CENTRAL SEQUENCES AND AUTOMORPHISMS OF C^*-ALGEBRAS

By

JOHN PHILLIPS

DM-435-IR FEBRUARY 1987

Central Sequences and Automorphisms of C^*-algebras

by

John Phillips

Research partially supported by N.S.E.R.C. grant A-4588 and a sabbatical leave grant from Dalhousie University.

Introduction.

In this paper we begin a more detailed study of the subtle relations between automorphisms and central sequences in (separable) C^*-algebras. This is an initial attempt to do for automorphisms of C^*-algebras what A. Connes did for automorphisms of von Neumann algebras. The notion of a central sequence in a C^*-algebra (i.e., a bounded sequence which commutes asymptotically with the algebra) has already been used to great advantage in the study of inner derivations on C^*-algebras by G.A. Elliott and C.A. Akemann and G.K. Pedersen. By combining the ideas of these workers, we showed in a previous paper that any separable C^*-algebra which does not have continuous trace must have an uncountable outer automorphism group. The automorphism groups of C^*-algebras with continuous trace had already been studied in some detail in joint work with I. Raeburn.

It turns out, that the results in the C^*-algebra setting are very different from those in the von Neumann setting. There are two reasons for this. One problem is the obvious lack of projections (and partial isometries) in a general C^*-algebra – thus, many of the techniques of proof used by A. Connes are simply not available. On the other hand, there is an almost obscene overabundance of nontrivial central sequences in general C^*-algebras. Even the usually conservative $C_r^*(\mathbb{F}_2)$ (the left regular C^*-algebra of the free group on two generators) which is simple and admits no nontrivial projections, conceals vast hordes of central sequences which stay well away from the centre. This lack of a simple (or even primitive) C^*-algebra with only trivial central sequences (i.e., central sequences asymptotically near the centre) follows from the work of C.A. Akemann and G.K. Pedersen and is a major stumbling block in the search for interesting automorphisms which behave trivially (in an asymptotic sense) on central sequences.

A point of notation: if A is a (separable, unital) C^*-algebra then we let Inn A, Inn A^-, Ct A, and Aut A denote respectively the inner automorphisms, the closure of the inner automorphisms in the topology of point-wise convergence on A, the automorphisms leaving all central sequences asymptotically fixed ("centrally trivial" automorphisms), and the group of all automorphisms of A. Also, we let ϵ: Aut A $\to$ Aut A/Inn A = Out A denote the canonical quotient and mapping, and let $\mathcal{X}(A) = \epsilon(\text{Ct } A \cap \text{Inn } A^-)$.

In section one, we use Connes' technique to show that ϵ(Ct A) commutes with ϵ(Inn A^-) and so deduce that $\mathcal{X}(A)$ is abelian. We then use the Akemann-Pedersen construction of central sequences to show that the induced action of Ct A on $\hat{A}$ (the space of unitary equivalence classes of irreducible representations of A) is trivial. Combining this with A. Kishimoto's result that outer automorphisms of simple C^*-algebras never act trivially on $\hat{A}$, we observe that Ct A = Inn A for any simple separable C^*-algebra, A.

In section two, we show that Ct A = Inn A for primitive (separable, unital) C^*-algebras A whenever A is an A.F.-algebra or contains the compact operators. The A.F. result is a fairly direct but somehwat tricky calculation using ideas from the theory of perturbations of C^*-algebras. The result for C^*-algebras containing the compact operators is an application of D. Voiculescu's double commutant theorem for separable C^*-subalgebras of the Calkin algebra together with two very careful choices of quasi-central approximate identities in the compact operators.

In section three, we study hypercentral sequences, that is, those central sequences which commute asymptotically with every other central sequence. We let $T_A \subseteq H_A \subseteq C_A$ denote the algebras of trivial, hypercentral, and central sequences in A, respectively. Again, by a more or less direct but somewhat tricky calculation we show that $H_A = T_A$ if A is a primitive A.F.-algebra. We then apply Connes' technique to show that if $H_A = C_A$, then $\overline{\text{Inn } A} \subseteq \text{Ct } A$. Combining this with previous results we show that any (separable, unital) C^*-algebra A that has a primitive infinite-dimensional quotient which is either simple, A.F. or contains the compact operators has central sequences which are not hypercentral. We close the section by giving examples showing that all possible equalities and inequalities can be realized in the containments, $T_A \subseteq H_A \subseteq C_A$. In particular, we construct a separable, unital A.F.-algebra, D, with trivial centre, for which $T_D \subsetneqq H_D \subsetneqq C_D$.

In section four, we compute the centrally trivial automorphisms in two very different non-trivial cases. For the previously mentioned A.F.-algebra D, we show that Ct D is the uniform closure of Inn D and that

$$\epsilon(\text{Ct } D) = \mathfrak{X}(D) \cong \frac{\{(\lambda_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} S^1 \mid \lim_{n\to\infty} \bar{\lambda}_n \lambda_{n+1} = 1\}}{\{(\lambda_n)_{n=1}^{\infty} \mid \lim_{n\to\infty} \lambda_n \text{ exists in } S^1\}} .$$

These calculations, while nontrivial and somewhat lengthy, are not difficult. For the second example, we let A be a (separable, unital) primitive A.F.-algebra and let $B = C(X) \otimes A$ where X is a separable, compact space. We then show that $\epsilon(\text{Ct}(B))$ can be embedded in the torsion subgroup of $H^2(X,\mathbb{Z})$, the second Čech cohomology group of X with coefficients in the integers. Most of the ideas for this result are already contained in previous joint work with I. Raeburn - the only new difficulty is in showing that a centrally trivial automorphism of B necessarily gives rise to a uniformly continuous map: $X \to \text{Inn } A$. In order for this to work we need both of our previous results for primitive A.F.-algebras, A; that is, $\text{Ct } A = \text{Inn } A$ and $H_A = T_A$. We close out section four by showing that there need be no containment relations between $\text{Ct } A$ and $\overline{\text{Inn } A}$ for general (separable, unital) C^*-algebras, A.

Finally, in section five we list a number of open problems and a few tentative conjectures.

Acknowledgements. We thank N.S.E.R.C. of Canada, Dalhousie University and the University of Victoria for financial support of this research. We thank Bob Miers for his mellow but slightly astringent hospitality and conversations at the University of Victoria where this work was completed.

§1. Preliminary definitions and results.

Throughout this work, A (and occasionally B), will denote a separable, unital C^*-algebra over the complex numbers. Aut A will denote the group of all $*$-automorphisms of A, and Inn A will denote the normal subgroup of all inner automorphisms. An automorphism, α, will be called inner if there is a unitary operator $u \in A$ so that $\alpha(a) = \mathrm{Ad}u(a) = uau^*$ for all $a \in A$. We will usually consider Aut A to be topologized with the topology of pointwise convergence on A. Occasionally we will consider the topology on Aut A given by uniform convergence on the unit ball of A: we will take care to indicate when we use this topology, however. We denote Aut A/Inn A by Out A and let ϵ: Aut A $\to$ Out A denote the quotient map.

A central sequence in A is a bounded sequence $\{x_n\}$ of elements of A with the property that $[x_n,a] = x_n a - ax_n \to 0$ in norm for each $a \in A$. A uniformly central sequence is a bounded sequence $\{x_n\}$ for which the operators (on A) $\mathrm{ad}x_n(\cdot) = [x_n,\cdot]$ converge to 0 in norm. A central sequence $\{x_n\}$ is called hypercentral if $\|[x_n,y_n]\| \to 0$ for every central sequence $\{y_n\}$ of A. A central sequence $\{x_n\}$ is called trivial if there is a sequence $\{\lambda_n\}$ of central elements in A so that $\|x_n-\lambda_n\| \to 0$. It is evident that any trivial sequence is uniformly central and any uniformly central sequence is hypercentral. Two central sequences are called equivalent if their difference converges to zero in norm.

If $\alpha \in$ Aut A, then we say that α is centrally trivial if $\|\alpha(x_n)-x_n\| \to 0$ for every central sequence $\{x_n\}$ of A. We denote the normal subgroup of centrally trivial automorphisms of A by Ct A and note that Inn A $\subseteq$ Ct A.

We let $\text{Inn } A^-$ denote the closure of $\text{Inn } A$ in the topology of pointwise convergence. Following A. Connes, [5], we define $\mathcal{X}(A) = \epsilon(\text{Ct } A \cap \text{Inn } A^-)$.

1.1 Proposition: If A is a separable unital C^*-algebra then $\mathcal{X}(A)$ is abelian. In fact, $\epsilon(\text{Ct } A)$ commutes with $\epsilon(\text{Inn } A^-)$.

proof: We adapt (to the uniform norm setting) V. Jones simplification (to the II_1 setting) of A. Connes argument, [5, lemma 2.2.2].

Let $\alpha \in \text{Ct } A$ and let $\beta \in \text{Inn } A^-$. then, for every $\epsilon > 0$ we can find a neighbourhood, V_ϵ, of the identity in $\text{Aut } A$ such that if $\text{Ad}u \in V_\epsilon$ then $\|\alpha(u)-u\| < \epsilon$. Otherwise, we could find an $\epsilon_0 > 0$ and a decreasing sequence of neighbourhoods V_n shrinking to the identity and unitaries $\{v_n\}$ with $\text{Ad}v_n \in V_n$ with $\|\alpha(v_n)-v_n\| \geq \epsilon_0$: but, then $\{v_n\}$ would be a central sequence and α would not be in $\text{Ct } A$, a contradiction. Now, choose $\{W_n\}$ a decreasing sequence of neighbourhoods of β in $\text{Aut } A$ so that $W_n^{-1}W_n \subseteq V_{\frac{1}{2^n}}$ and choose unitaries u_n so that $\text{Ad}u_n \in W_n$. Then, $\text{Ad}\left[u_{n+1}^* u_n\right] \in W_{n+1}^{-1}W_n \subseteq W_n^{-1}W_n \subseteq V_{\frac{1}{2^n}}$ so that $\|u_{n+1}\alpha(u_{n+1}^*)-u_n\alpha(u_n^*)\| = \|\alpha(u_{n+1}^*u_n)-u_{n+1}^*u_n\| < \frac{1}{2^n}$. Thus, $\{u_n\alpha(u_n^*)\}$ is a cauchy sequence of unitaries which converges to a unitary $w \in A$. Finally,

$$\beta\alpha\beta^{-1}\alpha^{-1} = \lim_n \text{Ad}u_n \circ \alpha \circ \text{Ad}u_n^* = \lim_n \text{Ad}u_n \circ \text{Ad}\alpha(u_n^*)$$

$$= \lim_n \text{Ad}u_n\alpha(u_n^*) = \text{Ad}w \text{ is inner. That is,}$$

$$\epsilon(\beta)\epsilon(\alpha) = \epsilon(\alpha)\epsilon(\beta) \text{ as required.} \blacksquare$$

1.2 Remarks: Proposition 1.1 has been observed by (many?) other people including R. Herman, V. Jones and J. Rosenberg. It is the obvious first step in trying to adapt Connes' study of centrally trivial automorphisms of von Neumann algebras to the C^*-algebra setting. Unfortunately, almost none of the other general techniques carry over to the C^*-algebra setting with the exception of lemma 3.5.

1.3 Corollary: Let A be a separable unital C^*-algebra and let $\mathrm{Inn}\, A^{-\|\cdot\|}$ denote the uniform closure of the inner automorphisms in $\mathrm{Aut}\, A$. Then $\epsilon\left[\mathrm{Inn}\, A^{-\|\cdot\|}\right]$ is abelian! In fact, $\epsilon\left[\mathrm{Inn}\, A^{-\|\cdot\|}\right] \subseteq \mathfrak{X}(A)$.

proof: Clearly, $\mathrm{Inn}\, A^{-\|\cdot\|} \subseteq \mathrm{Inn}\, A^{-}$. A simple $\frac{\epsilon}{2}$ - argument shows that $\mathrm{Inn}\, A^{-\|\cdot\|} \subseteq \mathrm{Ct}\, A$. Thus, $\epsilon\left[\mathrm{Inn}\, A^{-\|\cdot\|}\right] \subseteq \mathfrak{X}(A)$ as required. ∎

1.4 Remarks: Since $\mathrm{Inn}\, A$ is not always uniformly closed [11], $\mathfrak{X}(A)$ is not always trivial and thus $\mathrm{Ct}\, A$ is not always trivial. We will return to this in section 4.

1.5 Notation: If A is a C^*-algebra, we denote by $\mathrm{Aut}_Z\, A$ the normal subgroup of $\mathrm{Aut}\, A$ of those automorphisms which fix the centre of A (pointwise).

1.6 Proposition: Under the natural embedding of $\mathrm{Aut}\, A \to \mathrm{Aut}\, A^{**}$, $\mathrm{Ct}\, A$ gets sent into $\mathrm{Aut}_Z\, A^{**}$. A is assumed to be separable, of course.

proof: By the construction in the proof of lemma 1.1 of [1], given any element z in the centre of A^{**}, we can find a central sequence $\{x_n\}$ in A with $x_n \to z$ strongly. Thus, if $\alpha \in \mathrm{Ct}\ A$, $\|\alpha(x_n)-x_n\| \to 0$ and so $\alpha(x_n) \to z$ strongly. Finally, $\alpha^{**}(z) = \lim_n \alpha^{**}(x_n) = \lim_n \alpha(x_n) = z$. ∎

1.7 Corollary: If A is a separable, type I C^*-algebra, then $\mathrm{Ct}\ A \subseteq \pi(A)$, the group of π-inner automorphisms of A.

proof: By G. Elliott's result [7], $\pi(A)$ is the group of automorphisms which leave each ideal of A invariant. Since each ideal of A determines and is determined by a unique central projection in A^{**}, the result follows. ∎

1.8 Corollary: If A is a simple, unital separable C^*-algebra, then $\mathrm{Ct}\ A = \mathrm{Inn}\ A$.

proof: By A. Kishimoto's result [12], any outer automorphism, α, of A takes some irreducible representation, π, to an inequivalent representation $\pi \circ \alpha$. Thus, α^{**} of the central cover of π in A^{**} is the central cover of $\pi \circ \alpha$ and these elements are distinct (disjoint, in fact). Thus, we can have no outer autormophisms in $\mathrm{Ct}\ A$. ∎

1.9 Remarks:

(a) It is easy to show that $\mathrm{Ct}\ A \neq \pi(A)$, in general. For example, if $A = \{f\colon [0,1] \to M_2(\mathbb{C}) \mid f$ is continuous and $f(1) = \lambda 1$ for $\lambda \in \mathbb{C}\}$ then conjugation by any continuous unitary-valued function from $[0,1)$ into $M_2(\mathbb{C})$

defines an automorphism of A in a natural way. Such automorphisms are easily seen to be π-inner by Elliott's criterion; however, they are seldom centrally trivial. In fact, one can show that $\mathrm{Ct}\, A = \mathrm{Inn}\, A$, in this case.

We conjecture that $\mathrm{Ct}\, A \subseteq \pi(A)$ even if A is not type I.

(b) The fact that $\mathrm{Ct}\, A = \mathrm{Inn}\, A$ if A is simple, is disappointing. This immediately implies that $\mathfrak{X}(A) = 0$ for any simple (unital, separable) C^*-algebra, A. Thus, any hope that $\mathfrak{X}(A)$ will be a useful invariant for simple C^*-algebras dies here. Of course, if we restrict the allowable class of central sequences to be considered (i.e., change the definition) one might yet obtain a useful theory for simple C^*-algebras. This possibility was suggested to the author independently by A. Connes and V.F.R. Jones: we will follow up this line of research in a future publication.

§2. Primitive C^*-algebras.

2.1 Conjecture: If A is a primitive, unital, separable C^*-algebra, then $\mathrm{Ct}\, A = \mathrm{Inn}\, A$.

2.2 Remark: We already have this result in the special case that A is simple: corollary 1.8. We verify the conjecture for two other large classes of primitive C^*-algebras in what follows. We first consider primitive A.F.-algebras.

2.3 Lemma: Let A be a primitive, unital C^*-algebra acting irreducibly on the Hilbert space H. Let B be a finite-dimensional C^*-subalgebra of A containing the identity. Then, $B = B^{cc} = (B^c)'$ where $(\cdot)^c$ denotes relative commutant in A and $(\cdot)'$ denotes commutant in $\mathcal{B}(H)$.

proof: We will show that B^c is strongly dense in B'. For, once this is done, we will have $(B^c)'' = B''' = B'$ and so $(B^c)' = B'' = B$. However, $(B^c)^c \subseteq (B^c)' = B$ and so $B = B^{cc}$ since $B \subseteq B^{cc}$ trivially.

To this end, let $x \in B'$, let $\xi_1, \ldots, \xi_n \in H$ and let $\epsilon > 0$. Let G be a finite subgroup of the unitary group of B whose linear span is B. Since $A'' = \mathcal{B}(H)$, we can choose $a \in A$ so that $\|au\xi_i - xu\xi_i\| < \epsilon$ for each $i = 1, \ldots, n$ and for each $u \in G$. Now, the element $b = \frac{1}{|G|} \sum_{u \in G} u^* au$ is in B^c since $vb = bv$ for all unitaries $v \in G$ which spans B. Moreover, for each $i = 1, \ldots, n$ we have

$$\begin{aligned}\|b\xi_i - x\xi_i\| &= \left\|\frac{1}{|G|} \sum_{u \in G} u^* au\xi_i - x\xi_i\right\| \\ &= \left\|\frac{1}{|G|} \sum_{u \in G} (u^* au\xi_i - x\xi_i)\right\| \\ &= \left\|\frac{1}{|G|} \sum_{u \in G} (u^* au\xi_i - u^* xu\xi_i)\right\| \\ &\leq \frac{1}{|G|} \sum_{u \in G} \|au\xi_i - xu\xi_i\| < \epsilon \quad \text{as required.} \quad \blacksquare\end{aligned}$$

2.4 Theorem: Let A be a primitive, unital, separable A.F.-algebra. Then, $\mathrm{Ct}\, A = \mathrm{Inn}\, A$.

<u>proof</u>: Let $\alpha \in \mathrm{Ct}\, A$ and suppose that A is acting irreducibly on the Hilbert space, H. By proposition 1.6 there is a unitary $v \in \mathcal{B}(H)$ so that $\alpha = \mathrm{Adv}|_A$. Thus, we consider $\alpha = \mathrm{Adv}$ to be defined on all of $\mathcal{B}(H)$.

Now, let $A = \overline{\bigcup_{n=1}^{\infty} B_n}$ where each B_n is unital, finite-dimensional and $B_n \subseteq B_{n+1}$ for each n. Let $\epsilon_n = \|\alpha|_{B_n^c} - \mathrm{id}|_{B_n^c}\|$ for each n. Then, $\epsilon_n \to 0$ since α is centrally trivial. Since B_n^c is strongly dense in B_n' we have $\epsilon_n = \|\alpha|_{B_n'} - \mathrm{id}|_{B_n'}\|$ also. Since B_n' is a type I von Neumann algebra, we can apply [4, proposition 4.2] to obtain a unitary w_n in $(B_n' \cup \alpha(B_n'))''$ with $\|w_n - 1\| < \sqrt{2}\,\epsilon_n$ so that $\alpha|_{B_n'} = \mathrm{Adw}_n$, provided $\epsilon_n < 1$. Thus, $\mathrm{Adw}_n^* \circ \alpha$ considered as an automorphism of $\mathcal{B}(H)$ is the identity on B_n' and, therefore, leaves $B_n'' = B_n$ invariant. but, then for each n

$$\|B_n - \alpha(B_n)\| = \|\mathrm{Adw}_n^*(\alpha(B_n)) - \alpha(B_n)\| < 2\sqrt{2}\,\epsilon_n.$$

Now, provided $2\sqrt{2}\,\epsilon_n \leq \frac{1}{169}$, we can apply the last line of the proof of theorem 4.1 of [4] to obtain an isomorphism $\beta\colon B_n \to \alpha(B_n)$ with $\|\beta - \mathrm{id}\| < (12.8)(2\sqrt{2}\,\epsilon_n)^{\frac{1}{2}} < 22\epsilon_n^{\frac{1}{2}}$. Letting G_n be a finite subgroup of unitaries in B_n which span B_n and defining $x_n = \frac{1}{|G_n|} \sum_{u \in G_n} \beta(u^*)u$ we obtain $x_n v = \beta(v) x_n$ for all $v \in G_n$ and hence all $v \in B_n$. Moreover, $\|x_n - 1\| < 22\epsilon_n^{\frac{1}{2}}$ so that x_n is invertible and $\beta(b) = x_n b x_n^{-1}$ for all $b \in B_n$. We now polar decompose x_n

to obtain a unitary v_n in $C^*(B_n,\alpha(B_n)) \subseteq A$ so that $\beta = Adv_n|_{B_n}$ and $\|v_n-1\| < \sqrt{2}\, 22\epsilon_n^{\frac{1}{2}} < 32\epsilon_n^{\frac{1}{2}}$: see [3, lemma 2.7].

Now, we define $\alpha_n = Adv_n \circ \alpha$ so that α_n is an inner perturbation of α, $\|\alpha_n-\alpha\| < 64\epsilon_n^{\frac{1}{2}}$ and $\alpha_n(B_n) = B_n$ globally. Moreover, on B_n' we have $\|\alpha_n|_{B_n'}-id|_{B_n'}\| < 64\epsilon_n^{\frac{1}{2}} + \epsilon_n$. Thus, α_n fixes the centre of B_n^c and hence the centre of B_n and therefore $\alpha_n|_{B_n}$ is inner! Hence, we can find a unitary $u_n \in B_n \subseteq A$ so that $\alpha_n|_{B_n} = Adu_n$. We now define $\alpha_n' = Adu_n^* \circ \alpha_n$ which is still an inner perturbation of α. We note that $\alpha_n'|_{B_n} = id|_{B_n}$ and $\alpha_n'|_{B_n'} = \alpha_n|_{B_n'}$ so that $\|\alpha_n'|_{B_n'}-id|_{B_n'}\| < 64\epsilon_n^{\frac{1}{2}} + \epsilon_n$.

Let, $\mathcal{R}_n = (B_n \cup B_n')''$ so that $\mathcal{R}_n' = B_n' \cap B_n = Z(B_n)$ is abelian. Thus $Z(\mathcal{R}_n) = \mathcal{R}_n \cap \mathcal{R}_n' = Z(B_n)$. Also, α_n' is the identity on $Z(\mathcal{R}_n)$ and therefore on each minimal central summand of $\mathcal{R}_n$ we see that α_n' splits as a tensor product of the identity on the corresponding summand of B_n and the restriction of α_n' to the corresponding summand of B_n'. Hence, $\|\alpha_n'|_{\mathcal{R}_n}-id|_{\mathcal{R}_n}\| < 64\epsilon_n^{\frac{1}{2}} + \epsilon_n$ and so by [4, proposition 4.2] there is a unitary $y_n \in \mathcal{R}_n$ with $\|y_n-1\| < \sqrt{2}\left[64\epsilon_n^{\frac{1}{2}}+\epsilon_n\right] < 92\epsilon_n^{\frac{1}{2}}$ so that $\alpha_n'|_{\mathcal{R}_n} = Ady_n$. Now, recalling that $\alpha_n' = Adu_n^*v_nv$ on all of $\mathcal{B}(H)$ we see that $(u_n^*v_nv)y_n^* \in \mathcal{R}_n' = Z(B_n) \subseteq A$. Letting $z_n = (u_n^*v_nv)y_n^*$ we observe that $Ad(v_n^*u_nz_n)$ is an inner automorphism of A and that

$$\|\alpha-\mathrm{Ad}(v_n^*u_nz_n)\| = \|\mathrm{Ad}v-\mathrm{Ad}(v_n^*u_nz_n)\| = \|\mathrm{Ad}(z_n^*u_n^*v_nv)-\mathrm{id}\| = \|\mathrm{Ad}y_n-\mathrm{id}\| < 184\epsilon_n^{\frac{1}{2}}.$$

Since this is true for each n, we see that α is in $\mathrm{Inn}\ A^{-\|\cdot\|}$ and so α is inner by [11, corollary 5.5]. ∎

2.5 Remarks:

(a) The corresponding result for von Neumann algebras, namely, that $\mathrm{Ct}\ R = \mathrm{Inn}\ R$ for the hyperfinite II_1 factor, R, is much simpler for two reasons. One is the existence of a unique trace and two is that the finite-dimensional subalgebras can be chosen to be factors, B_n so that R splits: $R = B_n \otimes B_n^c$. These facts greatly expedite the analysis.

(b) In a personal communication, R. Herman and V.F.R. Jones informed the author that they had proven this result under the additional hypothesis that A is simple.

2.6 Theorem: Let A be an extension of a separable unital C^*-algebra by the compact operators (on a separable infinite-dimensional Hilbert space). Then $\mathrm{Ct}\ A = \mathrm{Inn}\ A$.

Before proving this theorem, we prove a very closely related result of independent interest which will imply theorem 2.6.

2.7 Theorem: Let A be a separable, unital C^*-algebra on the separable infinite-dimensional Hilbert space, H, and suppose $\mathcal{K}$, the algebra of compact operators, is contained in A. If $b \in \mathcal{B}(H)$ and $b \notin A$ then there is a bounded sequence $\{x_n\}$ in $\mathcal{K}$ so that $\|[x_n,a]\| \to 0$ for all $a \in A$, but $\|[x_n,b]\|$ is bounded away from zero.

<u>proof</u>: By D. Voiculescu's double commutant theorem [16, corollary 1.9] there is a $T \in \mathcal{B}(H)$, $\|T\| \leq 1$ so that $[T,a] \in \mathcal{K}$ for all $a \in A$, but $\|\pi([T,b])\| = \epsilon_0 > 0$ where $\pi\colon \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{K}$ is the Calkin map.

Let $\{a_i\}$ be a dense sequence in the unit ball of A. Now, by [2, Theorem 1] we can choose a quasi-central approximate unit $\{k_n\} \subseteq \mathcal{K}$ so that:

(1) $0 \leq k_n \leq 1$ for each n;

(2) $\|k_n[T,a_i]-[T,a_i]\| \leq \frac{1}{n}$ for each $i = 1, \ldots, n$; and

(3) $\|[k_n,a_i]\| \leq \frac{1}{n}$ for each $i = 1, \ldots, n$.

We define $T_n = (1-k_n)T$ so that $\|T_n\| \leq \|T\| \leq 1$ and $\pi(T_n) = \pi(T)$ for all n. Thus, $\|\pi([T_n,b])\| = \epsilon_0 > 0$ for all n. Now, using, (2) and (3) and the fact that $\|T\| \leq 1$ we see that $\|[T_n,a_i]\| \leq \frac{2}{n}$ for each $i = 1, \ldots, n$.

Now, for each n we can choose unit vectors ξ_n so that $\|[T_n,b]\xi_n\| > \frac{\epsilon_0}{2}$. Let e_n be the rank-one projection onto $[T_n,b]\xi_n$. Now, let $\{y_n\} \subseteq \mathcal{K}$ be an approximate unit which is quasi-central for $C^*(A,b)$ so that

(a) $\|y_n\| \leq 1$ for each n;

(b) $\|[y_n,a_i]\| \leq \frac{1}{n}$ for each $i = 1, \ldots, n$, for each n;

(c) $\|y_n[T_n,a_i]-[T_n,a_i]\| \leq \frac{1}{n}$ for each $i = 1, \ldots, n$, for each n;

(d) $\|y_n e_n - e_n\| \leq \frac{1}{n}$ for each n; and

(e) $\|[y_n,b]\| \leq \frac{1}{n}$ for each n.

Finally, we let $x_n = y_n T_n \in \mathcal{K}$ for each n. Then, clearly $\|x_n\| \leq 1$ for each n. Moreover, for each $i = 1, \ldots, n$ we get

$$\begin{aligned}\|[x_n,a_i]\| &= \|y_nT_na_i-a_iy_nT_n\| \\ &\le \|y_nT_na_i-y_na_iT_n\| + \|y_na_iT_n-a_iy_nT_n\| \\ &\le \|y_n[T_n,a_i]\| + \|[y_n,a_i]\| \\ &\le \|y_n[T_n,a_i]-[T_n,a_i]\| + \|[T_n,a_i]\| + \frac{1}{n} \\ &\le \frac{1}{n} + \frac{2}{n} + \frac{1}{n} = \frac{4}{n}.\end{aligned}$$

Thus, $\{x_n\}$ is quasi-central; for A. On the other hand,

$$\begin{aligned}\|[x_n,b]\| &= \|y_nT_nb-by_nT_n\| \\ &= \|y_n[T_n,b]-[b,y_n]T_n\| \\ &\ge \|y_n[T_n,b]\|-\|[b,y_n]\| \\ &\ge \|y_n[T_n,b]\xi_n\| - \frac{1}{n} \\ &= \|y_ne_n[T_n,b]\xi_n\| - \frac{1}{n} \\ &\ge \|e_n[T_n,b]\xi_n\| - \|(y_ne_n-e_n)[T_n,b]\xi_n\| - \frac{1}{n} \\ &> \frac{\epsilon_0}{2} - \frac{1}{n}2\|b\| - \frac{1}{n}\end{aligned}$$

and we're done. ■

<u>proof of Theorem 2.6</u>: We assume A acts irreducibly on a Hilbert space H so that $\mathcal{K} \subseteq A$. Now, if $\alpha \in \mathrm{Ct}\, A$ then by proposition 1.6, there is a unitary v in $\mathcal{B}(H)$ so that $\alpha = \mathrm{Ad}v|_A$. If $v \notin A$, then by 2.7 we can find a bounded sequence $\{x_n\}$ in $\mathcal{K} \subseteq A$ which is central for A, but so that $\|\alpha(x_n)-x_n\| = \|vx_nv^*-x_n\| = \|vx_n-x_nv\|$ is bounded away from 0, a contradiction. ■

2.8 <u>Corollary</u>: If A is a primitive (separable, unital) type I C^*algebra, then $\mathrm{Ct}\, A = \mathrm{Inn}\, A$.

proof: A is either finite-dimensional or contains the compact operators. ∎

§3. Hypercentral sequences.

In this section, we address the existence problem for nontrivial hypercentral sequences. In particular, we show that primitive A.F.-algebras never have nontrivial hypercentral sequences and that a general (separable, unital) C^*-algebra almost never has only hypercentral sequences. If $C_A \supseteq H_A \supseteq T_A$ stand for the central (respectively, hypercentral, trivial) sequence algebras, then we show, by examples, that all possible equalities and inequalities hold.

In the next section, we will show how the absence of hypercentral sequences can sometimes facilitate the calculation of the centrally trivial automorphisms in certain related C^*-algebras.

3.1 Lemma: Let B be a finite-dimensional C^*-algebra and let G be a finite subgroup of the unitary group of B which spans B. If $y \in B$ and $\|uy-yu\| < \delta$ for all $u \in G$ and some fixed $\delta > 0$, then there is a z in $z(B)$ the centre of B, with $\|z\| \leq \|y\|$ and $\|y-z\| < \delta$.

proof: Let $z = \frac{1}{|G|} \sum_{u \in G} u^*yu$. Then $zv = vz$ for all $v \in G$ and hence $z \in Z(B)$. Moreover, $\|z\| \leq \|y\|$ and $\|z-y\| \leq \frac{1}{|G|} \sum_{u \in G} \|u^*yu-y\| < \delta$. ∎

3.2 Lemma: Let A be a primitive (separable, unital) C^*-algebra and let B be a finite-dimensional $*$-subalgebra of A containing the identity. Let $\{x_n\}$

be a central sequence in A bounded by 1. If there is an $\epsilon_0 > 0$ so that $\text{dist}(x_n, \mathbb{C}1) \geq \epsilon_0$ for all n then there exists a positive integer N so that $\text{dist}(x_n, B_1) \geq \frac{1}{10}\epsilon_0$ for all $n \geq N$, where B_1 denotes the unit ball of B.

<u>proof</u>: If there exists no such positive integer N, then we can choose a subsequence $\{x_{n_k}\}$ so that $\text{dist}(x_{n_k}, B_1) < \frac{1}{10}\epsilon_0$ for all k. Thus, we can choose a sequence $\{y_k\}$ in B_1 with $\|x_{n_k} - y_k\| < \frac{1}{10}\epsilon_0$ for each k.

Let $e_1, \ldots, e_m$ be minimal central projections in B with $\sum_{i=1}^{n} e_i = 1$ and (assuming A acts irreducibly on the Hilbert space H) let $\xi_1, \ldots, \xi_m$ be unit vectors in $e_1, \ldots, e_m$ respectively. Now, by Kadison's transitivity theorem, [10], we can choose $a \in A_1$ so that $a\xi_1 = \xi_2$, $a\xi_2 = \xi_3$, $\ldots$, $a\xi_m = \xi_1$. Finally, let G be a finite subgroup of the unitary group of B which spans B and let N be a positive integer so that for all $k \geq N$ we have

(1) $\|[x_{n_k}, a^j]\| < \frac{\epsilon_0}{40}$ for each $j = 1, \ldots, m$ and

(2) $\|[x_{n_k}, u]\| < \frac{\epsilon_0}{40}$ for each $u \in G$.

Then, for all $k \geq N$, we have

$$\|[y_k, a^j]\| \leq \frac{1}{10}\epsilon_0 + \frac{\epsilon_0}{40} + \frac{1}{10}\epsilon_0 = \frac{9\epsilon_0}{40} \quad \text{for each } j = 1, \ldots, m.$$

Similarly, we have $\|[y_k, u]\| < \frac{9\epsilon_0}{40}$ for all $u \in G$. Now, by lemma 3.1 we can choose $z_k \in Z(B)_1$ so that $\|y_k - z_k\| < \frac{9\epsilon_0}{40}$ for all $k \geq N$. Then, for each $k \geq N$ and each $j = 1, \ldots, m$ we have $\|[z_k, a^j]\| < \frac{9\epsilon_0}{40} + \frac{9\epsilon_0}{40} + \frac{9\epsilon_0}{40} = \frac{27\epsilon_0}{40}$.

Now, let $z_k = \sum_{i=1}^{m} \lambda_i^k e_i$ for some $\lambda_i^k \in \mathbb{C}$. If we evaluate $[z_k, a^j]$ at the vector ξ_i we see that $\frac{27\epsilon_0}{40} > \|\lambda_{i+j}^k \xi_{i+j} - \lambda_i^k \xi_{i+j}\| = |\lambda_{i+j}^k - \lambda_i^k|$ for each $i, j = 1, \ldots, m$ where $i + j$ means $(i+j)(\text{mod } m)$, of course. Letting λ_0^k be the average of $\lambda_1^k, \lambda_2^k, \ldots, \lambda_m^k$ we see that $\|z_k - \lambda_0^k 1\| < \frac{27\epsilon_0}{40}$. Finally, we see that $\|x_{n_k} - \lambda_0^k 1\| \leq \|x_{n_k} - y_k\| + \|y_k - z_k\| + \|z_k - \lambda_0^k 1\| < \epsilon_0$ which contradicts the fact that $\text{dist}(x_{n_k}, \mathbb{C}1) \geq \epsilon_0$. ∎

3.3 <u>Theorem</u>: If A is a primitive (separable, unital) A.F.-algebra and $\{x_n\}$ is a nontrivial central sequence in A, then there exists a central sequence of unitaries $\{u_n\}$ in A so that $\|[x_n, u_n]\|$ does not converge to 0. That is, A has no nontrivial hypercentral sequences.

<u>proof</u>: By extracting a subsequence and normalizing, we can assume that $\|x_n\| \leq 1$ and $\text{dist}(x_n, \mathbb{C}1) \geq \epsilon_0 > 0$ for all n. Now, let $A = \left[\bigcup_{j=2}^{\infty} A_j\right]^{-}$ where each A_j is unital, finite-dimensional and $A_j \subseteq A_{j+1}$ (we start our numbering at $j = 2$ so that there will be no confusion with our notation for unit balls). By the previous lemma we can choose N_2 so that $\text{dist}(x_n, (A_2)_1) \geq \frac{\epsilon_0}{10}$ for all $n \geq N_2$. We now <u>claim</u> that there is a unitary $u \in A_2^c = A_2' \cap A$ so that $\|x_{N_2} u - u x_{N_2}\| \geq \frac{\epsilon_0}{20}$ (we have tacitly assumed that A is acting irreducibly on a Hilbert space H). For, if not, then for all unitaries u in A_2^c we have $\|x_{N_2} u - u x_{N_2}\| < \frac{\epsilon_0}{20}$. In particular, for each $j \geq 2$ and all unitaries in $A_2^c \cap A_j$ we have this

inequality. If we average the conjugates of x_{N_2} over a finite, spanning sub-group of unitaries in $A_2^c \cap A_j$ we get an $x_j \in \left[A_2^c \cap A_j\right]^c$ with $\|x_j\| \leq \|x_{N_2}\| \leq 1$ and $\|x_j - x_{N_2}\| < \frac{\epsilon_0}{20}$. Let x be a weak operator cluster point of the sequence $\{x_j\}$ so that $\|x\| \leq 1$ and $\|x - x_{N_2}\| \leq \frac{\epsilon_0}{20}$. Now, it is easy to see that $\bigcup_{j=2}^{\infty} \left[A_2^c \cap A_j\right]$ is norm dense in A_2^c and so $A_2 = \left[A_2^c\right]' = \bigcap_{j=2}^{\infty} \left[A_2^c \cap A_j\right]'$. Now, the unit balls of the sequence $\left\{\left[A_2^c \cap A_j\right]'\right\}$ form a <u>decreasing</u> family of weak-operator compact sets and for each j, x_j is in the j-th set. Thus, for fixed j_k we see that the limit x is in the j_kth set (where $x = \lim_{k\to\infty} x_{j_k}$). That is, $x \in \bigcap_{k=1}^{\infty} \left[A_2^c \cap A_{j_k}\right]' = \bigcap_{j=2}^{\infty} \left[A_2^c \cap A_j\right]' = A_2$. But, $\operatorname{dist}(x_{N_2}, (A_2)_1) \geq \frac{\epsilon_0}{10}$ and $\|x - x_{N_2}\| \leq \frac{\epsilon_0}{20}$, a contradiction. Therefore, our claim is proved and we can find a unitary $u_{N_2} \in A_2^c$ so that $\|x_{N_2} u_{N_2} - u_{N_2} x_{N_2}\| \geq \frac{\epsilon_0}{20}$. We continue by induction first choosing $N_2 < N_3 < \ldots$ so that $\operatorname{dist}(x_n, (A_k)_1) \geq \frac{\epsilon_0}{10}$ for all $n \geq N_k$ and each $k = 2, 3, \ldots$. Then we choose unitaries $u_{N_2}, u_{N_3}, \ldots$ so that $u_{N_k} \in A_k^c$ and $\|x_{N_k} u_{N_k} - u_{N_k} x_{N_k}\| \geq \frac{\epsilon_0}{20}$ for each $k = 2, 3, \ldots$. For those positive integers n <u>not</u> in the set $\{N_2, N_3, \ldots\}$ we define $u_n = 1$. Then, $\{u_n\}$ is clearly a central sequence in A and $[x_n, u_n]$ does not converge to 0. ∎

3.4 <u>Proposition</u>: If A is a separable, unital C^*-algebra with no nontrivial hypercentral sequences and X is a compact separable space, then $C(X) \otimes A$ has no nontrivial hypercentral sequences either.

<u>proof</u>: Let $\{f_n\}$ be a nontrivial central sequence in $B = C(X) \otimes A$ and let us suppose that $\mathrm{dist}(f_n, Z(B)) \geq \epsilon_0 > 0$ for all n. We note that $Z(B) = C(X) \otimes Z(A) = C(X, Z(A))$. Let $\{a_i\}$ be a dense sequence in A and let us also assume that $\|[f_n, 1\otimes a_i]\| < \frac{1}{n}$ for each $1, 2, \ldots, n$ and each n. Now, we <u>claim</u> that for each n there is an $x_n \in X$ so that $\mathrm{dist}(f_n(x_n), Z(A)) \geq \frac{\epsilon_0}{2}$. Otherwise, by the uniform continuity of f we can choose neighbourhoods $N_1, \ldots, N_k$ covering X so that $\|f(x)-f(y)\| < \frac{\epsilon_0}{2}$ for all $x, y \in N_i$ and elements $z_i \in Z(A)$ so that $\|f(x_i)-z_i\| < \frac{\epsilon_0}{2}$ for some $x_i \in N_i$. Now, let $\{\varphi_i\}_{i=1}^k$ be a partition of unity subordinate to the covering $\{N_i\}$ and let $g = \sum_{i=1}^{k} \varphi_i z_i$ so that $g \in Z(B)$. Now, for each $x \in N_j$ we have $\|z_j - f(x)\| \leq \|z_j - f(x_j)\| + \|f(x_j)-f(x)\| < \epsilon_0$. Therefore, $\|g-f\| = \sup_{x\in X} \left\| \sum_{i=1}^{n} \varphi_i(x)(z_i - f(x)) \right\| \leq \sup_{x\in X} \sum_{i=1}^{n} \varphi_i(x) \|z_i - f(x)\| < \sup_{x\in X} \sum_{i=1}^{n} \varphi_i(x)\, \epsilon_0 = \epsilon_0$, a contradiction.

Thus, for each n let $x_n \in X$ be chosen so that $\mathrm{dist}(f_n(x_n), Z(A)) \geq \frac{\epsilon_0}{2}$. Now, the sequence $\{f_n(x_n)\}$ is central in A since for each $i = 1, \ldots, n$ $\|[f_n(x_n), a_i]\| \leq \|[f_n, 1\otimes a_i]\| < \frac{1}{n}$. Since $\{f_n(x_n\}$ is not trivial it is not hypercentral and so we can find a central sequence $\{b_n\}$ in A with

$\|[f_n(x_n),b_n]\|$ not converging to 0. Then $\{1\otimes b_n\}$ is central in $B = C(X) \otimes A$ and $\|[f_n, 1\otimes b_n]\| \geq \|[f_n(x_n),b_n]\|$ does not converge to 0, as required. ∎

The following lemma is a simple adaptation of A. Connes' result [5, Theorem 2.2.1 ((b) ⇒ (c))] to the C^*-algebra setting.

3.5 <u>Lemma</u>: If A is a separable unital C^*-algebra in which all central sequences are hypercentral, then $\overline{\text{Inn } A} \subseteq \text{Ct } A$ so that $\epsilon(\overline{\text{Inn } A})$ is abelian ($= \mathfrak{X}(A)$, in fact).

<u>proof</u>: Since all central sequences are hypercentral, given $\epsilon > 0$, there are $a_1, \ldots, a_n \in A$ and $\delta > 0$ so that if $x, y \in A_1$ satisfy $\|[x,a_i]\| \leq \delta$, $\|[y,a_i]\| \leq \delta$ for each $i = 1, \ldots, n$ then $\|[x,y]\| \leq \epsilon$.

Now, let $\alpha \in \overline{\text{Inn } A}$ and let $\mathscr{V}$ be the following neighbourhood of id_A in $\text{Aut } A$:

$$\mathscr{V} = \{\beta \in \text{Aut } A \mid \|\beta(a_i)-a_i\| < \delta \quad \text{for} \quad i = 1, \ldots, n\}.$$

If $\beta \in \text{Inn } A \cap \mathscr{V}$, then whenever $x \in A_1$ and $\|[x,a_i]\| \leq \delta$ for each $i = 1, \ldots, n$, we have $\|\beta(x)-x\| \leq \epsilon$. A moment's thought shows that this is still true for $\beta \in \overline{\text{Inn } A} \cap \mathscr{V}$. Now, $\alpha = \lim_n \text{Ad}u_n = \lim_n \text{Ad}u_n u_N^* \circ \text{Ad}u_N = \beta \circ \text{Ad}u_N$ and for sufficiently large (fixed) N we see that $\beta \in \mathscr{V}$. Now, suppose $x \in A_1$, $\|[x,a_i]\| \leq \delta$ for each $i = 1, \ldots, n$ and $\|[x,u_N]\| \leq \epsilon$, then

$$\begin{aligned}\|\alpha(x)-x\| &= \|\beta \circ \text{Ad}u_N(x)-x\| \\ &\leq \|\beta \circ \text{Ad}u_N(x)-\beta(x)\| + \|\beta(x)-x\|\end{aligned}$$

$$\leq \|Adu_N(x)-x\| + \epsilon \leq 2\epsilon.$$

Thus, $\alpha \in Ct\ A$ as claimed. ∎

3.6 <u>Theorem</u>: Let A be a separable unital C^*-algebra which has an infinite-dimensional primitive quotient which is either simple, A.F., or contains the compact operators. Then not all central sequences of A are hypercentral.

<u>proof</u>: Let B be the quotient of A mentioned above, then $Inn\ B^- \neq Inn\ B$ by [14] while $Ct\ B = Inn\ B$ by 1.8, 2.4, or 2.6 and so B has a pair of central sequences which do not asymptotically commute by lemma 3.5. If we lift these to a pair of central sequences in A by [1, lemma 3.4], the lifted sequences do not asymptotically commute and so are not hypercentral. ∎

3.7 <u>Remark</u>: If conjecture 2.1 is true, then any separable, unital C^*-algebra A, not all of whose irreducible representations are finite-dimensional, possesses non-hypercentral central sequences: simply use the argument in 3.6.

3.8 <u>Example</u>: A separable unital C^*-algebra, A, for which all central sequences are hypercentral but <u>not</u> necessarily uniformly central and therefore certainly not trivial. Define:

$$A = \{f\colon [0,1] \to M_2 | f \text{ is continuous and } f(1) \text{ is diagonal}\}.$$

Let p and q be the projections in A with constant values $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$, respectively. Now, if $a \in A$ and $\|[a,p]\| \leq \epsilon$ and $\|[a,q]\| \leq \epsilon$, then

$\|a-(pap+qaq)\| \leq 2\epsilon$. Thus, a is with 2ϵ of a diagonal-valued function $\tilde{a} = pap + qaq$. Since any two diagonal-valued functions clearly commute, this implies that any central sequence is hypercentral. For each n let $a_n = \begin{bmatrix} 1 & 0 \\ 0 & f_n \end{bmatrix}$ where $f_n\colon [0,1] \to [0,1]$ is a continuous function such that $f_n\left[\left[0, 1-\frac{1}{n}\right]\right] \equiv 1$ and $f_n(1) = 0$. Then, one easily checks that $\{a_n\}$ is central but not uniformly central. Thus, for this example, we have $C_A = H_A \supsetneqq T_A$.

We note that for any primitive (infinite-dimensional) A.F.-algebra A, theorem 3.3 implies that $C_A \supsetneqq H_A = T_A$. Moreover, by [1, theorem 2.4] we see that for any unital continuous trace algebra, A, $C_A = H_A = T_A$.

3.9 <u>Example</u>: A separable, unital A.F.-algebra, D, with $Z(D) = \mathbb{C}\cdot 1$ for which $C_D \supsetneqq H_D \supsetneqq T_D$. This example is very similar to example 6.3 of [11].

For each positive integer n, let H_n be a separable infinite-dimensional Hilbert space and let $H = \bigoplus_{n=1}^{\infty} H_n$ be the Hilbert space sum of the H_n's. On each H_n let E_n, F_n be orthogonal infinite-rank projections so that $E_n + F_n = 1_n$, the identity on H_n. Let $\mathcal{J}$ denote the C^*-algebra on H, $\left\{\bigoplus_{n=1}^{\infty} k_n \,\middle|\, \text{each } k_n \text{ is compact on } H_n \text{ and } \lim_{n\to\infty} \|k_n\| = 0\right\}$. Let $\tilde{C}_0$ denote the commutative C^*-algebra of complex sequences $\{c_n\}_{n=1}^{\infty}$ for which $\lim_{n\to\infty} c_n$ exists in $\mathbb{C}$. We represent $\tilde{C}_0$ faithfully on H to obtain the C^*-algebra, $\mathcal{C} = \left\{\bigoplus_{n=1}^{\infty} \left[a_n E_n + a_{n+1} F_n\right] \,\middle|\, \{a_n\} \in \tilde{C}_0\right\}$. Let $D = \mathcal{C} + \mathcal{J}$. Then, we claim that D is a separable, unital, A.F.-algebra on H. To see that D is closed, we compute $\|c+k\|$ for $c + k \in \mathcal{C} + \mathcal{J}$:

$$\|c+k\| = \sup_n \|c_nE_n+c_{n+1}F_n+k_n\| \geq \sup_n \|c_nE_n+c_{n+1}F_n\| = \sup_n |c_n| = \|c\|.$$

From this it easily follows that D is a C^*-algebra. Clearly, D is separable and unital. Since both $\mathcal{J}$ and $\mathcal{C}$ are A.F.-algegras, one easily checks that D is an A.F.-algebra.

Now, clearly $D'' = \bigoplus_{n=1}^{\infty} \mathcal{B}(H_n)$ and $D' = \bigoplus_{n=1}^{\infty} \mathbb{C}1_n = Z(D'')$. To compute $Z(D)$, we first observe that $Z(D) \subseteq Z(D'')$. Thus, if $\bigoplus_{n=1}^{\infty} \lambda_n 1_n \in Z(D)$, then $\bigoplus_{n=1}^{\infty} \lambda_n 1_n = \bigoplus_{n=1}^{\infty} \left[c_nE_n+c_{n+1}F_n+k_n\right]$ and so $\lambda_n 1_n = c_nE_n + c_{n+1}F_n + k_n$ for each n. Thus, $k_n = 0$ for each n and $c_n = c_{n+1} = \lambda_n$ for each n. Thus, $\lambda_n = c_n = \lambda_{n-1}$ for each n and so $\lambda_n = \lambda_1$ for each n. Hence, $Z(D) = \mathbb{C}\cdot 1$. It is fairly clear that D is not primitive: however, this will follow from theorem 3.3 once we show that D has nontrivial hypercentral sequences.

For each positive integer n, let $1 = c_{n1} > c_{n2} > c_{n3} > \ldots \geq 0$ be a sequence so that $|c_{nk}-c_{nk+1}| \leq \frac{1}{n}$ for each $k = 1, 2, \ldots$ and $\lim_{k\to\infty} c_{nk} = 0$. Let $c_n = \bigoplus_{k=1}^{\infty} \left[c_{nk}E_k+c_{nk+1}F_k\right] \in D$. Then, clearly $\mathrm{dist}(c_n, Z(D'')) \leq \frac{1}{2n}$ and so for each $a \in D$ we have $\|[c_n,a]\| \leq \frac{1}{n}\|a\|$. Thus, $\{c_n\}$ is a uniformly central and hence hypercentral sequence in D. However, for each n $\mathrm{dist}(c_n, \mathbb{C}1) \geq \frac{1}{2}$ and so $\{c_n\}$ is not trivial. Therefore $H_D \neq T_D$.

To see that, $C_D \neq H_D$ we observe that D has a primitive, A.F. quotient which is infinite-dimensional (represent D on H_n in the obvious way) and so, by theorem 3.3, this quotient has a pair of central sequences which do not asymptotically commute. Lifting this pair to central sequences in D by [1], we obtain a pair of central sequences in D which do not asymptotically commute. Therefore, $C_D \neq H_D$ as claimed.

§4. Computations of Ct A and $\mathfrak{X}(A)$.

In this section, we explicitly compute $\mathrm{Ct}\, A$ and $\mathfrak{X}(A)$ for some of the previously described examples. There are two types of computations involving the centre of A. One is mainly analytical and is based on the relationship between $Z(A)$ and $Z(A'')$ when $Z(A)$ is "small" compared to $Z(A'')$. The other is mainly topological and involves the fibering of centrally trivial automorphism over $Z(A)^\wedge$ when $Z(A)$ is "large". We begin with example 3.9.

4.1 Theorem: Let D be the C^*-algebra of example 3.9. Then,

(1) $\mathrm{Inn}\ D \subsetneq \mathrm{Inn}\ D^{-\|\cdot\|} = \mathrm{Ct}\ D \subsetneq \mathrm{Inn}\ D^-$;

(2) $\alpha \in \mathrm{Ct}\ D \iff$ modulo inner automorphisms, $\alpha = \mathrm{Ad}u$ where

$$u = \bigoplus_{n=1}^{\infty} \left[\lambda_n E_n + \lambda_{n+1} F_n\right] \text{ where } \lambda_n \in S^1 \text{ and } \lim_{n\to\infty} \bar{\lambda}_n \lambda_{n+1} = 1;$$

(3) Such an α is inner $\iff \lim_{n\to\infty} \lambda_n$ exists in S^1;

(4) $$\mathfrak{X}(D) \cong \frac{\left\{(\lambda_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} S^1 \,\middle|\, \lim_{n\to\infty} \bar{\lambda}_n \lambda_{n+1} = 1\right\}}{\left\{(\lambda_n)_{n=1}^{\infty} \,\middle|\, \lim_{n\to\infty} \lambda_n \text{ exists in } S^1\right\}}.$$

We break the proof up into a series of more easily digested lemmas.

4.2 <u>Lemma</u>: Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in S^1 with $\lim_{n\to\infty} \bar{\lambda}_n\lambda_{n+1} = 1$. Then letting $u = \bigoplus_{n=1}^{\infty}\left[\lambda_n E_n + \lambda_{n+1}F_n\right]$, Adu defines an element of Inn $D^{-\|\cdot\|}$. Moreover, Adu is inner $\Leftrightarrow \lim_{n\to\infty}\lambda_n$ exists in S^1.

<u>proof</u>: For each N, let $v_N = \left[\bigoplus_{n=1}^{N}\left[\lambda_n E_n + \lambda_{n+1}F_n\right]\right] \oplus \left[\bigoplus_{n=N+1}^{\infty} \lambda_{N+1}1_n\right]$. Then, $v_N \in D$ for each N. If we choose K so that for $N \geq K$ we have $|\bar{\lambda}_N\lambda_{N+1}-1| < \epsilon$ for a prescribed $\epsilon > 0$, then we can show that $\|Adu-Adv_N\| \leq 2\epsilon$. Indeed, defining $w_N = \left[\bigoplus_{n=1}^{N} 1_n\right] \oplus \left[\bigoplus_{n=N+1}^{\infty} \bar{\lambda}_{N+1}\lambda_n 1_n\right]$ we see that $w_N \in Z(D'')$ and $\|uv_N^*-w_N\| \leq \epsilon$ and so $\|Adu-Adv_N\| = \|Aduv_N^*-id\| = \|Aduv_N^*-Adw_N\| \leq 2\epsilon$. Thus, $Adu \in$ Inn $D^{-\|\cdot\|}$.

Now, if there is a $\lambda \in S^1$ with $\lim_{n\to\infty}\lambda_n = \lambda$ then clearly $u \in D$ and so Adu is inner. On the other hand, if Adu is inner, then we have Adu = Adv where $v \in D$ and so $v = \bigoplus_{n=1}^{\infty}\left[\gamma_n E_n + \gamma_{n+1}F_n\right]$ where $\lim_{n\to\infty}\gamma_n = \gamma \in S^1$. Since $Aduv^* = id$ we have $uv^* \in D' = \bigoplus_{n=1}^{\infty} \mathbb{C}1_n$ and so $\bar{\gamma}_n\lambda_n = \bar{\gamma}_{n+1}\lambda_{n+1}$ for each n. That is, $\lambda_{n+1} = \gamma_{n+1}\bar{\gamma}_n\lambda_n$. From this, we see that $\lambda_{n+1} = \gamma_{n+1}\bar{\gamma}_1\lambda_1$ and so $\lim_{n\to\infty}\lambda_n = \gamma\bar{\gamma}_1\lambda_1 \in S^1$. ∎

4.3 <u>Lemma</u>: If $\alpha \in$ Inn $D^{-\|\cdot\|}$, then modulo Inn D we have $\alpha = Adu$ where $u = \bigoplus_{n=1}^{\infty}\left[\lambda_n E_n + \lambda_{n+1}F_n\right]$ and $\lim_{n\to\infty}\bar{\lambda}_n\lambda_{n+1} = 1$.

<u>proof</u>: Let $\{v_j\}$ be a sequence of unitaries in D with $\mathrm{Ad}v_j \to \alpha$ uniformly. Let $v_j = \bigoplus_{n=1}^{\infty}\left[\lambda_n^j E_n + \lambda_{n+1}^j F_n + k_n^j\right]$ where we may assume $\lambda_1^j = 1$ for each j. Since an inner automorphism leaves every ideal invariant, so does α and hence α induces an automorphism of every quotient which is clearly the limit of the corresponding inner automorphisms on the quotient. Thus, for fixed n, $\left\{\mathrm{Ad}\left[\lambda_n^j E_n + \lambda_{n+1}^j F_n + k_n^j\right]\right\}_{j=1}^{\infty}$ is a (uniformly) Cauchy sequence. In particular, $\mathrm{Ad}\left[E_1 + \lambda_2^j F_1 + k_1^j\right]$ is a Cauchy sequence (as automorphisms on the C^*-algebra $[\mathcal{K}(H_1)+\mathrm{span}\{E_1,F_1\}]$). We show that this implies that $\{\lambda_2^j\}_{j=1}^{\infty}$ and $\{k_1^j\}_{j=1}^{\infty}$ are Cauchy sequences. First, we observe that $\left[E_1+\lambda_2^j F_1 + k_1^j\right]$ unitary $\Rightarrow$ $\left[E_1 + \lambda_2^j F_1\right]$ is unitary $\Rightarrow$ $|\lambda_2^j| = 1$. Now, given $\epsilon > 0$ choose N so that $n, m \geq N \Rightarrow$ $\left\|\mathrm{Ad}\left[E_1+\lambda_2^n F_1 + k_1^n\right] - \mathrm{Ad}\left[E_1 + \lambda_2^m F_1 + k_1^m\right]\right\| < \epsilon$. Now, fixing $n, m \geq N$ let v be a compact partial isometry with $E_1 v F_1 = v$ and $\left\|v\left[k_1^n\right]^*\right\|$, $\left\|k_1^n v\right\|$, $\left\|v\left[k_1^m\right]^*\right\|$, $\left\|k_1^m v\right\| < \epsilon/6$. Then $\epsilon > \left\|\mathrm{Ad}\left[E_1+\lambda_2^n F_1+k_1^n\right]v - \mathrm{Ad}\left[E_1+\lambda_2^m F_1 + k_1^m\right]v\right\|$

$$= \left\|\left[\bar{\lambda}_2^n - \bar{\lambda}_2^m\right]v + v\left[k_1^n\right]^* - v\left[k_1^m\right]^* + k_1^n v \bar{\lambda}_2^n - k_1^m v \bar{\lambda}_2^m + k_1^n v\left[k_1^n\right]^* - k_1^m v\left[k_1^m\right]^*\right\|$$

$$\leq \left|\lambda_2^n - \lambda_2^m\right| - 6\,\frac{\epsilon}{6}.$$

Thus, for $n, m \geq N$ we have $\left|\lambda_2^n - \lambda_2^m\right| < 2\epsilon$ and so there exists a $\lambda_2 \in S^1$ where $\lambda_2 = \lim_j \lambda_2^j$.

Now, using the fact that the sequence of unitaries $\left\{U_j = \left[E_1 + \lambda_2^j F_1\right]\right\}$ is Cauchy together with the fact that $\left\{\mathrm{Ad}\left[U_j + k_1^j\right]\right\}$ is Cauchy we can show that $\left\{k_1^j\right\}$ is Cauchy by applying $\mathrm{Ad}\left[U_n + k_1^n\right]$ to a compact partial isometry whose range covers k_1^n, $\left[k_1^n\right]^*$, k_1^m, and $\left[k_1^m\right]^*$ to within ϵ and whose initial space annihilates these four operators to within ϵ. We omit the details. Let $k_1 = \lim_j k_1^j$.

Similarly, knowing that $\{\lambda_2^j\}$ and $\{Ad[\lambda_2^j E_2+\lambda_3^j F_2+k_2^j]\}$ are Cauchy sequences enables us to conclude that $\{\lambda_3^j\}$ and $\{k_2^j\}$ are Cauchy sequences. We let $\lambda_3 = \lim\limits_j \lambda_3^j$ and $k_2 = \lim\limits_j k_2^j$. We continue by induction to obtain $\lambda_n = \lim\limits_j \lambda_n^j$ and $k_n = \lim\limits_j k_n^j$ for each n.

Let $u = \bigoplus\limits_{n=1}^{\infty} [\lambda_n E_n+\lambda_{n+1}F_n+k_n]$ so that $\alpha = Adu$. Now, given $\epsilon > 0$ there exists v_j as above with $\|Adu-Adv_j\| < \epsilon$, or $\|Aduv^*-id\| < \epsilon$. Now, uv^* has the same general form as u and so the compact operators appearing there all have norm at most proportional to ϵ. To see this, one first shows that the coefficients of E_n and F_n are approximately the same and so the restriction of uv^* to H_n is approximately $\gamma 1_n+k$. Then, one easily concludes that $\|k\|$ is small. Thus, the compact pieces of uv^* on H_n are bounded by $C_1\epsilon$ (where C_1 is independent of ϵ and n). Since the corresponding compact pieces of v converge to zero as $n \to \infty$, we see that the compact pieces of u are bounded by $C_2\epsilon$ (where C_2 is independent of ϵ and n) as $n \to \infty$. Since this is true for every $\epsilon > 0$ we see that $u = \bigoplus\limits_{n=1}^{\infty} [\lambda_n E_n+\lambda_{n+1}F_n+k_n]$ where $\|k_n\| \to 0$ as $n \to \infty$. So $u = \left[\bigoplus\limits_{n=1}^{\infty} [\lambda_n E_n+\lambda_{n+1}F_n]\right]\left[\bigoplus\limits_{n=1}^{\infty} 1_n+[\bar{\lambda}_u E_n+\bar{\lambda}_{n+1}F_n]k_n\right]$ and this 2nd unitary is clearly in D.

It remains to see that $\bar{\lambda}_n\lambda_{n+1} \to 1$ as $n \to \infty$. Now, given $\epsilon > 0$ choose $v = \bigoplus\limits_{n=1}^{\infty} [\gamma_n E_n+\gamma_{n+1}F_n+h_n] \in D$ so that $\|Aduv^*-id\| < \epsilon$. Since $k_n \to 0$ and $h_n \to 0$ this implies that $\overline{\lim}|\lambda_n\bar{\gamma}_n-\lambda_{n+1}\bar{\gamma}_{n+1}| < \epsilon$ and so $\overline{\lim}|\bar{\gamma}_n\gamma_{n+1}-\bar{\lambda}_n\lambda_{n+1}| < \epsilon$. Since $\lim\limits_{n\to\infty} \bar{\gamma}_n\gamma_{n+1} = 1$ we have $\overline{\lim}|1-\bar{\lambda}_n\lambda_{n+1}| \leq \epsilon$. Thus, $\lim\limits_{n\to\infty} \bar{\lambda}_n\lambda_{n+1} = 1$ as required. ∎

4.4 Lemma: $\mathrm{Ct}\, D = \mathrm{Inn}\, D^{-\|\cdot\|}$.

proof: We have already observed in the proof of corollary 1.3 that $\mathrm{Inn}\, D^{-\|\cdot\|} \subseteq \mathrm{Ct}\, D$, in general. It suffices to show the reverse inclusion in this case. Now, if $\alpha \in \mathrm{Ct}\, D$, then α leaves each primitive quotient, D_n (acting on H_n) invariant and the induced automorphism is in $\mathrm{Ct}\, D_n$ since central sequences in D_n lift to central sequences in D by [1]. By theorem 2.4 (or 2.6), α is inner on that quotient and therefore $\alpha = \mathrm{Ad}u$ where $u = \bigoplus_{n=1}^{\infty} \left[\gamma_n E_n + \lambda_n F_n + k_n\right]$. If we multiply by $\mathrm{Ad}v$, where $v = \bigoplus_{n=1}^{\infty} \bar{\gamma}_n 1_n \in Z(D'')$, we do not change α and so we can assume $\alpha = \mathrm{Ad}u'$ where $u' = \bigoplus_{n=1}^{\infty} \left[E_n + \lambda'_n F_n + k'_n\right]$. Now, for each n, let v_n be a compact partial isometry on H_n with $E_n v_n F_n = v_n$ and $\left\|v_n (k'_n)^*\right\| < \frac{1}{n}$, $\|k'_n v_n\| < \frac{1}{n}$. Then $\{v_n\}$ is a central sequence in D and so $\|\alpha(v_n) - v_n\| \to 0$. But, $\alpha(v_n) = \bar{\lambda}'_n v_n + v_n (k'_u)^* + \bar{\lambda}'_n k'_n v_n + k'_n v_n (k'_n)^*$ and so $\lambda'_n \to 1$.

Now, define $\mu_1 = 1$ and $\mu_n = \lambda'_1 \ldots \lambda'_{n-1}$ for $n > 1$. Then, for each n $\lambda'_n = \bar{\mu}_n \mu_{n+1}$ and so multiplying α by $\mathrm{Ad}w$ where $w = \bigoplus_{n=1}^{\infty} \mu_n 1_n \in Z(D'')$ we see that $\alpha = \mathrm{Ad}u''$ where $u'' = \bigoplus_{n=1}^{\infty} \left[\mu_n E_n + \mu_{n+1} F_n + k''_n\right]$ and $\lim_n \bar{\mu}_n \mu_{n+1} = \lim_n \lambda'_n = 1$. Thus, to see that $\alpha \in \mathrm{Inn}\, D^{-\|\cdot\|}$, it suffices to see that $\|k''_n\| \to 0$ for then $u'' = \left[\bigoplus_{n=1}^{\infty} \left[\mu_n E_n + \mu_{n+1} F_n\right]\right] \left[\bigoplus_{n=1}^{\infty} \left[1_n + \left[\bar{\mu}_n E_n + \bar{\mu}_{n+1} F_n\right] k''_n\right]\right]$ where the 2^{nd} unitary is in D and Ad of the 1^{st} unitary is in $\mathrm{Inn}\, D^{-\|\cdot\|}$ by 4.2. To see that $\|k''_n\| \to 0$, it suffices to see that $k'''_n = \left[\bar{\mu}_n E_n + \bar{\mu}_{n+1} F_n\right] k''_n \to 0$. But, $\mathrm{Ad}\left[\bigoplus_{n=1}^{\infty} (1_n + k'''_n)\right]$ is in $\mathrm{Ct}\, D$ and so by a careful choice of compact partial isometries v_n on H_n we can obtain a central sequence $\{v_n\}$ so that knowing $\left\| Ad\left[\bigoplus_{m=1}^{\infty} (1_m+k_m''')\right]v_n-v_n\right\| \to 0$ as $n \to \infty$ will imply $\|k_n'''\| \to 0$ as $n \to \infty$. ■

proof of Theorem 4.1:

(1): By lemmas 4.2 and 4.4 we have $\text{Inn } D \subsetneq \text{Inn } D^{-\|\cdot\|} = \text{Ct } D$. To see that $\text{Inn } D^-$ is much larger than $\text{Ct } D = \text{Inn } D^{-\|\cdot\|}$, let U_1 be any unitary on H_1 which commutes with E_1 (and hence with F_1). Let $u = U_1 \oplus \left[\bigoplus_{n=2}^{\infty} 1_n\right]$. Then $\text{Ad}u$ is in $\text{Inn } D^-$ but not usually in $\text{Inn } D^{-\|\cdot\|}$.

(2) and (3): Lemmas 4.2, 4.3 and 4.4 immediately imply (2) and (3).

(4) By part (1), $\mathfrak{X}(D) = \text{Inn } D^{-\|\cdot\|}/\text{Inn } D$. Now, by the previous lemmas, the map $(\lambda_n)_{n=1}^{\infty} \to \text{Ad}u$ where $u = \bigoplus_{n=1}^{\infty} \left[\lambda_n E_n + \lambda_{n+1} F_n\right]$ is a well-defined homomorphism of $\left\{(\lambda_n)_{n=1}^{\infty} \middle| \lim_n \bar{\lambda}_n \lambda_{n+1} = 1\right\}$ into $\text{Inn } D^{-\|\cdot\|}$ which hits every coset modulo the subgroup Inn D. By lemma 4.2, the inverse image of Inn D is precisely $\left\{(\lambda_n)_{n=1}^{\infty} \middle| \lim_n \lambda_n \text{ exists}\right\}$. This proves (4). ■

4.5 <u>Theorem</u>: Let X be a separable compact space and let A be a separable unital C^*-algebra such that

1. $Z(A) = \mathbb{C}$
2. $\text{Ct } A = \text{Inn } A$
3. Every hypercentral sequence in A is uniformly central.

If $B = C(X) \otimes A$ then ;we have an exact sequence: $0 \to \text{Inn } B \to \text{Ct } B \xrightarrow{\eta} \check{H}^2(X,\mathbb{Z})$.

In fact, $\mathrm{Ct}\ B = \{\alpha | x \to \alpha_x : X \to \mathrm{Inn}\ A$ is uniformly continuous$\}$. (For example, any primitive A.F.-algebra satisfies 1., 2., 3.)

<u>proof</u>: If $\alpha \in \mathrm{Ct}\ B$ then clearly α preserves the centre of B and so α is a $C(X)$-module map. Thus, by [15, lemma 1.6] we have a map $x \longmapsto \alpha_x : X \to \mathrm{Aut}\ A$ continuous in the point-norm topology on $\mathrm{Aut}\ A$, so that $\alpha(f)(x) = \alpha_x(f(x))$ for all $x \in X$. Moreover, since every central sequence in A clearly lifts to one in B, we see that each $\alpha_x \in \mathrm{Ct}\ A = \mathrm{Inn}\ A$. Now, suppose the map $x \longrightarrow \alpha_x$: $X \to \mathrm{Inn}\ A$ is <u>not</u> uniformly continuous. Then, there is an $x_0 \in X$ and a sequence $\{x_n\}$ converging to x_0 so that $\alpha_{x_n} \to \alpha_{x_0}$ but not uniformly. Multiplying by $\mathrm{id} \otimes \alpha_{x_0}^{-1}$ which is inner we can assume that $\alpha_{x_0} = \mathrm{id}$. Now, each $\alpha_{x_n} = \mathrm{Ad}u_n$ for some unitaries $\{u_n\}$ in A. Since $\mathrm{Ad}u_n \to \mathrm{id}$ pointwise we see that $\{u_n\}$ is a central sequence in A which is not uniformly central and therefore not hypercentral. By property 3 we can choose a central sequence $\{a_n\}$ in A so that $\|[u_n, a_n]\|$ does not converge to 0. Now, $\{1 \otimes a_n\}$ is a central sequence in B and so $\|\alpha(1 \otimes a_n) - 1 \otimes a_n\| \to 0$. However, $\|\alpha(1 \otimes a_n) - 1 \otimes a_n\| = \|\alpha_{x_n}(a_n) - a_n\| = \|u_n a_n u_n^* - a_n\| = \|u_n a_n - a_n u_n\| = \|[u_n, a_n]\|$ does not converge to 0, a contradiction. Thus, the map $x \longrightarrow \alpha_x$: $X \to \mathrm{Inn}\ A$ is uniformly continuous.

Now, since $\mathrm{Ct}\ A = \mathrm{Inn}\ A$ and $\mathrm{Ct}\ A$ is clearly uniformly closed we see that $\mathrm{Ad}: U(A) \to \mathrm{Inn}\ A$ admits continuous local sections [13, theorem 2.1 and 11, theorem 5.3]. Thus, we can find an open cover $\{X_i\}$ of X and continuous maps $u_i: X_1 \to U(A)$ so that for elements $b \in C(X,A)$, supported on X_i, we have $\alpha(b) = u_i b u_i^*$. One easily verifies that $u_i u_j^*: X_{ij} = X_i \cap X_j \to S^1 1$ so that $u_i u_j^* = \lambda_{ij} 1$ and $\{\lambda_{ij}\}$ satisfies the 1-cocycle equation $\lambda_{ij}\lambda_{jk} = \lambda_{ik}$

on $X_i \cap X_j \cap X_k$. Thus, $\{\lambda_{ij}\}$ defines an element of the sheaf cohomology group $\check{H}^1(X,\mathcal{S})$ where $\mathcal{S}$ is the sheaf of continuous S^1-valued functions on X. The image of this element under the canonical isomorphism, $\check{H}^1(X,\mathcal{S}) \cong \check{H}^2(X,\mathbb{Z})$, gives our map $\mathrm{Ct}\, B \xrightarrow{\eta} \check{H}^2(X,\mathbb{Z})$. To see that this procedure gives a well-defined exact sequence as described in the theorem, see [15].

On the other hand, if α is given by a uniformly continuous map $x \to \alpha_x$: $X \to \mathrm{Inn}\, A$ then we can choose a finite number of closed neighbourhoods $X_1, \ldots, X_m$ covering X so that, α^i, the induced automorphism on the quotient $C(X_i) \otimes A$ is inner. Now, if $\epsilon > 0$ and $\{b_n\}$ is central in B then the quotient sequence $\{b_n^i\}$ is central in the quotient and so $\|\alpha_i(b_n^i) - b_n^i\| \to 0$. Thus we can choose N so that $\|\alpha^i(b_n^i) - b_n^i\| < \epsilon$ for all $n \geq N$ and $i = 1, \ldots, m$. Now, if $n \geq N$, $\|\alpha(b_n) - b_n\| = \sup_{i=1,\ldots,m} \|\alpha^i(b_n^i) - b_n^i\| < \epsilon$ and so $\alpha \in \mathrm{Ct}\, B$. ■

4.6 Remarks: 1. By theorems 2.4 and 3.3, any primitive unital A.F.-algebra satisfies the hypotheses of the theorem. In this case, we can show that the range of the map $\mathrm{Ct}\, B \xrightarrow{\eta} \check{H}^2(X,\mathbb{Z})$ is contained in the torsion subgroup of $\check{H}^2(X,\mathbb{Z})$. We briefly indicate how this is done (Iain Raeburn has also done this computation - private conversation, Algonquin park, 1980). Let $A = \left[\bigcup_{n=1}^{\infty} A_n\right]^-$ where each A_n is finite-dimensional and contains the unit of A. Let $U = U(A)$ and $U_n = U(A_n)$. Let $G_n = \mathrm{Ad}U_n \subseteq \mathrm{Inn}\, A = G$ so that $G = \left[\bigcup_{n=1}^{\infty} G_n\right]^-$. Since A is primitive, the map $\mathrm{Ad}\colon U \to G$ has local sections and so one can find a neighbourhood $\mathcal{O}_n$ of G_n in G so that projection of A onto A_n followed by polar decomposition induces a well-defined continuous map $p\colon \mathcal{O}_n \to G_n$ with $\|p(\alpha_0)-\alpha_0\| < \epsilon$ for all α_0 in $\mathcal{O}_n$.

Now, if $\alpha \in \mathrm{Ct}\, B$ so that $x \to \alpha_x\colon X \to \mathrm{Inn}\, A = G$ is continuous, then by compactness, we can find an n so that $\alpha_x \in \mathcal{O}_n$ for all $x \in X$. Then $x \to \beta_x = p(\alpha_x) \in G_n = \mathrm{Ad}U_n$ is continuous and $\|\beta\circ\alpha^{-1}-\mathrm{id}\| < \epsilon$ and is therefore inner (since it is locally inner). Thus, the images of α and β in $\check{H}^2(X,\mathbb{Z})$ are identical. We compute the image of β. Choose a finite open cover $\{X_i\}$ of X and continuous maps $v_i\colon X_i \to U_n$ so that $\beta = \mathrm{Ad}v_i$ for elements supported on X_i. Now, we unitally embed A_n in $M_k(\mathbb{C})$, the $k\times k$ matrices over $\mathbb{C}$, so that $\gamma_i = \det v_i\colon X_i \to S^1$ is continuous. By shrinking the cover, we can assume that each $\bar{\gamma}_i$ has a continuous k^{th} root, μ_i, in S^1. Let $\tilde{v}_i = \mu_i v_i$ so that $\mathrm{Ad}\tilde{v}_i = \mathrm{Ad}v_i = \beta$ on X_i. But, now, $\det \tilde{v}_i = \det(\mu_i v_i) = \mu_i^k \det v_i = \bar{\gamma}_i\gamma_i = 1$. Then, the 1-cocycle $\lambda_{ij}1_k = \tilde{v}_i\tilde{v}_j^*$ satisfies $\lambda_{ij}^k = \det(\lambda_{ij}1_k) = \det(\tilde{v}_i\tilde{v}_j^*) = 1$. Hence, $\eta(\alpha) = \eta(\beta)$ has order (a divisor of) k in $\check{H}^2(X,\mathbb{Z})$.

(2) If A is a U.H.F.-algebra and p is a prime number with p^∞ occurring in the "prime decomposition" of A, then one can show that the range of η contains the p-primary component of $\check{H}^2(X,\mathbb{Z})$.

(3) We observe that in example 3.7, we have $\mathrm{Inn}\, A \subsetneq \mathrm{Inn}\, A^- = \mathrm{Ct}\, A$. Moreover, one can easily find continuous trace C^*-algebras with $\mathrm{Inn}\, B = \mathrm{Inn}\, B^- \subsetneq \mathrm{Ct}\, B$. Finally, (by theorem 4.1) example 3.9 satisfies $\mathrm{Inn}\, D \subsetneq \mathrm{Ct}\, D \subsetneq \mathrm{Inn}\, D^-$. It is easy to check that the operations $\mathrm{Inn}(\cdot)$, $\mathrm{Ct}(\cdot)$ and $\mathrm{Inn}(\cdot)^-$ all commute with direct sums. Thus, letting $E = A \oplus B \oplus D$ we have

$$\mathrm{Inn}\ E \subsetneq (\mathrm{Ct}\ E \cap \mathrm{Inn}\ E^{-}) \subsetneq \mathrm{Ct}\ E \quad \text{and}$$

$$\mathrm{Inn}\ E \subsetneq (\mathrm{Ct}\ E \cap \mathrm{Inn}\ E^{-}) \subsetneq \mathrm{Inn}\ E^{-}.$$

§5. Open Problems.

We list a number of problems that we think are worth further study.

1. If A is primitive (separable, unital) do we have $\mathrm{Ct}\ A = \mathrm{Inn}\ A$? If the answer is no, then perhaps $\mathfrak{X}(A)$ may be an interesting invariant for primitive algebras.
2. In all our examples, $\epsilon(\mathrm{Ct}\ A)$ is abelian. Is there a nonabelian example?
3. If A is primitive (or even simple) is the centre of $\epsilon(\mathrm{Inn}\ A^{-})$ always trivial? The case of an A.F.-algebra should be tractable (where the answer is probably yes).
4. By proposition 1.1, we have $\mathfrak{X}(A) \subseteq$ centre $\epsilon(\mathrm{Inn}\ A^{-})$. When do we have equality? Always? In analogy with the case of II_1 factors, does $C_A \neq H_A$ imply $\mathfrak{X}(A) =$ centre $\epsilon(\mathrm{Inn}\ A^{-})$?
5. Does there exist a primitive infinite-dimensional algebra A with $C_A = H_A$? We guess that the answer is no, with theorem 3.6 offered as evidence. If there were such an example, then by 3.5 we'd have $\mathfrak{X}(A) = \epsilon(\mathrm{Inn}\ A^{-}) =$ centre $\epsilon(\mathrm{Inn}\ A^{-})$ very large, indeed! This would also yield negative answers to problems 1 and 3.
6. Does there exist a primitive (or even simple) algebra, A, with $H_A \neq T_A$? If so, such an algebra would have an interesting structure.

7. The direct analogue of Connes' short exact sequence [6, théorème 4] for computing $\mathfrak{X}(A)$ is not true, in general. For, if it were true, then we would have $\mathfrak{X}\left[C_r^*(\mathbb{F}_2) \times_\alpha \mathbb{Z}_2\right] \cong \mathbb{Z}_2$ where α acts on $C_r^*(\mathbb{F}_2)$ by interchanging the generators. However, $C_r^*(\mathbb{F}_2) \times_\alpha \mathbb{Z}_2$ is simple, and so by 1.8 $\mathfrak{X}\left[C_r^*(\mathbb{F}_2) \times_\alpha \mathbb{Z}_2\right] = \{0\}$. Is there a more subtle analogue of this exact sequence which can be made to work in interesting cases?

8. If A is non-unital, then there are two possible definitions of $\mathrm{Ct}\ A$. We define

$\mathrm{Ct}_n\ A = \{\alpha \in \mathrm{Aut}\ A \mid \|\alpha(x_n)-x_n\| \to 0$ if $\{x_n\}$ is central in $A\}$

$\mathrm{Ct}_s\ A = \{\alpha \in \mathrm{Aut}\ A \mid (\alpha(x_n)-x_n) \to 0$ strictly if $\{x_n\}$ is central in $A\}$.

Ct_n is easy to handle via the simple expedient of adding a unit to A. If $\tilde{A}$ is the algebra with the unit adjoined, then there is an obvious identification $\mathrm{Ct}_n\ A = \mathrm{Ct}\ \tilde{A}$. In this way, many of the results for the unital case translate directly to the nonunital setting. However, $\mathrm{Ct}_s\ A$ is the more natural object of study in that two central sequences $\{x_n\}$, $\{y_n\}$ are (usually) deemed to be equivalent if and only if $(x_n-y_n) \to 0$ strictly. Evidently, $\mathrm{Ct}_n\ A \subseteq \mathrm{Ct}_s\ A$. When are they equal? Always? We make one observation; it is not hard to show that $\mathrm{Inn}\ M(A) \subseteq \mathrm{Ct}_n\ A$, where $M(A)$ denotes the multiplier algebra of A. Thus, they are certainly equal when A is simple or has continuous trace.

REFERENCES

1. C.A. Akemann and G.K. Pedersen, Central sequences and inner derivations of separable C^*-algebras, Amer. J. Math. 101(1979), 1047-1061.

2. W.B. Arveson, Notes on extensions of C^*-algebras, Duke Math. J. 44(1977), 329-355.

3. E. Christensen, Perturbations of type I von Neumann algebras, J. London Math. Soc. 9(1975), 395-405.

4. ______________, Perturbation of operator algebras, Inv. Math. 43(1977), 1-13.

5. A. Connes, Outer conjugacy classes of automorphisms of factors, Ann. Sci. École Norm. Sup. 8(1975), 383-420.

6 ______________, Sur la classification des facteurs de type II, C.R. Acad. Sc. Paris 281A(1975), 13-15.

7. G.A. Elliott, Ideal-preserving automorphisms of postliminary C^*-algebras, Proc. Amer. Math. Soc. 27(1971), 107-109.

8. ______________, Some C^*-algebras with outer derivations, III, Annals of Math. 106(1977), 121-143.

9. V.F.R. Jones, Notes on Connes' invariant $\mathfrak{X}(M)$, University of Pennsylvania.

10. R.V. Kadison, Irreducible operator algebras, Proc. Natl. Acad. Sci. U.S.A. 43(1957), 273-276.

11. ______________, E.C. Lance and J.R. Ringrose, Derivations and automorphisms of operator algebras, II, J. Functional Anal. 1(1967), 204-221.

12. A. Kishimoto, Outer automorphisms and reduced crossed products of simple C^*-algebras, Comm. Math. Phys. 81(1981), 429-435.

13. J. Phillips, Automorphisms of C^*-algebra bundles, J. Functional Anal. 51(1983), 259-267.

14. ______________, Outer automorphisms of separable C^*-algebras, J. Functional Anal. 70(1987), 111-116.

15. ______________, and I. Raeburn, Automorphisms of C^*-algebras and second Čech cohomology, Indiana Univ. Math. J. 29(1980), 799-822.

16. D. Voiculescu, A non-commutative Weyl-von Neumann theorem, Rev. Roum. Math. Pures et Appl. 21(1976), 97-113.