Isolated Systems in General Relativity: The Gravitational–Electrostatic Two–Body Balance Problem and The Gravitational Geon

by

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Abstract

This dissertation examines two fundamentally different types of isolated systems in general relativity. In part 1, an exact solution of the Einstein–Maxwell equations representing the exterior field of two arbitrary charged essentially spherically symmetric (Reissner–Nordström) bodies in equilibrium is studied. Approximate solutions representing the gravitational-electrostatic balance of two arbitrary point sources in general relativity have led to contradictory arguments in the literature with respect to the condition of balance. Up to the present time, the only known exact solutions which can be interpreted as the nonlinear superposition of two Reissner–Nordström bodies without an intervening strut has been for critically charged masses, $M_i^2 = Q_i^2$. In this dissertation , the invariant physical charge for each source is found by direct integration of Maxwell's equations. The physical mass for each source is invariantly defined in a manner similar to which the charge was found. It is shown that balance without tension or strut can occur for non-critically charged bodies. It is demonstrated that other authors have not identified the correct physical parameters for the masses and charges of the sources. Examination of the fundamental parameters of the space-time suggests a refinement of the nomenclature used to describe the physical properties is necessary. Such a refinement is introduced. Further properties of the solution, including the multipole structure and comparison with other parameterizations, are examined. Part 2 investigates the viability of constructing gravitational and electromagnetic geons: zero-rest-mass field concentrations, consisting of gravitational or electromagnetic waves, held together for long periods of time by their gravitational attraction. In contrast to an exact solution, the method studied involves solving the Einstein or Einstein–Maxwell equations for perturbations on a static background metric in a self-consistent manner. The Brill–Hartle gravitational geon construct as a spherical shell of small amplitude, high-frequency gravitational waves is reviewed and critically analyzed. The spherical shell in the proposed Brill–Hartle geon cannot be regarded as an adequate geon construct because it does not meet the regularity conditions required for a non-singular source. An attempt is made to build a non-singular solution to meet the requirements of a gravitational geon. Construction of a geon requires gravitational waves of high-frequency and the field equations are decomposed accordingly. A geon must also possess the property of quasi-stability on a time-scale longer than the period of the comprising waves. It is found that only unstable equilibrium solutions to the gravitational and electromagnetic geon problem exist. A perturbation analysis to test the requirement of quasi-stability resulted in a contradiction. Thus it could not be concluded that either electromagnetic or gravitational geons meet all the requirements for existence. The broader implications of the result are discussed with particular reference to the problem of gravitational energy.

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To

Mom and Dad

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Notation

General relativity is notorious for its variety of notations and conventions. Since this dissertation consists of two parts, each will use a different notation in order to remain as consistent as possible with the current literature. Part I (chapters 1–5) follows the conventions and notations of Landau and Lifshitz [26]:

The metric signature is $(+ - - -)$.

Units are chosen in which $G = c = 1$.

Latin indices run over the four space-time coordinate labels 0, 1, 2, 3.

Greek indices run over the three space coordinate labels 1, 2, 3.

A comma and a semicolon denote, respectively, ordinary and covariant differentiation.

The flat-space Levi–Civita permutation symbol is e^{ijkl}, where $e^{0123} = 1$.

The Ricci tensor is given by $R_{ik} = \Gamma^l_{ik,l} - \Gamma^l_{il,k} + \Gamma^m_{ik}\Gamma^l_{ml} - \Gamma^l_{mk}\Gamma^m_{il}$.

Part II (chapters 6–11) follows the conventions and notations of Misner, Thorne and Wheeler [56]:

The metric signature is $(- + + +)$.

Units are chosen in which $G = c = 1$.

Greek indices run over the four space-time coordinate labels 0, 1, 2, 3.

Latin indices run over the three space coordinate labels 1, 2, 3 (apart from appendix C, where they assume the values 0, 2 and 3).

A comma and a semicolon denote, respectively, ordinary and covariant differentiation with respect to the background metric.

A dot over any quantity denotes the time derivative of that quantity. A prime denotes differentiation with respect to the argument (for a single variable function) or spatial coordinate (for a two variable function) of the quantity.

The Ricci tensor is given by $R_{\mu\nu} = \Gamma^\sigma_{\mu\nu,\sigma} - \Gamma^\sigma_{\mu\sigma,\nu} + \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} - \Gamma^\sigma_{\rho\nu}\Gamma^\rho_{\mu\sigma}$.

Chapter 1

Introduction

In this dissertation, two types of isolated systems in general relativity are studied. Part I investigates an exact solution of the Einstein–Maxwell equations which represents the exterior field of two arbitrarily charged spherically symmetric (Reissner–Nordström) bodies in static equilibrium. In the classical (Newtonian) analog of this system, the condition for equilibrium is found through the force balance equation

$$G\frac{M_1 M_2}{r^2} = k\frac{Q_1 Q_2}{r^2}. \tag{1.1}$$

Since the dependence on the radial separation r cancels, the separation between the sources is arbitrary. It is of considerable interest to compare the phenomena of classical (Newtonian) physics and general relativity. Simple systems in classical physics such as the one described above are often difficult to solve in a nonlinear theory such as general relativity. Such investigations are of great value for providing physical insight into nonlinear systems and they often yield new and unexpected results.

Both exact and approximate methods have previously been used to study the electrostatic two-body problem in general relativity. Methods involving the post-Newtonian approximation have led to contradictory arguments with respect to the condition of balance [1–3]. From the examination of a charged test body in the presence of a Reissner–Nordström source, Bonnor [4] has found several conditions for equilibrium. Some of the balance conditions found depend on the separation distance of the sources while others do not. With the plethora of possible balance conditions derived from the different approximation techniques, it is apparent that an exact solution is necessary to resolve the apparent discrepancies among the approximation methods.

Early formulation of the static, axially symmetric problem in general relativity was principally developed by Weyl [5]. Many solutions have been found (see, for example, [6–10] and others) which follow from [5]. These solutions (known as Weyl-class solutions) are characterized by the electrostatic potential being functionally related to the gravitational potential (metric component g_{00}). In all of the above solutions, a line singularity (interpreted as a physical strut or tension) is present to maintain the static configuration when two sources are present. Only under the added constraint that each source be 'critically charged,' i.e.[1]

$$M_i^2 = Q_i^2, \qquad i = 1, 2 \tag{1.2}$$

would the line singularity be removed.

To advance from the constraints of the Weyl-class solutions, several researchers have employed the use of generating techniques to solve the electrostatic two-body problem. In this method, new solutions are generated from

[1] Henceforth, units are chosen in which $G = c = 1$.

old ones using symmetry transformations. Although a wide variety of non-Weyl-class solutions have been found (see, for example, [11–15] and references therein), many cannot be interpreted as two spherically symmetric (Reissner–Nordström) bodies while others exhibit singularities outside of the sources (see, for example, [16]). Of those which can be interpreted as two Reissner–Nordström bodies, none have been found to exhibit balance conditions other than that found in Weyl-class solutions. Hence, up to the present time, the only known exact solutions which could be interpreted as the nonlinear superposition of two Reissner–Nordström bodies without an intervening strut or tension has been for critically charged masses, $M_i^2 = Q_i^2$.

The main results for part I of this dissertation are presented in [17]. In part I, an exact electrostatic solution of the Einstein–Maxwell equations representing the exterior field of two arbitrarily charged nonlinearly superposed Reissner–Nordström solutions in equilibrium is given. It is obtained with the aid of Sibgatullin's [18] method for constructing the complex Ernst potentials [19]. The metric functions and the electrostatic potential of the space-time can then be found with knowledge of the Ernst potentials. In Sibgatullin's method, the complete Ernst potentials are generated by specifying the structure of the Ernst potentials on the symmetry axis of the space-time. The solution presented in this dissertation is mathematically equivalent to the solutions of Manko *et al* [20] and Chamorro *et al* [21], henceforth referred to as papers I and papers II respectively, (with their spin parameters set to zero) and they are all special cases of the general mathematical solution given by Ernst [22]. It is of primary importance that the parameters in the solution be related to a *physical* set of parameters in order for any subsequent analysis of the solution to have any physical meaning. In this dissertation, we identify a physical set of parameters. The physical set of parameters one would like to use are the individual mass and charge of each body and the proper separation of the objects. The charge enclosed by a space-like hypersurface can be covariantly defined in general relativity. For space-times with a time-like Killing vector, a conserved quantity which can be interpreted as the contribution to the total mass from each body can be covariantly defined in analogy with the charge (see, for example, [12, 23, 24]). This dissertation follows the definition given by Kramer [12] for the definition of the individual mass of each body. It is demonstrated that the parameterizations employed in papers I and II do not represent the physical masses or charges of the individual sources even in the Weyl-class limit (except for the special case of identical bodies in paper I). The new parameterization introduced includes the physical Weyl-class parameterization as a special case. An analytical solution in terms of the individual physical masses and charges and the dependence of the balance conditions on the separation of the bodies is not yet known due to the complexities of the parameterization. The main result from this work is the discovery through numerical methods of three balance conditions (absence of a physical strut or tension) for which neither body is 'critically charged' nor is it the Newtonian balance condition. To the author's knowledge, this is the first demonstration of an exact solution which has such properties. One of the balanced cases requires the bodies to be oppositely charged. Such a configuration is not possible in Newtonian physics. The results also strongly suggest that these balanced cases are dependent on the separation distance. All the balanced cases found are in accordance with Bonnor's [4] test particle analysis. Examination of the fundamental parameters of the space-time suggests that a refinement of the nomenclature used to describe the physical properties is necessary. Such a refinement is introduced. Further properties of the solution, including the multipole structure and comparison with other parameterizations, are examined.

In contrast to the exact solution studied in part I, part II of this dissertation examines an approximate solution of the Einstein–Maxwell equations. It has been widely assumed that the gravitational field shares many of the essential properties of other physical fields. If one of these properties is energy, then it is conceivable to construct a near-spherical region with a measurable effective mass-energy content solely out of gravitational waves. Wheeler [36] conceived of analogous constructs formed from electromagnetic waves as well as neutrino concentrations, held together by their gravitational fields. The Brill–Hartle (henceforth referred to as BH) model [37] for the construction of a gravitational geon is critically analyzed. In their approach, BH considered a strongly curved static 'background geometry' $\gamma_{\mu\nu}$ on top of which a small ripple $h_{\mu\nu}$ resided, satisfying a linear wave equation. The wave frequency was assumed to be so high as to create a sufficiently large effective energy density which served as the source of the background $\gamma_{\mu\nu}$, taken to be spherically symmetric on a time-average. They claimed to have found a solution with a flat-space spherical interior, a Schwarzschild exterior and a thin shell separation meant to be created by high-frequency gravitational waves. With the mass M identified from the exterior metric, there would follow an unambiguous realization of the gravitational geon as described above.

The BH model does not implement the properties of high-frequency waves in their analysis. The properties of high-frequency waves were rigourously established by Isaacson [38] after the BH model was proposed. In addition, matching the flat interior metric to the Schwarzschild exterior metric at the thin shell boundary creates a discontinuity in the background metric. The presence of this discontinuity violates the Darmois junction conditions for regularity. Hence, the space-time cannot be taken as singularity-free. These deficiencies are sufficient cause for re-evaluating the gravitational geon problem.

The analysis in this dissertation presents a detailed and expanded study of the gravitational and electromagnetic geon problem based upon the papers of Cooperstock, Faraoni, and Perry [39–41] and Anderson and Brill [42]. It was proposed in [39–41] that a satisfactory gravitational geon model must be constructed and solved in a manner similar to that of Wheeler's [36] electromagnetic geon model. In such a model, the Einstein field equations are solved in a self-consistent manner while satisfying the regularity conditions for a singularity-free space-time. It must also be demonstrated that the geon be quasi-stable, i.e. the evolution in time of the background metric must take place on a time-scale much longer than the characteristic period of the constituent waves.

The proper decomposition of the Einstein field equations in reference to the gravitational geon problem was studied in [39–41] and [42]. In the former analysis, it is established that high-frequency waves are a necessary condition for constructing a gravitational geon. From these papers, it was found that in the high-frequency approximation, the gravitational geon problem and the electromagnetic geon problem are governed by the same set of ordinary differential equations (ODEs) and boundary conditions. These equations were derived assuming a static background metric. Thus any solutions are necessarily equilibrium solutions. As a consequence, they are only satisfactory for considering the regularity and self-consistency aspects of the geon problem. They are not sufficiently general for studying the evolution in time of geon constructs. Employing a phase portrait analysis, it is demonstrated that admissible equilibrium solutions are necessarily unstable. Therefore, one cannot claim existence of geon constructs until the property of quasi-stability is also demonstrated.

The property of quasi-stability has not been adequately demonstrated for the electromagnetic geon. The model used in [36] is based upon an analogy with the nuclear process of alpha-decay. It relies on the quantum mechanical effect of a photon tunnelling through a potential barrier to simulate leakage of electromagnetic radiation from a potential well. Such a quantum mechanical process cannot be reconciled with a classical object such as a geon. Another attempt [43] is based upon a thin-shell model for which singularities are present. Our approach is to apply a small amplitude time-dependent perturbation to an equilibrium solution of the electromagnetic geon field equations. The perturbations are designed in such a way as to induce the background metric to evolve in time. This can only be done in a meaningful way if it is assumed that the characteristic frequency of the perturbations vary on a time-scale much longer than that of the waves comprising the electromagnetic geon. This is in accordance with the requirement that the background metric be quasi-stable. However, in solving the perturbation equations, it is found that the perturbations must vary on the same time-scale as the constituent waves. This is a contradiction to the original assumption. The contradiction suggests that neither an electromagnetic geon nor a gravitational geon is a viable construct since not all of the requirements for the existence of a geon are met.

The nonexistence of a gravitational geon would be consistent with interpretations which question the conventional understanding of gravitational energy [44]. Determining the physically measurable properties of gravitational waves has important consequences in the design of gravitational wave detectors as well as in the interpretation of astrophysical phenomena such as the period change of the binary pulsar PSR 1913+16.

Chapter 2

Electrostatic Balance in General Relativity

2.1 Introduction

In a recent paper by Bonnor [4], the equilibrium conditions for a charged test particle in the field of a spherically symmetric charged mass (Reissner–Nordström solution) were investigated. He found that the classical condition for equilibrium

$$M_1 M_2 = Q_1 Q_2 \tag{2.1}$$

for which the separation between the particles is arbitrary, was neither necessary nor sufficient for electrostatic balance of two spherical masses. This is in conflict with the earlier results of Barker and O'Connell [1] and Kimura and Ohta [2] who used different approximation methods. Barker and O'Connell claimed that in the post-Newtonian approximation, the equation

$$(M_1 Q_2 - M_2 Q_1)(Q_1 - Q_2) = 0 \tag{2.2}$$

had to be satisfied in addition to (2.1). Kimura and Ohta claimed that in the post-post-Newtonian approximation, the necessary and sufficient condition for balance is that each mass be 'critically charged.'

$$M_i^2 = Q_i^2 \qquad i = 1,2 \tag{2.3}$$

and balance can occur for arbitrary separation of the sources. Up to the present time, the problem of gravitational–electrostatic balance of two spherical bodies in general relativity without an intervening Weyl line singularity (strut or tension) has been solved *exactly* only for critically charged masses [7, 10, 11]. A balanced solution was originally thought to have been found [15] within the Herlt-class for both sources having $M_i^2 > Q_i^2$, but it was subsequently shown that the intervening line singularity could not be removed [25]. Kramer [12] presented an exact solution for the electrostatic counterpart of the double Kerr–NUT solution with zero spin parameter. He found that condition (2.1) holds for electrostatic balance. However, he stated that his solution cannot be interpreted as the nonlinear superposition of two Reissner–Nordström solutions and thus the masses are not spherically symmetric.

Chapters 2 to 5 present and analyze an exact electrostatic solution of the Einstein–Maxwell equations representing the exterior field of two arbitrarily charged nonlinearly superposed Reissner–Nordström sources in equilibrium. In section 2.2, the integrals of charge and mass are introduced. In section 2.3 these integrals are applied to the Weyl-class solution for two Reissner–Nordström bodies and two Curzon bodies. Expressions for the charge decomposition for both the double Reissner–Nordström and double Curzon solutions are known [15]. The conjectured mass decomposition [10] for the double Curzon solution is verified and the mass decomposition for the double Reissner–

Nordström solution is presented for the first time. Chapter 3 presents the solution for a parameterization of the non-Weyl-class double Reissner–Nordström solution based on the Weyl-class parameterization. It is then compared to the parameterizations proposed in papers I and II. It is shown that the parameterizations employed in papers I and II do not represent the physical masses or charges of the individual sources even in the Weyl-class limit (except for the special case of identical bodies in paper I).

Chapter 4 examines equilibrium without a strut or tension for numerical values of the physical mass and charge. It is found that there are balance conditions for which neither body is critically charged and the Newtonian balance condition does not hold. The results are compared with Bonnor's [4] test particle analysis and are found to be consistent with them. A discussion of the results and conclusions are given in chapter 5.

2.2 Mass and charge

For a static axially symmetric space-time, the mass M_i and charge Q_i of a source inside a closed 2–surface σ_i are given by the integrals[1] [12]

$$M_i \equiv -\frac{1}{8\pi}\oint_{\sigma_i} K^{ab}\sqrt{-g}\,\mathrm{d}f^*_{ab} \tag{2.4}$$

$$Q_i = -\frac{1}{8\pi}\oint_{\sigma_i} F^{ab}\sqrt{-g}\,\mathrm{d}f^*_{ab} \tag{2.5}$$

[1]The metric signature is $+---$. Units are chosen in which $G = c = 1$. Latin indices run from 0 to 3 and Greek indices run from 1 to 3. A comma and a semicolon denote, respectively, ordinary and covariant differentiation. The Ricci tensor is given by $R_{ik} = \Gamma^l_{ik,l} - \Gamma^l_{il,k} + \Gamma^m_{ik}\Gamma^l_{ml} - \Gamma^l_{mk}\Gamma^m_{il}$. Notations and conventions used are those of [26].

where

$$K^{ab} \equiv \xi^{a;b} + \Phi F^{ab} \,. \tag{2.6}$$

The time-like Killing vector is ξ^a, F_{ab} is the electromagnetic field tensor. Φ is the electrostatic potential. g is the determinant of the metric and $\mathrm{d}f^*_{ab}$ is the dual to the surface element 2-form $\mathrm{d}f^{ab}$.

$$\mathrm{d}f^*_{ab} = \tfrac{1}{2} e_{abcd} \mathrm{d}f^{cd} \tag{2.7}$$

(here e_{abcd} is the flat space Levi–Civita permutation symbol). The above integral conservation laws follow from the local conservation laws

$$F^{ab}{}_{;b;a} = 0 \qquad K^{ab}{}_{;b;a} = 0, \tag{2.8}$$

the first, following from the conservation of charge and the second from the existence of the time-like Killing vector ξ^a and the restriction to a static axially symmetric space-time metric. Since the Einstein–Maxwell equations also imply

$$F^{ab}{}_{;b} = 0 \qquad K^{ab}{}_{;b} = 0, \tag{2.9}$$

in a source free region, any deformation of the surface σ_i in the electrovacuum region outside the sources does not change the values of the integrals M_i and Q_i.

2.3 The Weyl-class two-body solution

To investigate the structure of space-times with two sources, the Weyl-class double Reissner–Nordström solution provides a suitable yet mathematically elegant framework from which to proceed. The solution is easily found through the method presented in [10]. The metric for a static axially symmetric space-time can be written in the canonical form

$$ds^2 = e^w dt^2 - e^{v-w}\left(d\rho^2 + dz^2\right) - \rho^2 e^{-w} d\phi^2. \tag{2.10}$$

where w and v are functions of the cylindrical coordinates ρ and z. The Weyl-class solutions are characterized by the metric function w which is a function of the electrostatic potential, i.e. $w = w(\Phi)$ so that the gravitational and electrostatic equipotential surfaces overlap. For asymptotically flat boundary conditions, the unique functional relationship between e^w and Φ is [5]

$$e^w = 1 - 2\frac{m_T}{q_T}\Phi + \Phi^2, \tag{2.11}$$

where Φ is the electrostatic potential and m_T and q_T are the total mass and charge, respectively. The solution representing two 'undercharged' ($M_i^2 > Q_i^2$) Reissner–Nordström bodies (or 'black holes') is given by

$$\Phi = a\frac{f-1}{a^2 f - 1}, \tag{2.12}$$

where

$$f = \left(\frac{R_1 + R_2 - 2l_1}{R_1 + R_2 + 2l_1}\right)\left(\frac{R_3 + R_4 - 2l_2}{R_3 + R_4 + 2l_2}\right), \tag{2.13}$$

$$R_1^2 \equiv (z - d - 2l_1)^2 + \rho^2, \tag{2.14}$$

$$R_2^2 \equiv (z - d)^2 + \rho^2, \tag{2.15}$$

$$R_3^2 \equiv (z + d)^2 + \rho^2, \tag{2.16}$$

$$R_4^2 \equiv (z + d + 2l_2)^2 + \rho^2. \tag{2.17}$$

The constant parameters $2l_1, 2l_2$ and $2d$ are the 'lengths' of the horizons (Weyl 'rods') and the coordinate distance between the horizons, respectively (see figure 3.1). The parameter a is defined through the equation

$$\frac{1+a^2}{a} = \frac{2m_T}{q_T}. \tag{2.18}$$

The metric function e^w is found through equation (2.11). The metric function e^v is

$$\begin{aligned} e^v = \frac{(R_1+R_2)^2 - 4l_1^2}{4R_1R_2} \cdot \frac{(R_3+R_4)^2 - 4l_2^2}{4R_3R_4} \\ \times \left(\frac{((l_1+l_2+d)\,R_1 + (l_2+d)\,R_2 - l_1R_4)d}{((l_1+d)\,R_1 + R_2 d - l_1 R_3)\,(l_2+d)} \right)^2 . \end{aligned} \tag{2.19}$$

Choosing the surface σ_1 to encompass body 1 and the surface σ_2 to encompass body 2 of figure 3.1, the mass and charge integrals of equations (2.4) and (2.5) yield

$$\begin{aligned} M_1 &= \frac{1+a^2}{1-a^2}\,l_1 \qquad & Q_1 &= \frac{2a}{1-a^2}\,l_1 \\ M_2 &= \frac{1+a^2}{1-a^2}\,l_2 \qquad & Q_2 &= \frac{2a}{1-a^2}\,l_2 . \end{aligned} \tag{2.20}$$

The above form of the individual mass and charge for each Reissner–Nordström body is similar to the form proposed in [10] for the mass and charge decomposition of a system of two charged Curzon particles. It was stated in [15] that the conjectured charge decomposition for both the double Reissner–Nordström and double Curzon cases were verified by direct calculation through equation (2.5). It is straightforward to verify that equation (2.4) yields the conjectured mass decomposition for the double Curzon solution. The mass decomposition for the double Reissner–Nordström solution is presented for the first time in (2.20).

Because of the functional relationship between the gravitational potential and the electrostatic potential, not all of the parameters M_1, M_2, Q_1, Q_2 are independent. The Weyl-class is also characterized by the constraint

$$M_1 Q_2 = M_2 Q_1 \tag{2.21}$$

which is easily seen in (2.20). Removal of the line singularity (tension or strut) between the bodies yields equation (2.1) as an additional condition on the parameters. As a result, the parameters also satisfy equation (2.3). Thus equilibrium without a strut or tension occurs for 'critically charged' sources and this balance is found to be independent of the separation distance [10].

Chapter 3

Non-Weyl Parameterizations

3.1 Introduction

Generalizing the Weyl-class double Reissner–Nordström solution to the case in which the gravitational and electrostatic equipotential surfaces no longer overlap has usually been attempted through the means of generating techniques (see, for example, [11, 12] and [15]). In these techniques, new solutions are generated from old ones rather than by solving the equations directly. Recently, considerable interest has focused upon a method [18] which constructs the Ernst potentials [19] from initial data on the symmetry axis. The complex Ernst potentials $\mathcal{E}(\rho, z)$ and $\Psi(\rho, z)$ of all stationary axisymmetric electrovacuum space-times with axis data of the form

$$\mathcal{E}(z, \rho = 0) = \frac{U - W}{U + W}, \qquad \Psi(z, \rho = 0) = \frac{V}{U + W}, \tag{3.1}$$

where

$$U = z^2 + U_1 z + U_2 , \tag{3.2}$$

$$V = V_1 z + V_2 , \tag{3.3}$$

$$W = W_1 z + W_2 \tag{3.4}$$

and $U_1, U_2, V_1, V_2, W_1, W_2$ are complex constants, have been found [22]. However, a mathematical solution to the Einstein–Maxwell field equations does not imply a well understood physical interpretation of the solution. Sibgatullin's method of constructing the Ernst potentials aids in obtaining the physically meaningful parameterization which is sought for the two-body case in question.

In Sibgatullin's method, it is required that the Ernst potentials along the z-axis be specified. Our choice was [27, 28]

$$\begin{aligned}\mathcal{E}(\rho = 0, z) \equiv e(z) &= 1 - \frac{2\big(m_1(z+z_2) + m_2(z+z_1)\big)}{(z+z_1+m_1)(z+z_2+m_2) - q_1 q_2},\\ \Psi(\rho = 0, z) \equiv F(z) &= \frac{q_1(z+z_2) + q_2(z+z_1)}{(z+z_1+m_1)(z+z_2+m_2) - q_1 q_2}.\end{aligned} \tag{3.5}$$

It has the form of the Weyl-class double Reissner–Nordström axis data. If the additional Weyl-class constraint

$$m_1 q_2 - m_2 q_1 = 0 \tag{3.6}$$

is placed on the functions $e(z)$ and $F(z)$, then Sibgatullin's method yields the Weyl-class double Reissner-Nordström solution (in an alternate form to [10]) and the parameters m_1, m_2, q_1, q_2 are the physical masses and charges as defined by (2.4) and (2.5) (i.e. $M_1 = m_1, Q_1 = q_1, M_2 = m_2, Q_2 = q_2$). For the solution of two Weyl-class Reissner–Nordström black holes (given in chapter 2.3), figure 3.1 shows the coordinate positions of the centers of the 'rods' as $d + l_1$ for body 1 and $-d - l_2$ for body 2. The parameters z_1, z_2 identify the negative of the coordinate positions of the centers of the 'rods,' i.e.

$$z_1 = -d - l_1, \qquad z_2 = d + l_2.$$

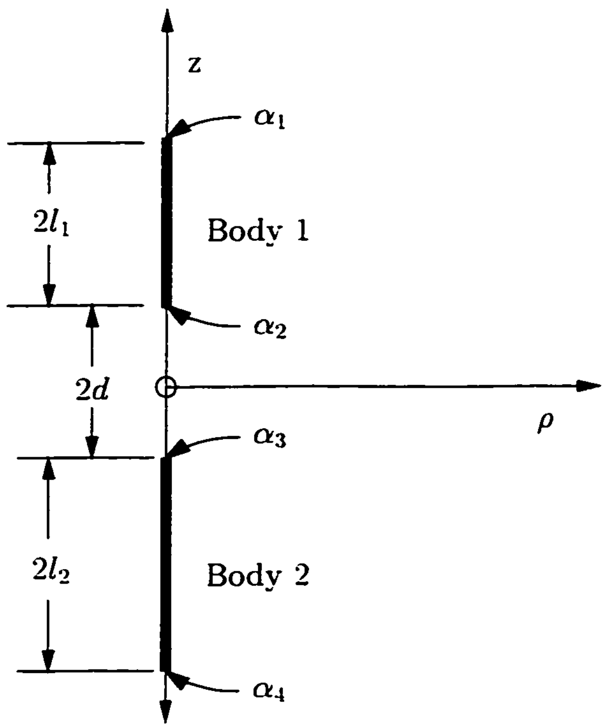

Figure 3.1: Two Reissner–Nordström black holes.

Schematic of two Reissner–Nordström black holes in cylindrical coordinates. The thick lines are the Weyl 'rods' which show the locations of the event horizon surfaces. In the Weyl-class, $\alpha_1 = d + 2l_1$, $\alpha_2 = d$, $\alpha_3 = -d$, $\alpha_1 = -d - 2l_2$ *for 'undercharged' bodies. For non-Weyl-class solutions, the aforementioned relationships for* α_n *are no longer valid. The* α_n *are defined as the roots of equation* (3.11).

If condition (3.6) is not imposed, $w \neq w(\Phi)$, i.e. the gravitational and electrostatic equipotential surfaces no longer overlap. In chapter 4 it will be shown that the parameters m_1, m_2, q_1, q_2 then no longer carry the suggested physical meaning and the parameters z_1, z_2 no longer coincide with the centers of the 'rods' when the Weyl-class constraint (3.6) is not imposed.

3.2 Construction of the Ernst potentials and metric functions

A brief outline of Sibgatullin's method [18] of constructing the full Ernst potentials $\mathcal{E}(\rho, z)$ and $\Psi(\rho, z)$ for the axis data of equation (3.5) will be given below. The extension of the method to the stationary case can be found in [18,20,29] and in the review article [30].

The Ernst potentials are found from the integrals

$$\mathcal{E}(\rho, z) = \frac{1}{\pi}\int_{-1}^{1} \frac{\mu_1(\sigma)\, e(\xi)\, \mathrm{d}\sigma}{\sqrt{1-\sigma^2}}, \qquad \Psi(\rho, z) = \frac{1}{\pi}\int_{-1}^{1} \frac{\mu_1(\sigma)\, F(\xi)\, \mathrm{d}\sigma}{\sqrt{1-\sigma^2}}, \tag{3.7}$$

with the unknown function $\mu_1(\sigma)$ satisfying the integral equations

$$\int_{-1}^{1} \frac{\mu_1(\sigma)\mathrm{d}\sigma}{\sqrt{1-\sigma^2}} = \pi \tag{3.8}$$

and

$$⨍_{-1}^{1} \frac{\left(e(\xi) + e(\eta) + 2\,F(\xi)F(\eta)\right)\mu_1(\sigma)\mathrm{d}\sigma}{(\xi-\eta)\sqrt{1-\sigma^2}} = 0, \tag{3.9}$$

where $⨍$ denotes the principal value of the integral.

In the above equations, $e(\xi)$ and $F(\xi)$ are the locally analytic continuations of $e(z)$ and $F(z)$ to the complex plane. They are obtained by applying the transformation $z \to \xi \equiv z + \mathrm{i}\rho\sigma$, $\sigma \in [-1, 1]$. The functions $e(\eta)$ and $F(\eta)$ are similarly found, with $z \to \eta \equiv z + \mathrm{i}\rho\tau$, $\tau \in [-1, 1]$.

In order to evaluate the integrals (3.9) and (3.8), $\mu_1(\sigma)$ should be sought in the form

$$\mu_1(\sigma) = A_0 + \sum_{n=1}^{4} A_n \left(\xi - \alpha_n\right)^{-1} . \tag{3.10}$$

where the coefficients A_0, A_n are functions of ρ and z and the constants α_n are the roots of the equation

$$e(\xi) + F^2(\xi) = 0 . \tag{3.11}$$

(The α_n can only be real or complex conjugate pairs.)

After substituting equation (3.10) and $e(\xi)$, $F(\xi)$ into (3.9) and (3.8), the integrals are evaluated with the aid of the following formulae:

$$\begin{aligned}
\int_{-1}^{1} \frac{d\sigma}{\sqrt{1-\sigma^2}} &= \pi , \\
\int_{-1}^{1} \frac{d\sigma}{(\xi-\alpha_i)\sqrt{1-\sigma^2}} &= \frac{\pi}{\sqrt{\rho^2 + (z-\alpha_i)^2}} . \\
\fint_{-1}^{1} \frac{d\sigma}{(\xi-\eta)\sqrt{1-\sigma^2}} &= 0 , \\
\fint_{-1}^{1} \frac{d\sigma}{(\xi-\eta)(\xi-\alpha_i)\sqrt{1-\sigma^2}} &= \frac{\pi}{(\alpha_i-\eta)\sqrt{\rho^2 + (z-\alpha_i)^2}} .
\end{aligned}$$

Once (3.9) and (3.8) have been evaluated, a closed system of five linear algebraic equations is obtained by performing a partial fraction decomposition of the resulting integrations and equating the coefficients of the independent partial fractions of η to zero. For the problem at hand, the system for the five unknowns A_0, A_n, $n = 1, \ldots, 4$ is

$$A_0 + \sum_{n=1}^{4} \frac{A_n}{r_n} = 1 , \tag{3.12}$$

$$A_0 + \sum_{n=1}^{4} \frac{A_n}{\beta_1 - \alpha_n} = 0 . \tag{3.13}$$

$$A_0 + \sum_{n=1}^{4} \frac{A_n}{\beta_2 - \alpha_n} = 0 . \tag{3.14}$$

$$\sum_{n=1}^{4} \frac{A_n}{(\beta_1 - \alpha_n)\, r_n} \left(K_1 + \frac{K_3^2}{\beta_1 - \alpha_n} + \frac{K_3 K_4}{\beta_2 - \alpha_n} \right) = 0 . \tag{3.15}$$

$$\sum_{n=1}^{4} \frac{A_n}{(\beta_2 - \alpha_n)\, r_n} \left(K_2 + \frac{K_4^2}{\beta_2 - \alpha_n} + \frac{K_3 K_4}{\beta_1 - \alpha_n} \right) = 0 . \tag{3.16}$$

where

$$r_n \equiv \sqrt{\rho^2 + (z - \alpha_n)^2}, \qquad n = 1, \ldots, 4 .$$

$$K_1 \equiv \frac{m_1 z_2 + m_2 z_1 + (m_1 + m_2)\beta_1}{\beta_1 - \beta_2}, \quad K_2 \equiv \frac{m_1 z_2 + m_2 z_1 + (m_1 + m_2)\beta_2}{\beta_2 - \beta_1} .$$

$$K_3 \equiv \frac{q_1 z_2 + q_2 z_1 + (q_1 + q_2)\beta_1}{\beta_1 - \beta_2}, \quad K_4 \equiv \frac{q_1 z_2 + q_2 z_1 + (q_1 + q_2)\beta_2}{\beta_2 - \beta_1} .$$

$$\beta_1 \equiv -\frac{1}{2} \left(z_1 + m_1 + z_2 + m_2 - \sqrt{(z_1 - z_2 + m_1 - m_2)^2 + 4 q_1 q_2} \right) ,$$

$$\beta_2 \equiv -\frac{1}{2} \left(z_1 + m_1 + z_2 + m_2 + \sqrt{(z_1 - z_2 + m_1 - m_2)^2 + 4 q_1 q_2} \right) .$$

The intermediate form of the Ernst potentials in terms of A_0 and A_n can be found by substituting (3.10) into (3.7) and utilizing (3.12)–(3.16). The results are

$$\mathcal{E} = 2 A_0 - 1 , \qquad \Psi = K_3 \sum_{n=1}^{4} \frac{A_n}{(\alpha_n - \beta_1)\, r_n} + K_4 \sum_{n=1}^{4} \frac{A_n}{(\alpha_n - \beta_2)\, r_n} . \tag{3.17}$$

Finally, the full Ernst potentials $\mathcal{E}(\rho, z)$ and $\Psi(\rho, z)$ for the axis data of equation (3.5), expressed in terms of the cylindrical coordinates (ρ, z), can be written in Kinnersley's [31] form

$$\mathcal{E} = \frac{A - B}{A + B}, \quad \Psi = \frac{C}{A + B}, \tag{3.18}$$

where

$$A \equiv \sum_{i<j}^{4} a_{ij} r_i r_j, \qquad B \equiv \sum_{i=1}^{4} b_i r_i, \qquad C \equiv \sum_{i=1}^{4} c_i r_i.$$

The explicit form of the constants α_n are given in appendix A. The remaining constants a_{ij}, b_i and c_i are defined as follows:

$$\begin{aligned}
a_{ij} &\equiv (-1)^{i+j+1} s_i s_j t_i t_j (s_i t_j - s_j t_i) \begin{vmatrix} s_k v_k & s_l v_l \\ t_k u_k & t_l u_l \end{vmatrix}, \\
&\qquad (i < j;\ k < l;\ k, l \neq i, j;\ i, k = 1, \ldots, 3;\ j, l = 2, \ldots, 4); \\
b_i &\equiv (-1)^i s_i t_i (s_i - t_i) \begin{vmatrix} s_k^2 t_k^2 & s_l^2 t_l^2 & s_m^2 t_m^2 \\ s_k v_k & s_l v_l & s_m v_m \\ t_k u_k & t_l u_l & t_m u_m \end{vmatrix}, \\
&\qquad (k < l < m;\ k, l, m \neq i;\ i = 1, \ldots, 4; \\
&\qquad\qquad k = 1, 2;\ l = 2, 3;\ m = 3, 4); \\
c_i &\equiv (-1)^{i+1} s_i t_i (s_i - t_i)(K_3 G_i + K_4 H_i), \\
G_i &\equiv \begin{vmatrix} s_k t_k^2 & s_l t_l^2 & s_m t_m^2 \\ s_k v_k & s_l v_l & s_m v_m \\ t_k u_k & t_l u_l & t_m u_m \end{vmatrix}, \qquad H_i \equiv \begin{vmatrix} s_k^2 t_k & s_l^2 t_l & s_m^2 t_m \\ s_k v_k & s_l v_l & s_m v_m \\ t_k u_k & t_l u_l & t_m u_m \end{vmatrix}, \\
&\qquad (k < l < m;\ k, l, m \neq i;\ i = 1, \ldots, 4; \\
&\qquad\qquad k = 1, 2;\ l = 2, 3;\ m = 3, 4); \\
s_i &\equiv \beta_1 - \alpha_i, \qquad t_i \equiv \beta_2 - \alpha_i, \\
u_i &\equiv K_1 s_i t_i + K_3^2 t_i + K_3 K_4 s_i, \qquad v_i \equiv K_2 s_i t_i + K_4^2 s_i + K_3 K_4 t_i.
\end{aligned} \tag{3.19}$$

where all of the subsequent quantities introduced are constants ultimately defined in terms of m_i, q_i, z_i, $i = 1, 2$, which specify the character and locations of the sources in the Weyl-class limit only.

The expressions for $\mathcal{E}$ and Ψ in Kinnersley's [31] form permits one to write the corresponding metric functions as

$$\mathrm{e}^{w} = \frac{A\overline{A} - B\overline{B} + C\overline{C}}{(A+B)(\overline{A}+\overline{B})}, \qquad \mathrm{e}^{v} = \frac{A\overline{A} - B\overline{B} + C\overline{C}}{K_0 r_1 r_2 r_3 r_4}. \tag{3.20}$$

where

$$K_0 = \left(\sum_{i<j}^{4} a_{ij}\right)\left(\sum_{i<j}^{4} \overline{a}_{ij}\right) \tag{3.21}$$

and a bar denotes complex conjugation. For a static metric, the electrostatic potential Φ is equal to the Ernst potential Ψ and this completes the solution.

3.3 Mass-charge integrals and multipole moments

With the knowledge of the full Ernst potentials and the metric functions, the next step would be to evaluate the true mass and charge integrals in terms of the parameters m_1, m_2, q_1, q_2, z_1, z_2. It is to be stressed that outside of the Weyl-class, these parameters no longer carry the suggested physical meaning. For the metric (2.10), the integrals (2.4) and (2.5) can be written as relations in flat 3-space ($i = 1, 2$) [12] :

$$M_i = \frac{1}{8\pi}\oint_{\sigma_i} \mathrm{e}^{-w}\mathcal{E}_{,\alpha}\, n^{\alpha}\, \mathrm{d}A\,, \tag{3.22}$$

$$Q_i = -\frac{1}{4\pi}\oint_{\sigma_i} \mathrm{e}^{-w}\Phi_{,\alpha}\, n^{\alpha}\, \mathrm{d}A\,, \tag{3.23}$$

where n^a (α runs from 1 to 3) is the unit vector orthogonal to the surface and dA denotes the invariant (flat) surface element (see also [12] and references therein).

We can extend the Weyl-class definitions of the coordinate positions of the bodies to the non-Weyl-class solution. There are three distinct types of sources of interest. They are characterized by the transition between a source with an event horizon to one without an event horizon. As mentioned previously, the constants α_n, $n = 1, \ldots, 4$ in equation (3.18) are either real or complex conjugate pairs. By definition, we choose[1] $Re(\alpha_1) \geq Re(\alpha_2) > Re(\alpha_3) \geq Re(\alpha_4)$. A Reissner–Nordström 'black hole' is characterized by a real pair of α_n. Figure 3.1 shows that in the Weyl canonical coordinate system, the pair α_1, α_2 indicates the end points of a Weyl 'rod,' which itself is the event horizon surface. An 'extreme' object is characterized by a real equal pair of α_n, e.g. $\alpha_1 = \alpha_2$ with $\alpha_1, \alpha_2 \in \mathbb{R}$. A 'superextreme' object [20] or 'naked singularity' is characterized by a complex conjugate pair of α_n. Body 2 of figure 3.2 illustrates the manifestation of a 'superextreme' body in the space-time. Therefore we have the following definitions for the coordinate positions of the sources:

(*i*) For a Reissner–Nordström 'black hole,' we define $-Z_i$ to be the coordinate position of the center of the Weyl 'rod.' For example, the coordinate position of body 1 of figure 3.2 is

$$-Z_1 = \tfrac{1}{2}(\alpha_1 + \alpha_2)\,.$$

(*ii*) For a 'superextreme' object, we define $-Z_i$ to be the coordinate position of the real part of α_n. For example, body 2 of figure 3.2 is a 'superextreme'

[1] If, for example, α_1, α_2 are a complex conjugate pair, then we define α_1 as the value with positive imaginary part.

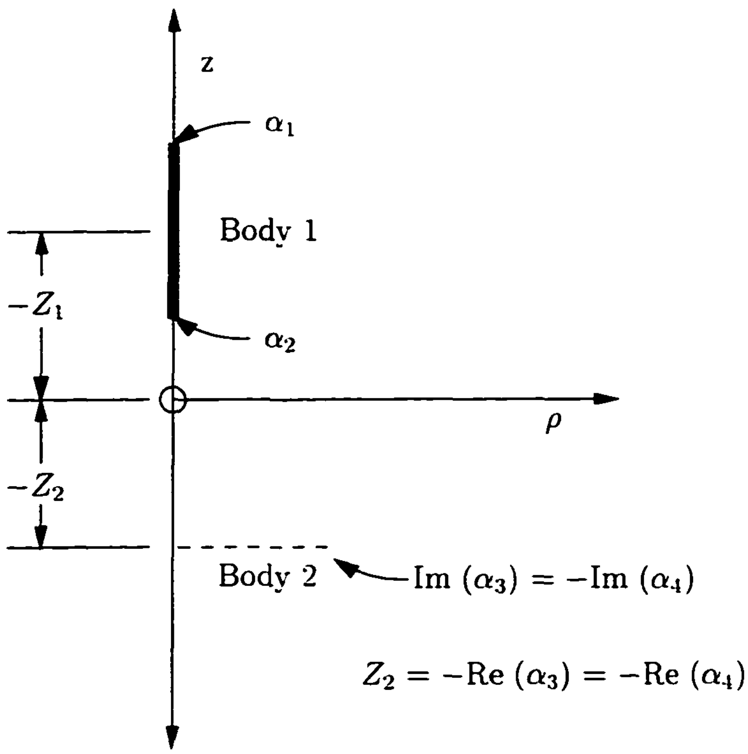

Figure 3.2: One black hole and one superextreme body.

Schematic of a Reissner-Nordström black hole and a Reissner-Nordström superextreme body. The dotted line is a 'complex Weyl rod.' The intersection of the 'rod' with the z-axis is defined as the coordinate position of body 2.

object. Therefore its coordinate position is

$$-Z_2 = \mathrm{Re}(\alpha_3) = \mathrm{Re}(\alpha_4).$$

(One could consider the imaginary part of α_n as the end points of a 'complex Weyl rod' with the coordinate position of this 'complex rod' being defined as its intersection with the real axis (z-axis).)

(*iii*) For an 'extreme' object, we define $-Z_i$ to be the coordinate position of the point locating the zero 'length' Weyl 'rod.' For example, if body 1 was an 'extreme' object, then $\alpha_1 = \alpha_2$ and $-Z_1 = \alpha_1$.

We also define

$$\mathrm{Re}(\alpha_2) > \mathrm{Re}(\alpha_3) \tag{3.24}$$

as the condition for having two separated bodies irrespective of the type of object.

With integrals (3.22), (3.23) and the coordinate positions as defined above evaluated in terms of m_1, m_2, q_1, q_2, z_1, z_2, it would then be possible, in principle, to invert these equations and hence write the solution (3.18)–(3.19) in terms of the true physical parameters M_i, Q_i and the coordinate positions Z_i, $i = 1, 2$. Ideally, the coordinate positions of the sources should be replaced with the proper separation of the sources. The complexity of the above Ernst potentials makes the analytic evaluation of the integrals (3.22), (3.23) and the proper separation difficult. As a consequence this goal has not yet been achieved. However, it is possible to numerically integrate equations (3.22) and (3.23) for a given set $\{m_1, m_2, q_1, q_2, z_1, z_2\}$. This will prove to be useful in studying balance conditions without a strut or tension in chapter 4.

Although the numerical evaluation of the physical mass and charge can be achieved from the parameterizations of paper I or paper II, it was hoped that the parameterization proposed in this paper, based on the Weyl-class solution, would facilitate the analytic evaluation of the integrals. It is not difficult to show that the parameterizations in papers I or II do not correctly identify the individual masses and charges of each source. We stated earlier that our parameterization $\{m_1, m_2, q_1, q_2, z_1, z_2\}$ only represents the physical masses and charges and coordinate positions of each source when the Weyl-class condition (equation (2.11) or (3.6)) is imposed (i.e. $\{M_1 = m_1, M_2 = m_2, Q_1 = q_1, Q_2 = q_2, Z_1 = z_1, Z_2 = z_2\}$). We can best demonstrate the problems with the parameterizations of papers I and II by comparing the representation of a properly parameterized Weyl-class solution with each of the other parameterizations. Let the set $\{m_1, m_2, q_1, q_2, z_1, z_2\}$ represent the physical Weyl-class parameters under the condition $m_1 q_2 = m_2 q_1$. Then the relationships between the three parameterizations is found by solving the set of equations (setting the spin parameters found in papers I and II to zero)

$$\begin{array}{ccccc}
\text{Weyl-class} & & \text{Paper I} & & \text{Paper II} \\
m_1 + m_2 & = & \tilde{m}_1 + \tilde{m}_2 & = & \hat{m}_1 + \hat{m}_2 \\
q_1 + q_2 & = & \tilde{q}_1 + \tilde{q}_2 & = & \hat{q}_1 + \hat{q}_2 \\
z_1 + z_2 & = & \tilde{z}_1 + \tilde{z}_2 & = & \hat{z}_1 + \hat{z}_2 \\
m_1 z_2 + m_2 z_1 & = & \tilde{m}_1 \tilde{z}_2 + \tilde{m}_2 \tilde{z}_1 & = & \hat{m}_1 \hat{z}_2 + \hat{m}_2 \hat{z}_1 + 2\hat{m}_1 \hat{m}_2 \\
q_1 z_2 + q_2 z_1 & = & \tilde{q}_1 \tilde{z}_2 + \tilde{q}_2 \tilde{z}_1 & = & \hat{q}_1 \hat{z}_2 + \hat{q}_2 \hat{z}_1 + \hat{q}_1 \hat{m}_2 + \hat{q}_2 \hat{m}_1 \\
z_1 z_2 + m_1 m_2 - q_1 q_2 & = & \tilde{z}_1 \tilde{z}_2 + \tilde{m}_1 \tilde{m}_2 & = & \hat{z}_1 \hat{z}_2 - \hat{m}_1 \hat{m}_2 \, .
\end{array} \tag{3.25}$$

The tilded and careted parameters are the parameterizations of papers I and II, respectively. Table 3.3 summarizes the results of solving the system (3.25)

Weyl-class	Paper I	Paper II
$m_1 = 8$	$\tilde{m}_1 = 7.52$	$\hat{m}_1 = 3.58$
$q_1 = 8$	$\tilde{q}_1 = 7.52$	$\hat{q}_1 = 3.58$
$m_2 = 3$	$\tilde{m}_2 = 3.48$	$\hat{m}_2 = 7.42$
$q_2 = 3$	$\tilde{q}_2 = 3.48$	$\hat{q}_2 = 7.42$
$z_1 = -7$	$\tilde{z}_1 = -8.67$	$\hat{z}_1 = -4.7$
$z_2 = 7$	$\tilde{z}_2 = 8.67$	$\hat{z}_2 = 4.7$

Table 3.1: Comparison of parameterizations for a Weyl-class solution.

The parameter values for the parameterizations of papers I and II are shown given the physical Weyl-class values. Neither the paper I nor the paper II parameterizations can be interpreted as the invariant physical parameters.

given the values shown in the first column. The solution represents two Weyl-class Reissner–Nordström 'critically charged' bodies without an intervening line singularity. It is clear that none of the parameter values in the latter two columns match the physical Weyl-class values. Thus, apart from one special case, neither the paper I nor the paper II parameterizations can be interpreted as the invariant physical parameters. The only exception is for identical bodies (with or without a line singularity) in the parameterization of paper I. In this very special case of the Weyl-class, the parameters $\tilde{m}_1 = \tilde{m}_2$, $\tilde{q}_1 = \tilde{q}_2$ are the physical masses and charges. However, $\tilde{z}_1$ and $\tilde{z}_2$ do not identify the coordinate positions of the bodies as defined earlier. The paper II parameterization is not physical even for identical bodies.

It is the demand for the inclusion of the Weyl-class solution in [10] which led to our form of $e(z)$ and $F(z)$. It should be emphasized that our parameterization contains as a special case, the simplest two-body balance solution of two critically charged bodies where the bodies are clearly individually spherical. This can be best illustrated by examining the Simon [32, 33] relativistic multipole moments of each parameterization. The first five Simon relativistic mass and charge multipole moments for our parameterization are

$$\begin{aligned}
\mathcal{M}_0 &= m_1 + m_2\,, \\
\mathcal{M}_1 &= -\,m_1 z_1 - m_2 z_2\,, \\
\mathcal{M}_2 &= m_1 z_1^2 + m_2 z_2^2 - (m_1 m_2 - q_1 q_2)(m_1 + m_2)\,, \\
\mathcal{M}_3 &= -\,m_1 z_1^3 - m_2 z_2^3 + (m_1 m_2 - q_1 q_2)(2m_1 z_1 + 2m_2 z_2 + z_1 m_2 + z_2 m_1)\,, \\
\mathcal{M}_4 &= m_1 z_1^4 + m_2 z_2^4 - (m_1 m_2 - q_1 q_2)\Big((m_1 + m_2)(q_1 q_2 - m_1 m_2) \\
&\qquad + 2\left(m_1 z_1^2 + m_2 z_2^2\right) + (m_1 + m_2)(z_1 + z_2)^2 \\
&\qquad + \frac{1}{7}(m_1 + m_2)\left((q_1 + q_2)^2 - (m_1 + m_2)^2\right)\Big) \\
&\qquad - \frac{1}{210}(z_1 - z_2)\Big(16(z_1 - z_2)(m_1 + m_2)(m_1 q_2 - m_2 q_1)^2 \\
&\qquad + z_1\left(30m_1\left(m_1 m_2 + m_2^2 - q_2^2\right) - 3q_1(3m_2 q_1 + 7q_2 m_1)\right) \\
&\qquad - z_2\left(30m_2\left(m_1 m_2 + m_1^2 - q_1^2\right) - 3q_2(3m_1 q_2 + 7q_1 m_2)\right)\Big)
\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
\mathcal{Q}_0 &= q_1 + q_2\,, \\
\mathcal{Q}_1 &= -\,q_1 z_1 - q_2 z_2\,, \\
\mathcal{Q}_2 &= q_1 z_1^2 + q_2 z_2^2 - (m_1 m_2 - q_1 q_2)(q_1 + q_2)\,, \\
\mathcal{Q}_3 &= -\,q_1 z_1^3 - q_2 z_2^3 + (m_1 m_2 - q_1 q_2)(2q_1 z_1 + 2q_2 z_2 + z_1 q_2 + z_2 q_1)\,, \\
\mathcal{Q}_4 &= q_1 z_1^4 + q_2 z_2^4 - (m_1 m_2 - q_1 q_2)\Big((q_1 + q_2)(q_1 q_2 - m_1 m_2) \\
&\qquad + 2\left(q_1 z_1^2 + q_2 z_2^2\right) + (q_1 + q_2)(z_1 + z_2)^2 \\
&\qquad + \frac{1}{7}(q_1 + q_2)\left((q_1 + q_2)^2 - (m_1 + m_2)^2\right)\Big) \\
&\qquad - \frac{1}{210}(z_1 - z_2)\Big(16(z_1 - z_2)(q_1 + q_2)(m_1 q_2 - m_2 q_1)^2
\end{aligned} \tag{3.27}$$

$$- z_1 \left(30q_2 \left(q_1q_2 - m_1m_2 + q_1^2\right) - 3m_1 \left(13m_1q_2 - 3m_2q_1\right)\right)$$
$$+ z_2 \left(30q_1 \left(q_1q_2 - m_1m_2 + q_1^2\right) - 3m_2 \left(13m_2q_1 - 3m_1q_2\right)\right)\Big).$$

respectively. In Newtonian physics, a system of two monopoles at positions z_1, z_2 has multipole moments

$$\mathcal{M}_n = m_1 z_1^n + m_2 z_2^n, \qquad \mathcal{Q}_n = q_1 z_1^n + q_2 z_2^n. \tag{3.28}$$

It is interesting to observe that this is also the relativistic multipole structure for two Weyl-class critically charged bodies, at least up to $\mathcal{M}_4$, $\mathcal{Q}_4$. There is an inherent asphericity imposed upon each, since the two bodies are interacting in a line. For nonlinearly interacting sources in a line, one would not expect to realize perfect sphericity of the individual sources. (It is yet to be explained why the sphericity is maintained in the Weyl-class, at least up to $\mathcal{M}_4$, $\mathcal{Q}_4$.) Once the solution is written analytically in terms of the physically meaningful constants M_i, Q_i and the coordinate positions Z_i, $i = 1, 2$, one will be able to examine the general multipole structure of nonlinearly interacting spherical bodies.

For comparison, the first four Simon relativistic mass and charge multipole moments for the parameterization of paper I (with their spin parameters $a_i = 0, i = 1, 2$) are

$$\begin{aligned}
\mathcal{M}_0 &= \widetilde{m}_1 + \widetilde{m}_2, \\
\mathcal{M}_1 &= -\widetilde{m}_1\widetilde{z}_1 - \widetilde{m}_2\widetilde{z}_2, \\
\mathcal{M}_2 &= \widetilde{m}_1\widetilde{z}_1^2 + \widetilde{m}_2\widetilde{z}_2^2 - \widetilde{m}_1\widetilde{m}_2\left(\widetilde{m}_1 + \widetilde{m}_2\right), \qquad (3.29)\\
\mathcal{M}_3 &= -\widetilde{m}_1\widetilde{z}_1^3 - \widetilde{m}_2\widetilde{z}_2^3 + \widetilde{m}_1\widetilde{m}_2\left(2\widetilde{m}_1\widetilde{z}_1 + 2\widetilde{m}_2\widetilde{z}_2 + \widetilde{z}_1\widetilde{m}_2 + \widetilde{z}_2\widetilde{m}_1\right)
\end{aligned}$$

and

$$
\begin{aligned}
\mathcal{Q}_0 &= \widetilde{q}_1 + \widetilde{q}_2 \, , \\
\mathcal{Q}_1 &= -\widetilde{q}_1 \widetilde{z}_1 - \widetilde{q}_2 \widetilde{z}_2 \, , \\
\mathcal{Q}_2 &= \widetilde{q}_1 \widetilde{z}_1^2 + \widetilde{q}_2 \widetilde{z}_2^2 - \widetilde{m}_1 \widetilde{m}_2 \left(\widetilde{q}_1 + \widetilde{q}_2 \right) , \\
\mathcal{Q}_3 &= -\widetilde{q}_1 \widetilde{z}_1^3 - \widetilde{q}_2 \widetilde{z}_2^3 + \widetilde{m}_1 \widetilde{m}_2 \left(2\widetilde{q}_1 \widetilde{z}_1 + 2\widetilde{q}_2 \widetilde{z}_2 + \widetilde{z}_1 \widetilde{q}_2 + \widetilde{z}_2 \widetilde{q}_1 \right) .
\end{aligned}
\tag{3.30}
$$

The first four Simon relativistic mass and charge multipole moments for the parameterization of paper II (with their spin parameters $a_i = 0, i = 1, 2$) are

$$
\begin{aligned}
\mathcal{M}_0 &= \widehat{m}_1 + \widehat{m}_2 \, , \\
\mathcal{M}_1 &= -\widehat{m}_1 \widehat{z}_1 - \widehat{m}_2 \widehat{z}_2 + 2\widehat{m}_1 \widehat{m}_2 \, , \\
\mathcal{M}_2 &= \widehat{m}_1 \widehat{z}_1^2 + \widehat{m}_2 \widehat{z}_2^2 + \widehat{m}_1 \widehat{m}_2 \left(\widehat{m}_1 + \widehat{m}_2 - 2\widehat{z}_1 - 2\widehat{z}_2 \right) , \\
\mathcal{M}_3 &= -\widehat{m}_1 \widehat{z}_1^3 - \widehat{m}_2 \widehat{z}_2^3 + \widehat{m}_1 \widehat{m}_2 \big(2\widehat{m}_1 \widehat{m}_2 + 2\widehat{z}_1 \widehat{z}_2 + 2\widehat{z}_1^2 + 2\widehat{z}_2^2 \\
&\qquad -\widehat{m}_1 \widehat{z}_2 - \widehat{m}_2 \widehat{z}_1 - 2\widehat{m}_1 \widehat{z}_1 - 2\widehat{m}_2 \widehat{z}_2 \big)
\end{aligned}
\tag{3.31}
$$

and

$$
\begin{aligned}
\mathcal{Q}_0 &= \widehat{q}_1 + \widehat{q}_2 \, , \\
\mathcal{Q}_1 &= -\widehat{q}_1 \widehat{z}_1 - \widehat{q}_2 \widehat{z}_2 + \widehat{m}_1 \widehat{q}_2 + \widehat{m}_2 \widehat{q}_1 \, , \\
\mathcal{Q}_2 &= \widehat{q}_1 \widehat{z}_1^2 + \widehat{q}_2 \widehat{z}_2^2 + \widehat{m}_1 \widehat{m}_2 \left(\widehat{q}_1 + \widehat{q}_2 \right) - \left(\widehat{q}_1 \widehat{m}_2 + \widehat{q}_2 \widehat{m}_1 \right) \left(\widehat{z}_1 + \widehat{z}_2 \right) , \\
\mathcal{Q}_3 &= -\widehat{q}_1 \widehat{z}_1^3 - \widehat{q}_2 \widehat{z}_2^3 - \widehat{m}_1 \widehat{m}_2 \left(2\widehat{q}_1 \widehat{z}_1 + 2\widehat{q}_2 \widehat{z}_2 + \widehat{z}_1 \widehat{q}_2 + \widehat{z}_2 \widehat{q}_1 \right) \\
&\qquad + \left(\widehat{q}_1 \widehat{m}_2 + \widehat{q}_2 \widehat{m}_1 \right) \left(\widehat{m}_1 \widehat{m}_2 + \widehat{z}_1 \widehat{z}_2 + \widehat{z}_1^2 + \widehat{z}_2^2 \right) .
\end{aligned}
\tag{3.32}
$$

If the above parameterizations did represent the physical mass and charge, it is evident that the multipole structure would not be that of Newtonian spherical bodies even for critically charged bodies. As stated earlier, it should be noted that in the parameterization of paper I. it can be shown that only in the case of identical bodies. the parameters $\tilde{m}_1 = \tilde{m}_2$, $\tilde{q}_1 = \tilde{q}_2$ are the physical mass and charge. However, in this case the multipoles still do not have the form of equation (3.28) since the parameters $\tilde{z}_1$ and $\tilde{z}_2$ do not identify the positions of the bodies as defined earlier. A simple transformation would correct the multipoles in this case.

Chapter 4

The Equilibrium Condition

In chapter 1 it was discussed that classically, two spherical charged bodies are found to be in equilibrium when the gravitational and electromagnetic forces are equal in magnitude but opposite in direction. Otherwise, there is relative motion between the bodies. The solution presented in chapter 3 is a static solution to the Einstein–Maxwell field equations. Hence, by definition, there is no motion between the bodies. Since this must be true for all values of the mass and charge, including in the limit of the charge of each source approaching zero simultaneously, it follows that there must, in general, be a material strut or tension present in order to prevent the system from becoming dynamic. The only source of stress-energy outside of the sources is the electromagnetic field. As a result, the strut or tension manifests itself as a (Weyl) line singularity between the sources. Removal of this singularity is known as the problem of elementary flatness (or the regularity condition). Elementary flatness demands that, for any infinitesimal space-like surface, the ratio of circumference to radius be 2π. If this Euclidian ratio is not satisfied, a singularity exists in the space-time. Examining the metric (2.10), it can be shown that [34]

$$\upsilon(z, \rho = 0) = 0 \tag{4.1}$$

is the condition for elementary flatness between the sources.

If the origin of the coordinate system is located between the sources (i.e. $\mathrm{Re}(\alpha_2) > 0$, $\mathrm{Re}(\alpha_3) < 0$), then application of equation (4.1) to equation (3.20), after some simplification, yields the balance equation

$$K \equiv \frac{a_{12}\left(\overline{a}_{13} + \overline{a}_{14}\right) + \overline{a}_{12}\left(a_{13} + a_{14}\right)}{K_0} = 0. \tag{4.2}$$

Three cases were examined: (*i*) two Reissner–Nordström black holes, (*ii*) two Reissner–Nordström superextreme bodies and (*iii*) one black hole and one superextreme body.

The procedure for testing for equilibrium without an intervening strut or tension will be as follows:

1. Assign numerical values to five of the six parameters from the unphysical set $\{m_1, m_2, q_1, q_2, z_1, z_2\}$.
2. Solve equation (4.2) for the unknown variable.
3. If a real root of equation (4.2) exists, then evaluate equations (3.22) and (3.23) to determine the physical mass and charge parameters.

The results for each of the three cases are as follows:

4.1 Two Reissner–Nordström black holes

Numerous sets of the parameters $\{m_1, m_2, q_1, q_2, z_1, z_2\}$, such that the constants α_n, $n = 1, \ldots, 4$ are real and distinct, were investigated. No roots of equation (4.2) were found. For example, choosing $m_1 = 9.0$, $q_1 = 3.0$, $z_1 = -15.0$, $m_2 = 8.0$, $q_2 = 2.0$, no balance for $0 \leq z_2 \leq 10^{10}$ was found. These findings are consistent with other results [11, 23, 25] that two Reissner–Nordström black holes cannot be found in equilibrium without an intervening strut or tension.

4.2 Two Reissner–Nordström superextreme bodies

Numerous sets of the parameters $\{m_1, m_2, q_1, q_2, z_1, z_2\}$, such that the constants α_n, $n = 1, \ldots, 4$ are complex conjugate pairs, were investigated. No roots of equation (4.2) were found. For example, in choosing $m_1 = 3.0$, $q_1 = 9.0$, $z_1 = -15.0$, $m_2 = 2.0$, $q_2 = 8.0$, no balance for $0 \leq z_2 \leq 10^{10}$ was found. These findings suggest that two Reissner–Nordström superextreme bodies cannot be found in equilibrium without a strut or tension.

4.3 One black hole and one superextreme body

The following three different cases were found for which equation (4.2) has a real root. Each case has the configuration illustrated in figure 3.2.

Case A. For $m_1 = 6.0$, $q_1 = 2.0$, $z_1 = -5.0$, $m_2 = -0.7$, $q_2 = 4.0$, balance at approximately $z_2 = 2.08$ was found.[1] The values of α_n are $\alpha_1 = 10.3$, $\alpha_2 = 1.74$, $\alpha_3 = -3.11 + i4.30$, $\alpha_4 = -3.11 - i4.30$. Using integrals (3.22) and (3.23), the physical masses and charges are $M_1 =$

[1] In cases A–C, equation (4.2) has been solved to a precision of $|K| < 10^{-50}$ using highly refined values of z_2.

3.95. $Q_1 = -0.887$, $M_2 = 1.35$, $Q_2 = 6.89$. Using the definitions of coordinate positions described in section 3.3, it was found that $Z_1 = -6.03$ and $Z_2 = 3.11$. Thus balance has occurred for $M_1 M_2 > Q_1 Q_2$, $Q_1 Q_2 < 0$ at a coordinate separation of $\mathcal{S} \equiv Z_2 - Z_1 = 9.13$. Note that the parameter m_2 is negative but both physical masses are positive. The parameterizations of papers I and II yield respectively

Paper I	Paper II
$\tilde{m}_1 = 4.96$	$\hat{m}_1 = 4.36$
$\tilde{q}_1 = 2.31$	$\hat{q}_1 = -1.05$
$\tilde{m}_2 = 0.34$	$\hat{m}_2 = 0.94$
$\tilde{q}_2 = 3.69$	$\hat{q}_2 = 7.05$
$\tilde{z}_1 = -6.60$	$\hat{z}_1 = -6.00$
$\tilde{z}_2 = 3.68$	$\hat{z}_2 = 3.08$

which do not agree with the integrated values of equations (2.4) and (2.5). This demonstrates that in general none of the analytic parameterizations proposed, including our own, are suitable choices for the individual masses and charges of the sources.

Case B. For $m_1 = 9.0$, $q_1 = 3.0$, $z_1 = -40.0$, $m_2 = 2.5$, $q_2 = 8.0$, balance was found at approximately $z_2 = 34.6$. The values of α_n are $\alpha_1 = 48.4$, $\alpha_2 = 31.61$, $\alpha_3 = -34.62 + i7.65$, $\alpha_4 = -34.62 - i7.65$. The physical masses and charges are $M_1 = 8.87$, $Q_1 = 2.00$, $M_2 = 2.63$, $Q_2 = 9.00$. The coordinate positions are $-Z_1 = 40.01$, $-Z_2 = -34.6$. Thus balance has occurred for $M_1 M_2 > Q_1 Q_2$, $Q_1 Q_2 > 0$ at a coordinate separation of $\mathcal{S} = 74.6$.

Case C. For $m_1 = 900.0$, $q_1 = 300.0$, $z_1 = -865.0$, $m_2 = 0.025$, $q_2 = 0.080$, balance was found at approximately $z_2 = 21.581$. The values of α_n are $\alpha_1 = 1713.5$, $\alpha_2 = 16.474$, $\alpha_3 = -21.582 + i0.26226$, $\alpha_4 = -21.582 - i0.26226$. The physical masses and charges are $M_1 = 899.71$, $Q_1 = 298.25$, $M_2 = 0.31897$, $Q_2 = 1.8254$. The coordinate positions are $-Z_1 = 865.00$, $-Z_2 = -21.582$. Thus balance has occurred for $Q_1Q_2 > M_1M_2$, $Q_1Q_2 > 0$ at a coordinate separation of $\mathcal{S} = 886.58$.

4.4 Comparison with test particle analysis

Bonnor's [4] examination of a test particle in the field of a Reissner-Nordström source yielded a wide variety of balance conditions. The following cases for separation-independent equilibrium were examined (note M, Q characterize the Reissner-Nordström space-time and m, q are the test body parameters):

Case 1. For $q = \epsilon m$, $Q = \eta M$, $\epsilon, \eta = \pm 1$, balance occurs if $\epsilon = \eta$.

Case 2. If $m = |q|$, $M \neq |Q|$, or $m \neq |q|$, $M = |Q|$, no equilibrium is possible.

Case 3. If $mM = qQ$ but $m \neq |q|$, then no equilibrium is possible.

Since the exact solution under study contains the Weyl-class solution as a special case, we also find Bonnor's case 1 as a separation-independent equilibrium condition. Case 2 or 3 cannot be readily tested by our numerical procedure. In order to do so, one would have to have the good fortune of correctly choosing the set $\{m_1, m_2, q_1, q_2, z_1, z_2\}$ such that the physical masses and charges satisfy the given conditions (i.e. $M_1 = |Q_1|$ etc.). Then, to test the dependence on separation, one would need to choose a new set of unphysical parameters such that the proper separation changes while the physical masses and charges remain the same.

The following separation-dependent cases were also found in [4]:

Case 4. If $|Q| > M$, $mM = -qQ$ and $m^2 \neq q^2$ with $qQ < 0$, then an equilibrium exists at

$$r = \frac{Q^2}{2M}.$$

Case 5. If $|Q| > M$, $|q| < m$, $qQ < 0$ or

Case 6. if $|Q| > M$, $|q| < m$, $qQ > 0$, $qQ < mM$, then an equilibrium position exists at

$$r = \frac{Q^2\left(M\left(m^2 - q^2\right) + q\sqrt{\left(m^2 - q^2\right)\left(Q^2 - M^2\right)}\right)}{m^2M^2 - q^2Q^2}.$$

Case 7. If $|Q| < M$, $|q| > m$, $qQ > 0$, $qQ > mM$, then an equilibrium position exists at

$$r = \frac{Q^2\left(M\left(m^2 - q^2\right) - q\sqrt{\left(m^2 - q^2\right)\left(Q^2 - M^2\right)}\right)}{m^2M^2 - q^2Q^2}.$$

Thus we have found a direct correspondence between cases A–C of the exact solution and cases 5–7 of Bonnor's test particle analysis. The separation dependence of cases 4–7 cannot be studied in the exact solution using the present methods for the same reasons cases 2–3 cannot be studied. Since the separation dependence cannot be tested using the present methods, there is little value in numerically calculating the proper separation of the sources in cases A–C.

The physical parameters in case C could approximate a test body in a strong gravitational field. Using these values in case 7 and transforming from spherical coordinates to cylindrical coordinates for a single Reissner–Nordström body using the transformation (with $\theta = 0$)

$$\begin{aligned} z &= (r - M)\cos\theta, \\ \rho &= \sqrt{r^2 - 2Mr + Q^2}\,\sin\theta. \end{aligned} \tag{4.3}$$

Bonnor's method yields a coordinate separation of $\mathcal{S} = 1465.5$. Since the separation of the bodies from these two methods are not consistent, it would appear that case C does not sufficiently approximate a test body.

Chapter 5

Discussion and Conclusions on the Two–Body Balance Problem

5.1 Discussion

The essential departure in the present paper from previous work is the attempt to parameterize the solution in terms of true physical constants of the space-time. For a static axially symmetric solution of the Einstein–Maxwell equations, the integrals of equations (2.4) and (2.5) provide the invariant parameters required for meaningful analysis of the properties of the solution.

The terms 'undercharged,' 'overcharged' and 'critically charged' are defined in [10] as follows ($i = 1, 2$):

$$M_i^2 > Q_i^2 \quad \text{'undercharged,'} \tag{5.1}$$

$$M_i^2 < Q_i^2 \quad \text{'overcharged,'} \tag{5.2}$$

$$M_i^2 = Q_i^2 \quad \text{'critically charged.'} \tag{5.3}$$

For the Weyl-class, the 'lengths' of the Weyl 'rods' are $2l_i = 2\sqrt{M_i^2 - Q_i^2}$, $i =$

1.2 [10]. If body 1 is 'critically charged,'[1] then $\alpha_1 = \alpha_2$ $(= d)$ since $l_1 = 0$ (see figure 3.1). This implies that the terminology 'critically charged' body and 'extreme' body may be used interchangeably for Weyl-class solutions.[2] If body 1 is 'undercharged,' α_1 $(= d + 2l_1)$ and α_2 $(= d)$ are real quantities. Thus 'undercharged body' and 'black hole' are synonymous terms in the Weyl-class. Finally, if body 1 is 'overcharged,' α_1 $(= d + l_1)$ and α_2 $(= d + \overline{l_1})$ are complex conjugates. Thus the terms 'overcharged' and 'superextreme' are equivalent descriptions in the Weyl-class. Unlike the Weyl-class solutions where the 'lengths' of the Weyl 'rods' (real or complex) depend only upon the mass and charge of the source in question, it is strongly suggested from the analysis of chapter 4 that for the general (non-Weyl-class) solution, the 'lengths' of the 'rods' also depend on the mass and charge of the other source and the distance separating the bodies as well. It would thus be possible to have a 'critically charged' body (according to equation (5.3)) for which the 'rod' is either of non-zero 'length' or 'complex.' This is important in terms of nomenclature for describing the physics of the space-time. Since the transition of a pair (e.g. (α_1, α_2)) from real values to a complex conjugate pair in Sibgatullin's [18] method defines a differentiation of an object with a

[1]It should be noted that having identical roots $\alpha_1 = \alpha_2$ is not sufficient for identifying critically charged bodies even in the Weyl-class. The Curzon particle is such an object with $\alpha_1 = \alpha_2$ but it is not necessarily critically charged (see [8, 10]).

[2]To appreciate the scale in magnitude for which a body must be charged in order to be 'critically charged' or 'extreme,' consider the following order of magnitude estimate (using SI units): The mass of the Earth is $M \sim 10^{24}$ kg. For the Earth to be 'critically charged,' $|Q| = \sqrt{4\pi\epsilon_0 G}\,M \sim 10^{13}$ Coulombs. The value of the elementary charge is $\sim 10^{-19}$ C. Therefore the excess number of elementary charged particles needed is $\sim 10^{32}$. If the Earth is assumed to be composed of $\sim 10^{51}$ atoms, the required number of ionized atoms would be $\sim 10^{32} \ll 10^{51}$. This does not mean that the Earth would be in the transition state between a between a black hole and a naked singularity. That would require the Earth to have a radius $\lesssim 1$ cm.

horizon to one without, it would seem that the appropriate description would be respectively, a black hole (horizon), 'extreme' body (zero 'length' Weyl 'rod') and 'superextreme' body (no horizon or naked singularity) as described in paper I. The descriptions 'under,' 'over' and 'critically charged' body should be reserved for the relations $M_i^2 > Q_i^2$, $M_i^2 < Q_i^2$ and $M_i^2 = Q_i^2$ respectively between the individual masses and charges. This classification scheme would describe equilibrium conditions more precisely once all are identified. The appropriateness of such a scheme would become apparent when the analytic physical parameterization of the solution is known.

There are three cases of the exact solution which have not been examined. They are an extreme body with respectively a Reissner–Nordström black hole, a superextreme body, and another extreme body for which the solution is not of the Weyl-class. Knowledge of the solution analytically in terms of the physical parameters is required to analyze these cases adequately. The results of the three cases that were studied are discussed below.

For the case of two Reissner–Nordström black holes, the numerical analysis suggets equilibrium cannot be achieved without an intervening strut or tension. If Newtonian physics has any resemblance to the physics of general relativity (which often it does not), this result is not surprising since the Newtonian balance condition (2.1) requires one 'undercharged' and one 'overcharged' body (borrowing the general relativistic terminology). For the same reason, it is not surprising that the analysis for two Reissner–Nordström superextreme bodies did not yield equilibrium without an intervening strut or tension. The case of one black hole and one superextreme body has the characteristic 'undercharged'–'overcharged' configuration of Newtonian physics. For like-signed charges, we found balance for both $M_1M_2 < Q_1Q_2$ (case B) and $M_1M_2 > Q_1Q_2$ (case C). These are different from condition (2.1). What is surprising from the Newtonian point of view is that balance is also possible if the charges have opposite signs. The explanation for this opposite-signed balance can be attributed to the phenomenon of Reissner–Nordström repulsion. For a single Reissner–Nordström source there is no event horizon when $Q^2 > M^2$. As $r \to 0$, a neutral test particle will experience a repulsive force when $r < Q^2/(2M)$ because the effective mass

$$m(r) \equiv M - \frac{Q^2}{2r} \tag{5.4}$$

the test particle interacts with becomes negative. Thus electric attraction is required to oppose the gravitational repulsion in order to achieve equilibrium.

The question of whether or not the equilibrium cases studied depend on the separation distance in the exact solution is still unresolved. The qualitative agreement with Bonnor's test particle analysis strongly suggests that the non-Weyl-class equilibrium cases given here are dependent on the separation distance. Expressing the exact solution in terms of the individual physical mass and charge of each source is necessary to adequately resolve this problem.

Bonnor's [4] test particle analysis has been modified [35] in such a way that the equilibrium conditions of a charged spinning test particle in the field of a Kerr–Newman source can be studied. The generalization of the mathematical solution to two spinning sources (Kerr–Newman sources) is already known [20–22]. One is able to invariantly define angular momentum for a stationary space-time in a manner similar to (2.4) and (2.5) because of the presence of an axial Killing vector (rotational symmetry) (see [23] and references therein for definitions of mass and angular momentum of stationary vacuum fields).

It is unknown how the subsequent analysis of two identical spinning bodies in paper I based on the invariant integrals of mass. charge and angular momentum will affect their results, if at all. However, it is clear that the parameterization given is inadequate for the physical analysis of the general case (nonidentical bodies).

5.2 Conclusions

The solution derived in papers I, II and this dissertation is a generalization of the Weyl-class double Reissner–Nordström solution. However, the analytic parameterizations presented in papers I, II and this dissertation cannot in all cases be interpreted as the true physical constants of the space-time. The invariant physical charge for each source is found by direct integration of Maxwell's equations. The physical mass is invariantly defined [12] in a manner similar to which the charge was found. Numerical methods were used to evaluate the invariant individual masses and charges for the axially symmetric superposition of two Reissner–Nordström bodies. Three balanced cases without a line singularity were found for the non-Weyl-class solution. It was found that neither the Newtonian balance condition nor critically charged bodies are necessary for electrostatic equilibrium. In one of the balanced cases the charges have opposite signs. The dependence of the balance conditions on the separation of the bodies is not yet known due to the complexities involved in expressing the solution analytically in terms of the true physical set of parameters. However, all the balance conditions found are consistent with Bonnor's test particle analysis. This suggests that there exist equilibrium conditions which depend on the separation of the sources. The parameterization of this dissertation is manifestly physical in the Weyl-class limit.

Chapter 6

The Gravitational Geon

6.1 Introduction and background

Over forty years ago, the geon concept was introduced [36]: zero-rest-mass field concentrations held together for long periods of time by their gravitational attraction. Such constructs were motivated by studies of the motion of bodies in general relativity. More recent interest arises from the study of the entropy of radiation [45] and from the analogy between electromagnetic geons and quark stars [46]. Electromagnetic, neutrino and mixed type geons were studied [36, 43, 47–51] and it was suggested that it should be possible to construct a geon from gravitational waves [52]. Brill and Hartle [37] (henceforth referred to as BH) attempted the construction of a gravitational geon model in detail. Later papers [53–55] assumed the correctness of the BH model. In their approach, BH considered a strongly curved static background geometry $\gamma_{\mu\nu}$ on top of which a small ripple $h_{\mu\nu}$ resided, satisfying a linear wave equation. The wave frequency was assumed to be so high as to create a sufficiently large effective energy density which served as the source of the background

$\gamma_{\mu\nu}$, taken to be spherically symmetric on a time-average. For their analysis, they took the Regge–Wheeler [52] (henceforth referred to as RW) decomposition of $h_{\mu\nu}$ in a spherical background in terms of waves characterized by the usual quantum numbers l, m related to the angular momentum operators and by the frequency Ω. They claimed to have found a solution with a flat-space spherical interior, a Schwarzschild exterior and a thin shell separation meant to be created by high-frequency gravitational waves. With the mass M identified from the exterior metric, there would follow an unambiguous realization of the gravitational geon as described above.

To be complete, however, three conditions must be satisfied. Firstly, the gravitational geon must be a nonsingular solution of the Einstein equations in vacuum. Any singularities present would indicate the presence of nongravitational sources $T_{\mu\nu}$ compactified into points, curves or surfaces, negating the desired nonsingular purely field structure. Secondly, the consistency of the solution must be demonstrated, namely that the background $\gamma_{\mu\nu}$ is consistent with the time-averaged effective density constructed from $h_{\mu\nu}$ as source in the region of nonvanishing $h_{\mu\nu}$. Thirdly, the solution must be stable on a time-scale longer than the period of the constituent waves. Regarding the first condition, it is straightforward to show that the junction conditions for regularity are not satisfied by the BH solution and hence as it stands, cannot be taken as singularity-free. With the first condition violated, there is no basis for proceeding with a consideration of the last two.

It was proposed by Cooperstock, Faraoni and Perry [39–41] (henceforth referred to as CFP) that a satisfactory gravitational geon model must be constructed and solved in a manner similar to that of Wheeler's [36] electromagnetic geon model. Such a model necessarily requires the Einstein field equations to be solved in a self-consistent manner while satisfying the regularity conditions.

Through a series of papers, it was established by CFP and by Anderson and Brill [42] that in the high-frequency approximation, the gravitational geon problem and the electromagnetic geon problem are governed by the same set of ordinary differential equations (ODEs) and boundary conditions. These equations are satisfactory for considering the regularity and self-consistency aspects of the geon problem but not the evolution in time. This part of the dissertation provides a detailed and expanded study of the gravitational and electromagnetic geon problem. In this chapter, the basic mathematical formalism for the construction of gravitational geons are reviewed. These concepts are used in chapter 7 to analyze the proposed BH solution. In chapter 8, the equations for the gravitational geon problem are derived in the high-frequency approximation and compared to the electromagnetic case. Any solutions to these equations are necessarily equilibrium solutions since the background metric is assumed static. Chapter 9 reexamines the numerical solutions to these equations studied in [36] and extends the analysis to obtain information on the stability of these solutions. Both the analytic behaviour at spatial infinity and the stability properties are found by constructing a phase portrait of the ODEs. It is explicitly shown that only unstable equilibrium solutions are possible. The evolution in time of unstable electromagnetic geon solutions is also studied in chapter 9. We chose to study the evolution of the electromagnetic geon over the gravitational geon because of the relative ease in computation for the former. The evolution is achieved by applying a small amplitude time-dependent perturbation to the unstable equilibrium solution and simultaneously solving for the time-dependence of the background metric functions. This can only be done in a meaningful way if it is assumed that the characteristic frequency of the perturbations vary on a time-scale much longer than that of the waves comprising the electromagnetic geon. This is in accordance with the requirement that the background metric be quasi-stable. However, the results of the analysis show that the perturbations must vary on the same time-scale as the constituent waves. This is a contradiction to the original assumption. In chapter 10, the possible interpretations and ramifications of this result are discussed. The contradiction suggests that neither an electromagnetic geon nor a gravitational geon is a viable construct since not all of the requirements for existence of a geon are met. We also review other approaches to the gravitational and electromagnetic geon problem as a comparison to the one presented in this dissertation. The conclusions are presented in chapter 11.

6.2 Gravitational geons

We consider the space-time metric given by[1]

$$g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}\,, \tag{6.1}$$

where we assume that $g_{\mu\nu}$ is asymptotically flat, that $\gamma_{\mu\nu}$ is a static, spherically symmetric, asymptotically flat metric and $h_{\mu\nu}$ are small perturbations

[1]The metric signature is $-+++$. Units are chosen in which $G = c = 1$. Greek indices run from 0 to 3 and Latin indices run from 1 to 3 (apart from appendix C, where they assume the values 0, 2 and 3). A comma and a semicolon denote, respectively, ordinary and covariant differentiation with respect to the background metric. The Ricci tensor is given by $R_{\mu\nu} = \Gamma^{\sigma}_{\mu\nu,\sigma} - \Gamma^{\sigma}_{\mu\sigma,\nu} + \Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma} - \Gamma^{\sigma}_{\rho\nu}\Gamma^{\rho}_{\mu\sigma}$. Notations and conventions used are those of [56].

($|h_{\mu\nu}| << 1$) representing gravitational waves. In a system of Schwarzschild-like coordinates $\{x^\alpha\} = \{t, r, \theta, \varphi\}$, the background metric is given by

$$(\gamma_{\mu\nu}) = \mathrm{diag}\left(-\mathrm{e}^{\nu}, \mathrm{e}^{\lambda}, r^2, r^2 \sin^2\theta\right) , \tag{6.2}$$

where

$$\lambda = \lambda(r) , \qquad \nu = \nu(r) \tag{6.3}$$

and

$$h_{\mu\nu} = h_{\mu\nu}(t, r, \theta, \varphi) . \tag{6.4}$$

Following BH, we represent the most general gravitational wave perturbation $h_{\mu\nu}$ of the spherical background as a superposition of tensor spherical harmonics:

$$h_{\mu\nu} = \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \int_0^{+\infty} \mathrm{d}\Omega \, h_{\mu\nu}^{(lm\Omega)}(r, \theta, \varphi) \, \mathrm{e}^{i\Omega t} \; + \; \mathrm{c.c.} \tag{6.5}$$

This is justified by the fact that the dynamics of the gravitational waves in the present context are governed by the *linearized* Einstein equations around the background $\gamma_{\mu\nu}$ and therefore a superposition principle holds. Due to linearity, we can restrict ourselves to a study of the evolution of the single tensor spherical modes. For ease of comparison with the BH paper, we will use the RW set of tensor spherical harmonics ([52,57–59]; see [60] for a review and for relations with other sets of tensor spherical harmonics). An 'even mode' (also called 'polar mode' by other authors [61]) in the RW formalism is factorized as the product of functions dependent only on time, radius and angles respectively. The angular part is determined by the numbers l and m related to the usual scalar spherical harmonics. The even modes have the form

$$h_{\mu\nu}^{(\text{even})} = \begin{pmatrix} -e^{\nu}H_0(r) & H_1(r) & 0 & 0 \\ H_1(r) & e^{\lambda}H_2(r) & 0 & 0 \\ 0 & 0 & r^2K(r) & 0 \\ 0 & 0 & 0 & r^2K(r)\sin^2\theta \end{pmatrix} Y_l^m e^{-i\Omega t}. \tag{6.6}$$

where $Y_l^m(\theta, \varphi)$ are the usual spherical harmonics.[2] These modes have parity $(-)^l$. The 'odd modes' (in the RW terminology—also called 'axial modes') are given by

$$h_{\mu\nu}^{(\text{odd})} = \begin{pmatrix} 0 & 0 & -h_0(r)\,(\sin\theta)^{-1}\dfrac{\partial}{\partial\varphi}Y_l^m & h_0(r)\sin\theta\,\dfrac{\partial}{\partial\theta}Y_l^m \\ 0 & 0 & -h_1(r)\,(\sin\theta)^{-1}\dfrac{\partial}{\partial\varphi}Y_l^m & h_1(r)\sin\theta\,\dfrac{\partial}{\partial\theta}Y_l^m \\ \text{Sym} & \text{Sym} & 0 & 0 \\ \text{Sym} & \text{Sym} & 0 & 0 \end{pmatrix} e^{-i\Omega t} \tag{6.7}$$

and have parity $(-)^{l+1}$. We will consider the case of odd modes only because of the relative ease in computations. It is unlikely that the even modes would produce a contrary result although it would be useful if a follow-up calculation were to be performed to verify this conjecture.

A *gravitational geon* is defined as a bounded configuration of gravitational waves whose gravity is sufficiently strong to keep them confined on a time-scale

[2]Strictly speaking, the radial functions in equations (6.6) and (6.7) depend on Ω, l and m and should be labelled accordingly. However, this would result in a cumbersome notation that is preferably avoided.

long compared to the characteristic composing wave period. It is required that no matter or fields other than the gravitational field be present. Although one may consider the possibility of strong gravitational waves, and the definition of gravitational geon allows for this possibility, in this paper we will restrict ourselves to the case in which the amplitude of gravitational waves is small. This permits us to apply the linearized Einstein theory to the propagation of each single wave in the background created by the average action of all the waves composing the geon. Furthermore, it is required that the configuration represented by the metric $\gamma_{\mu\nu}$ be quasi-stable over a time-scale much larger than the typical period of its gravitational wave constituents (i.e. a geon must maintain its bounded configuration for a sufficient length of time, otherwise it would not be possible to attribute a structural form to the gravitational geon) and that the gravitational field becomes asymptotically flat at spatial infinity. Gravitational geons were introduced on the basis of the analogy with electromagnetic and neutrino geons in the RW paper and were studied in greater detail by BH. Wheeler's method of building an electromagnetic geon was to replace the details of the electromagnetic field by the time-average of the components of the electromagnetic stress-energy tensor. Upon averaging over many modes of oscillation of the electromagnetic field, one obtains a stress-energy tensor and as a consequence, a gravitational field and metric which are spherically symmetric and time-independent. Any given mode of oscillation is taken to propagate in the spherically symmetric gravitational field created by the rest of the radiation. The attempt to build a geon resembles the construction, in other fields of physics, of a system with many (almost) identical components, each of which introduces a negligible perturbation in the dynamics of the whole system and has an evolution governed by the averaged action of all the other components. An example of such a system in Newtonian theory is a galaxy described by the potential created by the mass distribution of many stars (here we neglect dark matter and the fact that a potential-density pair usually describes only a single component of a galaxy and is adequate only for certain types of galaxies [62]). Each star gives a very small contribution to this potential and its orbit is determined by the global galactic potential.

Consistent with this idea, it is required that the time-average of the metric be equal to the background metric, i.e.

$$\gamma_{\mu\nu} = \langle g_{\mu\nu} \rangle \; , \tag{6.8}$$

where $\langle \cdot \rangle$ denotes an average over several periods of the typical gravitational wave wavelength λ ('Brill–Hartle average'). It follows that

$$\langle h_{\mu\nu} \rangle = \left\langle \frac{\partial h_{\mu\nu}}{\partial x^{\alpha}} \right\rangle = \left\langle \frac{\partial^2 h_{\mu\nu}}{\partial x^{\alpha} \partial x^{\beta}} \right\rangle = 0 \, . \tag{6.9}$$

A mathematically rigourous treatment of this concept is contained in the paper by MacCallum and Taub [63]. This idea has proved very valuable and the averaging process has been used by many authors after BH and is well defined only if it is assumed that the typical wavelength[3] λ is much smaller than the space and time-scale of variation L of the background metric $\gamma_{\mu\nu}$ (high-

[3]The term 'typical gravitational wavelength' λ may be source of confusion to some readers. Since we are decomposing the general wave form into an infinite set of Regge–Wheeler modes, one may think that λ represents the wavelength of each mode and that equation (6.10) is only valid if the geon was composed of one and only one mode. However, when one is analyzing a general wave form, it is justifiable to assign a *single* parameter describing the scale of variation of the wave form. In the present context, λ is the scale over which the wave form varies. Equation (6.10) is easily derived from equation (6.24) if one keeps in mind that $h_{\mu\nu,\alpha} \sim \epsilon/\lambda$ etc. (see, for example, [56]) and that λ represents the scale of variation of $h_{\mu\nu}$.

frequency approximation) [38]:

$$\epsilon \equiv \frac{\lambda}{L} << 1 \,. \tag{6.10}$$

This assumption provides us with a smallness parameter ϵ to be used as an expansion parameter. We measure times and lengths in units of L so that $\lambda = \epsilon$. Following [38], the order of magnitude of the metric components and their derivatives are

$$\mathrm{O}\,(\gamma_{\mu\nu}) = \mathrm{O}(\frac{\partial\gamma_{\mu\nu}}{\partial x^{\alpha}}) = \mathrm{O}(\frac{\partial^2\gamma_{\mu\nu}}{\partial x^{\alpha}\partial x^{\beta}}) = \mathrm{O}(1)\,, \tag{6.11}$$

$$\mathrm{O}\,(h_{\mu\nu}) = \mathrm{O}(\epsilon)\,, \tag{6.12}$$

$$\Omega = \frac{2\pi}{\lambda} = \mathrm{O}\left(\frac{1}{\epsilon}\right)\,, \tag{6.13}$$

$$\mathrm{O}\left(\frac{\partial h_{\mu\nu}}{\partial x^{\alpha}}\right) = \mathrm{O}(1)\,, \tag{6.14}$$

$$\mathrm{O}\left(\frac{\partial^2 h_{\mu\nu}}{\partial x^{\alpha}\partial x^{\beta}}\right) = \mathrm{O}\left(\frac{1}{\epsilon}\right)\,. \tag{6.15}$$

In our notation, $\mathrm{O}(1) \equiv \mathrm{O}(\epsilon^0)$. Equations (6.11)–(6.15) is derived in [26,38,56]. It is to be noted that, in the most general case of high-frequency gravitational waves on a curved space-time, two smallness parameters are involved: the dimensionless amplitude of the waves and the ratio λ/L. These two parameters coincide in the specific case under consideration, in which the only source of the background curvature are the gravitational waves. One can conceive of situations in which more than one parameter arises from the high-frequency approximation and these cases have been considered in the literature (see. for example, [64]). However, in these situations, gravitational waves are not the only source of curvature. When gravitational waves are the only source of curvature, as in the gravitational geon, these multiple parameters reduce to the single parameter ϵ. Equation (6.14) implies that the quantum numbers l and m are of order $O(1/\epsilon)$.

The Ricci tensor can be expanded in the form [37,38]

$$R_{\alpha\beta}(g) = R^{(0)}_{\alpha\beta}(\gamma) + R^{(1)}_{\alpha\beta}(\gamma, h) + R^{(2)}_{\alpha\beta}(\gamma, h) + \cdots , \tag{6.16}$$

where ([37,38] and references therein)

$$R^{(1)}_{\alpha\beta} = -\frac{1}{2}\gamma^{\rho\tau}\left(h_{\rho\tau;\alpha\beta} + h_{\alpha\beta;\rho\tau} - h_{\tau\alpha;\beta\rho} - h_{\tau\beta;\alpha\rho}\right) . \tag{6.17}$$

$$\begin{aligned} R^{(2)}_{\alpha\beta} = \frac{1}{2}\Bigg(&\frac{h^{\rho\tau}{}_{;\beta}}{2}h_{\rho\tau;\alpha} + h^{\rho\tau}\left(h_{\tau\rho;\alpha\beta} + h_{\alpha\beta;\rho\tau} - h_{\tau\alpha;\beta\rho} - h_{\tau\beta;\alpha\rho}\right) \\ &+ h_{\beta}{}^{\tau;\rho}\left(h_{\tau\alpha;\rho} - h_{\rho\alpha;\tau}\right) - \left(h^{\rho\tau}{}_{;\rho} - \frac{h^{;\tau}}{2}\right)\left(h_{\tau\alpha;\beta} + h_{\tau\beta;\alpha} - h_{\alpha\beta;\tau}\right)\Bigg) . \end{aligned} \tag{6.18}$$

and $h \equiv h^{\alpha}{}_{\alpha}$. The term $R^{(0)}_{\alpha\beta}(\gamma)$ is the Ricci tensor of the background metric $\gamma_{\mu\nu}$, whereas $R^{(1)}_{\alpha\beta}$ and $R^{(2)}_{\alpha\beta}$ are, respectively, the parts of the Ricci tensor linear and quadratic in $h_{\mu\nu}$ and their derivatives. In the absence of high-frequency waves (or on a flat background), $h_{\mu\nu}$ and their derivatives are all of order $O(\epsilon)$. In this case the superscripts on the expansion terms of equation (6.16) also indicate its order in powers of ϵ. However, in the high-frequency approximation it is clear that $R^{(1)}_{\mu\nu}$ contains terms of order $O(1/\epsilon)$ and $O(1)$ as well as $O(\epsilon)$ [38]. Similarly, $R^{(2)}_{\mu\nu}$ is comprised of terms of order $O(1)$, $O(\epsilon)$, etc. Solving the vacuum field equations

$$R_{\mu\nu}(g) = 0 \tag{6.19}$$

consistently to any order of approximation requires that we set each order in the expansion parameter ϵ equal to zero. We express equations (6.17) and (6.18) as

$$R^{(1)}_{\mu\nu}(\gamma, h) = R^{(1)}_{\mu\nu}\left[\epsilon^{-1}\right] + R^{(1)}_{\mu\nu}\left[\epsilon^{0}\right] + \cdots, \tag{6.20}$$

$$R^{(2)}_{\mu\nu}(\gamma, h) = R^{(2)}_{\mu\nu}\left[\epsilon^{0}\right] + R^{(2)}_{\mu\nu}\left[\epsilon\right] + \cdots, \tag{6.21}$$

where $R^{(k)}_{\mu\nu}\left[\epsilon^{n}\right]$ denotes the term of order $O(\epsilon^{n})$ in $R^{(k)}_{\mu\nu}$. The first order approximation is thus

$$R^{(1)}_{\mu\nu}\left[\epsilon^{-1}\right] = 0\,. \tag{6.22}$$

The second order approximation requires that terms of order $O(1)$ be set equal to zero. The field equations to this order are

$$R^{(0)}_{\mu\nu}(\gamma) + R^{(1)}_{\mu\nu}\left[\epsilon^{0}\right] + R^{(2)}_{\mu\nu}\left[\epsilon^{0}\right] = 0\,. \tag{6.23}$$

Performing the Brill–Hartle average on equation (6.23), one obtains

$$R^{(0)}_{\mu\nu}(\gamma) = -\left\langle R^{(2)}_{\mu\nu}\left[\epsilon^{0}\right]\right\rangle\,. \tag{6.24}$$

Note that from equation (6.9)

$$\left\langle R^{(1)}_{\mu\nu}\left[\epsilon^{-1}\right]\right\rangle = \left\langle R^{(1)}_{\mu\nu}\left[\epsilon^{0}\right]\right\rangle = \cdots = 0 \tag{6.25}$$

and hence

$$\left\langle R^{(1)}_{\mu\nu}(\gamma, h)\right\rangle = 0\,. \tag{6.26}$$

In equation (6.24) the part of the Ricci tensor quadratic in $h_{\mu\nu}$ and their derivatives has been taken to the right hand side and is seen as an effective source term due to the gravitational waves. It is important to note that equation (6.24) has the potential to lead to the description of a gravitational geon only by virtue of the high-frequency approximation. Under the assumption that gravitational waves are weak but not of high-frequency, equations (6.13)–(6.15) would not hold and the two terms in equation (6.24) would have different orders. $R^{(2)}_{\alpha\beta} = O(\epsilon^2)$ could never balance $R^{(0)}_{\alpha\beta}(\gamma) = O(1)$ in this equation. This would prevent *a priori* the construction of a gravitational geon. This point can be understood physically by noting that the effective energy density typically associated with gravitational waves with amplitude $h << 1$ and frequency Ω is roughly proportional to $(h\Omega)^2$. This quantity can be of order unity only if $\Omega \sim 1/h >> 1$. Therefore, it is clear that the high-frequency approximation is a necessary condition for geon construction in the present context.

We shall designate as the 'geon problem,' the problem of finding a solution $(\gamma_{\mu\nu}, h_{\mu\nu})$ to the Einstein equations (6.22), (6.23) and (6.24) with the aforementioned properties (nonsingular, self-consistent and be stable over a time-scale much larger than the typical period of its gravitational wave constituents[4]) and satisfying the boundary conditions describing asymptotic flatness

$$h_{\mu\nu} \to 0 \quad \text{as} \quad r \to +\infty \,. \tag{6.27}$$

[4]A modification to the field equations is necessary to investigate this latter condition.

Chapter 7

The Brill–Hartle Analysis

To the author's knowledge, the only explicit attempt at gravitational geon construction before CFP was that of BH. In this chapter we review their pioneering approach to the problem and critically analyze their work.

We follow BH in expressing the gravitational wave perturbations in terms of RW tensor spherical harmonics. For the sake of simplicity, as done by BH, we restrict ourselves to the case of odd modes with zero angular momentum along the z-axis (i.e. $m = 0$). The last assumption eliminates the φ-dependence from the $h_{\mu\nu}$ functions and considerably simplifies the Einstein equations. This can be seen from equation (6.7) and from the well-known form of the spherical harmonics that we present in equation (7.4) below. Thus, the metric perturbations are[1]

$$h_{\mu\nu}(t, r, \theta) = \mathcal{R}_{\mu\nu}(r)\,\Theta_l(\theta)\,\mathrm{e}^{-i\Omega t}, \tag{7.1}$$

[1] For ease of comparison with the BH paper, we use a complex exponential to describe the time-dependence of the metric perturbations in equation (7.1). This notation is adequate as long as linear quantities in $h_{\mu\nu}$ and their derivatives are considered, but it is incorrect when the part of the Ricci tensor quadratic in $h_{\mu\nu}$ and their derivatives enters the discussion. For future reference, we use a function of $\cos(\Omega t)$ and $\sin(\Omega t)$ instead of a complex exponential in our calculations of chapter 8.

where[2]

$$\mathcal{R}_{\mu\nu}(r) = h_0(r)\left(\delta^0_\mu\delta^3_\nu + \delta^3_\mu\delta^0_\nu\right) + h_1(r)\left(\delta^1_\mu\delta^3_\nu + \delta^3_\mu\delta^1_\nu\right) . \tag{7.2}$$

$$\Theta_l(\theta) = \sin\theta\,\frac{\mathrm{d}}{\mathrm{d}\theta}Y_l^0 = C_l^0\,\sin\theta\,\frac{\mathrm{d}}{\mathrm{d}\theta}P_l(\cos\theta) . \tag{7.3}$$

Here we use the expression of the spherical harmonics

$$Y_l^m(\theta,\varphi) = \begin{cases} C_l^m e^{im\varphi}P_l^m(\cos\theta) & m \geq 0 . \\ (-1)^m\left(Y_l^{|m|}\right)^* & m < 0 , \end{cases} \tag{7.4}$$

where C_l^m are normalization constants. Here * denotes complex conjugation and $P_l^m(x)$ are the associated Legendre polynomials (which can be expressed in terms of the Legendre polynomials $P_l(x)$). Using the relation $P_l^0(x) = P_l(x)$ we obtain

$$Y_l^0(\theta) = C_l^0 P_l(\cos\theta) , \tag{7.5}$$

from which equation (7.3) follows. One can now insert the form (7.1)–(7.3) of the metric perturbations into the Einstein equations (6.19), obtaining equations for the unknown functions $h_0(r)$ and $h_1(r)$. Simultaneously solving equations (6.22) and (6.24) for a pair $(\gamma_{\mu\nu}, h_{\mu\nu})$ then provides a solution to the geon problem.

The correct order of magnitude of the various terms in the Einstein equations is determined by equations (6.11)–(6.15). The correct order of magnitude decomposition of the Einstein equations is absent in [37]. While the high-frequency approximation was assumed in [37], it was not incorporated into the calculations. (The work of Isaacson [38] was completed after that of BH.)

[2] Note a misprint in the second of the equations (8) in [37], corresponding to our equation (7.2).

As a result, the authors did not obtain the two different orders $O(1/\epsilon)$ and $O(1)$ in the Einstein equations, using a parameter ϵ arising from the high-frequency approximation. This is evident from the fact that their final equations (10a)–(10c) and (14) contain terms of different orders in the high-frequency limit. In the remaining part of this chapter we will show how the BH results can be reproduced and we will comment on their proposed geon model.

The BH equations can only be reproduced in the absence of high-frequency waves. In terms of a parameter ϵ related to the weakness of the gravitational waves, equations (6.11)–(6.15) must be replaced by

$$\mathrm{O}\left(h_{\mu\nu}\right) = \mathrm{O}\left(\frac{\partial h_{\mu\nu}}{\partial x^{\alpha}}\right) = \mathrm{O}\left(\frac{\partial^2 h_{\mu\nu}}{\partial x^{\alpha}\partial x^{\beta}}\right) = \mathrm{O}(\epsilon) \qquad \alpha\,,\beta = 0,\ldots 3\,. \quad (7.6)$$

As a consequence of these equations, the Ricci tensor has the form given by equation (6.16), where $R^{(0)}_{\mu\nu}(\gamma)$ =O(1), $R^{(1)}_{\mu\nu} = \mathrm{O}(\epsilon)$ and $R^{(2)}_{\mu\nu} = \mathrm{O}\left(\epsilon^2\right)$. To order O(1) the Einstein equations give the well-known equations for a spherically symmetric, static background (see, for example, [26] p 300) with vanishing energy-momentum tensor. As far as the order $\mathrm{O}(\epsilon)$ is concerned, only the (0, 3), (1, 3) and (2, 3) components of the Ricci tensor give nontrivial results. These components are

$$R^{(1)}_{03} = \frac{\mathrm{e}^{-\lambda}}{2}\left(\dot{h}_{13}\left(\frac{2}{r} - \frac{\lambda'}{2} - \frac{\nu'}{2}\right) + \frac{h'_{03}}{2}\left(\lambda' + \nu'\right) + \dot{h}'_{13} - h''_{03} - \frac{2\nu'}{r}h_{03}\right) - \frac{1}{2r^2}\left(h_{03,22} - h_{03,2}\cot\theta\right)\,, \quad (7.7)$$

$$R^{(1)}_{13} = \frac{\mathrm{e}^{-\nu}}{2}\left(\ddot{h}_{13} - \dot{h}'_{03} + \frac{2\dot{h}_{03}}{r}\right) + \frac{\mathrm{e}^{-\lambda}}{r}h_{13}\left(\frac{\lambda'}{2} - \frac{\nu'}{2} - \frac{1}{r}\right) - \frac{1}{2r^2}\left(h_{13,22} - h_{13,2}\cot\theta\right)\,, \quad (7.8)$$

$$R^{(1)}_{23} = \frac{e^{-\lambda}}{2}\left(h'_{13,2} - 2h'_{13}\cot\theta + h_{13}\left(\lambda' - \nu'\right)\cot\theta + \frac{h_{13,2}}{2}\left(\nu' - \lambda'\right)\right)$$
$$+ e^{-\nu}\left(\dot{h}_{03}\cot\theta - \frac{\dot{h}_{03,2}}{2}\right). \quad (7.9)$$

where a dot and a prime denote differentiation with respect to t and r, respectively. We now insert the form of the metric perturbations (7.1)-(7.3) into the Einstein equations (6.22) and use the following property of the function Θ_l (see appendix B):

$$\frac{d^2\Theta_l}{d\theta^2} - \cot\theta\,\frac{d\Theta_l}{d\theta} + l(l+1)\,\Theta_l = 0\,. \quad (7.10)$$

After some manipulations we find[3]

$$i\Omega\left(h'_1 + h_1\left(\frac{2}{r} - \frac{\lambda'}{2} - \frac{\nu'}{2}\right)\right) - \frac{h'_0}{2}\left(\lambda' + \nu'\right) + h''_0$$
$$- h_0\left(l(l+1)\frac{e^{\lambda}}{r^2} - \frac{2\nu'}{r}\right) = 0\,, \quad (7.11)$$

$$i\Omega\,e^{-\nu}\left(h'_0 - \frac{2h_0}{r}\right) + h_1\left(\frac{l(l+1)}{r^2} - \Omega^2 e^{-\nu} + \frac{e^{-\lambda}}{r}\left(\lambda' - \nu' - \frac{2}{r}\right)\right) = 0\,. \quad (7.12)$$

$$i\Omega\,e^{-\nu}h_0 + e^{-\lambda}\left(h'_1 + \frac{h_1}{2}\left(\nu' - \lambda'\right)\right) = 0\,. \quad (7.13)$$

Following BH we can now use equation (7.13) to eliminate h_0 from equation (7.12), obtaining the second order differential equation for $h_1(r)$:

[3] See [65] for corrections to the radial equations in [37, 52]. Also note misprints in the BH equation (11) corresponding to our equation (7.14). One of the coefficients of Q in our equation (7.18) differs by a factor 1/2 from the corresponding one in BH equation (14). The sign of the right hand side of our equation (7.17) is opposite to that in the corresponding BH equation.

$$h_1'' + h_1' \left(\frac{3}{2}(\nu' - \lambda') - \frac{2}{r} \right) + h_1 \left(\frac{1}{2}(\nu' - \lambda')^2 + \frac{1}{2}(\nu'' - \lambda'') \right.$$
$$\left. - l(l+1)\frac{e^{\lambda}}{r^2} + \Omega^2 e^{\lambda-\nu} + \frac{2}{r^2} \right) = 0\,. \quad (7.14)$$

We introduce the variable Q and the Regge–Wheeler coordinate r^* defined by

$$h_1 \equiv r e^{(\lambda-\nu)/2} Q\,, \quad (7.15)$$

$$dr^* = e^{(\lambda-\nu)/2}\, dr\,. \quad (7.16)$$

In terms of these quantities we have

$$h_0 = -\frac{1}{i\Omega}\frac{d(rQ)}{dr^*} \quad (7.17)$$

and[4]

$$\frac{d^2 Q}{dr^{*2}} + \left[\Omega^2 + \frac{3}{2r}(\nu' - \lambda')\, e^{\nu-\lambda} - \frac{l(l+1)}{r^2} e^{\nu} \right] Q = 0\,. \quad (7.18)$$

This Schrödinger-like equation lends itself to the analogy with the dynamics of waves propagating in an effective potential [36, 37, 52].

At this point BH proceed with the specification of the background metric

$$e^{\nu} = \begin{cases} \dfrac{1}{9} & \text{if } r \leq a \\ 1 - \dfrac{2M}{r} & \text{if } r \geq a \end{cases} \quad (7.19)$$

$$e^{\lambda} = \begin{cases} 1 & \text{if } r < a \\ \left(1 - \dfrac{2M}{r}\right)^{-1} & \text{if } r > a \end{cases} \quad (7.20)$$

[4]An equation similar to equation (7.18) can be derived for the even modes with $m = 0$ [66].

where $a = 9M/4$ and M is the geon mass. This vacuum solution for the background metric implies that the effective energy density due to the gravitational waves vanishes for $r \neq a$. Since the effective energy is positive semi-definite, equations (7.19), (7.20) imply that

$$h_{\mu\nu} = 0 \qquad \text{for } r \neq a. \tag{7.21}$$

Conversely, if the condition (7.21) is satisfied, the Birkhoff theorem guarantees that the metric is Minkowskian for $r < a$ and the Schwarzschild metric for $r > a$.

Therefore, in the BH model, gravitational waves are confined to a spherical shell, the thickness of which is exactly zero. Apparently, BH meant to build a geon model in which the gravitational waves are trapped in a spherical shell which has a nonvanishing thickness which is much smaller than its radius. However, their equations do not allow for this possibility. To be complete, we examine the viability of a geon with gravitational waves confined to a shell whose thickness is exactly zero. The solutions of the radial equations (7.11)-(7.14) cannot be arbitrarily smooth functions but must be sought in some space of *distributions*. In equation (7.14), the coefficients proportional to $\nu' - \lambda'$ and $\nu'' - \lambda''$ (i.e. h_1', h_1)) are not arbitrarily smooth functions and have a mathematical meaning only if they are regarded as distributions. The first quantity $\nu' - \lambda'$ can be expressed as

$$\nu' - \lambda' = 4Mr^{-2}\left(1 - \frac{2M}{r}\right)^{-1} \theta_H(r - a)\ , \tag{7.22}$$

where

$$\theta_H(x) \equiv \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} \tag{7.23}$$

is the Heaviside step function. Clearly, the radial derivative of $\nu' - \lambda'$ can be taken only in a distributional sense. Therefore the solutions of the Einstein equations are distributions and their domain is some space of test functions which must be specified in such a way that the coefficients and the operations involved in the Einstein equations are well defined. There is no indication as to the manner in which this functional space should be determined. It seems almost certain that, if a meaningful and unambiguous mathematical formulation of the problem can be given, the distributional solutions $h_{\mu\nu}$ must have properties like a Dirac delta with support at $r = a$. Moreover, if the $h_{\mu\nu}$ are allowed to be distributions, the concept of linearization around the background $\gamma_{\mu\nu}$, the condition $|h_{\mu\nu}| << 1$, and the estimates of the different orders of magnitude in the Einstein equations become meaningless. The physical interpretation of a distributional metric and Riemann tensor is problematic. To appreciate this, one can consider the much simpler case of a metric which does not satisfy the appropriate junction conditions [67] on a space-like or time-like hypersurface. The case of the metric $\gamma_{\mu\nu}$ given by equations (7.19), (7.20) and the time-like hypersurface $r = a$ is considered in appendix C. It is found that the junction conditions are not satisfied. As suggested by Israel [68], and as can be seen from the computation of the Einstein tensor for the spherical metric specified by equations (7.19) and (7.20), a singular hypersurface S (in the sense [67] that the first, or the second fundamental form, or both are not continuous at S) is associated with nonvanishing $T_{\mu\nu}$, a source of stresses. The definition of a geon, a structure of pure gravitational waves in the absence of matter, excludes the use of a background metric which does not satisfy the proper junction conditions. Thus, we exclude the case in which gravitational waves are confined to a shell, the thickness of which is exactly zero, as interpretationally ambiguous at best and physically meaningless, mathematically ill-defined, and nonviable at worst.

A possible alternative for a geon model in which gravitational waves are confined to a spherical shell is the case in which the shell has a nonzero thickness. To be specific, let us consider a shell of radius a and thickness δa described by values of the radial coordinate in the range

$$a - \frac{\delta a}{2} \leq r \leq a + \frac{\delta a}{2}, \tag{7.24}$$

where $0 < \delta a << a$. In order for the geon to be a distribution of pure gravitational fields without matter, we must require that the metric tensor satisfies the appropriate junction conditions [67] at the two time-like hypersurfaces $S_\pm = \{(t, r, \theta, \varphi) : \ r = a \pm \delta a/2\}$. This guarantees the absence of a real (as opposed to 'effective,' i.e. generated by gravitational waves) stress-energy tensor $T_{\mu\nu}$ representing a matter distribution. In this model, the modified BH solution would be

$$\mathrm{e}^{\nu} = \begin{cases} \dfrac{1}{9} & \text{if } r \leq a - \dfrac{\delta a}{2} \\ 1 - \dfrac{2M}{r} & \text{if } r \geq a + \dfrac{\delta a}{2} \end{cases} \tag{7.25}$$

$$\mathrm{e}^{\lambda} = \begin{cases} 1 & \text{if } r \leq a - \dfrac{\delta a}{2} \\ \left(1 - \dfrac{2M}{r}\right)^{-1} & \text{if } r \geq a + \dfrac{\delta a}{2} \end{cases} \tag{7.26}$$

$$h_{\mu\nu} = 0 \qquad \text{if } r < a - \frac{\delta a}{2} \text{ or } r > a + \frac{\delta a}{2}. \tag{7.27}$$

The form of the background metric $\gamma_{\mu\nu}$ inside the spherical shell is not given by BH and must be determined by solving simultaneously the Einstein equations to the two lowest orders for a pair $(\gamma_{\mu\nu}, h_{\mu\nu})$ [64]. The proper orders of magnitude did not appear in [37] as a consequence of neglecting the high-frequency approximation, despite the fact that this was introduced at the beginning of the paper in order to define time-averages. These are the reasons why there is only one set of equations in [37] mixing different orders and a complete solution to the geon problem is not provided. However, it is not evident that the junction conditions at the new boundaries ($r = a \pm \delta a/2$) will be satisfied even if a self-consistent interior solution can be found. The electromagnetic geon model [36] guarantees the junction conditions to be satisfied because the wave equation (a Maxwell equation) and the background metric are solved simultaneously throughout all space. In essence, one is solving the interior geon problem throughout all space. The 'thin shell' is identified afterwards by the region where the amplitude of the electromagnetic energy is non-negligible compared to other regions. The adaptation of this method to solve the gravitational geon problem avoids the aforementioned inadequacies of the shell model. In the next chapter, the differential equations for the gravitational geon based on the electromagnetic geon model [36] are derived.

Chapter 8

Resolving the Geon Problem

In this chapter we derive the gravitational geon differential equations assuming the high-frequency approximation, as required, and take into account the orders of magnitude accordingly [39–42]. What follows closely parallels the derivation given in [36] for the electromagnetic geon. We consider odd RW modes only and specialize to $m = 0$. The first section presents the wave equation and the time-space averaged effective stress-energy tensor built from the gravitational waves. The last section presents the wave equation and the background equations in the limit of high angular momentum. In order to be self-consistent, these equations must be solved simultaneously for the pair $(\gamma_{\mu\nu}, h_{\mu\nu})$ while satisfying the boundary conditions. These equations are solved and analyzed in the subsequent chapter.

8.1 The wave equation and the effective stress-energy tensor

To avoid problems which can arise in nonlinear equations using a complex exponential to describe the time-dependence of the metric perturbations, it is advantageous to rewrite the individual modes of the RW spherical harmonics in their real form for $m = 0$:

$$h_{\mu\nu}^{(\mathrm{odd})}(t, r, \theta) = \begin{pmatrix} 0 & 0 & 0 & h_0(r)\,\Theta_l(\theta)T_0(t) \\ 0 & 0 & 0 & h_1(r)\,\Theta_l(\theta)T_1(t) \\ \mathrm{Sym} & \mathrm{Sym} & 0 & 0 \\ \mathrm{Sym} & \mathrm{Sym} & 0 & 0 \end{pmatrix}, \tag{8.1}$$

where

$$\Theta_l(\theta) = C_l^0 \sin\theta \frac{\mathrm{d}}{\mathrm{d}\theta} P_l(\cos\theta), \tag{8.2}$$

$$T_0(t) = \cos(\Omega t + \delta), \tag{8.3}$$

$$T_1(t) = \sin(\Omega t + \delta), \quad \delta = \mathrm{constant}. \tag{8.4}$$

The phase constant δ can be set to zero without loss of generality, because the phase-dependence no longer exists upon time-averaging.

The Ricci tensor is computed using equation (6.17) which, to the dominant order $O(1/\epsilon)$, is simplified to (see appendix D)

$$R_{\alpha\beta}^{(1)}\left[\epsilon^{-1}\right] = -\frac{1}{2}\gamma^{\rho\tau}\left(h_{\rho\tau,\alpha\beta} + h_{\alpha\beta,\rho\tau} - h_{\tau\alpha,\beta\rho} - h_{\tau\beta,\alpha\rho}\right). \tag{8.5}$$

Substituting equation (8.1) into equation (8.5) yields the three nontrivial equations

$$h_0''(r) - \Omega h_1'(r) - l^2 r^{-2} e^{\lambda} h_0(r) = 0\,. \tag{8.6}$$

$$\Omega h_0'(r) + \left(l^2 r^{-2} e^{\nu} - \Omega^2\right) h_1(r) = 0\,. \tag{8.7}$$

$$h_1'(r) + \Omega e^{\lambda-\nu} h_0(r) = 0\,, \tag{8.8}$$

from $(\alpha, \beta) = (0,3)$, $(1,3)$ and $(2,3)$ respectively. Eliminating $h_0(r)$ from equation (8.7) and (8.8) yields the second order radial wave equation

$$h_1''(r) + e^{\lambda}\left(\Omega^2 e^{-\nu} - \frac{l^2}{r^2}\right) h_1(r) = 0 \tag{8.9}$$

for the radial function $h_1(r)$. It should be noted that in the high-frequency limit, the angular momentum quantum number $l \gg 1$. Hence equation (7.10) takes the form

$$\frac{\mathrm{d}^2\Theta_l}{\mathrm{d}\theta^2} + l^2\,\Theta_l = 0 \tag{8.10}$$

and it is used in obtaining equations (8.6)–(8.9) (note that $\cot\theta\,\mathrm{d}\Theta_l/\mathrm{d}\theta$ in equation (7.10) is of higher order than the retained terms in equation (8.10) in the high-frequency limit).

The next step is to determine the order O(1) equations. Instead of evaluating equation (6.24), we can equivalently evaluate

$$R^{(0)}_{\mu\nu}(\gamma) - \frac{1}{2}\gamma_{\mu\nu}R^{(0)}(\gamma) = -\left\langle R^{(2)}_{\mu\nu}\left[\epsilon^0\right] - \frac{1}{2}\gamma_{\mu\nu}R^{(2)}\left[\epsilon^0\right]\right\rangle\,. \tag{8.11}$$

By defining the Brill–Hartle space-time averaged stress-energy tensor as

$$T^{\mathrm{BH}}_{\mu\nu} = \langle T_{\mu\nu}\rangle \equiv -\frac{1}{8\pi}\left\langle R^{(2)}_{\mu\nu}\left[\epsilon^0\right] - \frac{1}{2}\gamma_{\mu\nu}R^{(2)}\left[\epsilon^0\right]\right\rangle = -\frac{1}{8\pi}\left\langle G^{(2)}_{\mu\nu}\left[\epsilon^0\right]\right\rangle\,. \tag{8.12}$$

equation (8.11) takes the familiar form

$$G^{(0)}_{\mu\nu}(\gamma) = 8\pi T^{\mathrm{BH}}_{\mu\nu}\,. \tag{8.13}$$

of an effective stress-energy tensor generating the background metric $\gamma_{\mu\nu}$.

The procedure for finding the average over the angular dependence of $T_{\mu\nu}$ has been given in [36]. The results are directly applicable to the gravitational geon since the discussion covers general $T_{\mu\nu}$. Hence, we have for the time-angle average (denoted $\langle\,\cdot\,\rangle_{\mathrm{TA}}$) of T^{ν}_{μ} summed over all N active modes[1]

$$\left\langle 8\pi T^0_0 \right\rangle_{\mathrm{TA}} = -\frac{N}{2}\int_0^{\pi}\left\langle G^{(2)\,0}_{(\mathrm{I})\,0}\right\rangle_{\mathrm{T}} \sin\theta\,\mathrm{d}\theta\,, \tag{8.14}$$

$$\left\langle 8\pi T^1_1 \right\rangle_{\mathrm{TA}} = -\frac{N}{2}\int_0^{\pi}\left\langle G^{(2)\,1}_{(\mathrm{I})\,1}\right\rangle_{\mathrm{T}} \sin\theta\,\mathrm{d}\theta\,. \tag{8.15}$$

$$\left\langle 8\pi T^2_2 \right\rangle_{\mathrm{TA}} = \left\langle 8\pi T^3_3 \right\rangle_{\mathrm{TA}} = -\frac{N}{2}\int_0^{\pi}\left\langle G^{(2)\,2}_{(\mathrm{I})\,2} + G^{(2)\,3}_{(\mathrm{I})\,3}\right\rangle_{\mathrm{T}} \sin\theta\,\mathrm{d}\theta\,, \tag{8.16}$$

$$\left\langle 8\pi T^{\nu}_{\mu} \right\rangle_{\mathrm{TA}} = 0 \text{ for } \mu \neq \nu\,. \tag{8.17}$$

Here $\langle\,\cdot\,\rangle_{\mathrm{T}}$ denotes a time-average, and $G^{(2)\,\mu}_{(\mathrm{I})\,\nu}$ is mode number I of the disturbance under discussion. Using the mixed form of equation (6.18) in conjunction with equations (8.1) and (8.8), and performing the time and angle averaging (see appendix E), we obtain

$$\begin{aligned}\left\langle 8\pi T^0_0 \right\rangle_{\mathrm{TA}} = \frac{N\left(C^0_l\right)^2 l}{8r^2 e^{\lambda}}\Bigg(&-\frac{3}{2}\Omega^2 h_1^2 e^{-\nu} - e^{-\lambda}\left(2h_1 h_1'' - h_1'^2\right) \\ &- \Omega^{-2}e^{-\nu-2\lambda}\left(\frac{1}{2}h_1''^2 + h_1' h_1'''\right) + \frac{1}{2}l^2 r^{-2}\left(\Omega^{-2}e^{\nu-\lambda}h_1'^2 + h_1^2\right)\Bigg)\,.\end{aligned} \tag{8.18}$$

[1] The active modes are characterized by a sequence of parameter values (a family of modes) for which the effective stress-energy is concentrated in the same spatial region.

$$\langle 8\pi T^1_{\ 1}\rangle_{\rm TA} = \frac{N\left(C_l^0\right)^2 l}{8r^2e^\lambda}\left(\Omega^{-2}e^{\nu-2\lambda}\left(h_1' h_1''' - \frac{1}{2}h_1''^2\right) + e^{-\lambda}h_1'^2\right.$$
$$\left. + \frac{1}{2}e^{-\nu}\Omega^2 h_1^2 - \frac{1}{2}l^2r^{-2}\left(\Omega^{-2}e^{\nu-\lambda}h_1'^2 + h_1^2\right)\right) . \quad (8.19)$$

From equation (8.16), it is apparent that the (2, 2) field equation must be identical to the (3, 3) equation. Neither is necessary to solve the geon problem since the complete system of equations must be self consistent. The latter two equations may be derived from the (1, 1), (0, 0) and wave equations.

8.2 The gravitational geon field equations in the high angular momentum limit

The left hand side of equation (8.13) is

$$G^{(0)\,0}_{\ \ \ 0} = e^{-\lambda}\left(r^{-2} - r^{-1}\lambda'\right) - r^{-2}, \quad (8.20)$$

$$G^{(0)\,1}_{\ \ \ 1} = e^{-\lambda}\left(r^{-2} + r^{-1}\nu'\right) - r^{-2}. \quad (8.21)$$

The equations for the proposed gravitational geon have the same scale invariance as those for the electromagnetic counterpart. Therefore we can introduce the same dimensionless measure of radial coordinate, $\rho = \Omega r$. By analogy with the electromagnetic case, we can define a new measure of potential

$$f(\rho) = \sqrt{\kappa_l}\,\Omega h_1(r), \text{ where } \kappa_l \equiv \tfrac{1}{8} l N \left(C_l^0\right)^2 , \quad (8.22)$$

and two metric functions $L(\rho)$ and $Q(\rho)$ such that

$$e^{-\lambda} \equiv 1 - \frac{2L(\rho)}{\rho}, \quad (8.23)$$

$$e^{\lambda+\nu} \equiv Q^2(\rho)\,, \tag{8.24}$$

$$e^{\nu} = \left(1 - \frac{2L(\rho)}{\rho}\right) Q^2(\rho)\,. \tag{8.25}$$

Substituting the above expressions into equations (8.9) and (8.18)–(8.21) yields the wave equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\rho^{*2}} + \left[1 - \left(\frac{lQ}{\rho}\right)^2 \left(1 - \frac{2L}{\rho}\right)\right] f = 0\,, \tag{8.26}$$

and the two background equations

$$\begin{aligned} \frac{\mathrm{d}L}{\mathrm{d}\rho^*} = -\frac{(1-2L/\rho)}{2Q} \Bigg(& \frac{3}{2} f^2 + 2f \frac{\mathrm{d}^2 f}{\mathrm{d}\rho^{*2}} + \left(\frac{\mathrm{d}f}{\mathrm{d}\rho^*}\right)^2 + \frac{1}{2}\left(\frac{\mathrm{d}^2 f}{\mathrm{d}\rho^{*2}}\right)^2 \\ & + \frac{\mathrm{d}f}{\mathrm{d}\rho^*}\frac{\mathrm{d}^3 f}{\mathrm{d}\rho^{*3}} - \frac{1}{2}\rho^{-2} l^2 Q^2 \left(1 - \frac{2L}{\rho}\right)\left(\left(\frac{\mathrm{d}f}{\mathrm{d}\rho^*}\right)^2 + f^2\right)\Bigg)\,, \end{aligned} \tag{8.27}$$

$$\begin{aligned} \frac{\mathrm{d}Q}{\mathrm{d}\rho^*} = -\frac{(1-2L/\rho)}{\rho - 2L} \Bigg(& f^2 + f \frac{\mathrm{d}^2 f}{\mathrm{d}\rho^{*2}} + \left(\frac{\mathrm{d}f}{\mathrm{d}\rho^*}\right)^2 \\ & + \frac{\mathrm{d}f}{\mathrm{d}\rho^*}\frac{\mathrm{d}^3 f}{\mathrm{d}\rho^{*3}} - \frac{1}{2}\rho^{-2} l^2 Q^2 \left(1 - \frac{2L}{\rho}\right)\left(\left(\frac{\mathrm{d}f}{\mathrm{d}\rho^*}\right)^2 + f^2\right)\Bigg)\,. \end{aligned} \tag{8.28}$$

where

$$\mathrm{d}\rho^* = Q^{-1}\left(1 - \frac{2L}{\rho}\right)^{-1} \mathrm{d}\rho\,. \tag{8.29}$$

As in the electromagnetic case, the above equations permit a further change of scale without change of form:

$$\rho = b\rho_1\,,\ Q = bQ_1\,,\ \rho^* = \rho_1^*\,,\ L = bL_1\,,\ f = \sqrt{b}\, f_1\,. \tag{8.30}$$

This allows $Q_1(0) \equiv 1$ and then the scaling parameter can be found by demanding $Q(\infty) = 1$. Note that in equations (8.27) and (8.28) (and in Wheeler's electromagnetic study), an average over a length larger than the characteristic wavelength of the radial function $f(\rho)$ has *not* been performed. This would enormously complicate the system of equations (8.26)–(8.28) and was not believed to be necessary in [36]. Our approach parallels that of [36].

Since the wave equation for the proposed gravitational geon is identical to the one for the electromagnetic counterpart, the same arguments hold for applying another scaling law valid asymptotically for large l (the high-frequency of the gravitational waves guarantees large l). By making the transformation

$$x = (\rho^* - l)\, l^{-1/3}, \tag{8.31}$$

the entire 'active region' (region where the wave amplitude is non-negligible) of the proposed geon will be described by a range of x of order unity. Wheeler [36] has provided the expansion of the relevant quantities in inverse powers of $l^{1/3}$. They are

$$\begin{aligned} d\rho^* &\equiv l^{1/3} dx\,, \\ \rho_1 &= l + l^{1/3} r_0(x) + \cdots\,, \\ L_1 &= l\lambda_0(x) + l^{2/3}\lambda_1(x) + l^{1/3}\lambda_2(x) + \cdots\,, \\ Q_1 &= 1/k(x) + l^{-1/3} q_1(x) + l^{-2/3} q_2(x) + \cdots\,, \\ f_1 &= l^{1/3}\phi(x) + \phi_1(x) + l^{-1/3}\phi_2(x) + \cdots\,. \end{aligned} \tag{8.32}$$

After substituting equation (8.32) into (8.26), expanding in inverse powers of $l^{1/3}$ and setting the lowest three orders to zero, we find the two algebraic equations

$$\lambda_0 = \frac{1}{2}\left(1 - k^2\right), \tag{8.33}$$

$$\lambda_1 = q_1 k^3, \tag{8.34}$$

and the differential equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + j\,k\,\phi = 0\,. \tag{8.35}$$

Here, we have defined

$$j(x) \equiv \frac{1}{k^3}\left(-r_0 - 2q_2k^3 + 2\lambda_2 + 3q_1^2k^4 + 3k^2r_0\right) \tag{8.36}$$

after using equations (8.33) and (8.34). Repeating this procedure for equation (8.27), we obtain

$$\frac{\mathrm{d}k}{\mathrm{d}x} = -\frac{1}{2}k^2\phi^2\,, \tag{8.37}$$

$$\frac{\mathrm{d}q_1}{\mathrm{d}x} = \phi\phi_1\,, \tag{8.38}$$

$$\begin{aligned} -\frac{\mathrm{d}\lambda_2}{\mathrm{d}x} + \frac{1}{4}k^3r_0\phi + \frac{9}{4}q_1^2k^5\phi^2 - 3q_1k^4\phi\phi_1 + \frac{1}{2}k^3\phi_1^2 + \frac{1}{4}k^3\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 \\ + \phi\phi_2k^3 + k^3\phi\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - q_2k^4\phi^2 - \frac{1}{2}\lambda_2k\phi^2 + \frac{1}{4}kr_0\phi^2 = 0\,. \end{aligned} \tag{8.39}$$

The expansion of equation (8.28) yields equations (8.37), (8.38) and

$$\begin{aligned} -\frac{\mathrm{d}q_2}{\mathrm{d}x} + r_0\phi^2 - q_2k\phi^2 + \frac{1}{2}\phi_1^2 + \lambda_2k^{-2}\phi^2 + \frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 \\ + \phi\phi_2 + \phi\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \frac{3}{2}q_1^2k^2\phi^2 - \frac{1}{2}r_0\phi^2k^{-2} = 0\,. \end{aligned} \tag{8.40}$$

By differentiating equation (8.36), utilizing equations (8.33), (8.34), (8.37), (8.38)–(8.40) and

$$\frac{\mathrm{d}r_0}{\mathrm{d}x} = k \tag{8.41}$$

(derived from equations (8.29) and (8.32)), we obtain the differential equation

$$\frac{\mathrm{d}j}{dx} = 3 - \frac{1}{k^2}\left(1 + \frac{1}{2}k^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2\right) . \tag{8.42}$$

Solving equations (8.35), (8.37) and (8.42) simultaneously for the three functions $\phi(x)$, $j(x)$ and $k(x)$ is sufficient for determining the remaining leading terms in equation (8.32). These equations are qualitatively the same as those of the electromagnetic geon. It should be noted that since equations (8.35), (8.37) and (8.42) do not explicitly depend on x, the system of equations is autonomous. These equations can be brought into the same form as the electromagnetic geon equations [36] if equation (8.22) is rescaled similar to (7.15) [42], i.e.

$$f(\rho) = \frac{1}{\sqrt{2}}\sqrt{\kappa_l}\,\Omega\, r\, \mathrm{e}^{(\lambda-\nu)/2}\, h_1(r) . \tag{8.43}$$

Thus the gravitational geon problem is reduced to finding a solution to the system

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + j\,k\,\phi(x) = 0 , \tag{8.44}$$

$$\frac{\mathrm{d}k}{\mathrm{d}x} = -\phi^2 , \tag{8.45}$$

$$\frac{\mathrm{d}j}{\mathrm{d}x} = 3 - \frac{1}{k^2}\left(1 + \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2\right) , \tag{8.46}$$

with the following properties:

$$\begin{aligned} &\phi(x) \to 0 , k(x) \to 1 \text{ and } j(x) \to -\infty \text{ for } x \to -\infty , \\ &\phi(x) \to 0 , 0 < k(x) < 1 \text{ and } j(x) \to -\infty \text{ for } x \to \infty . \end{aligned} \tag{8.47}$$

These are the necessary conditions for asymptotic flatness and nonsingular behaviour at the boundaries.

In summary, the first and second order equations for the gravitational geon problem have been derived. They were determined using the high-frequency approximation [38]. Since the high-frequency approximation for the gravitational geon yields the same differential equations as the high angular momentum approximation for the electromagnetic geon, the properties of equations (8.44)–(8.46) are applicable to both types of geon. These equations have been studied in some detail [36]. In the next chapter, we will review the properties of this electromagnetic geon solution and extend them to address the requirements for the existence of any type of geon: nonsingular, self-consistent solutions which are stable over a time-scale much longer than the typical period of its wave constituents.

Chapter 9

Stability Analysis

The previous chapter established that in the high-frequency approximation, the gravitational and electromagnetic geon problem reduces to the same set of ordinary differential equations (ODEs) (8.44)–(8.46) and boundary conditions (8.47) [39–42]. Therefore any properties of the equations (8.44)–(8.46) apply equally well to both the gravitational and electromagnetic geon case. Any solutions to (8.44)–(8.46) are necessarily equilibrium solutions since the background metric is assumed static. In this chapter, the numerical solution presented in [36] will be reexamined. The results suggest further investigation of the numerical solutions is required in order to determine if the boundary conditions are satisfied. For this investigation a phase portrait of the ODEs is constructed. This is done in section 9.2. In the construction of this phase portrait, both the analytic form at spatial infinity and stability of any solutions will be obtained. Knowledge of solution stability provides a basis for investigating the evolution in time of these solutions. This is done in section 9.3 for the case of the electromagnetic geon. Unlike other investigations, we apply a small amplitude time-dependent perturbation to an equilibrium solution of

(8.44)–(8.46). Solving the time-dependent perturbation equations leads to a contradiction with one of the initial assumptions. The contradiction suggests that neither an electromagnetic geon nor a gravitational geon is a viable construct since not all of the requirements for existence of a geon are met. This interpretation of the results obtained from this investigation will be discussed in the next chapter.

9.1 Numerical integration

Wheeler [36] originally solved the system (8.44)–(8.46) by numerical methods in 1955. Since then computer algorithms have evolved considerably for solving differential equations. It is therefore worthwhile to utilize modern techniques[1] in reexamining those solutions. Even with the algorithm employed in [36] for solving the equations, Wheeler's results are remarkably accurate.

The geon problem (both electromagnetic and gravitational) is reduced to finding a solution to the autonomous system

$$\phi'' + j\,k\,\phi = 0\,, \tag{9.1}$$

$$k' = -\phi^2\,, \tag{9.2}$$

$$j' = 3 - k^{-2}\left(1 + \phi'^2\right)\,. \tag{9.3}$$

Admissible solutions to equations (9.1)–(9.3) are defined as those $\phi(x)$, $j(x)$ and $k(x)$ that satisfy the following criteria:

1. *For large negative* x: The wave function $\phi(x) \to 0$ and metric function $k(x) \to 1$. Under these conditions $j'(x) \to 2$. If $\phi(x)$, $j(x)$ and $k(x)$

[1]To perform the numerical integration, a Fehlberg fourth-fifth order Runge–Kutta method from the MAPLE V R4 program library is used.

are solutions to the autonomous system (9.1)-(9.3), then so are $\phi(x+a)$, $j(x+a)$ and $k(x+a)$ where a is a constant. Choosing the integration constant for $j(x)$ to be zero fixes a and consequently $j(x) \to 2x$. This removes any ambiguity in the start of the integration process. Thus for large negative x, $\phi(x)$ satisfies the equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = 2x\phi\,. \tag{9.4}$$

The approximate solution to (9.4) as given in [36] is

$$\phi(x) = \frac{A}{3}\left(-2x\right)^{-1/4}\exp\left(-(-2x)^{3/2}\right)\,. \tag{9.5}$$

2. *For large positive x*: It is required that $\phi(x) \to 0$, $0 < k(x) < 1$ and $j(x)$ approach large negative values.

The only free parameter is the amplitude A of the wave and this must be chosen so that the solution fits the boundary conditions. The nonlinearity of the problem makes it necessary to integrate the system of equations numerically. The integration is started at $x = -4$. The initial conditions are as follows:

$$\phi(-4) = \phi_0\,, \tag{9.6}$$

$$\left.\frac{\mathrm{d}\phi}{\mathrm{d}x}\right|_{x=-4} = \left(\frac{1}{16} + \sqrt{8}\right)\phi_0\,. \tag{9.7}$$

$$k(-4) = 1\,, \tag{9.8}$$

$$j(-4) = -8\,. \tag{9.9}$$

where ϕ_0 is to be chosen to give an admissible solution. Those ϕ_0 values which yield admissible solutions will be referred to as eigenvalues of the system (9.1)-(9.3) with initial conditions (9.6)-(9.9).

Solution set	ϕ_0
1	9.7910×10^{-5}
2	9.7908×10^{-5}
3	9.7906×10^{-5}
4	9.7904×10^{-5}
5	9.7902×10^{-5}
6	9.7900×10^{-5}

Table 9.1: Values of ϕ_0 for solution sets 1-6.

The behaviour for $\phi(x)$ as $x \to \infty$ depends upon the value of ϕ_0 (which translates into an initial choice of the amplitude A). Figure 9.1 illustrates numerically integrated solutions for values of ϕ_0 given in table 9.1. Solution set 1 shows that for sufficiently large values of ϕ_0, $\phi(x)$ reaches a positive minimum and then increases exponentially. As ϕ_0 is allowed to decrease, the exponential growth of $\phi(x)$ is delayed. This is depicted by solution sets 2 and 3. A further reduction in ϕ_0 results in $\phi(x) \to -\infty$ exponentially as shown in solution sets 4-6. A possible admissible solution lies between solution sets 3 and 4.

The mass of the geon inside radius ρ is related to the function $k(x)$ in the following way:

$$M(\rho(x)) = \frac{1}{b}\lambda_0(x) = \frac{1}{2b}\left(1 - k^2\right) , \tag{9.10}$$

with $b = 1/Q_1(\infty) = k(\infty)$. This implies that

$$0 < k^2(x) \leq 1 \text{ as } x \to \infty \tag{9.11}$$

in order to have a positive total mass. The mass factor $k(x)$ gives a positive mass throughout the integrable region and appears to have a $k(\infty)$ value of

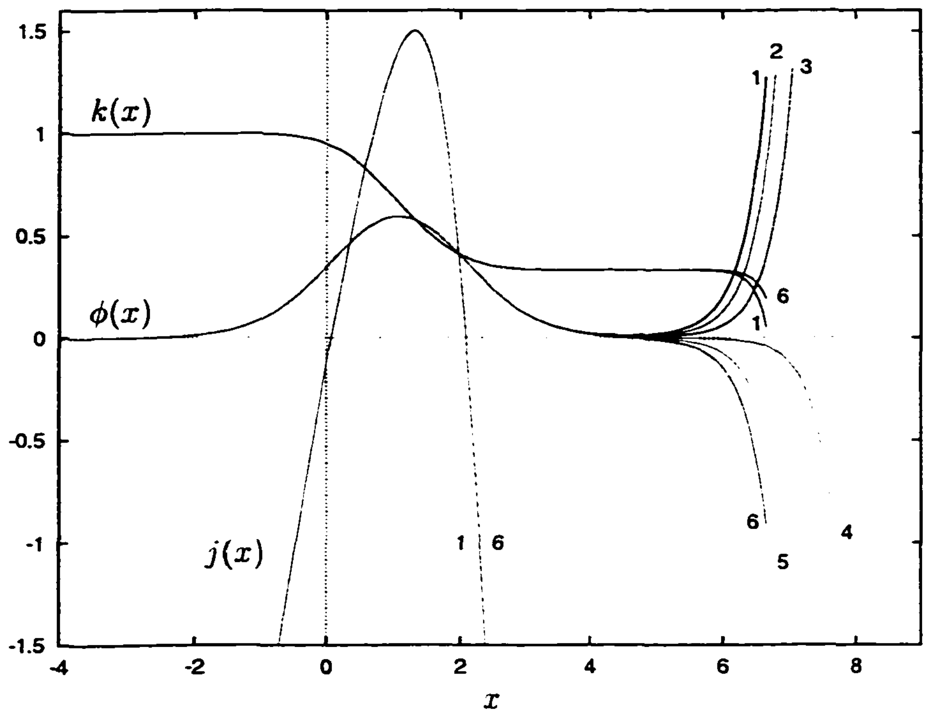

Figure 9.1: Numerical integration of the geon differential equations.

Results of the numerical integration for the geon differential equations (9.1)-(9.3). The values of ϕ_0 for solution sets 1-6 are summarized in table 9.1. The integration started at $x = -4$ and could not proceed beyond $x \simeq 7$ for the given initial values. The active region begins at $x \simeq 0.12$ and ends at $x \simeq 2.13$. A possible admissible solution lies between sets 3 and 4. Note that for $j(x)$, curves 1-6 are indistinguishable.

approximately 1/3. The 'active region' of the geon is defined to be the range of ρ where the square bracketed combination of terms in equation (8.26) is positive. In this region, the function $f(\rho)$ has oscillatory behaviour. Where the terms in square brackets are negative, the behaviour of $f(\rho)$ is exponential growth or decay. The active region can be identified in the x coordinate system as the region where the function $j(x)$ is positive. The function $j(x)$ is positive only for a limited range in the neighbourhood of $x = 1$, thus identifying the active region. In figure 9.1, the active region begins at $x \simeq 0.12$ and ends at $x \simeq 2.13$. The first admissible solution (characterized by $\phi(x)$ having one local maxima and no local minima) appears to lie between those values of ϕ_0 in the range $9.7904 \times 10^{-5} < \phi_0 < 9.7906 \times 10^{-5}$. Qualitatively, these results are similar to those in [36]. The only main difference between the calculation in [36] and the present one is that in [36], the first admissible solution appears to lie in the range $1.03000 \times 10^{-4} < \phi_0 < 1.03125 \times 10^{-4}$ and the active region starts at $x \simeq 4.05$ and ends at $x \simeq 6.02$.

9.2 Existence and stability of equilibrium states

We are interested in determining the analytical behaviour of the solutions shown in figure 9.1 as $x \to \infty$. By constructing the phase portrait for the differential equations, it will be possible to determine both the existence and stability properties of potential admissible solutions. We start by rewriting equations (9.1)–(9.3) as the set of first order equations

$$u' = -j\,k\,\phi\,, \tag{9.12}$$

$$\phi' = u\,, \tag{9.13}$$

$$k' = -\phi^2\,, \tag{9.14}$$

$$j' = 3 - k^{-2} + u^2 k^{-2}\,. \tag{9.15}$$

It is sufficient to look for solutions with the properties

$$\left.\begin{array}{l}\phi,\ u \to 0\\ k \to 1\end{array}\right\} \quad \text{as } x \to -\infty,$$
$$\left.\begin{array}{l}\phi,\ u \to 0\\ k \to \text{constant} > 0\end{array}\right\} \quad \text{as } x \to +\infty \tag{9.16}$$

and j remains finite for finite x. In a phase space, the critical points (or equilibrium points) are characterized by those points where the derivatives of u, ϕ, k and j are zero. The analytic behaviour of the solution about a critical point is determined by analyzing the corresponding linear system in a neighbourhood of that critical point [69].

The first step is to obtain the critical points of the system (9.12)–(9.15). One can easily verify that

$$u = 0\,, \tag{9.17}$$

$$\phi = 0\,, \tag{9.18}$$

$$k = \pm\frac{1}{\sqrt{3}} \tag{9.19}$$

is sufficient to satisfy $u' = \phi' = k' = j' = 0$. Thus the coordinates of the critical point in the phase space are

$$u = 0\,, \tag{9.20}$$

$$\phi = 0\,, \tag{9.21}$$

$$k = \frac{1}{\sqrt{3}}\,, \tag{9.22}$$

$$j = s, \qquad s \in \mathbb{R}, \tag{9.23}$$

where s is any value of $j(x)$. The function $k(x)$ cannot pass through zero since equation (9.3) becomes singular. The positive root of equation (9.19) is chosen to ensure positive $k(x)$, since initially $k(-\infty) = 1$. It is useful to shift the critical point to the origin using the following transformation:

$$\begin{aligned} \phi(x) &= f_1(x), & u(x) &= f_2(x), \\ k(x) &= f_3(x) + \frac{1}{\sqrt{3}}, & j(x) &= f_4(x) + s\,. \end{aligned} \tag{9.24}$$

Therefore the field equations become

$$f_1' = f_2\,, \tag{9.25}$$

$$f_2' = -\left(f_4 + s\right)\left(f_3 + \frac{1}{\sqrt{3}}\right) f_1\,, \tag{9.26}$$

$$f_3' = -f_1^2\,, \tag{9.27}$$

$$f_4' = 3 - \left(1 + f_2^2\right)\left(f_3 + \frac{1}{\sqrt{3}}\right)^{-2}\,, \tag{9.28}$$

with the critical point at $f_1 = f_2 = f_3 = f_4 = 0$. To linearize the field equations about this critical point, a MacLaurin series of $f_i' = f_i'(f_1, f_2, f_3, f_4)$, $i = 1, \ldots, 4$ is taken to first order and evaluated at the critical point (denoted c.p. below), i.e.

$$f_i' = f_i'\Big|_{\text{c.p.}} + \frac{\partial f_i'}{\partial f_1}\Big|_{\text{c.p.}} f_1 + \frac{\partial f_i'}{\partial f_2}\Big|_{\text{c.p.}} f_2 + \frac{\partial f_i'}{\partial f_3}\Big|_{\text{c.p.}} f_3 + \frac{\partial f_i'}{\partial f_4}\Big|_{\text{c.p.}} f_4 + \cdots\,. \tag{9.29}$$

Written in matrix form, the linearized field equations are

$$\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}x} = \mathsf{M}\,\mathbf{w}\,, \tag{9.30}$$

where

$$\mathbf{w} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} \quad \text{and} \quad \mathsf{M} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -s/\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6\sqrt{3} & 0 \end{pmatrix}\,. \tag{9.31}$$

The general solution to the above matrix differential equation is the eigenvector

$$\mathbf{w} = c_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c_2 \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ \sqrt{3}/18 \\ 0 \end{pmatrix} \right] + c_3 \begin{pmatrix} 1 \\ \alpha \\ 0 \\ 0 \end{pmatrix} \mathrm{e}^{\alpha x} + c_4 \begin{pmatrix} 1 \\ -\alpha \\ 0 \\ 0 \end{pmatrix} \mathrm{e}^{-\alpha x}\,, \tag{9.32}$$

where $\alpha \equiv \sqrt{-s/\sqrt{3}}$ and c_i, $i = 1, \ldots, 4$ are constants. Therefore the solution to the linear system is

$$f_1(x) = c_3\,\mathrm{e}^{\alpha x} + c_4\,\mathrm{e}^{-\alpha x}\,, \tag{9.33}$$

$$f_2(x) = c_3\,\alpha\,\mathrm{e}^{\alpha x} - c_4\,\alpha\,\mathrm{e}^{-\alpha x}\,, \tag{9.34}$$

$$f_3(x) = \frac{\sqrt{3}}{18}\,c_2\,, \tag{9.35}$$

$$f_4(x) = c_1 + c_2\,x\,. \tag{9.36}$$

The eigenvector $\mathbf{w}$ shows that in a neighbourhood of the critical point, the nonlinear system decouples into the disjoint subspaces (f_1, f_2) and (f_3, f_4).

The phase space projection of (f_3, f_4) (which is proportional to (k, j)) is illustrated in figure 9.2. Since s can take any value of j, the critical point lies at an arbitrary position $s \leq \max(j)$ on the curve $k(x) = \frac{\sqrt{3}}{18}c_2 + \frac{1}{\sqrt{3}}$ which

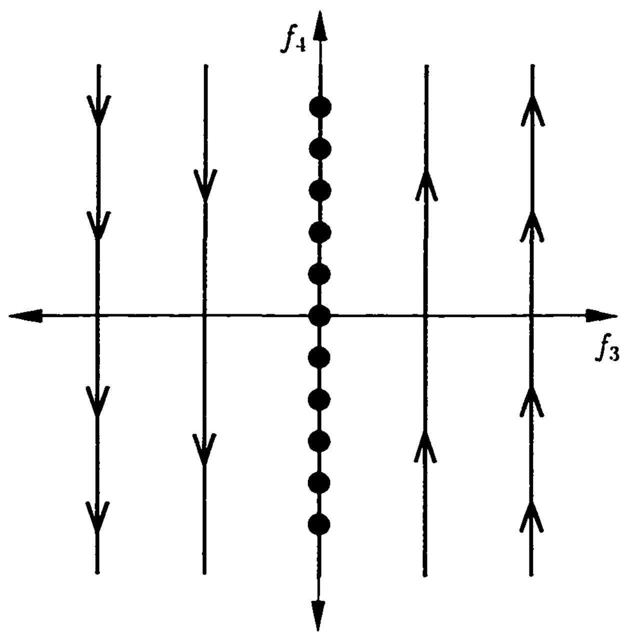

Figure 9.2: Nonisolated critical point in the phase plane.

The (f_3, f_4) phase space projection illustrates a nonisolated critical point. The critical point under investigation is located at the origin. The f_4-axis is a continuous set of critical points in a neighbourhood of the origin. The functions $f_4 \propto j$ and $f_3 \propto k$ behave as a linear function of x and a constant function respectively in a neighbourhood of the critical point not on the f_4-axis.

is transformed back to the origin in the (f_3, f_4) subspace. Equations (9.35) and (9.36) show that in a neighbourhood of the critical point, but not on the f_4-axis, f_3 and f_4 behave as a constant and a linear function of x respectively. If one is on the f_4-axis in a neighbourhood of the critical point, then $c_2 = 0$. Hence $f_4 = c_1$ defines a continuous set of critical points. These critical points are examples of nonisolated critical points. The (f_3, f_4) projection is insufficient for determining the existence and stability of admissible solutions since it only gives information about the functions j and k.

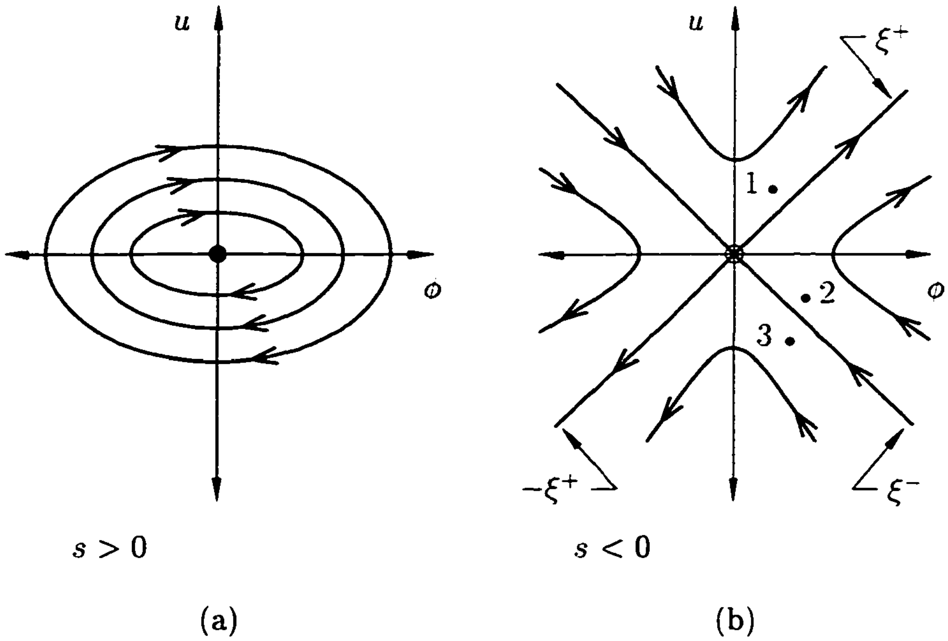

Figure 9.3: Illustration of a stable and an unstable critical point.
(a) Illustration of a stable critical point. This type of critical point occurs when the value of the parameter $s > 0$. (b) Illustration of an unstable critical point. This type of critical point occurs when the value of the parameter $s < 0$.

The nature of the (f_1, f_2) subspace[2] (or the (ϕ, u) subspace) depends upon the parameter $s \in j$. It will determine the stability properties of any admissible solutions. However, a 3-dimensional phase space projection in the coordinates (ϕ, u, j) is necessary to determine the existence of admissible solutions. An illustration of the two possible phase space projections of (ϕ, u) are shown in figure 9.3. Examining (9.33) and (9.34), if $s > 0$, then ϕ and u behave as sinusoidal functions of x. This type of critical point is described as a center (figure 9.3(a)) and is a *stable* critical point. If $s < 0$, then ϕ and u

[2]It is convenient to use the functions ϕ, u in place of f_1, f_2 for the remainder of this section. Recall $\phi = f_1$, $u = f_2$ from equation (9.24).

have an exponential behaviour and the critical point is *unstable*. This type of critical point is described as a saddle point (figure 9.3(b)). By following the integration procedure in the parameter x for solution sets 1 and 6 of figure 9.1, the behaviour of the complete nonlinear system can be described.

Figure 9.4 shows the 3-dimensional (ϕ, u, j) phase space projection of solution sets 1 and 6. The integration procedure starts in plane B of figure 9.4 at $j = -8$. In addition, the solution trajectories start somewhere along the line $u = (1/16 + \sqrt{8})\phi_0$ which must necessarily lie to the left of the unstable asymptote ξ^+. One such point is labelled "1" in figure 9.3(b). In order for the system to satisfy the boundary conditions (9.16), it is necessary for at least one solution to flow along the unstable asymptote ξ^- (in the $j = s \to -\infty$ plane). If figure 9.3(b) were a complete description of the phase space, then it would be impossible for a solution starting at position "1" to cross ξ^+. This is a consequence of the well-known property of autonomous systems that phase space trajectories do not cross. However, as the integration process in x continues, the value of j increases from a negative value to a positive value. Therefore the nature of the critical point in the 2-dimensional (ϕ, u) phase planes changes temporarily from a saddle point to that of a center. Plane A of figure 9.4 is an illustration of one such critical point. Soon afterward, j decreases to negative values and the critical points are saddle points once again. However, the solution trajectories have crossed ξ^+ and now follow the flow along the unstable asymptote ξ^-. Upon the transition of the critical point from centers to saddle points, the asymptotes have been reestablished with solution sets 1 and 6 on opposite sides of the ξ^- asymptote. The positions where solution sets 1 and 6 cut the (ϕ, u) planes for $j < 0$ are schematically illustrated

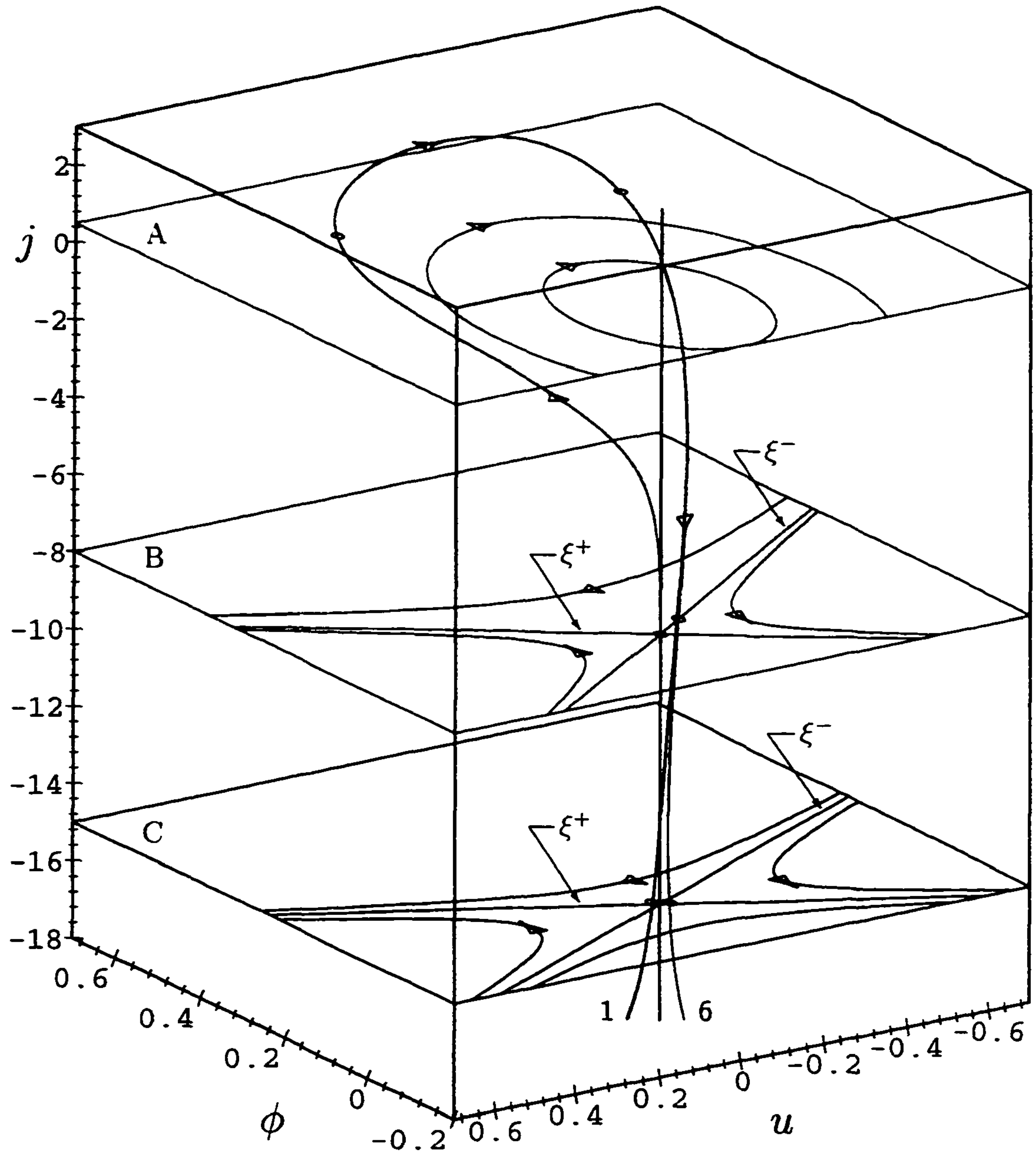

Figure 9.4: The 3-dimensional (ϕ, u, j) phase space projection.

The 3-dimensional (ϕ, u, j) projection of the phase space curves for the numerical solution sets 1 and 6 shown in figure 9.1. The numerical integration starts in plane B, proceeds up through plane A and continues down through plane C. The nature of the critical points along the j-axis (origin of the (ϕ, u) planes) change from centers ($j > 0$) to saddle points ($j < 0$) showing the system is unstable.

as points "2" and "3" respectively in figure 9.3(b). In figure 9.4, these positions are most clearly seen in plane C. Since trajectories for autonomous systems do not cross and the trajectories depend continuously on the initial data, there must be a value of ϕ_0 for which the trajectory approaches ξ^- as $j = s \to -\infty$. The existence of this trajectory shows that it is possible to find an eigenvalue of ϕ_0 which satisfies the boundary conditions (9.16). However, the nature of the critical point as $j = s \to -\infty$ ($x \to \infty$) is a saddle point and therefore this eigenvalue solution is an unstable solution. Since solution set 1 cuts plane C at position "2" of figure 9.3(b), the flow of the integration process requires this set to approach ξ^+. Similarly, solution set 6 must approach ξ^-, since it cuts plane C at position "3." Hence, any small perturbation of the eigenvalue of ϕ_0 implies the constant $c_3 \neq 0$ in equations (9.33) and (9.34). Thus the non-eigenvalue solutions do not satisfy the boundary conditions. Figure 9.5 shows the (ϕ, u) subspace for the six solution sets of figure 9.1. The projection is for $j < -15$. A comparison of figure 9.5 to figure 9.3(b) confirms that the admissible solution is unstable.

The existence of admissible solutions and instability of the electromagnetic geon system were discussed in [36], however, it was based on the numerical solution curves similar to figure 9.1. Performing a phase portrait analysis, we have formally shown that an eigenvalue does exist[3] which satisfies conditions (9.16). We have also shown that this solution must necessarily be *unstable* with respect to perturbations of the eigenvalue of ϕ_0. In [36], it was also suggested that a spherical geon would most likely collapse to a toroidal geon [70], presumably thought to be more stable. It was argued by Ernst [70] that the

[3]Existence of the eigenvalue was derived independently in [42] using an alternate method. However, stability aspects were not discussed in [42].

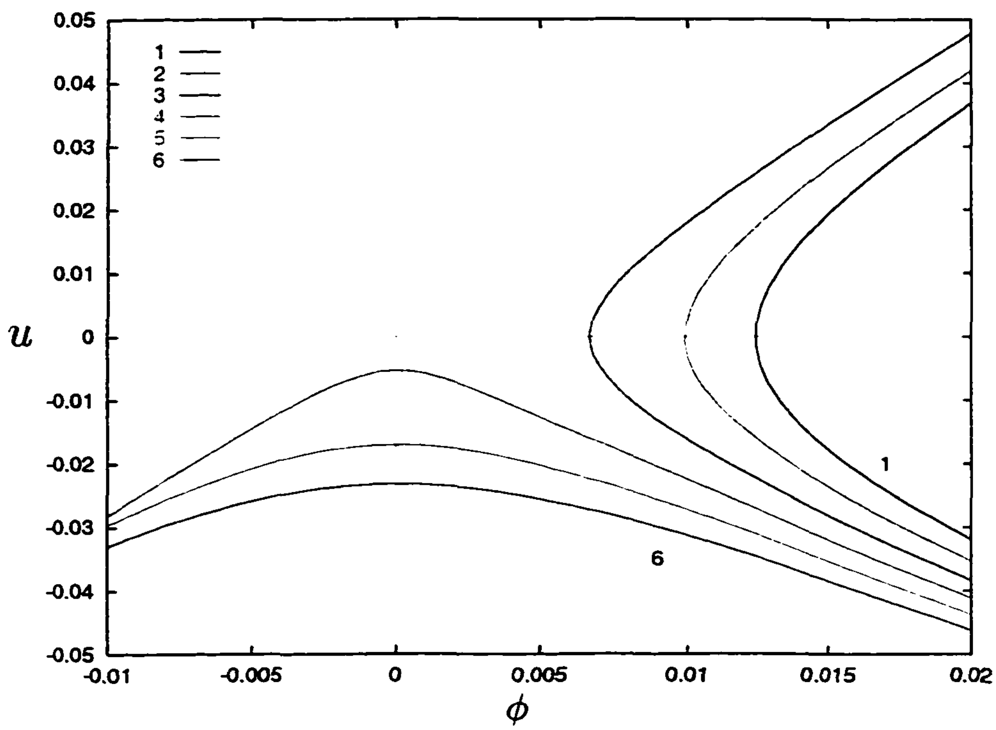

Figure 9.5: The (ϕ, u) phase space projection.

The (ϕ, u) phase space projection of solution sets 1-6 for $j < -15$. It has the characteristics of figure 9.3(b). This indicates that the critical point at the origin is unstable as $j \to -\infty$.

construction of a toroidal geon could be realized if one had complete knowledge of a linear geon (which approximates a small segment of the toroidal geon). Numerical evidence for amplitude eigenvalues in analogy with the spherical geon were presented in [70]. In appendix F we extend the work in [70] by performing a phase portrait analysis. We found that only unstable admissible solutions exist, as in the case of the spherical geon.

To this point, it has been established that a solution to equations (9.1)-(9.3) must be in a state of unstable equilibrium. It has been suggested [42] that this is sufficient for proclaiming the existence of both types of geons (electromagnetic and gravitational). However, in chapter 6.2. a geon was defined to have the property that the ensemble of waves comprising the geon be confined on a time-scale longer than the typical period of the constituent waves. Otherwise it would not be possible to attribute a structural form to the geon. The stability analysis showed that admissible solutions to equations (9.1)-(9.3) are unstable with respect to perturbations of the initial conditions (value of ϕ_0), i.e. the ensemble of waves must collapse or explode.

From the stability analysis on the (time-averaged) equilibrium solution, it is clear that a geon must collapse or explode. In order to determine the time-scale of the instability, the evolution in time of the unstable equilibrium solutions must be studied. The next section attempts a time-dependent perturbation analysis of the equations that describe the electromagnetic geon in order to determine this time-scale.

9.3 Time-evolution analysis of the electromagnetic geon

The electromagnetic geon model employed in [36] assumed a background metric independent of time. This precludes the possibility of studying the time-evolution of the individual modes coupled to a time-evolving background metric. Instead of introducing a time-dependence into the system of equations and solving the coupled wave–background system, an alternate method was employed in [36] for determining the time-scale of collapse or 'lifetime' of the electromagnetic geon. The approach was to study the behaviour of a single photon (borrowing particle physics terminology) in an effective potential created by the ensemble of photons comprising the geon. It was expected that photons would leak out of the potential well in an irreversible dissipation of energy. The model for the rate of photon leakage was based on the quantum mechanical process of radioactive alpha-decay. From this model, the probability of barrier penetration, α, (called the 'attrition' in [36]) has a value of the order

$$\alpha \sim e^{-2P}, \tag{9.37}$$

where P is the barrier penetration integral (c.f. the Gammow factor of alpha-decay [71]). The function P is roughly proportional to the height of the barrier as encountered by the escaping particle. A small attrition would translate to a small rate of radiation leakage out of the effective potential. The time to collapse or the lifetime of the geon in this model is inversely proportional to the product of the attrition and angular frequency of the vibrational mode under consideration. It was estimated using data from the numerical integration performed in [36] that the attrition $\alpha \sim e^{-1.52\sqrt{l(l+1)}}$. As the angular momentum l is increased, the attrition decreases exponentially which suppresses the photon leakage. Thus it was concluded that it was possible to construct a geon of virtually infinite lifetime.

The only similarity between the concept of a geon leaking photons and radioactive alpha-decay is the shape of the potential well found in each problem. However, there are important differences to consider. It is to be stressed that a geon is a classical object built from classical (massless) fields. Radioactive alpha-decay is a Coulomb repulsion effect involving massive, electrically charged particles. The mechanism for the emission of an alpha particle from a nucleus is the quantum mechanical effect of 'tunnelling' through a potential barrier. In a classical system, it is not possible for a particle to tunnel though a potential barrier. In addition, the concept of photon leakage implies the background metric (or potential barrier) evolves on some time-scale. An alpha-decay model cannot describe such an evolution. The quantum mechanical nature of alpha-decay brings into question the validity of using such a model for determining the lifetime of a classical object such as a geon. There are known phenomena which represent classical wave penetration of a potential barrier. For example, the optical phenomenon of frustrated total internal reflection [72] is such a process. Consider an electromagnetic wave incident on a glass to air interface at an angle greater than the critical angle. Under this circumstance, the wave undergoes total internal reflection. However, if a second block of glass is placed near enough to the first block, the air gap carries a flux of energy through to the second block. As a consequence, electromagnetic waves are found to propagate through the barrier region into the second block of glass. It is important to note that in both alpha-decay and frustrated total internal reflection, the potential is supplied by some material and not the waves themselves, as is the case for geons. Whether the analogy exists between these examples of barrier penetration and the time-evolution of a geon based upon the coupled Einstein–Maxwell equations is the subject of this section.

Another approach towards determining the lifetime of an electromagnetic geon is that of Brill [43]. The method is to study the evolution of the ensemble of photons which produce the effective potential using a thin-shell model for the electromagnetic geon. It was found that the radial position of the thin shell underwent a displacement towards collapse. It was also stated that the rate of collapse was 'slow.'

The junction condition problems associated with analyzing thin-shell geon models has previously been discussed in some detail in chapter 7. In addition, the evolution of the thin-shell model in [43] does not allow for leakage of radiation during the collapse nor does it allow an evolving shell thickness (evolving active region) as one might expect. The effect of correcting these deficiencies on the rate of collapse is not clear. A full understanding of the evolution of a geon must take into account the evolution of the typical individual modes of vibration coupled to the evolution of the collective ensemble of waves in a singularity-free model.

It is evident that models for studying the evolution of geons must be based on solving the Einstein (or Einstein–Maxwell) field equations. Intuitive models are not sufficient for describing the true physical system. To avoid the interpretation problems associated with the alpha-decay and thin-shell models and correct for the deficiencies of each model, the derivation of the electromagnetic geon equations [36] will be modified to permit the study of the time-evolution of the electromagnetic geon. The electromagnetic geon will be studied instead of the gravitational geon because of the relative ease in computation for the former. Equilibrium solutions for the gravitational and electromagnetic geon are governed by the same set of ODE's. Therefore it is not expected that the modified gravitational geon equations in the high-frequency (high angular momentum) approximation would yield a significantly different result from the electromagnetic case.

We are interested in following the evolution of the electromagnetic geon in the radial direction. Observing the time-evolution of the metric reflects the evolution of the ensemble of electromagnetic waves comprising the geon. However, it is not sufficient to simply perturb the background metric functions. It is the electromagnetic waves which are the source for the gravitational field, hence both the gravitational and electromagnetic quantities must be perturbed. This will be done by applying an amplitude perturbation on the electromagnetic waves comprising the geon in such a way as to induce the background metric to evolve in time. Frequency perturbations are not explicitly considered in the following derivation for two main reasons. Firstly, the stability analysis of the previous section indicates that the instability of the admissible equilibrium solution originates from changes in the amplitude eigenvalue (initial condition). Secondly, it can be shown that a perturbation of the form $\Omega \to \Omega + \delta\Omega$ is a special case of the amplitude perturbation studied below. Restricting study to the radial direction maintains the field equations in their simplest form. It should be emphasized that the time-space average of the electromagnetic disturbance must be incorporated into the background metric equations to maintain spherical symmetry but still allow for the solution to evolve in time. This time-averaging problem will be addressed when the perturbation is applied. Before the perturbation analysis is performed, the angle-averaged time-dependent electromagnetic geon field equations must first be derived. This part of the derivation follows closely that of [36].

The equations presented below are derived in greater detail in appendix G. Only an outline of the derivation is given here. We start by defining the electromagnetic vector potential for one mode of the electromagnetic waves

$$A_\mu = (0, 0, 0, A_\varphi) \ , \tag{9.38}$$

where

$$A_\varphi = a(r,t)B(\theta)\,, \qquad B(\theta) = \sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_l(\cos\theta)\,. \tag{9.39}$$

The function $a(r,t)$ has been left unspecified at this stage. The time-dependent metric is

$$\mathrm{d}s^2 = g_{\alpha\beta}\,\mathrm{d}x^\alpha\mathrm{d}x^\beta = -\mathrm{e}^\nu\mathrm{d}t^2 + \mathrm{e}^\lambda\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2\,, \tag{9.40}$$

where

$$\nu = \nu(r,t), \qquad \lambda = \lambda(r,t)\,.$$

In the absence of charges and currents, Maxwell's equations in a curved space-time are

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}F^{\beta\alpha}\right) = 0\,, \tag{9.41}$$

$$F_{\alpha\beta,\gamma} + F_{\gamma\alpha,\beta} + F_{\beta\gamma,\alpha} = 0\,, \tag{9.42}$$

where g is the determinant of the metric (9.40) and the Maxwell tensor, $F_{\alpha\beta}$ is related to the four-vector potential as $F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}$. The Einstein equations for the electromagnetic geon are

$$G_\mu{}^\nu = 8\pi\left\langle T_\mu{}^\nu\right\rangle\,, \tag{9.43}$$

where $\langle\,\cdot\,\rangle$ denotes a time-space average over all N active modes of the electromagnetic waves. Substituting (9.38) into equations (9.41) and (9.43), taking the angle average and finally transforming to the $\rho = \Omega\, r$ coordinate system yields the wave equation

$$\Omega^2\frac{\partial^2 a}{\partial\rho^{*2}} - \Omega^2 l^{*2}\rho^{-2}\left(1-\frac{2L}{\rho}\right)Q^2\,a - \left(1-\frac{2L}{\rho}\right)^2 Q^2\frac{\partial^2 a}{\partial t^{*2}} = 0 \tag{9.44}$$

and the background field equations

$$\frac{\partial L}{\partial \rho^*} = \frac{\kappa_l^2}{2}\left(Q^{-1}\left(\Omega^2\left\langle\left(\frac{\partial a}{\partial \rho^*}\right)^2\right\rangle_T + \left\langle\left(\frac{\partial a}{\partial t}\right)^2\right\rangle_T\right)\right.$$
$$\left. + \Omega^2 l^{*2}\rho^{-2}\left(1 - \frac{2L}{\rho}\right) Q\left\langle a^2\right\rangle_T\right). \quad (9.45)$$

$$\frac{\partial Q}{\partial \rho^*} = \frac{\kappa_l^2}{\rho - 2L}\left(\Omega^2\left\langle\left(\frac{\partial a}{\partial \rho^*}\right)^2\right\rangle_T + \left\langle\left(\frac{\partial a}{\partial \rho^*}\right)^2\right\rangle_T\right), \quad (9.46)$$

$$\frac{\partial L}{\partial t} = \kappa_l^2 \Omega^2 Q^{-1}\left\langle\frac{\partial a}{\partial \rho^*}\frac{\partial a}{\partial t}\right\rangle_T, \quad (9.47)$$

where

$$\kappa_l \equiv \sqrt{\frac{Nl(l+1)}{2l+1}}. \quad (9.48)$$

Equations (9.44)–(9.47) are the starting point for developing the dynamic perturbation (time-evolution) equations. The $\partial Q/\partial t$ equation (found from the $G_\theta{}^\theta = 8\pi\left\langle T_\theta{}^\theta\right\rangle$ equation) is not used in the subsequent analysis, but is given in appendix G for completeness.

The time-averaged equilibrium solution in [36] has the form

$$\kappa_l\,\Omega\, a(\rho, t) = f_0(\rho)\sin\Omega\, t, \quad (9.49)$$

$$L(\rho, t) = L_0(\rho), \quad (9.50)$$

$$Q(\rho, t) = Q_0(\rho). \quad (9.51)$$

where $f_0(\rho)$, $L_0(\rho)$ and $Q_0(\rho)$ are known functions.[4] We will designate this as the 'unperturbed solution.' The most general form for the radial perturbation

[4]These are known as numerical solutions to the system (9.1)–(9.3) with initial conditions (9.6)–(9.9) in the high angular momentum approximation.

of the wave function $a(\rho, t)$ is

$$\kappa_l \Omega\, a(\rho, t) = f_0(\rho) \sin \Omega t + \delta u_1(\rho, t) + \delta^2 u_2(\rho, t) + \mathrm{O}\left(\delta^3\right), \qquad \delta \ll 1. \tag{9.52}$$

where δ is the expansion parameter. As a result, a small time-dependent perturbation is introduced in the metric functions

$$L(\rho, t) = L_0(\rho) + \delta L_1(\rho, t) + \delta^2 L_2(\rho, t) + \mathrm{O}\left(\delta^3\right), \tag{9.53}$$

$$Q(\rho, t) = Q_0(\rho) + \delta Q_1(\rho, t) + \delta^2 Q_2(\rho, t) + \mathrm{O}\left(\delta^3\right). \tag{9.54}$$

The perturbation expansion will be carried out to the first order in δ. Before the perturbed system is solved in a self-consistent manner, the problem of time-averaging the functions on the right hand side of equations (9.45)–(9.47) must be addressed. From the definition of a time-average and equation (9.52),

$$\begin{aligned}
\kappa_l^2 \Omega^2 \left\langle a^2(\rho, t)\right\rangle_{\mathrm{T}} &\equiv \frac{1}{T}\int_0^T \kappa_l^2 \Omega^2 a^2(\rho, t)\, \mathrm{d}t \\
&= \frac{1}{T}\int_0^T \Big(f_0^2(\rho) \sin^2\Omega t + 2\delta\, u_1(\rho, t) f_0(\rho) \sin \Omega t \\
&\qquad + \delta^2 \left(u_1^2(\rho, t) + 2u_2(\rho, t) f_0(\rho) \sin \Omega t\right) + \mathrm{O}\left(\delta^3\right)\Big)\, \mathrm{d}t \\
&= \frac{1}{2} f_0^2(\rho) + \frac{1}{T}\int_0^T \Big(2\delta u_1(\rho, t) f_0(\rho) \sin \Omega t \\
&\qquad + \delta^2 \left(u_1^2(\rho, t) + 2u_2(\rho, t) f_0(\rho) \sin \Omega t\right) + \mathrm{O}\left(\delta^3\right)\Big)\, \mathrm{d}t.
\end{aligned}$$

where $T = 2\pi\Omega^{-1}$ is the time period of the electromagnetic waves. In order to make progress in the perturbation analysis, it is necessary to make some assumptions about the function $u_1(\rho, t)$. The presence of unevaluated integrals on the right hand side of the differential equations (9.45)–(9.47) would not make it possible to proceed with the analysis beyond this point. Let us suppose the time-dependence of $u_1(\rho, t)$ was sinusoidal and its characteristic frequency was of the order Ω of the electromagnetic waves. In this case, the time-dependence is lost to all orders in δ upon time-averaging. In essence, this assumption on $u_1(\rho, t)$ precludes the possibility of a time-dependent evolution of the system. This is not satisfactory. To maintain a time-dependence after time-averaging, another time-scale will be introduced into the problem. Suppose the time-dependence of $u_1(\rho, t)$ was again sinusoidal and its characteristic frequency was of the order $\omega \ll \Omega$. In this case, $u_1(\rho, t)$ is approximately constant over the short time period $T = 2\pi\Omega^{-1}$ of the electromagnetic waves and thus is constant in the time-average integral. Evaluating the time-average of $a^2(\rho, t)$ under this assumption, we find

$$\begin{aligned}\kappa_l^2\,\Omega^2 \left\langle a^2(\rho,\, t)\right\rangle_{\mathrm{T}} = \frac{1}{T}\int_0^T &\Big(f_0^2(\rho)\sin^2\!\Omega\, t + 2\delta\, u_1(\rho,\, t) f_0(\rho)\sin\Omega\, t \\ &+ \delta^2\left(u_1^2(\rho,\, t) + 2u_2(\rho,\, t)f_0(\rho)\sin\Omega\, t\right)\Big)\, \mathrm{d}t + \mathrm{O}\left(\delta^3\right) \\ = \frac{1}{2}f_0^2(\rho) &+ 2\delta u_1(\rho,\, t) f_0(\rho)\frac{1}{T}\int_0^T \sin\Omega\, t\,\mathrm{d}t \\ + \delta^2 u_1^2(\rho,\, t)&\frac{1}{T}\int_0^T \mathrm{d}t + 2u_2(\rho,\, t)f_0(\rho)\int_0^T \sin\Omega\, t\,\mathrm{d}t + \mathrm{O}\left(\delta^3\right) \\ = \frac{1}{2}f_0^2(\rho) &+ \delta^2 u_1^2(\rho,\, t) + \mathrm{O}\left(\delta^3\right)\,.\end{aligned} \tag{9.55}$$

Therefore the time-dependence is not present until the second order in δ. This is sufficient to proceed with the time-evolution of the system, since each order in the expansion parameter δ must be set to zero. To *first* order in δ

$$\kappa_l^2\,\Omega^2 \left\langle a^2(\rho,\, t)\right\rangle_{\mathrm{T}} = \frac{1}{2}f_0^2 + \mathrm{O}\left(\delta^2\right)\,. \tag{9.56}$$

Similarly, the remaining time-averages are

$$\kappa_l^2\Omega^2\left\langle\left(\frac{\partial a}{\partial\rho^*}\right)^2\right\rangle_T = \frac{1}{2}\left(\frac{\mathrm{d}f_0}{\mathrm{d}\rho^*}\right)^2 + \mathrm{O}\left(\delta^2\right). \tag{9.57}$$

$$\kappa_l^2\Omega^2\left\langle\left(\frac{\partial a}{\partial t}\right)^2\right\rangle_T = \frac{1}{2}\Omega^2 f_0^2 + \mathrm{O}\left(\delta^2\right). \tag{9.58}$$

$$\kappa_l^2\Omega^2\left\langle\frac{\partial a}{\partial t}\frac{\partial a}{\partial\rho^*}\right\rangle_T = \mathrm{O}\left(\delta^2\right). \tag{9.59}$$

Substitution of (9.52)–(9.59) into (9.44)–(9.47), expanding to first order in δ and setting each order in δ to zero yields the unperturbed equations (9.60)–(9.62) and the first order equations (9.63)–(9.67). The properties of the unperturbed equations

$$\frac{\mathrm{d}^2 f_0}{\mathrm{d}\rho^{*2}} + \left(1 - l^{*2}Q_0^2\rho^{-2}\left(1-\frac{2L_0}{\rho}\right)\right)f_0 = 0, \tag{9.60}$$

$$\frac{\mathrm{d}L_0}{\mathrm{d}\rho^*} = \frac{1}{2Q_0}\left(f_0^2 + \left(\frac{\mathrm{d}f_0}{\mathrm{d}\rho^*}\right)^2 + l^{*2}Q_0^2\rho^{-2}\left(1-\frac{2L_0}{\rho}\right)f_0^2\right), \tag{9.61}$$

$$\frac{\mathrm{d}Q_0}{\mathrm{d}\rho^*} = (\rho - 2L_0)^{-1}\left(f_0^2 + \left(\frac{\mathrm{d}f_0}{\mathrm{d}\rho^*}\right)^2\right), \tag{9.62}$$

are known from [36] and therefore can be used in the analysis of the first order equations. Setting the first order part of the wave equation (9.44) to zero yields

$$\mathfrak{A}(\rho,\, t)\sin\Omega t + \mathfrak{B}(\rho,\, t)\cos\Omega t + \mathfrak{C}(\rho,\, t) = 0, \tag{9.63}$$

where

$$\mathfrak{A}(\rho,\, t) \equiv \Omega\Bigg(\left(Q_1(\rho,\, t)\left(1-\frac{2L_0}{\rho}\right) - 2\rho^{-1}L_1(\rho,\, t)Q_0\right)\times$$
$$\times\left(2\rho^{-2}Q_0\frac{\mathrm{d}f_0}{\mathrm{d}\rho}\left(L_0 - \rho\frac{\mathrm{d}L_0}{\mathrm{d}\rho}\right) + Q_0\frac{\mathrm{d}^2 f_0}{\mathrm{d}\rho^2}\left(1-\frac{2L_0}{\rho}\right) + \frac{\mathrm{d}f_0}{\mathrm{d}\rho}\frac{\mathrm{d}Q_0}{\mathrm{d}\rho}\left(1-\frac{2L_0}{\rho}\right)\right)$$

$$
\begin{aligned}
&+ 2\rho^{-3} l^{*2} Q_0 \left(Q_0 L_1(\rho,\, t) - \rho Q_1(\rho,\, t) \left(1 - \frac{2L_0}{\rho}\right)\right) f_0 + Q_0 \left(1 - \frac{2L_0}{\rho}\right) \times \\
&\times \left(2\rho^{-2} \frac{\mathrm{d}f_0}{\mathrm{d}\rho} \left(Q_1 \left(L_0 - \rho \frac{\mathrm{d}L_0}{\mathrm{d}\rho}\right) + Q_0 \left(L_1(\rho,\, t) - \rho \frac{\partial}{\partial\rho} L_1(\rho,\, t)\right)\right)\right. \\
&+ \left(\left(1 - \frac{2L_0}{\rho}\right) \frac{\partial}{\partial\rho} Q_1(\rho,\, t) - 2\rho^{-1} L_1(\rho,\, t) \frac{\mathrm{d}Q_0}{\mathrm{d}\rho}\right) \frac{\mathrm{d}f_0}{\mathrm{d}\rho} \\
&+ \left(Q_1(\rho,\, t) \left(1 - \frac{2L_0}{\rho}\right) - 2\rho^{-1} L_1(\rho,\, t) Q_0 \right) \frac{\mathrm{d}^2 f_0}{\mathrm{d}\rho^2} \Bigg) \\
&+ \frac{f_0}{\rho - 2L_0} \left(2L_1(\rho,\, t) - \rho \left(1 - \frac{2L_0}{\rho}\right) Q_0^{-1} Q_1(\rho,\, t)\right) \\
&+ \left(1 - \frac{2L_0}{\rho}\right)^{-1} Q_0^{-1} \left(Q_1(\rho,\, t) \left(1 - \frac{2L_0}{\rho}\right) - 2\rho^{-1} L_1(\rho,\, t) Q_0 \right) f_0 \Bigg) .
\end{aligned}
\tag{9.64}
$$

$$
\mathfrak{B}(\rho,\, t) \equiv f_0 \left(2\rho^{-1} \left(1 - \frac{2L_0}{\rho}\right)^{-1} \frac{\partial}{\partial t} L_1(\rho,\, t) - Q_0^{-1} \frac{\partial}{\partial t} Q_1(\rho,\, t)\right) \tag{9.65}
$$

and

$$
\mathfrak{C}(\rho,\, t) \equiv \frac{\partial^2}{\partial\rho^{*2}} u_1(\rho,\, t) - l^{*2} Q_0^2 \rho^{-2} \left(1 - \frac{2L_0}{\rho}\right) u_1(\rho,\, t) - \Omega^{-2} \frac{\partial^2}{\partial t^2} u_1(\rho,\, t) . \tag{9.66}
$$

The first order equations obtained from (9.45) and (9.46) yield expressions for $\partial L_1(\rho,\, t)/\partial\rho^*$ and $\partial Q_1(\rho,\, t)/\partial\rho^*$. These are not used in the subsequent analysis and therefore are not shown. The derivative of $L_1(\rho,\, t)$ with respect to t, obtained from the first order part of equation (9.47), is

$$
\frac{\partial}{\partial t} L_1(\rho,\, t) = 0 . \tag{9.67}
$$

It is immediately integrable to yield

$$
L_1(\rho,\, t) = c_1(\rho) , \tag{9.68}
$$

where $c_1(\rho)$ is an unknown function of ρ. It is possible to obtain an equation for $\partial Q_1(\rho, t)/\partial t$ from equation (9.63). This is done by substituting (9.67) into the coefficient of $\cos\Omega t$ (*viz.* $\mathfrak{B}(\rho, t)$) and setting the expression to zero. This is justified since $\sin\Omega t$ and $\cos\Omega t$ are independent functions in the approximation where $\mathfrak{A}(\rho, t)$, $\mathfrak{B}(\rho, t)$ and $\mathfrak{C}(\rho, t)$ are slowly varying functions of time. Solving for $\partial Q_1(\rho, t)/\partial t$ and integrating yields

$$Q_1(\rho, t) = c_2(\rho)\,, \tag{9.69}$$

where $c_2(\rho)$ is an unknown function of ρ.

Up to this point the only first order field equation which has been satisfied is equation (9.67). To satisfy the first order wave equation (9.63), the additional conditions $\mathfrak{A}(\rho, t) = 0$ and $\mathfrak{C}(\rho, t) = 0$ must be imposed. We will focus upon the latter, since it is the simpler of the two equations. Setting (9.66) to zero yields the equation

$$\frac{\partial^2}{\partial \rho^{*2}} u_1(\rho, t) - l^{*2} Q_0^2 \rho^{-2}\left(1 - \frac{2L_0}{\rho}\right) u_1(\rho, t) - \Omega^{-2}\frac{\partial^2}{\partial t^2} u_1(\rho, t) = 0\,. \tag{9.70}$$

To solve this equation, we will use the method of separation of variables. Let

$$u_1(\rho, t) = u(\rho) T(t)\,. \tag{9.71}$$

Substituting (9.71) into (9.70) and dividing by $u(\rho)T(t)$ yields the two ordinary differential equations

$$\frac{\mathrm{d}^2 u(\rho)}{\mathrm{d}\rho^{*2}} - l^{*2} Q_0^2 \rho^{-2}\left(1 - \frac{2L_0}{\rho}\right) u(\rho) + \beta u(\rho) = 0\,, \tag{9.72}$$

$$\frac{\mathrm{d}^2 T(t)}{\mathrm{d}t^2} + \beta \Omega^2 T(t) = 0\,, \tag{9.73}$$

where $-\beta$ is the separation constant. The solution to (9.73) is

$$T(t) = c_3 \sin(\omega t + c_4)\,, \qquad c_3, c_4 \text{ - constants,} \tag{9.74}$$

where we have chosen $\beta \equiv \frac{\omega^2}{\Omega^2}$, $\omega \ll \Omega$. Making this choice for β satisfies the requirement of equation (9.55) for a meaningful time-average.

In order to solve equation (9.72), it is necessary to apply the high angular momentum approximation. It is therefore necessary to transform $\mathfrak{C}(\rho, t)$ to the x coordinate system and expand in inverse powers of $l^{*1/3}$ as is done for the unperturbed system [36]. The transformation in [36] for the unperturbed functions is repeated here for convenience

$$x = (\rho^* - l^*)\, l^{*-1/3}, \tag{9.75}$$

$$\begin{aligned} d\rho^* &\equiv l^{*1/3} dx\,, \\ \rho &= l^* + l^{*1/3} r_0(x) + \cdots, \\ L_0 &= l^* \lambda_0(x) + l^{*2/3}\lambda_1(x) + l^{*1/3}\lambda_2(x) + \cdots, \\ Q_0 &= 1/k(x) + l^{*-1/3} q_1(x) + l^{*-2/3} q_2(x) + \cdots, \\ f_0 &= l^{*1/3}\phi(x) + \phi_1(x) + l^{*-1/3}\phi_2(x) + \cdots. \end{aligned} \tag{9.76}$$

In addition, the function $u(\rho)$ must be expanded in a similar manner, i.e.

$$u = l^{*1/3}\mu_0(x) + \mu_1(x) + l^{*-1/3}\mu_2(x) + \cdots. \tag{9.77}$$

After a lengthy computation, the asymptotic expansion of (9.72) yields

$$l^{*1/3}\left(\frac{\omega^2}{\Omega^2} - \frac{1 - 2\lambda_0(x)}{k^2(x)}\right)\mu_0(x) + O(1) = 0\,. \tag{9.78}$$

In order for equation (9.78) to be satisfied for large arbitrary l^* (in the limit $l^* \to \infty$), each order of $l^{*1/3}$ must be set to zero. Since setting $\mu_0(x) = 0$ implies the absence of electromagnetic wave perturbations, the bracketed term must be zero. It is known from the unperturbed system that [36]

$$\lambda_0 = \frac{1}{2}\left(1 - k^2\right). \tag{9.79}$$

Substituting (9.79) into (9.78) and setting the bracketed term to zero yields the relation

$$\omega^2 = \Omega^2 \tag{9.80}$$

which must hold in order to satisfy the condition $\mathfrak{C}(\rho, t) = 0$. However, equation (9.80) is a *contradiction* to the original assumption $\omega \ll \Omega$. Because of the presence of the contradiction, it is not necessary to solve the remaining field equations.

We arrived at this contradiction through the assumption that (9.70) could be solved by separating variables (equation (9.71)). The same result is obtained if $u_1(\rho, t)$ is not separable as is shown below. Modifying (9.77) as follows

$$u_1 = l^{*1/3}\mu_0(x, t) + \mu_1(x, t) + l^{*-1/3}\mu_2(x, t) + \cdots . \tag{9.81}$$

and expanding (9.70) in inverse powers of $l^{*1/3}$ yields

$$-l^{*1/3}\left(\Omega^{-2}\frac{\partial^2}{\partial t^2}\mu_0(x, t) - \frac{1 - 2\lambda_0(x)}{k^2(x)}\right)\mu_0(x, t) + \mathrm{O}(1) = 0 \tag{9.82}$$

to lowest order in $l^{*1/3}$. Setting each order in $l^{*1/3}$ to zero requires $\mu_0(x, t)$ to satisfy the differential equation (using (9.79))

$$\frac{\partial^2}{\partial t^2}\mu_0(x, t) + \Omega^2\mu_0(x, t) = 0 . \tag{9.83}$$

The solution is

$$\mu_0(x, t) = c_5(x)\sin(\Omega t + c_6(x)) . \tag{9.84}$$

The characteristic frequency of $\mu_0(x, t)$ is Ω which contradicts the assumption $u_1(\rho, t) \sim \mu_0(x, t) \sim \omega \ll \Omega$.

Before we proceed with the discussion of the perturbation analysis based on equations (9.52)–(9.54), another form for the wave function perturbation should be considered. Consider the perturbation in the form

$$\kappa_l\,\Omega\,a(\rho,\,t) = (f_0(\rho) + \delta u_1(\rho,\,t))\sin\Omega\,t + \mathrm{O}\left(\delta^2\right) . \qquad (9.85)$$

where the characteristic frequency of $u_1(\rho,\,t)$ is of order $\omega \ll \Omega$. This form can be interpreted as a slowly evolving amplitude of the rapidly varying function $\sin\Omega\,t$. Equations (9.52) and (9.85), in addition to the assumptions placed on $u_1(\rho,\,t)$, cover the entire range of possibilities for these types of perturbations. Evaluation of the time-averages yields

$$\kappa_l^2\,\Omega^2\left\langle a^2(\rho,\,t)\right\rangle_{\mathrm{T}} = \frac{1}{2}f_0^2 + \delta f_0 u_1 + \mathrm{O}\left(\delta^2\right) . \qquad (9.86)$$

$$\kappa_l^2\,\Omega^2\left\langle\left(\frac{\partial a}{\partial\rho^*}\right)^2\right\rangle_{\mathrm{T}} = \frac{1}{2}\left(\frac{\mathrm{d}f_0}{\mathrm{d}\rho^*}\right)^2 + \delta\frac{\mathrm{d}f_0}{\mathrm{d}\rho^*}\frac{\partial u_1}{\partial\rho^*} + \mathrm{O}\left(\delta^2\right) . \qquad (9.87)$$

$$\kappa_l^2\,\Omega^2\left\langle\left(\frac{\partial a}{\partial t}\right)^2\right\rangle_{\mathrm{T}} = \frac{1}{2}\Omega^2 f_0^2 + \delta\,\Omega^2 f_0 u_1 + \mathrm{O}\left(\delta^2\right) . \qquad (9.88)$$

$$\kappa_l^2\,\Omega^2\left\langle\frac{\partial a}{\partial t}\frac{\partial a}{\partial\rho^*}\right\rangle_{\mathrm{T}} = \delta\frac{1}{2}\frac{\mathrm{d}f_0}{\mathrm{d}\rho^*}\frac{\partial u_1}{\partial t} + \mathrm{O}\left(\delta^2\right) . \qquad (9.89)$$

Hence, the time-dependence of the time-averaged functions becomes manifest at the first order in δ. This greatly increases the mathematical complexity of the system. The detailed analysis for this system is derived in appendix H for the case of $u_1(\rho,\,t)$ separable. From this analysis, the same contradiction results for the characteristic frequency of $u_1(\rho,\,t)$ as that found from the analysis based upon equation (9.52). For nonseparable $u_1(\rho,\,t)$, the differential equations become unmanageable. Hence, it was not possible to obtain a conclusive result. However, due to the similar nature of the systems based upon equations (9.52) and (9.85), there is no reason to suspect a different result for the nonseparable case.

Chapter 10

Discussion and Alternate Approaches to the Geon Problem

In the previous chapter, the time-evolution of an electromagnetic geon equilibrium solution was studied with the objective of determining the time-scale of collapse away from the equilibrium configuration. Perturbations of the form (9.52) and (9.85) were analyzed under certain assumptions. The problem of time-averaging the source terms (right hand side) of the field equations (9.45)–(9.47) requires the characteristic frequency (denoted ω) of the perturbation term of the wave function $u_1(\rho, t)$ to be much less than the characteristic frequency (denoted Ω) of the unperturbed solution. Without this assumption, the time-dependence of the perturbations is lost upon time-averaging, to all orders in the expansion. This is not possible given the unstable nature of the equilibrium solutions (see section 9.2). Hence, the assumption $u_1(\rho, t) \sim \omega \sim \Omega$ is not satisfactory for studying the time-evolution of geons.

The assumption $u_1(\rho, t) \sim \omega \ll \Omega$, for both forms of the perturbation, solves the time-averaging problem in a simple manner. With this assumption, the differential equations maintain a time-dependence after the time-average has been taken over the typical period of the high-frequency waves. The perturbation analysis leads to the requirement $\omega \sim \Omega$ in order for the field equations to be satisfied. This is a contradiction to the original assumption $\omega \ll \Omega$. Since all the possible combinations for the form of the perturbation (and assumptions placed on $u_1(\rho, t)$) have been explored, the possible interpretations of the results of chapter 9 are given below.

A reasonable interpretation of the contradiction in the time-evolution analysis is that the condition of slow evolution of the background cannot be satisfied. Since not all of the required conditions are satisfied, it is not possible to construct a geon comprised of high-frequency waves. It could be argued that an electromagnetic geon could be built from low-frequency waves. This has not been ruled out by our model since it only accommodates high-frequency waves. However, a gravitational geon necessarily must be constructed from high-frequency waves, otherwise the effective stress-energy would not be of the correct order of magnitude to create the background gravitational field binding the waves. Therefore the gravitational geon studied in this dissertation is subject to the same fate as its high-frequency electromagnetic counterpart.

A geon with a rapidly evolving background metric, where the background is somehow regarded as being distinct from the small amplitude waves on the background, is conceptually unsound. The definition presented in chapter 6 for this type of geon requires the background solution of the Einstein or Einstein–Maxwell field equations be quasi-stable on a time-scale much longer than the period of the constituent waves. If the background metric evolves away from the equilibrium configuration on the same time-scale as the constituent waves, one cannot speak of the waves binding gravitationally. Under these circumstances, there is no geon structure to identify. Apart from this argument, the realization of a geon with a rapidly varying background metric $\gamma_{\mu\nu}$ is problematic for another reason. If a spherically symmetric background is allowed to vary harmonically with frequency Ω comparable to the frequency of the gravitational or electromagnetic waves, one expects a parametric resonance [73] for the modes with $\omega_n = n\Omega/2$, with $n \in \mathbb{N}$. The strength of the resonance is a maximum for $n = 1$ and decreases rapidly as n increases. In the limit of a static background, the resonance phenomenon disappears. Accordingly, on the basis of studies of perturbations of black holes and relativistic stars [61], it is expected that in the case of a stationary axisymmetric background metric describing a rapidly rotating geon, the resonance phenomenon between the perturbations and the background metric occurs. In the general case of a time-dependent and rapidly varying background metric $\gamma_{\mu\nu}(t, \mathbf{x})$ without symmetries, it is not known how to decompose metric perturbations on a complete set playing the role of the tensor spherical harmonics in the spherical case, or even how to define frequencies in the strong curvature region. However, if such concepts can be given a meaning, it seems reasonable to expect some kind of resonance phenomena between the background metric and its wave perturbations. All these resonance phenomena certainly do not contribute to the realization of a stable configuration, but rather are associated with instabilities that tend to disrupt the system.

The contradiction which arises in the perturbation analysis of chapter 9 could be interpreted as a breakdown of the model, i.e. the perturbation model is not able to describe the evolution of the physical system. It might be argued that an alternative method of implementing the time-evolution (not necessarily using perturbative methods) may be better suited to determine the time-scale of evolution. For example, it may be possible to develop an exact numerical solution of the full Einstein (or Einstein–Maxwell) equations without the splitting of the metric into a background and waves on the background or taking time-averages. Considering the complexity involved in analyzing the simple perturbative model of section 9.3, any new model would undoubtedly be more complex to solve. However, it should be possible to approximate any exact method with an appropriate perturbation expansion (as is presented in chapter 9). Therefore, it is appropriate to consider the contradiction in a physical context, as was discussed earlier.

In the previous chapters, we have analyzed the BH construct and the models of gravitational and electromagnetic geons conceived by Wheeler. However, one can study different models of a gravitational geon and different, independent, approaches to the geon problem, which are given below.

The most intuitive model of a gravitational geon is that of a ball of high-frequency gravitational radiation, which behaves like a perfect fluid with a radiation equation of state. It appears that Wheeler was aware of this possibility, but discarded it as nonviable and therefore proceeded to study the more complicated models of [36] in which the waves do not propagate radially and the radiation is not isotropic:

> "... one naturally recalled that some stars derive their energy almost exclusively from particles; others, from a mixture of particles and radiation. The extreme limit of a system deriving its mass-energy from radiation alone therefore suggested itself. However, with no matter to provide opacity and to dam up the radiation against escape, stability could only be maintained by excluding all photon orbits in which the

motion is purely radial or largely radial ... " [74].

However plausible this argument may appear, it relies on the theory of the stability of Newtonian stars, and cannot be used for a relativistic fluid ball, since in general relativity the fluid pressure contributes to the energy density which may bind the fluid and this contribution cannot be neglected in a relativistic star dominated by radiation pressure. Therefore, it is important to reexamine the possibility of a fluid ball made only of gravitational or electromagnetic waves. There have been some papers (see, for example, [45,46] and references therein) analyzing the properties of self-gravitating electromagnetic radiation confined to a spherical box. The radiation is taken to be a perfect fluid with equation of state $p = \rho/3$. This model is as applicable to high-frequency gravitational waves as it is to electromagnetic waves, hence the results of these papers will hold true for gravitational radiation. In order to build a geon (both electromagnetic and gravitational), the constraint of the spherical box (with reflecting walls) would have to be removed. This requirement leads to the impossibility of constructing any type of geon using a relativistic perfect fluid model. Weinberg [75] has shown that a highly relativistic fluid with $p = \rho/3$ can never achieve hydrostatic equilibrium in a finite ball through gravity alone. The relativistic equation for hydrostatic equilibrium is

$$-\frac{\partial p}{\partial x^\lambda} = (p+\rho)\frac{\partial}{\partial x^\lambda}\ln\left((-g_{00})^{1/2}\right) . \tag{10.1}$$

For $p = \rho/3$, $\rho \propto (-g_{00})^{-(p+\rho)/2p}$. Since ρ must vanish outside the fluid, g_{00} would have to become singular at its surface. If the surface of the ball is allowed to extend to spatial infinity, then from Sokolov's work [46], it is not difficult to establish that the mass of the radiation ball diverges (also see [56], p 615, ex. 23.10).

Another work which casts doubt on the existence of gravitational geons is that of Gibbons and Stewart [76]. They conclude that the Einstein equations do not permit asymptotically flat solutions which are both periodic and empty near infinity. This precludes the existence of gravitational geons for at least the periodic case. They also suggest that the result may be extended to include the case when there is matter near infinity, such as electromagnetic or scalar radiation.

Traditionally, the geon was conceived as a structure of small-amplitude high-frequency gravitational waves compactified to the point where one could describe the resulting metric as the averaged 'background' metric induced by the totality of the waves plus a small perturbation due to the local wave presence. This is what was analyzed in the present work. It is natural to consider also waves of 'large' amplitude in which case linearization is no longer possible nor is it meaningful to envisage a splitting of the metric as before. In fact, to assign a measure to amplitude presupposes a standard for comparison and in the present work, the background metric served this role. To speak now of large amplitude is to consider waves for which there is no longer a discernible 'background' and hence no standard for comparison of amplitude measure. This leads to the realm of exact solutions. One might ask whether an exact wave-like solution of the Einstein equations, singularity-free with localized curvature and asymptotically flat, could exist. Existing exact wave-like solutions such as the plane waves of Bondi, Pirani and Robinson or the cylindrical waves of Bonnor or Einstein and Rosen [77,78] are not localized and in the latter case, are also not singularity-free. While it would appear doubtful that solutions with the geon-like properties can exist, to our knowledge they are not ruled out.

Implicit in the gravitational geon concept is the assumption that the gravitational field has some particular essential features shared by other fields. Other fields, even in their pure states, carry energy. Energy has a mass equivalent and all masses gravitate. Thus, given a sufficient concentration of field energy, one could imagine a gravitated concentration into a spherical region with the effective mass displayed unambiguously by the coefficient of the $1/r$ part of the asymptotic static vacuum metric. The gravitational geon concept is built upon the assumption that the gravitational field itself, even in its pure state, will gravitate and thus have the potential to behave as other concentrations of matter or fields. Through the years, various authors such as Isaacson [38] have dwelt upon the similarities between the gravitational and other fields. For example, Isaacson has attempted to establish that there is a basis for considering a certain construct of the metric as an energy-momentum tensor of the gravitational field which is as substantial as a true energy-momentum tensor. However, this requires averaging and under the appropriate limits, his construct merges with the energy-momentum pseudotensor, the shortcomings of which epitomize the gravitational energy problem. If the gravitational field in its pure form really did have all the properties which those authors have ascribed to it, then it would seem reasonable to expect that a gravitational geon could, at the very least in principle, be constructed. However, given the present results, it is worth considering alternative ideas.

It has been demonstrated by Nissani and Leibowitz [79,80] that there exist global preferred coordinate systems in which an ordinary continuity equation for the energy-momentum tensor holds globally. It has been put forth in

[79,80] the idea that these preferred frames have special physical significance. In this so-called 'nonrotating' class of geodesic coordinate systems, the local expression for conservation of energy-momentum

$$\sqrt{-g}\,T^{\alpha\beta}{}_{;\beta} = \left(\sqrt{-g}\,T^{\alpha\beta}\right)_{,\beta} + \sqrt{-g}\,\Gamma^{\alpha}_{\gamma\beta}\,T^{\beta\gamma} = 0 \tag{10.2}$$

transforms to the globally conserved expression

$$\left(\sqrt{-g}\,T^{\alpha\beta}\right)_{,\beta} = 0\,. \tag{10.3}$$

Whether the above expression for the conserved energy-momentum does or does not include a gravitational contribution depends on the physical interpretation of $T^{\alpha\beta}$ [81,82].

Recently, Cooperstock [44,83] introduced a new hypothesis that gravitational energy is localized in regions of nonvanishing energy-momentum tensor. The motivation derived from the fact that the traditional means by which physicists have identified gravitational energy was through the covariant energy-momentum conservation laws (10.2). While those laws were extrapolated to produce energy-momentum pseudotensors, implying densities and fluxes even in vacuum, the fact is that the laws themselves are devoid of content in vacuum, producing the empty identity $0 = 0$. Given that there is a plethora of possible pseudotensors and, as their name implies, they are not really tensors, it was suggested [44] that the root of the ambiguity lies in the extrapolation of the conservation laws to regions in which they are without actual content. The hypothesis goes on to propose that the true expression of the gravitational contribution to energy is confined to regions of nonvanishing $T_{\mu\nu}$. In a sense this is the opposite of the Isaacson approach in that rather than being satisfied with a construct which reduces to the pseudotensor, the new hypothesis suggests that proper localization is realized when the pseudotensor is removed.

Clearly, the realization of a gravitational geon would negate the new hypothesis as it would provide an example of a space totally free of true energy-momentum tensor $T_{\mu\nu}$ yet exhibit an unambiguous energy content via its asymptotic metric. While one might propose exact plane gravitational wave solutions as counter-examples to the hypothesis, it is to be noted that these are unbounded fields with questionable relevance to physical situations and more directly, these wave solutions can be expressed in Kerr–Schild form for which the pseudotensor vanishes in its entirety [84]. The gravitational geon is a direct challenge to the hypothesis and if the geon cannot exist, the hypothesis has passed another test.

Chapter 11

Conclusions on the Geon Problem

The construction of a satisfactory gravitational geon model requires an asymptotically flat, self-consistent solution of the Einstein field equations which meets the regularity conditions for a singularity-free space-time. Furthermore, it must be demonstrated that the evolution in time of the geon must take place on a time-scale much longer than the characteristic period of the constituent waves (quasi-stability property).

To satisfy these conditions, it was proposed [39–41] that a satisfactory gravitational geon model must be constructed in a manner similar to that of Wheeler's [36] electromagnetic geon. This type of model for the gravitational geon is in contrast to the thin-shell model of Brill and Hartle [37]. In order to construct a gravitational geon in principle, it was established [38–41] that gravitational waves of high-frequency were necessary. From a mathematical point of view, the main difference between our approach to the geon problem, as compared to that of BH, consists in our explicit use of the high-frequency approximation in conjunction with solving explicitly for the wave and background metric functions in a self-consistent manner. In addition, it was shown that the thin-shell model does not satisfy the Darmois junction conditions for regularity and thus the model cannot be considered singularity-free.

The application of the high-frequency approximation reduced the gravitational and electromagnetic geon problem to the same set of ordinary differential equations and boundary conditions. Since the background metric is initially assumed static, any solutions are necessarily equilibrium solutions. From a phase portrait analysis of the ordinary differential equations governing gravitational and electromagnetic geons, it was possible to determine both the existence and stability properties of equilibrium solutions. It was found that admissible equilibrium solutions were unstable to changes in the amplitude eigenvalues. Since a basic requirement for the existence of both types geon is the quasi-stability property, an investigation of the time-evolution of an electromagnetic geon was performed. In contrast to other investigations, a small amplitude time-dependent perturbation to an equilibrium solution was applied. The time-averaging problem is solved by assuming the characteristic frequency of the perturbations vary on a time-scale much longer than that of the waves comprising the electromagnetic geon. This is in accordance with the requirement that the background metric be quasi-stable. Solving the time-dependent perturbation equations leads to the characteristic frequency of the perturbations being of the same order in magnitude as the waves comprising the electromagnetic geon. This is a contradiction to the original assumption. Thus it could not be shown that the time-evolution of the electromagnetic geon proceeds on a slow time-scale. With not all of the requirements for the existence of an electromagnetic geon being satisfied, it cannot be concluded that an electromagnetic or a gravitational geon is a viable entity.

Bibliography

[1] B. M. Barker and R. F. O'Connell, *Phys. Lett. A* **61**, 297 (1977).

[2] T. Kimura and T. Ohta, *Phys. Lett.* **63A**, 193 (1977).

[3] W. B. Bonnor, *Phys. Lett.* **83A**, 414 (1981).

[4] W. B. Bonnor, *Class. Quantum Grav.* **10**, 2077 (1993).

[5] H. Weyl, *Ann. Phys. (Leipzig)***54**, 117 (1917).

[6] R. Bach and H. Weyl, *Math. Z.* **13**, 134 (1922).

[7] A. Papapetrou, *Proc. R. Irish Acad.* **51**, 191 (1947); S. D. Majumdar, *Phys. Rev.* **72**, 390 (1947).

[8] H. E. J. Curzon, *Proc. Lond. Math. Soc.* **23**, 477 (1924).

[9] G. Erez and N. Rosen, *Bull. Res. Council Israel* **8F**, 47 (1959).

[10] F. I. Cooperstock and V. de la Cruz, *Gen. Rel. Grav.* **10**, 681 (1979).

[11] A. Tomimatsu, *Prog. Theor. Phys.* **71**, 409 (1984).

[12] D. Kramer, *Class. Quantum Grav.* **5**, 1435 (1988).

[13] E. Herlt, *Gen. Rel. Grav.* **9**, 711 (1978); *Gen. Rel. Grav.* **11**, 337 (1979).

[14] J. Carminati, *Gen. Rel. Grav.* **13**, 1185 (1981).

[15] J. Carminati and F. I. Cooperstock, *Class. Quantum Grav.* **8**, L171 (1991); *Gen. Rel. Grav.* **24**, 881 (1992).

[16] J. Bičak and C. Hoenselaers, *Phys. Rev. D* **31**, 2476 (1985).

[17] G. P. Perry and F. I. Cooperstock, *Class. Quantum Grav.* **14**, 1329 (1991).

[18] N. R. Sibgatullin, *Oscillations and Waves in Strong Gravitational and Electromagnetic Fields* (Nauka, Moskow, 1984), in Russian (English translation: Springer-Verlag, Berlin, 1991).

[19] F. J. Ernst, *Phys. Rev.* **168**, 1415 (1968).

[20] V. S. Manko, J. Martín, and E. Ruiz, *J. Math. Phys.* **35**, 6644 (1994).

[21] A. Chamorro, V. S. Manko and N. R. Sibgatullin, *Lect. Notes Phys.* **423**, 119 (1993).

[22] F. J. Ernst, *Phys. Rev. D* **50**, 4993; 6179 (1994).

[23] W. Dietz and C. Hoenselaers, *Ann. Phys. (USA)* **165**, 319 (1985).

[24] H. Stephani, *General Relativity* (Cambridge University Press, Cambridge, 1990), 2nd edn.

[25] G. P. Perry, in *Proceedings of the 5th Canadian Conference on General Relativity and Relativistic Astrophysics*, ed. by R. B. Mann and R. G. McLenaghan (World Scientific, Singapore, 1994) p 449.

[26] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, Oxford, 1975), 4th revised English edn.

[27] G. P. Perry and F. I. Cooperstock, in *Proceedings of the 14th International Conference on General Relativity and Gravitation*, ed. by M. Francaviglia, G. Longhi, L. Lusanna and E. Sorace (World Scientific, Singapore, 1997) p 420; G. P. Perry and F. I. Cooperstock, in *Abstracts of contributed talks of the 14th International Conference on General Relativity and Gravitation*, Florence, Italy, Aug 1995, p A60.

[28] G. P. Perry and F. I. Cooperstock, in *Proceedings of the 6th Canadian Conference on General Relativity and Relativistic Astrophysics, Fields Inst. Comm.* **15**, ed. by S. P. Braham, J. D. Gegenberg and R. J. McKellar (American Mathematical Society, Providence, 1997) p 319.

[29] V. S. Manko and N. R. Sibgatullin, *Phys. Lett. A* **168**, 343 (1992); *J. Math. Phys.* **34**, 170 (1993).

[30] V. S. Manko and N. R. Sibgatullin, *Class. Quantum Grav.* **10**, 1390 (1993).

[31] W. Kinnersley, *J. Math. Phys.* **14**, 651 (1973).

[32] W. Simon, *J. Math. Phys.* **25**, 1035 (1984).

[33] C. Hoenselaers and Z. Perjés, *Class. Quantum Grav.* **7**, 1819 (1990).

[34] J. L. Synge, *Relativity: The General Theory* (North Holland, Amsterdam, 1966) p 313.

[35] J. M. Aguirregabiria, A. Chamorro, J. Suinaga and C. V. Vishveshwara, *Class. Quantum Grav.***12**, 699 (1995); **13**, 417 (1996).

[36] J. A. Wheeler, *Phys. Rev.* **97**, 511 (1955).

[37] D. R. Brill and J. B. Hartle, *Phys. Rev.* **135**, B271 (1964).

[38] R. A. Isaacson, *Phys. Rev.* **166**, 1263; 1272 (1968).

[39] F. I. Cooperstock, V. Faraoni and G. P. Perry, *Mod. Phys. Lett. A***10**, 359 (1995).

[40] F. I. Cooperstock, V. Faraoni and G. P. Perry, in *Proceedings of the 7th Marcel Grossmann Meeting on General Relativity*, ed. by R. T. Jantzen and G. MacKeiser (World Scientific, Singapore, 1996) p 351.

[41] F. I. Cooperstock, V. Faraoni and G. P. Perry, *Int. J. Mod. Phys. D***5**, 375 (1996).

[42] P. R. Anderson and D. R. Brill, *Phys. Rev. D***56**, 4824 (1997).

[43] D. R. Brill, in *Perspectives in Geometry and Relativity, Essays in honor of Václav Hlavatý*, ed. by B. Hoffmann (Indiana University Press, Bloomington, 1966) p 38.

[44] F. I. Cooperstock, *Found. Phys.* **22**, 1011 (1992).

[45] R. D. Sorkin, R. M. Wald and Z. Z. Jiu, *Gen. Rel. Grav.* **13**, 1127 (1981).

[46] S. N. Sokolov, *Gen. Rel. Grav.* **24**, 519 (1992).

[47] E. A. Power and J. A. Wheeler, *Rev. Mod. Phys.* **29**, 480 (1957).

[48] D. R. Brill and J. A. Wheeler, *Rev. Mod. Phys.* **29**, 465 (1957).

[49] F. J. Ernst, *Phys. Rev.* **105** 1662; 1665; *Rev. Mod. Phys.* **29**, 496 (1957).

[50] J. A. Wheeler, *Rev. Mod. Phys.* **33**, 63 (1961).

[51] J. A. Wheeler, *Geometrodynamics* (Academic Press, New York, 1962).

[52] T. Regge and J. A. Wheeler, *Phys. Rev.* **108**, 1063 (1957).

[53] U. H. Gerlach, *Phys. Rev. Lett.* **25**, 1771 (1970).

[54] J. M. Cohen and R. M. Wald, *J. Math. Phys.* **13**, 543 (1972).

[55] S. Coleman and L. Smarr, *Comm. Math. Phys.* **56**, 1 (1977).

[56] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[57] F. K. Manasse, *J. Math. Phys.* **4**, 746 (1963).

[58] F. J. Zerilli, *J. Math. Phys.* **11**, 2203 (1970).

[59] V. D. Sandberg, *J. Math. Phys.* **19**, 2441 (1978).

[60] K. S. Thorne, *Rev. Mod. Phys.* **52**, 299 (1980).

[61] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Clarendon Press, Oxford, 1983).

[62] J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton University Press, Princeton, 1987).

[63] M. A. H. MacCallum and A. H. Taub, *Comm. Math. Phys.* **30**, 153 (1973).

[64] M. E. de Araujo, *Gen. Rel. Grav.* **18**, 219 (1986); *Gen. Rel. Grav.* **21**, 323 (1989).

[65] L. A. Edelstein and C. V. Vishveshwara, *Phys. Rev. D***1**, 3514 (1970).

[66] F. J. Zerilli. *Phys. Rev. Lett.***24**, 737 (1970).

[67] W. B. Bonnor and P. A. Vickers, *Gen. Rel. Grav.* **13**, 29 (1981); G. Darmois. *Mémorial des Sciences Mathematiques* (Gauthier-Villars. Paris. 1927), Fasc. 25; S. O'Brien and J. L. Synge. *Jump Conditions at Discontinuities in General Relativity* (Dublin Institute for Advanced Studies, 1952); A. Lichnerowicz. *Théories Relativistes de la Gravitation et de l'Électromagnétisme* (Masson, Paris, 1955).

[68] W. Israel, *Nuovo Cimento B***44**, 1 (1966).

[69] D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations* (Oxford University Press, Oxford, 1987), 2nd edn.

[70] F. J. Ernst, *Phys. Rev.* **105**, 1665 (1957).

[71] S. K. Krane, *Introductory Nuclear Physics* (John Wiley & Sons, New York, 1988).

[72] R. Eisberg and R. Resnick, *Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles* (John Wiley & Sons, New York, 1985) 2nd edn. p 203.

[73] V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, New York, 1978).

[74] B. K. Harrison, K. S. Thorne, M. Wakano and J. A. Wheeler, *Gravitation Theory and Gravitational Collapse* (The University of Chicago Press, 1965) p 56.

[75] S. Weinberg, *Gravitation and Cosmology, Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, New York, 1972).

[76] G. W. Gibbons and J. M. Stewart, in *Classical General Relativity*, ed. by W. B. Bonnor, J. Islam and M. A. H. MacCallum (Cambridge University Press, 1984) p 77.

[77] W. B. Bonnor, *J. Math. and Mech.* **6**, 203 (1957).

[78] D. Kramer, H. Stephani, M. MacCallum and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, 1980).

[79] N. Nissani and E. Leibowitz, *Phys. Lett. A* **126**, 447 (1988).

[80] N. Nissani and E. Leibowitz, *Int. J. Theor. Phys.* **28**, 235 (1989).

[81] N. Nissani and E. Leibowitz, *Int. J. Theor. Phys.* **30**, 837 (1991).

[82] N. Nissani and E. Leibowitz, *Int. J. Theor. Phys.* **31**, 2065 (1992).

[83] F. I. Cooperstock, in *Topics in Quantum Gravity and Beyond*, ed. by F. Mansouri and J. J. Scanio (World Scientific, Singapore, 1993) p 201.

[84] M. Gürses and F. Gürsey, *J. Math. Phys.* **16**, 2385 (1975).

Appendix A

Explicit Form of α_n

Equation (3.11) can be written in the form

$$\xi^4 + a\,\xi^3 + b\,\xi^2 + c\,\xi + d = 0\,, \tag{A.1}$$

where

$$a = 2\,(z_1 + z_2)\;, \tag{A.2}$$

$$b = q_1^2 - m_1^2 + q_2^2 - m_2^2 + z_1^2 + z_2^2 + 4 z_1 z_2\,, \tag{A.3}$$

$$c = 2\left(z_1\left(q_2^2 - m_2^2\right) + z_2\left(q_1^2 - m_1^2\right) + z_1\,z_2\,(z_1 + z_2)\right)\;. \tag{A.4}$$

$$d = z_1^2\left(q_2^2 - m_2^2\right) + z_2^2\left(q_1^2 - m_1^2\right) + z_1^2 z_2^2 + \left(m_1 m_2 - q_1 q_2\right)^2\;. \tag{A.5}$$

Let α_n represent the roots of equation (A.1), i.e.

$$\prod_{n=1}^{4}\left(\xi - \alpha_n\right) = 0\,. \tag{A.6}$$

Solving equation (A.1) in terms of the constants a, b, c and d yields

$$\begin{aligned}\alpha_{1,2} &= -\frac{a}{4}+\frac{\sqrt{3}}{12}\sqrt{J_4}\pm\frac{\sqrt{6}}{12\,J_4^{1/4}}\left(\sqrt{J_4}\left(3\,a^2-8\,b-6\,J_3^{1/3}-6\,J_2^{1/3}\right)\right.\\&\quad\left.-\sqrt{3}\left(3\,a^2-12\,a\,b+24\,c\right)\right)^{1/2},\end{aligned}\tag{A.7}$$

$$\begin{aligned}\alpha_{3,4} &= -\frac{a}{4}-\frac{\sqrt{3}}{12}\sqrt{J_4}\pm\frac{\sqrt{6}}{12\,J_4^{1/4}}\left(\sqrt{J_4}\left(3\,a^2-8\,b-6\,J_3^{1/3}-6\,J_2^{1/3}\right)\right.\\&\quad\left.+\sqrt{3}\left(3\,a^2-12\,a\,b+24\,c\right)\right)^{1/2},\end{aligned}\tag{A.8}$$

with

$$J_4 \equiv 3\,a^2-8\,b+12\left(J_3^{1/3}+J_2^{1/3}\right),\tag{A.9}$$

$$J_3 \equiv -\frac{1}{6}\,a\,b\,c-\frac{3}{4}\,b\,d+\frac{1}{2}\,c^2+\frac{1}{2}\,d\,a^2+\frac{1}{27}\,b^3+\frac{1}{18}\,J_1,\tag{A.10}$$

$$J_2 \equiv -\frac{1}{6}\,a\,b\,c-\frac{3}{4}\,b\,d+\frac{1}{2}\,c^2+\frac{1}{2}\,d\,a^2+\frac{1}{27}\,b^3-\frac{1}{18}\,J_1.\tag{A.11}$$

$$\begin{aligned}J_1 &\equiv 27\,a^4\,d^2-18\,a^3\,b\,c\,d+4\,a^3\,c^3+4\,a^2\,b^3\,d-a^2\,b^2\,c^2-144\,a^2\,b\,d^2\\&\quad+6\,a^2\,c^2\,d+80\,a\,b^2\,c\,d18\,a\,b\,c^3-16\,b^4\,d+4\,b^3\,c^2+192\,a\,c\,d^2\\&\quad+128\,b^2\,d^2-144\,b\,c^2\,d+27\,c^4-256\,d^3.\end{aligned}\tag{A.12}$$

Using equations (A.2)-(A.5) in equations (A.7) and (A.8), we obtain α_n explicitly in terms of the constants m_1, m_2, q_1, q_2, z_1 and z_2.

Appendix B

Derivation of Equation (7.10)

We start from the Legendre equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(1-x^2\right)\frac{\mathrm{d}P_l(x)}{\mathrm{d}x}\right)+l(l+1)P_l(x)=0 \tag{B.1}$$

and note that

$$\Theta_l(\theta)=C_l^0\sin\theta\,\frac{\mathrm{d}}{\mathrm{d}\theta}P_l(\cos\theta)=C_l^0\left(x^2-1\right)\frac{\mathrm{d}P_l(x)}{\mathrm{d}x}\,. \tag{B.2}$$

where $x=\cos\theta$. Using

$$\frac{\mathrm{d}}{\mathrm{d}\theta}=-\sin\theta\,\frac{\mathrm{d}}{\mathrm{d}x}\,, \tag{B.3}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}=\sin^2\theta\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\cos\theta\,\frac{\mathrm{d}}{\mathrm{d}x}\,, \tag{B.4}$$

and the Legendre equation (B.1), we find the relations

$$\frac{\mathrm{d}\Theta_l}{\mathrm{d}\theta}=-l(l+1)C_l^0\sin\theta\,P_l(x)\,, \tag{B.5}$$

$$\frac{\mathrm{d}^2\Theta_l}{\mathrm{d}\theta^2}=-l(l+1)C_l^0\left(xP_l(x)+\left(x^2-1\right)\frac{\mathrm{d}P_l(x)}{\mathrm{d}x}\right)\,. \tag{B.6}$$

Using equations (B.5) and (B.2) in (B.6), equation (7.10) follows.

Appendix C

Junction Conditions for the BH Background Metric

We consider the Darmois junction conditions [67] for the BH background metric on the time-like hypersurface

$$S \equiv \{(t, r, \theta, \varphi) : \; r = a\} \tag{C.1}$$

separating the regions of the space-time manifold

$$U \equiv \{(t, r, \theta, \varphi) : \; r < a\} \; , \tag{C.2}$$

$$\overline{U} \equiv \{(t, r, \theta, \varphi) : \; r > a\} \; . \tag{C.3}$$

The sets $\{x^\alpha\} = \{\overline{x}^\alpha\} = \{t, r, \theta, \varphi\}$ and $\{u^i\}_{i=0,2,3} = \{t, \theta, \varphi\}$ are coordinate systems in U, $\overline{U}$ and S, respectively (note that in this appendix, Latin indices assume the values $0, 2, 3$ due to the time-like character of S). The unit normal to S is directed along the coordinate basis vector dual to $\mathrm{d}r$ and has components

$$n_\mu = \delta^1_\mu \, \mathrm{e}^{\lambda/2} \, . \tag{C.4}$$

The metric components $\gamma_{\mu\nu}$ in U and $\overline{\gamma}_{\mu\nu}$ in $\overline{U}$ are given by equations (6.2), (7.19) and (7.20). The first fundamental form of S has components $\gamma_{ij} = \overline{\gamma}_{ij}$. The second fundamental form $K_{\mu\nu} \equiv n_{\mu;\nu}$ of any hypersurface r = constant has components

$$K_{ij} = n_{\alpha;\beta}\,\frac{\partial x^\alpha}{\partial u^i}\,\frac{\partial x^\beta}{\partial u^j} = -\Gamma^1_{ij}\,\mathrm{e}^{\lambda/2} \tag{C.5}$$

in coordinates $\{u^i\}$. Using the Christoffel symbols of a spherically symmetric metric (see, for example, [26]), we obtain the only nonvanishing components

$$K_{00} = -\frac{\nu'}{2}\,\mathrm{e}^{\nu-\lambda/2}\,, \tag{C.6}$$

$$K_{22} = r\,\mathrm{e}^{-\lambda/2}\,, \tag{C.7}$$

$$K_{33} = r\,\mathrm{e}^{-\lambda/2}\sin^2\theta\,. \tag{C.8}$$

The Darmois conditions [67] require the continuity of the first and second fundamental form across S. The first condition is trivially satisfied, while the second is violated. In fact, we have

$$\lim_{r\to a^-} K_{00} = 0 \neq \lim_{r\to a^+} K_{00} = -\frac{16}{27M}\,, \tag{C.9}$$

$$\lim_{r\to a^-} K_{22} = a \neq \lim_{r\to a^+} K_{22} = \frac{a}{3}\,, \tag{C.10}$$

$$\lim_{r\to a^-} K_{33} = a\sin^2\theta \neq \lim_{r\to a^+} K_{33} = \frac{a}{3}\sin^2\theta\,, \tag{C.11}$$

where the BH relation $a = 9M/4$ was used.

Appendix D

Dominant Order in $R^{(1)}_{\alpha\beta}$

The second covariant derivatives appearing in equation (6.17) are

$$
\begin{aligned}
h_{\mu\nu;\alpha\beta} &= h_{\mu\nu,\alpha\beta} - \Gamma^{\sigma}_{\alpha\beta}h_{\mu\nu,\sigma} - \Gamma^{\sigma}_{\beta\mu}h_{\sigma\nu,\alpha} - \Gamma^{\sigma}_{\beta\nu}h_{\sigma\mu,\alpha} - \Gamma^{\sigma}_{\alpha\mu,\beta}h_{\sigma\nu} - \Gamma^{\sigma}_{\alpha\mu}h_{\sigma\nu,\beta} \\
&+\Gamma^{\sigma}_{\alpha\beta}\Gamma^{\rho}_{\sigma\mu}h_{\rho\nu} + \Gamma^{\sigma}_{\beta\mu}\Gamma^{\rho}_{\alpha\sigma}h_{\rho\nu} + \Gamma^{\sigma}_{\beta\nu}\Gamma^{\rho}_{\alpha\mu}h_{\rho\sigma} - \Gamma^{\sigma}_{\alpha\nu,\beta}h_{\sigma\mu} \\
&-\Gamma^{\sigma}_{\alpha\nu}h_{\sigma\mu,\beta} + \Gamma^{\sigma}_{\alpha\beta}\Gamma^{\rho}_{\sigma\nu}h_{\rho\mu} + \Gamma^{\sigma}_{\beta\nu}\Gamma^{\rho}_{\alpha\sigma}h_{\rho\mu} + \Gamma^{\sigma}_{\beta\mu}\Gamma^{\rho}_{\alpha\nu}h_{\rho\sigma} \ . \qquad \text{(D.1)}
\end{aligned}
$$

Symbolically, we express the various quantities in the last equation as follows:

$$\Gamma = \gamma\,\partial\gamma = \mathrm{O}(1) \ , \qquad \text{(D.2)}$$

$$\gamma\,\partial h = \mathrm{O}(1) \ , \qquad \text{(D.3)}$$

$$(\partial\gamma)h = \mathrm{O}(\epsilon) \ , \qquad \text{(D.4)}$$

$$h\,\partial h = \mathrm{O}(\epsilon) \ , \qquad \text{(D.5)}$$

$$\Gamma\,\partial h = \mathrm{O}(1) \ , \qquad \text{(D.6)}$$

$$(\partial\Gamma)h = \mathrm{O}(\epsilon) \ , \qquad \text{(D.7)}$$

$$\Gamma\,\Gamma\,h = \mathrm{O}(\epsilon) \ . \qquad \text{(D.8)}$$

By using equations (D.2)–(D.8) in (D.1) and then, in conjunction with (6.17), equation (8.1) follows. The quantity $(h_{\rho\tau,\alpha\beta} + h_{\alpha\beta,\rho\tau} - h_{\tau\alpha,\beta\rho} - h_{\tau\beta,\alpha\rho})$ in equation (8.1) contains terms of order $O(1/\epsilon)$ as well as terms of order $O(1)$. We retain only the former ones in the linearized Einstein equations to order $O(1/\epsilon)$.

Appendix E

Angle Average of $T_\mu{}^\nu$ in the High-Frequency Limit

Equations (8.14)–(8.16) includes integrating over the angle φ and dividing by the solid angle 4π, thus all that is left is evaluating the θ integrals (see, for example, [36]). The θ dependence of $T_\mu{}^\nu$ comes in three forms

$$\sin^{-2}\theta\,(\Theta_l(\theta))^2\,,\ \sin^{-2}\theta\,(\Theta_l(\theta)_{,\theta})^2 \text{ and } \sin^{-2}\theta\;\Theta_l(\theta)\Theta_l(\theta)_{,\theta\theta}\,, \tag{E.1}$$

where

$$\Theta_l(\theta) = C_l^0 B_l(\theta) \tag{E.2}$$

and

$$B_l(\theta) \equiv \sin\theta\,\frac{\mathrm{d}}{\mathrm{d}\theta}\,P_l(\cos\theta)\,. \tag{E.3}$$

The exact integrals are evaluated below with the last equality being the value used for the high-frequency approximation:

$$\int_0^\pi \sin^{-2}\theta\,(B_l(\theta))^2\sin\theta\,\mathrm{d}\theta = \frac{2l(l+1)}{2l+1} \approx l\,, \tag{E.4}$$

$$\int_0^{\pi} \sin^{-2}\theta\,(B_l(\theta)_{,\theta})^2 \sin\theta\,\mathrm{d}\theta = \frac{2l^2(l+1)^2}{2l+1} \approx l^3\,. \tag{E.5}$$

$$\int_0^{\pi} \sin^{-2}\theta B_l(\theta) B_l(\theta)_{,\theta\theta} \sin\theta\,\mathrm{d}\theta = -\frac{2l^3(l+1)}{2l+1} \approx -\,l^3\,. \tag{E.6}$$

The normalization constant for $\Theta_l(\theta)$ is found by requiring

$$\int_0^{2\pi}\int_0^{\pi} |\Theta_l(\theta)|^2 \sin\theta\,\mathrm{d}\theta\mathrm{d}\varphi = 1\,. \tag{E.7}$$

Therefore

$$(C_l^0)^2 = \frac{1}{2\pi}\left(\int_0^{\pi} (B_l(\theta))^2 \sin\theta\,\mathrm{d}\theta\right)^{-1} = \frac{1}{2\pi}\left(\frac{4l^2(l+1)^2}{(2l-1)(2l+1)(2l+3)}\right)^{-1}. \tag{E.8}$$

Thus the normalization constant is

$$C_l^0 = \left(\frac{(2l-1)(2l+1)(2l+3)}{8\pi l^2(l+1)^2}\right)^{1/2} \approx \frac{1}{\sqrt{\pi l}}\,. \tag{E.9}$$

Appendix F

Stability of a Linear Geon

The linear electromagnetic geon differential equations as given in the appendix of [70] are as follows:

$$B'' + \left(\mathrm{e}^{\delta} - 1\right)\mathrm{e}^{-2\lambda}B = 0\,, \tag{F.1}$$

$$l'\lambda' + l\lambda'' + \tfrac{1}{2}\left(B'\right)^2 + \tfrac{1}{2}\mathrm{e}^{-2\lambda}B^2 = 0\,, \tag{F.2}$$

$$l'' + \tfrac{1}{2}\left(B'\right)^2 - \tfrac{1}{2}\left(\mathrm{e}^{\delta} - 1\right)\mathrm{e}^{-2\lambda}B^2 = 0\,, \tag{F.3}$$

$$\begin{aligned}\delta'' + \lambda'\delta' + (2l)^{-1}\left(\left(B'\right)^2 + \left(1 + 3\mathrm{e}^{\delta}\right)\mathrm{e}^{-2\lambda}B^2\right)& \\ + 4l^{-1}l'\lambda' - \tfrac{3}{2}l^{-2}\left(l'\right)^2 - 2\left(\lambda'\right)^2 = 0\,,&\end{aligned} \tag{F.4}$$

where the functions B, l, δ and λ are dependent on the radial coordinate r^* and a prime denotes differentiation with respect to r^* (note: $r^* = r^*(r)$, where r is a radial coordinate. See equation (F.51) for the definition of r^* in terms of r). The above equations can be expressed as the set of first order equations

$$B' = \beta\,, \tag{F.5}$$

$$\lambda' = \alpha\,, \tag{F.6}$$

$$l' = L\,, \tag{F.7}$$

$$\delta' = \Delta\,, \tag{F.8}$$

$$\beta' = -\mathrm{e}^{-2\lambda} B\left(\mathrm{e}^{\delta} - 1\right)\,, \tag{F.9}$$

$$\alpha' = -l^{-1}\left(L\alpha + \tfrac{1}{2}\beta^2 + \tfrac{1}{2}\mathrm{e}^{-2\lambda}B^2\right)\,, \tag{F.10}$$

$$L' = \tfrac{1}{2}\left(\mathrm{e}^{\delta} - 1\right)\mathrm{e}^{-2\lambda}B^2 - \tfrac{1}{2}\beta^2\,, \tag{F.11}$$

$$\begin{aligned}\Delta' = {} & \tfrac{3}{2}l^{-2}L^2 + 2\alpha^2 - \big(\alpha\Delta \\ & + (2l)^{-1}\left(\beta^2 + \left(1 + 3\mathrm{e}^{\delta}\right)\mathrm{e}^{-2\lambda}B^2\right) + 4l^{-1}L\alpha\big)\,.\end{aligned} \tag{F.12}$$

The critical points of the system are characterized by those points where the derivatives B', λ', l', δ', β', α' and Δ' are zero. One set of critical points is found at the coordinates[1]

$$\begin{gathered}\alpha = 0, \quad \beta = 0, \quad L = 0, \quad \Delta = 0, \\ B = 0, \quad \lambda = s_1, \quad \delta = s_2, \quad l = s_3, \qquad s_1, s_2, s_3 \in \mathbb{R}\end{gathered} \tag{F.13}$$

This is the critical point of interest since it is required that the boundary condition $B = 0$ be satisfied for $r \to \infty$. There are critical points for $B \neq 0$ which influence the phase space trajectories. However, these critical points do not alter the stability property of the solutions we are interested in.

It is useful to shift the critical point to the origin with the following transformation:

$$\beta(r^*) = f_1(r^*)\,, \qquad \alpha(r^*) = f_2(r^*)\,, \tag{F.14}$$

$$L(r^*) = f_3(r^*)\,, \qquad \Delta(r^*) = f_4(r^*)\,, \tag{F.15}$$

$$B(r^*) = f_5(r^*)\,, \qquad \lambda(r^*) - s_1 = f_6(r^*)\,, \tag{F.16}$$

$$\delta(r^*) - s_2 = f_7(r^*)\,, \qquad l(r^*) - s_3 = f_8(r^*)\,. \tag{F.17}$$

[1]It is shown in the addendum at end of this appendix that $\lambda = s_1 \to -\infty$ and/or $l = s_3 \to 0$ are critical points in the asymptotic region $r \to \infty$.

Therefore the field equations become

$$f_5' = f_1\,, \tag{F.18}$$

$$f_6' = f_2\,, \tag{F.19}$$

$$f_8' = f_3\,, \tag{F.20}$$

$$f_7' = f_4\,, \tag{F.21}$$

$$f_1' = -\mathrm{e}^{-2(f_6+s_1)}\left(\mathrm{e}^{f_7+s_2} - 1\right) f_5\,, \tag{F.22}$$

$$f_2' = -\left(f_8 + s_3\right)^{-1}\left(f_3 f_4 + \tfrac{1}{2} f_1^2 + \tfrac{1}{2}\mathrm{e}^{-2(f_6+s_1)} f_5^2\right)\,, \tag{F.23}$$

$$f_3' = \tfrac{1}{2}\left(\mathrm{e}^{f_7+s_2} - 1\right)\mathrm{e}^{-2(f_6+s_1)} f_5^2 - \tfrac{1}{2} f_1^2\,, \tag{F.24}$$

$$\begin{aligned} f_4' = \tfrac{3}{2}\left(f_8 + s_3\right)^{-2} f_3^2 + 2 f_2^2 - \Big(f_2 f_4 + \tfrac{1}{2}\left(f_8 + s_3\right)^{-1}\Big(f_1^2 \\ + \left(1 + 3\mathrm{e}^{f_7+s_2}\right)\mathrm{e}^{-2(f_6+s_1)} f_5^2\Big) + 4 f_2 f_3 \left(f_8 + s_3\right)^{-1}\Big)\,. \end{aligned} \tag{F.25}$$

To linearize the field equations about the critical point, a MacLaurin series of $f_i' = f_i'\left(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\right)$, $i = 1, \ldots, 8$ is taken to first order and evaluated at the critical point. The result is

$$f_1' = -\mathrm{e}^{-2s_1}\left(\mathrm{e}^{s_2} - 1\right) f_5\,, \tag{F.26}$$

$$f_2' = 0\,, \tag{F.27}$$

$$f_3' = 0\,, \tag{F.28}$$

$$f_4' = 0\,, \tag{F.29}$$

$$f_5' = f_1\,, \tag{F.30}$$

$$f_6' = f_2\,, \tag{F.31}$$

$$f_7' = f_4\,, \tag{F.32}$$

$$f_8' = f_3\,. \tag{F.33}$$

The solution to the above set of differential equations is

$$f_1 = c_7\sqrt{-a_0}\,e^{\sqrt{-a_0}\,r^*} - c_8\sqrt{-a_0}\,e^{-\sqrt{-a_0}\,r^*}\,. \tag{F.34}$$

$$f_2 = c_4\,, \tag{F.35}$$

$$f_3 = c_6\,, \tag{F.36}$$

$$f_4 = c_7\,, \tag{F.37}$$

$$f_5 = c_7\,e^{\sqrt{-a_0}\,r^*} - c_8\,e^{-\sqrt{-a_0}\,r^*}\,, \tag{F.38}$$

$$f_6 = c_1 + c_4 r^*\,, \tag{F.39}$$

$$f_7 = c_2 + c_5 r^*\,, \tag{F.40}$$

$$f_8 = c_3 + c_6 r^*\,, \tag{F.41}$$

where $a_0 \equiv e^{-2s_1}\,(e^{s_2} - 1)$ and c_i, $i = 1, \ldots, 8$ are constants. To determine the stability of the phase space trajectories about the critical point, the sign of a_0 must be determined in the asymptotic region $r \to \infty$. Since $e^{-2s_1} \geq 0$ for all s_1, the sign of a_0 depends upon whether the value of e^{s_2} (or equivalently e^{δ}) is greater than or less than 1 in the asymptotic region.

We need to determine the behaviour of $e^{\delta(r^*)}$ as $r \to \infty$. Hence, we need to determine the relationship between r and r^*. The functions $l(r)$, $\lambda(r)$, $\delta(r)$ are defined in [70] as follows:

$$l(r) \equiv R^2(r)e^{-2\Psi(r)}\,, \tag{F.42}$$

$$\lambda(r) \equiv \ln R(r) - 2\Psi(r)\,, \tag{F.43}$$

$$\delta(r) \equiv \ln\left(1 + D(r)\right)\,, \tag{F.44}$$

where $R(r)$, $\Psi(r)$ and $D(r)$ are metric functions. The asymptotic forms of

$R(r)$, $\Psi(r)$ and $D(r)$ are [70]

$$R(r) \to r\,, \tag{F.45}$$

$$\Psi(r) \to 2M \ln(r/a)\,, \tag{F.46}$$

$$D(r) \to \Omega^2 (r/a)^{8M(M-1)} - 1\,, \tag{F.47}$$

where a, M and Ω are positive constants. Substitution of (F.45)–(F.47) into (F.42)–(F.44) yields the asymptotic forms

$$l(r) \to a^{4M} r^{2-4M}\,, \tag{F.48}$$

$$\lambda(r) \to \ln\left(a^{4M} r^{2-4M}\right)\,, \tag{F.49}$$

$$\delta(r) \to \ln\left(\Omega^2 a^{8M(M-1)} r^{8M(M-1)}\right)\,. \tag{F.50}$$

The relationship between r and r^* can be found through the defining equation of r^*

$$\frac{\mathrm{d}r^*}{\mathrm{d}r} = R(r) \mathrm{e}^{-2\Psi(r)}\,. \tag{F.51}$$

In the asymptotic region $r \to \infty$,

$$r^*(r) = \int R(r) \mathrm{e}^{-2\Psi(r)} \mathrm{d}r \to \int a^{4M} r^{2-4M} \mathrm{d}r\,. \tag{F.52}$$

Hence

$$r^*(r) \to \begin{cases} \dfrac{a^{4M}}{3-4M} r^{3-4M} + \text{constant} & M \neq \frac{3}{4}, \\ a^3 \ln r + \text{constant} & M = \frac{3}{4}\,. \end{cases} \tag{F.53}$$

The behavior of r^* as $r \to \infty$ depends on the value of the parameter M. There are three cases to examine:

Case 1 $M > \frac{3}{4}$

For this case, $r^* \to -\frac{a^{x+3}}{x} r^{-x} + \text{constant}$, where $x \equiv 4M - 3 > 0$. As $r \to \infty$, r^* becomes negative. This case is unphysical.

Case 2 $M = \frac{3}{4}$

For this case, $r^* \to a^3 \ln r + \text{constant}$. Therefore $r^* \to \infty$ as $r \to \infty$. This is acceptable behaviour for r^*.

Case 3 $0 < M < \frac{3}{4}$

Let $x \equiv 3 - 4M > 0$. For this case $r^* \to \frac{a^{x-3}}{x} r^x + \text{constant}$. Therefore $r^* \to \infty$ as $r \to \infty$. This is acceptable behaviour for r^*.

Since case 1 is unphysical, it is not necessary to determine the behaviour of $e^{\delta(r^*)}$ in the limit $r \to \infty$. For cases 2 and 3, the limit of $e^{\delta(r^*)}$ as $r \to \infty$ is equivalent to the limit of $e^{\delta(r^*)}$ as $r^* \to \infty$. Substituting in (F.53) into (F.50), we find

$$\lim_{r^* \to \infty} e^{\delta(r^*)} \to \begin{cases} \lim\limits_{r^* \to \infty} e^{-3a^{-3} r^*/2} \to 0 & \text{for } M = \frac{3}{4}, \\ \lim\limits_{r^* \to \infty} \dfrac{1}{r^{*z/x}} \to 0 & \text{for } 0 < M < \frac{3}{4}, \end{cases} \tag{F.54}$$

where $z \equiv 8M(1 - M) > 0$. Therefore

$$a_0 = e^{-2\lambda(r^*)} \left(e^{\delta(r^*)} - 1 \right) < 0 \tag{F.55}$$

in the asymptotic region. This indicates that the critical point is a saddle point. Hence, only unstable admissible equilibrium solutions are permitted. (The numerical solution presented in [70] satisfies the requirement $0 < M < \frac{3}{4}$ of case 3.)

Addendum We are interested in the nature of the critical points as $r \to \infty$ (or equivalently $r^* \to \infty$), since this determines the stability properties of admissible solutions. In equation (F.13) there were no restrictions placed on the parameters s_1, s_2, s_3 defining the coordinate position of the critical point. It will be shown below that in the asymptotic region, $\lambda = s_1 \to -\infty$ and/or $l = s_3 \to 0$ are critical points. It is unclear if $\delta = s_2 \to \infty$ represents a critical point. However, it was shown in cases 2 and 3 that $\delta = s_2 \to -\infty$ in the asymptotic region. Thus the point $\delta = s_2 \to \infty$ is of no importance since solutions do not approach this point.

A straightforward calculation yields

$$\lim_{r^* \to \infty} l\,(r^*) \to e^{-a^{-3}r^*} \to 0\,, \tag{F.56}$$

$$\lim_{r^* \to \infty} \lambda\,(r^*) \to -a^{-3}r^* \to -\infty\,, \tag{F.57}$$

for $M = \frac{3}{4}$. For $0 < M \leq \frac{3}{4}$

$$\lim_{r^* \to \infty} l\,(r^*) \to xr^{*y/x} \to \begin{cases} 0 & \text{for } \frac{1}{2} < M < \frac{3}{4}, \\ \text{constant} & \text{for } M = \frac{1}{2}, \\ \infty & \text{for } 0 < M < \frac{1}{2} \end{cases} \tag{F.58}$$

and

$$\lim_{r^* \to \infty} \lambda\,(r^*) \to \ln\left(xr^{*y/x}\right) \to \begin{cases} -\infty & \text{for } \frac{1}{2} < M < \frac{3}{4}, \\ \text{constant} & \text{for } M = \frac{1}{2}, \\ \infty & \text{for } 0 < M < \frac{1}{2}, \end{cases} \tag{F.59}$$

where $y \equiv 2(1-2M)$. Thus, there is a range of M values for which $l = s_3 \to 0$ and $\lambda = s_1 \to -\infty$. Substituting the asymptotic forms (F.54) and (F.58)–(F.59) for $M \neq \frac{3}{4}$ into the right hand side (RHS) of (F.9)–(F.12) (in addition to $\alpha = \beta = L = \delta = 0$) and noting that for admissible solutions $B(r^*) \sim e^{-r^*}$

(from equation (F.38)), we find in the limit $r^* \to \infty$

$$\begin{aligned}
\text{RHS of (F.9)} &= -e^{-2\lambda}\left(e^{\delta}-1\right)B \\
&\sim r^{*-y/x}\left(r^{*-z/x}-1\right)e^{-r^*} \to 0\,. \\
\text{RHS of (F.10)} &= -\tfrac{1}{2}l^{-1}e^{-2\lambda}B^2 \\
&\sim r^{*-y/x}r^{*-y/x}e^{-2r^*} \to 0\,. \\
\text{RHS of (F.11)} &= \tfrac{1}{2}\left(e^{\delta}-1\right)e^{-2\lambda}B^2 \\
&\sim \left(r^{*-z/x}-1\right)r^{*-y/x}e^{-2r^*} \to 0\,. \\
\text{RHS of (F.12)} &= -\tfrac{1}{2}l^{-1}\left(1+3e^{\delta}\right)e^{-2\lambda}B^2 \\
&\sim r^{*-y/x}\left(1+3r^{*-z/x}\right)r^{*-y/x}e^{-2r^*} \to 0\,.
\end{aligned}$$

It can be shown that the above result also holds for $M = \frac{3}{4}$. Therefore $l = s_3 \to 0$ and $\lambda = s_1 \to -\infty$ are critical points in the asymptotic region.

Appendix G

Time-Dependent Electromagnetic Geon Equations

We start by defining the electromagnetic vector potential for one mode of the electromagnetic waves

$$A_\mu = (0, 0, 0, A_\varphi) , \tag{G.1}$$

where

$$A_\varphi = a(r, t) B_l(\theta) , \qquad B_l(\theta) = \sin\theta \frac{\mathrm{d}}{\mathrm{d}\theta} P_l(\cos\theta) . \tag{G.2}$$

The time-dependent metric is

$$\mathrm{d}s^2 = g_{\alpha\beta}\, \mathrm{d}x^\alpha \mathrm{d}x^\beta = -\mathrm{e}^\nu \mathrm{d}t^2 + \mathrm{e}^\lambda \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta\, \mathrm{d}\varphi^2 . \tag{G.3}$$

where

$$\nu = \nu(r, t), \qquad \lambda = \lambda(r, t) .$$

In the absence of charges and currents, Maxwell's equations in a curved space-time are

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\alpha}}\left(\sqrt{-g}F^{\beta\alpha}\right)=0\,, \tag{G.4}$$

$$F_{\alpha\beta,\gamma}+F_{\gamma\alpha,\beta}+F_{\beta\gamma,\alpha}=0\,, \tag{G.5}$$

where g is the determinant of the metric (G.3) and the Maxwell tensor, $F_{\alpha\beta}$ is related to the four-vector potential as $F_{\alpha\beta}=A_{\beta,\alpha}-A_{\alpha,\beta}$. The only nontrivial equation is for $\alpha=\varphi$ in (G.4). It yields the wave equation

$$\frac{\partial^2 a}{\partial r^{*2}}-\frac{l(l+1)}{r^2}\mathrm{e}^{\nu}a-\mathrm{e}^{\nu-\lambda}\frac{\partial^2 a}{\partial t^{*2}}=0\,, \tag{G.6}$$

where

$$\frac{\partial}{\partial r^*}=\mathrm{e}^{(\nu-\lambda)/2}\frac{\partial}{\partial r}\,,\qquad \frac{\partial^2}{\partial r^{*2}}=\mathrm{e}^{(\nu-\lambda)/2}\frac{\partial}{\partial r}\left(\mathrm{e}^{(\nu-\lambda)/2}\frac{\partial}{\partial r}\right)\,, \tag{G.7}$$

$$\frac{\partial}{\partial t^*}=\mathrm{e}^{(\nu-\lambda)/2}\frac{\partial}{\partial t}\,,\qquad \frac{\partial^2}{\partial t^{*2}}=\mathrm{e}^{(\nu-\lambda)/2}\frac{\partial}{\partial t}\left(\mathrm{e}^{(\nu-\lambda)/2}\frac{\partial}{\partial t}\right)\,. \tag{G.8}$$

The Einstein equations for the electromagnetic geon are

$$G_{\mu}{}^{\nu}=8\pi\left\langle T_{\mu}{}^{\nu}\right\rangle\,, \tag{G.9}$$

where $\langle\,\cdot\,\rangle$ denotes a time-space average over all N modes of the electromagnetic waves. In the equations below, the energy-momentum tensor for a single mode of electromagnetic radiation is given by

$$T_{(1)}{}_{\mu}{}^{\nu}\equiv\frac{1}{4\pi}\left(F_{\mu\sigma}F^{\sigma\nu}-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\delta_{\mu}^{\nu}\right)\,, \tag{G.10}$$

with $F_{\alpha\beta}$ defined above. We will only evaluate the angle average of $T_{\mu}{}^{\nu}$. The time-average will be dealt with in the main text (chapter 9). In addition to the three angle averages[1] [36]

$$\left\langle T_t{}^t \right\rangle_{\mathrm{TA}} = \frac{N}{2}\int_0^\pi \left\langle T_{(l)\,t}{}^t \right\rangle_{\mathrm{T}} \sin\theta \, \mathrm{d}\theta \,. \tag{G.11}$$

$$\left\langle T_r{}^r \right\rangle_{\mathrm{TA}} = \frac{N}{2}\int_0^\pi \left\langle T_{(l)\,r}{}^r \right\rangle_{\mathrm{T}} \sin\theta \, \mathrm{d}\theta \,, \tag{G.12}$$

$$\left\langle T_\theta{}^\theta \right\rangle_{\mathrm{TA}} = \left\langle T_\varphi{}^\varphi \right\rangle_{\mathrm{TA}} = \frac{N}{2}\int_0^\pi \left\langle T_{(l)\,\theta}{}^\theta + T_{(l)\,\varphi}{}^\varphi \right\rangle_{\mathrm{T}} \sin\theta \, \mathrm{d}\theta \,, \tag{G.13}$$

there is an additional average which represents the radial flow of energy

$$\left\langle T_r{}^t \right\rangle_{\mathrm{TA}} = \frac{N}{2}\int_0^\pi \left\langle T_{(l)\,r}{}^t \right\rangle_{\mathrm{T}} \sin\theta \, \mathrm{d}\theta \,. \tag{G.14}$$

Evaluating (G.11)–(G.14) using (G.1) and the integrals of appendix E one obtains

$$\left\langle T_t{}^t \right\rangle_{\mathrm{TA}} = -\frac{Nl(l+1)}{8\pi r^2(2l+1)}\left(\mathrm{e}^{-\nu}\left\langle a_{,t}^2\right\rangle_{\mathrm{T}} + \mathrm{e}^{-\lambda}\left\langle a_{,r}^2\right\rangle_{\mathrm{T}} + \frac{l(l+1)}{r^2}\left\langle a^2\right\rangle_{\mathrm{T}}\right) \,, \tag{G.15}$$

$$\left\langle T_r{}^r \right\rangle_{\mathrm{TA}} = \frac{Nl(l+1)}{8\pi r^2(2l+1)}\left(\mathrm{e}^{-\nu}\left\langle a_{,t}^2\right\rangle_{\mathrm{T}} + \mathrm{e}^{-\lambda}\left\langle a_{,r}^2\right\rangle_{\mathrm{T}} - \frac{l(l+1)}{r^2}\left\langle a^2\right\rangle_{\mathrm{T}}\right) \,, \tag{G.16}$$

$$\left\langle T_\theta{}^\theta \right\rangle_{\mathrm{TA}} = \frac{Nl^2(l+1)^2}{8\pi r^4(2l+1)}\left\langle a^2\right\rangle_{\mathrm{T}} \,, \tag{G.17}$$

$$\left\langle T_r{}^t \right\rangle_{\mathrm{TA}} = -\frac{Nl(l+1)}{4\pi \mathrm{e}^{\nu} r^2(2l+1)}\left\langle a_{,r} a_{,t}\right\rangle_{\mathrm{T}} \,. \tag{G.18}$$

[1]The symbols $\langle\cdot\rangle_{\mathrm{TA}}$ and $\langle\cdot\rangle_{\mathrm{T}}$ denote a time-angle average and a time-average, respectively.

The components of the left hand side of equation (G.9) are

$$G_t{}^t = -r^{-2} + r^{-2}\mathrm{e}^{-\lambda} - r^{-1}\mathrm{e}^{-\lambda}\lambda_{,r} \tag{G.19}$$

$$G_r{}^r = -r^{-2} + r^{-2}\mathrm{e}^{-\lambda} + r^{-1}\mathrm{e}^{-\lambda}\nu_{,r} \tag{G.20}$$

$$\begin{aligned} G_\theta{}^\theta = G_\varphi{}^\varphi = \frac{1}{2}\Bigg(&\mathrm{e}^{-\lambda}\left(r^{-1}\nu_{,r} - r^{-1}\lambda_{,r} + \nu_{,rr} - \frac{1}{2}\lambda_{,r}\nu_{,r} + \frac{1}{2}\nu_{,r}^2\right) \\ &+ \mathrm{e}^{-\nu}\left(\frac{1}{2}\lambda_{,t}\nu_{,t} - \lambda_{,tt} - \frac{1}{2}\lambda_{,t}^2\right)\Bigg) \end{aligned} \tag{G.21}$$

$$G_r{}^t = -r^{-1}\mathrm{e}^{-\nu}\lambda_{,t}\,. \tag{G.22}$$

The final step in obtaining the time-dependent electromagnetic geon field equations is to make the transformation to the ρ coordinate system. In addition to the transformation

$$r = \frac{\rho}{\Omega}, \tag{G.23}$$

we introduce the two metric functions $L(\rho, t)$ and $Q(\rho, t)$ through the defining equations

$$e^{-\lambda} \equiv 1 - \frac{2L(\rho, t)}{\rho}, \tag{G.24}$$

$$e^{\lambda+\nu} \equiv Q^2(\rho, t)\,, \tag{G.25}$$

$$e^{\nu} = \left(1 - \frac{2L(\rho, t)}{\rho}\right) Q^2(\rho, t)\,. \tag{G.26}$$

The operator $\frac{\partial}{\partial r^*}$ has the following form in the ρ coordinate system

$$\frac{\partial}{\partial r^*} = \mathrm{e}^{(\nu-\lambda)/2}\frac{\partial}{\partial r} = \Omega\left(1 - \frac{2L}{\rho}\right) Q\frac{\partial}{\partial \rho}\,. \tag{G.27}$$

By defining

$$\frac{\partial}{\partial \rho^*} \equiv \left(1 - \frac{2L}{\rho}\right) Q\frac{\partial}{\partial \rho}, \tag{G.28}$$

the operators of (G.7) simply transform as

$$\frac{\partial}{\partial r^*} = \Omega \frac{\partial}{\partial \rho^*}, \qquad \frac{\partial^2}{\partial r^{*2}} = \Omega^2 \frac{\partial^2}{\partial \rho^{*2}}. \tag{G.29}$$

The operator $\frac{\partial^2}{\partial t^{*2}}$ of (G.8) transforms as

$$\frac{\partial^2}{\partial t^{*2}} = \left(1 - \frac{2L}{\rho}\right)^{-1} Q^{-1} \frac{\partial}{\partial t} \left(\left(1 - \frac{2L}{\rho}\right)^{-1} Q^{-1} \frac{\partial}{\partial t} \right). \tag{G.30}$$

After applying the transformation (G.23) and equations (G.24)–(G.30), a lengthy but straightforward computation yields the wave equation

$$\Omega^2 \frac{\partial^2 a}{\partial \rho^{*2}} - \Omega^2 l^{*2} \rho^{-2} \left(1 - \frac{2L}{\rho}\right) Q^2 a - \left(1 - \frac{2L}{\rho}\right)^2 Q^2 \frac{\partial^2 a}{\partial t^{*2}} = 0 \tag{G.31}$$

and the background field equations

$$\frac{\partial L}{\partial \rho^*} = \frac{\kappa_l^2}{2} \left(Q^{-1} \left(\Omega^2 \left\langle \left(\frac{\partial a}{\partial \rho^*}\right)^2 \right\rangle_{\mathrm{T}} + \left\langle \left(\frac{\partial a}{\partial t}\right)^2 \right\rangle_{\mathrm{T}} \right) \right.$$
$$\left. + \Omega^2 l^{*2} \rho^{-2} \left(1 - \frac{2L}{\rho}\right) Q \left\langle a^2 \right\rangle_{\mathrm{T}} \right). \tag{G.32}$$

$$\frac{\partial Q}{\partial \rho^*} = \frac{\kappa_l^2}{\rho - 2L} \left(\Omega^2 \left\langle \left(\frac{\partial a}{\partial \rho^*}\right)^2 \right\rangle_{\mathrm{T}} + \left\langle \left(\frac{\partial a}{\partial \rho^*}\right)^2 \right\rangle_{\mathrm{T}} \right). \tag{G.33}$$

$$\frac{\partial L}{\partial t} = \kappa_l^2 \Omega^2 Q^{-1} \left\langle \frac{\partial a}{\partial \rho^*} \frac{\partial a}{\partial t} \right\rangle_{\mathrm{T}} \tag{G.34}$$

and

$$\frac{\partial^2 L}{\partial t^2} + 4\rho^{-1} \left(\frac{\partial L}{\partial t}\right)^2 - Q^{-1} \frac{\partial L}{\partial t} \frac{\partial Q}{\partial t} = \frac{1}{2} \left(1 - \frac{2L}{\rho}\right)^2 Q^2 \rho \times$$
$$\times \left(\mathcal{A}(\rho) + \mathcal{B}(\rho) - 2\kappa^2 l^{*2} \Omega^4 \rho^{-4} \left\langle a^2 \right\rangle_{\mathrm{T}} \right). \tag{G.35}$$

In the above equations we have defined

$$\kappa_l \equiv \sqrt{\frac{Nl(l+1)}{2l+1}}, \tag{G.36}$$

$$\mathcal{A}(\rho) \equiv 2\Omega^2\rho^{-1}\left(Q^{-1}\frac{\partial Q}{\partial \rho}\left(1-\frac{2L}{\rho}\right)+2\rho^{-1}\left(\rho^{-1}L-\frac{\partial L}{\partial \rho}\right)\right) \tag{G.37}$$

and

$$\begin{aligned}\mathcal{B}(\rho) \equiv \Omega^2\left(1-\frac{2L}{\rho}\right)\Bigg(2\Bigg(&2Q^{-1}\left(\frac{\partial^2 Q}{\partial \rho^2}-Q^{-1}\left(\frac{\partial Q}{\partial \rho}\right)^2\right)\\
&-2\rho^{-2}\left(1-\frac{2L}{\rho}\right)^{-1}\left(\rho^{-1}L-\frac{\partial L}{\partial \rho}\right)\\
&+4\rho^{-2}\left(1-\frac{2L}{\rho}\right)^{-2}\left(\rho^{-1}L-\frac{\partial L}{\partial \rho}\right)^2\\
&+2\rho^{-2}\left(1-\frac{2L}{\rho}\right)^{-1}\left(\frac{\partial L}{\partial \rho}-\rho^{-1}L-\rho\frac{\partial^2 L}{\partial \rho^2}\right)\Bigg)\\
&-\left(2\rho^{-1}\left(1-\frac{2L}{\rho}\right)^{-1}\left(\frac{\partial L}{\partial \rho}-\rho^{-1}L\right)\right)\times\\
&\times\left(2Q^{-1}\frac{\partial Q}{\partial \rho}-2\rho^{-1}\left(1-\frac{2L}{\rho}\right)^{-1}\left(\frac{\partial L}{\partial \rho}-\rho^{-1}L\right)\right)\\
&+\left(2Q^{-1}\frac{\partial Q}{\partial \rho}-2\rho^{-1}\left(1-\frac{2L}{\rho}\right)^{-1}\left(\frac{\partial L}{\partial \rho}-\rho^{-1}L\right)\right)^2\Bigg) .\end{aligned} \tag{G.38}$$

Equations (G.31)–(G.34) are equations (9.44)–(9.47) of chapter 9.

Appendix H

Perturbation Analysis of a Slowly Varying Amplitude

To investigate the time-evolution of the unperturbed solution, the equilibrium solution (9.49)–(9.51) will be perturbed by allowing the coefficient of $\sin\Omega t$ to become a slowly varying function of time as compared to the period $2\pi\Omega^{-1}$ of the electromagnetic waves. In addition, it will be assumed that $u_1(\rho, t)$ is a separable function (i.e. $u_1(\rho, t) = u(\rho)T(t)$). Under this assumption, equation (9.85) becomes

$$\kappa_l \Omega\, a(\rho, t) = (f_0(\rho) + \delta u(\rho)T(t)) \sin\Omega t\,, \tag{H.1}$$

where the characteristic frequency of $u(\rho)T(t)$ is of order $\omega \ll \Omega$. This introduces a small time-dependent perturbation in the metric functions

$$L(\rho, t) = L_0(\rho) + \delta L_1(\rho, t)\,, \tag{H.2}$$

$$Q(\rho, t) = Q_0(\rho) + \delta Q_1(\rho, t)\,. \tag{H.3}$$

It is stated without proof that for $u_1(\rho, t)$ separable, the field equations impose the relations $u(\rho) = f_0(\rho)$ and $T(t) = \sin(\omega t)$. Therefore the analysis in this appendix is carried out for the perturbation

$$\kappa_l \Omega\, a(\rho, t) = (f_0(\rho) + \delta f_0(\rho) \sin \omega t) \sin \Omega t\,, \qquad \omega \ll \Omega\,, \quad \delta \ll 1\,. \tag{H.4}$$

The perturbation expansion will be taken to the first order in δ. We have assumed that the coefficient $f(\rho, t) \equiv f_0(\rho) + \delta f_0(\rho) \sin \omega t$ of $\sin \Omega t$ in equation (H.4) varies on a time-scale longer than that of $\sin \Omega t$. Therefore $f(\rho, t)$ is approximately constant over the short time period $T = 2\pi\Omega^{-1}$ of the electromagnetic waves. Therefore, evaluation of the time-averages (9.86)–(9.89) yields

$$\kappa_l^2 \Omega^2 \left\langle a^2(\rho, t) \right\rangle_T = \frac{1}{2} f_0^2 (1 + 2\delta \sin \omega t) + \mathrm{O}\left(\delta^2\right)\,, \tag{H.5}$$

$$\kappa_l^2 \Omega^2 \left\langle \left(\frac{\partial a}{\partial \rho^*} \right)^2 \right\rangle_T = \frac{1}{2} \left(\frac{\mathrm{d} f_0}{\mathrm{d}\rho^*} \right)^2 (1 + 2\delta \sin \omega t) + \mathrm{O}\left(\delta^2\right)\,. \tag{H.6}$$

$$\kappa_l^2 \Omega^2 \left\langle \left(\frac{\partial a}{\partial t} \right)^2 \right\rangle_T = \frac{1}{2} \Omega^2 f_0^2 (1 + 2\delta \sin \omega t) + \mathrm{O}\left(\delta^2\right)\,. \tag{H.7}$$

$$\kappa_l^2 \Omega^2 \left\langle \frac{\partial a}{\partial t} \frac{\partial a}{\partial \rho^*} \right\rangle_T = \frac{1}{2} \delta \omega f_0 \frac{\mathrm{d} f_0}{\mathrm{d}\rho^*} \cos \omega t + \mathrm{O}\left(\delta^2\right)\,. \tag{H.8}$$

Substitution of (H.4)–(H.8) into (9.44)–(9.47), expanding to first order in δ and setting each order to zero yields the unperturbed equations (9.60)–(9.62) and the first order equations (H.12)–(H.17). Setting the first order part of the wave equation (9.44) to zero yields

$$\mathfrak{A}(\rho, t) \sin \Omega t + \mathfrak{B}(\rho, t) \cos \Omega t = 0\,, \tag{H.12}$$

where

$$\mathfrak{A}(\rho, t) \equiv \Omega^2 \Bigg(\omega^2 \Omega^{-2} f_0 + \left(1 - l^{*2} \rho^{-2} Q_0^2 \left(1 - \frac{2L_0}{\rho} \right) \right) f_0 + Q_0 \left(1 - \frac{2L_0}{\rho} \right) \times$$

$$
\begin{aligned}
&\times\left(2\rho^{-2}Q_0\frac{\mathrm{d}f_0}{\mathrm{d}\rho}\left(L_0-\rho\frac{\mathrm{d}L_0}{\mathrm{d}\rho}\right)+Q_0\frac{\mathrm{d}^2f_0}{\mathrm{d}\rho^2}\left(1-\frac{2L_0}{\rho}\right)\right.\\
&\left.\left.+\frac{\mathrm{d}f_0}{\mathrm{d}\rho}\frac{\mathrm{d}Q_0}{\mathrm{d}\rho}\left(1-\frac{2L_0}{\rho}\right)\right)\right)\sin\omega t+\Omega^2\left(Q_0\left(1-\frac{2L_0}{\rho}\right)\times\right.\\
&\times\left(2\rho^{-2}\frac{\mathrm{d}f_0}{\mathrm{d}\rho}\left(Q_1\left(L_0-\rho\frac{\mathrm{d}L_0}{\mathrm{d}\rho}\right)+Q_0\left(L_1(\rho,\,t)-\rho\frac{\partial}{\partial\rho}L_1(\rho,\,t)\right)\right)\right.\\
&+\left(\left(1-\frac{2L_0}{\rho}\right)\frac{\partial}{\partial\rho}Q_1(\rho,\,t)-2\rho^{-1}L_1(\rho,\,t)\frac{\mathrm{d}Q_0}{\mathrm{d}\rho}\right)\frac{\mathrm{d}f_0}{\mathrm{d}\rho}\\
&\left.+\left(Q_1(\rho,\,t)\left(1-\frac{2L_0}{\rho}\right)-2\rho^{-1}L_1(\rho,\,t)Q_0\right)\frac{\mathrm{d}^2f_0}{\mathrm{d}\rho^2}\right)\\
&+2\rho^{-3}\left(l^{*2}Q_0^2L_1(\rho,\,t)-l^{*2}\rho Q_0Q_1(\rho,\,t)\left(1-\frac{2L_0}{\rho}\right)\right)f_0\\
&+\left(Q_1(\rho,\,t)\left(1-\frac{2L_0}{\rho}\right)-2\rho^{-1}L_1(\rho,\,t)Q_0\right)\left(2\rho^{-2}\frac{\mathrm{d}f_0}{\mathrm{d}\rho}Q_0\left(L_0-\rho\frac{\mathrm{d}L_0}{\mathrm{d}\rho}\right)\right.\\
&\left.\left.+\left(1-\frac{2L_0}{\rho}\right)\left(Q_0\frac{\mathrm{d}^2f_0}{\mathrm{d}\rho^2}+\frac{\mathrm{d}Q_0}{\mathrm{d}\rho}\frac{\mathrm{d}f_0}{\mathrm{d}\rho}\right)\right)\right)
\end{aligned}
\tag{H.13}
$$

and

$$
\mathfrak{B}(\rho,\,t)\equiv\Omega f_0\left(2\rho^{-1}\left(1-\frac{2L}{\rho}\right)^{-1}\frac{\partial}{\partial t}L_1(\rho,\,t)-Q_0^{-1}\frac{\partial}{\partial t}Q_1(\rho,\,t)+2\omega\cos\omega t\right).
\tag{H.14}
$$

The derivatives of $L_1(\rho,\,t)$ and $Q_1(\rho,\,t)$ with respect to ρ are found from the first order equations

$$
\begin{aligned}
\left(1-\frac{2L_0}{\rho}\right)Q_0\frac{\partial}{\partial\rho}L_1(\rho,\,t)&=-\left(\left(1-\frac{2L_0}{\rho}\right)Q_1(\rho,\,t)-2\rho^{-1}L_1(\rho,\,t)Q_0\right)\frac{\mathrm{d}L_0}{\mathrm{d}\rho}\\
&+\frac{1}{2}Q_0^{-1}\left(Q_0^2\left(1-\frac{2L_0}{\rho}\right)^2\left(\frac{\mathrm{d}f_0}{\mathrm{d}\rho}\right)^2\sin\omega t+\left(\left(1-\frac{2L_0}{\rho}\right)^2Q_0Q_1(\rho,\,t)\right.\right.\\
&\left.\left.-2\rho^{-1}Q_0^2\left(1-\frac{2L_0}{\rho}\right)L_1(\rho,\,t)\right)\left(\frac{\mathrm{d}f_0}{\mathrm{d}\rho}\right)^2+f_0^2\sin^2\omega t\right)\\
&-\frac{1}{2}Q_0^{-2}Q_1(\rho,\,t)\left(\frac{1}{2}\left(1-\frac{2L_0}{\rho}\right)^2Q_0^2\left(\frac{\mathrm{d}f_0}{\mathrm{d}\rho}\right)^2+\frac{1}{2}f_0^2\right)
\end{aligned}
$$

$$+\frac{1}{2}l^{*2}\rho^{-2}Q_0 f_0^2\left(1-\frac{2L_0}{\rho}\right)\sin\omega\, t$$
$$+\frac{1}{4}\left(l^{*2}\rho^{-2}\left(1-\frac{2L_0}{\rho}\right)Q_1(\rho,\, t)-2l^{*2}\rho^{-3}L_1(\rho,\, t)Q_0\right)f_0^2 \tag{H.15}$$

and

$$\left(1-\frac{2L_0}{\rho}\right)Q_0\frac{\partial}{\partial\rho}Q_1(\rho,\, t)=-\left(\left(1-\frac{2L_0}{\rho}\right)Q_1(\rho,\, t)-2\rho^{-1}L_1(\rho,\, t)Q_0\right)\frac{\mathrm{d}Q_0}{\mathrm{d}\rho}$$
$$+\frac{1}{\rho-2L_0}\left(\left(1-\frac{2L_0}{\rho}\right)^2 Q_0^2\left(\frac{\mathrm{d}f_0}{\mathrm{d}\rho}\right)^2\sin\omega\, t\right.$$
$$+\frac{1}{2}\left(2\left(1-\frac{2L_0}{\rho}\right)^2 Q_0Q_1(\rho,\, t)-4\rho^{-1}\left(1-\frac{2L_0}{\rho}\right)L_1(\rho,\, t)Q_0^2\right)\left(\frac{\mathrm{d}f_0}{\mathrm{d}\rho}\right)^2$$
$$\left.+f_0^2\sin^2\omega\, t\right)+\frac{L_1(\rho,\, t)}{(\rho-2L_0)^2}\left(\left(1-\frac{2L_0}{\rho}\right)^2 Q_0^2\left(\frac{\mathrm{d}f_0}{\mathrm{d}\rho}\right)^2+f_0^2\right), \tag{H.16}$$

respectively. The derivative of $L_1(\rho,\, t)$ with respect to t is given by the first order equation

$$\frac{\partial}{\partial t}L_1(\rho,\, t)=\frac{1}{2}\omega\left(1-\frac{2L_0}{\rho}\right)f_0\frac{\mathrm{d}f_0}{\mathrm{d}\rho}\cos\omega t\,. \tag{H.17}$$

The simplest of the first order equations is equation (H.17). It is immediately integrable to yield

$$L_1(\rho,\, t)=\frac{1}{2}\left(1-\frac{2L_0}{\rho}\right)f_0\frac{\mathrm{d}f_0}{\mathrm{d}\rho}\sin\omega t+c_1(\rho)\,. \tag{H.18}$$

where $c_1(\rho)$ is an unknown function of ρ. It is possible to obtain an equation for $\partial Q_1(\rho,\, t)/\partial t$ from equation (H.12). This is done by substituting (H.17) into the coefficient of $\cos\Omega\, t$ (*viz.* $\mathfrak{B}(\rho,\, t)$) and setting the expression to zero. This is justified since $\sin\Omega\, t$ and $\cos\Omega\, t$ are independent functions in the approximation that $\mathfrak{A}(\rho,\, t)$ and $\mathfrak{B}(\rho,\, t)$ are slowly varying functions of time.

Solving for $\partial Q_1(\rho, t)/\partial t$ and integrating yields

$$Q_1(\rho, t) = Q_0 \left(2 + \rho^{-1} f_0 \frac{\mathrm{d}f_0}{\mathrm{d}\rho} \right) \sin \omega t + c_2(\rho) \,. \tag{H.19}$$

where $c_2(\rho)$ is an unknown function of ρ.

Up to this point the only first order field equation which has been satisfied is equation (H.17). To satisfy the first order wave equation (H.12), the condition $\mathfrak{Q}(\rho, t) = 0$ must be imposed. By using the first order field equations (H.15) and (H.16), the ρ derivatives of $L_1(\rho, t)$ and $Q_1(\rho, t)$ found in $\mathfrak{Q}(\rho, t)$ (equation (H.13)) can be eliminated. After substitution of (H.18) and (H.19) into (H.13), $\mathfrak{Q}(\rho, t)$ no longer depends on the first order functions $L_1(\rho, t)$ and $Q_1(\rho, t)$. As a result of these substitutions, equation (H.13) takes the form[1]

$$\mathcal{A}(\rho) \sin \omega t + \mathcal{B}(c_1(\rho), c_2(\rho), \rho) = 0 \,, \tag{H.20}$$

where $\mathcal{A}(\rho)$ and $\mathcal{B}(\rho)$ depend only on ρ, the unperturbed functions $f_0(\rho)$, $L_0(\rho)$, $Q_0(\rho)$ (and their derivatives) and the two unknown functions $c_1(\rho)$ and $c_2(\rho)$. Note that the unknown functions $c_1(\rho)$ and $c_2(\rho)$ are only found in $\mathcal{B}(\rho)$. Equation (H.20) will be satisfied for all t only if $\mathcal{A}(\rho) = 0$ and $\mathcal{B}(\rho) = 0$. Since $c_1(\rho)$ and $c_2(\rho)$ are unknown, we will focus upon the equation $\mathcal{A}(\rho) = 0$.

The function $\mathcal{A}(\rho)$ is comprised of the known functions $f_0(\rho)$, $L_0(\rho)$ and $Q_0(\rho)$. It is therefore necessary to transform $\mathcal{A}(\rho)$ to the x coordinate system and expand in inverse powers of $l^{*1/3}$ as is done for the unperturbed system [36]. The transformation is given by equations (9.75)–(9.76). After a lengthy computation, the asymptotic expansion of $\mathcal{A}(\rho) = 0$ yields

$$l^{*1/3} \left(k^{-2}(x) \left(10\lambda_0(x) - 5 + k^2(x) \right) + \frac{\omega^2}{\Omega^2} \right) \phi(x) + \mathrm{O}(1) = 0 \,. \tag{H.21}$$

[1]Explicit forms of $\mathcal{A}(\rho)$ and $\mathcal{B}(\rho)$ will not be given due to their extreme length.

In order for equation (H.21) to be satisfied for large arbitrary l^* (in the limit $l^* \to \infty$), each order of $l^{*1/3}$ must be set to zero. Since setting $\phi(x) = 0$ implies the absence of electromagnetic wave perturbations, the bracketed term must be zero. It is known from the unperturbed system that [36]

$$\lambda_0 = \frac{1}{2}\left(1 - k^2\right) . \tag{H.22}$$

Substitution of (H.22) in (H.21) leads to the relation

$$\omega^2 = 4\Omega^2 \tag{H.23}$$

which must hold in order for the field equation (H.12) to be satisfied. However, equation (H.23) is a *contradiction* to the original assumption $\omega \ll \Omega$. This result is identical to that found for the perturbation analysis based upon equation (9.52).