Local *vs.* Global Real Best Rational Approximation with One Nonlinear Constraint

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1. Introduction

Several authors, for example, I. Barrodale [B] and M. Gugat [G] have shown that a local real best rational approximation, with linear constraints, is also globally best. We show here with an example that the addition of a single nonlinear constraint (in this case a derivative constraint) local best approximations can occur. Initially, the results in this direction were concerned with the question of uniqueness. In 1934, J.L. Walsh [W] gave an example of nonuniqueness of best rational Chebyshev approximation on a crescent-shaped Jordan region or an arc that is symmetric with respect to the unit circle. A more natural domain for this behavior was given by E.B. Saff and R.S. Varga [SV] who showed that nonuniqueness can occur when approximating a real function on a real interval. In a more recent paper, M.H. Gutknecht and L.N. Trefethen [GT] extended this result to complex valued functions on the unit disk. Finally, in 1993, M.-P. Istace and J.-P. Thiran [IT] gave an example of a local best Chebyshev rational approximation which is not a global best approximation.

We now turn to constructing our example of a local best approximation, on a finite point set, which is not globally best. Set

$$X = \left\{ \left(-\frac{2}{3}, \frac{1}{3} \right), (1,1), (1,2) \right\}$$

$$R_{11}(X) = \left\{ \frac{ax + by + c}{dx + 1} : dx + 1 > 0 \quad \text{for} \quad (x,y) \in X, \;\; a, b, c, d \text{ real} \right\}.$$

Define $f : X \to R$ by

$$f\left(-\frac{2}{3}, \frac{1}{3} \right) = -1, \quad f(1,1) = \frac{3}{2} \quad \text{and} \quad f(1,2) = -1.$$

Let

$$r(x,y) = \frac{ax + by + c}{dx + 1}$$

and

$$\|f-r\|_X = \max\{|f-r|,\ (x,y) \in X\}.$$

We shall show that, subject to the condition $r_x(0,0) \leq 0$, the constant rational function $s^*(x,y) = 0.25$ with coefficients $a = b = d = 0.0$, $c = 0.25$ is a local best approximation to f from $R_{11}(X)$ which is not globally best.

We now define the problem which we will solve in the next section.

Problem A: Find the coefficients a, b, c, and d which minimize

$$\|f-r\|_X \quad \text{for all} \quad r \in R_{11}(X)$$

subject to

$$\frac{\partial}{\partial x}\left[\frac{ax+by+c}{dx+1}\right]_{(0,0)} = a - cd \leq 0. \tag{1.1}$$

2. The Main Result

We now show that the rational function $s^* = 0.25$ defined in the Introduction is a local best approximation to f on X under the conditions of Problem A.

Theorem 2.1. *The constant function* $s^*(x,y) = 0.25$ *is a local best approximation for Problem A, that is,* $a = b = d = 0.0$, $c = 0.25$ *with* $\|f-s^*\|_X = 1.25$.

Proof: By direct computation, s^* satisfies the conditions of Problem A, and $\|f-s^*\|_X = 1.25$. Let

$$s(x,y) = \frac{s_1x + s_2y + s_3 + 0.25}{s_4x + 1} \tag{2.1}$$

also satisfy the conditions of Problem A. Setting $s_1 = s_2 = s_3 = s_4 = 0$ yields $s^*(x,y) = 0.25$. If we suppose that $|s_i| < \frac{2}{3}$, $i = 1, \cdots 4$, then we claim that $\|f-s\|_X \geq 1.25$, so $s(x,y) = 0.25$ is a local best approximation to f on X under the conditions of Problem A.

Suppose, on the contrary, that $\|f-s\|_X < 1.25$ for some $s \in R_{11}(X)$ with $r_x(0,0 \leq 0$. Then we have

$$s\left(-\frac{2}{3}, \frac{1}{3}\right) < 0.25, \tag{2.2}$$

$$s(1,1) > 0.25, \tag{2.3}$$

$$s(1,2) < 0.25, \tag{2.4}$$

$$s_1 - (0.25 + s_3)\, s_4 \leq 0 \quad \text{(derivative constraint).} \tag{2.5}$$

Inequalities (2.2), (2.3) and (2.4) give us

$$\frac{-\frac{2}{3}\, s_1 + \frac{1}{3}\, s_2 + s_3 + 0.25}{-\frac{2}{3}\, s_4 + 1} < 0.25, \tag{2.6}$$

$$\frac{s_1 + s_2 + s_3 + 0.25}{s_4 + 1} > 0.25, \tag{2.7}$$

$$\frac{s_1 + 2s_2 + s_3 + 0.25}{s_4 + 1} < 0.25. \tag{2.8}$$

Assuming, as we do, that $|s_4| < \frac{2}{3}$ guarantees a positive denominator, so inequality (2.6) gives us

$$-2s_1 + s_2 + 3s_3 + 0.75 < 0.25(-2s_4 + 3),$$

which is equivalent to

$$8s_1 - 4s_2 - 12s_3 - 2s_4 > 0. \tag{2.9}$$

Inequality (2.7) is equivalent to

$$4s_1 + 4s_2 + 4s_3 - s_4 > 0, \tag{2.10}$$

and inequality (2.8) is equivalent to

$$-4s_1 - 8s_2 - 4s_3 + s_4 > 0. \tag{2.11}$$

From inequality (2.5) (derivative constraint), we have

$$-4s_1 + (1 + 4s_3)\, s_4 \geq 0. \tag{2.12}$$

Now, adding inequalities (2.10) and (2.11) yields

$$s_2 < 0, \tag{2.13}$$

and adding inequalities (2.10) and (2.12) yields $4s_2 + 4s_3(1 + s_4) > 0$ and since $s_2 < 0$ and $1 + s_4 > 0$ we get $s_3 > 0$.

If we divide both sides of inequality (2.9) by two and then add it to inequality (2.12) we have $-2s_2 - 6s_3 + 4s_3s_4 > 0$ which is equivalent to

$$s_2 < -3s_3 + 2s_3s_4. \tag{2.14}$$

From (2.10) and (2.14) we get $s_4 - 4s_1 < 4(s_2 + s_3) < 4(-2s_3 + 2s_3s_4)$, or

$$s_4 - 4s_1 < -8s_3 + 8s_3s_4. \tag{2.15}$$

Using the left side of inequality (2.12) with (2.15), $s_3 < 0$ and $|s_4| < \frac{2}{3}$, we get

$$\begin{aligned} -4s_1 + (1 + 4s_3)\,s_4 = s_4 - 4s_1 + 4s_3s_4 &< -8s_3 + 12s_3s_4 \\ &= 4s_3(-2 + 3s_4) < 0. \end{aligned} \tag{2.16}$$

This contradicts (2.12), completing the proof. ▮

By direct computation, the rational function r^* defined by

$$r^*(x, y) = \frac{4x - 5y + 4}{x + 1}$$

satisfies the conditions of Problem A, and we have $\|f - r^*\|_X = 0.0 < 1.25$. Thus r^* is the best approximation to f subject to the conditions of Problem A, and s^* is not globally best.

It would be of interest to improve this result with an example which consists of a single (nonlinear) constraint and no additional linear constraints.

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References

[B] I. Barrodale: Best rational approximation and strict quasi-convesity, *SIAM J. Numer. Anal.* **10**(1973), 8–12.

[G] M. Gugat: An algorithm for Chebyshev approximation by rationals with constrained denominators, *Constr. Approx.* **12**(1996), 197–221.

[GT] M.H. Gutknecht and L.N. Trefethen: Nonuniqueness of best rational approximations on the unit disk, *J. Approx. Theory* **39**(1983), 275–288.

[IT] M.-P. Istace and J.-P. Thiran: On computing best Chebyshev complex rational approximants, *Numerical Algorithms* **5**(1993), 299–308.

[SV] E.B. Saff and R.S. Varga: Nonuniqueness of best complex rational approximations to real functions on real intervals, *J. Approx. Theory* **23**(1978), 78–85.

[W] On approximation to an analytic function by rational functions of best approximation, *Math Z.* **38**(1934), 163–176.