{T} \\ &= \frac{1}{T}\ell^\mu_{\text{non}-\text{eq.}} V_\mu - \frac{1}{T} f_{\text{non-eq.}} \nabla{\cdot}u - \frac{1}{2T}\mathcal{T}^{\mu\nu}_{\text{non-eq.}}\sigma_{\mu\nu}\,,\end{aligned}$$

where the frame invariants were defined in (2.20) and (2.21). Using the constitutive relations (2.23), this leads to

$$T\nabla_\mu S^\mu = \sigma V^2 + \tfrac{1}{2}\eta(\sigma^{\mu\nu})^2 + \zeta(\nabla{\cdot}u)^2\,. \tag{2.25}$$

Demanding $\nabla_\mu S^\mu \geqslant 0$ now gives

$$\sigma \geqslant 0\,, \quad \eta \geqslant 0\,, \quad \zeta \geqslant 0\,. \tag{2.26}$$

2.2.4 Kubo formulas

When the microscopic system is time-reversal invariant, transport coefficients can be further constrained by the Onsager relations. The retarded two-point functions of operators O_a and O_b in a time-reversal invariant theory in equilibrium obey

$$G^R_{ab}(\omega, \mathbf{k}) = \epsilon_a \epsilon_b\, G^R_{ba}(\omega, -\mathbf{k})\,, \tag{2.27}$$

where ϵ_a and ϵ_b are the time-reversal eigenvalues of the operators O_a and O_b. Equation (2.27) is given in Fourier space $G^R_{ab}(\omega,\mathbf{k}) = \int d^{d+1}x e^{-ik\cdot x} G^R_{ab}(x)$ where $k^\mu = (\omega, \mathbf{k})$.

More generally, if there are some time-reversal symmetry breaking parameters χ (such as an external magnetic field), the two-point functions of operators O_a and O_b in a time-reversal invariant microscopic theory in equilibrium obey

$$G^R_{ab}(\omega, \mathbf{k}, \chi) = \epsilon_a \epsilon_b\, G^R_{ba}(\omega, -\mathbf{k}, -\chi)\,. \tag{2.28}$$

We take our operators to be various components of $T^{\mu\nu}$ and J^μ, and evaluate the retarded two-point functions by varying one-point functions in the presence of the external source with respect to the source. Namely, we solve the hydrodynamic equations in the presence of fluctuating external sources $\delta A, \delta g$ (proportional to $\exp(-i\omega t + i\mathbf{k}\cdot\mathbf{x})$) to find $\delta T[A,g]$, $\delta\mu[A,g]$, $\delta u^\alpha[A,g]$, and then vary the resulting hydrodynamic expressions $T^{\mu\nu}[A,g]$ and $J^\mu[A,g]$ with respect to $g_{\alpha\beta}$, A_α to find the retarded functions. Specifically,

$$G^R_{T^{\mu\nu}T^{\alpha\beta}} = 2\frac{\delta}{\delta g_{\alpha\beta}}\left(\sqrt{-g}\,T^{\mu\nu}_{\text{on-shell}}[A,g]\right)\,, \quad G^R_{J^\mu T^{\alpha\beta}} = 2\frac{\delta}{\delta g_{\alpha\beta}}\left(\sqrt{-g}\,J^{\mu}_{\text{on-shell}}[A,g]\right)\,, \tag{2.29a}$$

$$G^R_{T^{\mu\nu}J^\alpha} = \frac{\delta}{\delta A_\alpha}T^{\mu\nu}_{\text{on-shell}}[A,g]\,, \qquad G^R_{J^\mu J^\alpha} = \frac{\delta}{\delta A_\alpha}J^{\mu}_{\text{on-shell}}[A,g]\,, \tag{2.29b}$$

where the subscript "on-shell" signifies that the corresponding hydrodynamic $T^{\mu\nu}[A,g]$ and $J^\mu[A,g]$ are evaluated on the solutions to (2.3), and the sources δA, δg are set to zero after the variation is taken. The expressions (2.29) are to be understood as

$$\delta(\sqrt{-g}\,T^{\mu\nu}_{\text{on-shell}}) = \tfrac{1}{2}G^R_{T^{\mu\nu}T^{\alpha\beta}}(\omega,\mathbf{k})\,\delta g_{\alpha\beta}(\omega,\mathbf{k})\,,$$

etc. This provides a direct method to evaluate the retarded functions, and allows both to check the Onsager relations and to derive Kubo formulas for transport coefficients. In the hydrodynamic regime we are working in, the Onsager relations don't impose any constraints on the transport coefficients. We will see in Chapter 4 that the Onsager relations do impose constraints on first order transport coefficients in the presence of strong magentic fields.

We next list the expressions for transport coefficients in terms of retarded functions evaluated in flat-space equilibrium. These expressions are known as the Kubo formulas for the transport coefficients [48]. In the limit $\mathbf{k}\to 0$ first, $\omega\to 0$ second we find the following Kubo formulas. For normal fluids in 3+1 dimension, the two-point function of the current J^i gives the conductivity,

$$\tfrac{1}{3\omega}\delta_{ij}\mathrm{Im}\,G^R_{J^iJ^j}(\omega,\mathbf{k}{=}0) = \sigma\,, \tag{2.30}$$

the shear viscositiy is given by

$$\tfrac{1}{\omega}\mathrm{Im}\,G^R_{T^{xy}T^{xy}}(\omega,\mathbf{k}{=}0) = \eta\,, \tag{2.31}$$

while the "bulk" viscosity may be expressed as

$$\tfrac{1}{9\omega}\delta_{ij}\delta_{kl}\mathrm{Im}\, G^R_{T^{ij}T^{kl}}(\omega,\mathbf{k}{=}0) = \zeta\,, \tag{2.32}$$

Correlation functions at non-zero momentum may be obtained in a straightforward way from the variational procedure described earlier.

2.2.5 Inequality constraints on transport coefficients

Finally, let us show that the inequality constraints on transport coefficients (2.26) derived from demanding that the entropy production is non-negative can also be obtained from hydrodynamic correlation functions, without using the entropy current. The argument is based on the fact that the imaginary part of the retarded function $G^R_{OO}(\omega,\mathbf{k})$ must be positive for any Hermitean operator O and $\omega > 0$,

$$\mathrm{Im}\, G^R_{OO}(\omega,\mathbf{k}) \geqslant 0\,. \tag{2.33}$$

See appendix D.4 for details. Then, using $\mathcal{O} = J^x$, T^{xx} and T^{xy} together with the Kubo formulas (2.30), (2.31) and (2.32) give the inequality constraints (2.26) derived by the entropy production argument. Now consider the operator $O = aO_1 + bO_2$, with real coefficients a and b, and Hermitean operators O_1, O_2. The inequality (2.33) implies

$$\mathrm{Im}\left[a^2 G^R_{O_1O_1} + abG^R_{O_1O_2} + abG^R_{O_2O_1} + b^2 G^R_{O_2O_2}\right] \geqslant 0\,,$$

for $\omega \geqslant 0$. This quadratic form in a, b must be non-negative for all a, b which implies $\mathrm{Im}G^R_{O_1O_1} \geqslant 0$, $\mathrm{Im}G^R_{O_2O_2} \geqslant 0$ together with

$$\left(\mathrm{Im}G^R_{O_1O_1}\right)\left(\mathrm{Im}G^R_{O_2O_2}\right) \geqslant \tfrac{1}{4}\left(\mathrm{Im}G^R_{O_1O_2} + \mathrm{Im}G^R_{O_2O_1}\right)^2\,. \tag{2.34}$$

The two terms in the right-hand side of (2.34) can be related by the Onsager relation (2.28). This argument can be expanded to $\mathcal{O} = \sum a_n \mathcal{O}_n$ to give more constraints when more transport coefficients appear in the correlation functions. The constraints found using this method are the same as the constraints found by the entropy production requirement for all systems studied thus far. If this is true in general, and the reason behind it remain an open question to this day.

Chapter 3

Anisotropic hydrodynamics

3.1 Introduction

The study of non-Fermi liquids is an area of great interest in recent years [49, 50, 51, 52]. Many new materials show behaviour uncharacteristic of Fermi liquid theory, and the search for appropriate descriptions of these materials without quasi-particle descriptions is a subject of great interest. High temperature cuprate superconductors are arguably the most famous example of these "strange metals" where the quasi-particle description of Fermi liquid theory breaks down. For the most part, quantum criticality has become a central avenue in the search of appropriate descriptions of these non-Fermi liquids [53, 10, 54, 52]. Studies of graphene have shown the formation of a Dirac fluid in which the quasi-particle description breaks down, exhibiting non-Fermi liquid behaviour [55]. The low energy description of graphene exhibits a pseudo-Lorentzian symmetry with an effective speed of light $v_{eff} \approx c/300$. This effective speed of light appears in the Coulomb coupling $\alpha_{eff} = \alpha/v_{eff} \approx 2.2$. Further studies into Dirac fluids have pointed towards the possibility of a hydrodynamic description of these systems [56, 57, 13].

Recent hydrodynamic descriptions have also been constructed for other exotic materials such a Weyl semi-metals [14]. An important aspect of Weyl semi-metals is the separation of the Fermi points for excitations of different chirality, connected through the boundary of the crystal by the exotic boundary states named Fermi arcs [58, 59, 60]. This makes the Weyl semi-metal an example of a topologically non-trivial phase of matter. In Weyl semi-metals, the Fermi points are commonly named Weyl nodes, as the quasi-particle excitations in those nodes are Weyl fermions. Being a theory of chiral fermions, the hydrodynamic description for Weyl semi-metals is that of hydrodynamics in the presence of anomalies, an active subject of research [16, 38, 61, 62] . Aside from both being examples of strange metals for which hydrodynamic descriptions have recently been studied, Weyl semi-metals and Dirac fluids in graphene have another thing in common. Namely, the low energy effective theory for both of these materials is not Lorentz invariant. To make this point clear, we write down their effective actions, first in the usual way and then in a manifestly covariant way.

The low energy effective description for graphene is that of massless fermions coupled to an electric potential with an effective speed of light $v_{eff} \approx c/300$. The low energy effective action for graphene is [63, 64]

$$S_{\text{graph.}} = \int dt d^2x \bar{\psi} \left(\tfrac{1}{v_{eff}} i\gamma^0(\partial_t + ieA_0) + i\gamma^i \partial_i \right) \psi \,, \tag{3.1}$$

where $\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2)$ obeys $\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}_{2+1}$. This effective action exhibits an emergent Lorentz symmetry with the effective speed of light v_{eff}. Consider coupling this theory to a dynamical electromagnetic field with Lagrangian $\mathcal{L}_{EM} = -\frac{1}{4}F^2$. The electromagnetic action is invariant under the true Lorentz group with speed of light c. To write this in a Lorentz covariant way, we introduce a unit vector pointing in the "time" direction $n^\mu = (1, \mathbf{0})$. We can then write

$$S_{\text{tot.}} = \int d^3x \bar{\psi} \left(i\gamma^\mu D_\mu + i\gamma^\nu n_\nu (1 - \tfrac{1}{v_{eff}}) n^\mu D_\mu \right) \psi - \int d^4x \tfrac{1}{4} F^2 = S_{\text{tot.}}[\psi, A, n] \,, \tag{3.2}$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the gauge covariant derivative. The last equality is there to emphasise that the effective action for graphene depends on the Dirac fields, the gauge field and a vector field n^μ, which roughly speaking corresponds to the velocity of the graphene lattice. The effective action for Weyl semi-metals has a similar Lorentz breaking term [65, 66, 67], which will require a space-like vector $b^\mu = (0, 0, 0, b)$ to be written in a Lorentz covariant way

$$\begin{aligned} S_{\text{Weyl}} &= \int d^4x \sqrt{-g} \bar{\psi} \left(i\gamma^\mu D_\mu - \gamma_z \gamma_5 b + M \right) \psi \\ &= \int d^4x \sqrt{-g} \bar{\psi} \left(i\gamma^\mu (D_\mu + ib_\mu \gamma_5) + M \right) \psi = S_{\text{Weyl}}[\psi, g, A, b] \,. \end{aligned} \tag{3.3}$$

In this case, the vector b^μ parametrizes the separation of the Weyl nodes in momentum space.

Roughly speaking, what is happening in these effective descriptions is that the full theory is Lorentz invariant, and in some particular cases (like the formation of a lattice) some degrees of freedom (the ones forming the lattice) can be integrated out, giving a new effective theory

$$\mathcal{Z} = \int \mathcal{D}\psi \mathcal{D}\chi e^{iS[\psi,\chi]} = \int \mathcal{D}\psi' e^{iS_{\text{eff}}[\psi',n]}\,, \tag{3.4}$$

where χ are the degrees of freedom that were integrated out and ψ' is some possible field redefinition of the degrees of freedom ψ that were not integrated out. The new vector $n^\mu = n^\mu[\langle\chi\rangle]$ can be a function of the expectation values of the χ degrees of freedom. Explicitly,

$$S_{\text{eff}}[\psi', n] = -i \log \int \mathcal{D}\chi e^{iS[\psi,\chi]}\,. \tag{3.5}$$

This raised an interesting theoretical question. How does the hydrodynamic framework change, when the generating functional depends not only on the external metric and gauge fields, but also on a new external vector? This is the question that will be addressed in this chapter.

3.2 Thermodynamics with an external vector n

Let us start with equilibrium thermodynamics of anisotropic theories, as the ones mentioned in the introduction of this chapter. To do this, we assume the equilibrium generating functional introduced in section 2.1 is now a function of the external metric $g_{\mu\nu}$, gauge field A_μ and vector n^μ

$$W_s = W_s[g, A, n], \quad \mathcal{L}_V = 0\,. \tag{3.6}$$

Recall that static equilibrium implies the existence of a timelike Killing vector field V^μ. With the extra external vector n^μ, there is a new auxiliary vector X_μ. The variation of the generating functional is then

$$\delta W = \int d^d x \sqrt{-g} \Big[\frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} + J^\mu \delta A_\mu + X_\mu \delta n^\mu\Big]\,, \tag{3.7}$$

which defines X_μ. We also assume the external vector is gauge independent. This ensures the current conservation equation (2.3b) remains unaffected. On the other hand, the energy-momentum non-conservation (2.3a) receives corrections coming from the action of diffeomorphisms on the external vector

$$\nabla_\mu T^{\mu\nu} = F^{\nu\mu} J_\mu + X_\mu \nabla^\nu n^\mu + \nabla_\mu \Big(X^\nu n^\mu \Big), \quad \nabla_\mu J^\mu = 0\,. \tag{3.8}$$

See appendix A for a derivation of the equations of motion. Note that the auxiliary vector doesn't come with a new conservation equation. As was discussed in section 2.1, we can expand the free energy density $\mathcal{F}$ (defined in (2.1)) order by order in derivatives and take variations with respect to the sources to get the equilibrium constitutive relations. Note that there will be a third constitutive relation for the auxiliary vector

$$X_\mu = \frac{1}{\sqrt{-g}} \frac{\delta W_s}{\delta n^\mu}\,. \tag{3.9}$$

The $O(1)$ part of the free energy density $\mathcal{F}$, the equilibrium pressure p, can now be a function of three scalars

$$T = \frac{\beta_0^{-1}}{\sqrt{-V^2}}, \quad \mu = u^\mu A_\mu + \frac{\Lambda_V}{\sqrt{-V^2}}, \quad n{\cdot}u = n^\mu u^\nu g_{\mu\nu}\,, \tag{3.10}$$

where Λ_V is a gauge parameter which ensures that μ is gauge-invariant [38]. Explicitly,

$$\mathcal{F} = p(T, \mu, n{\cdot}u) + O(\partial)\,, \tag{3.11}$$

from which we find the $O(1)$ constitutive relations

$$T^{\mu\nu} = \Big(\epsilon - g n{\cdot}u\Big) u^\mu u^\nu + p\Delta^{\mu\nu} + g(u^\mu n_\perp^\nu + u^\nu n_\perp^\mu)\,, \tag{3.12a}$$

$$J^\mu = \rho u^\mu\,, \quad X^\mu = g u^\mu\,, \tag{3.12b}$$

where $n_\perp^\mu = \Delta^{\mu\nu} n_\nu$, the part of n^μ orthogonal to u^μ. We also have $dp = sdT + \rho d\mu + g dn{\cdot}u$ and $\epsilon = sT + \rho\mu - p$. As in chapter 2, s is the entropy density. Note the different notation for charge density ρ to avoid confusion with the external vector n^μ. We also have a new thermodynamic function g, which contains the pressure's dependence on the new scalar $n{\cdot}u$. In the fluid's rest frame (i.e. where $u^\mu = (1, \mathbf{0})$), the new term gives a non-zero momentum density along the $n_\perp^\nu$ direction. As for normal fluids, p correspond to the pressure in this reference frame, while the energy density receives a new contribution $g n{\cdot}u$.

We now turn to the $O(\partial)$ equilibrium scalars in the derivative expansion. There are 11 $O(\partial)$ scalars, but $\mathcal{L}_V = 0$ imposes 7 constrains between them, leaving only four independent $O(\partial)$ equilibrium scalars. The 11 $O(\partial)$ scalars are listed in table 3.1. The equilibrium constraints can be found in appendix B.1. In 3+1 dimensions, there are 3 $O(\partial)$ equilibrium pseudo-scalars, while in 2+1 dimensions there are 6. The $O(\partial)$ equilibrium scalars and pseudo-scalars in 3+1 dimensions and 2+1 dimensions are listed in tables 3.2 and 3.3 respectively.

First order scalars										
1	2	3	4	5	6	7	8	9	10	11
$\partial_n T$	$\partial_n \frac{\mu}{T}$	$\partial_n \frac{n\cdot u}{T}$	$\nabla_\mu n^\mu$	$n \cdot (E - T\partial \frac{\mu}{T})$	$n \cdot (a + \frac{\partial T}{T})$	$\mathbb{N}^{\mu\nu}\nabla_\mu u_\nu$	$\partial_u T$	$\partial_u \frac{\mu}{T}$	$\partial_u \frac{n\cdot u}{T}$	$\nabla_\mu u^\mu$

Table 3.1: First order scalars in d+1 dimensions. The first four will correspond to the equilibrium scalars, the last seven vanish in equilibrium. We have used $\mathbb{N}^{\mu\nu} = \Delta^{\mu\nu} - n_\perp^\mu n_\perp^\nu / n_\perp^2$ and $\partial_n = n^\mu \partial_\mu$.

Zero order equilibrium terms			
	1	2	3
scalars ($s_{0,i}$)	T	μ	$n \cdot u$

First order equilibrium terms				
	1	2	3	4
scalars ($s_{1,i}$)	$\partial_n T$	$\partial_n \frac{\mu}{T}$	$\partial_n \frac{n\cdot u}{T}$	$\nabla_\mu n^\mu$
pseudoscalars ($\tilde{s}_{1,i}$)	$B \cdot n$	$\omega \cdot n$	$\tilde{\omega} \cdot n$	

Table 3.2: Zero and first order independent scalars and pseudoscalars in 3+1 dimensions. We have defined $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} u_\nu F_{\rho\sigma}$, $\omega^\mu = \epsilon^{\mu\nu\rho\sigma} u_\nu \nabla_\rho u_\sigma$ and $\tilde{\omega}^\mu = \epsilon^{\mu\nu\rho\sigma} u_\nu \nabla_\rho n_\sigma$.

Zero order equilibrium terms			
	1	2	3
scalars ($s_{0,i}$)	T	μ	$n \cdot u$

First order equilibrium terms						
	1	2	3	4	5	6
scalars ($s_{1,i}$)	$\partial_n T$	$\partial_n \frac{\mu}{T}$	$\partial_n \frac{n\cdot u}{T}$	$\nabla_\mu n^\mu$		
pseudoscalars ($\tilde{s}_{1,i}$)	B	ω	$\tilde{\omega}$	$\tilde{\omega}_n$	$\epsilon_{\frac{n\cdot u}{T}}$	$\epsilon^{\mu\nu\rho} n_\mu u_\nu \nabla_n u_\rho$

Table 3.3: Zero and first order independent scalars and pseudoscalars in 2+1 dimensions. We have defined $B = \frac{1}{2}\epsilon^{\nu\rho\sigma} u_\nu F_{\rho\sigma}$, $\omega = \epsilon^{\nu\rho\sigma} u_\nu \partial_\rho u_\sigma$, $\tilde{\omega} = \epsilon^{\mu\nu\rho} u_\mu \partial_\nu n_\rho$, $\tilde{\omega}_n = \epsilon^{\mu\nu\rho} n_\mu \partial_\nu n_\rho$ and $\epsilon_s = \epsilon^{\mu\nu\rho} n_\nu u_\nu \partial_\rho$.

Similar to the decompositions of the energy-momentum tensor (2.5) and the current (2.6) with respect to the fluid velocity

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + u^\mu \mathcal{Q}^\nu + \mathcal{Q}^\mu u^\nu + \mathcal{T}^{\mu\nu}\,, \tag{3.13a}$$

$$J^\mu = \mathcal{N} u^\mu + \mathcal{J}^\mu\,, \tag{3.13b}$$

we can decompose the auxiliary vector

$$X^\mu = \mathcal{G} u^\mu + \mathcal{X}^\mu \,, \tag{3.14}$$

where $\mathcal{G} \equiv -u^\mu X_\mu$ and $\mathcal{X}^\mu \equiv \Delta^{\mu\nu} X_\nu$. Just like equations (3.13), the decomposition (3.14) is an identity. These decmopositions will also be used when studying the out-of equilibrium energy-momentum tensor, conserved current and auxiliary vector. With the addition of the external vector n^μ, it is useful to further decompose $\mathcal{Q}^\mu$

$$\mathcal{Q}^\mu = g n_\perp^\mu + q^\mu, \tag{3.15}$$

where $n_\perp^\mu = \Delta^{\mu\nu} n_\nu$. Equation (3.15) defines q^μ, which contains the $O(\partial)$ part of $\mathcal{Q}^\mu$. Compared to the hydrodynamic framework for normal fluids introduced in chapter 2, the facts that there is a new auxiliary vector X_μ and that the momentum current $\mathcal{Q}^\mu$ has an $O(1)$ component require a careful treatment of frame transformations and the entropy current. These will be the focus of the next section.

3.3 Consequences of anisotropy

3.3.1 Frame transformations

As was mentioned in chapter 2, the out of equilibrium definition of the hydrodynamic variables is not unique. These redefinitions of fluid variables are known as hydrodynamic frame transformations. Refer to section 2.2.1 for an introduction to hydrodynamic frame transformations. In this discussion, we focus on decomposing the $O(\partial)$ structures that appear in the constitutive relations of anisotropic relativistic hydrodynamics in terms of the SO(d) rotation subgroup of SO(d,1) that leaves the fluid velocity u^μ invariant. In the constitutive relations for a fluid coupled to an external vector field in d+1 dimensions, there are four SO(d) scalars $\mathcal{E}\,,\ \mathcal{P}\,,\ \mathcal{N}$ and $\mathcal{G}$, three transverse vectors $\mathcal{Q}^\mu\,,\ \mathcal{J}^\mu$ and $\mathcal{X}^\mu$ and one transverse traceless symmetric tensor $\mathcal{T}^{\mu\nu}$. We will be interested in the $O(\partial)$ parts of these terms, and how they vary under frame transformations ($T \to T + \delta T\,,\ \mu \to \mu + \delta\mu$ and $u^\mu \to u^\mu + \delta u^\mu$). As we will soon see, the scalar and vector variations under frame transformations will get coupled because the transformation of $n{\cdot}u$ will depend on the transformation of u^μ. There are three hydrodynamic variables to redefine (temperature, chemical potential and fluid velocity), so we can recover $4+3+1-3 = 5$ frame invariant $O(\partial)$

terms that transform under a representation of $SO(d)$. In comparison, for normal fluids we have three $O(\partial)$ frame invariants, as was shown in section 2.2.1. Since there are no tensor variables to transform, one of these invariants is expected to be a tensor. By the same reasoning, we can expect to have two linearly independent frame invariant vectors. This means there will be two frame invariant scalars. However, because of the coupling of vector and scalar variation, at least one of these scalars will depend on the first order vectors as well. Using the ideal constitutive relations and requiring that $T^{\mu\nu}$, J^μ , X^μ remain invariant to $O(\partial)$ under a change of frame $T \to T + \delta T$, $\mu \to \mu + \delta\mu$, $u^\mu \to u^\mu + \delta u^\mu$ leads to

$$\delta\mathcal{E} = -2gn_\mu\delta u^\mu, \quad \delta\mathcal{P} = -\frac{2}{d-1}gn_\mu\delta u^\mu, \quad \delta\mathcal{N} = \delta\mathcal{G} = 0 \tag{3.16}$$

We define the $O(\partial)$ functions $f_\mathcal{E}$, $f_\mathcal{P}$, $f_\mathcal{N}$ and $f_\mathcal{G}$ by

$$\mathcal{E} = \epsilon - gn\cdot u + f_\mathcal{E}, \quad \mathcal{P} = p + f_\mathcal{P}, \quad \mathcal{N} = \rho + f_\mathcal{N}, \quad \mathcal{G} = g + f_\mathcal{G}\,, \tag{3.17}$$

Equation (3.16) implies the following transformation of the $O(\partial)$ functions

$$\begin{aligned}
f'_\mathcal{E} &= f_\mathcal{E} - \Big(\frac{\partial\epsilon}{\partial T} - n{\cdot}u\frac{\partial g}{\partial T}\Big)\delta T - \Big(\frac{\partial\epsilon}{\partial\mu} - n{\cdot}u\frac{\partial g}{\partial\mu}\Big)\delta\mu - \Big(\frac{\partial\epsilon}{\partial n{\cdot}u} - n{\cdot}u\frac{\partial g}{\partial n{\cdot}u} + g\Big)n_\mu\delta u^\mu\,,\\
f'_\mathcal{P} &= f_\mathcal{P} - s\delta T - \rho\delta\mu - \frac{d+1}{d-1}gn_\mu\delta u^\mu\,,\\
f'_\mathcal{N} &= f_\mathcal{N} - \frac{\partial\rho}{\partial T}\delta T - \frac{\partial\rho}{\partial\mu}\delta\mu - \frac{\partial\rho}{\partial n{\cdot}u}n_\mu\delta u^\mu\,,\\
f'_\mathcal{G} &= f_\mathcal{G} - \frac{\partial g}{\partial T}\delta T - \frac{\partial g}{\partial\mu}\delta\mu - \frac{\partial g}{\partial n{\cdot}u}n_\mu\delta u^\mu\,,
\end{aligned} \tag{3.18}$$

where the thermodynamic variables are T , μ and $n{\cdot}u$. That is, $\frac{\partial\epsilon}{\partial T} = \left(\frac{\partial\epsilon}{\partial T}\right)_{\mu,n\cdot u}$, and so on. The frame invariant scalar made out of these is

$$f_{inv} = f_\mathcal{P} - \alpha_\mathcal{E}f_\mathcal{E} - \alpha_\mathcal{N}f_\mathcal{N} - \alpha_\mathcal{G}f_\mathcal{G}\,, \tag{3.19}$$

where[1]

$$\begin{aligned}
\alpha_{\mathcal{E}} &= \frac{1}{T}\left(\left(\frac{\partial p}{\partial s}\right)_{\rho,g} + \frac{2g}{d-1}\left(\frac{\partial n{\cdot}u}{\partial s}\right)_{\rho,g}\right) , \\
\alpha_{\mathcal{N}} &= \left(\frac{\partial p}{\partial \rho}\right)_{s,g} + \frac{2g}{d-1}\left(\frac{\partial n{\cdot}u}{\partial \rho}\right)_{s,g} - \frac{\mu}{T}\left(\left(\frac{\partial p}{\partial s}\right)_{\rho,g} + \frac{2g}{d-1}\left(\frac{\partial n{\cdot}u}{\partial s}\right)_{\rho,g}\right) , \qquad (3.20)\\
\alpha_{\mathcal{G}} &= \left(\frac{\partial p}{\partial g}\right)_{s,\rho} + \frac{2g}{d-1}\left(\frac{\partial n{\cdot}u}{\partial g}\right)_{s,\rho} + \frac{n{\cdot}u}{T}\left(\left(\frac{\partial p}{\partial s}\right)_{\rho,g} + \frac{2g}{d-1}\left(\frac{\partial n{\cdot}u}{\partial s}\right)_{\rho,g}\right) .
\end{aligned}$$

The linear combination

$$\begin{aligned}
f &= f_{\mathcal{P}} - \left(\frac{\partial p}{\partial \epsilon}\right)_{\rho,g}(f_{\mathcal{E}} + n{\cdot}u f_{\mathcal{G}}) - \left(\frac{\partial p}{\partial \rho}\right)_{\epsilon,g} f_{\mathcal{N}} - \left(\frac{\partial p}{\partial g}\right)_{\epsilon,\rho} f_{\mathcal{G}} \\
&\equiv f_{\mathcal{P}} - \beta_{\mathcal{E}} f_{\mathcal{E}} - \beta_{\mathcal{N}} f_{\mathcal{N}} - \beta_{\mathcal{G}} f_{\mathcal{G}}
\end{aligned} \qquad (3.21)$$

transforms by

$$\delta f = \left(\frac{\partial P}{\partial \epsilon} - \frac{2}{d-1}\right) g n_\mu \delta u^\mu , \qquad (3.22)$$

and will need to be mixed with some vector to give a frame invariant scalar. We will return to this below. The vectors transform as

$$\begin{aligned}
\delta\mathcal{Q}^\mu &= -(\mathcal{E}+\mathcal{P})\delta u^\mu + \mathcal{G} u^\mu n_\lambda \delta u^\lambda, \quad \delta\mathcal{J}^\mu = -\mathcal{N}\delta u^\mu, \quad \delta\mathcal{X}^\mu = -\mathcal{G}\delta u^\mu, \\
\delta q^\mu &= -(\mathcal{E}+\mathcal{P}+\mathcal{G} n{\cdot}u)\delta u^\mu - \left(\frac{\partial g}{\partial T}\delta T + \frac{\partial g}{\partial \mu}\delta\mu + \frac{\partial g}{\partial n{\cdot}u} n_\lambda \delta u^\lambda\right) n_\perp^\mu ,
\end{aligned} \qquad (3.23)$$

so that

$$\delta(q^\mu - f_{\mathcal{G}} n_\perp^\mu) = -(\mathcal{E}+\mathcal{P}+\mathcal{G} n{\cdot}u)\delta u^\mu . \qquad (3.24)$$

We can thus find two linearly independent frame invariant vectors

$$\ell^\mu = \mathcal{J}^\mu - \frac{\rho}{\epsilon+p}(q^\mu - f_{\mathcal{G}} n_\perp^\mu), \quad k^\mu = \mathcal{J}^\mu - \frac{\rho}{g}\mathcal{X}^\mu . \qquad (3.25)$$

Of course, a linear combination of the two will also be frame invariant. One particularly useful combination of these for a neutral fluid coupled to an external field

[1] The thermodynamic derivatives appearing in equations (3.20) and (3.21) can be related to the derivatives using $(T\,,\ \mu\,,\ n{\cdot}u)$ as the thermodynamic variables using the mnemonic $\left(\frac{\partial p}{\partial s}\right)_{\rho,g} = \frac{\partial(p\,,\rho\,,g)}{\partial(s\,,\rho\,,g)} = \frac{\left(\frac{\partial(p\,,\rho\,,g)}{\partial(T\,,\mu\,,n\cdot u)}\right)}{\left(\frac{\partial(s\,,\rho\,,g)}{\partial(T\,,\mu\,,n\cdot u)}\right)}$, etc., where $\frac{\partial(x^1\,,x^2\,,x^3)}{\partial(y^1\,,y^2\,,y^3)} = \det\left(\frac{\partial x^i}{\partial y^j}\right)$.

is

$$m^\mu = \mathcal{X}^\mu - \frac{g}{\epsilon + p}(q^\mu - f_{\mathcal{G}} n_\perp^\mu)\,. \tag{3.26}$$

Finally, the transverse symmetric tensor $\mathcal{T}^{\mu\nu}$ will change by

$$\delta\mathcal{T}^{\mu\nu} = -g\left(\delta u^\mu n_\perp^\nu + \delta u^\nu n_\perp^\mu - \frac{2}{d-1}\Delta^{\mu\nu} n_\lambda \delta u^\lambda\right)\,. \tag{3.27}$$

Using the transformation rules for the first order vectors, we can form three invariant scalar and three tensors

$$f + \Big(\frac{\partial p}{\partial \epsilon} - \frac{2}{d-1}\Big)\frac{g}{\mathcal{V}} n_\mu v^\mu, \quad \mathcal{T}^{\mu\nu} - \frac{g}{\mathcal{V}}(v^\mu n_\perp^\nu + v^\nu n_\perp^\mu - \frac{2}{d}\Delta^{\mu\nu} n_\lambda v^\lambda)\,, \tag{3.28}$$

where $(v^\mu, \mathcal{V}) \in \{(\mathcal{X}^\mu, g), (\mathcal{J}^\mu, \rho), (q^\mu - f_{\mathcal{G}} n_\perp^\mu, \epsilon + p)\}$. From here on, we will label these scalars and tensors $f_{\mathcal{V}}$ and $\tau_{\mathcal{V}}^{\mu\nu}$. As an example, we have

$$f_\rho = f + \Big(\frac{\partial p}{\partial \epsilon} - \frac{2}{d-1}\Big)\frac{g}{\rho}\mathcal{J}\cdot n\,. \tag{3.29}$$

Only one of the three scalars in (3.28) and only one of the three tensors in (3.28) will be linearly independent from the rest of the frame invariants. For example, we have $f_\rho = f_g + \left(\frac{\partial p}{\partial \epsilon} - \frac{2}{d-1}\right)\frac{g}{\rho} k\cdot n$.

In this discussion we relied on the $SO(d)$ subgroup of the Lorentz group that leaves u^μ invariant to keep track of our frame invariants. Decomposing only with respect to u^μ gives the frame invariants in the $SO(d)$ basis: $f_{inv}, f_{\epsilon+p}, \ell^\mu, m^\mu, \tau_{\epsilon+p}^{\mu\nu}$. This decomposition has two scalars, two transverse vectors and one transverse symmetric traceless tensor of SO(d) for a total of 2 + 2d + (d+2)(d-1)/2 = d+ (d+1)(d+2)/2 degrees of freedom. Due to the presence a new vector n^μ, we can choose instead to decompose objects with respect to both u^μ and $n_\perp^\mu$. We can thus rewrite the frame invariants in terms of the scalars, vectors and tensors of a smaller group SO(d-1) that leaves both u^μ and n^μ invariant by using the projector $\mathbb{N}^{\mu\nu} = \Delta^{\mu\nu} - n_\perp^\mu n_\perp^\nu / n_\perp^2$. This basis has five $SO(d-1)$ scalars, three transverse vectors and one transverse symmetric traceless tensor for a total of 5 + 3(d-1) + (d+1)(d-2)/2 = d + (d+1)(d+2)/2 degrees of freedom. These two choices of basis appear in table 3.4. Moving forward, we will continue to use the thermodynamic Landau-Lifshitz frame as explained in section 2.2.1, some information on how to go from any frame to the Landau-Lifshitz frame can be found in appendix B.2.

First order frame invariants								
SO(d)	f_{inv}	$f_{\epsilon+p}$	ℓ^μ	m^μ	$\tau^{\mu\nu}_{\epsilon+p}$				
SO(d-1)	f_{inv}	$f_{\epsilon+p}$	ℓ	m	$\tau_{\epsilon+p}$	$\ell^\mu_\perp$	$m^\mu_\perp$	$\tau^\mu_{\epsilon+p}$	$\tau^{\mu\nu}_{\epsilon+p,\perp}$

Table 3.4: First order independent scalars decomposed in terms of two different Lorentz subgroups. Here, we defined $\tau^{\mu\nu}_{\epsilon+p,\perp} = \mathbb{N}^\mu{}_\alpha \mathbb{N}^\nu{}_\beta \tau^{\alpha\beta}_{\epsilon+p}$, $\tau^\mu_{\epsilon+p} = \mathbb{N}^\mu{}_\nu \tau^{\nu\sigma}_{\epsilon+p} n_\sigma$, $\ell = n{\cdot}\ell$, $\ell^\mu_\perp = \mathbb{N}^{\mu\nu}\ell_\nu$ and similarly for m^μ and $\tau^\mu_{\epsilon+p}$. The degrees of freedom for both decompositions will be the same. The only difference is the way we decided to collect them. For example, $\ell^\mu = \ell n^\mu_\perp / n^2_\perp + \ell^\mu_\perp$.

3.3.2 Entropy current

When the effective action (and hence the generating functional W) depends on an external vector n^μ, the canonical entropy current discussed in section 2.2.3 will need to be modified. Using the conservation equations (3.8) and the $O(1)$ constitutive relations (3.12) we find

$$\nabla_\mu\Big[(\epsilon+p)u^\mu\Big] = \nabla_u p - g\nabla_u n{\cdot}u\,, \qquad \nabla_\mu\big(\rho u^\mu\big) = 0\,, \tag{3.30a}$$

$$\begin{aligned} E\cdot n = n^2\nabla_\mu(gu^\mu) + n{\cdot}u\nabla_\mu\Big[\big(\epsilon+p+gn{\cdot}u\big)u^\mu\Big] + \nabla_n p \\ - gu^\lambda\nabla_n n_\lambda + (\epsilon+p+gn{\cdot}u)n^\lambda\nabla_u u_\lambda\,. \end{aligned} \tag{3.30b}$$

Together with the thermodynamic definitions $dp = sdT + \rho d\mu + gd(n{\cdot}u)$ and $\epsilon = sT + \rho\mu - p$, these lead to

$$\nabla_\mu\big(su^\mu\big) = 0\,. \tag{3.31}$$

Thus, as expected, the entropy current is conserved in ideal ($O(1)$) hydrodynamics. The canonical entropy current (2.24) however, gives $S_{\text{canon.}} = su^\mu + \frac{g}{T}n^\mu$. To amend this, we need a modified entropy current

$$S^\mu_{\text{anisotropic}} = \frac{1}{T}\Big(pu^\mu - T^{\mu\nu}u_\nu - \mu J^\mu + u^\nu X_\nu n^\mu\Big), \tag{3.32}$$

so that $S^\mu_{\text{anisotropic}} = su^\mu$ at leading order in derivatives. Direct calculation shows this modified form of the canonical entropy current is frame invariant. This is no longer the case for the usual form of the canonical entropy current which doesn't include the last term.

3.4 First order constitutive relation

3.4.1 Equilibirum terms

We now proceed to find the equilibrium constitutive relations to $O(\partial)$ in derivatives. For a system in equilibrium in the thermodynamic frame, we can expand the constitutive relations to $O(\partial)$

$$\begin{aligned} \bar{f}_{\mathcal{E}} &= \sum_{n=1}^{4} \bar{\epsilon}_n s_n\,, \quad \bar{f}_{\mathcal{P}} = \sum_{n=1}^{4} \bar{\pi}_n s_n\,, \quad \bar{f}_{\mathcal{N}} = \sum_{n=1}^{4} \bar{\phi}_n s_n\,, \quad \bar{f}_{\mathcal{G}} = \sum_{n=1}^{4} \bar{\chi}_n s_n\,, \\ \bar{q}^\mu &= \sum_{n=1}^{7} \bar{\gamma}_n v_n^\mu\,, \quad \bar{\mathcal{J}}^\mu = \sum_{n=1}^{7} \bar{\delta}_n v_n^\mu\,, \quad \bar{\mathcal{X}}^\mu = \sum_{n=1}^{7} \bar{\xi}_n v_n^\mu\,, \quad \bar{\mathcal{T}}^{\mu\nu} = \sum_{n=1}^{m} \sum_{i=1}^{N_n} \bar{\theta}_{n,i} t_{n,i}^{\mu\nu}\,. \end{aligned} \tag{3.33}$$

The coefficients $\bar{\epsilon}_n$, etc. are given below in equation (3.35). The first order scalars, vectors and tensors relevant for hydrostatics are listed in table 3.5.

First order							
	1	2	3	4	5	6	7
scalars (s_n)	$\partial_n T$	$\partial_n \frac{\mu}{T}$	$\partial_n \frac{n\cdot u}{T}$	$\nabla_\mu n^\mu$			
vectors (v_n^μ)	$\partial_n T n_\perp^\mu$	$\partial_n \frac{\mu}{T} n_\perp^\mu$	$\partial_n \frac{n\cdot u}{T} n_\perp^\mu$	$\nabla_\lambda n^\lambda n_\perp^\mu$	$\Delta^{\mu\nu}\partial_\nu T$	$\Delta^{\mu\nu}\partial_\nu \frac{\mu}{T}$	$\Delta^{\mu\nu}\partial_\nu \frac{n\cdot u}{T}$
tensors ($t_n^{\mu\nu}$)	$\partial_n T \sigma_n^{\mu\nu}$	$\partial_n \frac{\mu}{T} \sigma_n^{\mu\nu}$	$\partial_n \frac{n\cdot u}{T} \sigma_n^{\mu\nu}$	$\nabla_\lambda n^\lambda \sigma_n^{\mu\nu}$	$\sigma_T^{\mu\nu}$	$\sigma_{\frac{\mu}{T}}^{\mu\nu}$	$\sigma_{\frac{n\cdot u}{T}}^{\mu\nu}$

Table 3.5: Equilibrium first order independent scalars, vectors and tensors in a parity invariant hydrodynamic theory. Here, we defined $\partial_v = v^\mu \partial_\mu$ for any vector v^μ, $\sigma_n^{\mu\nu} = (n_\perp^\mu n_\perp^\nu - \frac{1}{d}\Delta^{\mu\nu} n_\perp^2)$ and $\sigma_s^{\mu\nu} = (n_\perp^\mu \partial^\nu s + n_\perp^\nu \partial^\mu s - \frac{2}{d}\Delta^{\mu\nu}\partial_{n_\perp} s)$ for any scalar s.

For a parity invariant theory in arbitrary dimensions, the free energy density can be expanded to $O(\partial)$

$$\mathcal{F} = p(T,\mu,n\cdot u) + \sum_{n=1}^{4} M_n(T,\mu,n\cdot u) s_n + O(\partial^2)\,, \tag{3.34}$$

from which we can get the $O(\partial)$ equilibrium constitutive relations in the thermodynamic frame by varying the generating functional (3.6) with respect to the sources.

The resulting constitutive relations using the expansion (3.33) are

$$
\begin{aligned}
&\bar\pi_1 = \bar\xi_5 = M_1 - M_{4,T} - \frac{\mu}{T}M_{4,\mu} - \frac{n{\cdot}u}{T}M_{4,n\cdot u}\,, \quad \bar\pi_2 = \bar\xi_6 = M_2 - TM_{4,\mu}\,,\\
&\bar\pi_3 = \bar\xi_7 = M_3 - TM_{4,n\cdot u}\,, \quad \bar\phi_1 = M_{1,\mu} - \frac{1}{T}\Big(M_{2,T} + \frac{\mu}{T}M_{2,\mu} + \frac{n{\cdot}u}{T}M_{2,n\cdot u}\Big)\,,\\
&\bar\phi_2 = M_{2,\mu}\,, \quad \bar\phi_3 = -\bar\chi_2 = -\bar\gamma_2 = M_{3,\mu} - M_{2,n\cdot u}\,, \bar\phi_4 = M_{4,\mu} - \frac{M_2}{T}\,,\\
&\bar\chi_1 = \bar\gamma_1 = M_{1,n\cdot u} - \frac{1}{T}\Big(M_{3,T} + \frac{\mu}{T}M_{3,\mu} + \frac{n{\cdot}u}{T}M_{3,n\cdot u}\Big)\,,\\
&\bar\chi_4 = \bar\gamma_4 = M_{4,n\cdot u} - \frac{M_3}{T}\,, \quad \bar\epsilon_1 = -2n{\cdot}u\bar\chi_1 - \bar\pi_1\,, \quad \bar\epsilon_2 = 2n{\cdot}u\bar\phi_3 - T^2\bar\phi_1 - \bar\pi_2\,,\\
&\bar\epsilon_3 = -T^2\bar\chi_1 - \bar\pi_3\,, \quad \bar\epsilon_4 = -T\bar\pi_1 + 2\frac{n{\cdot}u}{T}\bar\pi_3\,, \quad \bar\pi_4 = \bar\chi_3 = \bar\gamma_3 = \bar\delta_i = \bar\theta_i = 0\,.
\end{aligned}
\tag{3.35}
$$

The comma denotes a thermodynamic derivative with respect to the following argument i.e., $M_{,T} = \frac{\partial M}{\partial T}$. The constitutive relations have 20 non vanishing coefficients, from which only 9 are linearly independent (for example, $\bar\pi_1$ and $\bar\xi_5$ are not linearly independent) and are related through 4 functions and their derivatives. The bar is used to specify that these are thermodynamic frame coefficients. The constraints in (3.35) can be brought to the Landau frame using (B.11).

3.4.2 Non-equilibrium terms

We have found the equilibrium constitutive relations to $O(\partial)$ in the previous section. The next step is to find the non-equilibrium terms appearing in the constitutive relations to first order in derivatives. This amounts to including all possible contributions to the energy-momentum, current and auxiliary vector made up of independent terms appearing in Table B.1. Although there are 7 scalars, 4 vectors and two tensors in Table B.1, these are related by three scalar equations ($\nabla_\mu T^{\mu\nu} + F^{\mu\nu}J_\mu - X_\mu\nabla^\nu n^\mu - \nabla_\mu(X^\nu n^\mu) = 0$ contracted with u^μ and with $n^\mu_\perp$, as well as $\nabla_\mu J^\mu = 0$) and one transverse vector equation ($\mathbb{N}_{\rho\nu}(\nabla_\mu T^{\mu\nu} + F^{\mu\nu}J_\mu - X_\mu\nabla^\nu n^\mu - \nabla_\mu(X^\nu n^\mu)) = 0$). We can therefore include only 4 scalars, 3 vectors and two tensors from Table B.1 in the constitutive relations.

To fix the frame ambiguities in the constitutive relations, we pick the thermodynamic Landau-Lifshitz frame, that is

$$
\begin{aligned}
&f_{\mathcal{E}} = \bar{f}_{\mathcal{E}}\,, \quad f_{\mathcal{P}} = \bar{f}_{\mathcal{P}} + f_{\mathcal{P},\text{non-eq.}}\,, \quad f_{\mathcal{N}} = \bar{f}_{\mathcal{E}}, \quad f_{\mathcal{G}} = \bar{f}_{\mathcal{G}} + f_{\mathcal{G},\text{non-eq.}}\,,\\
&q^{\mu} = \bar{q}^{\mu}\,, \quad \mathcal{J}^{\mu} = \bar{\mathcal{J}}^{\mu} + \mathcal{J}^{\mu}_{\text{non-eq.}}, \quad \mathcal{X}^{\mu} = \bar{\mathcal{X}}^{\mu} + \mathcal{X}^{\mu}_{\text{non-eq.}}\,,\\
&\mathcal{T}^{\mu\nu} = \bar{\mathcal{T}}^{\mu\nu} + \mathcal{T}^{\mu\nu}_{\text{non-eq.}}\,.
\end{aligned}
\tag{3.36}
$$

The non-equilibrium terms in this frame can be related to the non-equilibrium terms in any frame by (B.10). They can also be related to the non-equilibrium part of the frame invariants $f_{P+\epsilon}$, f_{inv}, ℓ^{μ},m^{μ} and $\tau^{\mu\nu}_{P+\epsilon}$ by the equations preceding (B.10). With these things in mind, the non-equilibrium terms in the constitutive relations take the form

$$
f_{\mathcal{P},\text{non-eq.}} = c_1 s_{1,\text{non-eq.}} + c_2 s_{3,\text{non-eq.}} + c_3 s_{4,\text{non-eq.}} + c_4 s_{7,\text{non-eq.}}\,,
\tag{3.37a}
$$

$$
f_{\mathcal{G},\text{non-eq.}} = c_5 s_{1,\text{non-eq.}} + c_6 s_{3,\text{non-eq.}} + c_7 s_{4,\text{non-eq.}} + c_8 s_{7,\text{non-eq.}}\,,
\tag{3.37b}
$$

$$
\begin{aligned}
\mathcal{J}^{\mu}_{\text{non-eq.}} = {} & (c_9 s_{1,\text{non-eq.}} + c_{10} s_{3,\text{non-eq.}} + c_{11} s_{4,\text{non-eq.}} + c_{12} s_{7,\text{non-eq.}}) n^{\mu}_{\perp}\\
& + c_{13} v^{\mu}_{1,\text{non-eq.}} + c_{14} v^{\mu}_{3,\text{non-eq.}} + c_{15} v^{\mu}_{4,\text{non-eq.}} + c_{16} \tilde{v}^{\mu}_{1,\text{non-eq.}}\\
& + c_{17} \tilde{v}^{\mu}_{3,\text{non-eq.}} + c_{18} \tilde{v}^{\mu}_{4,\text{non-eq.}}\,,
\end{aligned}
\tag{3.37c}
$$

$$
\begin{aligned}
\mathcal{X}^{\mu}_{\text{non-eq.}} = {} & (c_{19} s_{1,\text{non-eq.}} + c_{20} s_{3,\text{non-eq.}} + c_{21} s^{(1)}_{4,\text{non-eq.}} + c_{22} s_{7,\text{non-eq.}}) n^{\mu}_{\perp}\\
& + c_{23} v^{\mu}_{1,\text{non-eq.}} + c_{24} v^{\mu}_{3,\text{non-eq.}} + c_{25} v^{\mu}_{4,\text{non-eq.}} + c_{26} \tilde{v}^{\mu}_{1,\text{non-eq.}}\\
& + c_{27} \tilde{v}^{\mu}_{3,\text{non-eq.}} + c_{28} \tilde{v}^{\mu}_{4,\text{non-eq.}}\,,
\end{aligned}
\tag{3.37d}
$$

$$
\begin{aligned}
\mathcal{T}^{\mu\nu}_{\text{non-eq.}} = {} & (c_{29} s_{1,\text{non-eq.}} + c_{30} s_{3,\text{non-eq.}} + c_{31} s_{4,\text{non-eq.}} + c_{32} s_{7,\text{non-eq.}}) \sigma^{\mu\nu}_{n}\\
& + c_{33} v^{(\mu}_{1,\text{non-eq.}} n^{\nu)}_{\perp} + c_{34} v^{(\mu}_{3,\text{non-eq.}} n^{\nu)}_{\perp} + c_{35} v^{(\mu}_{4,\text{non-eq.}} n^{\nu)}_{\perp} + c_{36} \sigma^{\mu\nu}_{\perp}\\
& + c_{37} \tilde{v}^{(\mu}_{1,\text{non-eq.}} n^{\nu)}_{\perp} + c_{38} \tilde{v}^{(\mu}_{3,\text{non-eq.}} n^{\nu)}_{\perp} + c_{39} \tilde{v}^{(\mu}_{4,\text{non-eq.}} n^{\nu)}_{\perp} + c_{40} \tilde{\sigma}^{\mu\nu}_{\perp}\,.
\end{aligned}
\tag{3.37e}
$$

To further restrict these non-equilibrium transport coefficients, the transformation properties of n^{μ} have to be specified. If n^{μ} behaves as a regular vector under $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$, parity invariance of the microscopic theory fixes $c_{16} = c_{17} = c_{18} = c_{26} = c_{27} = c_{28} = c_{37} = c_{38} = c_{39} = c_{40} = 0$. In a conformal theory, the tracelessness of the energy-momentum tensor imposes $c_1 = c_2 = c_3 = c_3 = c_4 = 0$. A natural step from here would be to find the constraints on these coefficients due to the positivity of entropy production and the corresponding Kubo formulas. This would amount to following the procedures outlined in sections 2.2.3 and 2.2.4 with the modified anisotropic entropy current (3.32), the modified hydrodynamic equations (3.8) and the anisotropic constitutive relations (3.12), (3.35) and (3.37).

Chapter 4

Relativistic magnetohydrodynamics

4.1 Introduction

In a macroscopic system, near-equilibrium phenomena can often be described by classical hydrodynamics. When the microscopic theory contains weakly coupled $U(1)$ gauge fields, long-range correlations mediated by those fields are possible. Maxwell's equations in matter give an effective description of such correlations in terms of classical gauge fields. These equations are useful when the coupling between electromagnetic and thermal/mechanical degrees of freedom can be neglected. We would like to understand the effective description of relativistic systems in which macroscopic electromagnetic degrees of freedom are coupled to the macroscopic thermal and mechanical degrees of freedom. This amounts to coupling Maxwell's equations in matter to hydrodynamic equations. When the matter is electrically conducting and electric fields are neglected, such classical effective theory is usually called magnetohydrodynamics (MHD).

Our motivation it two-fold. From a fundamental point of view, a number of recent developments in relativistic hydrodynamics have pushed the boundaries of the "traditional" theory, as described for example in the classic textbook [1]. These include: a systematic derivative expansion in hydrodynamics [15], an equivalence between hydrodynamics and black hole dynamics [68], the manifestation of chiral anomalies in hydrodynamic equations [16], the relevance of partition functions [19, 18], elucidation of the role of the entropy current [24, 25], new insights into relativistic hydrodynamic turbulence [69], convergence properties of the hydrodynamic expansion [20], and a classification of hydrodynamic transport coefficients [26]. It is reasonable to expect that the above insights will also lead to an improved understanding of the "traditional" MHD. For example, there does not appear to be an agreement in the current literature on such basic question as the number of transport coefficients in MHD.

From an applied point of view, recent years have seen relativistic hydrodynamics expand from its traditional areas of astrophysical plasmas and hot subnuclear matter into the domain of condensed matter physics. Examples include transport near relativistic quantum critical points [10], in graphene [57, 13] and in Weyl semi-metals [14]. For conducting matter, MHD is a natural extension of such hydrodynamic models.

In what follows, we will outline the construction of classical relativistic hydrodynamics with dynamical electromagnetic fields, starting from equilibrium thermodynamics. In order to write down the hydrodynamic equations, we will assume that the system is locally in thermal equilibrium. We will further assume that the departures from local equilibrium may be implemented through a derivative expansion such that the parameters which characterize the equilibrium (temperature, chemical potential, magnetic field, fluid velocity) vary slowly in space and time. At one-derivative order, transport coefficients such as viscosity and electrical conductivity appear in the constitutive relations. We are not aware of previous treatments that list all one-derivative terms in the constitutive relations of magnetohydrodynamics.

For parity-preserving conducting fluids in magnetic field, we find eleven transport coefficients at one-derivative order. One transport coefficient is thermodynamic, and determines the angular momentum of charged fluid induced by the magnetic field. Three transport coefficients are non-equilibrium and non-dissipative: these are the two Hall viscosities (transverse and longitudinal), and one Hall conductivity. There are also seven non-equilibrium dissipative transport coefficients: two electrical conductivities (transverse and longitudinal), two shear viscosities (transverse and longitudinal), and three bulk viscosities. The constitutive relations for the energy-momentum tensor are given in eqs. (4.9), (4.19), and for the current in eqs. (4.10), (4.20). The dissipative coefficients have to satisfy the inequalities in eq. (4.27) imposed by the positivity of entropy production, or alternatively by the positivity of the spectral function. As a simple application of the hydrodynamic equations, we study eigenmodes of small oscillations near thermal equilibrium in constant magnetic field.

We start in Section 4.2 with a discussion of equilibrium thermodynamics in the presence of external electromagnetic and gravitational fields. In Section 4.3, we will discuss hydrodynamics, again when electromagnetic and gravitational fields are external. The magnetic fields are taken as "large" and electric fields as "small" in the sense of the derivative expansion. The smallness of the electric field is due to electric screening. We expect the hydrodynamic description only to be valid for $B \ll T^2$, otherwise new non-hydrodynamic degrees of freedom (such as those associated with Landau levels) must be taken into account. Our procedure will improve on existing studies by taking into account the effects of polarization (magnetic, electric, or both), electric fields, and by enumerating all transport coefficients at leading order in derivatives. In Section 4.4 we discuss hydrodynamics with dynamical electromagnetic fields, as an extension of hydrodynamics with fixed electromagnetic fields. As a simple example, one can study Alfvén and magnetosonic waves in a neutral state (including their damping and polarization), and waves in a dynamically charged (but overall electrically neutral) state. We compare our results with the recent "dual" formulation of MHD in Section 4.5, and with some of the previous studies of transport coefficients of relativistic fluids in magnetic field in the Appendix.

4.2 Thermodynamics

Let us start with equilibrium thermodynamics. For a system in equilibrium subject to an external non-dynamical gauge field A_μ and an external non-dynamical metric $g_{\mu\nu}$, we write the logarithm of the partition function $W_s = -i \ln Z$ as

$$W_s[g, A] = \int d^{d+1}x \sqrt{-g}\, \mathcal{F}\,, \tag{4.1}$$

and we will call $\mathcal{F}$ the free energy density. [Conventions: metric is mostly plus, $\epsilon^{0123}{=}1/\sqrt{-g}$.] For a system with short-range correlations in equilibrium and for external sources A and g which only vary on scales much longer than the correlation length, $\mathcal{F}$ is a local function of the external sources, and W_s is extensive in the thermodynamic limit. The density $\mathcal{F}$ may then be written as an expansion in derivatives of the external sources [19, 18]. The current J^μ (defined by varying W_s with respect to the gauge field) and the energy-momentum tensor $T^{\mu\nu}$ (defined by varying W_s with respect to the metric) automatically satisfy

$$\nabla_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda \,, \tag{4.2a}$$

$$\nabla_\mu J^\mu = 0 \,. \tag{4.2b}$$

owing to gauge- and diffeomorphism-invariance of $W_s[g, A]$. The object $W_s[g, A]$ is the generating functional of static (zero frequency) correlation functions of $T^{\mu\nu}$ and J^μ in equilibrium. Of course, the conservation laws (4.2) are also true out of equilibrium, being a consequence of gauge- and diffeomorphism-invariance in the microscopic theory.

Being in equilibrium means that there exists a timelike Killing vector V such that the Lie derivative of the sources with respect to V vanishes. The equilibrium temperature T, velocity u^α and the chemical potential μ are functions of the Killing vector and the external sources [19, 18]

$$T = \frac{1}{\beta_0 \sqrt{-V^2}} \,, \qquad u^\mu = \frac{V^\mu}{\sqrt{-V^2}} \,, \qquad \mu = \frac{V^\mu A_\mu + \Lambda_V}{\sqrt{-V^2}} \,. \tag{4.3}$$

Here β_0 is a constant setting the normalization of temperature, and Λ_V is a gauge parameter which ensures that μ is gauge-invariant [38]. The electromagnetic field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ can be decomposed in 3+1 dimensions as

$$F_{\mu\nu} = u_\mu E_\nu - u_\nu E_\mu - \epsilon_{\mu\nu\rho\sigma} u^\rho B^\sigma \,, \tag{4.4}$$

where $E_\mu \equiv F_{\mu\nu} u^\nu$ is the electric field, and $B^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}$ is the magnetic field, satisfying $u{\cdot}E = u{\cdot}B = 0$. The decomposition (4.4) is just an identity, true for any antisymmetric $F_{\mu\nu}$ and any timelike unit u^μ. Electric and magnetic fields are not independent, but are related by the "Bianchi identity" $\epsilon^{\mu\nu\alpha\beta}\nabla_\nu F_{\alpha\beta} = 0$, which in equilibrium becomes

$$\nabla{\cdot}B = B{\cdot}a - E{\cdot}\Omega \,, \tag{4.5a}$$

$$u_\mu \epsilon^{\mu\nu\rho\sigma} \nabla_\rho E_\sigma = u_\mu \epsilon^{\mu\nu\rho\sigma} E_\rho a_\sigma \,. \tag{4.5b}$$

Here $\Omega^\mu \equiv \epsilon^{\mu\nu\alpha\beta} u_\nu \nabla_\alpha u_\beta$ is the vorticity and $a^\mu \equiv u^\lambda \nabla_\lambda u^\mu$ is the acceleration. In equilibrium, the acceleration is related to temperature by $\partial_\lambda T = -T a_\lambda$. Relations (4.5) are curved-space versions of the familiar flat-space equilibrium identities $\boldsymbol{\nabla}{\cdot}\mathbf{B} =$

0 and $\nabla \times \mathbf{E} = 0$.

In order to write down the density $\mathcal{F}$ in the derivative expansion, we need to specify the derivative counting of the external sources A and g. The natural derivative counting for the metric is $g \sim O(1)$ (assuming we are interested in transport phenomena in flat space), while the derivative counting for A depends on the physical system under consideration.

As an example, consider an insulator, such as a system made out of particles which carry electric/magnetic dipole moments, but no electric charges. In such a system, there is no conserved electric charge, and the above μ is not a relevant thermodynamic variable. If we are interested in thermodynamics of such a system subject to external electric and magnetic fields, we are free to choose $B \sim O(1)$ and $E \sim O(1)$ in the derivative expansion. The free energy density is then

$$\mathcal{F} = p(T, E^2, E{\cdot}B, B^2) + O(\partial)\,. \tag{4.6}$$

The leading-order term is the pressure [46], whose dependence on E and B encodes the electric, magnetic, and mixed susceptibilities. For the list of $O(\partial)$ contributions to $\mathcal{F}$, see ref. [70].

As another example, consider a system that has electrically charged degrees of freedom (a conductor), such that μ gives a non-negligible contribution to thermodynamics. In equilibrium, $\partial_\lambda \mu = E_\lambda - \mu a_\lambda$ is satisfied identically, which suggests that counting $\mu \sim O(1)$ leads to $E \sim O(\partial)$. This is a manifestation of electric screening. The magnetic field, on the other hand, may still be counted as $O(1)$. The counting $B \sim O(1)$ and $E \sim O(\partial)$ is the relevant derivative counting for MHD. The free energy density is then

$$\mathcal{F} = p(T, \mu, B^2) + \sum_{n=1}^{5} M_n(T, \mu, B^2) s_n^{(1)} + O(\partial^2)\,, \tag{4.7}$$

where $s_n^{(1)}$ are $O(\partial)$ gauge- and diffeomorphism-invariants, and the coefficients M_n need to be determined by the microscopic theory, just like the pressure p. Following ref. [70], we list the invariants $s_n^{(1)}$ in Table 4.1. The rows labeled C, P, T indicate the eigenvalue of the invariant under charge conjugation, parity, and time reversal. The last row shows the weight w of the invariant under a local rescaling of the metric: $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{-2\varphi} g_{\mu\nu}$, and $s_n \to \tilde{s}_n = e^{w\varphi} s_n$. The invariant $s_3^{(1)}$ does not transform homogeneously under the rescaling, and can not appear in a conformally invariant generating functional. Hence, we expect that in a conformal theory $M_3 = 0$.

n	1	2	3	4	5
$s_n^{(1)}$	$B^\mu \partial_\mu(\frac{B^2}{T^4})$	$\epsilon^{\mu\nu\rho\sigma} u_\mu B_\nu \nabla_\rho B_\sigma$	$B{\cdot}a$	$B{\cdot}\Omega$	$B{\cdot}E$
C	$-$	$+$	$-$	$-$	$+$
P	$-$	$-$	$-$	$+$	$-$
T	$-$	$+$	$-$	$+$	$-$
W	3	5	n/a	3	4

Table 4.1: Independent non-zero $O(\partial)$ invariants in equilibrium in 3+1 dimensions.

The coefficient M_5 is the usual magneto-electric (or electro-magnetic) susceptibility; similarly M_4 may be termed magneto-vortical susceptibility. For the rest of this chapter, we will adopt the derivative counting $B \sim O(1)$ and $E \sim O(\partial)$, as is appropriate for MHD.

As an example, consider a parity-invariant theory in magnetic field. The only $O(\partial)$ thermodynamic coefficient is the magneto-vortical susceptibility $M_\Omega \equiv M_4$, which affects $\langle T^{\mu\nu}\rangle$ and $\langle J^\mu\rangle$ when there is non-zero vorticity, and higher-point equilibrium correlation functions of $T^{\mu\nu}$ and J^μ when there is no vorticity. We define static (zero frequency) correlation functions of $T^{\mu\nu}$ and J^μ by varying the generating functional (4.1) with respect to $g_{\mu\nu}$ and A_μ in the standard fashion. For example, in flat space at constant temperature T_0, constant chemical potential μ_0, and constant magnetic field B_0 in the z-direction, one finds the following static correlation functions at small momentum

$$\langle T^{tx} J^z\rangle = -k_x k_z M_\Omega\,, \qquad \langle T^{tx} T^{yz}\rangle = -iB_0 k_z M_\Omega\,. \tag{4.8}$$

The first expression may be used to evaluate the magneto-vortical susceptibility M_Ω in a system that is not subject to magnetic field, and is not rotating.

4.3 Hydrodynamics with external electromagnetic fields

4.3.1 Constitutive relations

Hydrodynamics is conventionally formulated as an extension of thermodynamics, in the sense that hydrodynamic variables are inherited from the thermodynamic parameters. This is a strong assumption, and we expect the hydrodynamic description only to be valid for $B \ll T^2$, otherwise new non-hydrodynamic degrees of freedom (such as those associated with Landau levels) must be taken into account. Let us start by taking E and B fields as external and non-dynamical. In hydrodynamics, the thermodynamic variables T, u^α, and μ are promoted to time-dependent quantities. Out of equilibrium, they no longer have a microscopic definition, but are merely auxiliary variables used to build the non-equilibrium energy-momentum tensor and the current. The expressions of $T^{\mu\nu}$ and J^μ in terms of the auxiliary variables T, u^α, and μ are called constitutive relations; they contain both thermodynamic contributions (coming from the variation of $\mathcal{F}$), and non-equilibrium contributions (such as the viscosity). It is worth noting that thermodynamic contributions and non-equilibrium contributions to the constitutive relations may appear at the same order in the derivative expansion. The constitutive relations are then used together with the conservation laws (4.2) to find the energy-momentum tensor and the current. While in thermodynamics Eqs. (4.2) are mere identities reflecting the symmetries of W_s, solving Eqs. (4.2) in hydrodynamics can be a challenging endeavour leading to rich physics.

We will write the energy-momentum tensor using the decomposition with respect to the timelike velocity vector u^μ,

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + \mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu + \mathcal{T}^{\mu\nu}\,, \tag{4.9}$$

where $\Delta^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$ is the transverse projector, $\mathcal{Q}^\mu$ is transverse to u_μ, and $\mathcal{T}^{\mu\nu}$ is transverse to u_μ, symmetric, and traceless. Explicitly, the coefficients are $\mathcal{E} \equiv u_\mu u_\nu T^{\mu\nu}$, $\mathcal{P} \equiv \frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu}$, $\mathcal{Q}_\mu \equiv -\Delta_{\mu\alpha}u_\beta T^{\alpha\beta}$ and $\mathcal{T}_{\mu\nu} \equiv \frac{1}{2}(\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\nu\alpha}\Delta_{\mu\beta} - \frac{2}{3}\Delta_{\mu\nu}\Delta_{\alpha\beta})T^{\alpha\beta}$. Similarly, we will write the current as

$$J^\mu = \mathcal{N} u^\mu + \mathcal{J}^\mu\,, \tag{4.10}$$

where the charge density is $\mathcal{N} \equiv -u_\mu J^\mu$, and the spatial current is $\mathcal{J}_\mu \equiv \Delta_{\mu\lambda}J^\lambda$.

Using the equilibrium free energy (4.7), one can isolate $O(1)$ and $O(\partial)$ contributions to the energy-momentum tensor and the current:

$$\begin{aligned}
\mathcal{E} &= \epsilon(T,\mu,B^2) + f_{\mathcal{E}}\,,\\
\mathcal{P} &= \Pi(T,\mu,B^2) + f_{\mathcal{P}}\,,\\
\mathcal{N} &= n(T,\mu,B^2) + f_{\mathcal{N}}\,,\\
\mathcal{T}^{\mu\nu} &= \alpha_{\rm BB}(T,\mu,B^2)\left(B^\mu B^\nu - \tfrac{1}{3}\Delta^{\mu\nu}B^2\right) + f_{\mathcal{T}}^{\mu\nu}\,,
\end{aligned}$$

where $\epsilon = -p + T(\partial p/\partial T) + \mu(\partial p/\partial\mu)$, $\Pi = p - \frac{2}{3}\alpha_{\rm BB}B^2$, $n = \partial p/\partial\mu$, and the magnetic susceptibility is $\alpha_{\rm BB} = 2\partial p/\partial B^2$. The terms $f_{\mathcal{E}}$, $f_{\mathcal{P}}$, $f_{\mathcal{N}}$, $f_{\mathcal{T}}^{\mu\nu}$, $\mathcal{Q}^\mu$, and $\mathcal{J}^\mu$ are all $O(\partial)$, and contain both equilibrium and non-equilibrium contributions, $f_{\mathcal{E}} = \bar{f}_{\mathcal{E}} + f_{\mathcal{E}}^{\text{non-eq.}}$ etc, where the bar denotes $O(\partial)$ contributions coming from the variation of W_s.

4.3.2 Field redefinitions

Out of equilibrium, the variables T, u^α, and μ may be redefined. Such a redefinition is often referred to as a choice of "frame", see section 2.2.1 for an introduction to frame transformations. Consider changing the hydrodynamic variables to $T' = T + \delta T$, $u'^\alpha = u^\alpha + \delta u^\alpha$, $\mu' = \mu + \delta\mu$, where δT, δu^α, and $\delta\mu$ are $O(\partial)$. The same energy-momentum tensor and the current may be expressed either in terms of T, u^α, μ, or in terms of T', u'^α, μ' (note that $B^2 = B'^2 + O(\partial^2)$). Physical transport coefficients must be derived from $O(\partial)$ quantities which are invariant under such changes of hydrodynamic variables. A direct evaluation shows that the following combinations are invariant under "frame" transformations:

$$f \equiv f_{\mathcal{P}} - \left(\frac{\partial\Pi}{\partial\epsilon}\right)_n f_{\mathcal{E}} - \left(\frac{\partial\Pi}{\partial n}\right)_\epsilon f_{\mathcal{N}}\,, \tag{4.11a}$$

$$\ell \equiv \frac{B^\alpha}{B}\left(\mathcal{J}_\alpha - \frac{n}{\epsilon+p}\mathcal{Q}_\alpha\right)\,, \tag{4.11b}$$

$$\ell_\perp^\mu \equiv \mathbb{B}^{\mu\alpha}\left(\mathcal{J}_\alpha - \frac{n}{\epsilon+p-\alpha_{\rm BB}B^2}\mathcal{Q}_\alpha\right)\,, \tag{4.11c}$$

$$t^{\mu\nu} \equiv f_{\mathcal{T}}^{\mu\nu} - \left(B^\mu B^\nu - \tfrac{1}{3}\Delta^{\mu\nu}B^2\right)\left[\left(\frac{\partial\alpha_{\rm BB}}{\partial\epsilon}\right)_n f_{\mathcal{E}} + \left(\frac{\partial\alpha_{\rm BB}}{\partial n}\right)_\epsilon f_{\mathcal{N}}\right]\,. \tag{4.11d}$$

Here $\mathbb{B}^{\mu\nu} \equiv \Delta^{\mu\nu} - B^\mu B^\nu/B^2$ is the projector onto a plane orthogonal to both u^μ and B^μ, all thermodynamic derivatives are evaluated at fixed B^2, and $B \equiv \sqrt{B^2}$. When the magnetic susceptibility $\alpha_{\rm BB}$ is T- and μ-independent, the stress $f_{\mathcal{T}}^{\mu\nu}$ is frame-invariant.

As an example, one can choose δT and $\delta\mu$ such that $\mathcal{E}' = \epsilon(T', \mu', B'^2)$, $\mathcal{N}' = n(T', \mu', B'^2)$, and further choose δu^α such that $\mathcal{Q}'_\alpha = 0$. This corresponds to the Landau-Lifshitz frame [1]. The components of energy-momentum tensor and the current take the following form in the Landau-Lifshitz frame:

$$\mathcal{P}' = \Pi(T', \mu', B'^2) + f \,, \tag{4.12a}$$

$$\mathcal{J}'^\mu = \ell_\perp^\mu + \frac{B'^\mu}{B'}\ell \,, \tag{4.12b}$$

$$\mathcal{T}'^{\mu\nu} = \alpha_{\rm BB}(T', \mu', B'^2)\left(B'^\mu B'^\nu - \tfrac{1}{3}\Delta'^{\mu\nu}B'^2\right) + t^{\mu\nu} \,, \tag{4.12c}$$

where the frame invariants are given by eq. (4.11). In the Landau-Lifshitz frame, a non-zero value of the pseudoscalar frame-invariant ℓ indicates a current flowing along the magnetic field. In a constant external magnetic field such currents arise as consequences of chiral anomalies [16]; in an inhomogeneous external field, an electric current flowing along the magnetic field can arise without chiral anomalies, owing to a non-zero magnetic susceptibility.

4.3.3 Thermodynamic frame

The energy-momentum tensor and the current derived from the static generating functional W_s correspond to a different frame, termed in [18] the thermodynamic frame. Taking the variation of the free energy (4.7), one finds the following equilibrium $O(\partial)$ contributions in the thermodynamic frame:

$$\begin{aligned}
&\bar{f}_{\mathcal{E}} = \sum_{n=1}^{5} \epsilon_n s_n^{(1)} \,, \quad \bar{f}_{\mathcal{P}} = \sum_{n=1}^{5} \pi_n s_n^{(1)} \,, \quad \bar{f}_{\mathcal{N}} = \sum_{n=1}^{5} \phi_n s_n^{(1)} \,, \\
&\bar{\mathcal{Q}}^\mu = \sum_{n=1}^{4} \gamma_n v_n^{(1)\mu} \,, \quad \bar{\mathcal{J}}^\mu = \sum_{n=1}^{4} \delta_n v_n^{(1)\mu} \,, \quad \bar{f}_{\mathcal{T}}^{\mu\nu} = \sum_{n=1}^{10} \theta_n t_n^{(1)\mu\nu} \,,
\end{aligned} \tag{4.13}$$

where the bar signifies equilibrium contributions, and the coefficients ϵ_n, π_n, ϕ_n, γ_n, δ_n, θ_n are all $O(1)$ functions of the five thermodynamic coefficients $M_n(T, \mu, B^2)$ and

n	1	2	3	4
$v_n^{(1)\mu}$	$\epsilon^{\mu\nu\rho\sigma}u_\nu\partial_\sigma B_\rho$	$\epsilon^{\mu\nu\rho\sigma}u_\nu B_\rho\partial_\sigma T/T$	$\epsilon^{\mu\nu\rho\sigma}u_\nu B_\rho\partial_\sigma B^2$	$\epsilon^{\mu\nu\rho\sigma}u_\nu E_\rho B_\sigma$

n	$1-5$	6	7	8	9	10
$t_n^{(1)\mu\nu}$	$s_n^{(1)}B^{\langle\mu}B^{\nu\rangle}$	$v_1^{(1)\langle\mu}B^{\nu\rangle}$	$v_2^{(1)\langle\mu}B^{\nu\rangle}$	$v_3^{(1)\langle\mu}B^{\nu\rangle}$	$v_4^{(1)\langle\mu}B^{\nu\rangle}$	$\Omega^{\langle\mu}B^{\nu\rangle}$

Table 4.2: Top: Non-zero transverse $O(\partial)$ vectors that appear in the equilibrium energy flux $\mathcal{Q}^\mu$ and in the equilibrium spatial current $\mathcal{J}^\mu$. The vector $v_4^{(1)\mu}$ is the Poynting vector. Bottom: Non-zero symmetric transverse traceless $O(\partial)$ tensors that appear in the equilibrium stress $\mathcal{T}^{\mu\nu}$. For any two transverse vectors X^μ and Y^μ, the angular brackets stand for $X^{\langle\mu}Y^{\nu\rangle} \equiv X^\mu Y^\nu + X^\nu Y^\mu - \frac{2}{3}\Delta^{\mu\nu}X{\cdot}Y$.

of the magnetic susceptibility $\alpha_{\rm BB} = 2\partial p/\partial B^2$. The explicit expressions are given in Appendix C.1. The one-derivative scalars $s_n^{(1)}$ are given in Table 4.1. The one-derivative vectors $v_n^{(1)\mu}$ and tensors $t_n^{(1)\mu\nu}$ are listed in Table 4.2. The table does not list all $O(\partial)$ vectors and tensors, but only those that appear in the equilibrium $\mathcal{Q}^\mu$ and $\mathcal{T}^{\mu\nu}$. The frame invariants (4.11) then become

$$f = \sum_{n=1}^{5}\Phi_n s_n^{(1)} + f_{\text{non-eq.}}\,, \qquad \ell = \sum_{n=1}^{5}\Lambda_n s_n^{(1)} + \ell_{\text{non-eq.}}\,, \tag{4.14a}$$

$$\ell_\perp^\mu = \sum_{n=1}^{5}\Gamma_n v_n^{(1)\mu} + \ell^\mu_{\perp\text{non-eq.}}\,, \qquad t^{\mu\nu} = \sum_{n=1}^{10}\Theta_n t_n^{(1)\mu\nu} + t^{\mu\nu}_{\text{non-eq.}} \tag{4.14b}$$

In the vector invariant, we have defined $v_5^{(1)\mu} \equiv s_2^{(1)}B^\mu$. The subscript "non-eq" denotes non-equilibrium contributions which by definition vanish in equilibrium. The functions $\Phi_n(T,\mu,B^2)$, $\Lambda_n(T,\mu,B^2)$, $\Gamma_n(T,\mu,B^2)$, $\Theta_n(T,\mu,B^2)$ are non-dissipative thermodynamic transport coefficients. Explicitly,

$$\Phi_n = \pi_n - \epsilon_n\left(\frac{\partial\Pi}{\partial\epsilon}\right)_n - \phi_n\left(\frac{\partial\Pi}{\partial n}\right)_\epsilon\,, \qquad \Lambda_{n\neq 2} = 0\,, \qquad \Lambda_2 = \frac{1}{B}\big(\delta_1 - \frac{n}{\epsilon+p}\gamma_1\big)\,,$$

$$\Gamma_{n\leqslant 4} = \delta_n - \frac{n}{\epsilon{+}p{-}\alpha_{\rm BB}B^2}\gamma_n\,, \qquad \Gamma_5 = -\frac{1}{B^2}\left(\delta_1 - \frac{n}{\epsilon{+}p{-}\alpha_{\rm BB}B^2}\gamma_1\right)\,,$$

$$\Theta_{n\leqslant 5} = \theta_n - \tfrac{1}{2}\epsilon_n\left(\frac{\partial\alpha_{\rm BB}}{\partial\epsilon}\right)_n - \tfrac{1}{2}\phi_n\left(\frac{\partial\alpha_{\rm BB}}{\partial n}\right)_\epsilon\,, \qquad \Theta_{n\geqslant 6} = \theta_n\,.$$

We see that the constitutive relations for energy-momentum tensor and the current contain twenty-one thermodynamic transport coefficients Φ_n, Λ_2, Γ_n, Θ_n. These twenty-one coefficients are not independent, but can all be expressed in terms of only five parameters M_n of the equilibrium generating functional.

Let us now write down the constitutive relations in the thermodynamic frame that is a natural generalization of the Landau-Lifshitz frame. We will define the thermodynamic frame (primed variables) by redefinitions of T, μ, and u^α that give

$$\mathcal{E}' = \epsilon(T', \mu', B'^2) + \bar{f}_{\mathcal{E}} \,, \tag{4.15a}$$

$$\mathcal{N}' = n(T', \mu', B'^2) + \bar{f}_{\mathcal{N}} \,, \tag{4.15b}$$

$$\mathcal{Q}'_\alpha = \bar{\mathcal{Q}}_\alpha \,. \tag{4.15c}$$

In other words, in this thermodynamic frame the coefficients $\mathcal{E}$, $\mathcal{N}$, and $\mathcal{Q}_\alpha$ in the decompositions (4.9), (4.10) take their equilibrium values, derived from the equilibrium generating functional W_s. The other coefficients take the following form in the thermodynamic frame:

$$\mathcal{P}' = \Pi(T', \mu', B'^2) + \bar{f}_{\mathcal{P}} + f_{\text{non-eq.}} \,, \tag{4.15d}$$

$$\mathcal{J}'^\mu = \bar{\mathcal{J}}^\mu + \ell^\mu_{\perp\text{non-eq.}} + \frac{B'^\mu}{B'} \ell_{\text{non-eq.}} \,, \tag{4.15e}$$

$$\mathcal{T}'^{\mu\nu} = \alpha_{\text{BB}}(T', \mu', B'^2) \left(B'^\mu B'^\nu - \tfrac{1}{3}\Delta'^{\mu\nu} B'^2\right) + \bar{f}^{\mu\nu}_{\mathcal{T}} + t^{\mu\nu}_{\text{non-eq.}} \,. \tag{4.15f}$$

4.3.4 Non-equilibrium contributions

With the equilibrium contributions out of the way, the next task is to find the non-equilibrium terms in the constitutive relations (4.14). This amounts to finding one-derivative scalars, vectors (orthogonal both to B_μ and to u_μ), and transverse traceless symmetric tensors that vanish in equilibrium. Note that non-equilibrium contributions (those that vanish in equilibrium) are not the same as dissipative contributions (those that contribute to hydrodynamic entropy production). Every dissipative contribution is non-equilibrium, but not every non-equilibrium contribution is dissipative.

The six independent non-equilibrium one-derivative scalars are given in Table 4.3. The scalar $u^\lambda \partial_\lambda B^2$ is not independent as a consequence of the electromagnetic Bianchi identity, and can be expressed as a combination of $\nabla{\cdot}u$ and $B^\mu B^\nu \nabla_\mu u_\nu$. Three scalar

n	1	2	3	4	5	6
$s^{(1)}_{n\ \text{non-eq.}}$	$u^\lambda\partial_\lambda T$	$u^\lambda\partial_\lambda\mu$	$\nabla{\cdot}u$	$b^\mu b^\nu\nabla_\mu u_\nu$	$b^\lambda E_\lambda - Tb^\lambda\partial_\lambda(\mu/T)$	$b^\lambda a_\lambda + b^\lambda\partial_\lambda T/T$
P	$+$	$+$	$+$	$+$	$-$	$-$

n	1	2	3
$v^{(1)\mu}_{n\ \text{non-eq.}}$	$E^\mu - T\Delta^{\mu\nu}\partial_\nu(\mu/T)$	$a^\mu + \Delta^{\mu\nu}\partial_\nu T/T$	$\sigma^{\mu\nu}b_\nu$
P	$-$	$-$	$+$

Table 4.3: Non-equilibrium scalars and transverse non-equilibrium vectors at $O(\partial)$, written in terms of $b^\mu \equiv B^\mu/B$. In addition to the vectors listed in the table, there are corresponding transverse non-equilibrium vectors $\tilde{v}^{(1)\mu}_{\text{non-eq.}} \equiv \epsilon^{\mu\nu\rho\sigma}u_\nu b_\rho v^{(1)}_{\text{non-eq.}\,\sigma}$. The table also shows the parity of non-equilibrium scalars and vectors. Under time-reversal, the scalars $s^{(1)}_{n\ \text{non-eq.}}$ are T-odd, the vectors $v^{(1)\mu}_{n\ \text{non-eq.}}$ are T-even, and the vectors $\tilde{v}^{(1)\mu}_{n\ \text{non-eq.}}$ are T-odd.

equations of motion $\nabla_\mu J^\mu = 0$, $u_\nu \nabla_\mu T^{\mu\nu} + E_\mu J^\mu = 0$, and $B_\nu \nabla_\mu T^{\mu\nu} + (E\cdot B)(u\cdot J) = 0$ taken at zeroth order provide three relations among the scalars. We choose to eliminate $s^{(1)}_{1\ \text{non-eq.}}$, $s^{(1)}_{2\ \text{non-eq.}}$, and $s^{(1)}_{6\ \text{non-eq.}}$ and write the scalar and pseudo-scalar constitutive relations as

$$f_{\text{non-eq.}} = c_1 s^{(1)}_{3\ \text{non-eq.}} + c_2 s^{(1)}_{4\ \text{non-eq.}} + c_3 s^{(1)}_{5\ \text{non-eq.}}\,,$$

$$\ell_{\text{non-eq.}} = c_4 s^{(1)}_{3\ \text{non-eq.}} + c_5 s^{(1)}_{4\ \text{non-eq.}} + c_6 s^{(1)}_{5\ \text{non-eq.}}\,,$$

with some undetermined transport coefficients c_n.

The independent non-equilibrium transverse one-derivative vectors are given in Table 4.3, where the shear tensor is $\sigma^{\mu\nu} \equiv \Delta^{\mu\alpha}\Delta^{\nu\beta}(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{3}\Delta_{\alpha\beta}\nabla\cdot u)$. We use the vector equation of motion (4.2a) projected with $\mathbb{B}^{\mu\nu}$ at zeroth order to eliminate one of the vectors,[1] and write the vector constitutive relation as

$$\ell^\mu_{\perp\text{non-eq.}} = c_7\, \mathbb{B}^\mu{}_\nu\, v^{(1)\nu}_{1\ \text{non-eq.}} + c_8\, \mathbb{B}^\mu{}_\nu\, v^{(1)\nu}_{3\ \text{non-eq.}} + c_9\, \tilde{v}^{(1)\mu}_{1\ \text{non-eq.}} + c_{10}\, \tilde{v}^{(1)\mu}_{3\ \text{non-eq.}}\,,$$

The tilded vectors are defined as $\tilde{v}^\mu \equiv \epsilon^{\mu\nu\rho\sigma} u_\nu B_\rho v_\sigma / B$.

There is a number of symmetric transverse traceless non-equilibrium one-derivative tensors besides the shear tensor $\sigma^{\mu\nu}$. One such tensor is

$$\tilde{\sigma}^{\mu\nu} \equiv \frac{1}{2B}\left(\epsilon^{\mu\lambda\alpha\beta} u_\lambda B_\alpha \sigma_\beta{}^\nu + \epsilon^{\nu\lambda\alpha\beta} u_\lambda B_\alpha \sigma_\beta{}^\mu\right)\,. \tag{4.16}$$

Other tensors can be formed by $B^{\langle\mu} B^{\nu\rangle} s^{(1)}_{n\ \text{non-eq.}}$, or by symmetrizing B^μ with a transverse non-equilibrium vector. Again, we eliminate three scalars and one vector by the zeroth order equations of motion and write the tensor constitutive relation in terms of $b^\mu \equiv B^\mu / B$ as

$$\begin{aligned} t^{\mu\nu}_{\text{non-eq.}} = c_{11}\sigma^{\mu\nu} &+ b^{\langle\mu} b^{\nu\rangle}\left(c_{12} s^{(1)}_{3\ \text{non-eq.}} + c_{13} s^{(1)}_{4\ \text{non-eq.}} + (c_{14} - c_{15}) s^{(1)}_{5\ \text{non-eq.}}\right) \\ &+ c_{15} b^{\langle\mu} v^{(1)\nu\rangle}_{1\ \text{non-eq.}} + c_{16} b^{\langle\mu} v^{(1)\nu\rangle}_{3\ \text{non-eq.}} + c_{17} b^{\langle\mu} \tilde{v}^{(1)\nu\rangle}_{1\ \text{non-eq.}} + c_{18} b^{\langle\mu} \tilde{v}^{(1)\nu\rangle}_{3\ \text{non-eq.}} + c_{19}\, \tilde{\sigma}^{\mu\nu}\,, \end{aligned}$$

with some undetermined transport coefficients c_n. Thus there are five equilibrium functions $M_n(T, \mu, B^2)$, and nineteen non-equilibrium functions $c_n(T, \mu, B^2)$ that determine one-derivative contributions to the energy-momentum tensor and the current

[1] Namely, using the equation of motion (4.2a) with the constitutive relations for $T^{\mu\nu}$ and J^μ derived from the generating functional $W = \int \sqrt{-g}\, p(T, \mu, B^2) + O(\partial)$. The relation among the vectors that one finds is $v^{(1)\mu}_{2\ \text{non-eq.}} = v^{(1)\mu}_{1\ \text{non-eq.}} n/(\epsilon + p) + O(\partial^2)$.

in strong magnetic field. If the microscopic system is parity-invariant, all thermodynamic coefficients M_n vanish except for M_4. In addition, the dynamical coefficients c_3, c_4, c_5, c_8, c_{10}, c_{14}, c_{15}, c_{17} must vanish by parity invariance. Thus a conducting parity-invariant system in magnetic field has one thermodynamic coefficient M_4, three "electrical conductivities" c_6, c_7, and c_9, and eight "viscosities" c_1, c_2, c_{11}, c_{12}, c_{13}, c_{16}, c_{18}, and c_{19}. We will see later that the Onsager relations impose a relation between c_2, c_{12}, and c_{13}, plus four more relations among the parity-violating coefficients. This leaves eleven transport coefficients (one thermodynamic and ten non-equilibrium) for a conducting parity-invariant system in magnetic field in 3+1 dimensions. In a conformal theory, the tracelessness condition[2] will in addition impose $c_1 = c_2 = 0$.

The constitutive relations may be simplified further if we note that the shear tensor can be decomposed with respect to the magnetic field as

$$\sigma^{\mu\nu} = \sigma_\perp^{\mu\nu} + (b^\mu \Sigma^\nu + b^\nu \Sigma^\mu) + \tfrac{1}{2} b^{\langle\mu} b^{\nu\rangle} \left(3S_4 - S_3\right) \, . \tag{4.17}$$

Here $\sigma_\perp^{\mu\nu} \equiv \frac{1}{2}\left(\mathbb{B}^{\mu\alpha}\mathbb{B}^{\nu\beta} + \mathbb{B}^{\nu\alpha}\mathbb{B}^{\mu\beta} - \mathbb{B}^{\mu\nu}\mathbb{B}^{\alpha\beta}\right)\sigma_{\alpha\beta}$ is traceless, $\Sigma^\mu \equiv \mathbb{B}^{\mu\lambda}\sigma_{\lambda\rho}b^\rho$, and both are orthogonal to the magnetic field B_μ. The scalars are $S_3 \equiv \nabla{\cdot}u$ and $S_4 \equiv b^\mu b^\nu \nabla_\mu u_\nu$. The tensor (4.16) then becomes

$$\tilde{\sigma}^{\mu\nu} = \tilde{\sigma}_\perp^{\mu\nu} + \tfrac{1}{2}\left(b^\mu \tilde{\Sigma}^\nu + b^\nu \tilde{\Sigma}^\mu\right) \, , \tag{4.18}$$

where $\tilde{\sigma}_\perp^{\mu\nu}$ is transverse to both u_μ and B_μ, symmetric, and traceless.

For completeness, let us summarize the constitutive relations for a parity-invariant theory in the thermodynamic frame. Defining $M_\Omega \equiv M_4$, the energy-momentum

[2] In a conformal theory subject to external fields $g_{\mu\nu}$ and A_μ, the trace of the energy-momentum tensor receives an anomalous contribution $T^\mu_{\ \mu} = \kappa F^2 + O(\partial^4)$, where κ is a theory-dependent constant that counts the number of charged degrees of freedom, and the terms $O(\partial^4)$ are due to curvature invariants. It was shown in ref. [61] that the conformal anomaly may be captured by a certain local term in the hydrostatic generating functional, which for our purposes amounts to a term in $p(T, \mu, B^2)$ proportional to κ.

tensor is given by eq. (4.9) with the following coefficients:

$$\mathcal{E} = -p + T\, p_{,T} + \mu\, p_{,\mu} + \big(TM_{\Omega,T} + \mu M_{\Omega,\mu} - 2M_\Omega\big)\, B{\cdot}\Omega\,, \tag{4.19a}$$

$$\mathcal{P} = p - \tfrac{4}{3}\, p_{,B^2} B^2 - \tfrac{1}{3}(M_\Omega + 4M_{\Omega,B^2}B^2) B{\cdot}\Omega - \zeta_1 \nabla{\cdot}u - \zeta_2 b^\mu b^\nu \nabla_\mu u_\nu\,, \tag{4.19b}$$

$$\begin{aligned}\mathcal{Q}^\mu = & -M_\Omega \epsilon^{\mu\nu\rho\sigma} u_\nu \partial_\sigma B_\rho + (2M_\Omega - TM_{\Omega,T} - \mu M_{\Omega,\mu})\epsilon^{\mu\nu\rho\sigma} u_\nu B_\rho \partial_\sigma T/T \\ & - M_{\Omega,B^2} \epsilon^{\mu\nu\rho\sigma} u_\nu B_\rho \partial_\sigma B^2 + (-2p_{,B^2} + M_{\Omega,\mu} - 2M_{\Omega,B^2} B{\cdot}\Omega)\epsilon^{\mu\nu\rho\sigma} u_\nu E_\rho B_\sigma \\ & + M_\Omega \epsilon^{\mu\nu\rho\sigma} \Omega_\nu E_\rho u_\sigma\,,\end{aligned} \tag{4.19c}$$

$$\begin{aligned}\mathcal{T}^{\mu\nu} = & \, 2p_{,B^2}\left(B^\mu B^\nu - \tfrac{1}{3}\Delta^{\mu\nu}B^2\right) + M_{\Omega,B^2} B^{\langle\mu}B^{\nu\rangle} B{\cdot}\Omega + M_\Omega B^{\langle\mu}\Omega^{\nu\rangle} \\ & - \eta_\perp \sigma_\perp^{\mu\nu} - \eta_\parallel (b^\mu \Sigma^\nu + b^\nu \Sigma^\mu) - b^{\langle\mu}b^{\nu\rangle}\left(\eta_1 \nabla{\cdot}u + \eta_2 b^\alpha b^\beta \nabla_\alpha u_\beta\right) \\ & - \tilde{\eta}_\perp \tilde{\sigma}_\perp^{\mu\nu} - \tilde{\eta}_\parallel (b^\mu \tilde{\Sigma}^\nu + b^\nu \tilde{\Sigma}^\mu)\,,\end{aligned} \tag{4.19d}$$

and the current is given by eq. (4.10) with the following coefficients:

$$\mathcal{N} = p_{,\mu} + M_{\Omega,\mu} B{\cdot}\Omega - m{\cdot}\Omega\,, \tag{4.20a}$$

$$\mathcal{J}^\mu = \epsilon^{\mu\nu\rho\sigma} u_\nu \nabla_\rho m_\sigma + \epsilon^{\mu\nu\rho\sigma} u_\nu a_\rho m_\sigma + \left(\sigma_\perp \mathbb{B}^{\mu\nu} + \sigma_\parallel \frac{B^\mu B^\nu}{B^2}\right) V_\nu + \tilde{\sigma}\, \tilde{V}^\mu\,. \tag{4.20b}$$

The current is written in terms of the magnetic polarization vector

$$m^\mu = \big(2\, p_{,B^2} + 2M_{\Omega,B^2} B{\cdot}\Omega\big)\, B^\mu + M_\Omega \Omega^\mu\,, \tag{4.21}$$

while the electric polarization vector vanishes at leading order in a parity-invariant system. The comma subscript denotes the derivative with respect to the argument that follows. Note that we are keeping $O(\partial^2)$ thermodynamic terms in the constitutive relations (coming from the variation of $M_4 s_4^{(1)}$) that are needed to ensure that the conservation laws (4.2) are satisfied identically for time-independent background fields. In writing down the constitutive relations (4.19), (4.20), we have relabeled the non-equilibrium transport coefficients as $\zeta_1 \equiv -c_1$, $\zeta_2 \equiv -c_2$, $\sigma_\parallel \equiv c_6$, $\sigma_\perp \equiv c_7$, $\tilde{\sigma} \equiv c_9$, $\eta_\perp \equiv -c_{11}$, $\eta_\parallel \equiv -c_{11} - c_{16}$, $\eta_1 \equiv -c_{12} + \frac{1}{2}c_{11} + \frac{2}{3}c_{16}$, $\eta_2 \equiv -c_{13} - \frac{3}{2}c_{11} - 2c_{16}$, $\tilde{\eta}_\parallel \equiv -c_{18} - \frac{1}{2}c_{19}$, $\tilde{\eta}_\perp \equiv -c_{19}$, and defined $V^\mu \equiv E^\mu - T\Delta^{\mu\nu}\partial_\nu(\mu/T)$. The coefficients $\sigma_\perp$, $\sigma_\parallel$ are the transverse and longitudinal conductivities, and $\eta_\perp$, $\eta_\parallel$ are the transverse and longitudinal shear viscosities. The coefficients ζ_1, ζ_2, η_1 and η_2 may all be called "bulk viscosities", of which only three are independent due to the Onsager relation.

The coefficients $\tilde{\eta}_\perp$, $\tilde{\eta}_\parallel$ are the two Hall viscosities, and $\tilde{\sigma}$ is the Hall conductivity.[3]

When the external electromagnetic field vanishes, the system becomes isotropic, and we expect to recover the constitutive relations of the standard isotropic hydrodynamics, with shear viscosity η, bulk viscosity ζ, and electrical conductivity σ. Thus as $B \to 0$ we expect $\eta_\perp = \eta_\parallel = -2\eta_1 = \frac{2}{3}\eta_2 = \eta$, $\tilde{\eta}_\perp = \tilde{\eta}_\parallel = 0$, $\zeta_1 = \zeta$, $\zeta_2 = 0$, $\sigma_\perp = \sigma_\parallel = \sigma$, $\tilde{\sigma} = 0$.

4.3.5 Eigenmodes

As a simple application of the hydrodynamic equations (4.2) together with the constitutive relations (4.19), (4.20), one can study the eigenmodes of small oscillations about the thermal equilibrium state. We set the external sources to zero, and linearize the hydrodynamic equations near the flat-space equilibrium state with constant $T = T_0$, $\mu = \mu_0$, $u^\alpha = (1, \mathbf{0})$, and $B^\alpha = (0, 0, 0, B_0)$. Taking the fluctuating hydrodynamic variables proportional to $\exp(-i\omega t + i\mathbf{k}{\cdot}\mathbf{x})$, the source-free system admits five eigenmodes, two gapped ($\omega(\mathbf{k}{\to}0) \neq 0$), and three gapless ($\omega(\mathbf{k}{\to}0) = 0$). The frequencies of the gapped eigenmodes are

$$\omega = \pm \frac{B_0 n_0}{w_0} - \frac{iB_0^2}{w_0}\,(\sigma_\perp \pm i\tilde{\sigma}) - iD_c k^2\,, \tag{4.22}$$

where $w_0 \equiv \epsilon_0 + p_0$ is the equilibrium enthalpy density, and we have taken $\alpha_{\rm BB} B_0^2 \ll w_0$, $M_{\Omega,\mu} B_0^2 \ll w_0$ in the hydrodynamic regime $B_0 \ll T_0^2$. As the imaginary part of the eigenfrequency must be negative for stability, this implies $\sigma_\perp > 0$. The mode has a circular polarization (at $k = 0$), with δu_x and δu_y oscillating with a $\pi/2$ phase difference. The analogous mode in 2+1 dimensional hydrodynamics was christened the hydrodynamic cyclotron mode in ref. [10], which also explored its implications for transport near two-dimensional quantum critical points.

For momenta $\mathbf{k} \parallel \mathbf{B}_0$, the three gapless eigenmodes are the two sound waves, and

[3] The actual Hall conductivity, measured as a response to external electric field, must be obtained after the hydrodynamic equations with the constitutive relations (4.19), (4.20) have been solved. Doing so in a state with constant charge density n_0 and magnetic field B_0 gives the Hall conductivity n_0/B_0, as expected from elementary considerations of boosting the state in the plane transverse to $\mathbf{B}_0$. See eq. (4.32c) below.

one diffusive mode. The eigenfrequencies in the small momentum limit are

$$\omega = \pm k v_s - i\frac{\Gamma_{s,\parallel}}{2}k^2\,, \tag{4.23a}$$

$$\omega = -iD_\parallel k^2\,, \tag{4.23b}$$

where v_s is the speed of sound. As in ref. [4], we can write the coefficients in terms of the elements of the susceptibility matrix in the grand canonical ensemble. The non-zero elements of the 3×3 susceptibility matrix are $\chi_{11} = T(\partial\epsilon/\partial T)_{\mu/T}$, $\chi_{13} = \chi_{31} = (\partial\epsilon/\partial\mu)_T$, $\chi_{33} = (\partial n/\partial\mu)_T$, and $\chi_{22} = w_0$, with derivatives evaluated at constant B^2 in equilibrium. The longitudinal diffusion constant is

$$D_\parallel = \frac{\sigma_\parallel\, w_0^2}{n_0^2\chi_{11} + w_0^2\chi_{33} - 2n_0 w_0\chi_{13}}\,.$$

The positivity of the diffusion constant implies $\sigma_\parallel > 0$. The speed of sound squared expressed in terms of the elements of the susceptibility matrix is given by

$$v_s^2 = \frac{n_0^2\chi_{11} + w_0^2\chi_{33} - 2n_0 w_0\chi_{13}}{\det(\chi)}\,,$$

and the damping coefficient is

$$\Gamma_{s,\parallel} = \frac{1}{w_0}\left(\tfrac{4}{3}(\eta_1+\eta_2) + \zeta_1 + \zeta_2\right) + \frac{\sigma_\parallel\, w_0}{\det(\chi)}\frac{(n_0\chi_{11} - w_0\chi_{13})^2}{n_0^2\chi_{11} + w_0^2\chi_{33} - 2n_0 w_0\chi_{13}}\,.$$

The expression for v_s and $D_\parallel$ in terms of the thermodynamic functions formally look the same as in hydrodynamics without external $O(1)$ magnetic fields [4]. All of v_s, $\Gamma_{s,\parallel}$, and $D_\parallel$ depend on B_0 through $p = p(T,\mu,B^2)$ and the transport coefficients.

For momenta $\mathbf{k}\perp\mathbf{B}_0$, the three gapless eigenmodes include two diffusive modes, and one "subdiffusive" mode with a quartic dispersion relation,

$$\omega = -iD_\perp k^2\,, \tag{4.24a}$$

$$\omega = -i\frac{\eta_\parallel k^2}{w_0}\,, \tag{4.24b}$$

$$\omega = -i\frac{\eta_\perp k^4}{B_0^2\,\chi_{33}}\,. \tag{4.24c}$$

The transverse diffusion constant is determined by the transverse resistivity. We define the 2×2 conductivity matrix in the plane transverse to $\mathbf{B}_0$ as $\sigma_{ab} \equiv \sigma_\perp \delta_{ab} + \left(\frac{n_0}{|\mathbf{B}_0|} + \tilde{\sigma}\right) \epsilon_{ab}$, and the corresponding resistivity matrix as $\rho_{ab} \equiv (\sigma^{-1})_{ab} = \rho_\perp \delta_{ab} + \tilde{\rho}_\perp \, \epsilon_{ab}$, which defines $\rho_\perp$ and $\tilde{\rho}_\perp$. The transverse diffusion constant is then

$$D_\perp = \frac{w_0^3 \chi_{33}}{\det(\chi) B_0^2} \, \rho_\perp \,,$$

again using $M_{\Omega,\mu} B_0^2 \ll w_0$. Stability of the equilibrium state now implies $\eta_\perp > 0$, $\eta_\parallel > 0$.

For modes propagating at an angle θ with respect to $\mathbf{B}_0$, the gapless modes include sound waves (unless $\theta = \pi/2$), and a diffusive mode. For a fixed value of θ, the small-momentum eigenfrequencies are $\omega = \pm k v_s \cos\theta - \frac{i}{2}\Gamma_s(\theta) k^2$, and $\omega = -iD(\theta)k^2$, where

$$D(\theta) = D_\parallel \cos^2\theta + \frac{n_0^2}{v_s^2 w_0 \chi_{33}} D_\perp \sin^2\theta \,,$$

$$\Gamma_s(\theta) = \Gamma_{s,\parallel} \cos^2\theta + \left(\frac{\eta_\parallel}{w_0} + \frac{(n_0\chi_{13} - w_0\chi_{33})^2}{\chi_{33}\, v_s^2 \, \det(\chi)} D_\perp\right) \sin^2\theta \,.$$

The coefficient D_c in the cyclotron mode eigenfrequency (4.22) at small B_0 is

$$D_c = \left(\pm \frac{i v_s^2 w_0}{2 n_0 B_0} + \frac{(n_0^2 \chi_{11} - w_0^2 \chi_{33}) w_0}{2 n_0^2 \det(\chi)} \sigma + \frac{3\zeta + 7\eta}{6 w_0}\right) \sin^2\theta + \frac{\eta}{w_0} \cos^2\theta + O(B_0) \,.$$

Note that the limits $\theta \to \pi/2$ and $k \to 0$ in the eigenfrequencies do not commute.

4.3.6 Entropy production

The simple flat-space eigenfrequency analysis in the previous subsection imposes certain constraints on non-equilibrium transport coefficients. In order to find more general constraints, one method is to impose a local version of the second law of thermodynamics: the existence of a local entropy current with positive semi-definite divergence for every non-equilibrium configuration consistent with the hydrodynamic equations. We will not attempt to construct the most general entropy current from scratch. Rather, we will use the result of [24, 25] saying that the constraints on transport coefficients derived from the entropy current are the same as those derived from the equilibrium generating functional, plus the inequality constraints on dissipative transport coefficients. We take the entropy current to be

$$S^\mu = S^\mu_{\text{canon}} + S^\mu_{\text{eq.}}\,,$$

where the canonical part of the entropy current is

$$S^\mu_{\text{canon}} = \frac{1}{T}\left(pu^\mu - T^{\mu\nu}u_\nu - \mu J^\mu\right)\,, \tag{4.25}$$

and $S^\mu_{\text{eq.}}$ is found from the equilibrium partition function, as described in [24, 25]. The constraints on transport coefficients follow by demanding $\nabla_\mu S^\mu \geqslant 0$. Using conservation laws (4.2), the divergence of the canonical entropy current is

$$\nabla_\mu S^\mu_{\text{canon}} = \nabla_\mu\left(\frac{p}{T}u^\mu\right) - T^{\mu\nu}\nabla_\mu\frac{u_\nu}{T} + J^\mu\left(\frac{E_\mu}{T} - \partial_\mu\frac{\mu}{T}\right)\,.$$

The $S^\mu_{\text{eq.}}$ part of the entropy current is explicitly built to cancel out the part of $\nabla_\mu S^\mu_{\text{canon}}$ that arises from the equilibrium terms in the constitutive relations, i.e. the terms in $T^{\mu\nu}$ and J^μ derived from the equilibrium generating functional. In fact, ref. [25] has already found $S^\mu_{\text{eq.}}$ in the case when the generating functional contains a contribution proportional to $B{\cdot}\Omega$. We thus focus on non-equilibrium terms, and write the thermodynamic frame constitutive relations (4.15) as $T^{\mu\nu} = T^{\mu\nu}_{\text{eq.}} + T^{\mu\nu}_{\text{non-eq.}}$ and $J^\mu = J^\mu_{\text{eq.}} + J^\mu_{\text{non-eq.}}$. The divergence of the entropy current is then

$$\begin{aligned}
\nabla_\mu S^\mu &= \frac{1}{T}J^\mu_{\text{non-eq.}}\left(E_\mu - T\partial_\mu\frac{\mu}{T}\right) - T^{\mu\nu}_{\text{non-eq.}}\nabla_\mu\frac{u_\nu}{T}\\
&= \frac{1}{T}\left(\ell^\mu_{\perp\text{non-eq.}} + \frac{B^\mu}{B}\ell_{\text{non-eq.}}\right)V_\mu - \frac{1}{T}f_{\text{non-eq.}}\nabla{\cdot}u - \frac{1}{2T}t^{\mu\nu}_{\text{non-eq.}}\sigma_{\mu\nu}\,.
\end{aligned}$$

Using the constitutive relations (4.19), (4.20), this leads to

$$\begin{aligned}
T\nabla_\mu S^\mu = \sigma_\parallel\frac{(B{\cdot}V)^2}{B^2} &+ \sigma_\perp(\mathbb{B}^{\mu\nu}V_\nu)^2 + \tfrac{1}{2}\eta_\perp(\sigma^{\mu\nu}_\perp)^2 + \eta_\parallel\Sigma^2\\
&+ (\zeta_1 - \tfrac{2}{3}\eta_1)S_3^2 + 2\eta_2 S_4^2 + (2\eta_1 + \zeta_2 - \tfrac{2}{3}\eta_2)S_3S_4\,,
\end{aligned} \tag{4.26}$$

where again $S_3 \equiv \nabla{\cdot}u$ and $S_4 \equiv b^\mu b^\nu\nabla_\mu u_\nu$. Demanding $\nabla_\mu S^\mu \geqslant 0$ now gives

$$\sigma_\parallel \geqslant 0\,,\qquad \sigma_\perp \geqslant 0\,,\qquad \eta_\perp \geqslant 0\,,\qquad \eta_\parallel \geqslant 0\,, \tag{4.27a}$$

together with the condition that the quadratic form made out of S_3, S_4 in the second line of eq. (4.26) is non-negative, which implies

$$\eta_2 \geqslant 0\,, \qquad \zeta_1 - \tfrac{2}{3}\eta_1 \geqslant 0\,, \tag{4.27b}$$

$$2\eta_2(\zeta_1 - \tfrac{2}{3}\eta_1) \geqslant \tfrac{1}{4}(2\eta_1 + \zeta_2 - \tfrac{2}{3}\eta_2)^2\,. \tag{4.27c}$$

The coefficients $\tilde{\eta}_\perp$, $\tilde{\eta}_\parallel$, and $\tilde{\sigma}$ do not contribute to entropy production, and are not constrained by the above analysis. Thus, $\tilde{\eta}_\perp$, $\tilde{\eta}_\parallel$, and $\tilde{\sigma}$ are non-equilibrium non-dissipative coefficients.

4.3.7 Kubo formulas

We now follow the argument presented in section 2.2.4. When the microscopic system is time-reversal invariant (i.e. the only source of time-reversal breaking is due to the external magnetic field), transport coefficients can be further constrained by the Onsager relations. The retarded two-point functions of operators O_a and O_b in a time-reversal invariant theory in equilibrium obey

$$G^R_{ab}(\omega, \mathbf{k}, B) = \epsilon_a \epsilon_b\, G^R_{ba}(\omega, -\mathbf{k}, -B)\,, \tag{4.28}$$

where ϵ_a and ϵ_b are time-reversal eigenvalues of the operators O_a and O_b. We take our operators to be various components of $T^{\mu\nu}$ and J^μ, and evaluate the retarded two-point functions by varying one-point functions in the presence of the external source with respect to the source, as explained in section 2.2.4, to find

$$G^R_{T^{\mu\nu}T^{\alpha\beta}} = 2\frac{\delta}{\delta g_{\alpha\beta}}\left(\sqrt{-g}\,T^{\mu\nu}_{\text{on-shell}}[A,g]\right)\,, \quad G^R_{J^{\mu}T^{\alpha\beta}} = 2\frac{\delta}{\delta g_{\alpha\beta}}\left(\sqrt{-g}\,J^{\mu}_{\text{on-shell}}[A,g]\right)\,, \tag{4.29a}$$

$$G^R_{T^{\mu\nu}J^{\alpha}} = \frac{\delta}{\delta A_\alpha} T^{\mu\nu}_{\text{on-shell}}[A,g]\,, \qquad G^R_{J^{\mu}J^{\alpha}} = \frac{\delta}{\delta A_\alpha} J^{\mu}_{\text{on-shell}}[A,g]\,, \tag{4.29b}$$

where the subscript "on-shell" signifies that the corresponding hydrodynamic $T^{\mu\nu}[A,g]$ and $J^\mu[A,g]$ are evaluated on the solutions to (4.2), and the sources δA, δg are set to zero after the variation is taken. The expressions (4.29) are to be understood as

$$\delta(\sqrt{-g}\,T^{\mu\nu}_{\text{on-shell}}) = \tfrac{1}{2} G^R_{T^{\mu\nu}T^{\alpha\beta}}(\omega, \mathbf{k})\,\delta g_{\alpha\beta}(\omega, \mathbf{k})\,,$$

etc. This provides a direct method to evaluate the retarded functions, and allows both to check the Onsager relations and to derive Kubo formulas for transport coefficients.[4] The constraint on transport coefficients we find by demanding that eq. (4.28) holds is

$$3\zeta_2 - 6\eta_1 - 2\eta_2 = 0\,. \tag{4.30}$$

For the rest of this chapter, we will assume that (4.30) holds, which leaves us with ten non-equilibrium transport coefficients for a parity-invariant microscopic system. Using eq. (4.30) to eliminate ζ_2, the inequality constraint in eq. (4.27c) turns into

$$2\eta_2(\zeta_1 - \tfrac{2}{3}\eta_1) \geqslant 4\eta_1^2\,. \tag{4.31}$$

We next list the expressions for transport coefficients in terms of retarded functions evaluated in flat-space equilibrium with external magnetic field in the z direction, as in sec. 4.3.5. In the limit $\mathbf{k} \to 0$ first, $\omega \to 0$ second we find the following Kubo formulas. The two-point function of the longitudinal current J^z gives the longitudinal conductivity,

$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{J^zJ^z}(\omega, \mathbf{k}{=}0) = \sigma_\parallel\,, \tag{4.32a}$$

while the two-point functions of the transverse currents J^x, J^y give the transverse resistivities,

$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{J^xJ^x}(\omega, \mathbf{k}{=}0) = \omega^2\rho_\perp \frac{w_0^2}{B_0^4}\,, \tag{4.32b}$$

$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{J^xJ^y}(\omega, \mathbf{k}{=}0) = \frac{n_0}{B_0} - \omega^2\tilde{\rho}_\perp \frac{w_0^2}{B_0^4}\,\mathrm{sign}(B_0)\,, \tag{4.32c}$$

where the resistivities $\rho_\perp$ and $\tilde{\rho}_\perp$ were defined below eq. (4.24). Alternatively, the resistivities can be found from correlation functions of momentum density,

$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{T_{0x}T_{0x}}(\omega, \mathbf{k}{=}0) = \rho_\perp \frac{w_0^2}{B_0^2}\,, \tag{4.33a}$$

$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{T_{0x}T_{0y}}(\omega, \mathbf{k}{=}0) = -\tilde{\rho}_\perp \mathrm{sign}(B_0)\frac{w_0^2}{B_0^2}\,, \tag{4.33b}$$

[4] Taken at face value, hydrodynamic correlation functions violate Onsager relations at non-zero ω and non-zero k. However these violations do not affect the Kubo formulas and disappear in the limit $B \ll T^2$, which corresponds to the validity regime of hydrodynamics.

assuming $B_0^2 \ll w_0$. The shear viscosities are given by

$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{T^{xy}T^{xy}}(\omega, \mathbf{k}{=}0) = \eta_\perp\,, \tag{4.34a}$$
$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{T^{xy}T^{xx}}(\omega, \mathbf{k}{=}0) = \tilde{\eta}_\perp\,\mathrm{sign}(B_0)\,, \tag{4.34b}$$
$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{T^{xz}T^{xz}}(\omega, \mathbf{k}{=}0) = \eta_\parallel\,, \tag{4.34c}$$
$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{T^{yz}T^{xz}}(\omega, \mathbf{k}{=}0) = \tilde{\eta}_\parallel\,\mathrm{sign}(B_0)\,, \tag{4.34d}$$

while the “bulk” viscosities may be expressed as

$$\tfrac{1}{\omega}\delta_{ij}\mathrm{Im}\, G^R_{T^{ij}T^{xx}}(\omega, \mathbf{k}{=}0) = 3\zeta_1\,, \tag{4.34e}$$
$$\tfrac{1}{3\omega}\delta_{ij}\delta_{kl}\,\mathrm{Im}\, G^R_{T^{ij}T^{kl}}(\omega, \mathbf{k}{=}0) = 3\zeta_1 + \zeta_2\,, \tag{4.34f}$$
$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{O_1O_1} = \zeta_1 - \tfrac{2}{3}\eta_1\,, \tag{4.34g}$$
$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{O_2O_2} = 2\eta_2\,, \tag{4.34h}$$

where $O_1 = \frac{1}{2}(T^{xx} + T^{yy})$, and $O_2 = T^{zz} - \frac{1}{2}(T^{xx} + T^{yy})$. Correlation functions at non-zero momentum may be obtained in a straightforward way from the variational procedure described earlier.

4.3.8 Inequality constraints on transport coefficients

Let us show that the inequality constraints on transport coefficients derived from demanding that the entropy production is non-negative can also be obtained from hydrodynamic correlation functions as outlined in section 2.2.5, without using the entropy current. The argument is based on the fact that the imaginary part of the retarded function $G^R_{OO}(\omega, \mathbf{k})$ must be positive for any Hermitean operator O and $\omega > 0$,

$$\mathrm{Im}\, G^R_{OO}(\omega, \mathbf{k}) \geqslant 0\,. \tag{4.35}$$

See appendix D.4 for details. Now consider the operator $O = aO_1 + bO_2$, with real coefficients a and b, and Hermitean operators O_1, O_2. The inequality (4.35) implies

$$\mathrm{Im}\left[a^2 G^R_{O_1O_1} + ab G^R_{O_1O_2} + ab G^R_{O_2O_1} + b^2 G^R_{O_2O_2}\right] \geqslant 0\,,$$

for $\omega \geqslant 0$. This quadratic form in a, b must be non-negative for all a, b which implies $\mathrm{Im}G^R_{O_1O_1} \geqslant 0$, $\mathrm{Im}G^R_{O_2O_2} \geqslant 0$ together with

$$\left(\mathrm{Im}G^R_{O_1O_1}\right)\left(\mathrm{Im}G^R_{O_2O_2}\right) \geqslant \tfrac{1}{4}\left(\mathrm{Im}G^R_{O_1O_2} + \mathrm{Im}G^R_{O_2O_1}\right)^2 . \tag{4.36}$$

The two terms in the right-hand side of (4.36) can be related by the Onsager relation (4.28). As an example, take $O_1 = \frac{1}{2}(T^{xx} + T^{yy})$, and $O_2 = T^{zz} - \frac{1}{2}(T^{xx} + T^{yy})$. Evaluating the correlation functions at $\mathbf{k} = 0$ and $\omega \to 0$, the inequalities (4.35), (4.36) immediately imply the entropy current constraint (4.27c). The constraints (4.27a), (4.27b) follow directly from the Kubo formulas given in the previous subsection.

4.3.9 Parity violating sector

For a theory that breaks parity microscopically, new terms appear in the hydrodynamic description. A study of parity violating fluids in 2+1 dimensions was done in [17]. We now turn to the study of hydrodynamics of parity-violating hydrodynamics in 3+1 dimensions in the presence of a strong magnetic field.

Constitutive relations

The parity-violating equilibrium terms come from the variation of the parity-violating terms in the generating functional, $M_i(T, \mu, B^2)s_i^{(1)}$ for $i \neq 4$ and appear in Appendix C.1. The spatial current and heat vector do not receive parity-violating contributions to $O(\partial)$.

The hydrodynamic equations take the same form as (4.2), (4.2a) due to gauge and diffeomorphism invariance of the generating functional. The definition of the equilibrium energy-momentum tensor and conserved currents ensure that the equations of motion are satisfied in equilibrium.

For convenience, we rewrite the non-equilibrium parity-violating terms appearing in the constitutive relations

$$f_{\text{non-eq. PV}} = c_3 b \cdot V \,, \tag{4.37a}$$

$$\ell_{\text{non-eq. PV}} = c_4 \nabla \cdot u + c_5 b^\mu b^\nu \nabla_\mu u_\nu \,, \tag{4.37b}$$

$$\ell^\mu_{\perp,\text{non-eq. PV}} = c_8 \Sigma^\mu + c_{10} \tilde{\Sigma}^\mu \,, \tag{4.37c}$$

$$\tau^{\mu\nu}_{\text{non-eq. PV}} = c_{14} b^{\langle\mu} b^{\nu\rangle} b \cdot V + c_{15} b^{(\mu} V_\perp^{\nu)} + c_{17} b^{(\mu} \tilde{V}^{\nu)} \,, \tag{4.37d}$$

where $A^{(\mu}B^{\nu)} = A^\mu B^\nu + A^\nu B^\mu$ and $V_\perp^\mu = \mathbb{B}^{\mu\nu}V_\nu$ where [[equation]] . These coefficients will be subject to four equality constraints coming from the Onsager relations, as well as some inequality constraints coming from the entropy and correlation function arguments. Let us summarize the constitutive relations for a parity-violating theory in the thermodynamic frame

$$
\begin{aligned}
\mathcal{E} = &-p + T\,p_{,T} + \mu\,p_{,\mu} + \left(TM_{\Omega,T} + \mu M_{\Omega,\mu} - 2M_\Omega\right) B{\cdot}\Omega \\
&+ \left(TM_{1,T} + \mu M_{1,\mu} + 4B^2 M_{1,B^2} + T^4 M_{3,B^2} - M_1\right) s_1 \\
&+ \left(TM_{2,T} + \mu M_{2,\mu} - M_2\right) s_2 \\
&+ \frac{4B^2}{T^4}\left(M_1 - TM_{1,T} - \mu M_{1,\mu} - 4B^2 M_{1,B^2} - T^4 M_{3,B^2}\right) s_3 \\
&+ \left(TM_{5,T} + \mu M_{5,\mu} + \frac{4B^2}{T^4}M_{1,\mu} + M_{3,\mu}\right) s_5\,,
\end{aligned}
\tag{4.38}
$$

$$
\begin{aligned}
\mathcal{P} = &\; p - \tfrac{4}{3}p_{,B^2}B^2 - \tfrac{1}{3}(M_\Omega + 4M_{\Omega,B^2}B^2)B{\cdot}\Omega - \tfrac{2}{3}\left(M_2 + 2B^2 M_{2,B^2}\right) s_2 \\
&+ \frac{4B^2}{3T^4}\left(M_1 - TM_{1,T} - \mu M_{1,\mu} - 4B^2 M_{1,B^2} - T^4 M_{3,B^2}\right) s_3 \\
&+ \frac{4B^2}{3T^4}\left(M_{1,\mu} - T^4 M_{5,B^2}\right) s_5 - \zeta_1 \nabla{\cdot}u - \zeta_2 b^\mu b^\nu \nabla_\mu u_\nu + c_3 b\cdot V\,,
\end{aligned}
\tag{4.39}
$$

$$
\begin{aligned}
\mathcal{Q}^\mu = &-M_\Omega \epsilon^{\mu\nu\rho\sigma}u_\nu \partial_\sigma B_\rho + (2M_\Omega - TM_{\Omega,T} - \mu M_{\Omega,\mu})\epsilon^{\mu\nu\rho\sigma}u_\nu B_\rho \partial_\sigma T/T \\
&- M_{\Omega,B^2}\epsilon^{\mu\nu\rho\sigma}u_\nu B_\rho \partial_\sigma B^2 + (M_{\Omega,\mu} - 2p_{,B^2})\epsilon^{\mu\nu\rho\sigma}u_\nu E_\rho B_\sigma\,,
\end{aligned}
\tag{4.40}
$$

$$
\begin{aligned}
\mathcal{T}^{\mu\nu} = &\; 2p_{,B^2}\left(B^\mu B^\nu - \tfrac{1}{3}\Delta^{\mu\nu}B^2\right) + B^{\langle\mu}B^{\nu\rangle}\left(M_{\Omega,B^2}B{\cdot}\Omega + M_{2,B^2}s_2 + (M_{5,B^2} - \tfrac{1}{T}M_{1,\mu})s_5\right) \\
&+ B^{\langle\mu}B^{\nu\rangle}\frac{1}{T^4}\left(TM_{1,T} + \mu M_{1,\mu} + 4B^2 M_{1,B^2} - M_1 + T^4 M_{3,B^2}\right) s_3 + M_\Omega B^{\langle\mu}\Omega^{\nu\rangle} \\
&+ 2M_2 B^{\langle\mu}\epsilon^{\nu\rangle\rho\sigma\alpha}u_\rho \partial_\sigma B_\alpha + \left(TM_{2,T} + \mu M_{2,\mu} - M_2\right) B^{\langle\mu}\epsilon^{\nu\rangle\rho\sigma\alpha}u_\rho \partial_\sigma T/T \\
&+ M_{2,B^2}B^{\langle\mu}\epsilon^{\nu\rangle\rho\sigma\alpha}u_\rho \partial_\sigma B^2 - M_{2,\mu}B^{\langle\mu}\epsilon^{\nu\rangle\rho\sigma\alpha}u_\rho E_\sigma B_\alpha \\
&- \eta_\perp \sigma_\perp^{\mu\nu} - \eta_\parallel (b^\mu \Sigma^\nu + b^\nu \Sigma^\mu) - b^{\langle\mu}b^{\nu\rangle}\left(\eta_1 \nabla{\cdot}u + \eta_2 b^\alpha b^\beta \nabla_\alpha u_\beta - c_{14} b\cdot V\right) \\
&- \tilde{\eta}_\perp \tilde{\sigma}_\perp^{\mu\nu} - \tilde{\eta}_\parallel (b^\mu \tilde{\Sigma}^\nu + b^\nu \tilde{\Sigma}^\mu) + c_{15}(b^\mu V_\perp^\nu + b^\nu V^\mu) + c_{17}(b^\mu \tilde{V}^\nu + b^\nu \tilde{V}^\mu)\,,
\end{aligned}
\tag{4.41}
$$

and the current is given by

$$\mathcal{N} = p_{,\mu} + \nabla{\cdot}p - p{\cdot}a - m{\cdot}\Omega + M_{\Omega,\mu}B{\cdot}\Omega + \left(M_{1,\mu} - T^4 M_{5,B^2}\right) s_1 + M_{2,\mu}s_2$$
$$\left(M_{3,\mu} + TM_{5,T} + \mu M_{5,\mu} + 4B^2 M_{5,B^2}\right) s_3 \,, \tag{4.42a}$$
$$\mathcal{J}^\mu = \epsilon^{\mu\nu\rho\sigma}u_\nu\nabla_\rho m_\sigma + \epsilon^{\mu\nu\rho\sigma}u_\nu a_\rho m_\sigma + \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu\Omega_\rho p_\sigma - \Delta^{\mu\nu}u^\rho\nabla_\rho p_\nu + \sigma_\perp V_\perp^\mu + \tilde{\sigma}\,\tilde{V}^\mu$$
$$+ b^\mu(\sigma_{||} b\cdot V + c_4\nabla\cdot u + c_5 b^\mu b^\nu \nabla_\mu u_\nu) + c_8\Sigma^\mu + c_{10}\tilde{\Sigma}^\mu \,. \tag{4.42b}$$

The current is written in terms of the magnetic polarization vector $m_\mu = \frac{1}{\sqrt{-g}}\frac{\delta W_s}{\delta B^\mu}$

$$\begin{aligned} m^\mu = {} & \left(2\,p_{,B^2} + 2\sum_{n=2}^{5} M_{n,B^2}s_n + \frac{2}{T^4}\left(M_1 - TM_{1,T} - \mu M_{1,\mu} - 4B^2 M_{1,B^2}\right) B{\cdot}\partial T/T\right) B^\mu \\ & + M_\Omega\Omega^\mu + M_3 a^\mu + M_5 E^\mu + M_1\Delta^{\mu\nu}\partial_\nu\frac{B^2}{T^4} + M_{2,\mu}\epsilon^{\mu\nu\rho\sigma}u_\nu B_\rho E_\sigma \\ & + M_{2,B^2}\epsilon^{\mu\nu\rho\sigma}u_\nu B_\rho\partial_\sigma B^2 + (TM_{2,T} + \mu M_{2,\mu} - M_2)\,\epsilon^{\mu\nu\rho\sigma}u_\nu B_\rho\partial_\sigma T/T \\ & + 2M_2\epsilon^{\mu\nu\rho\sigma}u_\nu\partial_\rho B_\sigma \,, \end{aligned} \tag{4.43}$$

and the electric polarization vector $p_\mu = \frac{1}{\sqrt{-g}}\frac{\delta W_s}{\delta E^\mu} = M_5 B_\mu$. The comma subscript denotes the derivative with respect to the argument that follows. Note that we are keeping $\mathrm{O}(\partial^2)$ thermodynamic terms in the current (coming from the variation of $\sum_{n=1}^{5} M_n s_n$ in the generating functional) that are needed to ensure that the conservation laws (2.2) are satisfied to $O(\partial^2)$ for time-independent background fields. Including the $O(\partial^2)$ thermodynamic terms in the energy-momentum tensor will ensure these are satisfied identically, but we omit them here for simplicity.

Kubo formulas

The two point correlation functions of energy-momentum and conserved currents can be found by varying the one-point functions given by the constitutive relations in the presence of external sources with respect to the external sources, as was outlined in 4.3.7. This provides a direct method to evaluate the retarded functions, and allows both to find constrains due to the Onsager relations and to derive Kubo formulas for transport coefficients.

The Kubo fomulas can be found by evaluating the zero spatial momentum, low frequency limit of the retarded functions in flat space equilibrium. The parity violating coefficients are given by

$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{T^{tx}T^{xz}}(\omega,\mathbf{k}{=}0) = \frac{w_0}{B_0}(c_8\tilde{\rho}_\perp + c_{10}\rho_\perp)\,, \tag{4.44a}$$

$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{T^{tx}T^{yz}}(\omega,\mathbf{k}{=}0) = \frac{w_0}{|B_0|}(c_8\rho_\perp - c_{10}\tilde{\rho}_\perp)\,, \tag{4.44b}$$

$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{T^{xz}T^{tx}}(\omega,\mathbf{k}{=}0) = \frac{w_0}{B_0}(c_{15}\tilde{\rho}_\perp + c_{17}\rho_\perp)\,, \tag{4.44c}$$

$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{T^{yz}T^{tx}}(\omega,\mathbf{k}{=}0) = \frac{w_0}{|B_0|}(c_{15}\rho_\perp - c_{17}\tilde{\rho}_\perp)\,, \tag{4.44d}$$

$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{J^zT^{xx}}(\omega,\mathbf{k}{=}0) = c_4\,\mathrm{sign}(B_0)\,, \tag{4.44e}$$

$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{J^zT^{zz}}(\omega,\mathbf{k}{=}0) = (c_4 + c_5)\,\mathrm{sign}(B_0) \tag{4.44f}$$

$$\tfrac{1}{\omega}\delta_{ij}\mathrm{Im}\, G^R_{T^{ij}J^z}(\omega,\mathbf{k}{=}0) = -3c_3\,\mathrm{sign}(B_0)\,, \tag{4.44g}$$

$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{O_2J^z}(\omega,\mathbf{k}{=}0) = -2c_{14}\,\mathrm{sign}(B_0)\,. \tag{4.44h}$$

The Onsager relations give constraints on the transport coefficients. The constraints on the parity-violating coefficients depend on whether the microscopic theory is $\mathcal{T}$ or $\mathcal{PT}$ symmetric. Here, $\mathcal{T}$ refers to time-reversal and $\mathcal{P}$ to parity. For a $\mathcal{T}$ preserving theory, the constraints are the following

$$c_3 = -c_4 - \tfrac{1}{3}c_5\,, \quad c_{14} = -\tfrac{1}{2}c_5\,, \quad c_{15} = -c_8\,, \quad c_{17} = -c_{10}\,, \tag{4.45}$$

while for a $\mathcal{PT}$ preserving theory, we find

$$c_3 = c_4 + \tfrac{1}{3}c_5\,, \quad c_{14} = \tfrac{1}{2}c_5\,, \quad c_{15} = c_8\,, \quad c_{17} = c_{10}\,. \tag{4.46}$$

As one might expect, with a given theory preserves $\mathcal{T}$ and $\mathcal{PT}$, it follows that it preserves parity and therefore all parity-violating transport coefficients vanish.

Entropy constraints

To find constraints on the transport coefficients, one method is to impose a local version of the second law of thermodynamics: the existence of a local entropy current with positive semi-definite divergence for every non-equilibrium configuration consistent with the hydrodynamic equations. As was shown in [24, 25], the constraints on transport coefficients derived from the entropy current are the same as those derived from the equilibrium generating functional, plus the inequality constraints on dissipative transport coefficients. We take the entropy current to be

$$S^\mu = S^\mu_{\rm canon} + S^\mu_{\rm eq.}\,,$$

where the canonical part of the entropy current is

$$S^\mu_{\rm canon} = \frac{1}{T}\left(p u^\mu - T^{\mu\nu}u_\nu - \mu J^\mu\right)\,, \tag{4.47}$$

and $S^\mu_{\rm eq.}$ is found from the equilibrium partition function, as described in [24, 25]. The constraints on transport coefficients follow by demanding $\nabla_\mu S^\mu \geqslant 0$. Using the hydrodynamic equations (4.2), the divergence of the modified canonical entropy current is

$$\nabla_\mu S^\mu_{\rm canon} = \nabla_\mu\left(\frac{p}{T}u^\mu\right) - T^{\mu\nu}\nabla_\mu\frac{u_\nu}{T} + J^\mu\left(\frac{E_\mu}{T} - \partial_\mu\frac{\mu}{T}\right)\,.$$

The $S^\mu_{\rm eq.}$ part of the entropy current is explicitly built to cancel out the part of $\nabla_\mu S^\mu_{\rm canon}$ that arises from the equilibrium terms in the constitutive relations, i.e. the terms in $T^{\mu\nu}$ and J^μ derived from the equilibrium generating functional. We thus focus on non-equilibrium terms, and write the thermodynamic frame constitutive relations as $T^{\mu\nu} = T^{\mu\nu}_{\rm eq.} + T^{\mu\nu}_{\rm non\text{-}eq.}$ and $J^\mu = J^\mu_{\rm eq.} + J^\mu_{\rm non\text{-}eq.}$. The divergence of the entropy current is then

$$\begin{aligned}
\nabla_\mu S^\mu &= \frac{1}{T}J^\mu_{\rm non\text{-}eq.}\left(E_\mu - T\partial_\mu\frac{\mu}{T}\right) - T^{\mu\nu}_{\rm non\text{-}eq.}\nabla_\mu\frac{u_\nu}{T}\\
&= \frac{1}{T}\left(\ell^\mu_{\perp\rm non\text{-}eq.} + \frac{B^\mu}{B}\ell_{\rm non\text{-}eq.}\right)V_\mu - \frac{1}{T}f_{\rm non\text{-}eq.}\nabla{\cdot}u - \frac{1}{2T}t^{\mu\nu}_{\rm non\text{-}eq.}\sigma_{\mu\nu}\,.
\end{aligned}$$

Using the constitutive relations (4.19), (4.42), this leads to

$$\begin{aligned}
T\nabla_\mu S^\mu = {}& \tfrac{1}{2}\eta_\perp(\sigma^{\mu\nu}_\perp)^2 + \sigma_\perp V_\perp^2 + \eta_\parallel\Sigma^2 + (c_8 - c_{15})\,\Sigma{\cdot}V_\perp\\
&+ (\zeta_1 - \tfrac{2}{3}\eta_1)S_3^2 + 2\eta_2 S_4^2 + \sigma_\parallel S_5^2 + (2\eta_1 + \zeta_2 - \tfrac{2}{3}\eta_2)S_3S_4\\
&+ \left(c_4 - c_3 + \tfrac{2}{3}c_{14}\right)S_3S_5 + (c_5 - 2c_{14})\,S_4S_5\,,
\end{aligned} \tag{4.48}$$

where $S_3 \equiv \nabla{\cdot}u$, $S_4 \equiv b^\mu b^\nu\nabla_\mu u_\nu$ and $S_5 \equiv b{\cdot}V$. Demanding $\nabla_\mu S^\mu \geqslant 0$ now gives $\eta_\perp \geqslant 0$ together with the condition that the quadratic forms made out of $V_\perp$, and Σ

and S_3, S_4 and S_5 are non-negative, which implies

$$\sigma_\perp \geqslant 0\,,\quad \eta_\parallel \geqslant 0 \quad \eta_2 \geqslant 0\,,\quad \zeta_1 - \tfrac{2}{3}\eta_1 \geqslant 0\,,\quad \sigma_\parallel \geqslant 0\,, \tag{4.49a}$$

$$\begin{aligned} &\sigma_\perp \eta_\parallel \geqslant \tfrac{1}{4}\left(c_8 - c_{15}\right)^2\,,\quad 2\eta_2(\zeta_1 - \tfrac{2}{3}\eta_1) \geqslant \tfrac{1}{4}(2\eta_1 + \zeta_2 - \tfrac{2}{3}\eta_2)^2\,,\\ &\sigma_\parallel(\zeta_1 - \tfrac{2}{3}\eta_1) \geqslant \tfrac{1}{4}(c_4 - c_3 + \tfrac{2}{3}c_{14})^2\,,\quad 2\eta_2\sigma_\parallel \geqslant \tfrac{1}{4}(c_5 - 2c_{14})^2\,, \end{aligned} \tag{4.49b}$$

$$\det(M) \geqslant 0\,, \tag{4.49c}$$

where

$$M = \begin{bmatrix} \zeta_1 - \frac{2}{3}\eta_1 & \eta_1 + \frac{1}{2}\zeta_2 - \frac{1}{3}\eta_2 & \frac{1}{2}c_4 - \frac{1}{2}c_3 + \frac{1}{3}c_{14} \\ \eta_1 + \frac{1}{2}\zeta_2 - \frac{1}{3}\eta_2 & 2\eta_2 & \frac{1}{2}c_5 - c_{14} \\ \frac{1}{2}c_4 - \frac{1}{2}c_3 + \frac{1}{3}c_{14} & \frac{1}{2}c_5 - c_{14} & \sigma_\parallel \end{bmatrix}.$$

The coefficients $\tilde{\eta}_\perp$, $\tilde{\eta}_\parallel$, $\tilde{\sigma}$, c_{10} and c_{17} do not contribute to entropy production, and are not constrained by the above analysis. Thus, $\tilde{\eta}_\perp$, $\tilde{\eta}_\parallel$, $\tilde{\sigma}$ c_{10} and c_{17} are non-equilibrium non-dissipative coefficients. Note that using the $\mathcal{PT}$ symmetry Onsager relations these constraints reduce to the ones for a parity preserving theory (4.27)

$$\begin{aligned} &\eta_\perp \geqslant 0\,,\quad \sigma_\perp \geqslant 0\,,\quad \eta_\parallel \geqslant 0\,,\quad \sigma_\parallel \geqslant 0\,,\quad \eta_2 \geqslant 0\,,\\ &\zeta_1 - \tfrac{2}{3}\eta_1 \geqslant 0\,,\quad 2\eta_2(\zeta_1 - \tfrac{2}{3}\eta_1) \geqslant 4\eta_1^2\,. \end{aligned} \tag{4.50}$$

Alternatively, if the theory is time reversal invariant, we have

$$\begin{aligned} &\eta_\perp \geqslant 0\,,\quad \sigma_\perp \geqslant 0\,,\quad \eta_\parallel \geqslant 0 \quad \eta_2 \geqslant 0\,,\quad \zeta_1 - \tfrac{2}{3}\eta_1 \geqslant 0\,,\quad \sigma_\parallel \geqslant 0\,,\\ &\sigma_\perp\eta_\parallel \geqslant c_8^2\,,\quad 2\eta_2(\zeta_1 - \tfrac{2}{3}\eta_1) \geqslant 4\eta_1^2\,,\quad \sigma_\parallel(\zeta_1 - \tfrac{2}{3}\eta_1) \geqslant c_4^2\,,\quad 2\eta_2\sigma_\parallel \geqslant c_5^2\,,\\ &\det(M) \geqslant 0\,, \end{aligned} \tag{4.51}$$

where now

$$M = \begin{bmatrix} \zeta_1 - \frac{2}{3}\eta_1 & 2\eta_1 & c_4 \\ 2\eta_1 & 2\eta_2 & c_5 \\ c_4 & c_5 & \sigma_\parallel \end{bmatrix}.$$

4.4 Hydrodynamics with dynamical electromagnetic fields

4.4.1 Dynamical gauge field

We now move on to systems where the gauge field A_μ is dynamical rather than external, which will lead us to MHD. In external metric g, the (microscopic) generating functional is

$$Z[g] = \int DA\; e^{iS[g,A]}\,,$$

where S is the action. Let us couple the gauge field to an external conserved current $J^\mu_{\rm ext}$. We do this so that the new generating functional is

$$Z[g, J_{\rm ext}] = \int DA\, D\varphi\; e^{iS[g,A]+i\int\sqrt{-g}\,(A_\mu-\partial_\mu\varphi)J^\mu_{\rm ext}}\,, \tag{4.52}$$

and $W \equiv -i\ln Z$. The new field φ is a Lagrange multiplier which shifts under gauge transformations and ensures that the external current is conserved. We define the energy-momentum tensor and the current by the variation of the action:

$$\delta_g S[g,A] = \tfrac{1}{2}\int\sqrt{-g}\,T^{\mu\nu}\delta g_{\mu\nu}\,,\qquad \delta_A S[g,A] = \int\sqrt{-g}\,J^\mu\delta A_\mu\,.$$

Diffeomorphism invariance of $W[g, J_{\rm ext}]$ implies $\nabla_\mu\langle T^{\mu\nu}\rangle = \langle F^{\lambda\nu}\rangle J_{{\rm ext}\,\lambda}$. In what follows, we will omit the angular brackets, writing the (non)-conservation of the energy-momentum tensor simply as

$$\nabla_\mu T^{\mu\nu} = F^{\lambda\nu}J_{{\rm ext}\,\lambda}\,. \tag{4.53}$$

In the standard hydrodynamic approach, $T^{\mu\nu}$ and $F_{\mu\nu}$ will then be taken as dynamical variables in the classical hydrodynamic theory. Note that the sign in the right-hand side of eq. (4.53) is opposite compared to eq. (4.2a), owing to the fact that the current, rather than the gauge field, is now external. In order to proceed with hydrodynamics, we need to specify *a)* the constitutive relations for the energy-momentum tensor to be used in eq. (4.53), and *b)* the equations which determine the evolution of the dynamical gauge field $F_{\mu\nu}$.

4.4.2 Maxwell's equations in matter

Classical equations specifying the dynamics of electric and magnetic fields are usually referred to as Maxwell's equations in matter. While we don't have a recipe of deriving them in a most general form in a model-independent way, a useful starting point is provided by matter in thermal equilibrium. Maxwell's equations for equilibrium matter may be then amended to include the non-equilibrium and dissipative effects, such as the electrical conductivity. To this end, as advocated in [71], we take the static generating functional $W_s[g, A]$ to be the effective action for gauge fields in equilibrium,

$$S_{\text{eff}}[g, A] = \int d^4x \, \sqrt{-g}\, \mathcal{F} \,, \tag{4.54}$$

where $\mathcal{F}$ is a local gauge-invariant function of the sources $g_{\mu\nu}$ and A_μ, and we have ignored the surface terms. To leading order in the derivative expansion, $\mathcal{F}$ is simply the pressure. We can always write $\mathcal{F} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{F}_{\text{m}}$, where the vacuum action is $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(E^2 - B^2)$, and $\mathcal{F}_{\text{m}}$ is the "matter" contribution. The isolation of the vacuum term is arbitrary, but it will allow us to make contact with the textbook form of Maxwell's equations in matter. Our (equilibrium) effective theory is then given by the partition function (4.52), with S replaced by S_{eff}, and the total action is

$$S_{\text{tot}}[A, \varphi] = W_s[g, A] + \int \sqrt{-g}\, (A_\mu - \partial_\mu \varphi) J^\mu_{\text{ext}} \,.$$

The current derived by varying the total action with respect to A_μ is $J^\mu_{\text{tot}} = J^\mu + J^\mu_{\text{ext}}$, or

$$J^\mu_{\text{tot}} = -\nabla_\nu (F^{\mu\nu} - M^{\mu\nu}_{\text{m}}) + n u^\mu + J^\mu_{\text{ext}} \,,$$

where the polarization tensor $M^{\mu\nu}_{\text{m}}$ is defined by $\delta_F \int d^4x \, \sqrt{-g}\, \mathcal{F}_{\text{m}} = \frac{1}{2} \int d^4x \, \sqrt{-g}\, M^{\mu\nu}_{\text{m}} \, \delta F_{\mu\nu}$, and the density of "free" charges is $n \equiv \partial \mathcal{F}_{\text{m}} / \partial \mu$. The equation of motion for the gauge field follows from $\delta_A S_{\text{tot}} = 0$, or equivalently $J^\mu_{\text{tot}} = 0$, and becomes

$$\nabla_\nu H^{\mu\nu} = n u^\mu + J^\mu_{\text{ext}} \,, \tag{4.55}$$

where $H^{\mu\nu} \equiv F^{\mu\nu} - M^{\mu\nu}_{\text{m}}$. This is the desired equation that must be satisfied by electromagnetic fields in equilibrium. Following the standard hydrodynamic lore and assuming that eq. (4.55) also holds for small departures away from equilibrium, one obtains hydrodynamics of "perfect fluids", now with dynamical electric and magnetic fields. For these perfect fluids, equations (4.55) have to be solved together with the stress tensor (non)-conservation (4.53), where $T^{\mu\nu}$ is derived from the effective action (4.54).

In fact, eq. (4.55) is nothing but the standard Maxwell's equations in matter. The polarization tensor $M_{\rm m}^{\mu\nu}$ defines electric and magnetic polarization vectors P^μ and M^μ through the decomposition

$$M_{\rm m}^{\mu\nu} = P^\mu u^\nu - P^\nu u^\mu - \epsilon^{\mu\nu\rho\sigma} u_\rho M_\sigma \,. \tag{4.56}$$

The antisymmetric tensor $H_{\mu\nu}$ can be decomposed in the same way as the field strength $F_{\mu\nu}$,

$$H_{\mu\nu} = u_\mu D_\nu - u_\nu D_\mu - \epsilon_{\mu\nu\rho\sigma} u^\rho H^\sigma \,,$$

which defines $D_\mu \equiv H_{\mu\nu} u^\nu$ and $H^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu H_{\alpha\beta}$, so that

$$D^\mu = E^\mu + P^\mu \,,$$
$$H^\mu = B^\mu - M^\mu \,.$$

It is then clear that eq. (4.55) is the covariant form of Maxwell's equations in matter: the currents of 'free charges' are in the right-hand side, while the effects of polarization appear in the left-hand side through the substitution $E^\mu \to D^\mu$, $B^\mu \to H^\mu$ in the vacuum Maxwell's equations. Action (4.54) is the action for Maxwell's equations in matter.

As an example, consider the following "matter" contribution: $\mathcal{F}_{\rm m} = p_{\rm m}(T, \mu, E^2, B^2, E{\cdot}B)$, where $p_{\rm m}$ is the "matter" pressure. The polarization tensor is then $M_{\rm m}^{\mu\nu} = 2\partial p_{\rm m}/\partial F_{\mu\nu}$, and the polarization vectors are

$$P^\mu = \chi_{\rm EE} E^\mu + \chi_{\rm EB} B^\mu \,, \tag{4.57a}$$

$$M^\mu = \chi_{\rm EB} E^\mu + \chi_{\rm BB} B^\mu \,, \tag{4.57b}$$

where the susceptibilities $\chi_{\rm EE} \equiv 2\partial p_{\rm m}/\partial E^2$, $\chi_{\rm EB} \equiv \partial p_{\rm m}/\partial (E{\cdot}B)$, and $\chi_{\rm BB} \equiv 2\partial p_{\rm m}/\partial B^2$ all depend on T, μ, E^2, B^2, and $E{\cdot}B$. This gives the standard constitutive relations, expressing D and B in terms of E and H,

$$D^\mu = \varepsilon_{\rm m} E^\mu + \beta_{\rm m} H^\mu \,,$$

$$B^\mu = \beta_{\rm m} E^\mu + \mu_{\rm m} H^\mu \,,$$

where $\varepsilon_{\rm m} \equiv 1 + \chi_{\rm EE} + \chi_{\rm EB}^2/(1-\chi_{\rm BB})$ is the electric permittivity, $\mu_{\rm m} \equiv 1/(1-\chi_{\rm BB})$ is the magnetic permeability, and $\beta_{\rm m} \equiv \chi_{\rm EB}/(1-\chi_{\rm BB})$. We will also use $\varepsilon_{\rm e} \equiv 1+\chi_{\rm EE}$, which coincides with the electric permittivity if $\chi_{\rm EB} = 0$.

4.4.3 Hydrodynamics

We take the MHD equations to be as follows:

$$\nabla_\mu T^{\mu\nu} = F^{\lambda\nu} J_{{\rm ext}\,\lambda} \,, \tag{4.58a}$$

$$J^\mu + J^\mu_{\rm ext} = 0 \,, \tag{4.58b}$$

$$\epsilon^{\mu\nu\alpha\beta} \nabla_\nu F_{\alpha\beta} = 0 \,. \tag{4.58c}$$

The last equation is the electromagnetic "Bianchi identity", expressing the fact that the electric and magnetic fields are derived from the vector potential A_μ. The second equation (Maxwell's equations in matter) can be rewritten as $\nabla_\nu (F^{\mu\nu} - M^{\mu\nu}_{\rm m}) = J^\mu_{\rm free} + J^\mu_{\rm ext}$ which defines $J^\mu_{\rm free}$, the current of "free charges". While eqs. (4.58a) and (4.58c) are true microscopically, the Maxwell's equations in matter (4.58b) are written based on the above intuition of the equilibrium effective action. Note that $\nabla_\mu J^\mu_{\rm free} = 0$ is a consequence of (4.58b), and is not an independent equation. The hydrodynamic variables are T, u^α, μ, as well as the electric and magnetic fields which satisfy $u_\alpha E^\alpha = 0$, $u_\alpha B^\alpha = 0$. Hydrodynamic equations (4.58) must be supplemented by constitutive relations, which express $T^{\mu\nu}$, J^μ (or $J^\mu_{\rm free}$ and $M^{\mu\nu}_{\rm m}$) in terms of the hydrodynamic variables. These constitutive relations will contain equilibrium contributions coming from the equilibrium effective action (4.54). In addition, the constitutive relations will contain non-equilibrium contributions, such as the electrical conductivity and the shear viscosity.

Taking the divergence of eq. (4.58b) and using $J^\mu_{\rm ext} = -J^\mu$ gives

$$\nabla_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda \,,$$
$$\nabla_\mu J^\mu = 0 \,,$$

which shows that the variables T, u^α, and μ satisfy exactly the same equations (4.2) as they did in the theory with a non-dynamical, external A_μ. Thus in order to "solve" the MHD theory (4.58) one can *i)* solve the hydrodynamic equations with an external gauge field (4.58) to find $T[A,g]$, $u^\alpha[A,g]$, $\mu[A,g]$, and *ii)* solve $J^\mu[T[A,g],u^\alpha[A,g],\mu[A,g],A,g] + J^\mu_{\rm ext} = 0$ in order to find $A_\mu[J_{\rm ext},g]$, and *iii)* use the constitutive relations to find the energy-momentum tensor $T^{\mu\nu}[J_{\rm ext},g] = T^{\mu\nu}[T[A[J_{\rm ext},g],g],u^\alpha[A[J_{\rm ext},g],g],\mu[A[J_{\rm ext},g],g],A[J_{\rm ext},g],g]$. MHD correlation functions may then be obtained through variations with respect to the external sources $J^\lambda_{\rm ext}$ and $g_{\mu\nu}$.

An equivalent way to understand the classical effective theory (4.58) is to promote the real-time generating functional to the non-equilibrium effective action [71], i.e. to write

$$S_{\rm tot}[A,\varphi] = W_r[A,g] + \int\!\sqrt{-g}\,(A_\mu - \partial_\mu\varphi) J^\mu_{\rm ext} \,,$$

where $W_r[A,g]$ is low-energy, real-time generating functional for retarded correlation functions in the theory with a non-dynamical A_μ. The functional $W_r[g,A]$ is non-local due to the gapless low-energy degrees of freedom (sound waves etc). However, for the purposes of MHD we do not need the actual generating functional, but only the equations of motion for the effective action $S_{\rm tot}$. These equations of motion are $J^\mu[A,g] + J^\mu_{\rm ext} = 0$, where $J^\mu[A,g]$ is the on-shell current in the theory with a non-dynamical A_μ. One can then solve the theory as described in the previous paragraph.

We will thus adopt the simplest hydrodynamic effective theory (4.58) where the constitutive relations for $T^{\mu\nu}$ and J^μ are the same as in the case of external non-dynamical electromagnetic fields. Under this "mean-field" assumption, transport coefficients which are naively independent would still be related by the conditions originating from the static generating functional.

Further, any solution $T[A,g]$, $u^\alpha[A,g]$, $\mu[A,g]$ to the MHD equations is also a solution to the hydrodynamic equations (4.2) in the theory with a non-dynamical A_μ. Thus the entropy current with a non-negative divergence on the solutions to (4.2) will also have non-negative divergence when evaluated on the solutions to the

MHD equations (4.58). This means that the entropy current in MHD may be taken the same as the entropy current in the theory with a non-dynamical gauge field [71], and we do not need to perform a separate entropy current analysis beyond what was already done in sec. 4.3.

To sum up, with the MHD scaling $B \sim O(1)$, $E \sim O(\partial)$, the equilibrium effective action is given by eq. (4.7),

$$S_{\text{eff}} = \int \sqrt{-g} \left(-\tfrac{1}{2}B^2 + p_{\text{m}}(T,\mu,B^2) + \sum_{n=1}^{5} M_n(T,\mu,B^2) s_n^{(1)} + O(\partial^2) \right) . \tag{4.59}$$

For a parity-invariant theory, only the M_4 term in the sum contributes. The constitutive relations for the energy-momentum tensor and the current were already found in the previous section, where now we have $p(T,\mu,B^2) = -\frac{1}{2}B^2 + p_{\text{m}}(T,\mu,B^2)$. The energy-momentum tensor appearing in eq. (4.58) and the current J^μ satisfying $J^\mu + J^\mu_{\text{ext}} = 0$ take the form (4.9), (4.10), and the constitutive relations for a parity-invariant theory in the thermodynamic frame are given by Eqs. (4.19), (4.20).

We will find it useful to modify the above effective theory by giving dynamics to the electric field. To do so, we add an $O(\partial^2)$ term $\frac{1}{2}\varepsilon_{\text{e}}E^2$ to the effective action (4.59), where ε_{e} is the electric permittivity which we take constant. This term is one of the many $O(\partial^2)$ terms, and we add it as a "ultraviolet regulator" which improves the high-frequency behaviour of the theory. When studying the near-equilibrium eigenmodes of the system, this term will affect the frequency gaps, but not the leading-order dispersion relations of the gapless modes. With this new term, the following contributions have to be added to the constitutive relations (4.19), (4.20):

$$\begin{aligned} T^{\mu\nu}_{\text{El.}} &= \varepsilon_{\text{e}} \left(\tfrac{1}{2}E^2 g^{\mu\nu} + E^2 u^\mu u^\nu - E^\mu E^\nu \right) , \\ J^\mu_{\text{El.}} &= -\varepsilon_{\text{e}} \nabla_\lambda \left(E^\lambda u^\mu - E^\mu u^\lambda \right) . \end{aligned}$$

The current $J^\mu_{\text{El.}}$ contains the kinetic term for the electric field in Maxwell's equations, as well as the "bound" current due to electric polarization.

4.4.4 Eigenmodes

As a simple application of the above MHD theory, one can study the eigenmodes of small oscillations about the thermal equilibrium state. As we did earlier, we set the external sources to zero, and linearize the hydrodynamic equations near the flat-space equilibrium state with constant $T = T_0$, $\mu = \mu_0$, $u^\alpha = (1, \mathbf{0})$, and $B^\alpha = (0, 0, 0, B_0)$. For simplicity, we will take the magnetic permeability $\mu_{\rm m}$ constant, though it is straightforward to find how the eigenfrequencies below are modified for non-constant $\mu_{\rm m} = \mu_{\rm m}(T, \mu, B^2)$.

Neutral state

We begin with the neutral state at $\mu_0 = 0$ and $n_0 = 0$. The system admits nine eigenmodes, three gapped, and six gapless.

Let us start with the familiar case of vanishing magnetic field in equilibrium. The system is then isotropic, with shear viscosity η, bulk viscosity ζ, and conductivity $\sigma \equiv \sigma_\perp = \sigma_\parallel$. The fluctuations of δT, δu_i decouple from the fluctuations of $\delta\mu$, δE_i, δB_i. The eigenmodes include two transverse shear modes with eigenfrequency $\omega = -i\eta k^2/(\epsilon_0{+}p_0)$, and longitudinal sound waves with $v_s^2 = \partial p/\partial\epsilon$ and $\Gamma_s = (\frac{4}{3}\eta + \zeta)/(\epsilon_0+p_0)$. In addition, there is a longitudinal charge diffusion mode which becomes gapped because of non-zero electrical conductivity,

$$\omega = -\frac{i\sigma}{\varepsilon_{\rm e}} - i\left(\frac{\sigma}{\partial n/\partial\mu}\right)k^2\,.$$

Thus, charge fluctuations in a neutral conducting medium do not diffuse. Instead, what diffuses are the transverse magnetic and electric fields: there are two sets of transverse conductor modes whose eigenfrequencies are determined by

$$\omega\left(\omega + \frac{i\sigma}{\varepsilon_{\rm e}}\right) = \frac{k^2}{\varepsilon_{\rm e}\mu_{\rm m}}\,.$$

Recall that $\varepsilon_{\rm e}$ is the electric permittivity and $\mu_{\rm m} = 1/(1{-}2\partial p_{\rm m}/\partial B^2)$ is the magnetic permeability, so $\sqrt{\varepsilon_{\rm e}\mu_{\rm m}}$ is the elementary index of refraction. The conductor modes have the following frequencies at small momenta:

$$\omega = -\frac{i\sigma}{\varepsilon_{\rm e}} + \frac{ik^2}{\sigma\mu_{\rm m}}\,, \qquad \omega = -\frac{ik^2}{\sigma\mu_{\rm m}}\,.$$

The gapless conductor mode is responsible for the skin effect in metals.

We now turn on a non-zero magnetic field and consider modes propagating at an angle θ with respect to $\mathbf{B}_0$. Thermal and mechanical fluctuations now no longer decouple from electromagnetic fluctuations. There is one longitudinal gapped mode, and two transverse gapped modes,

$$\omega = -\frac{i\sigma_\parallel}{\varepsilon_{\rm e}} + O(k^2)\,, \qquad \omega = -\frac{i\sigma_\perp \pm \tilde{\sigma}}{\varepsilon_{\rm e}} + O(k^2)\,.$$

In writing down the transverse eigenfrequencies, we have assumed $B_0^2 \ll \epsilon_0 + p_0$.

All six gapless modes have linear dispersion relation at small momenta. Two of the gapless modes are the Alfvén waves,

$$\omega = \pm v_{\rm A} k \cos\theta - \frac{i\Gamma_{\rm A}}{2} k^2\,, \tag{4.60a}$$

whose speed and damping are determined by

$$v_{\rm A}^2 = \frac{B_0^2}{\mu_{\rm m}(\epsilon_0{+}p_0) + B_0^2}\,, \quad \Gamma_{\rm A} = \frac{1}{\epsilon_0{+}p_0}\left(\eta_\perp \sin^2\theta + \eta_\parallel \cos^2\theta\right) + \frac{1}{\mu_{\rm m}}\Big(\rho_\perp \cos^2\theta + \rho_\parallel \sin^2\theta\Big)\,, \tag{4.60b}$$

where $\rho_\parallel \equiv 1/\sigma_\parallel$, and $\rho_\perp$ was defined below eq. (4.24). In writing down the damping coefficient, we have taken $B_0^2 \ll \epsilon_0{+}p_0$, the corrections of order $B_0^2/(\epsilon_0{+}p_0)$ are straightforward to write down. The other four gapless modes are the two branches of magnetosonic waves,

$$\omega = \pm v_{\rm ms} k - \frac{i\Gamma_{\rm ms}}{2} k^2\,, \tag{4.61a}$$

whose speed is determined by the quadratic equation

$$(v_{\rm ms}^2)^2 - v_{\rm ms}^2(v_A^2 + v_s^2 - v_A^2 v_s^2 \sin^2\theta) + v_A^2 v_s^2 \cos^2\theta = 0\,, \tag{4.61b}$$

where $v_s^2 = (s/T)/(\partial s/\partial T) = \partial p/\partial\epsilon$ is the speed of sound at $n_0 = 0$. The two solutions of (4.61b) correspond to the sound-type (or "fast") branch, and the Alfvén-type (or "slow") branch. At $\theta = 0$, the slow branch turns into a second set of Alfvén waves, while the fast branch becomes the sound wave. See e.g. ref. [72] for an early derivation of $v_{\rm A}$ and $v_{\rm ms}$ in relativistic MHD. The damping coefficients of the magnetosonic waves are straightforward to evaluate, but are quite lengthy to write down in general, and we will only present them in the limits of small B_0 and small θ.

As $B_0 \to 0$, the damping coefficients become

$$\text{slow: } \Gamma_{\rm ms} = \frac{\eta}{\epsilon_0{+}p_0} + \frac{1}{\sigma\mu_{\rm m}}\,, \tag{4.61c}$$

$$\text{fast: } \Gamma_{\rm ms} = \frac{1}{\epsilon_0{+}p_0}\left(\tfrac{4}{3}\eta + \zeta\right)\,. \tag{4.61d}$$

On the other hand, as $\theta \to 0$, the damping coefficients become

$$\text{slow: } \Gamma_{\rm ms} = \frac{\eta_\parallel}{\epsilon_0{+}p_0} + \frac{\rho_\perp}{\mu_{\rm m}}\,, \tag{4.61e}$$

$$\text{fast: } \Gamma_{\rm ms} = \frac{1}{\epsilon_0{+}p_0}\left(\tfrac{10}{3}\eta_1 + 2\eta_2 + \zeta_1\right)\,. \tag{4.61f}$$

We have again taken $B_0^2 \ll \epsilon_0 + p_0$, the corrections of order $B_0^2/(\epsilon_0{+}p_0)$ are straightforward to write down. At $\theta = 0$, both polarizations of Alfvén waves have the same damping.

Let us now consider the gapless modes propagating perpendicularly to the magnetic field, i.e. taking $\theta \to \pi/2$ first, $k \to 0$ second. These include sound waves

$$\omega = \pm k v_{\pi/2} - \frac{i\Gamma_{\pi/2}}{2}k^2\,, \tag{4.62a}$$

where $v_{\pi/2}$ is the non-zero solution of eq. (4.61b) at $\theta = \pi/2$. In the limit of small B_0 it reduces to $v_{\pi/2}^2 = v_s^2 = (s/T)/(\partial s/\partial T) = \partial p/\partial\epsilon$, in equilibrium. The damping coefficient is

$$\Gamma_{\pi/2} = \frac{1}{\epsilon_0{+}p_0}\left(\zeta_1 - \tfrac{2}{3}\eta_1 + \eta_\perp\right)\,, \tag{4.62b}$$

assuming $B_0^2 \ll \epsilon_0{+}p_0$. The other four gapless modes at $\theta = \pi/2$ are purely diffusive,

$$\omega = -\frac{i\eta_{\|}}{\epsilon_0+p_0}k^2\,, \tag{4.63a}$$

$$\omega = -\frac{i\rho_{\|}}{\mu_{\rm m}}k^2\,, \tag{4.63b}$$

$$\omega = -\frac{i\eta_{\perp}}{\epsilon_0+p_0}k^2\,, \tag{4.63c}$$

$$\omega = -\frac{i\rho_{\perp}}{\mu_{\rm m}}k^2\,. \tag{4.63d}$$

In writing down (4.63c) and (4.63d) we have again taken $B_0^2 \ll \epsilon_0+p_0$.

Charged state offset by background charge

We now consider a state with a non-zero value of μ_0, which gives rise to a constant non-zero charge density n_0. In order to ensure that the equilibrium state is stable, we will offset this equilibrium value of the dynamical charge density by a constant non-dynamical external background charge density $-n_0$. This can be achieved by choosing the external current in the hydrodynamic equations (4.58) as $J^\mu_{\rm ext} = (-n_0, \mathbf{0})$. In the particle language, this would correspond to a state where the excess of electrically charged particles over antiparticles (or vice versa) is compensated by a constant charge density of immobile background "ions". Even though the system is overall electrically neutral, its dynamics is not equivalent to that of the system with $\mu_0 = 0$, $n_0 = 0$: for example, the fluctuation of the spatial electric current has a convective contribution $n_0\,\delta u_i$. More formally, when analyzing hydrodynamic modes, the limits $n_0 \to 0$ and $k \to 0$ do not commute. We now find six gapped modes and three gapless modes.

To get some intuition about the gapped modes, let us set all transport coefficients to zero, as well as set $B_0 = 0$. Then at small momenta there are two longitudinal gapped modes whose frequencies are determined by

$$\omega^2 = \Omega_p^2 + v_s^2 k^2\,,$$

where $\Omega_p^2 \equiv n_0^2/[(\epsilon_0+p_0)\varepsilon_{\rm e}]$, and v_s is the speed of sound that the charged fluid would have, if the electromagnetic fields were not dynamical, see Sec 4.3.5. These modes are the relativistic analogues of Langmuir oscillations, and Ω_p is the relativistic "plasma frequency" which gaps out the sound waves. In addition, there are four transverse gapped modes whose frequencies are determined by

$$\omega^2 = \Omega_p^2 + \frac{k^2}{\varepsilon_{\rm e}\mu_{\rm m}}\,.$$

These are electromagnetic waves in the fluid, gapped by the same plasma frequency Ω_p as the sound waves. If we now turn on the transport coefficients, the gaps are determined by

$$\omega\left(\omega + \frac{i\sigma_\parallel}{\varepsilon_{\rm e}}\right) = \Omega_p^2\,, \qquad \omega\left(\omega + \frac{i(\sigma_\perp \pm i\tilde{\sigma})}{\varepsilon_{\rm e}}\right) = \Omega_p^2\,,$$

indicating the damping of plasma oscillations. At non-zero $B_0^2 \ll \epsilon_0 + p_0$, the gaps will receive dependence on the magnetic field.

At $B_0 = 0$ the system is isotropic. The gapless modes ($B_0 \to 0$ first, $k \to 0$ second) include two transverse shear modes with quartic dispersion relation, and one longitudinal diffusive mode,

$$\omega = -\frac{i\eta k^4}{n_0^2\mu_{\rm m}}\,, \qquad \omega = -\frac{i\sigma\chi_{33}w_0^3}{n_0^2\det(\chi)}k^2\,,$$

where again $w_0 \equiv T_0 s_0 + \mu_0 n_0$, and the susceptibility matrix χ was defined below eq. (4.23).

At non-zero B_0, the three gapless modes all have quadratic dispersion relation at small momenta. There are two propagating waves with real frequencies

$$\omega = \pm\frac{B_0\cos\theta}{n_0\mu_{\rm m}}\,k^2\,, \tag{4.64}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{B}_0$, and one diffusive mode. For $B_0^2 M_{\Omega,\mu} \ll \epsilon_0 + p_0$, the diffusive frequency is

$$\omega = -i\frac{\chi_{33}w_0^3}{\det(\chi)}\left(\frac{\sigma_\parallel\cos^2\theta}{n_0^2} + \frac{\rho_\perp\sin^2\theta}{B_0^2}\right)k^2\,. \tag{4.65}$$

For gapless modes propagating at $\theta = \pi/2$ at small momenta ($\theta \to \pi/2$ first, $k \to 0$ second), we again find the diffusive mode $\omega = -iD_\perp k^2$, with the same coefficient $D_\perp$ as in sec. 4.3.5. In addition, at $\theta = \pi/2$ there are two "subdiffusive" modes with quartic dispersion relation,

$$\omega = -i\frac{\eta_\perp k^4}{n_0^2\mu_{\rm m}}\,, \qquad \omega = -i\frac{\eta_\parallel k^4}{n_0^2\mu_{\rm m}}\,.$$

The eigenfrequencies are noticeably different from the ones in a theory with fixed, non-dynamical electromagnetic field discussed in sec. 4.3.5. Compared to the case of $n_0 = 0$ earlier in this section, one can say that non-vanishing dynamical charge density gaps out the magnetosonic waves, and turns Alfvén waves into waves whose frequency is quadratic in momentum.

4.4.5 Kubo formulas

We can find MHD correlation functions following the same variational procedure outlined in sec. 4.3.7. As the total current vanishes by the equations of motion, the objects whose correlation functions it makes sense to evaluate in MHD are the energy-momentum tensor $T^{\mu\nu}$ and the electromagnetic field strength tensor $F_{\mu\nu}$. It is straightforward to evaluate retarded functions in flat space, in an equilibrium state with constant $T = T_0$, $\mu = \mu_0$, $u^\alpha = (1, \mathbf{0})$, and constant magnetic field. We solve the hydrodynamic equations in the presence of fluctuating external sources $\delta J_{\rm ext}$, δg (proportional to $\exp(-i\omega t + i\mathbf{k}{\cdot}\mathbf{x})$) to find $\delta T[J_{\rm ext}, g]$, $\delta\mu[J_{\rm ext}, g]$, $\delta u^\alpha[J_{\rm ext}, g]$, $\delta F_{\mu\nu}[J_{\rm ext}, g]$ and then vary the resulting hydrodynamic expressions $T^{\mu\nu}[J_{\rm ext}, g]$ and $F_{\mu\nu}[J_{\rm ext}, g]$ with respect to $g_{\alpha\beta}$, $J^\alpha_{\rm ext}$ to find the retarded functions. The metric variations are performed as usual,

$$G^R_{T^{\mu\nu}T^{\alpha\beta}} = 2\frac{\delta}{\delta g_{\alpha\beta}}\left(\sqrt{-g}\,T^{\mu\nu}_{\text{on-shell}}[J_{\rm ext}, g]\right)\,, \quad G^R_{F_{\mu\nu}T^{\alpha\beta}} = 2\frac{\delta}{\delta g_{\alpha\beta}}\left(\sqrt{-g}\,F^{\text{on-shell}}_{\mu\nu}[J_{\rm ext}, g]\right)\,.$$

The subscript "on-shell" signifies that $T^{\mu\nu}$ and $F_{\mu\nu}$ are evaluated on the solutions to (4.58) with the constitutive relations (4.19), (4.20). Further, recall that the external current must be conserved, which can be implemented by choosing $\delta J^0_{\rm ext} = k_i\,\delta J^i_{\rm ext}/\omega + \frac{1}{2}n_0\delta g_\mu{}^\mu$. The coupling $A_\mu J^\mu_{\rm ext}$ then implies that $i\omega\,\delta/\delta J^l_{\rm ext}(k)$ produces an insertion of $F_{0l}(-k)$, while $ik_m\epsilon^{nml}\delta/\delta J^l_{\rm ext}(k)$ produces an insertion of $\frac{1}{2}\epsilon^{nml}F_{lm}(-k)$. For example, for electric field correlation functions we have

$$G^R_{T^{\mu\nu}F_{0l}} = i\omega\frac{\delta}{\delta J^l_{\rm ext}}T^{\mu\nu}_{\text{on-shell}}[J_{\rm ext}, g]\,, \qquad G^R_{F_{\mu\nu}F_{0l}} = i\omega\frac{\delta}{\delta J^l_{\rm ext}}F^{\text{on-shell}}_{\mu\nu}[J_{\rm ext}, g]\,,$$

and similarly for the magnetic field.[5]

Choosing the external magnetic field in the z-direction, we find the same Kubo formulas (4.33) and (4.34). The electrical resistivities may also be expressed in terms of correlation functions of the electric field. In the zero-density state with $\mu_0 = 0$, $n_0 = 0$ we find

$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{F_{z0}F_{z0}}(\omega, \mathbf{k}{=}0) = \rho_\parallel\,, \tag{4.66a}$$

at small frequency, where $\rho_\parallel \equiv 1/\sigma_\parallel$. Similarly, for the transverse resistivities we find

$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{F_{x0}F_{x0}}(\omega, \mathbf{k}{=}0) = \rho_\perp\,, \tag{4.66b}$$
$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{F_{x0}F_{y0}}(\omega, \mathbf{k}{=}0) = -\tilde\rho_\perp\,\mathrm{sign}(B_0)\,, \tag{4.66c}$$

where again $w_0 \equiv \epsilon_0{+}p_0$, and $\rho_\perp$, $\tilde\rho_\perp$ were defined below eq. (4.24). We have taken $B_0^2 \ll w_0$, otherwise there is a multiplicative factor of $w_0(w_0{-}B_0^2 M_{\Omega,\mu})\mu_{\rm m}^2/(w_0\mu_{\rm m}{+}B_0^2)^2$ in the right-hand side of (4.66b), (4.66c). In a charged state (offset by non-dynamical $-n_0$), the correlation functions change, for example $G^R_{F_{x0}F_{y0}}(\omega, \mathbf{k}{=}0) = i\omega\frac{B_0}{n_0}$, while $\sigma_\parallel$ can be found from

$$\tfrac{1}{\omega}\mathrm{Im}\, G^R_{T_{0z}T_{0z}}(\omega, \mathbf{k}{=}0) = \sigma_\parallel\,. \tag{4.67}$$

Retarded functions at non-zero momentum may be found from the above variational procedure. For example, the function $G^R_{F_{x0}F_{x0}}(\omega, \mathbf{k})$ in a state with $n_0 = 0$ and with $\mathbf{k} \parallel \mathbf{B}_0$ has singularities at the eigenfrequencies of Alfvén waves for small momenta.

4.5 A dual formulation

As this work was being completed, an interesting article [37] (abbreviated below as GHI) came out which approached magnetohydrodynamics from a different perspective. The dual electromagnetic field strength tensor $J^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ was taken as a conserved current, and the constitutive relations were written down for $J^{\mu\nu}$, rather than for the electric current J^μ as was done in MHD historically. This "dual" construction follows the earlier work of ref. [73] which studied a similar MHD-like setup

[5] Alternatively, one can introduce an antisymmetric "polarization source" $M^{\mu\nu}_{\rm ext}$, by taking the conserved current as $J^\mu_{\rm ext} = \nabla_\nu M^{\mu\nu}_{\rm ext}$. The coupling $A_\mu J^\mu_{\rm ext}$ then becomes $\frac{1}{2}M^{\mu\nu}_{\rm ext}F_{\mu\nu}$ upon integration by parts, and correlation functions of $F_{\mu\nu}$ may be obtained as variations with respect to $M^{\mu\nu}_{\rm ext}$.

for "string fluids". The paper [37] identifies six transport coefficients in MHD, compared to eleven transport coefficients (in a parity-preserving system) found here. In this section we revisit the analysis of GHI, and show that the dual formulation allows for the same eleven transport coefficients we described earlier in Sections 4.3 and 4.4.

4.5.1 Constitutive relations

The conservation laws are taken as follows:

$$\nabla_\mu T^{\mu\nu} = H^\nu{}_{\rho\sigma} J^{\rho\sigma}\,, \qquad \nabla_\mu J^{\mu\nu} = 0\,. \tag{4.68}$$

These are the same equations (4.58a), (4.58c) we had earlier. The conserved external current is taken as $J^\mu_{\text{ext}} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu \Pi^{\text{ext}}_{\rho\sigma}$, where $\Pi^{\text{ext}}_{\mu\nu}$ may be viewed as the dual of the external polarization tensor $M^{\mu\nu}_{\text{ext}}$. The coupling $A_\mu J^\mu_{\text{ext}}$ then becomes $\frac{1}{2}\Pi^{\text{ext}}_{\mu\nu} J^{\mu\nu}$ upon integration by parts, and correlation functions of $J^{\mu\nu}$ may be obtained as variations with respect to $\Pi^{\text{ext}}_{\mu\nu}$. The tensor H in (4.68) is $H = \frac{1}{2} d\Pi^{\text{ext}}$, or in components $H_{\alpha\beta\gamma} = \frac{1}{4}\partial_\alpha \Pi^{\text{ext}}_{\beta\gamma}$ + (signed permutations).

In order to relate the GHI thermodynamic parameters to ours, we can compare equilibrium currents. The currents at zeroth order in derivatives are given by

$$T^{\mu\nu} = (\varepsilon_{\text{d}} + p_{\text{d}}) u^\mu u^\nu + p_{\text{d}}\, g^{\mu\nu} - \mu_{\text{d}}\, \rho_{\text{d}}\, h^\mu h^\nu + O(\partial)\,, \tag{4.69a}$$

$$J^{\mu\nu} = \rho_{\text{d}} (u^\mu h^\nu - u^\nu h^\mu) + O(\partial)\,. \tag{4.69b}$$

The subscript "d" for "dual" is used to differentiate the parameters from those used earlier in this chapter. The currents can be compared with our eq. (4.19) and the dual of eq. (4.4) at zeroth order:

$$T^{\mu\nu} = \left(w_{\text{m}} + \frac{B^2}{\mu_{\text{m}}} \right) u^\mu u^\nu + \left(-\tfrac{1}{2} B^2 + p_{\text{m}} + \frac{B^2}{\mu_{\text{m}}} \right) g^{\mu\nu} - \frac{B^\mu B^\nu}{\mu_{\text{m}}} + O(\partial)\,, \tag{4.70a}$$

$$J^{\mu\nu} = u^\mu B^\nu - u^\nu B^\mu + O(\partial)\,, \tag{4.70b}$$

where $w_{\text{m}} \equiv T p_{\text{m},T} + \mu p_{\text{m},\mu} = Ts + \mu n$ is the enthalpy density, and $\mu_{\text{m}} = 1/(1 - 2\partial p_{\text{m}}/\partial B^2)$ is the magnetic permeability. Using $h^2 = 1$, we can identify $\rho_{\text{d}} = B$, $\mu_{\text{d}} = B/\mu_{\text{m}}$, $h^\mu = B^\mu/B$, $p_{\text{d}} = -\frac{1}{2}B^2 + p_{\text{m}} + B^2/\mu_{\text{m}}$, up to $O(\partial)$ terms. Out of equilibrium, h^μ and μ_{d} are auxiliary dynamical variables (without a unique microscopic definition) designed to capture the dynamics of the magnetic field. The entropy density is $s_{\text{d}} =$

$p_{\mathrm{m},T} + \frac{\mu}{T} p_{\mathrm{m},\mu}$, as follows from $\varepsilon_{\mathrm{d}} + p_{\mathrm{d}} = Ts_{\mathrm{d}} + \mu_{\mathrm{d}}\rho_{\mathrm{d}}$. The energy densities coincide, $\varepsilon_{\mathrm{d}} = -p + Ts + \mu n = \epsilon$, again with $p = -\frac{1}{2}B^2 + p_{\mathrm{m}}(T, \mu, B^2)$.

At order $O(\partial)$, our constitutive relations can not be directly compared to those of GHI because of different hydrodynamic variables. However, we can compare the number of transport coefficients. The comparison may be done based on the entropy current argument which we review below.

In a particular hydrodynamic "frame", the one-derivative contributions to the GHI constitutive relations are given in eq. (3.4), (3.5) of ref. [37],

$$T^{\mu\nu}_{(1)} = \delta f_{\mathrm{d}}\, \Delta^{\mu\nu}_{\mathrm{d}} + \delta\tau_{\mathrm{d}}\, h^\mu h^\nu + \ell^\mu_{\mathrm{d}} h^\nu + \ell^\nu_{\mathrm{d}} h^\mu + t^{\mu\nu}_{\mathrm{d}}\,, \tag{4.71a}$$

$$J^{\mu\nu}_{(1)} = m^\mu_{\mathrm{d}} h^\nu - m^\nu_{\mathrm{d}} h^\mu + s^{\mu\nu}_{\mathrm{d}}\,, \tag{4.71b}$$

where $\Delta^{\mu\nu}_{\mathrm{d}} = g^{\mu\nu} + u^\mu u^\nu - h^\mu h^\nu$, and the coefficients δf_{d}, $\delta\tau_{\mathrm{d}}$, ℓ^μ_{d}, $t^{\mu\nu}_{\mathrm{d}}$, m^μ_{d}, $s^{\mu\nu}_{\mathrm{d}}$ are all $O(\partial)$. The quantities ℓ^μ_{d}, $t^{\mu\nu}_{\mathrm{d}}$, m^μ_{d}, $s^{\mu\nu}_{\mathrm{d}}$ are all transverse to both u_μ and h_μ, the tensor $t^{\mu\nu}_{\mathrm{d}}$ is symmetric and traceless, and the tensor $s^{\mu\nu}_{\mathrm{d}}$ is anti-symmetric. We do not write the subscript on the temperature and fluid velocity, even though the GHI's T and u^μ differ from ours at $O(\partial)$. Further, GHI impose charge conjugation as a constraint on the dynamics.

4.5.2 Entropy production

The "canonical" entropy current in the GHI formulation is analogous to eq. (4.25),

$$S^\mu_{\mathrm{d}} = \frac{1}{T}\left(p_{\mathrm{d}} u^\mu - T^{\mu\nu} u_\nu - \mu_{\mathrm{d}} J^{\mu\nu} h_\nu\right) . \tag{4.72}$$

This does not take into account the $O(\partial)$ contributions to thermodynamics: as we have seen earlier, the only non-trivial thermodynamic susceptibility in a parity-invariant theory is odd under charge charge conjugation C, and gets eliminated if C is imposed as a symmetry of hydrodynamics.

Upon using the conservation equations (4.68) together with the zeroth-order constitutive relations (4.69), the divergence of the entropy current (4.72) is

$$\nabla_\mu S^\mu_{\mathrm{d}} = -T^{\mu\nu}_{(1)}\, \nabla_\mu \left(\frac{u_\nu}{T}\right) - J^{\mu\nu}_{(1)} \left[\nabla_\mu \left(\frac{\mu_{\mathrm{d}} h_\nu}{T}\right) + \frac{u_\alpha H^\alpha{}_{\mu\nu}}{T}\right] .$$

Substituting the first-order constitutive relations (4.71), we find

$$T\nabla_\mu S^\mu_{\rm d} = -\delta f_{\rm d}\,(S_3 - S_4) - \delta\tau_{\rm d} S_4 - \ell^\mu_{\rm d}\Sigma_\mu - \tfrac{1}{2}t^{\rm d}_{\mu\nu}\sigma^{\mu\nu}_\perp - m^{\rm d}_\alpha Y^\alpha - \tfrac{1}{2}s^{\rm d}_{\rho\sigma}Z^{\rho\sigma}\,. \tag{4.73}$$

Using the notation similar to sec. 4.3.6, we have the scalars $S_3 \equiv \nabla\cdot u$, $S_4 \equiv h^\mu h^\nu \nabla_\mu u_\nu$, as well as $\sigma^{\mu\nu}_\perp \equiv \frac{1}{2}\left(\Delta^{\mu\alpha}_{\rm d}\Delta^{\nu\beta}_{\rm d} + \Delta^{\nu\alpha}_{\rm d}\Delta^{\mu\beta}_{\rm d} - \Delta^{\mu\nu}_{\rm d}\Delta^{\alpha\beta}_{\rm d}\right)\sigma_{\alpha\beta}$ and $\Sigma^\mu \equiv \Delta^{\mu\lambda}_{\rm d}\sigma_{\lambda\rho}h^\rho$. We have further defined

$$\begin{aligned} Y^\lambda &\equiv \Delta^{\lambda\rho}_{\rm d}\left[T\partial_\rho(\mu_{\rm d}/T) + 2u_\alpha H^\alpha{}_{\rho\sigma}h^\sigma - \mu_{\rm d} h^\alpha \nabla_\alpha h_\rho\right]\,, \\ Z^{\alpha\beta} &\equiv \Delta^{\alpha\rho}_{\rm d}\Delta^{\beta\sigma}_{\rm d}\left[\mu_{\rm d}(\nabla_\rho h_\sigma - \nabla_\sigma h_\rho) + 2u_\alpha H^\alpha{}_{\rho\sigma}\right]\,. \end{aligned}$$

In order to ensure that the entropy production in eq. (4.73) is non-negative, GHI demand

$$\begin{aligned} &\delta f_{\rm d} = -\zeta_\perp(S_3 - S_4)\,, \qquad \delta\tau_{\rm d} = -2\zeta_\parallel S_4\,, \qquad \ell^\mu_{\rm d} = -\eta_\parallel \Sigma^\mu\,, \\ &t^{\mu\nu}_{\rm d} = -\eta_\perp \sigma^{\mu\nu}_\perp\,, \qquad m^\alpha_{\rm d} = -r_\perp Y^\alpha\,, \qquad s^{\rho\sigma}_{\rm d} = -r_\parallel Z^{\rho\sigma}\,, \end{aligned} \tag{4.74}$$

with six non-negative coefficients $\zeta_\perp$, $\zeta_\parallel$, $\eta_\perp$, $\eta_\parallel$, $r_\perp$, $r_\parallel$. This clearly gives $\nabla_\mu S^\mu_{\rm d} \geqslant 0$.

Note however that while demanding eq. (4.74) is sufficient to ensure non-negative entropy production, there are more ways besides eq. (4.74) to make the right-hand side of eq. (4.73) non-negative. These other options will give rise to extra transport coefficients. Indeed, consider the following coefficients of the $O(\partial)$ constitutive relations:

$$\delta f_{\rm d} = -f_1 S_3 - f_2 S_4\,, \tag{4.75a}$$

$$\delta\tau_{\rm d} = -\tau_1 S_3 - \tau_2 S_4\,, \tag{4.75b}$$

$$\ell^\mu_{\rm d} = -\eta_\parallel \Sigma^\mu - \tilde\eta_\parallel \tilde\Sigma^\mu\,, \tag{4.75c}$$

$$t^{\mu\nu}_{\rm d} = -\eta_\perp \sigma^{\mu\nu}_\perp - \tilde\eta_\perp \tilde\sigma^{\mu\nu}_\perp\,, \tag{4.75d}$$

$$m^\alpha_{\rm d} = -r_\perp Y^\alpha - \tilde r_\perp \tilde Y^\alpha\,, \tag{4.75e}$$

$$s^{\rho\sigma}_{\rm d} = -r_\parallel Z^{\rho\sigma}\,. \tag{4.75f}$$

The tilded vectors are defined as $\tilde V^\mu = \epsilon^{\mu\nu\alpha\beta}u_\nu h_\alpha V_\beta$, and the tilded shear tensor is

$$\tilde\sigma^{\mu\nu}_\perp \equiv \tfrac{1}{2}\left(\epsilon^{\mu\lambda\alpha}{}_\beta u_\lambda h_\alpha \sigma^{\beta\nu}_\perp + \epsilon^{\nu\lambda\alpha}{}_\beta u_\lambda h_\alpha \sigma^{\beta\mu}_\perp\right)\,,$$

as in eq. (4.16). The tensor $s^{\rho\sigma}_{\rm d}$ has only one degree of freedom, hence it contains only one transport coefficient. The divergence of the entropy current (4.73) is then

$$\begin{aligned} T\nabla_\mu S^\mu_{\rm d} &= f_1 S_3^2 + (\tau_1{+}f_2{-}f_1) S_3 S_4 + (\tau_2{-}f_2) S_4^2 \\ &\quad + \eta_\| \Sigma_\mu \Sigma^\mu + \tfrac{1}{2}\eta_\perp (\sigma_\perp^{\mu\nu})^2 + r_\perp Y_\mu Y^\mu + \tfrac{1}{2} r_\| (Z^{\rho\sigma})^2 \,. \end{aligned} \tag{4.76}$$

The three tilded coefficients do not contribute to entropy production in eq. (4.73) due to $\tilde{V}^\mu V_\mu = 0$ and $\sigma_{\perp\mu\nu}\,\tilde{\sigma}_\perp^{\mu\nu} = 0$, and can take any real values,

$$\tilde{\eta}_\| \in \mathbb{R}\,, \qquad \tilde{\eta}_\perp \in \mathbb{R}\,, \qquad \tilde{r}_\perp \in \mathbb{R}\,. \tag{4.77}$$

Demanding that $\nabla_\mu S^\mu_{\rm d}$ in eq. (4.76) is non-negative now implies

$$\eta_\perp \geqslant 0\,, \qquad \eta_\| \geqslant 0\,, \qquad r_\perp \geqslant 0\,, \qquad r_\| \geqslant 0\,, \tag{4.78a}$$

together with the condition that the quadratic form in the first line of eq. (4.76) is positive semi-definite. The latter gives

$$f_1 \geqslant 0\,, \qquad \tau_2 - f_2 \geqslant 0\,, \qquad f_1(\tau_2 - f_2) \geqslant \tfrac{1}{4}(\tau_1 - f_1 + f_2)^2\,. \tag{4.78b}$$

Thus there are eleven apriori independent non-equilibrium transport coefficients listed in Eqs. (4.75) that are consistent with non-negative entropy production, provided the constraints (4.78) are satisfied. The coefficients $\tilde{r}_\perp$, $\tilde{\eta}_\perp$, $\tilde{\eta}_\|$ are odd under charge conjugation C, and can be eliminated if one demands C-invariance of hydrodynamics. An implicit assumption of ref. [37] amounts to choosing $f_1 = -f_2 = \zeta_\perp$, $\tau_1 = 0$, $\tau_2 = 2\zeta_\|$.

4.5.3 Kubo formulas

Assuming time-reversal covariance, the above transport coefficients can be further constrained by the Onsager relation (4.28). In order to find the retarded functions, we can use exactly the same variational procedure as in sec. 4.4.5:

$$G^R_{T^{\mu\nu}T^{\alpha\beta}} = \frac{2\,\delta}{\delta g_{\alpha\beta}} \left(\sqrt{-g}\, T^{\mu\nu}_{\text{on-shell}}[\Pi^{\rm ext}, g]\right)\,, \quad G^R_{J^{\mu\nu}T^{\alpha\beta}} = \frac{2\,\delta}{\delta g_{\alpha\beta}} \left(\sqrt{-g}\, J^{\mu\nu}_{\text{on-shell}}[\Pi^{\rm ext}, g]\right)\,, \tag{4.79a}$$

as well as

$$G^R_{T^{\mu\nu}J^{\alpha\beta}} = 2\frac{\delta}{\delta\Pi^{\text{ext}}_{\alpha\beta}} T^{\mu\nu}_{\text{on-shell}}[\Pi^{\text{ext}}, g]\,, \qquad G^R_{J^{\mu\nu}J^{\alpha\beta}} = 2\frac{\delta}{\delta\Pi^{\text{ext}}_{\alpha\beta}} J^{\mu\nu}_{\text{on-shell}}[\Pi^{\text{ext}}, g]\,. \tag{4.79b}$$

Again, the subscript "on-shell" signifies that $T^{\mu\nu}$ and $J^{\mu\nu}$ are evaluated on the solutions to the conservation equations (4.68) with the constitutive relations (4.75). We use the above prescription to evaluate correlation functions at zero spatial momentum, which gives rise to Kubo formulas. Demanding that the correlation functions satisfy (4.28) now gives the Onsager relation

$$\tau_1 = f_1 + f_2\,. \tag{4.80}$$

We further find the following Kubo formulas for transport coefficients in the constitutive relations (4.75). The resistivities are given by

$$\tfrac{1}{\omega}\text{Im}\, G^R_{J^{xy}J^{xy}}(\omega, \mathbf{k}{=}0) = r_\parallel\,, \tag{4.81a}$$

$$\tfrac{1}{\omega}\text{Im}\, G^R_{J^{xz}J^{xz}}(\omega, \mathbf{k}{=}0) = r_\perp\,, \tag{4.81b}$$

$$\tfrac{1}{\omega}\text{Im}\, G^R_{J^{yz}J^{xz}}(\omega, \mathbf{k}{=}0) = \tilde{r}_\perp\, \text{sign}(B_0)\,, \tag{4.81c}$$

the "shear viscosities" are given by

$$\tfrac{1}{\omega}\text{Im}\, G^R_{T^{xz}T^{xz}}(\omega, \mathbf{k}{=}0) = \eta_\parallel\,, \qquad \tfrac{1}{\omega}\text{Im}\, G^R_{T^{xy}T^{xy}}(\omega, \mathbf{k}{=}0) = \eta_\perp\,, \tag{4.81d}$$

$$\tfrac{1}{\omega}\text{Im}\, G^R_{T^{yz}T^{xz}}(\omega, \mathbf{k}{=}0) = \tilde{\eta}_\parallel\, \text{sign}(B_0)\,, \qquad \tfrac{1}{\omega}\text{Im}\, G^R_{T^{xy}T^{xx}}(\omega, \mathbf{k}{=}0) = \tilde{\eta}_\perp\, \text{sign}(B_0)\,, \tag{4.81e}$$

and the "bulk viscosities" are given by

$$\tfrac{1}{\omega}\text{Im}\, G^R_{T^{xx}T^{xx}}(\omega, \mathbf{k}{=}0) = f_1 + \eta_\perp\,, \tag{4.81f}$$

$$\tfrac{1}{\omega}\text{Im}\, G^R_{T^{xx}T^{zz}}(\omega, \mathbf{k}{=}0) = f_1 + f_2\,, \tag{4.81g}$$

$$\tfrac{1}{\omega}\text{Im}\, G^R_{T^{zz}T^{zz}}(\omega, \mathbf{k}{=}0) = \tau_1 + \tau_2\,. \tag{4.81h}$$

Correlation functions at non-zero momentum may also be found by using the above variational procedure.

4.5.4 Mapping of transport coefficients

We can compare the correlation functions of $T^{\mu\nu}$ and $J^{\mu\nu}$ evaluated using (4.79) with the correlation functions found in sec. 4.4.5. If the two approaches to MHD (section 4.4 and section 4.5) compute the same physical objects $G^R_{T^{\mu\nu}T^{\alpha\beta}}$ etc, the results should agree. Comparing correlation functions at zero spatial momentum allows one to relate the transport coefficients in the constitutive relations (4.75) to transport coefficients introduced in section 4.3, see eq. (4.19), (4.20). Doing so in the (dynamically) neutral state with $n_0 = 0$ gives the following relations. The resistivities are related by

$$r_\parallel = \frac{1}{\sigma_\parallel}\,, \qquad r_\perp = \frac{\sigma_\perp}{\sigma_\perp^2 + \tilde{\sigma}^2}\,, \qquad \tilde{r}_\perp = -\frac{\tilde{\sigma}}{\sigma_\perp^2 + \tilde{\sigma}^2}\,, \tag{4.82a}$$

the "shear viscosities" $\eta_\perp$, $\tilde{\eta}_\perp$, $\eta_\parallel$, $\tilde{\eta}_\parallel$ agree, and the "bulk viscosities" are related by

$$f_1 = \zeta_1 - \tfrac{2}{3}\eta_1\,, \qquad f_2 = \zeta_2 - \tfrac{2}{3}\eta_2\,, \tag{4.82b}$$

$$\tau_1 = \zeta_1 + \tfrac{4}{3}\eta_1\,, \qquad \tau_2 = \zeta_2 + \tfrac{4}{3}\eta_2\,. \tag{4.82c}$$

The Onsager relation (4.30) maps to the Onsager relation (4.80), as expected. The entropy current constraints (4.27) map to the entropy current constraints (4.78), as expected.

Finally, the mapping of transport coefficients (4.82) can be used to compare the eigenfrequencies of small oscillations of the (dynamically) neutral state found in eq. (4.60), (4.61) to those found in ref. [37]. Using the map of thermodynamic parameters spelled out below eq. (4.70), the speed of Alfvén waves agrees with ref. [37]. The damping coefficient of Alfvén waves in eq. (4.60) agrees with ref. [37] when $B^2/\mu_{\rm m} \ll \epsilon + p$. The speed of magnetosonic waves in eq. (4.61b) agrees with ref. [37]: in order to see this, note that the assumption of constant magnetic permeability amounts to assuming that the equation of state takes the form $p_{\rm d} = \frac{1}{2}\mu_{\rm m}\mu_{\rm d}^2 + F(T)$, or $p = -\frac{1}{2\mu_{\rm m}}B^2 + F(T)$, with some $F(T)$. In general, the speed of magnetosonic waves derived from the formalisms of sec. 4.4 and sec. 4.5 will not agree, except when $B^2/\mu_{\rm m} \ll (\epsilon + p)$. One reason is that the chemical potential for the electric charge is treated as a thermodynamic variable in sec. 4.4, hence the magnetosonic wave speed will in general depend on the charge susceptibility $(\partial n/\partial\mu)_{\mu=0}$. This thermodynamic derivative is not present in the formalism of sec. 4.5. Finally, note that the transport coefficient τ_1 contributes to damping of fast magnetosonic waves, for example at $\theta = 0$

we have $\Gamma_{\rm ms} = (\tau_1 + \tau_2)/(Ts_{\rm d})$, in agreement with eq. (4.61f).

4.6 Discussion

In this chapter we have presented the equations of relativistic magnetohydrodynamics, by which we mean the hydrodynamics of a conducting fluid in local thermal equilibrium, with dynamical electromagnetic fields. MHD is naturally formulated in a derivative expansion with magnetic field $B \sim O(1)$. Electric screening does not imply that the electric field vanishes: rather, it implies $E \sim O(\partial)$ is subleading in the derivative expansion. We have adopted the simplest "mean-field" formulation in which the constitutive relations in the theory with dynamical electromagnetic fields are inherited from the theory with external electromagnetic fields. Our main focus was on transport coefficients. For a parity-symmetric microscopic system, we find eleven transport coefficients at one-derivative order. One transport coefficient is thermodynamic: it is a part of the equation of state in curved space, and contributes to flat-space correlations. Transport coefficients of this type in relativistic hydrodynamics were first identified in [15] where they appeared at second order in derivatives. In 2+1 dimensional hydrodynamics, thermodynamic transport coefficients can already appear at first order in derivatives [17]. Of the remaining ten transport coefficients, three are non-equilibrium and non-dissipative, and seven are non-equilibrium and dissipative. There are more transport coefficients for parity-violating fluids, as listed in sec. 4.3. We now comment on questions not discussed in detail in the main body of the chapter.

> *Angular momentum generated by the magnetic field.*— The thermodynamic transport coefficient M_Ω determines the response of equilibrium magnetic polarization to vorticity, as can be seen from eq. (4.21). One way to view M_Ω is to note that a system of charged particles in external magnetic field will develop angular momentum. One can see this in the thermodynamic framework of sec. 4.2. For a bounded system, the equilibrium energy-momentum tensor obtained by varying the equilibrium free energy (4.1), (4.7) with respect to the metric will have a boundary contribution after the variation $M_\Omega B{\cdot}\delta_g\Omega$ is integrated by parts [70]. The surface momentum density $\mathcal{Q}_{\rm s}^\alpha = M_\Omega \epsilon^{\alpha\mu\nu\rho} u_\mu B_\nu n_\rho$ (where n^μ is the unit spacelike normal vector to the boundary) will give rise to angular momentum induced by the magnetic field. Consider a system at rest in

flat space at constant temperature, charge density, and constant magnetic field $\mathbf{B}$. The angular momentum $\mathbf{L}$ derived from the energy-momentum tensor only receives a boundary contribution, and one finds

$$\frac{\mathbf{L}}{V} = 2M_\Omega \mathbf{B} \,,$$

where V is the spatial volume. In this sense M_Ω determines "angular momentum density". As the coefficient M_Ω is odd under charge conjugation C, this generation of angular momentum only happens in a C-invariant theory if the equilibrium state has non-zero charge density. Similarly, for a system not subject to the magnetic field, in flat space, which rotates uniformly with small (namely $|\boldsymbol{\omega}|R \ll 1$ where R is the size of the system) angular velocity $\boldsymbol{\omega}$, the magnetization density is $\mathbf{m} = 2M_\Omega \boldsymbol{\omega}$. More generally, the susceptibility M_Ω provides a macroscopic parametrization of gyromagnetic phenomena such as the Barnett and Einstein-de Haas effects.

Previous work on transport coefficients.— Papers [74, 75] studied transport coefficients for relativistic fluids subject to an external magnetic field. While this does not correspond to MHD in the sense described in this chapter (we define MHD as a theory in which magnetic field or its auxiliary is a dynamical degree of freedom), a fluid in external field is a fundamental building block for MHD. Parts of Refs. [74, 75] overlap with our Section 4.3. Some of our results differ from those in Refs. [74, 75]: the analysis of thermodynamics, the number of transport coefficients, constraints on transport coefficients imposed by the positivity of entropy production, and some of the Kubo formulas. The details are given in Appendix C.2.

Dual formulation of magneto-hydrodynamics.— In sec. 4.5 we compared our results with the recent "dual" formulation of MHD in ref. [37]. We found the same number of transport coefficients in the two approaches, provided the bulk viscosity missed in ref. [37] is restored, and the constraint of C-invariance imposed in ref. [37] is lifted. It would be interesting to investigate the relation between the "dual" and "conventional" formulations of MHD further, in particular with regard to the description of electric charge fluctuations.

Applicability regime.— The MHD described in this chapter treats electromagnetic fields classically. This means that the electromagnetic coupling constant must be small so that quantum fluctuations of the electromagnetic field can be ignored. The applicability regime of MHD also includes $B \ll T^2$ (or restoring the fundamental constants $\hbar c e B \ll (k_{\rm B}T)^2$), as is necessary to restrict the hydrodynamic degrees of freedom to those inherited from thermodynamics. We do not have a method to systematically incorporate the effects of larger magnetic fields within the MHD description of sec. 4.4. A possible way to approach this problem would be to study the hydrodynamic description for the D'Hoker-Kraus magnetic brane [76] in the $B > T^2$ regime. These magnetic branes solutions in AdS_5 are dual to 4-dimensional gauge theories in the presence of constant background magnetic fields. The classical hydrodynamic theory also ignores statistical fluctuations, which are known to invalidate classical second-order hydrodynamics in 3+1 dimensions (and classical first-order hydrodynamics in 2+1 dimensions). Understanding the effects of statistical fluctuations in magnetic field requires further work.

Transport coefficients at strong coupling.— While the small electromagnetic coupling allows one to treat magnetic fields classically, other interactions in the theory do not have to be small. For strongly interacting non-abelian gauge theories in external $U(1)$ magnetic field, methods of gauge-gravity duality provide a window into non-equilibrium physics, both within and outside the hydrodynamic regime. Some of the hydrodynamic transport coefficients discussed in this chapter were evaluated in holographic models in refs. [77, 75]. The full set of transport coefficients for fluids in external magnetic field has not yet been explored holographically.

Higher-order terms.— We have not taken into account the terms beyond first order in the derivative expansion. In conventional hydrodynamics, higher-order terms are required to render the theory causal [78] (see e.g. [15, 79] for more recent discussions). We expect that a causal formulation of MHD will involve higher-order relaxation times as well as the electric field dynamics.

Chapter 5

Conclusion

In this work, we employed the equilibrium generating functional formulation of relativistic hydrodynamics postulated in [18, 19] to explore three new regimes of relativistic hydrodynamics: relativistic hydrodynamics in the presence of an external vector field, relativistic hydrodynamics in strong external magnetic fields and relativistic hydrodynamics in strong dynamical magnetic fields.

Previous approaches to anisotropic hydrodynamics have consisted of modifications to the isotropic hydrodynamic framework by the use of Boltzmann's equation with an anisotropic distribution function and the addition of new hydrodynamic variables characterizing the anisotropy [80, 81, 82, 83]. The derivation presented in chapter 3 is the first to start from a generating functional that depends on an external vector field characterizing the anisotropy. Consequently, the anisotropic constitutive relations (3.35) and (3.37), the modified equations of motions (3.8) and entropy current (3.32) and the use of the auxiliary vector X_μ are all new to the approach presented here. We assumed that the external vector n^μ has a microscopic description and thus is not a hydrodynamic variable subject to frame transformations. This formulation of anisotropic hydrodynamics presents several directions for further studies. An eigenmode analysis similar to the ones presented in sections 4.3.5 and 4.4.4 for the new system of equations can provide experimentally testable departures of this framework from an isotropic hydrodynamic description of the materials introduced in section 3.1. Following the prescription of chapter 2, the modified entropy current of section 3.3.2 can be found to $O(\partial)$, from which the constraints on transport coefficients due to entropy production can be derived. Similarly, the Onsager relations coming from hydrodynamic correlation functions can impose further constraints on the transport coefficients. The Kubo formulas for these transport coefficients can also be derived following the prescription presented in 2.2.4.

We presented the constitutive relations for relativistic fluids in strong magnetic fields to first order in derivatives including the effects of polarization , and derived the constraints on transport coefficients as well as their corresponding Kubo formulas in chapter 4. For parity preserving systems, these include one thermodynamic transport coefficient M_Ω and eleven non-equilibrium transport coefficients: three charge conductivities, four bulk viscosities and four shear viscosities. Prior to [70], the magnetovortical susceptibility M_Ω doesn't appear in previous literature of relativistic magnetohydrodynamics. This thermodynamic function determines the response of the magnetic polarization to vorticity and as well as the angular momentum generated due to an external magnetic field. Ten of the eleven non-equilibrium transport coeffieients were known for some time [74]. The combination of bulk viscosities $3\zeta_2 - 6\eta_1 - 2\eta_2$ doesn't appear in the literature, and can be non-zero for hydrodynamic descriptions of systems that don't preserve time reversal invariance. The eight parity violating transport coefficients studied in section 4.3.9 and the four parity-violating thermodynamic functions $M_n\,,\ n = 1, 2, 3, 5$ coming from the generating functional as outlined in section 4.2 are also new. The Kubo formulas for the complete set of transport coefficients for relativistic fluids subject to a constant magnetic field was presented in sections 4.3.7, 4.4.5 and 4.3.9, and includes the ones presented earlier in [75] and [37] respectively. The full set of Kubo formulas for parity-respecting relativistic fluids in a constant magnetic first appeared in [47]. See appendix C.2.2 for a comparison with [75] and section 4.5 for a comparison with [37]. The Kubo formulas for parity violating transport coefficients can be used to calculate them explicitly for theories that break parity.

We also studied the hydrodynamic fluctuations of fluids in strong external magnetic fields as well as for fluids in strong dynamical magnetic fields. The analysis of hydrodynamic fluctuations in strong dynamical magnetic fields is new, and was presented in section 4.4.4. For the neutral state in a magnetic field, we find three gapped modes and six gapless modes. Two gapped modes are transverse and one is longitudinal. The gapless modes include Alfvén waves and two branches of magnetosonic waves. For fluctuations parallel to the magnetic field, the fast branch of magnetosonic waves turn into sound waves and the slow branch into a second set of Alfvén waves. For fluctuations perpendicular to the magnetic field, Alfvén waves and the slow branch of magnetosonic waves become diffusive and the fast magnetosonic waves turn to sound waves. For the state with a non-zero dynamical charge density in a magnetic field, plasma oscillations gap out the magnetosonic waves, and the Alfvén waves get a quadratic dispersion relation. We thus find 6 gapped modes and three gapless modes. For fluctuations perpendicular to the magnetic field, the gapless modes are all diffusive. For fluctuations parallel to the magnetic field, we find two Alfvén-like modes with quadratic dispersion relation and one diffusive mode. The dispersion relations for these hydrodynamic fluctuations may in principle provide a way to experimentally measure the transport coefficients presented in this thesis.

With new insights into effective actions for out of equilibrium hydrodynamicss [27, 28, 29, 30, 31, 32], it would be interesting to see if the new transport coefficients presented in this work naturally arise in the derivative expansion of such an effective action if strong magnetic fields are assumed. If that were the case, one might consider the constraints that such a formalism would impose on the transport coefficients, and compare them with the ones derived here.

An important open question in hydrodynamics is the relation between the, in general, out of equilibrium generating functional and the entropy current. It was shown in [24, 25] that the constraints on transport coefficients derived from the entropy current are the same as those derived from the equilibrium generating functional up to a frame transformation from the thermodynamic frame, plus the inequality constraints on dissipative transport coefficients. These inequality constraints can be found through the demand of entropy production, as outlined in 2.2.3 or by the positivity constraints on hydrodynamic correlation functions, as outlined in 2.2.5. These inequality constraints on dissipative transpot coefficients have so far been the same for all systems studied, though the connection between these two is still not well understood. This connection has been studied in the language of non-relativistic hydrodynamics in flat space [2, 84] but a proper generalization to relativistic fluids in curved space-time is still lacking. This question could be explored in the effective action approach to hydrodynamics of [27, 28, 29, 30, 31, 32]. This question has recently been tackled in [85, 86].

Appendix A

Deriving the equations of motion

We want to show how gauge and diffeomorphism invariance of the generating functional $W[g, A]$ lead to the conservation equations (2.3):

$$\nabla_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda \,, \quad \nabla_\mu J^\mu = 0 \,.$$

Under a gauge transformation $A_\mu \to A_\mu - \partial_\mu \alpha$, the generating functional remains invariant. That is

$$\delta W = -\int d^{d+1}x \sqrt{-g} J^\mu \partial_\mu \alpha = \int d^{d+1}x \sqrt{-g} \alpha \nabla_\mu J^\mu = 0 \,.$$

Since α is arbitrary, it follows that the current is conserved, $\nabla_\mu J^\mu = 0$. To find the other equation we consider diffeomorphisms $x^\mu \to x^\mu - \xi^\mu$. The external metric and gauge field change by $\delta g_{\mu\nu} = g_{\mu\lambda}\partial_\nu \xi^\lambda + g_{\nu\lambda}\partial_\mu \xi^\lambda + \xi^\lambda \partial_\lambda g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ and $\delta A_\mu = A_\lambda \partial_\mu \xi^\lambda + \xi^\lambda \partial_\lambda A_\mu = A_\lambda \nabla_\mu \xi^\lambda + \xi^\lambda \nabla_\lambda A_\mu$. Again, the fact that the generating functional is invariant under this transformation leads to

$$\begin{aligned} 0 &= \int d^{d+1}x \sqrt{-g} \left(T^{\mu\nu} \nabla_\mu \xi_\nu + J^\mu \left(A_\nu \partial_\mu \xi^\nu + \xi^\nu \partial_\nu A_\mu \right) \right) \\ &= \int d^{d+1}x \sqrt{-g} \left(F^{\nu\mu} J_\mu - \nabla_\mu T^{\mu\nu} - \nabla_\mu J^\mu A^\nu \right) \xi_\nu \,, \end{aligned}$$

and since ξ^μ is arbitrary, the energy-momentum is conserved up to the influx term $\nabla_\mu T^{\mu\nu} = F^{\nu\mu} J_\mu$.

Now let's assume that the generating functional depends on a gauge invariant external vector n^μ, and we define X_μ as the auxiliary vector that couples to this external vector. The variation of the generating functional is then

$$\delta W = \int d^{d+1}x\sqrt{-g}\left(\tfrac{1}{2}T^{\mu\nu}\delta g_{\mu\nu} + J^{\mu}\delta A_{\mu} + X_{\mu}\delta n^{\mu}\right) \,.$$

This defines X_μ. For a gauge transformation $\delta n^\mu = 0$ and the current conservation equation remains unchanged, $\nabla_\mu J^\mu = 0$. For a diffeomorphism, however, we have $\delta n^\mu = \xi^\nu \partial_\nu n^\mu - n^\nu \partial_\nu \xi^\mu = \xi^\nu \nabla_\nu n^\mu - n^\nu \nabla_\nu \xi^\mu$. Diffeomorphism invariance of the generating functional then implies

$$\begin{aligned} 0 &= \int d^{d+1}x\sqrt{-g}\left[\left(F^{\nu\mu}J_\mu - \nabla_\mu T^{\mu\nu} - \nabla_\mu J^\mu A^\nu\right)\xi_\nu + X_\mu\left(\xi^\nu \nabla_\nu n^\mu - n^\nu \nabla_\nu \xi^\mu\right)\right] \,, \\ &= \int d^{d+1}x\sqrt{-g}\left(F^{\nu\mu}J_\mu - \nabla_\mu T^{\mu\nu} - \nabla_\mu J^\mu A^\nu + X_\mu \nabla^\nu n^\mu + \nabla_\mu\left(n^\mu X^\nu\right)\right)\xi_\nu \end{aligned}$$

from which we get the anisotropic equations of motion (3.8)

$$\nabla_\mu T^{\mu\nu} = F^{\nu\mu}J_\mu + X_\mu \nabla^\nu n^\mu + \nabla_\mu\Big(X^\nu n^\mu\Big), \quad \nabla_\mu J^\mu = 0 \,.$$

Appendix B

More on anisotropic hydrodynamics

B.1 Equilibrium constraints

The requirement that $\mathcal{L}_V g = 0$ gives one symmetric $SO(d,1)$ tensor equation. Since we now have two $O(1)$ vectors u^μ and n^μ in the hydrodynamic description, we can use the projector $\mathbb{N}^{\mu\nu} = \Delta^{\mu\nu} - n_\perp^\mu n_\perp^\nu / n_\perp^2$ and the two vectors u^μ and $n_\perp^\mu$ to decompose this equation into one $SO(d-1)$ tensor equation, two $SO(d-1)$ vector equations and four $SO(d-1)$ scalar equations

$$\sigma_\perp^{\mu\nu} \equiv \mathbb{N}^{\mu\alpha}\mathbb{N}^{\nu\beta}\Big(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{d-1} g_{\alpha\beta}\mathbb{N}^{\rho\sigma}\nabla_\rho u_\sigma\Big) = 0\,, \tag{B.1a}$$

$$\mathbb{N}^{\mu\nu}\left(\partial_\nu T + T a_\nu\right) = 0, \quad \mathbb{N}^{\mu\nu}\sigma_{\nu\rho} n^\rho = 0\,, \tag{B.1b}$$

$$\partial_u T = 0, \quad \nabla_\mu u^\mu = 0, \quad \partial_n T = -T n\cdot a, \quad \mathbb{N}^{\mu\nu}\nabla_\mu u_\nu = 0. \tag{B.1c}$$

We use the shorthand $\partial_v = v^\mu \partial_v$ for any vector v^μ. Similarly, the requirement $\mathcal{L}_V A = 0$ gives one $SO(d-1)$ vector equation and two $SO(d-1)$ scalar equations

$$\mathbb{N}^{\mu\nu}\left(E_\mu - T\partial_\nu \frac{\mu}{T}\right) = 0\,, \tag{B.2a}$$

$$\partial_u \frac{\mu}{T} = 0, \quad \partial_n \frac{\mu}{T} = \frac{1}{T} E\cdot n. \tag{B.2b}$$

Finally, $\mathcal{L}_V n = 0$ gives one $SO(d-1)$ vector equation and two $SO(d-1)$ scalar equations. One scalar equation is similar to that found by $\mathcal{L}_V g = 0$

$$\nabla_n u_\mu = \Delta_{\mu\nu} \nabla_u n^\nu \,, \tag{B.3a}$$

$$\partial_u \frac{n{\cdot}u}{T} = 0, \quad \mathbb{N}^{\mu\nu} \nabla_\mu u_\nu = 0. \tag{B.3b}$$

We have sued the shorthand $\nabla_v = v^\mu \nabla_\mu$ for any vector v^μ. Equations (B.1), (B.2) and (B.3) give in total 7 scalar constraints, 4 vector constraints and 1 tensor constraint. Using the Levi-Civita tensor $\epsilon^{\mu\nu\rho\sigma}$ and the constraints (B.1), (B.2) and (B.3), we can form 4 new vectors and 1 new tensor that must vanish in equilibrium. The scalars, vectors and tensors that vanish in equilibrium but can in principle appear in the constitutive relations when we allow our system to be time dependent are the non-equilibrium structures. These are listed listed in table B.1.

First order non-equilibrium structures							
	1	2	3	4	5	6	7
scalars ($s_{i,\text{non-eq.}}$)	$n \cdot (E - T\partial \frac{\mu}{T})$	$n \cdot (a + \frac{\partial T}{T})$	$\mathbb{N}^{\mu\nu}\nabla_\mu u_\nu$	$\partial_u T$	$\partial_u \frac{\mu}{T}$	$\partial_u \frac{n\cdot u}{T}$	$\nabla_\mu u^\mu$
vectors ($v^\mu_{i,\text{non-eq.}}$)	$\mathbb{N}^{\mu\nu}(E_\nu - T\partial_\nu \frac{\mu}{T})$	$\mathbb{N}^{\mu\nu}(a_\nu - \frac{\partial_\nu T}{T})$	$\mathbb{N}^{\mu\nu}\sigma_{\nu\rho}n^\rho$	$\mathbb{N}^{\mu\nu}(\nabla_u n_\nu - \nabla_n u_\mu)$			
tensors ($t^{\mu\nu}_{i,\text{non-eq.}}$)	$\sigma^{\mu\nu}_\perp$	$\tilde{\sigma}^{\mu\nu}_\perp$					

Table B.1: Non-equilibrium first order independent scalars, vectors and tensors in a parity invariant hydrodynamic theory. We have decomposed them in terms of the SO(d-1) subgroup of the Lorentz group leaving u^μ and n^μ invariant. We have defined $\sigma^{\mu\nu}_\perp = \mathbb{N}^{\mu\alpha}\mathbb{N}^{\nu\beta}(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{d-1} g_{\alpha\beta}\mathbb{N}^{\rho\sigma}\nabla_\rho u_\sigma)$ and $\tilde{\sigma}^{\mu\nu}_\perp = \frac{1}{2}(\epsilon^{\mu\rho\sigma\alpha}u_\rho n_\sigma \sigma^\nu_{\perp\alpha} + \epsilon^{\nu\rho\sigma\alpha}u_\rho n_\sigma \sigma^\mu_{\perp\alpha})$. For every vector there is an additional $\tilde{v}^{(1)\mu}_{i,\text{non-eq.}} = \epsilon^{\mu\nu\rho\sigma}u_\nu n_\rho v^{(1)}_{i,\text{non-eq.}\sigma}$.

B.2 More on frame transformations

In this appendix, we look at frame invariants and frame transformations for anisotropic hydrodynamics. The first order scalars, vectors and tensors appearing in the constitutive relations were defined in sections 3.2 and 3.3.1. For an arbitrary frame, these $O(\partial)$ terms have the expansion

$$
\begin{aligned}
f_{\mathcal{E}} &= \sum_{n=1}^{m}\sum_{i=1}^{N_n} \epsilon_{n,i} s_{n,i}, \qquad & f_{\mathcal{P}} &= \sum_{n=1}^{m}\sum_{i=1}^{N_n} \pi_{n,i} s_{n,i}, \qquad & f_{\mathcal{N}} &= \sum_{n=1}^{m}\sum_{i=1}^{N_n} \phi_{n,i} s_{n,i}, \\
f_{\mathcal{G}} &= \sum_{n=1}^{m}\sum_{i=1}^{N_n} \chi_{n,i} s_{n,i}, \qquad & q^{\mu} &= \sum_{n=1}^{m}\sum_{i=1}^{N_n} \gamma_{n,i} v^{\mu}_{n,i}, \qquad & \mathcal{J}^{\mu} &= \sum_{n=1}^{m}\sum_{i=1}^{N_n} \delta_{n,i} v^{\mu}_{n,i}, \\
\mathcal{X}^{\mu} &= \sum_{n=1}^{m}\sum_{i=1}^{N_n} \xi_{n,i} v^{\mu}_{n,i}, \qquad & \mathcal{T}^{\mu\nu} &= \sum_{n=1}^{m}\sum_{i=1}^{N_n} \theta_{n,i} t^{\mu\nu}_{n,i}\,,
\end{aligned}
\tag{B.4}
$$

where $s_{n,i}$ are the independent scalars of $O(\partial^n)$, $v^{\mu}_{n,i}$ are the independent vectors of $O(\partial^n)$ and $t^{\mu\nu}_{n,i}$ are the independent tensors of $O(\partial^n)$. The first order scalars, vectors and tensors relevant for hydrostatics are listed in table 3.5.

The closest thing to a Landau-Lifshitz frame is $\mathcal{E} = \epsilon - g n{\cdot}u, \quad \mathcal{N} = \rho, \quad \mathcal{Q}^{\mu} = g n^{\mu}_{\perp}$. Which implies $f_{\mathcal{E}} = 0, \quad f_{\mathcal{N}} = 0, \quad q^{\mu} = 0$. In this frame, using the definition of the frame invariants (3.19) and (3.28) we find

$$
\begin{aligned}
f_{\epsilon+p} &= f^{L}_{\mathcal{P}} - \gamma_{\mathcal{G}} f^{L}_{\mathcal{G}}\,, \\
f_{inv} &= f^{L}_{\mathcal{P}} - \alpha_{\mathcal{G}} f^{L}_{\mathcal{G}}\,,
\end{aligned}
\tag{B.5}
$$

where the superscript L is used to specify that these are the Landau-Lifshitz terms $f^{L}_{\mathcal{P}}$, etc. Here, $\gamma_{\mathcal{G}} = \beta_{\mathcal{G}} + g\frac{n^2+(n\cdot u)^2}{\epsilon+p}\left(\frac{\partial p}{\partial \epsilon} - \frac{2}{d-1}\right)$ and $\alpha_{\mathcal{G}}$ and $\beta_{\mathcal{G}}$ were defined in (3.20) and (3.21). Inverting these, we get

$$
\begin{aligned}
f^{L}_{\mathcal{P}} &= \frac{\alpha_{\mathcal{G}} f_{\epsilon+p} - \gamma_{\mathcal{G}} f_{inv}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\,, \\
f^{L}_{\mathcal{G}} &= \frac{f_{\epsilon+p} - f_{inv}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\,.
\end{aligned}
\tag{B.6}
$$

From the definitions of ℓ^{μ} (3.25) and m^{μ} (3.26), and substituting $f^{L}_{\mathcal{G}}$ we get

$$
\begin{aligned}
\mathcal{J}^{\mu}_{L} &= \ell^{\mu} - \frac{\rho}{\epsilon+p}\frac{f_{\epsilon+p} - f_{inv}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\Delta^{\mu\nu} n_{\nu}\,, \\
\mathcal{X}^{\mu}_{L} &= m^{\mu} - \frac{g}{\epsilon+p}\frac{f_{\epsilon+p} - f_{inv}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\Delta^{\mu\nu} n_{\nu}\,.
\end{aligned}
\tag{B.7}
$$

We also have, from (3.28)

$$
\tau^{\mu\nu}_{\epsilon+p} = \mathcal{T}^{\mu\nu}_{L} + 2\frac{g}{\epsilon+p} f^{L}_{\mathcal{G}} \sigma^{\mu\nu}_{n}\,, \tag{B.8}
$$

where $\sigma^{\mu\nu}_{n} = (n^{\mu}_{\perp} n^{\nu}_{\perp} - \frac{1}{d}\Delta^{\mu\nu} n^2_{\perp})$, so that

$$\mathcal{T}_L^{\mu\nu} = \tau_{\epsilon+p}^{\mu\nu} - 2\frac{g}{\epsilon+p}\frac{f_{\epsilon+p} - f_{inv}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\sigma_n^{\mu\nu}\,. \tag{B.9}$$

Using the definition of these frame invariants, we can find the Landau frame structures in terms of those in an arbitrary frame

$$\begin{aligned}
f_{\mathcal{P}}^L &= f_{\mathcal{P}} - \frac{\alpha_{\mathcal{G}}\beta_{\mathcal{E}} - \gamma_{\mathcal{G}}\alpha_{\mathcal{E}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}f_{\mathcal{E}} - \frac{\alpha_{\mathcal{G}}\beta_{\mathcal{N}} - \gamma_{\mathcal{G}}\alpha_{\mathcal{N}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}f_{\mathcal{N}} + \frac{\alpha_{\mathcal{G}}}{n_t^2}\frac{\gamma_{\mathcal{G}} - \beta_{\mathcal{G}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\frac{g}{\epsilon+p}q{\cdot}n \\
f_{\mathcal{G}}^L &= f_{\mathcal{G}} + \frac{\alpha_{\mathcal{E}} - \beta_{\mathcal{E}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}f_{\mathcal{E}} + \frac{\alpha_{\mathcal{N}} - \beta_{\mathcal{N}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}f_{\mathcal{N}} + \frac{\gamma_{\mathcal{G}} - \beta_{\mathcal{G}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\frac{g}{\epsilon+p}\frac{q{\cdot}n}{n_t^2} \\
\mathcal{J}_L^{\mu} &= \mathcal{J}^{\mu} - \frac{\rho}{\epsilon+p}\Big(q^\mu + \Big(\frac{\alpha_{\mathcal{E}} - \beta_{\mathcal{E}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}f_{\mathcal{E}} + \frac{\alpha_{\mathcal{N}} - \beta_{\mathcal{N}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}f_{\mathcal{N}} + \frac{\gamma_{\mathcal{G}} - \beta_{\mathcal{G}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\frac{g}{\epsilon+p}\frac{q{\cdot}n}{n_t^2}\Big)n_\perp^\mu\Big) \\
\mathcal{X}_L^{\mu} &= \mathcal{X}^{\mu} - \frac{g}{\epsilon+p}\Big(q^\mu + \Big(\frac{\alpha_{\mathcal{E}} - \beta_{\mathcal{E}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}f_{\mathcal{E}} + \frac{\alpha_{\mathcal{N}} - \beta_{\mathcal{N}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}f_{\mathcal{N}} + \frac{\gamma_{\mathcal{G}} - \beta_{\mathcal{G}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\frac{g}{\epsilon+p}\frac{q{\cdot}n}{n_t^2}\Big)n_\perp^\mu\Big) \\
\mathcal{T}_L^{\mu\nu} &= \mathcal{T}^{\mu\nu} - 2\frac{g}{\epsilon+p}\Big(\frac{1}{2}\sigma_q^{\mu\nu} + \Big(\frac{\alpha_{\mathcal{E}} - \beta_{\mathcal{E}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}f_{\mathcal{E}} + \frac{\alpha_{\mathcal{N}} - \beta_{\mathcal{N}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}f_{\mathcal{N}} + \frac{\gamma_{\mathcal{G}} - \beta_{\mathcal{G}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\frac{g}{\epsilon+p}\frac{q{\cdot}n}{n_t^2}\Big)\sigma_n^{\mu\nu}\Big)
\end{aligned} \tag{B.10}$$

where $\sigma_q^{\mu\nu} = (n_\perp^\mu q^\nu + n_\perp^\nu q^\mu - \frac{2}{d}\Delta^{\mu\nu}n{\cdot}q)$ Now, using (B.4) and the definition of these frame invariants, we can find the expansion coefficients in the Landau frame in terms of those in an arbitrary frame

$$\pi^L_{1,i} = \pi_{1,i} - \frac{\alpha_{\mathcal{G}}\beta_{\mathcal{E}} - \gamma_{\mathcal{G}}\alpha_{\mathcal{E}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\epsilon_{1,i} - \frac{\alpha_{\mathcal{G}}\beta_{\mathcal{N}} - \gamma_{\mathcal{G}}\alpha_{\mathcal{N}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\phi_{1,i}$$

$$+ \frac{\alpha_{\mathcal{G}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\frac{g}{\epsilon + p}\Big(\gamma_{\mathcal{G}} - \beta_{\mathcal{G}}\Big)\Big(n_t^2\gamma_{1,i} + \gamma_{1,i+4}\Big)\,, \tag{B.11a}$$

$$\chi^L_{1,i} = \chi_{1,i} + \frac{\alpha_{\mathcal{E}} - \beta_{\mathcal{E}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\epsilon_{1,i} + \frac{\alpha_{\mathcal{N}} - \beta_{\mathcal{N}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\phi_{1,i} + \frac{\gamma_{\mathcal{G}} - \beta_{\mathcal{G}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\Big(n_t^2\gamma_{1,i} + \gamma_{1,i+4}\Big)\,, \tag{B.11b}$$

$$\delta^L_{1,i} = \delta_{1,i} - \frac{\rho}{\epsilon + p}\Big(\frac{\alpha_{\mathcal{E}} - \beta_{\mathcal{E}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\epsilon_{1,i} + \frac{\alpha_{\mathcal{N}} - \beta_{\mathcal{N}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\phi_{1,i}$$

$$+ \frac{\gamma_{\mathcal{G}} - \beta_{\mathcal{G}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\Big(\Big(n_t^2 + \frac{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}{\gamma_{\mathcal{G}} - \beta_{\mathcal{G}}}\Big)\gamma_{1,i} + \gamma_{1,i+4}\Big)\Big), \quad i = 1, 2, 3, 4\,, \tag{B.11c}$$

$$\delta^L_{1,i} = \delta_{1,i} - \frac{\rho}{\epsilon + p}\gamma_{1,i}, \quad i = 5, 6, 7\,, \tag{B.11d}$$

$$\xi^L_{1,i} = \xi_{1,i} - \frac{g}{\epsilon + p}\Big(\frac{\alpha_{\mathcal{E}} - \beta_{\mathcal{E}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\epsilon_{1,i} + \frac{\alpha_{\mathcal{N}} - \beta_{\mathcal{N}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\phi_{1,i}$$

$$+ \frac{\gamma_{\mathcal{G}} - \beta_{\mathcal{G}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\Big(\Big(n_t^2 + \frac{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}{\gamma_{\mathcal{G}} - \beta_{\mathcal{G}}}\Big)\gamma_{1,i} + \gamma_{1,i+4}\Big)\Big), \quad i = 1, 2, 3, 4\,, \tag{B.11e}$$

$$\xi^L_{1,i} = \xi_{1,i} - \frac{g}{\epsilon + p}\gamma_{1,i}, \quad i = 5, 6, 7\,, \tag{B.11f}$$

$$\theta^L_{1,i} = \theta_{1,i} - 2\frac{g}{\epsilon + p}\Big(\frac{\alpha_{\mathcal{E}} - \beta_{\mathcal{E}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\epsilon_{1,i} + \frac{\alpha_{\mathcal{N}} - \beta_{\mathcal{N}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\phi_{1,i}$$

$$+ \frac{\gamma_{\mathcal{G}} - \beta_{\mathcal{G}}}{\alpha_{\mathcal{G}} - \gamma_{\mathcal{G}}}\Big(\frac{\alpha_{\mathcal{G}} - \beta_{\mathcal{G}}}{\gamma_{\mathcal{G}} - \beta_{\mathcal{G}}}\gamma_{1,i} + \gamma_{1,i+4}\Big)\Big), \quad i = 1, 2, 3, 4\,, \tag{B.11g}$$

$$\theta^L_{1,i} = \theta_{1,i} - \frac{g}{\epsilon + p}\gamma_{1,i}, \quad i = 5, 6, 7\,. \tag{B.11h}$$

Can be used to bring the thermodynamic frame equilibrium constraints to the Landau frame.

Appendix C

More on relativistic magnetohydrodynamics

C.1 Equilibrium $T^{\mu\nu}$ and J^μ

The coefficients ϵ_n, π_n, ϕ_n, γ_n, δ_n, θ_n in the equilibrium energy-momentum tensor and the current (4.13) have the following expressions in terms of the five parameters $M_n(T, \mu, B^2)$ of the generating functional (4.7). The $O(\partial)$ correction to the energy density is determined by

$$
\begin{aligned}
\epsilon_1 &= -M_1 + TM_{1,T} + \mu M_{1,\mu} + 4B^2 M_{1,B^2} + T^4 M_{3,B^2} \,, \\
\epsilon_2 &= -M_2 + TM_{2,T} + \mu M_{2,\mu} \,, \\
\epsilon_3 &= \frac{4B^2}{T^4}\left(M_1 - TM_{1,T} - \mu M_{1,\mu} - 4B^2 M_{1,B^2}\right) - 4B^2 M_{3,B^2} \,, \\
\epsilon_4 &= -2M_4 + TM_{4,T} + \mu M_{4,\mu} \,, \\
\epsilon_5 &= TM_{5,T} + \mu M_{5,\mu} + \frac{4B^2}{T^4} M_{1,\mu} + M_{3,\mu} \,,
\end{aligned}
$$

where the comma denotes the partial derivative: $M_{1,T} \equiv (\partial M_1/\partial T)$ evaluated at fixed μ and B^2, etc. The $O(\partial)$ correction to the pressure is determined by

$$
\begin{aligned}
\pi_1 &= 0\,,\\
\pi_2 &= -\tfrac{2}{3}M_2 - \tfrac{4}{3}B^2 M_{2,B^2}\,,\\
\pi_3 &= -\tfrac{4}{3}B^2 M_{3,B^2} + \frac{4B^2}{3T^4}\left(M_1 - TM_{1,T} - \mu M_{1,\mu} - 4B^2 M_{1,B^2}\right)\,,\\
\pi_4 &= -\tfrac{1}{3}M_4 - \tfrac{4}{3}B^2 M_{4,B^2}\,,\\
\pi_5 &= -\tfrac{4}{3}B^2 M_{5,B^2} + \frac{4B^2}{3T^4}M_{1,\mu}\,.
\end{aligned}
$$

The $O(\partial)$ correction to the charge density is determined by

$$
\begin{aligned}
\phi_1 &= M_{1,\mu} - T^4 M_{5,B^2}\,,\\
\phi_2 &= M_{2,\mu}\,,\\
\phi_3 &= M_{3,\mu} + TM_{5,T} + \mu M_{5,\mu} + 4B^2 M_{5,B^2}\,,\\
\phi_4 &= -\alpha_{\rm BB} + M_{4,\mu}\,,\\
\phi_5 &= 0\,.
\end{aligned}
$$

The $O(\partial)$ correction to the energy flux is determined by

$$
\begin{aligned}
\gamma_1 &= -M_4\,,\\
\gamma_2 &= 2M_4 - TM_{4,T} - \mu M_{4,\mu}\,,\\
\gamma_3 &= -M_{4,B^2}\,,\\
\gamma_4 &= -\alpha_{\rm BB} + M_{4,\mu}\,.
\end{aligned}
$$

The $O(\partial)$ correction to the spatial current is determined by the magnetic susceptibility,

$$
\begin{aligned}
\delta_1 &= -\alpha_{\rm BB}\,,\\
\delta_2 &= \alpha_{\rm BB} - T\alpha_{{\rm BB},T} - \mu\alpha_{{\rm BB},\mu}\,,\\
\delta_3 &= -\alpha_{{\rm BB},B^2}\,,\\
\delta_4 &= \alpha_{{\rm BB},\mu}\,.
\end{aligned}
$$

The $O(\partial)$ correction to the stress is determined by

$$\begin{aligned}
\theta_1 &= 0\,,\\
\theta_2 &= M_{2,B^2}\,,\\
\theta_3 &= M_{3,B^2} - \frac{1}{T^4}\left(M_1 - TM_{1,T} - \mu M_{1,\mu} - 4B^2 M_{1,B^2}\right)\,,\\
\theta_4 &= M_{4,B^2}\,,\\
\theta_5 &= M_{5,B^2} - \frac{1}{T^4}M_{1,\mu}\,,\\
\theta_6 &= 2M_2\,,\\
\theta_7 &= -M_2 + TM_{2,T} + \mu M_{2,\mu}\,,\\
\theta_8 &= M_{2,B^2}\,,\\
\theta_9 &= -M_{2,\mu}\,,\\
\theta_{10} &= M_4\,.
\end{aligned}$$

C.2 Comparison with previous work

C.2.1 Comparison with Huang et al

In this appendix we will comment on how our work relates to some earlier studies of transport coefficients, for the benefit of the reader who might want to compare different approaches. Ref. [74], abbreviated below as HSR, studied relativistic hydrodynamics of parity-invariant fluids in external non-dynamical magnetic field. HSR enumerated the transport coefficients, giving a relativistic version of the classification in the book [84], §13, and derived the Kubo formulas for transport coefficients in an operator formalism. Parts of the HSR paper overlap with our Section 4.3.

Our counting of non-equilibrium transport coefficients for parity-invariant systems agrees with HSR. Denoting the transport coefficients in ref. [74] with the subscript HSR, the relations to our transport coefficients are as follows:

$$\begin{aligned}
&\eta_\perp = \eta_{0,\text{HSR}}\,, \qquad \tilde\eta_\perp = -2\eta_{3,\text{HSR}}\,, \qquad \eta_\parallel = \eta_{0,\text{HSR}} + \eta_{2,\text{HSR}}\,, \qquad \tilde\eta_\parallel = -\eta_{4,\text{HSR}}\,,\\
&\eta_1 = -\tfrac{1}{2}\eta_{0,\text{HSR}} - \tfrac{3}{8}\eta_{1,\text{HSR}} - \tfrac{3}{4}\zeta_{\perp,\text{HSR}}\,, \qquad \zeta_1 = \zeta_{\perp,\text{HSR}}\,,\\
&\eta_2 = \tfrac{3}{2}\eta_{0,\text{HSR}} + \tfrac{9}{8}\eta_{1,\text{HSR}} + \tfrac{3}{4}\zeta_{\perp,\text{HSR}} + \tfrac{3}{2}\zeta_{\parallel,\text{HSR}}\,, \qquad \zeta_2 = \zeta_{\parallel,\text{HSR}} - \zeta_{\perp,\text{HSR}}\,,\\
&\sigma_\perp = \kappa_{\perp,\text{HSR}}\,, \qquad \sigma_\parallel = \kappa_{\parallel,\text{HSR}}\,, \qquad \tilde\sigma = -\kappa_{\times,\text{HSR}}\,,
\end{aligned} \tag{C.1}$$

assuming the convention $\epsilon^{0123} = 1$. This lists eleven transport coefficients compared to ten HSR coefficients, hence under this mapping the eleven transport coefficients are not independent. Indeed, the comparison (C.1) implies $\zeta_2 = 2\eta_1 + \frac{2}{3}\eta_2$, which is precisely our Onsager constraint (4.30). Thus our counting of non-equilibrium transport coefficients in Section 4.3 agrees with that of HSR.

There are also some differences between our Section 4.3 and HSR. In terms of the setup, the HSR treatment neglects electric fields, while we include them and explain how to do so systematically. Related to that, the treatment of polarization effects in HSR was incomplete. A direct way to obtain the equilibrium energy-momentum tensor and the current in the presence of external fields is by varying the corresponding generating functional with respect to the metric and the gauge field, as was done for example in ref. [70]. As a result, HSR did not include the thermodynamic transport coefficient, denoted in Section 4.3 as M_Ω, and did not distinguish between the Landau-Lifshitz and thermodynamic frames. In the Landau-Lifshitz frame, M_Ω would contribute to all frame invariants in eq. (4.14) inducing $O(\partial)$ contributions to pressure, electric current, and spatial stress.

We also find that our constraints on transport coefficients imposed by the positivity of entropy production differ somewhat from those presented in HSR. Rewriting our constraints (4.27) in terms of the HSR coefficients, we find

$$\begin{aligned}
&\eta_{0,\text{HSR}} \geqslant 0\,, \qquad \eta_{0,\text{HSR}} + \eta_{2,\text{HSR}} \geqslant 0\,, \qquad \tfrac{1}{3}\eta_{0,\text{HSR}} + \tfrac{1}{4}\eta_{1,\text{HSR}} + \tfrac{3}{2}\zeta_{\perp,\text{HSR}} \geqslant 0\,,\\
&3\eta_{0,\text{HSR}} + \tfrac{9}{4}\eta_{1,\text{HSR}} + \tfrac{3}{2}\zeta_{\perp,\text{HSR}} + 3\zeta_{\parallel,\text{HSR}} \geqslant 0\,,\\
&18\zeta_{\parallel,\text{HSR}}\zeta_{\perp,\text{HSR}} + 4\zeta_{\parallel,\text{HSR}}\eta_{0,\text{HSR}} + 3\zeta_{\parallel,\text{HSR}}\eta_{1,\text{HSR}} + 8\zeta_{\perp,\text{HSR}}\eta_{0,\text{HSR}} + 6\zeta_{\perp,\text{HSR}}\eta_{1,\text{HSR}} \geqslant 0\,,\\
&\kappa_{\perp,\text{HSR}} \geqslant 0\,, \qquad \kappa_{\parallel,\text{HSR}} \geqslant 0\,.
\end{aligned} \tag{C.2}$$

On the other hand, the constraints coming from the second law in ref. [74] state that all the dissipative HSR transport coefficients must be positive. We find that the constraints on dissipative transport coefficients (C.2) are in fact weaker. In other words, the constraints of ref. [74] are too restrictive: some of the dissipative transport coefficients in the HSR notation can be negative, while still satisfying (C.2), and therefore still leading to positive entropy production.

Finally, there are differences between our Kubo formulas and those of HSR. In particular our Kubo formulas for conductivities transverse to the external magnetic field are markedly different. Comparing the correlation functions in the neutral state ($n_0 = 0$), the HSR Kubo formulas give the conductivities $\kappa_{\perp,\text{HSR}}$ and $\kappa_{\times,\text{HSR}}$ in terms of the $i\omega$ coefficient of the retarded current-current correlation functions at zero momentum. On the other hand, our Kubo formulas (4.32b), (4.32c) show that the coefficient of $i\omega$ vanishes, while the subleading coefficient in the small-ω expansion is determined by the resistivity rather than the conductivity. In the charged state, the term n_0/B_0 in our eq. (4.32c) describes the standard Hall effect in the plane transverse to the magnetic field. The Hall effect appears to be missing from correlation functions in ref. [74].

C.2.2 Comparison with Finazzo et al

In ref. [75] (abbreviated below as FCRN), the authors considered hydrodynamics with fixed non-dynamical magnetic field, and derived Kubo formulas for transport coefficients that appear in the energy-momentum tensor in the Landau-Lifshitz frame. FCRN use a variational approach to find the retarded functions of the energy-momentum tensor, and Appendix B of FCRN overlaps with our Section 4.3. FCRN follow ref. [74] in their constitutive relations for the energy-momentum tensor, so the comments in Section C.2.1 apply to FCRN as well, where FCRN agree with ref. [74]. In particular, FCRN did not include the thermodynamic transport coefficient M_Ω that appears in the equilibrium free energy at one-derivative order.

FCRN use mostly the same convention for transport coefficients as HSR: $\eta_{0,\text{FCRN}} = \eta_{0,\text{HSR}}$, $\eta_{1,\text{FCRN}} = \eta_{1,\text{HSR}}$, $\eta_{4,\text{FCRN}} = \eta_{4,\text{HSR}}$, $\zeta_{\perp,\text{FCRN}} = \zeta_{\perp,\text{HSR}}$, $\zeta_{\parallel,\text{FCRN}} = \zeta_{\parallel,\text{HSR}}$, while $\eta_{2,\text{FCRN}} = -\eta_{2,\text{HSR}}$, $\eta_{3,\text{FCRN}} = -2\eta_{3,\text{HSR}}$, assuming the convention $\epsilon^{0123} = 1$. The translation to our convention for transport coefficients can be done through eq. (C.1). The convention for the variational retarded correlation functions used by FCRN differs from ours by an overall minus sign.

We agree with FCRN's Kubo formulas for $\eta_{0,\text{FCRN}}$, $\zeta_{\perp,\text{FCRN}}$, and $\zeta_{\parallel,\text{FCRN}}$. Our Kubo formulas for $\eta_{2,\text{FCRN}}$ and $\eta_{3,\text{FCRN}}$ differ from those in ref. [75] by a minus sign. Our Kubo formula for $\eta_{4,\text{FCRN}}$ differs from that in ref. [75] by a factor of 1/4. Our Kubo formula for $\eta_{1,\text{FCRN}} + \frac{4}{3}\eta_{0,\text{FCRN}}$ differs from that in ref. [75] by a factor of 2. Ref. [75] does not derive Kubo formulas for electrical conductivities in external magnetic field, so we can not compare those.

Appendix D

Green's functions

We are interested in the symmetries of the retarded two point correlation functions of conserved currents in Poincare invariant theories. We use the mostly plus flat metric for this discussion. Let's begin with the non-mixed Green functions

$$G^R_{J^\mu J^\nu}(x-y) = -i\theta(x^0 - y^0)\langle[J_\mu(x), J_\nu(y)]\rangle \tag{D.1}$$

$$G^R_{T^{\mu\nu}T^{\rho\sigma}}(x-y) = -i\theta(x^0 - y^0)\langle[T_{\mu\nu}(x), T_{\rho\sigma}(y)]\rangle \tag{D.2}$$

Or, in Fourier space

$$G^R_{J^\mu J^\nu}(k) = \int d^D x e^{-ik(x-y)} G^R_{J^\mu J^\nu}(x-y) \tag{D.3}$$

$$G^R_{T^{\mu\nu}T^{\rho\sigma}}(k) = \int d^D x e^{-ik(x-y)} G^R_{T^{\mu\nu}T^{\rho\sigma}}(x-y) \tag{D.4}$$

Then, CPT invariance of density operator leads to the symmetry relations

$$G^R_{J^\mu J^\nu}(k) = G^R_{J^\nu J^\mu}(k) \tag{D.5}$$

$$G^R_{T^{\mu\nu}T^{\rho\sigma}}(k) = G^R_{T^{\rho\sigma}T^{\mu\nu}}(k) \tag{D.6}$$

And the current conservation relations $\nabla_\mu J^\mu = 0$ and $\nabla_\mu T^{\mu\nu} = 0$ lead to the Ward identites

$$k^\mu G^R_{J^\mu J^\nu}(k) = 0 \tag{D.7}$$

$$k^\mu G^R_{T^{\mu\nu}T^{\rho\sigma}}(k) = 0 \tag{D.8}$$

Symmetry	Momentum-Momentum	Current-Current	Mixed
CPT	$G^R_{T^{\mu\nu}T^{\rho\sigma}}(t,\mathbf{x}) = \theta_{(\mu\nu)}\theta_{(\rho\sigma)}G^R_{T^{\rho\sigma}T^{\mu\nu}}(t,-\tilde{\mathbf{x}})$	$G^R_{J^\mu J^\rho}(t,\mathbf{x}) = \theta_{(\mu)}\theta_{(\rho)}G^R_{J^\rho J^\mu}(t,-\tilde{\mathbf{x}})$	$G^R_{T^{\mu\nu}J^\rho}(t,\mathbf{x}) = -\theta_{(\mu\nu)}\theta_{(\rho)}G^R_{J^\rho T^{\mu\nu}}(t,-\tilde{\mathbf{x}})$
PT	$G^R_{T^{\mu\nu}T^{\rho\sigma}}(t,\mathbf{x}) = \theta_{(\mu\nu)}\theta_{(\rho\sigma)}G^R_{T^{\rho\sigma}T^{\mu\nu}}(t,-\tilde{\mathbf{x}})$	$G^R_{J^\mu,J^\rho}(t,\mathbf{x}) = \theta_{(\mu)}\theta_{(\rho)}G^R_{J^\rho,J^\mu}(t,-\tilde{\mathbf{x}})$	$G^R_{T^{\mu\nu}J^\rho}(t,\mathbf{x}) = \theta_{(\mu\nu)}\theta_{(\rho)}G^R_{J^\rho T^{\mu\nu}}(t,-\tilde{\mathbf{x}})$
C	$G^R_{T^{\mu\nu}T^{\rho\sigma}}(t,\mathbf{x}) = G^R_{T^{\mu\nu}T^{\rho\sigma}}(t,\mathbf{x})$	$G^R_{J^\mu,J^\rho}(t,\mathbf{x}) = G^R_{J^\mu,J^\rho}(t,\mathbf{x})$	$G^R_{T^{\mu\nu},J^\rho}(t,\mathbf{x}) = G^R_{J^\rho T^{\mu\nu}}(t,\mathbf{x}) = 0$
T	$G^R_{T^{\mu\nu}T^{\rho\sigma}}(t,\mathbf{x}) = \epsilon_{(\mu\nu)}\epsilon_{(\rho\sigma)}G^R_{T^{\rho\sigma}T^{\mu\nu}}(t,-\mathbf{x})$	$G^R_{J^\mu J^\rho}(t,\mathbf{x}) = \epsilon_{(\mu)}\epsilon_{(\rho)}G^R_{J^\rho J^\mu}(t,-\mathbf{x})$	$G^R_{T^{\mu\nu}J^\rho}(t,\mathbf{x}) = \epsilon_{(\mu\nu)}\epsilon_{(\rho)}G^R_{J^\rho T^{\mu\nu}}(t,-\mathbf{x})$
P	$G^R_{T^{\mu\nu}T^{\rho\sigma}}(t,\mathbf{x}) = \eta_{(\mu\nu)}\eta_{(\rho\sigma)}G^R_{T^{\mu\nu}T^{\rho\sigma}}(t,\tilde{\mathbf{x}})$	$G^R_{J^\mu J^\rho}(t,\mathbf{x}) = \eta_{(\mu)}\eta_{(\rho)}G^R_{J^\mu J^\rho}(t,\tilde{\mathbf{x}})$	$G^R_{T^{\mu\nu}J^\rho}(t,\mathbf{x}) = \eta_{(\mu\nu)}\eta_{(\rho)}G^R_{T^{\mu\nu}J^\rho}(t,\tilde{\mathbf{x}})$
CT	$G^R_{T^{\mu\nu}T^{\rho\sigma}}(t,\mathbf{x}) = \epsilon_{(\mu\nu)}\epsilon_{(\rho\sigma)}G^R_{T^{\rho\sigma}T^{\mu\nu}}(t,-\mathbf{x})$	$G^R_{J^\mu J^\rho}(t,\mathbf{x}) = \epsilon_{(\mu)}\epsilon_{(\rho)}G^R_{J^\rho J^\mu}(t,-\mathbf{x})$	$G^R_{T^{\mu\nu}J^\rho}(t,\mathbf{x}) = -\epsilon_{(\mu\nu)}\epsilon_{(\rho)}G^R_{J^\rho T^{\mu\nu}}(t,-\mathbf{x})$
CP	$G^R_{T^{\mu\nu}T^{\rho\sigma}}(t,\mathbf{x}) = \eta_{(\mu\nu)}\epsilon_{(\rho\sigma)}G^R_{T^{\mu\nu}T^{\rho\sigma}}(t,\tilde{\mathbf{x}})$	$G^R_{J^\mu J^\rho}(t,\mathbf{x}) = \eta_{(\mu)}\eta_{(\rho)}G^R_{J^\mu J^\rho}(t,\tilde{\mathbf{x}})$	$G^R_{T^{\mu\nu}J^\rho}(t,\mathbf{x}) = -\eta_{(\mu\nu)}\eta_{(\rho)}G^R_{T^{\mu\nu}J^\rho}(t,\tilde{\mathbf{x}})$

Table D.1: Constraints of the retarded Green functions for different symmetries of the equilibrium state.

Furthermore, the symmetry of the energy-momentum tensor leads to

$$G^R_{T^{\mu\nu}T^{\rho\sigma}}(k) = G^R_{T^{\nu\mu}T^{\rho\sigma}}(k) \tag{D.9}$$

D.1 Symmetries and constraints

The symmetry of the state in which the expectation value is taken will restrict the form of the Green functions. The constraints that follows from $\langle \mathcal{O} \rangle = \langle \theta \mathcal{O} \theta^{-1} \rangle$ for unitary θ satisfying $[\theta, \rho] = 0$, or $\langle \mathcal{O} \rangle = \langle \theta \mathcal{O}^\dagger \theta^{-1} \rangle$ for antiunitary θ satisfying $[\theta, \rho] = 0$, are given in table D.1.

Here, $\tilde{\mathbf{x}} = P\mathbf{x}P^{-1}$, and the coefficients ϵ, η and θ are given by

$$TT^{\mu\nu}(t,\mathbf{x})T^{-1} = \epsilon_{(\mu\nu)}T^{\mu\nu}(-t,\mathbf{x}) \tag{D.10}$$

$$TJ^\mu(t,\mathbf{x})T^{-1} = \epsilon_{(\mu)}J^\mu(-t,\mathbf{x}) \tag{D.11}$$

$$PT^{\mu\nu}(t,\mathbf{x})P^{-1} = \eta_{(\mu\nu)}T^{\mu\nu}(t,\tilde{\mathbf{x}}) \tag{D.12}$$

$$PJ^\mu(t,\mathbf{x})P^{-1} = \eta_{(\mu)}J^\mu(t,\tilde{\mathbf{x}}) \tag{D.13}$$

and

$$\theta = \epsilon\eta \tag{D.14}$$

These coefficients are $\epsilon = (-1)^n$ where n is the number indices taking spatial values, $\eta = (-1)^n$ where n is the number of indices taking the spatial values that are being reversed by P, and $\theta = (-1)^n$ where n is the number of indices taking the spatial values that are not being reversed by P.

Note that for an odd number of spatial dimensions, $\theta = 1$ and $\tilde{\mathbf{x}} = -\mathbf{x}$. Fur-

Symmetry	Momentum-Momentum	Current-Current	Mixed
RCT	$G^R_{T^{\mu\nu}T^{\rho\sigma}}(t,\mathbf{x}) = G^R_{T^{\rho\sigma}T^{\mu\nu}}(t,\mathbf{x})$	$G^R_{J^\mu J^\rho}(t,\mathbf{x}) = G^R_{J^\rho J^\mu}(t,\mathbf{x})$	$G^R_{T^{\mu\nu}J^\rho}(t,\mathbf{x}) = -G^R_{J^\rho T^{\mu\nu}}(t,\mathbf{x})$
RT	$G^R_{T^{\mu\nu},T^{\rho\sigma}}(t,\mathbf{x}) = G^R_{T^{\rho\sigma},T^{\mu\nu}}(t,\mathbf{x})$	$G^R_{J^\mu J^\rho}(t,\mathbf{x}) = G^R_{J^\rho J^\mu}(t,\mathbf{x})$	$G^R_{T^{\mu\nu}J^\rho}(t,\mathbf{x}) = G^R_{J^\rho T^{\mu\nu}}(t,\mathbf{x})$
RC	$G^R_{T^{\mu\nu},T^{\rho\sigma}}(t,\mathbf{x}) = \epsilon_{(\mu\nu)}\epsilon_{(\rho\sigma)}G^R_{T^{\mu\nu},T^{\rho\sigma}}(t,-\mathbf{x})$	$G^R_{J^\mu J^\rho}(t,\mathbf{x}) = \epsilon_{(\mu)}\epsilon_{(\rho)}G^R_{J^\mu J^\rho}(t,-\mathbf{x})$	$G^R_{T^{\mu\nu}J^\rho}(t,\mathbf{x}) = -\epsilon_{(\mu\nu)}\epsilon_{(\rho)}G^R_{T^{\mu\nu}J^\rho}(t,-\mathbf{x})$
RCPT	$G^R_{T^{\mu\nu},T^{\rho\sigma}}(t,\mathbf{x}) = \eta_{(\mu\nu)}\eta_{(\rho\sigma)}G^R_{T^{\rho\sigma},T^{\mu\nu}}(t,\tilde{\mathbf{x}})$	$G^R_{J^\mu J^\rho}(t,\mathbf{x}) = \eta_{(\mu)}\eta_{(\rho)}G^R_{J^\rho J^\mu}(t,\tilde{\mathbf{x}})$	$G^R_{T^{\mu\nu}J^\rho}(t,\mathbf{x}) = -\eta_{(\mu\nu)}\eta_{(\rho)}G^R_{J^\rho T^{\mu\nu}}(t,\tilde{\mathbf{x}})$
RPT	$G^R_{T^{\mu\nu},T^{\rho\sigma}}(t,\mathbf{x}) = \eta_{(\mu\nu)}\eta_{(\rho\sigma)}G^R_{T^{\rho\sigma},T^{\mu\nu}}(t,\tilde{\mathbf{x}})$	$G^R_{J^\mu J^\rho}(t,\mathbf{x}) = \eta_{(\mu)}\eta_{(\rho)}G^R_{J^\rho J^\mu}(t,\tilde{\mathbf{x}})$	$G^R_{T^{\mu\nu}J^\rho}(t,\mathbf{x}) = \eta_{(\mu\nu)}\eta_{(\rho)}G^R_{J^\rho T^{\mu\nu}}(t,\tilde{\mathbf{x}})$
RP	$G^R_{T^{\mu\nu},T^{\rho\sigma}}(t,\mathbf{x}) = \theta_{(\mu\nu)}\theta_{(\rho\sigma)}G^R_{T^{\mu\nu},T^{\rho\sigma}}(t,-\tilde{\mathbf{x}})$	$G^R_{J^\mu J^\rho}(t,\mathbf{x}) = \theta_{(\mu)}\theta_{(\rho)}G^R_{J^\mu J^\rho}(t,-\tilde{\mathbf{x}})$	$G^R_{T^{\mu\nu}J^\rho}(t,\mathbf{x}) = \theta_{(\mu\nu)}\theta_{(\rho)}G^R_{T^{\mu\nu}J^\rho}(t,-\tilde{\mathbf{x}})$
RCP	$G^R_{T^{\mu\nu},T^{\rho\sigma}}(t,\mathbf{x}) = \theta_{(\mu\nu)}\theta_{(\rho\sigma)}G^R_{T^{\mu\nu},T^{\rho\sigma}}(t,-\tilde{\mathbf{x}})$	$G^R_{J^\mu J^\rho}(t,\mathbf{x}) = \theta_{(\mu)}\theta_{(\rho)}G^R_{J^\mu J^\rho}(t,-\tilde{\mathbf{x}})$	$G^R_{T^{\mu\nu}J^\rho}(t,\mathbf{x}) = -\theta_{(\mu\nu)}\theta_{(\rho)}G^R_{T^{\mu\nu}J^\rho}(t,-\tilde{\mathbf{x}})$

Table D.2: Constraints of the retarded Green functions for different symmetries of the equilibrium state in even spatial dimensions.

thermore, for an even number of spatial dimensions, there is a rotation R such that $R\mathbf{x}R^{-1} = -\mathbf{x}$. So that

$$RT^{\mu\nu}(t,\mathbf{x})R^{-1} = \epsilon_{(\mu\nu)}T^{\mu\nu}(t,-\mathbf{x}) \tag{D.15}$$

$$RJ^{\mu}(t,\mathbf{x})R^{-1} = \epsilon_{(\mu)}J^{\mu}(t,-\mathbf{x}) \tag{D.16}$$

Then for a rotationally symmetric state, we have the constraints in table D.2.

Note that R in even spatial dimensions plays the same role as P in odd spatial dimensions for the constraints of the retarded Green function.

D.2 Linear response theory

Here we outline how to use linear response theory to find the hydrodynamic correlation functions from the equations of motion. These follow section 2 of ref. [4], with an overall minus sign in the definition of the retarded Green's function to keep the same convention as the body of the thesis. Adding sources to the Hamiltonian by

$$\delta H(t) = -\int d^d x \lambda_a(t,\mathbf{x})\varphi_a(t,\mathbf{x})\,, \tag{D.17}$$

yields a change in the expectation value of φ_a given by

$$\delta\langle\varphi_a(t,\mathbf{x})\rangle = -i\int_{-\infty}^{t} dt'\langle[\varphi_a(t,\mathbf{x}),\delta H(t')]\rangle = \int d^{d+1}x' G^R_{ab}(t-t',\mathbf{x}-\mathbf{x}')\lambda_b(t',\mathbf{x}')\,, \tag{D.18}$$

where

$$G^R_{ab}(t-t',\mathbf{x}-\mathbf{x}') = i\theta(t-t')\langle[\varphi_a(t,\mathbf{x}),\varphi_b^\dagger(t',\mathbf{x}')]\rangle\,, \tag{D.19}$$

is the retarded Green's function. In fourier basis this is

$$\delta\langle\varphi_a(\omega,\mathbf{k})\rangle = G^R_{ab}(\omega,\mathbf{k})\lambda_b(\omega,\mathbf{k})\,. \tag{D.20}$$

The hydrodynamical variables φ_a obey some set of equations. For small fluctuations, these equation can be linearised

$$\big(\partial_t\delta_{ab} + M_{ab}(\mathbf{k})\big)\varphi_b(t,\mathbf{k}) = 0\,. \tag{D.21}$$

Laplace transforming these equations yields

$$K_{ab}\varphi_b(z,\mathbf{k}) \equiv \big(-iz\delta_{ab} + M_{ab}(\mathbf{k})\big)\varphi_b(z,\mathbf{k}) = \varphi_a(t,\mathbf{k})|_{t=0} \equiv \varphi^0_a(\mathbf{k})\,. \tag{D.22}$$

Then, using $\varphi^0_a(\mathbf{k}) = \chi_{ab}(\mathbf{k})\lambda_b(\mathbf{k})$ for $\mathbf{k}\to 0$, where $\chi_{ab}(\mathbf{k}) = \left(\frac{\partial\varphi_a}{\partial\lambda_b}\right) = G^R_{ab}(z,\mathbf{k})|_{z=0}$, together with $\varphi_a(z,\mathbf{k}) = \frac{1}{iz}\big(G^R_{ab}(z,\mathbf{k}) - G^R_{ab}(z,\mathbf{k})|_{z=0}\big)\lambda^0_b(\mathbf{k})$ yields

$$G^R(z,\mathbf{k}) = (1 + izK^{-1})\chi\,. \tag{D.23}$$

The second equation comes from taking $\lambda_b(t,\mathbf{k}) = \theta(-t)e^{\epsilon t}\lambda^0_b(\mathbf{k})$ and Laplace transforming $\varphi_a(t,\mathbf{k}) = \int_{-\infty}^0 dt' G^R_{ab}(t-t',\mathbf{k})\lambda_b(t',\mathbf{k})$.

D.3 Green's function identities

In real time, there are several definitions of Green's functions. Let's look at some general identities of the Green's functions. These can all be found in chapter 8 of [87]. First we define the advanced, symmetric and time ordered Green's functions,

$$G^A_{ab}(x) = -i\theta(-t)\langle[\mathcal{O}_a(x),\mathcal{O}^\dagger_b(0)]\rangle\,, \tag{D.24a}$$

$$G_{ab}(x) = \frac{1}{2}\langle\{\mathcal{O}_a(x),\mathcal{O}^\dagger_b(0)\}\rangle\,, \tag{D.24b}$$

$$G^F_{ab}(x) = i\langle\mathcal{T}\mathcal{O}_a(x)\mathcal{O}^\dagger_b(0)\rangle = \frac{1}{2}(G^R_{ab}(x) + G^A_{ab}(x)) + iG_{ab}(x)\,. \tag{D.24c}$$

Using $G_\alpha(k) = \int d^Dx e^{-ik\cdot x}G_\alpha(x)$, we find

$$(G^R_{ab}(k))^* = G^R_{ab}(-k) = G^A_{ab}(k)\,, \tag{D.25}$$

from which we find

$$\mathrm{Im}G^R_{ab}(k) = \tfrac{1}{2}\rho_{ab}(k), \tag{D.26}$$

where $\rho_{ab}(x) = \langle[\mathcal{O}_a(x), \mathcal{O}^\dagger_b(0)]\rangle$ is the antisymmetric Green's function. The Fourier transform $\rho_{ab}(k)$ of the antisymmetric Green's function is called the spectral function. In a thermal state, using $\langle\mathcal{O}\rangle = \mathrm{Tr}\,(\rho\mathcal{O})$ with $\rho = e^{-\beta H}$ and inserting a set of energy eigenstates $\mathbb{I} = \sum_n |n\rangle\langle n|$ we can get

$$\rho_{ab}(k) = 2\tanh(\frac{\beta\omega}{2})G_{ab}(k)\,, \tag{D.27}$$

or equivalently, $\mathrm{Im}G^R_{ab}(k) = \tanh(\frac{\beta\omega}{2})G_{ab}(k)$. We can then write

$$G^F_{ab}(k) = \mathrm{Re}G^R_{ab}(k) + i\coth(\frac{\beta\omega}{2})\mathrm{Im}G^R_{ab}(k), \tag{D.28}$$

and any real-time Green function can be found from the retarded Green function in a thermal state. Explicitly, we have

$$G^A_{ab}(k) = \left(G^R_{ab}(k)\right)^*\,, \tag{D.29a}$$

$$G_{ab}(k) = \coth\left(\beta\omega/2\right)\mathrm{Im}G^R_{ab}\,, \tag{D.29b}$$

$$\rho_{ab}(k) = 2\mathrm{Im}G^R_{ab}(k)\,, \tag{D.29c}$$

$$G^F_{ab}(k) = \mathrm{Re}G^R_{ab}(k) + i\coth(\frac{\beta\omega}{2})\mathrm{Im}G^R_{ab}(k)\,. \tag{D.29d}$$

These real-time Green's functions can also be related to the Euclidean Green's function

$$G^E_{ab}(\tau, \mathbf{x}) = \langle\mathcal{O}_a(\tau, \mathbf{x})\mathcal{O}^\dagger_b(0)\rangle_E \tag{D.30}$$

evaluated on the Wick rotated statistical field theory $\langle ...\rangle_E = \int_{\mathcal{O}(\tau,\mathbf{x})=\mathcal{O}(\tau+\beta,\mathbf{x})} D\mathcal{O}[...]e^{-S_E}$ via

$$G^R_{ab}(k) = G^E_{ab}(k^0_E \to -i(k^0 - i0^+), \mathbf{k})\,. \tag{D.31}$$

This provides a useful way to derive real-time correlation functions from the Euclidean correlation function, which is often easier to find.

D.4 Spectral decomposition

The spectral decomposition of the two point functions leads to a positivity constraint on some Green's functions in a thermal ensemble. Here, we review how these constraints arise. We follow a generalized version of the argument presented in chapter 2 of ref. [88]. We use $P^\mu = (H, \mathbf{P})$ as the generator of space-time translations. The energy eigenstates are labelled by energy and momentum numbers $|E_p, \mathbf{p}\rangle$.

We begin by defining the following out of time order Green's functions

$$G^{>}_{ab}(t, \mathbf{x}) = \langle \mathcal{O}_a(t, \mathbf{x}) \mathcal{O}^\dagger_b(0) \rangle_\beta \,, \tag{D.32a}$$

$$G^{<}_{ab}(t, \mathbf{x}) = \langle \mathcal{O}^\dagger_b(0) \mathcal{O}_a(t, \mathbf{x}) \rangle_\beta \,, \tag{D.32b}$$

where $\langle [...] \rangle_\beta = \frac{1}{Z(\beta)} \mathrm{tr}\left(e^{-\beta H}[...]\right)$. $Z(\beta) = \mathrm{tr}\left(e^{-\beta H}\right)$ is the thermal partition function. From the definition of the antisymmetric Green's function ρ_{ab}, it is clear that

$$\rho_{ab}(x) = G^{>}_{ab}(x) - G^{<}_{ab}(x) \,. \tag{D.33}$$

Now, using the cyclic property of the trace and the Heisenberg picture identity $\mathcal{O}(t, \mathbf{x}) = e^{iHt} \mathcal{O}(0, \mathbf{x}) e^{-iHt}$, we find

$$G^{<}(t, \mathbf{x}) = G^{>}(t - i\beta, \mathbf{x}) \,. \tag{D.34}$$

Looking at the Fourier transforms of these Green's functions, we find

$$G^{<}_{ab}(\omega, \mathbf{k}) = e^{-\beta\omega} G^{>}_{ab}(\omega, \mathbf{k}) \,. \tag{D.35}$$

Now, from the trace definition of $G^{>}_{ab}(t, \mathbf{x})$, using $\mathcal{O}(0, \mathbf{x}) = e^{i\mathbf{x}\cdot\mathbf{P}} \mathcal{O}(0, 0) e^{-i\mathbf{x}\cdot\mathbf{P}}$ and inserting $\mathbb{I} = \sum_{E_q, \mathbf{q}} |E_q, \mathbf{q}\rangle\langle E_q, \mathbf{q}|$ we can write

$$G^{>}_{ab}(t, \mathbf{x}) = \frac{1}{Z(\beta)} \sum_{E_p, \mathbf{p}} \sum_{E_q, \mathbf{q}} e^{-\beta E_p} e^{it(E_p - E_q)} e^{i\mathbf{x}\cdot(\mathbf{q}-\mathbf{p})} \langle E_p, \mathbf{p}|\mathcal{O}_a(0)|E_q, \mathbf{q}\rangle \langle E_q, \mathbf{q}|\mathcal{O}^\dagger_b(0)|E_p, \mathbf{p}\rangle \tag{D.36}$$

or, in Fourier basis

$$G^{>}_{ab}(\omega,\mathbf{k}) = \frac{1}{Z(\beta)} \sum_{E_p\,,\mathbf{p}} \sum_{E_q\,,\mathbf{q}} e^{-\beta E_p} \delta^{(4)}(k+p-q) \langle E_p,\mathbf{p}|\mathcal{O}_a(0)|E_q,\mathbf{q}\rangle \langle E_q,\mathbf{q}|\mathcal{O}^{\dagger}_b(0)|E_p,\mathbf{p}\rangle \tag{D.37}$$

where $p^\mu = (E_p,\mathbf{p})$ and similarly for q^μ, and $k^\mu = (\omega,\mathbf{k})$. Note that the diagonal elements of $G^{>}(\omega,\mathbf{k})$ are positive

$$G^{>}_{aa}(\omega,\mathbf{k}) = \frac{1}{Z(\beta)} \sum_{E_p\,,\mathbf{p}} \sum_{E_q\,,\mathbf{q}} e^{-\beta E_p} \delta^{(4)}(k+p-q) |\langle E_p,\mathbf{p}|\mathcal{O}_a(0)|E_q,\mathbf{q}\rangle|^2 \,. \tag{D.38}$$

From (D.35) we see that the diagonal elements of $G^{<}(\omega,\mathbf{k})$ are also positive. Furthermore, from (D.33) we see that

$$\rho_{ab}(\omega,\mathbf{k}) = (1-e^{-\beta\omega}) G^{>}_{ab}(\omega,\mathbf{k}) \,, \tag{D.39}$$

from which we find the positivity condition on the spectral function

$$\mathrm{sign}(\omega)\rho_{aa}(\omega,\mathbf{k}) \geq 0 \,. \tag{D.40}$$

Finally, relating the spectral function to the retarded Green's function in Fourier space by (D.26), we get the positivity condition used in the hydrodynamic correlation functions

$$\mathrm{sign}(\omega)\mathrm{Im}G^{R}_{aa}(\omega,\mathbf{k}) \geq 0 \,. \tag{D.41}$$

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