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Article

Faber Polynomial Coefficient Estimates of Bi-Close-to-Convex Functions Associated with Generalized Hypergeometric Functions

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Abstract: A new subclass of bi-close-to-convex functions associated with the generalized hypergeometric functions defined in $\triangle = \{z \in \mathbb{C} : |z| < 1\}$ is introduced. The estimates for the general Taylor–Maclaurin coefficients of the functions in the introduced subclass are obtained by making use of Faber polynomial expansions. In particular, several previous results are generalized.

Keywords: analytic function; bi-univalent function; subordination; schwarz function; bi-close-to-convex; generalized hypergeometric function; faber polynomial expansion

MSC: 30C45; 05A30

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1. Introduction

Denote by $\mathcal{A}$ the class of analytic functions in $\triangle := \{z \in \mathbb{C} : |z| < 1\}$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \triangle). \tag{1}$$

Likewise, denote by $\mathcal{S}(\subset \mathcal{A})$ the class of analytic functions that are univalent in $\triangle$. For the functions $f \in \mathcal{A}$ and $h \in \mathcal{A}$ given by

$$h(z) = z + \sum_{n=2}^{\infty} \psi_n z^n \quad (z \in \triangle), \tag{2}$$

we define the Hadamard product of f and h as the following:

$$(f * h)(z) := z + \sum_{n=2}^{\infty} a_n \psi_n z^n =: (h * f)(z) \quad (z \in \triangle).$$

Let f_1 and f_2 be two analytic functions in $\triangle$. Then, the function f_1 is subordinate to the function f_2 and written as follows:

$$f_1(z) \prec f_2(z),$$

if there is a Schwarz function $u(z)$, so that

$$f_1(z) = f_2\big(u(z)\big).$$

Furthermore, if the function f_2 is univalent in $\triangle$, then it follows that

$$f_1(z) \prec f_2(z) (z \in \triangle) \iff f_1(0) = f_2(0) \;\text{ and }\; f_1(\triangle) \subset f_2(\triangle).$$

Let $\mathcal{P}$ denote the class of analytic functions φ having the following form:

$$\varphi(z) = 1 + P_1 z + P_2 z^2 + P_3 z^3 + \cdots \quad (P_1 > 0) \tag{3}$$

and $\Re\varphi(z) > 0$ $(z \in \triangle)$.

For $f \in \mathcal{A}$ and $0 \leqq \alpha < 1$, we recall here the following well-known subclasses of the analytic function class $\mathcal{A}$:

(i) f is called to be a starlike function of the order α if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \triangle).$$

We denote this subclass with $\mathcal{S}^*(\alpha)$.

(ii) f is called to be a convex function of the order α if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \triangle).$$

We denote this subclass with $\mathcal{C}(\alpha)$.

(iii) f is called to be a close-to-convex function of the order α if

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha \quad (z \in \triangle),$$

where $g \in \mathcal{S}^*(0) =: \mathcal{S}^*$. We denote this subclass with $\mathcal{K}(\alpha)$.

For $s_1, s_2, s_3 \in \mathbb{C}$, the generalized Gauss hypergeometric function ${}_2F_1(s_1, s_2, s_3, k; z)$ is given here by

$$\begin{aligned} {}_2F_1(s_1, s_2, s_3, k; z) &= \frac{\Gamma(s_3)}{\Gamma(s_2)} \sum_{n=0}^{\infty} \frac{(s_1)_n \Gamma(kn + s_2)}{\Gamma(kn + s_3)} \frac{z^n}{n!} \\ &= 1 + \frac{\Gamma(s_3)}{\Gamma(s_2)} \sum_{n=2}^{\infty} \frac{\Gamma\big(k(n-1) + s_2\big)(s_1)_{n-1}}{\Gamma\big(k(n-1) + s_3\big)(n-1)!} z^{n-1}, \end{aligned} \tag{4}$$

where $(x)_n$ is the Pochhammer symbol, $k > 0$, $\Re(s_3 - s_2 - 1) > 0$, $s_3 \neq 0, -1, -2, \cdots$ and $z \in \triangle$.

According to the generalized Gauss hypergeometric function defined in (4), Hussain et al. [1] considered the operator $\mathcal{J}(s_1, s_2, s_3, k)$ as the following:

$$\mathcal{J}(s_1, s_2, s_3, k) f(z) = f(z) * z\, {}_2F_1(s_1, s_2, s_3, k; z) = z + \sum_{n=2}^{\infty} \gamma_n a_n z^n, \tag{5}$$

where $f \in \mathcal{A}$ and

$$\gamma_n = \gamma(s_1, s_2, s_3, k, n) = \frac{(s_1)_{n-1} \Gamma\big(k(n-1) + s_2\big) \Gamma(s_3)}{(n-1)! \Gamma\big(k(n-1) + s_3\big) \Gamma(s_2)}. \tag{6}$$

It is well-known that a function $g \in \mathcal{S}$ has its inverse g^{-1}, which meets the following equality:

$$g(g^{-1}(\zeta)) = \zeta \quad \left(|\zeta| < r_0(g);\ r_0(g) \geqq \frac{1}{4}\right).$$

We say that a function $g \in \mathcal{S}$ is bi-univalent in $\triangle$ if g and g^{-1} are univalent in $\triangle$, and we denote the subclass with $\Sigma(\subset \mathcal{S})$. A history of the functions in Σ can be found in [2,3]. Lewin considered the class Σ in [4] and obtained that $|a_2| < 1.51$. In [5], Brannan and Clunie proved that $|a_2| < \sqrt{2}$. In [6], Netanyahu improved the results above to $|a_2| < \frac{4}{3}$.

Some elements of functions in the class Σ are presented below (see [2]):

$$f_1(z) = \frac{z}{1-z}, \;\; f_2(z) = -\log(1-z) \;\text{ and }\; f_3(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

and their corresponding inverses are given by

$$f_1^{-1}(\omega) = \frac{\omega}{1+\omega}, \;\; f_2^{-1}(\omega) = \frac{e^{\omega}-1}{e^{\omega}} \;\text{ and }\; f_3^{-1}(\omega) = \frac{e^{2\omega}-1}{e^{2\omega}+1}.$$

Certain subclasses of the bi-univalent function class Σ, considered by Brannan and Taha [7], are similar to the subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ (see [8]). The authors of [7] introduced the subclasses $\mathcal{S}^*_\Sigma(\alpha)$ of bi-starlike functions of the order α, and $\mathcal{C}^*_\Sigma(\alpha)$ of bi-convex functions of the order α, as presented below:

$$\mathcal{S}^*_\Sigma(\alpha) := \left\{ f : f \in \Sigma, \; \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (z \in \Delta) \right.$$
$$\left. \text{and } \left|\arg\left(\frac{\omega\mathcal{F}'(\omega)}{\mathcal{F}(\omega)}\right)\right| < \frac{\alpha\pi}{2} \quad (\omega \in \Delta) \right\}$$

and

$$\mathcal{C}^*_\Sigma(\alpha) := \left\{ f : f \in \Sigma, \; \left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (z \in \Delta) \right.$$
$$\left. \text{and } \left|\arg\left(1+\frac{\omega\mathcal{F}''(\omega)}{\mathcal{F}'(\omega)}\right)\right| < \frac{\alpha\pi}{2} \quad (\omega \in \Delta) \right\},$$

where

$$\begin{aligned} \mathcal{F}(\omega) := f^{-1}(\omega) &= \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots \\ &= \omega + \sum_{n=2}^{\infty} A_n\omega^n. \end{aligned} \tag{7}$$

For each of the above bi-univalent function subclasses, $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{C}^*_\Sigma(\alpha)$, non-sharp bounds of the first two coefficients $|a_2|$ and $|a_3|$ are given in [7]. The widely cited paper by Srivastava et al. [3] not only represents one of the most important studies of bi-univalent functions, but it also resuscitated the study of bi-univalent functions in recent years. Many subsequent papers investigated the problems concerned with bi-univalent functions, such as [9–12].

Next, we introduce a new subclass, $\mathfrak{K}_\Sigma(\beta, \gamma_n)$, of bi-close-to-convex functions.

Definition 1. *For $0 \leqq \beta \leqq 1$ and γ_n given by (6), a function $f \in \Sigma$ is said to be in the class $\mathfrak{K}_\Sigma(\beta, \gamma_n)$ if there exists a function $g \in \mathcal{S}^*$ and if it satisfies the following subordination conditions:*

$$\frac{z\big(\mathcal{J}(s_1,s_2,s_3,k)f(z)\big)' + \beta z^2\big(\mathcal{J}(s_1,s_2,s_3,k)f(z)\big)''}{(1-\beta)\mathcal{J}(s_1,s_2,s_3,k)g(z) + \beta z\big(\mathcal{J}(s_1,s_2,s_3,k)g(z)\big)'} \prec \varphi(z) \quad (z \in \Delta) \tag{8}$$

and

$$\frac{\omega\big(\mathcal{J}(s_1,s_2,s_3,k)\mathcal{F}(\omega)\big)' + \beta\omega^2\big(\mathcal{J}(s_1,s_2,s_3,k)\mathcal{F}(\omega)\big)''}{(1-\beta)\mathcal{J}(s_1,s_2,s_3,k)\mathcal{G}(\omega) + \beta\omega\big(\mathcal{J}(s_1,s_2,s_3,k)\mathcal{G}(\omega)\big)'} \prec \varphi(\omega) \quad (\omega \in \Delta), \tag{9}$$

where $\varphi \in \mathcal{P}$, the function $\mathcal{F}$ given by (7) is the analytic extension of f^{-1}, and the function $\mathcal{G}$ is an extension of g^{-1} as the following:

$$\mathcal{G}(\omega) = \omega - b_2\omega^2 + (2b_2^2 - b_3)\omega^3 - (5b_2^3 - 5b_2b_3 + b_4)\omega^4 + \cdots \quad (\omega \in \Delta). \tag{10}$$

By setting $b_n = a_n (n \in \mathbb{N} \setminus \{1\})$, *we can define the bi-starlike function class* $\mathfrak{S}_\Sigma(\beta, \gamma_n)$ *given below:*

$$\frac{z\big(\mathcal{J}(s_1,s_2,s_3,k)f(z)\big)' + \beta z^2\big(\mathcal{J}(s_1,s_2,s_3,k)f(z)\big)''}{(1-\beta)\mathcal{J}(s_1,s_2,s_3,k)f(z) + \beta z\big(\mathcal{J}(s_1,s_2,s_3,k)f(z)\big)'} \prec \varphi(z) \quad (z \in \triangle) \tag{11}$$

and

$$\frac{\omega\big(\mathcal{J}(s_1,s_2,s_3,k)\mathcal{F}(\omega)\big)' + \beta \omega^2\big(\mathcal{J}(s_1,s_2,s_3,k)\mathcal{F}(\omega)\big)''}{(1-\beta)\mathcal{J}(s_1,s_2,s_3,k)\mathcal{F}(\omega) + \beta \omega\big(\mathcal{J}(s_1,s_2,s_3,k)\mathcal{F}(\omega)\big)'} \prec \varphi(\omega) \quad (\omega \in \triangle). \tag{12}$$

Remark 1. *If we set* $\varphi(z) = \frac{1+(1-2\eta)z}{1-z}$ $(0 \le \eta < 1)$ *and replace* $\mathcal{J}(s_1,s_2,s_3,k)$ *by* $\mathfrak{B}(\lambda,\alpha,\beta)$ *in (8), (9), (11), and (12), where*

$$\begin{aligned}
&\mathfrak{B}(\lambda,\alpha,\beta)f(z) \\
&= z + \sum_{n=2}^{\infty} \frac{\Gamma(1+(n-1)\lambda)[(n-1)\lambda]^{n-2}e^{-(n-1)\lambda}}{(n-1)!\mathfrak{E}_{\alpha,\beta}((n-1)\lambda)\Gamma(\beta+\alpha(n-1))} a_n z^n \\
&= z + \sum_{n=2}^{\infty} \phi_n a_n z^n \quad (0 < \lambda \le 1; \alpha \in \mathbb{C}, \mathrm{Re}(\alpha) > 0; \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-),
\end{aligned}$$

$$\mathfrak{E}_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha(n-1))} z^n$$

and

$$\phi_n = \frac{\Gamma(1+(n-1)\lambda)[(n-1)\lambda]^{n-2}e^{-(n-1)\lambda}}{(n-1)!\mathfrak{E}_{\alpha,\beta}((n-1)\lambda)\Gamma(\beta+\alpha(n-1))},$$

then we obtain the function classes $\mathfrak{U}_\Sigma^{\alpha,\beta,\lambda}(\eta,\nu)$ *and* $\mathcal{P}_\Sigma^{\alpha,\beta,\lambda}(\eta,\nu)$ *given by Srivastava et al. [11].*

Applying Faber polynomial expansions to $f \in \mathcal{A}$, we get the coefficient expansion of the inverse mapping, as follows (see [13]; also see the recent developments [14–18], each of which is based upon the Faber polynomial expansions):

$$\mathcal{F}(\omega) = f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} \frac{1}{n} \mathcal{K}_{n-1}^{-n}(a_2, a_3, \cdots)\omega^n = \omega + \sum_{n=2}^{\infty} A_n \omega^n, \tag{13}$$

where

$$\begin{aligned}
\mathcal{K}_{n-1}^{-n}(a_2,a_3,\cdots) &= \frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3 \\
&+ \frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4 \\
&+ \frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}[(-n+2)a_3^2 + a_5] \\
&+ \frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}[(-2n+5)a_3a_4 + a_6] \\
&+ \sum_{j \geqq 7} a_2^{n-j} U_j.
\end{aligned} \tag{14}$$

In this paper, an expression such as $(-n)!$ is to be symbolically explained by

$$(-n)! := (-n)(-n-1)(-n-2)\cdots = \Gamma(1-n) \quad (n \in \mathbb{N}_0),$$

and U_j $(7 \leqq j \leqq n)$ is a homogeneous polynomial of $a_2, a_3, \cdots, a_n$.

In particular, $\mathcal{K}_1^{-2} = -2a_2$, $\mathcal{K}_2^{-3} = 3(2a_2^2 - a_3)$, and $\mathcal{K}_3^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4)$. In general, an expansion of $\mathcal{K}_n^p$ is as follows (see [2]):

$$\mathcal{K}_n^p = pa_n + \frac{p(p-1)}{2}\mathcal{D}_n^2 + \frac{p!}{(p-3)!3!}\mathcal{D}_n^3 + \cdots + \frac{p!}{(p-n)!n!}\mathcal{D}_n^n,$$

where

$$\mathcal{D}_n^p = \mathcal{D}_n^p(a_2, a_3, \cdots),$$

and (see [19])

$$\mathcal{D}_n^m(a_1, a_2, \cdots, a_n) = \sum_{n=1}^{\infty} \frac{m!}{i_1! \cdots i_n!} a_1^{i_1} \cdots a_n^{i_n}.$$

While $a_1 = 1$, the above sum is taken over by the non-negative integers $i_1, \cdots, i_n$ satisfying

$$\begin{cases} i_1 + 2i_2 + \cdots + ni_n = n \\ i_1 + i_2 + \cdots + i_n = m. \end{cases}$$

Finally, we get

$$\mathcal{D}_n^n(a_1, a_2, \cdots, a_n) = a_1^n.$$

Lemma 1 (see [20]). *Let the function $s(z)$ given by*

$$s(z) = \sum_{n=1}^{\infty} s_n z^n \in \mathcal{A} \qquad (|z| < 1)$$

be a Schwarz function, then $|s_n| \leqq 1$. Moreover, if $\vartheta \geqq 0$, then

$$|s_2 + \vartheta s_1^2| \leqq 1 + (\vartheta - 1)|s_1|^2.$$

In the investigation of bi-univalent functions, estimates for the first two coefficients are usually obtained. Furthermore, bounds of the first three Taylor–Maclaurin coefficients are given in [21]. Essentially motivated by some recent works (for example, see [11,21–23]), in this paper, we investigate the estimates for the Taylor–Maclaurin coefficients of the functions in $\mathfrak{K}_\Sigma(\beta, \gamma_n)$ and $\mathfrak{S}_\Sigma(\beta, \gamma_n)$. Several previous results are generalized.

2. Main Results

In this section, we find estimates for the Taylor–Maclaurin coefficients of the functions in $\mathfrak{K}_\Sigma(\beta, \gamma_n)$ by using the Faber polynomial expansion method.

Theorem 1. *Let $0 \leqq \beta \leqq 1$ and $f \in \mathfrak{K}_\Sigma(\beta, \gamma_n)$. If $a_k = 0$ and $b_k = 0$ for $2 \leqq k \leqq n-1$, then*

$$|a_n| \leqq 1 + \frac{P_1}{n[1 + (n-1)\beta]|\gamma_n|}.$$

Proof. Suppose that $f \in \mathfrak{K}_\Sigma(\beta, \gamma_n)$, then we have a function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*$, such that

$$\frac{z\big(\mathcal{J}(s_1, s_2, s_3, k)f(z)\big)' + \beta z^2\big(\mathcal{J}(s_1, s_2, s_3, k)f(z)\big)''}{(1-\beta)\mathcal{J}(s_1, s_2, s_3, k)g(z) + \beta z\big(\mathcal{J}(s_1, s_2, s_3, k)g(z)\big)'} \prec \varphi(z) \quad (z \in \triangle).$$

According to the Faber polynomial expansion, we get

$$
\begin{aligned}
&\frac{z\big(\mathcal{J}(s_1,s_2,s_3,k)f(z)\big)'+\beta z^2\big(\mathcal{J}(s_1,s_2,s_3,k)f(z)\big)''}{(1-\beta)\mathcal{J}(s_1,s_2,s_3,k)g(z)+\beta z\big(\mathcal{J}(s_1,s_2,s_3,k)g(z)\big)'}\\
&\quad=1+\sum_{n=2}^{\infty}\big([1+\beta(n-1)]\gamma_n(na_n-b_n)+\sum_{t=1}^{n-2}\gamma_{n-t}[1+(n-t-1)\beta]\\
&\quad\cdot\mathcal{K}_t^{-1}[(1+\beta)\gamma_2 b_2,(1+2\beta)\gamma_3 b_3,\cdots,(1+t\beta)\gamma_{t+1}b_{t+1}]\cdot[(n-t)a_{n-t}-b_{n-t}]z^{n-1}.
\end{aligned}
\tag{15}
$$

Moreover, for the function $\mathcal{F}=f^{-1}$, we have a function $\mathcal{G}(\omega)=\omega+\sum_{n=2}^{\infty}B_n\omega^n\in\mathcal{S}^*$, such that

$$
\frac{\omega\big(\mathcal{J}(s_1,s_2,s_3,k)\mathcal{F}(\omega)\big)'+\beta\omega^2\big(\mathcal{J}(s_1,s_2,s_3,k)\mathcal{F}(\omega)\big)''}{(1-\beta)\mathcal{J}(s_1,s_2,s_3,k)\mathcal{G}(\omega)+\beta\omega\big(\mathcal{J}(s_1,s_2,s_3,k)\mathcal{G}(\omega)\big)'}\prec\varphi(\omega)\quad(\omega\in\triangle).
$$

Since

$$
F(\omega)=\omega+\sum_{n=2}^{\infty}A_n\omega^n,
$$

we get

$$
\begin{aligned}
&\frac{\omega\big(\mathcal{J}(s_1,s_2,s_3,k)\mathcal{F}(\omega)\big)'+\beta\omega^2\big(\mathcal{J}(s_1,s_2,s_3,k)\mathcal{F}(\omega)\big)''}{(1-\beta)\mathcal{J}(s_1,s_2,s_3,k)\mathcal{G}(\omega)+\beta\omega\big(\mathcal{J}(s_1,s_2,s_3,k)\mathcal{G}(\omega)\big)'}\\
&\quad=1+\sum_{n=2}^{\infty}\big([1+\beta(n-1)]\gamma_n(nA_n-B_n)+\sum_{t=1}^{n-2}\gamma_{n-t}[1+(n-t-1)\beta]\\
&\quad\cdot\mathcal{K}_t^{-1}[(1+\beta)\gamma_2 B_2,(1+2\beta)\gamma_3 B_3,\cdots,(1+t\beta)\gamma_{t+1}B_{t+1}]\cdot[(n-t)A_{n-t}-B_{n-t}]\omega^{n-1}.
\end{aligned}
\tag{16}
$$

Since both f and $\mathcal{F}=f^{-1}$ are in $\mathfrak{K}_\Sigma(\beta,\gamma_n)$, there exist the following two Schwarz functions:

$$
u(z)=\sum_{n=1}^{\infty}c_n z^n
$$

and

$$
\nu(\omega)=\sum_{n=1}^{\infty}d_n\omega^n,
$$

so that

$$
\begin{aligned}
&\frac{z\big(\mathcal{J}(s_1,s_2,s_3,k)f(z)\big)'+\beta z^2\big(\mathcal{J}(s_1,s_2,s_3,k)f(z)\big)''}{(1-\beta)\mathcal{J}(s_1,s_2,s_3,k)g(z)+\beta z\big(\mathcal{J}(s_1,s_2,s_3,k)g(z)\big)'}=\varphi(u(z))\\
&\quad=1+P_1c_1z+(P_1c_2+P_2c_1^2)z^2+\cdots\\
&=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}P_kD_n^k(c_1,c_2,\cdots,c_n)z^n
\end{aligned}
\tag{17}
$$

and

$$
\begin{aligned}
&\frac{\omega\big(\mathcal{J}(s_1,s_2,s_3,k)\mathcal{F}(\omega)\big)'+\beta\omega^2\big(\mathcal{J}(s_1,s_2,s_3,k)\mathcal{F}(\omega)\big)''}{(1-\beta)\mathcal{J}(s_1,s_2,s_3,k)\mathcal{G}(\omega)+\beta\omega\big(\mathcal{J}(s_1,s_2,s_3,k)\mathcal{G}(\omega)\big)'}=\varphi(\nu(\omega))\\
&\quad=1+P_1d_1\omega+(P_1d_2+P_2d_1^2)\omega^2+\cdots\\
&\quad=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}P_kD_n^k(d_1,d_2,\cdots,d_n)\omega^n.
\end{aligned}
\tag{18}
$$

Then, from (15) and (17), we obtain

$$\sum_{k=1}^{n-1} P_k D_{n-1}^k(c_1, c_2, \cdots, c_{n-1})$$
$$= ([1+\beta(n-1)]\gamma_n(na_n - b_n) + \sum_{t=1}^{n-2} \gamma_{n-t}[1+(n-t-1)\beta] \tag{19}$$
$$\cdot \mathcal{K}_t^{-1}[(1+\beta)\gamma_2 b_2, (1+2\beta)\gamma_3 b_3, \cdots, (1+t\beta)\gamma_{t+1} b_{t+1}] \cdot [(n-t)a_{n-t} - b_{n-t}].$$

Under the assumption that $a_k = 0$ and $b_k = 0$ for $2 \leqq k \leqq n-1$, we have

$$[1+\beta(n-1)]\gamma_n(na_n - b_n) = P_1 c_{n-1}. \tag{20}$$

Similarly, by using (16) and (18), we obtain

$$\sum_{k=1}^{n-1} P_k D_{n-1}^k(d_1, d_2, \cdots, d_{n-1})$$
$$= ([1+\beta(n-1)]\gamma_n(nA_n - B_n) + \sum_{t=1}^{n-2} \gamma_{n-t}[1+(n-t-1)\beta] \tag{21}$$
$$\cdot \mathcal{K}_t^{-1}[(1+\beta)\gamma_2 B_2, (1+2\beta)\gamma_3 B_3, \cdots, (1+t\beta)\gamma_{t+1} B_{t+1}] \cdot [(n-t)A_{n-t} - B_{n-t}].$$

Applying the hypothesis, we have $A_n = -a_n$. Thus,

$$[1+\beta(n-1)]\gamma_n(-na_n - B_n) = P_1 d_{n-1}. \tag{22}$$

By making moduli of each member in (20) and (22) for $|B_n| \leqq n$ and $|b_n| \leqq n$, and using Lemma 1, we find that

$$|a_n| \leqq 1 + \frac{P_1}{n[1+(n-1)\beta]|\gamma_n|}.$$

Now, the proof of Theorem 1 is completed. □

Setting

$$\varphi(z) = \frac{1+(1-2\eta)z}{1-z} \quad (0 \leqq \eta < 1) \quad \text{and} \quad \gamma_n = \phi_n$$

in Theorem 1, we get the following corollary.

Corollary 1. *Let* $0 < \lambda \leqq 1$, $0 \leqq \beta \leqq 1$, $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ *and* $\beta \in \mathbb{C} \setminus \mathbb{Z}_0^-$. *In addition, let* $f \in \mathfrak{U}_\Sigma^{\alpha,\beta,\lambda}(\eta, \nu)$. *If* $a_k = 0$ *and* $b_k = 0$ *for* $2 \leqq k \leqq n-1$, *then*

$$|a_n| \leqq 1 + \frac{2(1-\eta)}{n[1+(n-1)\beta]|\phi_k|}.$$

Theorem 2. *Let* $0 \leqq \beta \leqq 1$ *and* $f \in \mathfrak{S}_\Sigma(\beta, \gamma_n)$. *Suppose that* $P_2 = \alpha P_1$ $(0 < \alpha \leqq 1)$. *Then,*

$$|a_2| \leqq \begin{cases} \frac{P_1}{(1+\beta)|\gamma_2|} & \left(P_1 \geqq \frac{(1+\beta)^2|\gamma_2|^2}{|2(1+2\beta)|\gamma_3|-(1+\beta)^2|\gamma_2|^2|}\right) \\ \sqrt{\frac{P_1}{|2(1+2\beta)|\gamma_3|-(1+\beta)^2|\gamma_2|^2|}} & \left(0 < P_1 \leqq \frac{(1+\beta)^2|\gamma_2|^2}{|2(1+2\beta)|\gamma_3|-(1+\beta)^2|\gamma_2|^2|}\right) \end{cases} \tag{23}$$

and

$$|a_3| \leqq \begin{cases} \frac{P_1(1+P_1)}{2(1+2\beta)|\gamma_3|} & \left(P_1 \geqq \frac{(1+\beta)^2|\gamma_2|^2}{|2(1+2\beta)|\gamma_3|-(1+\beta)^2|\gamma_2|^2|}\right) \\ \frac{P_1}{|2(1+2\beta)|\gamma_3|-(1+\beta)^2|\gamma_2|^2|} & \left(0 < P_1 \leqq \frac{(1+\beta)^2|\gamma_2|^2}{|2(1+2\beta)|\gamma_3|-(1+\beta)^2|\gamma_2|^2|}\right). \end{cases} \tag{24}$$

Proof. Setting $n = 2$ and $n = 3$ in (19) and (21), one can see that

$$(1+\beta)\gamma_2 a_2 = P_1 c_1, \tag{25}$$

$$2(1+2\beta)\gamma_3 a_3 - (1+\beta)^2\gamma_2^2 a_2^2 = P_1 c_2 + \alpha P_1 c_1^2, \tag{26}$$

$$-(1+\beta)\gamma_2 a_2 = P_1 d_1 \tag{27}$$

and

$$-2(1+2\beta)\gamma_3 a_3 + \{4(1+2\beta)\gamma_3 - (1+\beta)^2\gamma_2^2\}a_2^2 = P_1 d_2 + \alpha P_1 d_1^2. \tag{28}$$

From (25), (26), and Lemma 1, we have

$$\begin{aligned} |a_2| &= \frac{P_1|c_1|}{(1+\beta)|\gamma_2|} = \frac{P_1|d_1|}{(1+\beta)|\gamma_2|} \\ &\leqq \frac{P_1}{(1+\beta)|\gamma_2|}. \end{aligned} \tag{29}$$

Now, by adding (26)–(28), we obtain

$$\{4(1+2\beta)\gamma_3 - 2(1+\beta)^2\gamma_2^2\}a_2^2 = P_1[(c_2+\alpha c_1^2) + (d_2 + \alpha d_1^2)]$$

such that

$$a_2^2 = \frac{P_1[(c_2+\alpha c_1^2) + (d_2+\alpha d_1^2)]}{4(1+2\beta)\gamma_3 - 2(1+\beta)^2\gamma_2^2}. \tag{30}$$

By using Lemma 1 , we obtain

$$\begin{aligned} |a_2^2| &\leqq \frac{P_1[1+(\alpha-1)|c_1|^2 + 1 + (\alpha-1)|d_1|^2]}{|4(1+2\beta)|\gamma_3| - 2(1+\beta)^2|\gamma_2|^2|} \\ &\leqq \frac{P_1}{|2(1+2\beta)|\gamma_3| - (1+\beta)^2|\gamma_2|^2|}. \end{aligned}$$

Therefore, we have

$$|a_2| \leqq \sqrt{\frac{P_1}{|2(1+2\beta)|\gamma_3| - (1+\beta)^2|\gamma_2|^2|}}. \tag{31}$$

Combining (29) and (31), we derive the estimate on $|a_2|$ as presented in (28).
By subtracting (28) from (26), we get

$$4(1+2\beta)\gamma_3(a_3 - a_2^2) = P_1[(c_2+\alpha c_1^2) - (d_2+\alpha d_1^2)],$$

so that

$$a_3 = a_2^2 + \frac{P_1[(c_2+\alpha c_1^2) - (d_2+\alpha d_1^2)]}{4(1+2\beta)\gamma_3}. \tag{32}$$

Plugging (30) into (32) and using Lemma 1, we have

$$|a_3| \leqq \frac{P_1}{|2(1+2\beta)|\gamma_3| - (1+\beta)^2|\gamma_2|^2|}. \tag{33}$$

Moreover, if we substitute the value of a_2^2 from (26) into (27), we obtain

$$a_3 = \frac{P_1[(c_2 + \alpha c_1^2 + P_1 c_1^2]}{2(1+2\beta)\gamma_3}.$$

Thus, based on Lemma 1, we find that

$$|a_3| \leqq \frac{P_1(1+P_1)}{2(1+2\beta)|\gamma_3|}. \tag{34}$$

From (33) and (34), we get the estimate on $|a_3|$, as presented in (24). This proves Theorem 2. □

Putting

$$\varphi(z) = \frac{1+(1-2\eta)z}{1-z} \quad (0 \leqq \eta < 1) \quad \text{and} \quad \gamma_n = \phi_n$$

in Theorem 2, we get the following corollary.

Corollary 2. *Let* $0 < \lambda \leqq 1$, $0 \leqq \beta \leqq 1$, $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ *and* $\beta \in \mathbb{C} \setminus \mathbb{Z}_0^-$. *Additionally, let* $f \in \mathcal{P}_{\Sigma}^{\alpha,\beta,\lambda}(\eta, \nu)$. *Then,*

$$|a_2| \leqq \begin{cases} \frac{2(1-\eta)}{(1+\beta)|\phi_2|} & \left(0 \leqq \eta < 1 - \frac{(1+\beta)^2|\phi_2|^2}{2|2(1+2\beta)|\phi_3| - (1+\beta)^2|\phi_2|^2|}\right) \\ \sqrt{\frac{2(1-\eta)}{|2(1+2\beta)|\phi_3| - (1+\beta)^2|\phi_2|^2|}} & \left(1 - \frac{(1+\beta)^2|\phi_2|^2}{2|2(1+2\beta)|\phi_3| - (1+\beta)^2|\phi_2|^2|} \leqq \eta < 1\right) \end{cases}$$

and

$$|a_3| \leqq \begin{cases} \frac{(1-\eta)(1+2(1-\eta))}{(1+2\beta)|\phi_3|} & \left(0 \leqq \eta < 1 - \frac{(1+\beta)^2|\phi_2|^2}{2|2(1+2\beta)|\phi_3| - (1+\beta)^2|\phi_2|^2|}\right) \\ \frac{2(1-\eta)}{|2(1+2\beta)|\phi_3| - (1+\beta)^2|\phi_2|^2|} & \left(1 - \frac{(1+\beta)^2|\phi_2|^2}{2|2(1+2\beta)|\phi_3| - (1+\beta)^2|\phi_2|^2|} \leqq \eta < 1\right). \end{cases}$$

Theorem 3. *Let* $0 \leqq \beta \leqq 1$ *and* $f \in \mathfrak{S}_{\Sigma}(\beta, \gamma_n)$. *Moreover, let* $P_2 = \alpha P_1$ $(0 < \alpha \leqq 1)$. *Then,*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \frac{P_1}{2(1+2\beta)|\gamma_3|} & \left(0 \leqq |y(\mu)| \leqq \frac{P_1}{4(1+2\beta)|\gamma_3|}\right) \\ 2|y(\mu)| & \left(|y(\mu)| \geqq \frac{P_1}{4(1+2\beta)|\gamma_3|}\right), \end{cases} \tag{35}$$

where

$$y(\mu) = \frac{(1-\mu)P_1}{4(1+2\beta)\gamma_3 - 2(1+\beta)^2\gamma_2^2} \quad (\mu \in \mathbb{C}).$$

Proof. According to (30) and (32), we have

$$\begin{aligned} a_3 - \mu a_2^2 =& (1-\mu)\frac{P_1[(c_2+\alpha c_1^2) + (d_2 + \alpha d_1^2)]}{4(1+2\beta)\gamma_3 - 2(1+\beta)^2\gamma_2^2} \\ &+ \frac{P_1[(c_2+\alpha c_1^2) - (d_2+\alpha d_1^2)]}{4(1+2\beta)\gamma_3}, \end{aligned}$$

so that

$$\begin{aligned} a_3 - \mu a_2^2 =& \left(\frac{(1-\mu)P_1}{4(1+2\beta)\gamma_3 - 2(1+\beta)^2\gamma_2^2} + \frac{P_1}{4(1+2\beta)\gamma_3}\right)(c_2 + \alpha c_1^2) \\ &+ \left(\frac{(1-\mu)P_1}{4(1+2\beta)\gamma_3 - 2(1+\beta)^2\gamma_2^2} - \frac{P_1}{4(1+2\beta)\gamma_3}\right)(d_2 + \alpha d_1^2), \end{aligned}$$

that is, that

$$a_3 - \mu a_2^2 = \left(y(\mu) + \frac{P_1}{4(1+2\beta)\gamma_3}\right)(c_2 + \alpha c_1^2) + \left(y(\mu) - \frac{P_1}{4(1+2\beta)\gamma_3}\right)(d_2 + \alpha d_1^2), \tag{36}$$

where

$$y(\mu) = \frac{(1-\mu)P_1}{4(1+2\beta)\gamma_3 - 2(1+\beta)^2\gamma_2^2}.$$

Taking the moduli of each member of (36), we have

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \frac{P_1}{2(1+2\beta)|\gamma_3|} & \left(0 \leqq |y(\mu)| \leqq \frac{P_1}{4(1+2\beta)|\gamma_3|}\right) \\ 2|y(\mu)| & \left(|y(\mu)| \geqq \frac{P_1}{4(1+2\beta)|\gamma_3|}\right). \end{cases} \tag{37}$$

This completes the proof of Theorem 3. □

Setting

$$\varphi(z) = \frac{1+(1-2\eta)z}{1-z} \quad (0 \leqq \eta < 1) \quad \text{and} \quad \gamma_n = \phi_n$$

in Theorem 3, we get the following corollary.

Corollary 3. *Let $0 < \lambda \leqq 1$, $0 \leqq \beta \leqq 1$, $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ and $\beta \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Furthermore, let $f \in \mathcal{P}_\Sigma^{\alpha,\beta,\lambda}(\eta,\nu)$. Then,*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \frac{(1-\eta)}{(1+2\beta)|\phi_3|} & \left(0 \leqq |h(\mu)| \leqq \frac{(1-\eta)}{2(1+2\beta)|\phi_3|}\right) \\ 2|y(\mu)| & \left(|h(\mu)| \geqq \frac{(1-\eta)}{2(1+2\beta)|\phi_3|}\right), \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)(1-\eta)}{2(1+2\beta)\phi_3 - (1+\beta)^2\phi_2^2}.$$

By taking $\mu = 1$ in Theorem 3, we have the following corollary.

Corollary 4. *Let $0 \leqq \beta \leqq 1$ and $f \in \mathfrak{S}_\Sigma(\beta, \gamma_n)$. Then,*

$$|a_3 - a_2^2| \leqq \frac{P_1}{2(1+2\beta)|\gamma_3|}.$$

3. Conclusions

In the investigation of bi-univalent functions, estimates of the first two coefficients are usually obtained. However, there are bounds of the first three Taylor–Maclaurin coefficients, which are given in [21]. In this paper, we introduced a new subclass of bi-close-to-convex functions associated with the generalized hypergeometric functions defined in the open unit disk. By using the Faber polynomial expansions, the estimates for the general Taylor–Maclaurin coefficients of the functions in this subclass were derived. In particular, several previous results were generalized.

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