PermuNim: An impartial game of permutation avoidance

by

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Abstract

PermuNim is an impartial combinatorial game played on a board of squares where each player takes turns playing in rows and columns of the board which have not been played in, avoiding zero or more permutations. The game comes to an end when neither player can move. The first player unable to move on his or her turn loses the game.

Many researchers have investigated combinatorial game theory as well as the idea of permutation pattern avoidance. PermuNim combines both of these ideas.

When (12) or (1) is the forbidden permutation in PermuNim, or when the forbidden permutation is 'close' in size to that of the smallest of the two dimensions of the board, we can say a great deal about the value of the game. For other permutations, the values of the options seem much more chaotic. Even (123) is chaotic as evidenced by our data in the appendix. We investigate the trend for even height boards which are 'wide enough' to have options with all odd values and vice versa but we don't believe that this to be true in general. If a PermuNim board is stretched by adding columns, sometimes the value of the position is affected. We find that when any permutation is avoided and t moves have been made, as long at 2^{m-t} columns are available together, there is a place where any number of columns may be added to the board without affecting the value of the position. We suspect that the number of columns necessary may be much lower for some permutations.

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Chapter 1

Preliminaries

1.1 Impartial Games

A *combinatorial game* has just two players: Player One or Alice and Player Two or Bob. There are *positions*, which the players take turns changing in defined ways to achieve a defined winning condition. All combinatorial games will come to an end after finitely many turns because one player is unable to move. In the *normal play* convention a player who is unable to play on his turn loses. There is also the *misère play* convention in which a player unable to move wins. For the duration of this paper we will assume that all games are played in the normal play convention.

In combinatorial game theory, an *impartial game* is one in which Alice and Bob have the same moves available from any position.

Every position in an impartial game may be identified with the set of positions which can be reached in one move. These positions are called the *options* of the game. At any position of an impartial game, if $A_1, A_2, \ldots, A_m$ are the values of the options, then the value of the position can be denoted by $\{A_1, A_2, \ldots, A_m\}$.

If G is a game, and position H is a game reachable by one or more successive moves from the starting position of G, then the game H is called *simpler* than G.

Nim is an impartial combinatorial game which plays a central role in the study of impartial game theory. It is a two-player mathematical game of strategy in which players take turns removing objects from distinct heaps. On each turn, a player must remove at least one object, and may remove any number of objects provided that they all come from the same heap. Every nim heap has a value called a *nimber* or *Grundy number*. The value of a nim heap of size n is denoted by the nimber $*n$.

Since the value of an impartial game can be defined as the set of its options, we may define

$$*0 = \{\}$$

$$*1 = \{*0\}$$

$$*2 = \{*0, *1\}$$

$$\vdots$$

$$*n = \{*0, *1, *2, \ldots, *(n-1)\}$$

If the options of a position of some impartial game are the nim heaps $*a, *b, *c, \ldots$ then the game at this position can be regarded (using a "reversibility" argument) as a nim heap $*m$, where m is the *Minimal-EXcluded* natural number or *mex* of the values $*a, *b, *c, \ldots$. For example

$$\text{mex}\{*1, *2, *3\} = *0 \quad \text{and} \quad \text{mex}\{*0, *1, *4, *7, *18\} = *2.$$

Theorem 1.1 (Sprague 1935, Grundy 1939) *Every position in every impartial game is equivalent to some nim heap.*

Proof. Assume the result is true for all games which are simpler than G, in particular the options of G. So the options are equivalent to nim heaps whose sizes form a set S (which we assume is not the set of all nonnegative integers). From the mex rule we then get $G \equiv *m$ for $m = \text{mex}(S)$. ■

The *nim value* of an impartial game is then defined as the unique nimber equivalent to that game. If an impartial game has all the nonnegative integers as its set of options we say that the game has value $*\omega$.

Two impartial games G and H can be added to make a new game $G+H$ in which a player can chose either to move in G or in H. More formally if $G = \{G_1, G_2, \ldots, G_n\}$ and $H = \{H_1, H_2, \ldots, H_m\}$ then

$$G + H = \{G_1 + H, G_2 + H, \ldots, G_n + H, G + H_1, G + H_2, \ldots, G + H_m\}.$$

The sum of two nimbers $*m$ and $*n$ can then be defined to be the size of the single nim heap equivalent to the two heaps $*m$ and $*n$. It can be shown that this sum is equivalent to binary addition without carry or exclusive or and can be defined recursively by

$$*a \oplus *0 = *0 \oplus *a = *a$$

and

$$*a \oplus *b = \text{mex}\{\{*a' \oplus *b \mid *a' < *a\} \cup \{*a \oplus *b' \mid *b' < *b\}\}.$$

The following table demonstrates that nim addition is equivalent to binary addition without carrying.

*	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
7	7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
12	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
14	14	14	12	13	10	11	8	9	6	7	4	5	2	3	0	1
15	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

By associativity, the impartial game nim is mathematically solved for any number of initial heaps and objects. The following can be found in [1].

Theorem 1.2 (Bouton 1902) *In a normal nim game, Alice has a winning strategy if and only if the nim sum of the sizes of the heaps is nonzero. Otherwise, Bob has a winning strategy.*

Therefore a winning strategy is to finish every move with a nim sum of 0. This is always possible if the nim sum is not zero before the move. To find out which move to make let X be the nim sum of all the heap sizes. Find a heap with value Y such that $X \oplus Y \leq Y$. A winning strategy is to reduce that heap to the nim sum of its original size with X.

For more information and examples, [1] gives an excellent introduction to combinatorial game theory.

1.2 Permutation patterns

A *permutation* τ of length k is written as $(a_1, a_2, ..., a_k)$ where $\tau(i) = a_i$ for $1 \leq i \leq k$. For $k < 10$ we omit the commas from the notation. As usual, let $\mathcal{S}_n$ denote the symmetric group on $[n] = \{1, 2, ..., n\}$.

Let $\tau \in \mathcal{S}_k$ and $\pi \in \mathcal{S}_n$ for some $k \leq n$. π is said to be τ*-avoiding* if there are no distinct elements $i_{\tau(1)}, i_{\tau(2)}, ..., i_{\tau(k)}$ of $[n]$ such that

$$\pi(i_1) < \pi(i_2) < \cdots < \pi(i_k).$$

If there is such a subsequence, the subsequence $\pi(i_{\tau(1)}), \pi(i_{\tau(2)}), \ldots, \pi(i_{\tau(k)})$ is said to be of type τ .

For example, the permutation $\pi = (52687431)$ is (2413)-avoiding. π is not (3142)-avoiding though because of its subsequence (5283). An equivalent, but perhaps more insightful, definition is the following reformulation in terms of matrices.

Let $\tau \in \mathcal{S}_n$. The permutation matrix $M(\tau)$ is the $n \times n$ matrix having a 1 in position $(i, \tau(i))$ for $1 \leq i \leq n$, and having 0 elsewhere. Given two permutation matrices M and N, M is said to avoid N if no submatrix of M is identical to N.

Let $\mathcal{S}_n(\tau)$ denote the set of τ-avoiding permutations in $\mathcal{S}_n$. Two permutations τ and σ are called *Wilf equivalent* if $|\mathcal{S}_n(\tau)| = |\mathcal{S}_n(\sigma)|$ for all $n \in \mathbb{N}$. Wilf equivalence defines an equivalence relation on sets of permutations, and we call the resulting equivalence classes Wilf classes.

A sequence which admits a partition into k sequences is said to be a shuffling of those k sequences. The permutations which avoid (12) are exactly the decreasing sequences. The permutations which avoid (123) consist of those which have no ascending subsequence of length three. These are exactly those sequences obtained by shuffling two decreasing sequences together. Similarly sequences avoiding $(123 \ldots k)$ are avoided by sequences obtained by shuffling any $k-1$ decreasing sequences.

All permutations of size 3 are equivalent to (123) or (132) by symmetries of their associated permutation matrices. That is, any size 3 permutation matrix can be reflected or rotated to achieve one or (123) or (132). The first major result in the theory of forbidden subsequences states that (123) and (132) are Wilf equivalent. It then follows that S_3 is one Wilf class. In fact, the number of permutations of length n which avoid each of (123) or (132) is the nth Catalan number. By comparison, the partition of $\mathcal{S}_4$ into Wilf classes is more complicated but has been completely solved. See [3] for details.

A permutation is called an involution if it is its own inverse. We call a permutation half- or quarter-turn invariant if the corresponding permutation matrices are not changed by rotation by 180 or 90 degrees respectively.

Chapter 2

Introduction

2.1 PermuNim

PermuNim is an impartial game played on a board of squares where each player takes turns playing in rows and columns of the board which have not been played in, where zero or more permutations are avoided. The game comes to an end when neither player can move. The first player unable to move is the loser. We denote the value of the PermuNim game in which the permutation σ is avoided on a board with m rows and n columns by $P^{m,n}(\sigma)$ when no moves have been made and by $P^{m,n}_{(i,j)}(\sigma)$ when a move has been made in cell (i,j). The boards which these games are played on are similarly denoted by $\Pi^{m,n}(\sigma)$ and $\Pi^{m,n}_{(i,j)}(\sigma)$ respectively and display the options of the position. Not every cell in $\Pi^{m,n}_{(i,j)}(\sigma)$ would have a value though because some cells may become unplayable after each move. These are indicated by empty cells. Cells which have been played in are denoted by a •. When more than one move has been made you may replace (i, j) in the notation by a list of cells separated by commas. For example $P^{5,4}_{(1,1),(3,4)}(123) = *2$ because it denotes the value of the PermuNim game played on a board with 5 rows and 4 columns avoiding the identity permutation of size three where there has been a play made in the top left corner and in the right most column, three rows from the top. These two plays make some cells unplayable but also leave *0 and *1 as options and thus the position has value *2.

$\Pi^{5,4}_{(1,1),(3,4)}(123)$

•			
			•
	0	1	
	1	0	

The permutation σ may also be replaced in the notation by a list of permutations when more than one permutation is forbidden or by a blank space when no permutations are being avoided.

A PermuNim board may also have infinitely many rows or columns. $\Pi^{m,\mathbb{Z}^+}$ is a board with m rows and infinitely many columns which may be indexed from left to right by the set

$$\mathbb{Z}^+ = \{1, 2, \dots\}.$$

$\Pi^{m,\mathbb{Z}}$ is a board with m rows and infinitely many columns which may be indexed by the set

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2 \ldots\}.$$

Similarly $\Pi^{m,\mathbb{Q}^+}$ and $\Pi^{m,\mathbb{Q}}$ have rows which may be indexed by the sets $\mathbb{Q}^+$ and $\mathbb{Q}$ respectively, where $\mathbb{Q}$ is the set of rational numbers and $\mathbb{Q}^+$ is the set of rational numbers which are greater than or equal to 1.

2.2 Elementary results

The following result holds regardless of the permutation(s) being avoided.

Lemma 2.1 *Consider the values $P^{m,n}(\sigma)$ or $P^{l,S}(\sigma)$ where $l = min\{m, n\}$, σ is a permutation, and $S \in \{\mathbb{Z}, \mathbb{Z}^+, \mathbb{Q}, \mathbb{Q}^+\}$. If Bob can take a strategy so that an even number of moves are made, then the value of the position is $*0$. Conversely if Alice can take a strategy so that an odd number of moves are made, then the value of the position is non zero. If Alice is in a position where she has no choice but to play so that an odd number of moves remain, then the value of the game is $*1$.*

Proof. If an even number of moves are made, then Bob makes the last move winning the game. Thus all the options for Alice are non-zero, losing positions and therefore the original game has value *0. Conversely if Alice can take a strategy so that an odd number of moves are made, then *0 is an option and thus the value of the original game is non zero. If Alice has no choice but to take an odd number of moves then all her options will be *0 and thus the value of the game will be *1. ■

The most trivial cases of PermuNim are when no permutations are being avoided, and when the forbidden permutation is (1). When (1) is the forbidden permutation, no moves may be made and thus Alice loses and the value of the game is $*0$ regardless of the dimensions of the board. The following result demonstrates that when no permutation is being avoided the value of the board is dependent only on the parity of the smallest dimension. In this case we call the winner the parity winner.

Theorem 2.2 *Let $l = min\{m, n\}$ and $S \in \{\mathbb{Z}, \mathbb{Z}^+, \mathbb{Q}, \mathbb{Q}^+\}$. Then*

$$P^{m,n}() = P^{l,S}() = \begin{cases} *0 & \textit{if } l \textit{ is even;} \\ *1 & \textit{if } l \textit{ is odd.} \end{cases}$$

Proof. When no permutations are being avoided players will move until there are no available rows or columns left to move in. Thus exactly l moves will be made and the result follows by Theorem 2.1 ■

With any instance of PermuNim on a finite board there is, as the next result expresses, always an upper bound on the possible Nim Value.

Theorem 2.3 *For any permutation σ, $P_S^{m,n}(\sigma) \leq d = min\{r, c\}$ where r and c are the number of rows and columns which are available to be moved into, and S is any feasible set of plays which might have been made.*

Proof. For any position of an instance of PermuNim where the board only has one available row or only one available column, all options have value *0 since they end the game. Therefore the value of the game is *1 as required. Let $d \geq 1$. Assume that in any position if there are less than or equal to d available rows or columns, that the value of the game is no more than d. Consider an instance of PermuNim played on a board with either no more than $(d+1)$ available rows or no more than $(d+1)$ available columns. Its options have value no more than $*d$ by the induction hypothesis and thus the value of the position is no more than $*(d+1)$. ■

If a permutation is an involution or if it has some rotational symmetry we can make some observations about the nim values of the options.

If σ is an involution then $P^{m,n}(\sigma) = P^{n,m}(\sigma)$. For example $P^{5,7}(132) = P^{7,5}(132)$.

$\Pi^{5,7}(132)$

4	4	2	0	0
4	0	0	2	2
2	0	2	4	0
0	0	0	2	1
0	0	2	4	0
0	2	1	0	2
0	4	0	4	0

$\Pi^{7,5}(132)$

4	4	2	0	0	0	0
4	0	0	0	0	2	4
2	0	2	0	2	1	0
0	2	4	2	4	0	4
0	2	0	1	0	2	0

More generally if S is any set of occupied cells on an $n \times m$ PermuNim board where the permutation being avoided is an involution σ, then the transposed set of occupied cells $S^\top$ on an $m \times n$ PermuNim board contains the same values. This is not however true for permutations which are not involutions.

If σ is half turn invariant then $P^{m,n}_{(i,j)}(\sigma) = P^{m,n}_{(m+1-i,n+1-j)}(\sigma)$. In other words the board is also half turn invariant. In situations such as this, only half of the options would need to be analyzed to determine the value of the game. For example $\Pi^{6,7}_{(i,j)}(2143) = \Pi^{6,7}_{(6-i+1,7-j+1)}(2143)$ for $1 \leq i \leq 6$ and $1 \leq j \leq 7$

$\Pi^{6,7}(2143)$

1	1	1	1	3	1	1
1	1	1	1	1	1	1
1	1	1	1	1	1	3
3	1	1	1	1	1	1
1	1	1	1	1	1	1
1	1	3	1	1	1	1

Similarly if σ is quarter turn invariant, then so is $\Pi^{m,m}(\sigma)$ for any m. In this situation only about a quarter of the options would need to be analyzed to determine the value of the game. For example consider an 8×8 board where (2413) is the permutation being avoided.

$\Pi^{8,8}(2413)$

1	3	3	5	3	1	1	1
1	1	3	1	2	1	1	3
1	1	1	1	1	1	3	3
3	2	1	1	1	1	1	5
5	1	1	1	1	1	2	3
3	3	1	1	1	1	1	1
3	1	1	2	1	3	1	1
1	1	1	3	5	3	3	1

The following result is an important tool for analyzing any instance of PermuNim.

Lemma 2.4 *Consider a board in a position in which all available cells may be divided into m sections in such a way that no two sections contain the same row or column and that any permutation that is being avoided which can fit in the available cells would be entirely contained in one section. The value of a board in this situation is the nim sum of the values of the sections, each avoiding the forbidden permutation(s).*

Proof. Each of the m sections are completely independent of each other since they do not share any rows or columns and any instance of the forbidden permutation would be contained entirely in one section. So plays made in once section can not affect the other sections. ∎

We will use this result throughout the thesis, in particular in Chapter 3 to completely identify the value of every subposition of PermuNim on a finite board where (12) is the forbidden permutation. An example illustrating this result follows the proof of Theorem 3.1

2.3 Summary of results

Chapter 3 will discuss instances of PermuNim in which (12) is the forbidden permutation. On the finite board when no moves have been made, this game has value equal to the smallest dimension of the board. We are able to recursively calculate the value of each cell of this game. $P^{m,\mathbb{Z}}(12)$, $P^{m,\mathbb{Z}^+}(12)$ have value m but $P^{m,\mathbb{Q}}(12)$ has value 0 for m even and 1 for m odd, and

$P^{m,\mathbb{Q}^+}(12)$ has value 1 for $m = 1$, 2 for m even and 3 for m odd larger than 1.

In Chapter 4, games where (123) is the forbidden permutation are discussed. Our analysis of this instance of the game is much less complete than that for (12). The options seem much more chaotic and complex. For small boards there seems to be a trend for even height boards to have all odd entries if they are 'wide enough' and for odd height boards to have all even entries if they are 'wide enough' however we have an example which suggests this is not true in general. We are able to make conclusions about half of the cells on the even by even board by using a strategy whereby one player could take the winning strategy of mirroring his opponents moves. We will also demonstrate in this chapter that a similar mirroring strategy confirms that $P^{m,\mathbb{Z}}(123)$ has value 0 for m even. Finally in chapter 4 we show that $P^{m,\mathbb{Q}}(123)$ has value 0 if m is even and 1 if m is odd.

Chapter 5 investigates the instance of PermuNim where the permutation being avoided is close in size with that of the board. We will demonstrate that for any positive integer k there exists M such that for any $m \geq M$ and any permutation of size $m-k$, $P^{m,n}(\sigma)$ is 0 if m is even and 1 if m is odd. We then end chapter 5 by demonstrating a bound for M when the permutation is an identity.

If a PermuNim board is stretched by adding columns, sometimes the value of the position is affected and sometimes not. In the case where the permutation is (12), and the board is square or wider, adding columns to the board does not affect the value of the position when no moves have been made because the value of the position is only dependent on the smallest dimension of the board. Chapter 6 extends this idea to show that even after any number of moves have been made, columns may be added without affecting the value of the board. For an arbitrary permutation we are able to show that on a board of height m when t moves have been made, if there is a set of at least 2^{m-t} consecutive columns not played in, then columns may be added and deleted from this set without affecting the value of the position as long as $2^{m-t} - 1$ consecutive columns which have not been played in remain in the set. The results from this chapter give us that no more than 2^m columns are needed on a board of height m avoiding (123) before adding more columns will not affect the value of the position but our computer-generated data suggest that much fewer than that many columns are necessary before this is guaranteed. We conclude chapter 6 by demonstrating that although 2^m columns may not be needed, there is no constant c, such that having c more columns than rows would guarantee this.

Chapter 3

The Permutation (12)

We will now look at PermuNim where (12) is the permutation being avoided on finite and infinite boards.

3.1 The Finite Board

Theorem 3.1 $P^{m,n}(12) = min\{*m, *n\}$

Proof. Since (12) is an involution, it is enough to prove the result for boards which have at least as many columns as rows. We proceed by strong induction on the number of rows. For any positive integer n, $P^{1,n}(12) = *1$ because every option is clearly $*0$. Now assume for some fixed m, that for any $n \geq m$, $P^{k,n}(12) = *k$ for every $1 \leq k \leq m$. Consider $P^{m+1,n}(12)$, where n is now any integer at least $(m+1)$. The option of moving into cell $(i, 1)$ has value $*(i-1)$ for each $1 \leq i \leq (m+1)$. This follows by the induction hypothesis because the position at this stage is equivalent to $P^{(i-1),n}(12)$, since rows $i, i+1, \ldots, n$ have become unplayable. Theorem 2.3 gives us that the largest value that the position can be is $*(m+1)$. It then follows that the value of the position must be $\text{mex}\{i-1 \mid 1 \leq i \leq (m+1)\} = \text{mex}\{0, 1, \ldots, m\} = *(m+1)$. ∎

We can say more about this game then just the value before any moves have been made. In fact we can completely identify the value of every sub-position by applying Lemma 2.4 and Theorem 3.1 . All moves made must be in an off-diagonal pattern to avoid creating an instance of the forbidden permutation. Cells above and to the left, or below and to the right of any move become unplayable and the remaining playable region is composed of rectangular blocks of cells in an off-diagonal pattern. For example

$$P^{13,14}_{(4,8),(10,5)}(12) = P^{3,4}(12) \oplus P^{5,2}(12) \oplus P^{3,6}(12) = *3 \oplus *2 \oplus *3 = *2.$$

–	–	–	–	$\Pi^{3,6}(12)$
–	–	–	•	–
–	–	$\Pi^{5,2}(12)$	–	–
–	•	–	–	–
$\Pi^{3,4}(12)$	–	–	–	–

In this way we can calculate the value of each of the options of any instance of PermuNim on a finite board where (12) is the forbidden permutation

For example:

$\Pi^{4,4}(12)$

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

$\Pi^{5,5}(12)$

0	1	2	3	4
1	0	3	2	3
2	3	0	3	2
3	2	3	0	1
4	3	2	1	0

$\Pi^{6,6}(12)$

0	1	2	3	4	5
1	0	3	2	5	4
2	3	0	1	2	3
3	2	1	0	3	2
4	5	2	3	0	1
5	4	3	2	1	0

This game, although nice to analyze is not very interesting to play because Alice can always end the game on her first move by playing in the top left or bottom right cell of the board. For $P^{m,\mathbb{Z}^+}(12)$ it is also the case that Alice can end the game on her first move by playing in cell $(1,1)$.

3.2 Infinite Boards

Theorem 3.2

$$P^{m,\mathbb{Z}^+}(12) = *m$$

Proof. If $m = 1$ then every option of this game has value *0 and thus $P^{1,\mathbb{Z}^+}(12) = *1$. Suppose for some positive integer m, that $P^{k,\mathbb{Z}^+}(12) = *k$ for all $1 \leq k \leq m$. $P^{m+1,Z^+}_{(i,1)}(12) = P^{i-1,\mathbb{Z}^+}(12)$ for $i \in \{1, 2, 3, \ldots, m+1\}$ because rows i through $(m+1)$ become unplayable. Since $P^{i-1,\mathbb{Z}^+}(12) = *(i-1)$ by the induction hypothesis, it then follows that $P^{m+1,Z^+}_{(i,1)}(12) = *(i-1)$ for $i \in \{1, 2, 3, \ldots, m+1\}$. Therefore since this position has options {0,1,2, ..., m}, Theorem 2.3 gives us that $P^{m+1,\mathbb{Z}^+}(12) = *(m+1)$. ■

From Theorem 3.2 we can prove a similar result for $\Pi^{m,\mathbb{Z}}$. This game is a little more interesting to play because there is no play that Alice could make which would end the game on her first move. The starting position no longer has value equal to the height of the board. The value becomes dependent on the parity of the height.

Theorem 3.3

$$P^{m,\mathbb{Z}}(12) = \begin{cases} *0 & \textit{if } m \textit{ is even} \\ *1 & \textit{if } m \textit{ is odd} \end{cases}$$

Proof. If Alice moves into cell (i, j) of the board, then cells

$$\{(x, y)|(x > i \text{ and } y > j) \text{ or } (x < i \text{ and } y < j)\}$$

become unplayable. This leaves two regions left to play in (either of which may be empty), an upper right region R_1 and a lower left region R_2. These regions share no rows or columns, and any instance of (12) that may be formed would be entirely in one of these regions. R_1 has value $P^{i-1,Z^+}(12) = *(i-1)$, by Theorem 3.2 and since (12) is half turn invariant, you can rotate R_2 180 degrees to see that it is similarly equivalent to $P^{m-i,Z^+}(12) = *(m-i)$. So by Theorem 2.4, $P^{m,\mathbb{Z}}_{(i,j)}(12) = (i-1) \oplus (m-i)$. When you nim-sum two even nimbers you get an even, when you nim-sum two odd nimbers you get an even and when you nim-sum an even nimber with an odd nimber you get an odd nimber. This is because nim-addition is bitwise, and parity is determined precisely by the least-significant binary digit. It then follows that for $1 \leq i \leq m$, $(m-i) \oplus (i-1)$ is odd for m even and even for m odd and thus

$$P^{m,\mathbb{Z}}(12) = \text{mex}\{(i-1) \oplus (m-i) |\; i = 1, 2, \ldots, m\}.$$

This is $*0$ for m even because all of the options are odd. Similarly the value is $*1$ for m odd because all of the options are even and include $*0$ for $i = \frac{m+1}{2}$.
■ Again for the board $\Pi^{m,\mathbb{Q}^+}$, Alice can end the game on her first move into cell $(1,1)$.

Theorem 3.4

$$P^{m,\mathbb{Q}^+}(12) = \begin{cases} *1 & \textit{if } m \textit{ is } 1 \\ *2 & \textit{if } m \textit{ is } 2 \\ *3 & \textit{if } m \textit{ is } 3 \end{cases}$$

Proof. $P^{1,\mathbb{Q}^+}(12) = *1$ because all options are $*0$. $P^{2,\mathbb{Q}^+}(12) = *2$ because it has $*0$ and $*1$ as options in the first column. $P^{3,\mathbb{Q}^+}(12) = *3$ because it has $*0$ and $*1$ as options in the first column and $P^{3,\mathbb{Q}^+}_{1,i}(12) = P^{2,\mathbb{Q}^+}(12) = *2$.

Suppose for some $m \geq 3$ the result holds for all $1 \leq k \leq m$.

For m even or odd, $P^{m+1,\mathbb{Q}^+}_{1,1}(12) = *0$ because it ends the game and $P^{m+1,\mathbb{Q}^+}_{1,i}(12) = P^{i-1,\mathbb{Q}}(12)$ which is $*0$ for i odd and $*1$ for i even when i is at least 2. So the first column contains only 1s and 0s.

When $m+1$ is even we will show that for any column j other than column 1, the options are only 3s and 1s.

$$P_{m+1,j}^{m+1,\mathbb{Q}^+}(12) = P^{m,\mathbb{Q}}(12) = *1$$

$$P_{m,j}^{m+1,\mathbb{Q}^+}(12) = P^{1,\mathbb{Q}^+}(12) \oplus P^{m-2,\mathbb{Q}}(12) = *1 \oplus *0 = *1$$

$$P_{1,j}^{m+1,\mathbb{Q}^+}(12) = P^{m,\mathbb{Q}^+} = *3$$

$$P_{i,j}^{m+1,\mathbb{Q}^+}(12) = P^{m+1-i,\mathbb{Q}^+}(12) \oplus P^{i-1,\mathbb{Q}}(12)$$

which is $*3 \oplus *0 = *3$ if i odd and $*2 \oplus *1 = *3$ if i even. It then follows that $P^{m+1,\mathbb{Q}^+}(12)$ is $*2$ for $m+1$ even since the only options are $*0$, $*1$ and $*2$.

When $m+1$ is odd we will show that for any column j other than column 1, the options are only 0s, 1s, and 2s.

$$P_{m+1,j}^{m+1,\mathbb{Q}^+}(12) = P^{m,\mathbb{Q}}(12) = *0$$

$$P_{m,j}^{m+1,\mathbb{Q}^+}(12) = P^{1,\mathbb{Q}^+}(12) \oplus P^{m-1,\mathbb{Q}}(12) = *1 \oplus *1 = *0$$

$$P_{1,j}^{m+1,\mathbb{Q}^+}(12) = P^{m,\mathbb{Q}^+}(12) = *2$$

$$P_{i,j}^{m+1,\mathbb{Q}^+}(12) = P^{m+1-j,\mathbb{Q}^+}(12) \oplus P^{i-1,\mathbb{Q}}(12)$$

which is $*3 \oplus *1 = *2$ when i is even and $*2 \oplus *0 = *2$ when i is odd for $2 \leq i \leq m-1$. It then follows that $P^{m+1,\mathbb{Q}^+}(12) = *3$ for $m+1$ odd since the only options are $*0$, $*1$ and $*2$.

■

Again for $\Pi^{m,\mathbb{Q}}$ the board no longer has value equal to its height. The value is dependent on the parity of the height. We can see from this that having a least column plays a significant role in determining the outcome of the game.

Theorem 3.5

$$P^{m,\mathbb{Q}}(12) = \begin{cases} *0 & \text{if } m \text{ is even} \\ *1 & \text{if } m \text{ is odd} \end{cases}$$

Proof. Suppose at some position after the first move of the game there is a row r which has not yet been played in. If row r is below all rows which have been played in, then the next player may move into row r in any position to the left of all other previous moves. If row r is above all rows which have been played in, then the next player may move into row r in any position to the right of all other previous moves. Otherwise row r is between two rows which have already been played in. Let row s be the closest row above r which has been played in and row t be the closest row below r which has been played in. The next player can then move in row r in any column (position) which is in between the moves in rows s and t. This move will clearly not create an instance of the forbidden permutation and such a position will always exist. Since at any position of the game, players may move until all rows have been moved in, the result follows by Lemma 2.1 ■

In the next chapter we will generalize this result to larger identity permutations and even to arbitrary permutations of size k which begin with a

1 or end with a k. We will also investigate the trend for even height boards to have value $*0$ and odd height boards to have value $*1$. We will now take a look at boards which have both infinitely many rows and infinitely many columns.

When avoiding the permutation (12) on a board with infinitely many rows and columns there are many instances which will not satisfy the ending condition. If either dimension is indexed by the rationals then the ending condition is not satisfied. Similarly if the board does not have a least column and a greatest row (Southeast quadrant) or a least row and a greatest column (Northwest quadrant), then play may go on forever.

We prove the following result for the Southeast quadrant board when both dimensions are indexed by the integers.

Theorem 3.6 *A board with rows indexed by the integers and columns indexed by the integers where there is a leftmost column and a highest row has value $*\omega$ when* (12) *is the forbidden permutation.*

Proof. Playing in position $(1, i)$ for any positive integer i makes columns to the right of column $(i-1)$ unplayable. Therefore this position has value $(i-1)$ by Theorems 3.2. So the board has every non-negative integer as an option and thus the value of the board is $*\omega$. ∎

Chapter 4

(123) and Larger Identities

PermuNim where (123) is the forbidden permutation is much more complicated to analyze than the cases of (), (1), or (12). The values of the options appear much more chaotic.

At least for boards with relatively small heights, if they are 'wide enough' there seems to be a trend for even height boards to have value zero and more specifically all odd entries and similarly for odd height boards to have value one and more specifically all even entries. This is not true in general. For an example, it can be seen in the appendix that $P^{7,n}(123)$ appears to have *3 as an option even as n gets large. This trend may be attributed to Lemma 2.1 and also to the fact that on a small board the permutation is closer in size to the board. We discuss this further in chapter 5 when we look at situations where the permutation is close in size to that of the board. For now we will just explore this trend for the permutation (123).

Examples

The data in the appendix suggest that $P^{3,n}(123)$ for $n \geq 3$ has only 0s and 2s as options.

$\Pi^{3,3}(123)$

2	0	0
0	2	0
0	0	2

$\Pi^{3,4}(123)$

2	0	0	0
0	2	2	0
0	0	0	2

$\Pi^{3,5}(123)$

2	0	0	0	0
0	2	0	2	0
0	0	0	0	2

$\Pi^{3,6}(123)$

2	0	0	0	0	0
0	2	0	0	2	0
0	0	0	0	0	2

The data suggest that $P^{4,n}(123)$ for $n \geq 4$ has only 1s and 3s as options.

$\Pi^{4,4}(123)$

3	3	1	1
3	1	3	1
1	3	1	3
1	1	3	3

$\Pi^{4,5}(123)$

3	3	1	1	1
3	1	3	3	1
1	3	3	1	3
1	1	1	3	3

$\Pi^{4,6}(123)$

3	3	1	1	1	1
3	1	3	1	3	1
1	3	1	3	1	3
1	1	1	1	3	3

$\Pi^{4,7}(123)$

3	3	1	1	1	1	1
3	1	3	1	1	3	1
1	3	1	1	3	1	3
1	1	1	1	1	3	3

$P^{5,5}(123)$ has 1 as an option but for $n \geq 6$ the data suggest that $P^{5,n}(123)$ has only even options.

$\Pi^{5,5}(123)$

4	4	4	0	0
4	2	2	4	0
4	2	1	2	4
0	4	2	2	4
0	0	4	4	4

$\Pi^{5,6}(123)$

4	4	4	0	0	0
4	2	0	4	4	0
4	0	4	4	0	4
0	4	4	0	2	4
0	0	0	4	4	4

$\Pi^{5,7}(123)$

4	4	4	0	0	0	0
4	2	0	4	2	4	0
4	0	4	2	4	0	4
0	4	2	4	0	2	4
0	0	0	0	4	4	4

See the appendix for some larger examples. You may notice that there seems to be some consistency when more columns are added to a board. We will discuss this phenomenon further in Chapter 6.

The following result supports the tendency for even-width boards to have value zero by demonstrating that games on boards with even width and height avoiding (123) do not have zero options at least in the Northwest and Southeast quadrants of the board.

Theorem 4.1 *Consider an even by even board avoiding* (123)*. Options in the Northwest and Southeast quadrants of the board have non-zero values.*

Proof. Suppose that Alice makes her first move in cell a in the Northwest or Southeast quadrant of the board. We will show that Bob has a winning response by moving on every turn into the 180-degree rotated image of Alice's move. For the duration of this proof if a cell of the board is labeled by a letter then the prime of that letter will be used to label the cell in the 180-degree reflected image.

Suppose Alice moves into the Northwest or Southeast quadrants of the board and that Bob takes this strategy in reflecting Alice's moves and at some point Alice moves into cell c but Bob is unable to move into cell c'. Then it must be that either some player has already moved into the cell c', or moving into this cell would create an instance of the forbidden permutation.

If moving into cell c' is not possible because the cell already contains a move, then it would not have been possible for Alice to have even moved into cell c because the board is half turn invariant after each of Bob's moves.

Conversely if moving into cell c' would create an instance of (123), then every instance of (123) at this stage in the game must contain cell c'. Also

Alice's last move in c must be contained in every instance of (123) because the board is half-turn invariant after each of Bob's moves.

If c is in the Northeast or Southwest quadrants then c and c' would be positioned in a way that could not contribute to an instance of the permutation (123). So c and c' must be in the Northwest or Southeast quadrants of the board.

Consider an instance of the forbidden permutation (123). It must contain cells c, c' and some third cell e. The following will show that Alice's move into cell c must have created an instance of (123).

Without loss of generally let a and c be in the Northwest quadrant of the board.

If c is above and to the left of a or below and to the right, then c, a and a' would make up an instance of (123).

Suppose c is above and to the right of a. Then c and a' makes up an instance of (123) with which ever of e and e' is in the top half of the board.

Finally suppose that c is below and to the left of a. Then c and a' makes up an instance of (123) with whichever of e and e' is in the left half of the board.

So If Bob continues this strategy for the remainder of the game he will win because he will always have a response to each of Alice's moves. ∎

The following result reveals the value of $P^{m,\mathbb{Q}}(\sigma)$ where σ is of size k beginning with 1 or ending with k, in particular any identity permutation.

Theorem 4.2 *For any permutation σ of size k, where $\sigma(1) = 1$ or $\sigma(k) = k$,*

$$P^{m,\mathbb{Q}}(\sigma) = \begin{cases} 0 & \textit{if } m \textit{ is even;} \\ 1 & \textit{if } m \textit{ is odd.} \end{cases}$$

Proof. We will prove the result for $\sigma(1) = 1$. The case where $\sigma(k) = k$ follows by (rotational) symmetry. By Lemma 2.1 it is enough to show that at any stage in the game, in every row which has not been occupied, there is a cell which is available to be moved in. Suppose there exists a row r which has not been played in. If r may be played in to the left of all existing plays then we're done so assume there is no such column. Then it must be that there are cells

$$(r_1, c_1), (r_2, c_2), \ldots, (r_{k-1}, c_{k-1})$$

such that $(r, c), (r_1, c_1), (r_2, c_2), \ldots, (r_{k-1}, c_{k-1})$ is an instance of the forbidden permutation for any rational $c < c_1$ where $c_1 \leq c_2 \leq, \ldots, c_{k-1}$. Out of all sets of $(k-1)$ cells $(r_1, c_1), (r_2, c_2), \ldots, (r_{k-1}, c_{k-1})$ that satisfy this criteria, choose a set which has its leftmost element (r_1, c_1) as far right on the board as possible.

The player whose turn it is may move in (r, c'), where c' is any column to the right of c_1, but to the left of all columns to the right of c_1 which have been played in. This move will be allowed because there can not be any plays made in cells above and to the left of (r, c') because they would also be above and to the left of each of $(r_1, c_1), (r_2, c_2), \ldots, (r_{k-1}, c_{k-1})$ which would be a contradiction. The only way that moving in (r, c') could contribute to an instance of the forbidden permutation of size k is if there were plays in cells

$$(r'_1, c'_1), (r'_2, c'_2), \ldots, (r'_{k-1}, c'_{k-1})$$

such that $(r, c'), (r'_1, c'_1), (r'_2, c'_2), \ldots, (r'_{k-1}, c'_{k-1})$ form an instance of the forbidden permutation where $c' < c'_1 < \ldots c'_{k-1}$. So $c'_1 > c'$ and by the choice of c', $c' > c_1$. Thus $c'_1 > c_1$ contradicting the choice of $(r_1, c_1), (r_2, c_2), \ldots, (r_{k-1}, c_{k-1})$ to be so that c_1 is as far to the right as possible. ■

played cells

$$(r_1, c_1), (r_2, c_2), \ldots (r_{k-1}, c_{k-1})$$

Although Theorem 4.2 applies only to σ such that $\sigma(1) = 1$ or $\sigma(k) = k$ we have yet to find any sigma for which this result is not true.

In the next chapter we will look at games where the forbidden permutation is close in size with that of the board.

Chapter 5

Relatively Large Permutations

Theorem 5.1 *For each positive integer k, there exists M such that for any $m \geq M$ and any permutation σ of size $m-k$*

$$P^{m,n}(\sigma) = \begin{cases} *0 & \text{if } m \text{ is even;} \\ *1 & \text{if } m \text{ is odd.} \end{cases}$$

for all $n \geq m$.

Proof. Let $n = m+c$ for some non-negative integer c and σ be a permutation of size $m-k$. For an $m \times n$ board there exist

$$[m-(m-k)+1] \times [n-(m-k)+1] = (k+1) \times (c+k+1)$$

ways in which σ could fit on the board without any unused rows or columns in between the cells used to form it. Let K be the set of cells which could be covered by plays in this way. Since any other way that σ could fit on the board would use only cells from K, it then follows that K is in fact the set of all cells on the board that could possibly contribute to an instance of σ. K has no more than $(k+1) \times (c+k+1) \times (m-k)$ cells in it. Therefore there exist at least $mn - [(k+1) \times (c+k+1) \times (m-k)]$ cells that could not contribute to an instance of σ. Let R be the set of these cells.

A move on the board into position (r, c) renders row r and column c unplayable. On move i, at most $[m+n-2i+1] = [2m+c-2i+1]$ cells in R become unplayable in this way.

Suppose m is even. If Bob can make $(k+1)$ moves in R it would yield $(k+1)$ rows unplayable without contributing to an instance of σ. Thus there would not be enough rows available in K for an instance of σ to occur (need $m-k$). Bob would leave the game in a zero position after his $(k+1)$st move in R, because it will no longer be possible for an instance of σ to be created and thus exactly $[m-2(k+1)]$ moves will be made in the remaining game since now every row of the board must be played in for the game to come to an end. This position has value $*0$ by Lemma 2.1 since $[m-2(k+1)]$ is even. It would then follow that every option of $P^{m,n}(\sigma)$ must be non zero since Bob will always have a winning strategy and thus $P^{m,n}(\sigma)$ would be the mex of some non zero numbers which is $*0$.

$n - 2(k+1)$ is even.

Recall the number of cells in R is at least

$$mn-[(k+1)\times(c+k+1)\times(m-k)] = m^2+mc-[(k+1)\times(c+k+1)\times(m-k)].$$

The maximum number of cells in R left unplayable by moves 1 through $2(k+1)-1$ is

$$\sum_{i=1}^{2k+1}(2m+c-2i+1).$$

Therefore as long as

$$m^2+mc-[(k+1)\times(c+k+1)\times(m-k)]) > \sum_{i=1}^{2k+1}(2m+c-2i+1) \quad (5.1)$$

Bob will be able to make $(k+1)$ moves in R and the result will follow for m even.

There clearly exists M such that for all $m \geq M$, inequality (5.1) is satisfied because the left hand side grows faster than the right hand side with m.

Suppose conversely that m is odd. Let Alice make her first move in any cell of the board. If Alice can make her next $(k+1)$ moves in cells from R, then after her $(k+2)$nd move, there would be $(k+1)$ rows removed from the playable region by moves which can not contribute to an instance of σ. After her $(k+2)$nd move, there would be less than $(m-k)$ rows left on the board which could possibly contribute to an instance of σ which is not enough to create an instance σ and thus exactly $[m-2(k+1)-1]$ moves will be made in the remaining game since now every row of the board must be played in for the game to come to an end. This position has value *0 by Lemma 2.1 since $[m-2(k+1)-1]$ is even. It would then follow that every option of $P^{m,n}(\sigma)$ must be zero since Alice will always have a winning strategy no matter where her first move is made and thus $P^{m,n}(\sigma)$ would be $\text{mex}\{0\} = *1$ in this case.

Again the number of cells in R is at least

$$m^2 + mc - [(k+1) \times (c+k+1) \times (m-k)],$$

and the maximum number of cells in R left unplayable by moves 1 through $2(k+1)$ is

$$\sum_{i=1}^{2k+2} (2m + c - 2i + 1).$$

Therefore as long as

$$m^2 + mc - [(k+1) \times (c+k+1) \times (m-k)] > \sum_{i=1}^{2k+2} (2m + c - 2i + 1) \quad (5.2)$$

then after Alice's first move, there will enough cells left in the playable region of R so that Alice may in fact make her next $(k+1)$ moves in R.

There clearly exists M such that inequality (5.2) holds for all $m \geq M$ because the left hand side grows faster than the right hand side with m.

■

If k is any integer and σ is the identity permutation of size $m-k$ then we can give a more precise result.

Theorem 5.2 *For any positive integer k and $m \geq 4k+4$*

$$P^{m,n}(1,2,3,\ldots,m-k) = \begin{cases} 0 & \textit{if } m \textit{ is} \quad \textit{even;} \\ 1 & \textit{if } m \textit{ is} \quad \textit{odd.} \end{cases}$$

for all $n \geq m$.

Proof. Let R be the union of the following two sets of positions from an $m \times n$ board

$$\{(i,j) \mid i = 1,2,\ldots m-k-1,\ j = n-m+k+i+1, n-m+k+i+2,\ldots,n\}$$

$$\{(i,j) \mid i = k+j+1, k+j+2,\ldots,m,\ j = 1,2,\ldots,m-k-1\}.$$

Moving in any of these positions clearly can not contribute to an instance of $(1,2,3,\ldots,m-k)$. To illustrate this, let $k=1$, $m=4$ and $n=5$. Then $R = \{(1,4),(1,5),(2,5),(3,1),(4,1),(4,2)\}$. These cells are labeled with an asterisk in the following diagram.

			*	*
				*
*				
*	*			

Suppose m is even and $m \geq 4k+4$. If Bob could make his first $(k+1)$ moves in positions from R, then after his $(k+1)$st move, there would be at most $(m-k-1)$ rows left available on the board that could possibly contribute to an instance of $(1, 2, 3, \ldots, m-k)$ which is not enough room to fit an identity permutation of size $(m-k)$. Since no instance of $(1, 2, 3, \ldots, m-k)$ may be formed, then every row of the board must be played in for the game to come to an end and thus there must be exactly $[m-2(k+1)]$ moves remaining. This position thus has value *0 by Lemma 2.1 since $[m-2(k+1)]$ is even. It would then follow that every option of the original game would be non zero since Bob always has a winning strategy and thus $P^{m,n}(1, 2, 3, \ldots, m-k) = *0$ for m even.

Each row of the board contains at least $(m-2k-1)$ elements from R. As long as each row of the board has more than one cell from R remaining in the playable region, each move on the board rules out at most one row from the playable region inside R. Since $m \geq 4k+4$, we get that

$$m - 2k - 1 \geq 2(k+1)$$

and thus there are enough elements in each row of R so that Bob may in fact make his first $k+1$ moves in R.

Suppose now that m is odd and $m \geq 4k+4$. Let Alice make her first move in any position of the board. If Alice can make her next $(k+1)$ moves in positions from R, then after her $(k+2)$nd move, there would be $(k+1)$ rows removed from the playable region by moves which can not contribute to an instance of $(1, 2, 3, \ldots, m-k)$. So after her $(k+2)$nd move, there would be at most $[m-k-1]$ rows left on the board that could possibly contribute to an instance of $(1, 2, 3, \ldots, m-k)$ which is not enough. As above, every row of the board must be played in for the game to come to an end and thus there must be exactly $[m-2(k+1)-1]$ moves remaining. This position thus has value *0 by Lemma 2.1 since $[m-2(k+1)]-1]$ is even. It would then follow that every option of the original game is $*0$ since no matter where Alice makes her first move, she always has a winning strategy and thus $P^{m,n}(1, 2, 3, \ldots, m-k) = *1$ for m odd.

After Alice's first move, each row of the playable region of the board contains at least $(m-2k-2)$ elements from R. As long as each column and row of the board has $2(k+1)$ positions from R remaining in the playable region, each of the next $2(k+1)$ moves on the board will rule out at most one row and one column from the playable region inside R. Since $m \geq 4k+4$, we get that

$$m - 2k - 2 \geq 2(k+1).$$

and thus, after Alice's first move, there are enough elements in each row of R so that Alice may in fact make her next $(k+1)$ moves in R.

■

Both of the results from this section are examples of how we can apply Lemma 2.1 by delaying the game until the forbidden configuration will no longer fit in the available cells of the board. This allows the parity winner to force a particular parity of moves to be made in the game.

Chapter 6

Stretching results

If a board is stretched by adding columns, the value of the game on that board is sometimes affected. We saw in Chapter 5 that for a fixed k, if m is large enough that on an $m \times n$ board with $n \geq m$ and no moves made, where a permutation of size $m - k$ is being avoided, columns may be added without affecting the value of the position. We also know by Theorem 3.1 that if the forbidden permutation is (12), then any board which is square or wider, with no plays made, may have columns added to it without affecting the value of the game. These statements are true because the value of these games are only dependent on the smallest dimension of the board. What if some moves have been made?

Theorem 6.1 *Consider the game board* $\Pi_S^{m,n}(12)$ *where* $m \leq n$ *and* S *is any set of cells which could feasibly be occupied during a game of PermuNim on this board. There exists a place on the board where adding any number of* *columns will not change the value of the game.*

Proof. Suppose k moves have been made for $1 \leq k \leq m$. The remaining unoccupied cells at this stage form rectangles in diagonal pattern against the permutation. By Lemma 2.4 the value of this position is equal to the nim sum of the values of each of those rectangles, each avoiding (12).

Suppose that at this stage, in every column there is either a rook or an available cell. Then there must be $(n - k)$ columns all together in the rectangles of available cells. But there are at most $(m - k) \leq (n - k)$ available rows in these rectangles, which means there must be at least one of these rectangles which has at least as many columns as it has rows. Adding any number of columns to this small rectangle will not affect the value of it by Theorem 3.1 and Theorem 3.2, and thus will not affect the value of the game composed of these rectangles at this stage.

Conversely suppose that there exists a column c on the board which is unoccupied and can not be played in. In this situation, adding any number of columns adjacent to column c will not affect the nim value of the position because these new columns will not be available to be played in either.

In any case there is always a place at any position of the game where any number of columns may be added without affecting the nim value of the position. ■

If $m < n$ then there will also always be a place where a column can be deleted without changing the value of the game, even after any number of moves have been made. This is because one of the small rectangles in the proof of Theorem 6.1 will have strictly more columns than rows and thus a column may be deleted by Theorem 3.1 and Theorem 3.2.

6.1 Very wide boards

For more complicated permutations it is not as clear how wide exactly a board needs to be before columns may be added or deleted without affecting the value of the position. The following result gives a bound on the number of columns necessary after a given number of plays have been made. In particular it demonstrates that for a board of height m, when an arbitrary permutation is being avoided and no moves have been made, no more than 2^m columns are necessary before columns may be added and deleted. We suspect that the actual number necessary is much lower in many cases. In particular, we have seen that for the permutation (12) only m columns are needed. As demonstrated in Chapter 5, the same is true for permutations which are close in size to that of the board provided m is large enough.

Theorem 6.2 *On a board of height m avoiding any permutation, if t moves have been made and there exists a set S of at least 2^{m-t} (and possibly infinitely many) consecutive columns not played in then any number of columns may be added to or deleted from S without changing the value of the position provided $2^{m-t}-1$ columns remain in S.*

Proof. If $m-1$ moves have been made and there exists a set S of $|S| \geq 2^{m-(m-1)} = 2$ consecutive columns not played in then deleting any of them or some columns adjacent to them will not affect the value of the position as long as one column remains. This is because there is at most one move left and thus moving in any column of S is equivalent.

Suppose as an induction hypothesis that for some $t < m$, if t moves have been made, and there exists a set S of $|S| \geq 2^{(m-t)}$ adjacent columns not played in, then any finite number of columns may be added to or deleted from S without changing the value of the position as long as $2^{m-t} - 1$ consecutive unoccupied columns remain.

Now consider the position P_1 of some game on a board of height m where only $t-1$ moves have been made and where there exists a set S_1 of $|S_1| \geq 2^{m-(t-1)}$ adjacent columns which have not been played in. Also consider the position P_2 of the corresponding game where some number of columns have been added to S_1 to give a set S_2, and the position P_3 of the corresponding game where some number of columns have been deleted from S_1 to give a set S_3 where $|S_3| \geq 2^{m-(t-1)} - 1$ and of course $|S_2| > 2^{m-(t-1)}$.

Regardless of where the t^{th} move is made in P_1, there will still be a set of at least 2^{m-t} adjacent columns in S_1 which have not been played in.

By the induction hypothesis after this t^{th} move is made, any number of columns may be added to or deleted from this set of adjacent available columns in S_1 without changing the value of the position provided that at least $2^{m-t} - 1$ columns remain. In this way we can add or delete as many columns as required to verify that this option of P_1 has value equal to both an option of P_2 and an option of P_3. So all the options of P_1 are options of

P_2 and P_3 and in fact I will show that every option of P_2 and of P_3 is also an option of P_1.

Each option P_2' of P_2 is a position where there are t moves made and there is a set of at least 2^{m-t} adjacent, unoccupied columns in S_2 either to the right of the t^{th} move, to the left of the t^{th} move or both. If there is one such set call it S_2' and if there is a second set call it S_2''. Columns may be deleted from S_2' (and S_2'' if applicable) to make this position an option P_1' of P_1 leaving at least $2^{m-t} - 1$ columns in each of S_2' and S_2''. Do P_1' and P_2' have the same value? The induction hypothesis gives us that this removal of columns from the position P_2' to the position P_1' will not change the value of the position and thus every option of P_2 has value equal to some option of P_1.

Similarly each option P_3' of P_3 is a position where there are t moves made and a set S_3' of at least $2^{m-t}-1$ adjacent columns in S_3 which are unoccupied. Columns may be added to S_3' in such a way as to obtain an option P_1' of P_1 from P_3'. Like all options of P_1, P_1' must have a set S_1' of at least 2^{m-t} adjacent columns in S_1 which are unoccupied and thus by the induction hypothesis the removal of columns to obtain P_3' from P_1' results in an equivalent game. Therefore every option of P_3 is equal to some option of P_1. It then follows that P_2 and P_3 both have the same options as P_1 and thus have the same value. The result follows by induction. ■

So the limit of $P^{m,n}(\sigma)$ as n approaches infinity exists. Is this limit equal to the value of some infinitely wide board? Perhaps an infinitely wide board where there is still a least and greatest column. In the case where (12) is the forbidden permutation, the very wide board does have the same value as $P^{m,\mathbb{Z}^+}$ but this is not necessarily true in general. They do, however, have some similar structure.

Corollary 6.3 *For* $(2^{m-1}+1) \leq c \leq (n - 2^{m-1} - 1)$*,* $P^{m,n}_{r,c} = P^{m,n}_{r,c+1}$ *and for* $c \geq (2^{m-1}+1)$*,* $P^{m,\mathbb{Z}^+}_{r,c} = P^{m,\mathbb{Z}^+}_{r,c+1}$*, and for any* c*,* $P^{m,\mathbb{Z}}_{r,c} = P^{m,\mathbb{Z}}_{r,c+1}$*.*

Proof. By Theorem 6.2, for $(2^{m-1}+1) \leq c \leq (n - 2^{m-1} - 1)$ a column may be added to the left of column c and deleted from the right in $\Pi^{m,n}_{r,c}$ to obtain $\Pi^{m,n}_{r,c+1}$ without affecting the value of the position. Similarly if $c \geq 2^{m-1}+1$ a column may be added to the left of column c in $P^{m,\mathbb{Z}^+}_{r,c}$ to obtain $P^{m,\mathbb{Z}^+}_{r,c+1}$ without affecting the value of the position. In the last case for any c a column may be added to the left of column c and deleted from the right in $\Pi^{m,\mathbb{Z}}_{r,c}$ to obtain $\Pi^{m,\mathbb{Z}}_{r,c+1}$. ∎

So for a wide enough board or $\Pi^{m,\mathbb{Z}}$ or $\Pi^{m,\mathbb{Z}^+}$, many of the options will have horizontal sterility when no moves have been made.

6.2 Non-Stretchability

The computer-generated data indicates that for small m, a board with m rows, where (123) is the forbidden permutation, needs many fewer than 2^m columns before some may be added or deleted without affecting the value of the game. So how many columns are needed before this is guaranteed? The following result verifies that there is no constant x so that having x more columns than rows would guarantee this.

Theorem 6.4 *Let $x \geq 1$ and $m = 20x-2$. There is a position in $\Pi^{m,m+x}(123)$ where no column may be added between any other columns without changing the value of the game.*

Proof. Consider the position $\Pi_S^{m,m+x}(123)$ for $S = \{(m-9, 9), (m-8, 10), (m-20, 20), (m-19, 21), \ldots (9, m+x-10), (10, m+x-9)\}$. The available region consists of a pattern of non overlapping alternating 8×8, and 8×9 boards with two plays in between each, making all cells above and to the left as well as those below and right unplayable. This position is depicted below for the case where $x = 2$ with bullets for cells which have been played in.

–	–	–	–	–	–	–	–	–	$\Pi^{8,9}(123)$
–	–	–	–	–	–	–	•	–	–
–	–	–	–	–	–	–	–	•	–
–	–	–	–	–	–	$\Pi^{8,8}(123)$	–	–	–
–	–	–	–	•	–	–	–	–	–
–	–	–	–	–	•	–	–	–	–
–	–	–	$\Pi^{8,9}(123)$	–	–	–	–	–	–
–	•	–	–	–	–	–	–	–	–
–	–	•	–	–	–	–	–	–	–
$\Pi^{8,8}(123)$	–	–	–	–	–	–	–	–	–

Let P be the value of this position. By 2.4, P is the nim sum

$$\Sigma_{i=1}^{x}[P^{8,8}(123) \oplus P^{8,9}(123)].$$

This sum is clearly $*0$ if x is even and $P^{8,8}(123) \oplus P^{8,9}(123)$ if x is odd which from the computer-generated data is $*3$ since $P^{8,8}(123) = *1$ and $P^{8,9}(123) = *2$.

If a column is inserted at position P within one of the 8×8 boards it will change it to an 8×9 board which has value *2 instead of *1 and thus the value of P would also be affected. Similarly if a column is inserted within one of the 8×9 boards it will change it to an 8×10 board which has value

*0 instead of *2 and thus the value of P would also be affected. Finally consider the situation where a column is added in between one of the sets of two adjacent plays. Without loss of generality let it be added between the rooks in cells $(m-8,10)$ and $(m-9,9)$. Before this column is added, the bottom left 18×19 rectangle of cells had value $*1 \oplus *2 = *3$. After the column is added, this now 18×20 rectangle of cells does not have value $*3$. To demonstrate this we will show that the option of moving into cell $(m-7,10)$ has value $*3$. The move into cell $(m-7,10)$ makes the bottom left 18×20 rectangle of cells equivalent to the nim sum of $P^{8,9}(123) = *2$ and $P^{8,9}_{(1,9)}(123) = P^{7,8}(123) = *1$. The result follows because $*1 \oplus *2 = *3$ as required. ■

Appendix A

Computer Data

The following data for PermuNim where (123) is the forbidden permutation was calculated with a fast, two-bin, (123) permutation pattern checking algorithm. Positions in the game are stored as strings of numbers $a_1, a_2, \ldots, a_n$ where a_i has value j if there is a rook in row j, column i and value -1 if column i has not been played in. The string of numbers are sorted from left to right into two bins whereby the numbers are put into a bin only if its smaller than all other elements in that bin. If the string of numbers can not be sorted into two bins in this way then the position is not valid.

From the data, it appears that for any n at least three, $\Pi^{3,n}(123)$ has only 0s and 2s which supports the tendency for odd height boards to have only even values.

$\Pi^{3,3}(123)$

2	0	0
0	2	0
0	0	2

$\pi^{3,4}(123)$

2	0	0	0
0	2	2	0
0	0	0	2

$\Pi^{3,5}(123)$

2	0	0	0	0
0	2	0	2	0
0	0	0	0	2

$\Pi^{3,6}(123)$

2	0	0	0	0	0
0	2	0	0	2	0
0	0	0	0	0	2

$\Pi^{3,7}(123)$

2	0	0	0	0	0	0
0	2	0	0	0	2	0
0	0	0	0	0	0	2

$\Pi^{3,8}(123)$

2	0	0	0	0	0	0	0
0	2	0	0	0	0	2	0
0	0	0	0	0	0	0	2

$\Pi^{3,9}(123)$

2	0	0	0	0	0	0	0	0
0	2	0	0	0	0	0	2	0
0	0	0	0	0	0	0	0	2

From the data, it appears that for any n at least four, $\Pi^{4,n}(123)$ has only 1s and 3s which supports the tendency for even height boards to have only odd values.

$\Pi^{4,4}(123)$

3	3	1	1
3	1	3	1
1	3	1	3
1	1	3	3

$\Pi^{4,5}(123)$

3	3	1	1	1
3	1	3	3	1
1	3	3	1	3
1	1	1	3	3

$\Pi^{4,6}(123)$

3	3	1	1	1	1
3	1	3	1	3	1
1	3	1	3	1	3
1	1	1	1	3	3

$\Pi^{4,7}(123)$

3	3	1	1	1	1	1
3	1	3	1	1	3	1
1	3	1	1	3	1	3
1	1	1	1	1	3	3

$\Pi^{4,8}(123)$

3	3	1	1	1	1	1	1
3	1	3	1	1	1	3	1
1	3	1	1	1	3	1	3
1	1	1	1	1	1	3	3

$\Pi^{4,9}(123)$

3	3	1	1	1	1	1	1	1
3	1	3	1	1	1	1	3	1
1	3	1	1	1	1	3	1	3
1	1	1	1	1	1	1	3	3

$\Pi^{4,10}(123)$

3	3	1	1	1	1	1	1	1	1
3	1	3	1	1	1	1	1	3	1
1	3	1	1	1	1	1	3	1	3
1	1	1	1	1	1	1	1	3	3

From the data $\Pi^{5,5}(123)$ has a 1 in it but it appears that for any n at least six, $\Pi^{5,n}(123)$ has only even values.

$\Pi^{5,5}(123)$

4	4	4	0	0
4	2	2	4	0
4	2	1	2	4
0	4	2	2	4
0	0	4	4	4

$\Pi^{5,6}(123)$

4	4	4	0	0	0
4	2	0	4	4	0
4	0	4	4	0	4
0	4	4	0	2	4
0	0	0	4	4	4

$\Pi^{5,7}(123)$

4	4	4	0	0	0	0
4	2	0	4	2	4	0
4	0	4	2	4	0	4
0	4	2	4	0	2	4
0	0	0	0	4	4	4

$\Pi^{5,8}(123)$

4	4	4	0	0	0	0	0
4	2	0	4	0	2	4	0
4	0	4	2	2	4	0	4
0	4	2	0	4	0	2	4
0	0	0	0	0	4	4	4

$\Pi^{5,9}(123)$

4	4	4	0	0	0	0	0	0
4	2	0	4	0	0	2	4	0
4	0	4	2	0	2	4	0	4
0	4	2	0	0	4	0	2	4
0	0	0	0	0	0	4	4	4

$\Pi^{5,10}(123)$

4	4	4	0	0	0	0	0	0	0
4	2	0	4	0	0	0	2	4	0
4	0	4	2	0	0	2	4	0	4
0	4	2	0	0	0	4	0	2	4
0	0	0	0	0	0	0	4	4	4

$\Pi^{5,11}(123)$

4	4	4	0	0	0	0	0	0	0	0
4	2	0	4	0	0	0	0	2	4	0
4	0	4	2	0	0	0	2	4	0	4
0	4	2	0	0	0	0	4	0	2	4
0	0	0	0	0	0	0	0	4	4	4

From the data $\Pi^{6,6}(123)$ has some 2s in it, but it appears that for any n at least seven, $\Pi^{6,n}(123)$ has only odd values.

$\Pi^{6,6}(123)$

5	5	5	5	3	3
5	5	5	3	2	3
5	5	3	2	3	5
5	3	2	3	5	5
3	2	3	5	5	5
3	3	5	5	5	5

$\Pi^{6,7}(123)$

5	5	5	5	3	1	1
5	5	5	1	5	5	1
5	5	1	5	5	1	5
5	1	5	5	1	5	5
1	5	5	1	5	5	5
1	1	3	5	5	5	5

$\Pi^{6,8}(123)$

5	5	5	5	1	1	1	1
5	5	5	1	5	3	5	1
5	5	1	5	3	5	1	5
5	1	5	3	5	1	5	5
1	5	3	5	1	5	5	5
1	1	1	1	5	5	5	5

$\Pi^{6,9}(123)$

5	5	5	5	1	1	1	1	1
5	5	5	1	5	1	3	5	1
5	5	1	5	3	3	5	1	5
5	1	5	3	3	5	1	5	5
1	5	3	1	5	1	5	5	5
1	1	1	1	1	5	5	5	5

$\Pi^{6,10}$

5	5	5	5	1	1	1	1	1	1
5	5	5	1	5	1	1	3	5	1
5	5	1	5	3	1	3	5	1	5
5	1	5	3	1	3	5	1	5	5
1	5	3	1	1	5	1	5	5	5
1	1	1	1	1	1	5	5	5	5

$\Pi^{6,11}(123)$

5	5	5	5	1	1	1	1	1	1	1
5	5	5	1	5	1	1	1	3	5	1
5	5	1	5	3	1	1	3	5	1	5
5	1	5	3	1	1	3	5	1	5	5
1	5	3	1	1	1	5	1	5	5	5
1	1	1	1	1	1	1	5	5	5	5

$\Pi^{6,12}(123)$

5	5	5	5	1	1	1	1	1	1	1	1
5	5	5	1	5	1	1	1	1	3	5	1
5	5	1	5	3	1	1	1	3	5	1	5
5	1	5	3	1	1	1	3	5	1	5	5
1	5	3	1	1	1	1	5	1	5	5	5
1	1	1	1	1	1	1	1	5	5	5	5

$\Pi^{6,13}(123)$

5	5	5	5	1	1	1	1	1	1	1	1	1
5	5	5	1	5	1	1	1	1	1	3	5	1
5	5	1	5	3	1	1	1	1	3	5	1	5
5	1	5	3	1	1	1	1	3	5	1	5	5
1	5	3	1	1	1	1	1	5	1	5	5	5
1	1	1	1	1	1	1	1	1	5	5	5	5

From the data, $\Pi^{7,7}(123)$ has some odd entries but it appears that for any n at least eight, $\Pi^{7,n}(123)$ has only even entries.

$\Pi^{7,7}(123)$

6	6	6	6	1	0	0
6	4	4	1	4	1	0
6	4	1	4	3	4	1
6	1	4	3	4	1	6
1	4	3	4	1	4	6
0	1	4	1	4	4	6
0	0	1	6	6	6	6

$\Pi^{7,8}(123)$

6	6	6	6	2	0	0	0
6	4	4	4	0	2	2	0
6	4	4	2	6	6	0	2
6	4	2	4	4	2	4	6
2	0	6	6	2	4	4	6
0	2	2	0	4	4	4	6
0	0	0	2	6	6	6	6

$\Pi^{7,9}(123)$

6	6	6	6	2	0	0	0	0
6	4	4	4	0	2	2	2	0
6	4	4	2	6	4	6	0	2
6	4	2	4	4	4	2	4	6
2	0	6	4	6	2	4	4	6
0	2	2	2	0	4	4	4	6
0	0	0	0	2	6	6	6	6

$\Pi^{7,10}(123)$

6	6	6	6	2	0	0	0	0	0
6	4	4	4	0	2	0	2	2	0
6	4	4	2	6	4	4	3	0	2
6	4	2	4	4	4	4	2	4	6
2	0	3	4	4	6	2	4	4	6
0	2	2	0	2	0	4	4	4	6
0	0	0	0	0	2	6	6	6	6

$\Pi^{7,11}(123)$

6	6	6	6	2	0	0	0	0	0	0
6	4	4	4	0	2	0	0	2	2	0
6	4	4	2	6	0	2	4	3	0	2
6	4	2	4	4	0	4	4	2	4	6
2	0	3	4	2	0	6	2	4	4	6
0	2	2	0	0	2	0	4	4	4	6
0	0	0	0	0	0	2	6	6	6	6

$\Pi^{7,12}(123)$

6	6	6	6	2	0	0	0	0	0	0	0
6	4	4	4	0	2	0	0	0	2	2	0
6	4	4	2	6	0	0	2	4	3	0	2
6	4	2	4	4	0	0	4	4	2	4	6
2	0	3	4	3	0	0	6	2	4	4	6
0	2	2	0	0	0	2	0	4	4	4	6
0	0	0	0	0	0	0	2	6	6	6	6

$\Pi^{7,13}(123)$

6	6	6	6	2	0	0	0	0	0	0	0	0
6	4	4	4	0	2	0	0	0	0	2	2	0
6	4	4	2	6	0	0	0	2	4	3	0	2
6	4	2	4	4	0	0	0	4	4	2	4	6
2	0	3	4	2	0	0	0	6	2	4	4	6
0	2	2	0	0	0	0	2	0	4	4	4	6
0	0	0	0	0	0	0	0	2	6	6	6	6

From the data, $\Pi^{8,8}(123)$ and $\Pi^{8,9}(123)$ each have two 0s and thus two winning first moves for Alice, but it appears that for any n at least ten, $\Pi^{8,n}(123)$ has only odd entries and Alice does not have a winning move.

$\Pi^{8,8}(123)$

7	7	7	7	2	2	2	2
7	7	7	7	7	5	2	2
7	7	5	5	5	0	5	2
7	7	5	5	7	5	7	2
2	7	5	7	5	5	7	7
2	5	0	5	5	5	7	7
2	2	5	7	7	7	7	7
2	2	2	2	7	7	7	7

$\Pi^{8,9}(123)$

7	7	7	7	7	3	5	1	1
7	7	7	7	7	1	3	7	1
7	7	5	5	3	7	0	1	3
7	7	5	3	7	7	3	7	7
7	7	3	7	7	3	5	7	7
3	1	0	7	3	5	5	7	7
1	7	3	1	7	7	7	7	7
1	1	5	3	7	7	7	7	7

$\Pi^{8,10}(123)$

7	7	7	7	7	3	1	1	1	1
7	7	7	7	7	1	3	3	7	1
7	7	5	5	3	7	5	7	1	3
7	7	5	3	7	5	7	3	7	7
7	7	3	7	5	7	3	5	7	7
3	1	7	5	7	3	5	5	7	7
1	7	3	3	1	7	7	7	7	7
1	1	1	1	3	7	7	7	7	7

$\Pi^{8,11}(123)$

7	7	7	7	7	3	1	1	1	1	1
7	7	7	7	7	1	3	3	3	7	1
7	7	5	5	3	7	5	1	7	1	3
7	7	5	3	7	5	1	7	3	7	7
7	7	3	7	1	5	7	3	5	7	7
3	1	7	1	5	7	3	5	5	7	7
1	7	3	3	3	1	7	7	7	7	7
1	1	1	1	1	3	7	7	7	7	7

$\Pi^{8,12}(123)$

7	7	7	7	7	3	1	1	1	1	1	1
7	7	7	7	7	1	3	3	1	3	7	1
7	7	5	5	3	7	5	3	1	7	1	3
7	7	5	3	7	5	1	1	7	3	7	7
7	7	3	7	1	1	5	7	3	5	7	7
3	1	7	1	3	5	7	3	5	5	7	7
1	7	3	1	3	3	1	7	7	7	7	7
1	1	1	1	1	1	3	7	7	7	7	7

$\Pi^{8,13}(123)$

7	7	7	7	7	3	1	1	1	1	1	1	1
7	7	7	7	7	1	3	3	1	1	3	7	1
7	7	5	5	3	7	5	1	3	1	7	1	3
7	7	5	3	7	5	1	1	1	7	3	7	7
7	7	3	7	1	1	1	5	7	3	5	7	7
3	1	7	1	3	1	5	7	3	5	5	7	7
1	7	3	1	1	3	3	1	7	7	7	7	7
1	1	1	1	1	1	1	3	7	7	7	7	7

$\Pi^{8,14}(112)$

7	7	7	7	7	3	1	1	1	1	1	1	1	1
7	7	7	7	7	1	3	3	1	1	1	3	7	1
7	7	5	5	3	7	5	1	1	3	1	7	1	3
7	7	5	3	7	5	1	1	1	1	7	3	7	7
7	7	3	7	1	1	1	1	5	7	3	5	7	7
3	1	7	1	3	1	1	5	7	3	5	5	7	7
1	7	3	1	1	1	3	3	1	7	7	7	7	7
1	1	1	1	1	1	1	1	3	7	7	7	7	7

$\Pi^{8,15}(123)$

7	7	7	7	7	3	1	1	1	1	1	1	1	1	1
7	7	7	7	7	1	3	3	1	1	1	1	3	7	1
7	7	5	5	3	7	5	1	1	1	3	1	7	1	3
7	7	5	3	7	5	1	1	1	1	1	7	3	7	7
7	7	3	7	1	1	1	1	1	5	7	3	5	7	7
3	1	7	1	3	1	1	1	5	7	3	5	5	7	7
1	7	3	1	1	1	1	3	3	1	7	7	7	7	7
1	1	1	1	1	1	1	1	1	3	7	7	7	7	7

$\Pi^{8,16}(123)$

7	7	7	7	7	3	1	1	1	1	1	1	1	1	1	1
7	7	7	7	7	1	3	3	1	1	1	1	1	3	7	1
7	7	5	5	3	7	5	1	1	1	1	3	1	7	1	3
7	7	5	3	7	5	1	1	1	1	1	1	7	3	7	7
7	7	3	7	1	1	1	1	1	1	5	7	3	5	7	7
3	1	7	1	3	1	1	1	1	5	7	3	5	5	7	7
1	7	3	1	1	1	1	1	3	3	1	7	7	7	7	7

$\Pi^{8,17}(123)$

7	7	7	7	7	3	1	1	1	1	1	1	1	1	1	1	1
7	7	7	7	7	1	3	3	1	1	1	1	1	1	3	7	1
7	7	5	5	3	7	5	1	1	1	1	1	3	1	7	1	3
7	7	5	3	7	5	1	1	1	1	1	1	1	7	3	7	7
7	7	7	3	1	1	1	1	1	1	1	5	7	3	5	7	7
3	1	7	1	3	1	1	1	1	1	5	7	3	5	5	7	7
1	7	3	1	1	1	1	1	1	3	3	1	7	7	7	7	7
1	1	1	1	1	1	1	1	1	1	1	3	7	7	7	7	7

$\Pi^{9,9}(123)$

8	8	8	8	8	8	1	1	1
8	8	8	8	8	1	6	1	1
8	8	8	8	8	2	3	6	1
8	8	8	3	6	1	2	1	8
8	8	8	6	1	6	8	8	8
8	1	2	1	6	3	8	8	8
1	6	3	2	8	8	8	8	8
1	1	6	1	8	8	8	8	8
1	1	1	8	8	8	8	8	8

For the permutations (132) and (1234) the fast two-bin permutation pattern checking algorithm can no longer be used. The following data was calculated with a brute force algorithm.

$\Pi^{3,3}(132)$

2	0	0
0	0	2
0	2	0

$\Pi^{3,4}(132)$

2	0	0	0
0	0	0	2
0	2	2	0

$\Pi^{3,5}(132)$

2	0	0	0	0
0	0	0	0	2
0	2	0	2	0

$\Pi^{3,6}(132)$

2	0	0	0	0	0
0	0	0	0	0	2
0	2	0	0	2	0

$\Pi^{3,7}(132)$

2	0	0	0	0	0	0
0	0	0	0	0	0	2
0	2	0	0	0	2	0

$\Pi^{3,8}(132)$

2	0	0	0	0	0	0	0
0	0	0	0	0	0	0	2
0	2	0	0	0	0	2	0

$\Pi^{3,9}(132)$

2	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	2
0	2	0	0	0	0	0	2	0

$\Pi^{4,4}(132)$

3	3	1	1
3	1	3	3
1	3	3	3
1	3	3	1

$\Pi^{4,5}(132)$

3	3	1	1	1
3	1	1	3	3
1	3	1	3	3
1	3	0	3	1

$\Pi^{4,6}(132)$

3	3	1	1	1	1
3	1	1	1	3	3
1	3	1	1	3	3
1	3	3	3	3	1

$\Pi^{4,7}(132)$

3	3	1	1	1	1	1
3	1	1	1	1	3	3
1	3	1	1	1	3	3
1	3	3	1	3	3	1

$\Pi^{4,8}(132)$

3	3	1	1	1	1	1	1
3	1	1	1	1	1	3	3
1	3	1	1	1	1	3	3
1	3	3	1	1	3	3	1

$\pi^{4,9}(132)$

3	3	1	1	1	1	1	1	1
3	1	1	1	1	1	1	3	3
1	3	1	1	1	1	1	3	3
1	3	3	1	1	1	3	3	1

From the data, $\Pi^{5,5}(132)$ has a 1 in it but it appears that for any n at least six, $\Pi^{5,n}(123)$ has only 0s, 2s and 4s which supports the tendency for odd height boards to have all even values.

$\Pi^{5,5}(132)$

4	4	2	0	0
4	0	4	2	4
2	4	2	1	0
0	2	1	4	4
0	4	0	4	0

$\Pi^{5,6}(132)$

4	4	2	0	2	2
4	0	0	0	2	4
2	0	2	2	1	0
2	2	4	4	0	4
2	2	4	4	2	2

$\Pi^{5,7}(132)$

4	4	2	0	0	0	0
4	0	0	0	0	2	4
2	0	2	0	2	1	0
0	2	4	2	4	0	4
0	2	0	1	0	2	0

From the data, $\Pi^{6,6}(123)$ has four 2s in it, but it appears that for any n

at least seven, $\Pi^{6,n}(123)$ has only odd entries which supports the tendency for even height boards to have all odd values.

$\Pi^{6,6}(132)$

5	5	5	3	3	3
5	5	5	1	3	5
5	5	3	5	5	5
3	1	5	5	5	5
3	3	5	5	5	5
3	5	5	5	5	3

$\Pi^{6,7}(132)$

5	5	5	3	1	3	3
5	5	1	1	5	3	5
5	1	1	3	5	5	5
3	1	5	3	5	5	5
3	3	1	5	5	5	5
3	3	5	1	5	3	3

$\Pi^{6,8}(132)$

5	5	5	3	3	1	3	3
5	5	1	1	1	1	3	5
5	1	1	3	3	5	5	5
3	1	5	3	3	5	5	5
3	3	5	5	5	1	5	5
3	3	5	5	5	5	3	3

$\Pi^{7,7}(132)$

6	6	6	4	4	0	0
6	4	1	0	4	4	6
6	1	2	6	4	1	0
4	0	6	4	3	2	1
4	4	4	3	6	6	0
0	4	1	2	6	6	6
0	6	0	1	0	6	0

$\Pi^{7,8}(132)$

6	6	6	4	0	2	0	0
6	4	6	0	0	4	4	6
6	6	2	2	6	4	1	0
4	6	6	2	6	3	2	1
1	0	2	6	6	6	6	0
0	2	4	6	6	6	0	6
0	2	6	6	6	6	2	0

$\Pi^{8,8}(132)$

7	7	7	7	2	5	5	5
7	5	7	7	7	1	5	2
7	7	7	7	7	2	7	7
7	7	7	0	7	7	7	7
2	7	7	7	1	7	7	7
5	1	2	7	7	5	7	7
5	5	7	7	7	7	7	4
5	2	7	7	7	7	4	5

The following data is for permutations of length 4. For small boards the permutation is now closer in size with the board.

$\Pi^{4,4}(1234)$

1	1	1	1
1	1	1	1
1	1	1	1
1	1	1	1

$\Pi^{4,5}(1234)$

1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1

$\Pi^{4,6}(1234)$

1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1

$\Pi^{4,7}(1234)$

1	1	1	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	1	1	1

$\Pi^{5,5}(1234)$

0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

$\Pi^{5,6}(1234)$

0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$\Pi^{5,7}(1234)$

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

$\Pi^{5,8}(1234)$

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

$\Pi^{6,6}(1234)$

3	1	1	1	1	1
1	1	1	1	1	1
1	1	3	1	1	1
1	1	1	3	1	1
1	1	1	1	1	1
1	1	1	1	1	3

$\Pi^{6,7}(1234)$

1	1	1	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	1	1	1

$\Pi^{7,7}(1234)$

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	6	0	0	0
0	0	6	0	6	0	0
0	0	0	6	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

$\Pi^{7,8}(1234)$

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

$\Pi^{8,8}(1234)$

2	1	3	1	3	1	1	1
1	1	1	3	1	3	1	1
3	1	3	3	5	1	3	1
1	3	3	5	1	5	1	3
3	1	5	1	5	3	3	1
1	3	1	5	3	3	1	3
1	1	3	1	3	1	1	1
1	1	1	3	1	3	1	2

Bibliography

[1] E.R. Berlekamp, J.H. Conway, and R.K. Guy, *Winning Ways for your mathematical plays, Second edition.* Volumes **1-4**, A K Peteres, Ltd., Natick, MA, 2003.

[2] S. Byrnes, Poset game periodicity. *Integers* **3** (2003), G3, 19pp.

[3] Z. Stankova, Classification of forbidden subsequences of length 4. *European J. Combin.*, **17** 501-517 (1996).

[4] J. West, Permutation trees and the Catalan and Schröder numbers. *Discrete Math*, **146** 247-262 (1995).