CYCLES IN THE BLOCK–INTERSECTION GRAPH OF PAIRWISE BALANCED DESIGNS

by

DONOVAN R. HARE

DMS–573–IR January 1991

This paper was written while the author was a sessional instructor at the University of Victoria, 1990–91.

Cycles in the Block-Intersection Graph of Pairwise Balanced Designs

Donovan R. Hare
Department of Mathematics and Statistics
Simon Fraser University
Burnaby, B.C.
Canada V5A 1S6

January 28, 1991

Abstract

It is shown that the block-intersection graph of pairwise balance design with $\lambda = 1$ has cycles of all possible lengths given that its minimum block size is at least 3.

1 Introduction

Let K be a finite set of positive integers, and let λ and v be positive integers such that $v > \max K$. A *pairwise balanced design*, denoted $PBD(v, K, \lambda)$, is a pair $(V, \mathcal{B})$ where V is a finite set whose elements are called *points*, and $\mathcal{B}$ is a collection of subsets of V, called *blocks*, such that $|V| = v$, the blocks have their sizes (cardinalities) from K and any pair of distinct points is contained in exactly λ blocks. If $K = \{k\}$, then $(V, \mathcal{B})$ is called a *balanced incomplete block design* and we denote it by $BIBD(v, k, \lambda)$. When $k = 3$ and $\lambda = 1$, the pair $(V, \mathcal{B})$ is called a *Steiner triple system*.

Let G be a graph. G is *pancyclic* if for every integer n, $3 \leq n \leq |V(G)|$, there is a cycle of length n in G. G is *edge-pancyclic* (*vertex-pancyclic*) if for every edge e (vertex v) of G, there is a cycle of length n in G using e (v).

There are many ways of defining graphs from designs. The interested reader is referred to the survey found in [3]. In this paper we use the following graph: for each i, $0 \leq i \leq \max K$, the *block-intersection graph* of a $PBD(v, K, \lambda)$, $(V, \mathcal{B})$, denoted by $B_i(\mathcal{B})$, has vertex-set $\mathcal{B}$ and has two vertices adjacent if and only if their corresponding blocks have i points in common. For balanced incomplete block designs, block-intersection graphs of these types have been used as effective isomorphism invariants to distinguish non-isomorphic designs that have the same parameters (see [3]).

When $i > 1$ and $\lambda = 1$, the block-intersection graph $B_i(\mathcal{B})$, of a $PBD(v, K, \lambda)$, $(V, \mathcal{B})$, is just the empty graph of order $|\mathcal{B}|$. Some attention however, has been given to the cycle properties of $B_1(\mathcal{B})$. P. Horák and A. Rosa were the first to show that if $(V, \mathcal{B})$ is a $BIBD(v, k, 1)$, $k \geq 3$, then $B_1(\mathcal{B})$ is hamiltonian [4]. B. Alspach, K. Heinrich and B. Mohar subsequently proved that if $(V, \mathcal{B})$ is a $PBD(v, K, 1)$ such that $\max K \leq 2 \min K$, then $B_1(\mathcal{B})$ is hamiltonian [2]. Even more recently, B. Alspach and D. Hare prove that if $(V, \mathcal{B})$ is a $BIBD(v, k, 1)$, $k \geq 3$, then $B_1(\mathcal{B})$ is edge-pancyclic and that the same is true for transversal designs [1].

The purpose of this paper is to prove:

Theorem 1 *If $(V, \mathcal{B})$ is a $PBD(v, K, 1)$ with $\min K \geq 3$, then the block-intersection graph $B_1(\mathcal{B})$ is pancyclic.*

The proof of Theorem 1 generalizes the proof in [1] for balanced incomplete block designs. It gives the following corollaries:

Corollary 2 (Alspach and Hare [1]) *If $(V, \mathcal{B})$ is a $BIBD(v, k, 1)$, $k \geq 3$, then the block-intersection graph $B_1(\mathcal{B})$ is edge-pancyclic.*

Corollary 3 *If $(V, \mathcal{B})$ is a $PBD(v, K, 1)$ with $K = \{k, k+1\}$, $k \geq 3$, then the block-intersection graph $B_1(\mathcal{B})$ is vertex-pancyclic.*

We close this section with an example of a design and one of its block-intersection graphs that is not even hamiltonian (note that $i, \lambda > 1$).

Example *A $(6, 3, 2)$-design $(V, \mathcal{B})$ such that the block-intersection graph $B_2(\mathcal{B})$ is non-hamiltonian.*

Let $V = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{B} = \{\{1,2,3\}, \{1,2,4\}, \{1,3,5\}, \{1,4,6\}, \{1,5,6\}, \{2,3,6\}, \{2,4,5\}, \{2,5,6\}, \{3,4,5\}, \{3,4,6\}\}$. Then $B_2(\mathcal{B})$ is isomorphic to the Petersen graph.

2 Proof of Main Theorem

As much as possible, lower case letters are used for positive integers or for points of a design, capital letters are used for blocks of a design (or equivalently vertices of the block-intersection graph) or sets of points of the design, and script style letters denote designs or sets of blocks of a design. The following notation will be used throughout the paper. Let $(V, \mathcal{B})$ be a $PBD(v, K, 1)$, let $\ell = \min K$ and let $u = \max K$. Moreover, let $B^* \in \mathcal{B}$, $B^* = \{b_1, b_2, \ldots, b_k\}$ for some $k \in K$, and for $i = 1, 2, \ldots, k$, let

$$\mathcal{B}_i = \{B \in \mathcal{B} : B \cap B^* = \{b_i\}\} .$$

Finally, let $\mathcal{B}^* = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$ and let $G = B_1(\mathcal{B})$.

We first start with a lemma about the neighbourhood of B^*, $\mathcal{B}^*$.

Lemma 4 *For $i, j \in \{1, 2, \ldots, k\}$, $i \neq j$, the number of edges between $\mathcal{B}_i$ and $\mathcal{B}_j$ in G is $v - k$. Furthermore,*

$$\frac{v-k}{u-1} \leq |\mathcal{B}_i| \leq \frac{v-k}{\ell - 1}.$$

Proof: Let $B \in \mathcal{B}_i$. For each $x \in B \setminus \{b_i\}$ there is a unique block B_x of $\mathcal{B}_j$ containing $\{x, b_j\}$. Moreover, if $\{x, y\} \subset B \setminus \{b_i\}$, $x \neq y$, then $B_x \neq B_y$. Thus the number of edges in G between B and $\mathcal{B}_j$ is $|B| - 1$ and hence the number of edges between $\mathcal{B}_i$ and $\mathcal{B}_j$ is

$$\sum_{B \in \mathcal{B}_i} (|B| - 1) .$$

On the other hand, there are $v - k$ pairs $\{x, b_i\}$, $x \in V \setminus B^*$, all of these pairs are contained in blocks from $\mathcal{B}_i$, and every $B \in \mathcal{B}_i$ contains $|B| - 1$ of the pairs. Thus

$$\sum_{B \in \mathcal{B}_i} (|B| - 1) = v - k$$

and the first part of the lemma is proved. For all $B \in \mathcal{B}_i$, $\ell \leq |B| \leq u$ and so

$$(\ell - 1) |\mathcal{B}_i| \leq \sum_{B \in \mathcal{B}_i} (|B| - 1) \leq (u - 1) |\mathcal{B}_i| .$$

The lemma is thus proved. ■

Each $\mathcal{B}_i$ is a clique in G. Lemma 4 ensures that there is at least one edge between any two of these cliques. This gets us started in forming a cycle of a desired length.

Proof of Theorem 1: Let $B^* \in \mathcal{B}$ such that $|B^*| = k \in \{u-1, u\}$ and let $p \in \{3, 4, \ldots, |\mathcal{B}|\}$. We will construct a p-cycle in G which uses B^*. Since $v > u$ we have at least one edge between each $\mathcal{B}_i$ by Lemma 4. Thus for each $p = 3, 4, \ldots, n$, where $n = |\{B^*\} \cup \mathcal{B}^*|$, the closed neighbourhood of B^* in G contains a p-cycle which uses B^*.

Suppose therefore that $p > n$. We will use $n - p$ vertices of G that are not in the closed neighbourhood of B^* with the n vertices of the closed neighbourhood. To do this, a sequence of paths in the rest of G having $n - p$ vertices are joined to the closed neighbourhood of B^*.

Let

$$\mathcal{D} = \{B \in \mathcal{B} : B \cap B^* = \emptyset\}$$

($\mathcal{D}$ is the set of all vertices in G not in the closed neighbourhood of B^*). Let $\mathcal{L}_1$ be a path of maximum length in $G[\mathcal{D}]$ (the subgraph of G induced by the vertices of $\mathcal{D}$). For $j \geq 1$, let $\mathcal{L}_{j+1}$ be a path of maximum length in $G[\mathcal{D}_{j+1}]$ where

$$\mathcal{D}_{j+1} = \mathcal{D} \setminus \left(V\left(\mathcal{L}_1\right) \cup V\left(\mathcal{L}_2\right) \cup \ldots \cup V\left(\mathcal{L}_j\right)\right).$$

Moreover, let s be the first integer j such that $\mathcal{D}_{j+1} = \emptyset$. For each $t \in \{1, 2, \ldots, s\}$, we say $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_t$ is a *truncation* of $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_t$ if for $i = 1, 2, \ldots, t-1$, $\mathcal{N}_i = \mathcal{L}_i$ and if $\mathcal{N}_t$ is a subpath of $\mathcal{L}_t$ having the same initial vertex. Choose a truncation so that $|V(\mathcal{N}_1) \cup V(\mathcal{N}_2) \cup \cdots \cup V(\mathcal{N}_t)| = n - p$.

For $j = 1, 2, \ldots, t$, choose $a_j \in V$ from the first vertex (a block) of $\mathcal{N}_j$ and $c_j \in V$ from the last vertex of $\mathcal{N}_j$ so that $c_j \neq a_j$. Define H to be the bipartite graph (X, Y) where $X = \{a_1, c_1, a_2, c_2, \ldots, a_t, c_t\}$, $Y = \mathcal{B}^*$, and for all $x \in X$ and $B \in Y$, $xB \in E(H)$ if and only if $x \in B$. Since $d_H(x) = k$ for all $x \in X$, and $d_H(B) \leq u - 1 \leq k$ for all $B \in Y$, H has a matching that saturates X. Let

$$\{a_1A_1, c_1C_1, a_2A_2, c_2C_2, \ldots, a_tA_t, c_tC_t\}$$

be this matching (note that because of the maximality of the $\mathcal{L}_j$ paths, $|X| = 2t$). Then for each $j \in \{1, 2, \ldots, t\}$, $\mathcal{P}_j = A_j \mathcal{N}_j C_j$ is a path in G that starts and ends in Y.

We need an orderly way to create the p-cycle using these paths. Let $\mathcal{M}$ be the multigraph that has vertex-set $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k\}$ and edge-set $\{e_1, e_2, \ldots, e_t\}$ where $e_j = \{\mathcal{B}_u, \mathcal{B}_v\}$, $A_j \in \mathcal{B}_u$, and $C_j \in \mathcal{B}_v$, for $j = 1, 2, \ldots, t$. Let q be the number of connected components of $\mathcal{M}$, and for $i = 1, 2, \ldots, q$, let $2o_i$ be the number of odd vertices of component i. For each $i = 1, 2, \ldots, q$, if $o_i > 0$, then decompose component i of $\mathcal{M}$ into o_i edge-disjoint trails, and if $o_i = 0$, then component i has an Euler trail. Let $\mathcal{T}$ be the set of all these trails.

Note that $E(\mathcal{M}) = \bigcup_{T \in \mathcal{T}} E(T)$ and that if $o_i > 0$ for some $i \in \{1, 2, \ldots, q\}$, then the edge-disjoint trails of component i must begin and end in the $2o_i$ vertices of odd degree . Thus if two trails $T_1, T_2 \in \mathcal{T}$ have a common end-vertex in $\mathcal{M}$, then $T_1 = T_2$.

Each trail $T \in \mathcal{T}$, $T = \mathcal{B}_{i_1} e_{j_1} \mathcal{B}_{i_2} e_{j_2} \cdots e_{j_{r-1}} \mathcal{B}_{i_r}$ is easily transformed into a path $\mathcal{Q}_T$ in G: $\mathcal{Q}_T = \mathcal{Q}_{j_1} \mathcal{Q}_{j_2} \cdots \mathcal{Q}_{j_r}$ where $\mathcal{Q}_j$ is either $\mathcal{P}_j$ or its reverse, for $j = 1, 2, \ldots, t$. The rest of the proof involves changing the $\mathcal{Q}_T$ into cycles and combining these cycles into a single cycle. Before we continue, though, we need a definition.

We define a *clique-edge* to be an edge BC of G such that for some $i \in \{1, 2, \ldots, k\}$, $B, C \in V(\mathcal{B}_i)$. We say a path or cycle $\mathcal{R}$ in G is *clique-edge extendible avoiding* $\mathcal{A}$ for some $\mathcal{A} \subset V(\mathcal{R}) \cap \mathcal{B}^*$, if every $B \in (V(\mathcal{R}) \cap \mathcal{B}^*) \setminus \mathcal{A}$ is incident to a clique-edge from $E(\mathcal{R})$. With this definition, we have for each $T \in \mathcal{T}$ that $\mathcal{Q}_T$ is clique-edge extendible avoiding the set of its end-vertices.

We now perform extensions on the $\mathcal{Q}_T$ to turn them into cycles in G which are clique-edge extendible avoiding a special set of vertices. We proceed step by step through the cliques $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$, looking in step j for a path that has an end-vertex in $\mathcal{B}_j$. If such a path exists, then we extend it to a cycle or to a longer path.

Let $\mathcal{R}_0 = \{(\mathcal{Q}_T, \mathcal{A}_T) : T \in \mathcal{T}\}$ where $\mathcal{A}_T$ is the set of end-vertices of $\mathcal{Q}_T$, for all $T \in \mathcal{T}$. At the end of each step j, $0 \leq j \leq k$, the newly formed set $\mathcal{R}_j$ of paths and cycles with their special sets of vertices will have the following properties:

P1. Each $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_j$ is a path or cycle in G which is clique-edge extendible avoiding $\mathcal{A}$.

P2. If $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_j$ and $\mathcal{Q}$ is a cycle, then $\mathcal{Q}$ has a clique-edge (i.e. $\mathcal{A} \neq V(\mathcal{Q}) \cap \mathcal{B}^*$).

P3. If $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_j$ and $\mathcal{Q}$ is a path, then $\mathcal{Q}$ has both its end-vertices in $(\mathcal{B}_{j+1} \cup \mathcal{B}_{j+2} \cup \cdots \cup \mathcal{B}_k) \cap \mathcal{A}$.

P4. For each $i = 1, 2, \ldots, k$,

$$|\{(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_j : \mathcal{B}_i \cap \mathcal{A} \neq \emptyset\}| \leq 1,$$

and for all $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_j$, and $|\mathcal{B}_i \cap \mathcal{A}| \leq 2$. Moreover, $\mathcal{Q}$ is a path with both its end-vertices in $\mathcal{B}_i$ if and only if $|\mathcal{B}_i \cap \mathcal{A}| = 2$.

Property **P4** may be the least understandable of the properties. It ensures that in a clique $\mathcal{B}_i$, there is at most one path or cycle having avoidable vertices, that a cycle has at most one avoidable vertex, and that a path has at most two avoidable vertices and it has two if and only if these vertices are the end-vertices of the path.

Properties **P1**–**P4** are true for $\mathcal{R}_0$. Suppose that for some j, $0 < j \leq k$, $\mathcal{R}_{j-1}$ satisfies **P1**–**P4**. We define $\mathcal{R}_j$ using the following cases with the intention that $\mathcal{R}_j$ will also satisfy **P1**–**P4**. Since $\mathcal{R}_{j-1}$ satisfies **P4**, we may use the following three main cases.

Case 1. There does not exist an end-vertex S of a path $\mathcal{Q}$ avoiding a set $\mathcal{A}$ such that $S \in \mathcal{B}_j \cap \mathcal{A}$ and $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_{j-1}$.

In this case we let $\mathcal{R}_j = \mathcal{R}_{j-1}$. By the induction hypothesis $\mathcal{R}_{j-1}$ satisfies **P1**–**P4**. Thus $\mathcal{R}_j$ satisfies **P1**, **P2**, and **P4**. $\mathcal{R}_j$ satisfies **P3** since $\mathcal{R}_{j-1}$ satisfies **P3** and Case 1.

Case 2. There exists end-vertices $S, F \in \mathcal{B}_j$, $S \neq F$, of a path $\mathcal{Q}$ avoiding a set $\mathcal{A}$ such that $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_{j-1}$.

In this case we replace a path avoiding a set with a cycle avoiding a set. Without loss of generality, let $\mathcal{Q}$ start in S and end in F. Let $\mathcal{Q}' = \mathcal{Q}S$, let $\mathcal{A}' = \mathcal{A} \setminus \{S, F\}$, and let $\mathcal{R}_j = (\mathcal{R}_{j-1} \setminus \{(\mathcal{Q}, \mathcal{A})\}) \cup \{(\mathcal{Q}', \mathcal{A}')\}$. Then the cycle $\mathcal{Q}'$ is clique-edge extendible avoiding $\mathcal{A}'$ and so $\mathcal{R}_j$

satisfies **P1**. Since $SF \in E(\mathcal{Q}')$ is a clique-edge, $\mathcal{R}_j$ satisfies **P2**. Any path (avoiding a set) in $\mathcal{R}_j$ is also in $\mathcal{R}_{j-1}$ and $\mathcal{R}_j$ has no path with an end-vertex in $\mathcal{B}_j$ (there can only be one since $\mathcal{R}_{j-1}$ satisfies **P4** and we have dealt with it here in Case 2), and so $\mathcal{R}_j$ satisfies **P3**. Since $\mathcal{R}_{j-1}$ satisfies **P4**, $S, F \in \mathcal{A}$ and hence $\mathcal{B}_j \cap \mathcal{A}' = \emptyset$. Thus $\mathcal{R}_j$ satisfies **P4**.

Case 3. There exists a path $\mathcal{Q}$ avoiding a set $\mathcal{A}$ which starts in a vertex $S \in \mathcal{B}_j$ such that $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_{j-1}$ and $\mathcal{B}_j \cap \mathcal{A} = \{S\}$.

Let $\mathcal{Q}$ end in a vertex $F \in \mathcal{B}_m$, for some $m \in \{1, 2, \ldots, k\}$. Since $\mathcal{R}_{j-1}$ satisfies **P3**, $m \geq j$, and since $F \neq S$ and $\mathcal{B}_j \cap \mathcal{A} = \{S\}$, $F \notin \mathcal{B}_j$. Hence $m > j$. Let $N_m(S) = \{B \in \mathcal{B}_m : S \cap B \neq \emptyset\}$.

We choose a vertex Q from $N_m(S)$ carefully. Let $\{Q_1, Q_2, \ldots, Q_r\} = V(\mathcal{Q}) \cap \mathcal{B}_m$ so that the indices of the Q_i correspond to the vertices' order on the path $\mathcal{Q}$ (from S to F). Note that $Q_r = F$ and so $r \geq 1$. Moreover, since $\mathcal{Q}$ is clique-edge extendible and $F \in \mathcal{B}_m \cap \mathcal{A}$ by **P3**, r is odd, and $Q_1Q_2, Q_3Q_4, \ldots, Q_{r-2}Q_{r-1} \in E(\mathcal{Q})$.

Since $\ell \geq 3$, $|N_m(S)| \geq 2$, and so we choose $Q \in N_m(S) \setminus \{Q_1\}$. We will use the fact that $Q \neq Q_1$ in Case 3(f).

We have several subcases depending on where the vertex Q is within $\mathcal{B}_m$.

Case 3(a). $Q \in \mathcal{B}_m \setminus \{A_1, C_1, A_2, C_2, \ldots, A_t, C_t\}$.

Let $\mathcal{Q}' = \mathcal{Q}QS$, let $\mathcal{A}' = \mathcal{A} \setminus \{F\}$, and let $\mathcal{R}_j = (\mathcal{R}_{j-1} \setminus \{(\mathcal{Q}, \mathcal{A})\}) \cup \{(\mathcal{Q}', \mathcal{A}')\}$. Since $\mathcal{Q}$ is clique-edge extendible avoiding $\mathcal{A}$ and $QF \in E(\mathcal{Q}')$ is a clique-edge, $\mathcal{Q}'$ is clique-edge extendible avoiding $\mathcal{A}'$. Thus $\mathcal{R}_j$ satisfies **P1** and **P2**. Since $(\mathcal{Q}, \mathcal{A}) \notin \mathcal{R}_j$, and $\mathcal{Q}'$ is a cycle, $\mathcal{R}_j$ satisfies **P3**. Finally, $\mathcal{R}_j$ satisfies **P4** because $\mathcal{R}_{j-1}$ does and because $\mathcal{B}_m \cap \mathcal{A}' = \emptyset$.

Case 3(b). $Q \in V(\mathcal{R})$ for some cycle $\mathcal{R}$ avoiding a set $\mathcal{A}_1$ such that $(\mathcal{R}, \mathcal{A}_1) \in \mathcal{R}_{j-1}$.

Since $F \in \mathcal{B}_m \cap \mathcal{A}$, and since $\mathcal{R}_{j-1}$ satisfies **P4**, we can conclude that $\mathcal{A}_1 \cap \mathcal{B}_m = \emptyset$. Moreover, since $\mathcal{R}$ is clique-edge extendible avoiding $\mathcal{A}_1$, there exists $U \in \mathcal{B}_m$ such that $QU \in E(\mathcal{R})$. Let $\mathcal{R}'$ be the path that starts in U and ends in Q such that $\mathcal{R} = \mathcal{R}'U$. The cycle $\mathcal{Q}'$ in this case is $\mathcal{Q}\mathcal{R}'S$, $\mathcal{A}'$ is $(\mathcal{A} \setminus \{F\}) \cup \mathcal{A}_1 \cup \{Q\}$, and $\mathcal{R}_j = (\mathcal{R}_{j-1} \setminus \{(\mathcal{Q},\mathcal{A}),(\mathcal{R},\mathcal{A}_1)\}) \cup \{(\mathcal{Q}',\mathcal{A}')\}$.

Since $\mathcal{Q}$ is clique-edge extendible avoiding $\mathcal{A}$, $\mathcal{R}$ is clique-edge extendible avoiding $\mathcal{A}_1$, and $FU \in E(\mathcal{Q}')$ is a clique-edge, $\mathcal{Q}'$ is clique-edge extendible avoiding $\mathcal{A}'$. Thus $\mathcal{R}_j$ satisfies **P1** and **P2**. Since $(\mathcal{Q},\mathcal{A}) \notin \mathcal{R}_j$, and $\mathcal{Q}'$ is a cycle, $\mathcal{R}_j$ satisfies **P3**. Finally, since Q replaces F in $\mathcal{A}'$, $\mathcal{R}_j$ satisfies **P4**.

Case **3(c).** $Q \in V(\mathcal{R})$ for some path $\mathcal{R}$ avoiding a set $\mathcal{A}_1$ such that $(\mathcal{R},\mathcal{A}_1) \in \mathcal{R}_{j-1}$.

In this case we replace the paths $\mathcal{Q}$ and $\mathcal{R}$ with a longer path $\mathcal{Q}'$. As in Case 3(b), there exists $U \in \mathcal{B}_m$ such that $QU \in E(\mathcal{R})$. Let $\mathcal{W}_1$ be the path that ends in Q and $\mathcal{W}_2$ be the path that starts in U such that $\mathcal{W}_1\mathcal{W}_2 = \mathcal{R}$ (or the reversed path of $\mathcal{R}$). We form the longer path $\mathcal{Q}' = \mathcal{W}_1\mathcal{Q}\mathcal{W}_2$, let $\mathcal{A}' = (\mathcal{A} \setminus \{F\}) \cup \mathcal{A}_1 \cup \{Q\}$, and let $\mathcal{R}_j = (\mathcal{R}_{j-1} \setminus \{(\mathcal{Q},\mathcal{A}),(\mathcal{R},\mathcal{A}_1)\}) \cup \{(\mathcal{Q}',\mathcal{A}')\}$.

Since $\mathcal{Q}$ is clique-edge extendible avoiding $\mathcal{A}$, $\mathcal{R}$ is clique-edge extendible avoiding $\mathcal{A}_1$, and $FU \in E(\mathcal{Q}')$ is a clique-edge, $\mathcal{Q}'$ is clique-edge extendible avoiding $\mathcal{A}'$. Thus $\mathcal{R}_j$ satisfies **P1** and **P2**. Since $(\mathcal{Q},\mathcal{A}) \notin \mathcal{R}_j$, and $\mathcal{Q}'$ is a path that has the same end-vertices as $\mathcal{R}$ neither of which are in $\mathcal{B}_j$ (a condition of Case 3 is that $\mathcal{B}_j \cap \mathcal{A} = \{S\}$), $\mathcal{R}_j$ satisfies **P3**. Finally, since Q replaces F in $\mathcal{A}'$, $\mathcal{R}_j$ satisfies **P4**.

Case **3(d).** $Q = Q_j$ for some $j \in \{3, 5, \ldots, r-4, r-2\}$.

In this case we break up $\mathcal{Q}$ into two cycles $\mathcal{Q}_1'$ and $\mathcal{Q}_2'$. Let $\mathcal{Q} = \mathcal{W}_1\mathcal{W}_2$ so that $\mathcal{W}_1$ starts in S and ends in Q_j, and $\mathcal{W}_2$ starts in Q_{j+1} and ends in F. Let $\mathcal{Q}_1' = \mathcal{W}_1 S$ and $\mathcal{A}_1' = (\mathcal{A} \cap V(\mathcal{W}_1)) \cup \{Q_j\}$. Moreover, let $\mathcal{Q}_2' = \mathcal{W}_2 Q_{j+1}$, $\mathcal{A}_2' = (\mathcal{A} \cap V(\mathcal{W}_2)) \setminus \{F\}$, and $\mathcal{R}_j = (\mathcal{R}_{j-1} \setminus \{(\mathcal{Q},\mathcal{A})\}) \cup \{(\mathcal{Q}_1',\mathcal{A}_1'),(\mathcal{Q}_2',\mathcal{A}_2')\}$.

Since $\mathcal{Q}$ is clique-edge extendible avoiding $\mathcal{A}$, and $Q_j \in \mathcal{A}_1'$, $\mathcal{Q}_1'$ is clique-edge extendible avoiding $\mathcal{A}_1'$. Also, since $\mathcal{Q}$ is clique-edge extendible avoiding $\mathcal{A}$ and $FQ_{j+1} \in E(\mathcal{Q}_2')$ is a clique-edge, $\mathcal{Q}_2'$ is clique-edge extendible avoiding $\mathcal{A}_2'$. $Q_1Q_2 \in E(\mathcal{Q}_1')$ is a clique-edge as well (note that $j \geq 3$). Thus, $\mathcal{R}_j$ satisfies **P1** and **P2**. Since $(\mathcal{Q},\mathcal{A}) \notin \mathcal{R}_j$, and $\mathcal{Q}'_1$ and $\mathcal{Q}'_2$ are cycles, $\mathcal{R}_j$ satisfies **P3**. Finally, since Q_j replaces F in $\mathcal{A}'_1 \cup \mathcal{A}'_2$ and $\mathcal{R}_{j-1}$ satisfies **P4**, $\mathcal{R}_j$ satisfies **P4**.

Case 3(e). $Q = Q_j$ for some $j \in \{2, 4, \ldots, r-3, r-1\}$.

In this case, form a longer cycle from $\mathcal{Q}$. Let $\mathcal{Q} = \mathcal{W}_1\mathcal{W}_2$ so that $\mathcal{W}_1$ starts in S and ends in Q_{j-1}, and $\mathcal{W}_2$ starts in Q_j and ends in F. We define $\mathcal{Q}' = \mathcal{W}_1 F \overline{\mathcal{W}_2} S$ where $\overline{\mathcal{W}_2}$ is the reverse of $\mathcal{W}_2$. Moreover, let $\mathcal{A}' = (\mathcal{A} \setminus \{F\}) \cup \{Q_j\}$, and $\mathcal{R}_j = (\mathcal{R}_{j-1} \setminus \{(\mathcal{Q}, \mathcal{A})\}) \cup \{(\mathcal{Q}', \mathcal{A}')\}$.

Since $\mathcal{Q}$ is clique-edge extendible avoiding $\mathcal{A}$, and $FQ_{j-1} \in E(\mathcal{Q}')$ is a clique-edge and $Q_j \in \mathcal{A}'$, $\mathcal{Q}'$ is clique-edge extendible avoiding $\mathcal{A}'$. Thus $\mathcal{R}_j$ satisfies **P1** and **P2**. Since $(\mathcal{Q}, \mathcal{A}) \notin \mathcal{R}_j$, and $\mathcal{Q}'$ is a cycle, $\mathcal{R}_j$ satisfies **P3**. Finally, since Q_j replaces F in $\mathcal{A}'$, $\mathcal{R}_j$ satisfies **P4**.

Case 3(f). $Q = Q_r$.

Since $Q \neq Q_1$, $r \geq 3$. Let $\mathcal{Q}' = \mathcal{Q}S$, and let $\mathcal{R}_j = (\mathcal{R}_{j-1} \setminus \{(\mathcal{Q}, \mathcal{A})\}) \cup \{(\mathcal{Q}', \mathcal{A})\}$. Since $Q_1Q_2 \in E(\mathcal{Q}')$ is a clique-edge, $\mathcal{R}_j$ satisfies **P1** and **P2**. Since $(\mathcal{Q}, \mathcal{A}) \notin \mathcal{R}_j$, and $\mathcal{Q}'$ is a cycle, $\mathcal{R}_j$ satisfies **P3**. $\mathcal{R}_j$ satisfies **P4** since $\mathcal{R}_{j-1}$ does.

Thus $\mathcal{R}_j$ satisfies **P1**–**P4** and hence by induction, $\mathcal{R}_k$ satisfies **P1**–**P4**. By **P3**, if $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_k$ then $\mathcal{Q}$ is a cycle. From now on, let

$$\mathcal{A}_i = \bigcup_{(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_k} \mathcal{A} \cap \mathcal{B}_i,$$

for $i = 1, 2, \ldots, k$, and note that since $\mathcal{R}_k$ satisfies **P4**, $|\mathcal{A}_i| \leq 1$.

We extend the cycles in $\mathcal{R}_k$ even further. First, suppose for some $i \in \{1, 2, \ldots, k\}$, there exist $(\mathcal{Q}_1, \mathcal{A}_1), (\mathcal{Q}_2, \mathcal{A}_2) \in \mathcal{R}_k$, $\mathcal{Q}_1 \neq \mathcal{Q}_2$, such that $\mathcal{Q}_1$ has a clique-edge U_1V_1 in $\mathcal{B}_i$, and $\mathcal{Q}_2$ also has a clique-edge U_2V_2 in $\mathcal{B}_i$. We can then replace $(\mathcal{Q}_1, \mathcal{A}_1)$ and $(\mathcal{Q}_2, \mathcal{A}_2)$ in $\mathcal{R}_k$ with $(\mathcal{W}_1\mathcal{W}_2V_1, \mathcal{A}_1 \cup \mathcal{A}_2)$, where $\mathcal{Q}_1 = \mathcal{W}_1U_1V_1$ and $\mathcal{Q}_2 = \mathcal{W}_2U_2V_2$. Let $\mathcal{R}_{k+1}$ be the set resulting from performing this type of replacement as many times as possible on each $\mathcal{B}_i$ (some cycles may be extended more than once). Then $\mathcal{R}_{k+1}$ satisfies **P1**–**P4**, and the additional property:

P5. For $i = 1, 2, \ldots, k$,

$$|\{\mathcal{Q} : (\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_j \text{ for some } \mathcal{A}, \mathcal{Q} \text{ has a clique-edge in } \mathcal{B}_i\}| \leq 1.$$

Second, if there exists a vertex $B \in \mathcal{B}_i$, for some $i \in \{1, 2, \ldots, k\}$, such that B is not in any cycle $\mathcal{Q}$ avoiding a set $\mathcal{A}$ with $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_{k+1}$, then B may be included on a cycle that has a clique-edge in $\mathcal{B}_i$ (if one exists). Since $\mathcal{R}_{k+1}$ satisfies **P5**, there is at most one cycle $\mathcal{Q}$ avoiding a set $\mathcal{A}$ (with $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_{k+1}$) that has a clique-edge UV in $\mathcal{B}_i$. If there is one, we extend $\mathcal{Q}$ by replacing UV with UBV (the set $\mathcal{A}$ remains the same). Do this for any such B in $\mathcal{B}^*$ and let the $\mathcal{R}_{k+2}$ be the resulting set of pairs of modified cycles and sets. Then $\mathcal{R}_{k+2}$ satisfies **P1**–**P5**, and the additional property:

P6. For all $i \in \{1, 2, \ldots, k\}$, if there exists a cycle $\mathcal{Q}$ avoiding a set $\mathcal{A}$ with $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_j$ such that $\mathcal{Q}$ has a clique-edge in $\mathcal{B}_i$, then every vertex $B \in \mathcal{B}_i \setminus \mathcal{A}_i$ is in $V(\mathcal{Q})$.

We are now ready to form the p-cycle through $\mathcal{B}^*$. Start with a vertex $U_1 \in \mathcal{B}_1 \setminus \mathcal{A}_1$. If U_1 is on some cycle $\mathcal{Q}$ avoiding a set $\mathcal{A}$ such that $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_{k+2}$, then $U_1 \notin \mathcal{A}$ and so there exist a $V_1 \in \mathcal{B}_1 \setminus \mathcal{A}_1$ such that $U_1V_1 \in E(\mathcal{Q})$ since $\mathcal{Q}$ is clique-extendible avoiding $\mathcal{A}$. We then let $\mathcal{S}_1$ be the path defined by $\mathcal{Q} = \mathcal{S}_1 U_1$. Note that $\mathcal{B}_1 \subset \mathcal{A}_1 \cup V(\mathcal{S}_1)$ by **P6**. If U_1 is not on some cycle, then let $\mathcal{S}_1$ be the path which starts in U_1 and includes all the vertices of $\mathcal{B}_1 \setminus \mathcal{A}_1$ in any order (note that none of these vertices are on a cycle avoiding a set in $\mathcal{R}_{k+2}$ since $\mathcal{R}_{k+2}$ satisfies **P6**). Let $\mathcal{S}_1$ end in the vertex V_1.

If $\mathcal{B}_2 \subset \mathcal{A}_2 \cup V(\mathcal{S}_1)$, then let $\mathcal{S}_2 = \mathcal{S}_1$ and $V_2 = V_1$. Otherwise, $V(\mathcal{S}_1) \cap (\mathcal{B}_2 \setminus \mathcal{A}_2) = \emptyset$ by **P6**, and so let U_2 be a neighbour of V_1 such that $U_2 \in \mathcal{B}_2 \setminus \mathcal{A}_2$. This is possible since V_1 has $|V_1| - 1 \geq \ell - 1 \geq 2$ neighbours in $\mathcal{B}_2$ and $|\mathcal{A}_2| \leq 1$. Using the above method we form a path that starts at U_2, ends in some vertex $V_2 \in \mathcal{B}_2 \setminus \mathcal{A}_2$, and includes all of the vertices of $\mathcal{B}_2 \setminus \mathcal{A}_2$. Adjoining this path to the end of $\mathcal{S}_1$ we form a path $\mathcal{S}_2$ which starts in U_1 and ends in V_2 and contains all of the vertices of $(\mathcal{B}_1 \setminus \mathcal{A}_1) \cup (\mathcal{B}_2 \setminus \mathcal{A}_2)$.

Suppose that for some $j \in \{2, 3, \ldots, k-1\}$, we have formed a path $\mathcal{S}_j$ which starts in U_1, ends in a vertex V_j, and which contains all of the vertices from $(\mathcal{B}_1 \setminus \mathcal{A}_1) \cup (\mathcal{B}_2 \setminus \mathcal{A}_2) \cup \cdots \cup (\mathcal{B}_j \setminus \mathcal{A}_j)$. If $\mathcal{B}_{j+1} \subset \mathcal{A}_{j+1} \cup V(\mathcal{S}_j)$, then let $\mathcal{S}_{j+1} = \mathcal{S}_j$ and $V_{j+1} = V_j$. Otherwise, $V(\mathcal{S}_j) \cap (\mathcal{B}_{j+1} \setminus \mathcal{A}_{j+1}) = \emptyset$ by **P6**, and so let U_{j+1} be a neighbour of V_j such that $U_{j+1} \in \mathcal{B}_{j+1} \setminus \mathcal{A}_{j+1}$. We extend $\mathcal{S}_j$ to a path $\mathcal{S}_{j+1}$ using the method of the last paragraph. Let $\mathcal{S}_{j+1}$ end in a vertex V_{j+1}.

In both cases, we have formed a path $\mathcal{S}_{j+1}$ which starts in U_1, ends in a vertex V_{j+1}, and which contains all of the vertices from $(\mathcal{B}_1 \setminus \mathcal{A}_1) \cup (\mathcal{B}_2 \setminus \mathcal{A}_2) \cup$

$\cdots \cup (\mathcal{B}_{j+1} \setminus \mathcal{A}_{j+1})$. Now the path $\mathcal{S}_k$ which we get by induction actually contains all the vertices of $\mathcal{B}^*$. This is because each cycle avoiding a set in $\mathcal{R}_{k+2}$ has a clique-edge by **P2** and hence will be used to form some $\mathcal{S}_j$. Moreover, for all $i = 1, 2, \ldots, k$, any element in $\mathcal{A}_i$ is in some cycle avoiding a set in $\mathcal{R}_{k+2}$ and thus in some $\mathcal{S}_j$.

The p-cycle we want is just $B^*\mathcal{S}_k B^*$. ∎

With some additional restrictions on the designs, more can be obtained from the proof of the theorem. We now give the proofs of these corollaries.

Proof of Corollary 2:

Let B^*B_1 be any edge of G. Without loss of generality we may suppose $B_1 \in \mathcal{B}_1$. Since $|B^*| = k = u$ we can let $\mathcal{B}^* = \mathcal{B}_2 \cup \mathcal{B}_3 \cup \cdots \cup \mathcal{B}_k$ and the proof of Theorem 1 is still valid. To form the p-cycle using B^*B_1 we start with B^*B_1, use all the vertices in $\mathcal{B}_1$, jump to a vertex in $\mathcal{B}_2 \setminus \mathcal{A}_2$ and continue as in Theorem 1. This gives the desired p-cycle using B^*B_1. ∎

Proof of Corollary 3:

Since $K = \{k, k+1\} = \{u-1, u\}$ we may choose any $B^* \in \mathcal{B}$ and the cycle constructed from Theorem 1 uses this B^*. ∎

3 Acknowledgements

The author wishes to thank the Department of Mathematics and Statistics, University of Victoria, B. C., for their hospitality during the preparation of this paper.

References

[1] B. Alspach and D. Hare. Edge-pancyclic block-intersection graphs. Discrete Mathematics, to appear.

[2] B. Alspach, K. Heinrich, and B. Mohar. A note on hamilton cycles in block-intersection graphs. In E. S. Kramer and S. Magliveras, editors, *Contemporary Mathematics*, pages 1–4, AMS, 1990.

[3] M. Behzad and E. S. Mahmoodian. Graphs versus designs — a quasisurvey. Proceedings of the Sixth International Conference on the Theory and Applications of Graphs (Wiley), to appear.

[4] P. Horák and A. Rosa. Decomposing steiner triple systems into small configurations. *Ars Combinatoria*, 26:91–105, 1988.