Renormalization procedures for C^*-algebras

by

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HBSc, University of Toronto, 2019

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ABSTRACT

Renormalization procedures for families of dynamical systems have been used to prove many interesting results. Examples of results include that the bifurcation rate for the attractors of an analytic one-parameter family of quadratic-like maps is universal for all such families, unique ergodicity for almost every interval exchange mapping, a unique ergodicity criterion for the vertical translation flow of a flat surface in terms of its “renormalization dynamics”, known as Masur’s criterion, and the classification of circle diffeomorphisms up to C^∞ conjugation. We introduce renormalization procedures for C^*-algebras and étale groupoids using the concepts of $C_0(X)$-algebras and Morita equivalence for the former, and groupoid bundles and groupoid equivalence, in the sense of Muhly, Renault and Williams, for the latter. We focus on proving analogs to Masur’s criterion in both cases using C^*-algebraic methods. Applying our criterion to our examples of renormalization procedures provides a unique trace criterion for unital AF algebras extending the one provided by Treviño in the setting of flat surfaces and the one provided by Veech in the setting of interval exchange mappings. Also, we recover the old fact that rotation of the circle by an irrational angle is uniquely ergodic, and the new fact that interesting groupoids associated to certain iterated function systems, recently introduced by Korfanty, have unique invariant probability measures whenever they are minimal. Lastly, we show how an étale groupoid renormalization procedure arises from an étale groupoid which factors down onto a groupoid associated to its renormalization dynamics, whenever it is a local homeomorphism.

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Section 1

Introduction

To renormalize a dynamical system $T : X \mapsto X$ is to restrict to a region of space $Y \subseteq X$ that defines a new system $T_Y : Y \mapsto Y$ by sending a point y in Y to the first (positive) iterate $T^n(y)$ returning to Y, and such that Y has a "re-scaling" by a change of coordinates to a space X' so that the resulting dynamical system $T' : X' \mapsto X'$ is "of the same type" as T. By "of the same type", we shall mean that there is a family $(T_s : X_s \mapsto X_s)_{s\in S}$ of dynamical systems varying over a space S (continuously, or measurably) to which T and T' belong. This is essentially the definition of a renormalization procedure appearing in Sullivan [50]. The concept of a first return $T_Y : Y \mapsto Y$ was originally considered by Poincare in the context of flows [39], so it as old as the field of dynamics. However, the interpretation of a collection of first returns like the above as a renormalization procedure, in analogy to renormalization procedures in physics, came about from the work of Feigenbaum [12] and independently Coullet and Tresser [5]. The basic insight from this interpretation is that if the dynamical systems at parameters s in a large enough subset $S_1 \subseteq S$ can be renormalized to parameters $\sigma(s)$, then it is possible to consider the iterations of the dynamical system $\sigma : S_1 \mapsto S$, called the renormalization dynamics, and the properties of the renormalization dynamics can be used to infer properties of the dynamical systems $(T_s)_{s\in S}$.

The works of Feigenbaum [12] and Coullet and Tresser [5] were concerned with studying how the structure of attractors for families $(f_t)_{t\in\mathbb{R}}$ of 2-to-1 mappings of the interval, varying analytically over the real parameter t, changed as the parameter varied. They observed (independently) that in any family under consideration, there are parameter values t_n at which the structure of the attractors changes dramatically in comparison to the neighbouring parameters, and that the difference of these parameters $|t_n - t_{n+1}|$ converges exponentially fast to zero at a rate δ which is universal (independent of the choice of family). This constant is now called the Feigenbaum constant. They conjectured that this universality was due to the fact that there should be a renormalization procedure on the space of certain (at the time not known) class of 2-to-1 mappings for which a limit point of a bifcurcation sequence $(t_n)_{n\in\mathbb{N}}$ corresponds to a fixed point of the renormalization dynamics σ and, at these fixed points,

σ is hyperbolic with hyperbolicity constant on the stable set of a fixed point equal to the Feigenbaum constant. The search for a proof of this universality result, and the space of dynamical systems in which it applied, was a driving force for much of the theory in complex dynamics used today, and culminated in the work of Lyubich [26] in which the right class of systems (analytically varying one-parameter families of 2-to-1 quasiconformal mappings), and the renormaization procedure (based off the Douady and Hubbard tuning procedure developed in [8]) was found. The hyperbolicity and more properties (see Lyubich pg. 321 [26]) of the renormalization procedure were used to prove the universality theorem (Lyubich pg. 322 [26]).

Herman [13] studied a renormalization procedure on the space of orientation preserving diffeomorphisms of the circle to show that almost every such dynamical system is C^∞ conjugate to a rotation. The procedure has the effect that succesive renormalizations become increasingly like rotations, and that if one of these renormalizations is C^∞ conjugate to a rotation, then the original system is as well. Once the diffeomorphism is close enough to being a rotation, KAM theory is applied to find a conjugation.

Rauzy [43] and Veech [53] constructed a renormalization procedure for the space of interval exchange mappings. The combinatorics of the renormalizations of such a mapping leads to a combinatorical description of the space of its invariant measures. Veech showed that if the combinatorics of the renormalizations is recurrent in some sense, then the interval exchange mapping is uniquely ergodic (Veech proposition 3.30 [53]). Veech later showed this unique ergodicity criterion, along with the existence of a certain ergodic measure for the renormalization dynamics (theorem 13.8 [52]), imply the conjecture of Keane, that almost every interval exchange mapping is uniquely ergodic (Veech theorem 13.10 [52]).

Masur [29] independently proved Keane's conjecture as well by first suspending an interval exchange mapping to a translation flow on a flat surface. Translation flows on flat surfaces have a continuous time renormalization procedure known as Teichmüller flow, which acts on Teichmüller space, the space of flat surfaces of a fixed genus that are identfied up to conformal isotopy. This procedure can be thought of as a suspension of the Rauzy-Veech renormalization procedure (see Yoccoz [54] chapter 9 for this description). Masur's proof of Keane's conjecture followed a similar strategy to Veech's by proving a unique ergrodicity condition for the translation flow of a flat surface in terms of a recurrence condition of the surface's orbit under the Teichmüller flow (Masur proposition 6.2 [29]), and used this, along with the fact that Teichmüller flow is ergodic with respect to a certain measure (Masur proposition 4.1 [29]), to prove almost every flat surface is uniquely ergodic. This result was then translated back down to the underlying interval exchange mappings to prove Keane's conjecture (Masur pg 194 [29]). Masur's work eventually led to what is known as Masur's criterion (see [28] or Theorem 3 in [27]), which states that if the orbit of a minimal flat surface under the Teichmüller flow has an accumulation point in Teichmüller space, then its translation flow is uniquely ergodic.

Thus, in all the above instances, renormalization procedures and their induced dynamics have been used to understand the dynamical systems in which they act upon. Given that renormalization procedures have had such wide applicability in the setting of dynamical systems, it is of interest to try and bring some of these ideas into the setting of C^*-algebras. To see how to do this, it is useful to think of a a first return T_Y of a dynamical system in a different way.

Let's assume $T : X \mapsto X$ is invertible, so that we can view it as an action of the group of integers $\mathbb{Z}$, and therefore make sense of its transformation groupoid $G_T = X \times \mathbb{Z}$. We will consider the definition of a groupoid in more detail in section 2, but we will just say for now that a groupoid is a set equipped with an involution and a partially defined multiplication. In the case of G_T, the involution of a point (x, n) in $X \times \mathbb{Z}$ is just $(x, n)^{-1} = (T^{-n}x, -n)$ and (y, m) in $X \times \mathbb{Z}$ is composable with (x, n) if and only if $T^n y = x$, in which case the product is $(x, n)(y, m) = (x, n+m)$. The mappings sending (x, n) to $s(x, n) = (x, n)^{-1}(x, n) = (T^{-n}x, 0)$ and to $r(x, n) = (x, n)(x, n)^{-1} = (x, 0)$ are called the source and range map, respectively. It is sometimes useful to think of a groupoid element (x, n) as an "arrow" with tail equal to $s(x, n) = (T^{-n}x, 0)$ and head equal to $r(x, n) = (x, 0)$. These arrows represent iterating points in X by the dynamical system T. Clearly then the groupoid G_T contains the information of the dynamical system T. Now, given a space $Y \subseteq X$, we can consider the restriction $G_T|_Y$ to all the arrows beginning and ending in Y. More concretely, $G_T|_Y = \{(y, m) \in X \times \mathbb{Z} : r(y, m) = (y, 0) \in Y \times 0 \text{ and } s(y, m) = (T^{-m}(y), 0) \in Y \times 0\}$. Notice that $G_T|_Y$ is the same as G_{T_Y} when T_Y is well defined and a bijection, so we can think of the restriction $G_T|_Y$ as the first return of G_T on Y. Now, consider $G_T^Y = \{(y, n) \in Y \times \mathbb{Z} : r(y, n) = (y, 0) \in Y \times 0\}$. G_T^Y contains all the information about which orbits pass through Y and at what time, so it is important in understanding the relationship between the original dynamical system T and the first return T_Y. Just as the dynamics of T is encoded in the groupoid product, the relationship between the dynamics of T_Y and the dynamics of T is encoded in a pair of groupoid actions $G_T|_Y \to G_T^Y \leftarrow G_T$, where $G_T|_Y$, G_T act on the left, right, respectively, by multiplication with the groupoid product. When every orbit of T meets the space Y, and some topological conditions are satisfied, then this pair of actions is known as a groupoid equivalence, first considered by Muhly, Renault and Williams [31]. Therefore, the information of a renormalization procedure for a family of dynamical systems $(T_s)_{s \in S}$, defined on a subset of $S_1 \subseteq S$ of parameters with Y_s, s in S_1, the first return domain and renormalization dynamics $\sigma : S_1 \mapsto S$, is encoded in a family of groupoid actions

$$G_{T_{\sigma(s)}} \simeq G_{T_s}|_{Y_s} \to G_{T_s}^{Y_s} \leftarrow G_{T_s},$$

for s in S_1. This picture is useful, because when the T_s are continuous dynamical systems and the actions satisfy the topological conditions in Muhly, Renault and Williams [31], these equivalences, as shown by the above authors, induce a family of Morita equivalences

$$C^*(G_{T_{\sigma(s)}}) \to C^*(G_{T_s}^{Y_s}) \leftarrow C^*(G_{T_s}),$$

where, $C^*(G_{T_s})$ is the groupoid C^*-algebra of G_{T_s} (see the introduction of section 2.2 for the definition of this C^*-algebra). Morita equivalence for C^*-algebras is a concept introduced by Rieffel [47], which we will present in definition 3.1.2, along with groupoid equivalences and their associated Morita equivalences. Since first returns T_{Y_s} are rarely continuous, and we want to introduce renormalization procedures for C^*-algebras, which are defined with continuous data it will be important to move away from these specific instances of groupoid equivalences, and consider arbitrary families of groupoid equivalences $(G_{\sigma(s)} \to E_s \leftarrow G_s)_{s\in S}$ for étale groupoids G_s, which are generalizations of discrete time dynamical systems. We will require that these equivalences have certain scaling properties, outlined in definition 4.2.7. Also, the family $(G_s)_{s\in S}$ should vary continuously. Such a family is known as a groupoid bundle, which we introduce in definition 2.2.3. Therefore, by the Muhly, Renault and Williams construction, we are led to consider Morita equivalence between the C^*-algebraic duals of groupoid bundles, which are section algebras of a canonically associated field (bundle) of C^*-algebras ($(C^*(G_s))_{s\in S}$ being the field).

The goal of this thesis is to establish a first approximation of what renormalization procedures are for section algebras of fields of C^*-algebras, and for bundles of topological groupoids, as well as the properties they should have so that the link between properties of the renormalization dynamics and the properties of the parameter algebras, or groupoids, in the corresponding bundle is strong. In section 2 we will introduce the non-commutative analog of a bundle of groupoids, and present some basic constructions and facts about them that will be useful to us in the later sections. We also introduce groupoid bundles more formally, and consider the analogous constructions and facts. Section 3 follows in a similar vein and is the presentation of Morita equivalence for C^*-algebras and groupoid equivalence, as well as the analogous notions of equivalence for section algebras and groupoid bundles. In section 4, we provide a first approximation to what a renormalization procedure should be in the setting of C^*-algebras and topological groupoids, which in the C^*-setting will be a Morita equivalence between a section algebra of a field of C^*-algebras $(A_x)_{x\in X}$ and its pullback by the renormalization dynamics $\sigma : X_1 \mapsto X$, and analogously for groupoids. These equivalences should be contractions in a certain sense, which we make precise for topological groupoids.

Throughout the whole thesis, we will work through our introduced concepts with some examples, namely the section algebra of the field of unital AF algebras (introduced in section 2.1.1), AF groupoid bundles (section 2.2.1), the rotation groupoid bundle (section 2.2.2), and groupoids associated to iterated function systems constructed recently by Korfanty in [20] (section 2.2.3).

We consider properties of renormalization procedures and their iterations that will imply analogs to Masur's criterion. Whenever a section algebra of a field of C^*-algebras (or groupoid bundle) has a renormalization procedure, our criterion will in general provide a strategy to proving unique tracial state (or unique groupoid invariant probability measure) for a C^*-algebra in the field (or a fibre groupoid in the bundle). In some cases, our theory will automatically imply unique invariant probability measure, as in the case of the renormalization procedure for the groupoid bundle associated to rotations of the circle (section 3.4.2) at an irrational paramater value (corollary 4.2.22). In the case of the renormalization procedure for groupoids associated to certain iterated function systems (section 3.4.3), the unique invariant probability measure criterion will reduce to an easy to verify property of the singularities of such a groupoid when it is not minimal, and will imply unique invariant probability measure when the groupoid is minimal and satisfies the strong open set condition (see definition 4.2.23) (corollary 4.2.34). The renormalization procedure for the field of all unital AF algebras (section 3.2.1) and our theory will imply a unique trace criterion that covers a couple pre-existing criterion considered by other authors. For instance the Masur's criterion for infinite genus flat surfaces constructed out of bi-infinite Bratteli diagrams in Treviño (theorem 1 [51]) can be seen as a corollary of our criterion. Veech's criterion of unique ergodicity for an interval exchange mapping in terms of its induced Bratteli diagram, mentioned in the above discussion, will also be a corollary.

The proofs for our main results (theorem 4.1.5 and theorem 4.2.13) rely on elementary C^*-algebra and groupoid techniques, as well as a limit argument using the operators $S_x : T^{<\infty}(A_x) \mapsto T^{<\infty}(A_{\sigma x})$, from the finite traces on A_x to the finite traces on $A_{\sigma x}$, induced from the Morita equivalences $A_{\sigma x} \to F_x \leftarrow A_x$ (see definition 3.1.10), for x in X_1, coming from a renormalization procedure on the section algebra of a field $(A_x)_{x \in X}$ of C^*-algebras. Specifically, we use properties of the iterates of these "renormalization operators" to infer the unique trace criterion. This argument can be seen in our proof of Masur's criterion (theorem 4.1.5 and also theorem 4.2.5) and, as far as the author knows, is new.

In section 5 we will show how renormalization procedures for étale groupoid bundles all come from a general construction involving a larger groupoid (called the renormalization groupoid) and a factor map down to the groupoid associated to the renormalization dynamics, whenever it is a local homeomorphism.

Section 2

$C_0(X)$-algebras

In the introduction we were motivated to look at the C*-algebraic dual to a bundle of groupoids, which is the algbera of sections $\mathcal{A}$ of a field (bundle) of C*-algebras $\{\mathcal{A}_x\}_{x \in X}$, where X is some space. These section algebras have an abstract definition (making no explicit mention of a field), due to Kasparov [18], and are known as $C_0(X)$-algebras. This section is devoted entirely to the introduction of these algebras, their basic properties and constructions useful to renormalization, and the examples which will appear throughout this thesis. In section 2.1 we prove basic facts about $C_0(X)$-algebras and their associated constructions. In section 2.1.1, we parameterize the set of all unital AF algebras into a field of C*-algebras and construct its section algebra. In 2.2 topological groupoid bundles are introduced formally and it is shown that the C*-algebra of a groupoid bundle over a locally compact Hausdorff space X is naturally a $C_0(X)$ algebra, and we introduce groupoid analogues to the constructions in section 2.1, as well as provide a few examples of topological groupoid bundles.

2.1 General Theory and Constructions

We define $C_0(X)$-algebras (definition 2.1.1) as in Kasparov [18]. For renormalization it will be sometimes be useful to replace $C_0(X)$ with an abstract commutative C*-algebra D not necessarily represented as the continuous functions on its Gelfand spectrum $\hat{D}$, but rather as a sub-algebra of bounded continuous functions on *some* space X. So, we make the relevant adjustments to the definitions and constructions and call them D-algebras to emphasize our view. Specifically, this point of view will be needed in section 2.1.1 when we construct the section algebra of the field of unital AF-algebras. We show how to interpret any D-algebra as the section algebra of an "upper semi-continuous" field of C*-algebras (definition 2.1.5 and corollary 2.1.6). We construct the pullback by a continuous map $f : Y \mapsto X$ of a D-algebra when D is represented as $D = C_0(X)$, $X = \hat{D}$ (definition 2.1.8), and prove a couple facts about this construction and convergence of states in fields of C*-algebras (proposition 2.1.9 and 2.1.11).

First, let us recall the multiplier algebra $\mathcal{M}(A)$ of a C^*-algebra A. If we regard A as a right A-module, with right multiplication by elements of A as the module structure, and as a Banach space, with the C*-norm as the Banach space norm, then $\mathcal{M}(A)$ is the set of all bounded A-module homomorphisms $T : A \mapsto A$ which are adjointable in the sense that there is another A-module homomorphism T^* such that, for all a and b in A, $(T(a))^*b = a^*T^*(b)$. $\mathcal{M}(A)$ is a C^*-algebra, and when A is unital, the map sending T in $\mathcal{M}(A)$ to $T(1)$ in A is an isomorphism of C*-algebras. When A is unital, we will identify $\mathcal{M}(A)$ with A. For more details about multiplier algebras, see Lance [24].

Definition 2.1.1. *Let D be a commutative C*-algebra. A **D-algebra**, denoted $(\mathcal{A}, \alpha)$, is a C^*-algebra $\mathcal{A}$ together with an injective *-homomorphism $\alpha : D \mapsto Z(\mathcal{M}(\mathcal{A}))$ mapping into the centre of the multiplier algebra of $\mathcal{A}$ such that $\alpha(D)\mathcal{A}$ linearly spans a dense sub-algebra in $\mathcal{A}$ (i.e. α is non-degenerate). The product of a function g in D and an element a $\mathcal{A}$ will be denoted $ga := \alpha(g)a$.*

Definition 2.1.2. *We say two D-algebras $(\mathcal{A}, \alpha)$, $(\mathcal{B}, \beta)$ are **isomorphic** if there is a *-isomorphism $\gamma : \mathcal{A} \mapsto \mathcal{B}$ such that for all a in $\mathcal{A}$ and h in D, $\gamma(ha) = h\gamma(a)$.*

A D-algebra $\mathcal{A}$ determines a bundle of C^*-algebras and a representation of $\mathcal{A}$ as sections of said bundle. This construction appears in Nilsen [34], though the idea of disintegrating a C*-algebra over its spectra goes back to Kaplansky [17].

Definition 2.1.3. *Let D be a commutative C^*-algebra along with an embedding $D \subseteq C_b(X)$ into the continuous bounded functions on a Hausdorff space X (not necessarily locally compact). Let $(\mathcal{A}, \alpha)$ be a D-algebra. For a point x in X, define $\mathcal{A}^x$ to be the closed linear span of the products ga of all elements a in $\mathcal{A}$ and functions g in D vanishing at x. Then $\mathcal{A}^x$ is a closed two-sided ideal in $\mathcal{A}$. Define $\mathcal{A}_x := \mathcal{A}/\mathcal{A}^x$, called the **fibre of $\mathcal{A}$ at** x. For a in $\mathcal{A}$, denote $a(x)$ to be the image of a under the quotient map, and we will prefer to call this **the evaluation of** a **at** x.*

Hence, we can represent an element a in $\mathcal{A}$ as a section of the bundle $(\mathcal{A}_x)_{x \in X}$ by evaluating along X. Two elements may determine the same section, and these sections may not be continuous (see remark 2.1.7). We make the following auxiliary definitions:

Definition 2.1.4. *Let $(\mathcal{A}, \alpha)$ be a D-algebra with $D \subseteq C_b(X)$. We say $\mathcal{A}$ is **injective over** X if for all distinct elements a, b in $\mathcal{A}$ there is a point x_0 in X for which $a(x_0) \neq b(x_0)$. We say $\mathcal{A}$ is **continuous over** X if for all elements a in $\mathcal{A}$, the function $\|a\|_X : X \mapsto \mathbb{R}_+$ defined by $\|a\|_X(x) = \|a(x)\|$, x in X, is continuous.*

If D is represented as $D = C_0(\hat{D})$, then the section representation is injective, and the sections are at least upper semi-continuous. This is a result of Nilsen [34]. We provide a proof for completeness.

Proposition 2.1.5. *If $(\mathcal{A}, \alpha)$ is a D-algebra with $D = C_0(\hat{D}) \subseteq C_b(\hat{D})$, then $\mathcal{A}$ is injective over $\hat{D}$, and for every a in $\mathcal{A}$, $\|a\|_{\hat{D}}$ is upper semi-continuous.*

Proof. We prove upper semi-continuity first. Let a be in $\mathcal{A}$, y be in $\hat{D}$ and δ be such that $\|a(y)\| < \delta$. Let $(g_\lambda)_{\lambda \in \Lambda}$ be a compactly supported approximate unit of D^y, indexed over the directed set Λ, such that $g_\lambda = 0$ in an open neighbourhood U_λ of y. Since $\mathcal{A}^y = \overline{\text{span}(D^y \mathcal{A})}$, it follows that $\|a(y)\| = \lim_\lambda \|(1 - g_\lambda)a\|$. Therefore, there is λ_0 in Λ such that $\|(1 - g_{\lambda_0})a\| < \delta$. Since $g_{\lambda_0} = 0$ on U_{λ_0}, it follows from the definition of the quotient norm that $\|a(\tilde{y})\| \leq \|(1 - g_{\lambda_0})a\| < \delta$ for $\tilde{y}$ in U_{λ_0}. Therefore, $\|a\|_Y$ is upper semi-continuous.

For injectivity, suppose $a(y) = 0$ for all points y in $\hat{D}$. By non-degeneracy of α and the C^* norm condition, it suffices to prove $ga^*a = 0$ for all positive compactly supported functions g in $C_0(\hat{D})$. Fix $\epsilon > 0$. Let $\text{supp}(g) = K$. From the above argument, for every y in K there is an open neighbourhood U_y of y and a function $0 \leq \phi_y \leq 1$ compactly supported such that $\phi = 1$ on U_y and $\phi_y g a^* a \leq \epsilon$. Let $\{U_{y_i}\}_{i=1}^n$ be a finite subcover of K. Let ψ_i be positive functions supported on U_{y_i} such that $\sum_{i=1}^n \psi_i = 1$ on K and $\sum_{i=1}^n \psi_i \leq 1$ on $\hat{D}$. Since $C^*(C_0(\hat{D}), a^*a)$ is commutative, it follows that $\psi_i g a^* a = \psi_i \phi_{y_i} g a^* a \leq \psi_i \epsilon$. Hence,

$$ga^*a = \sum_{i=1}^n \psi_i g a^* a \leq \sum_{i=1}^n \psi_i \epsilon \leq \epsilon.$$

Since ϵ was arbitrary, $ga^*a = 0$. □

Now, we show the sections in the section representation of a D-algebra for an arbitrary representation of D, $D \subseteq C_b(X)$, are upper semi-continuous and provide an injectivity result.

First, recall from Pedersen corollary 4.3.14 [36] that a commutative C*-algebra A is isometrically $*$-isomorphic to $C_0(\hat{A})$, where $\hat{A}$ is the locally compact Hausdorff space of $*$-homomorphisms from A to $\mathbb{C}$ topologized by the weak* topology. This *-isomorphism $\Gamma_A : A \mapsto C_0(\hat{A})$ sends a in A to the function $\hat{a} : \hat{A} \mapsto \mathbb{C}$, defined for χ in $\hat{A}$ as $\hat{a}(\chi) = \chi(a)$. For a C^*-algebra $A = C_b(X)$, where X is a Hausdorff space, $\hat{A}$ is typically denoted βX, and is known as the **Stone-Čech compactification of** X. It is easy to see the map $i_X : X \mapsto \beta X$ sending a point x in X to the character $\delta_x : C_b(X) \mapsto \mathbb{C}$, defined for f in $C_b(X)$ as $\delta_x(f) = f(x)$, is continuous. Moreover, the image of i_X is dense, since $\hat{f}(\delta_x) = f(x) = 0$ for all x in X implies $f = 0$. If X is a Tychonoff space, then it can be shown (Pedersen proposition 4.3.18) that i_X is injective, though this need not be the case in general.

Corollary 2.1.6. *If $D \subseteq C_b(X)$ and $(\mathcal{A}, \alpha)$ is a D-algebra, then $\mathcal{A}$ is upper semi-continuous over X. If $\mathcal{A}$ is continuous over $\hat{D}$, then $\mathcal{A}$ is injective over X.*

Proof. By non-degeneracy of α, $\mathcal{A}$ extends to a $\tilde{D}$-algebra $(\mathcal{A}, \tilde{\alpha})$, where $\tilde{D}$ and $\tilde{\alpha}$ denote the respective unitizations of D, α. By proposition 2.1.5, for every a in $\mathcal{A}$, $\|a\|_{\hat{D}\cup\{\infty\}}$ is upper semi-continuous. Let $i : \tilde{D} \mapsto C_b(X)$ be the inclusion map, and let $\hat{D}$, βX be as in the above discussion. Then, $\eta = \Gamma_{C_b(X)} \circ i \circ \Gamma_{\tilde{D}}^{-1}$ is an injective $*$-homomorphism from $C(\hat{D}\cup\{\infty\})$ to $C(\beta X)$. By the Gelfand correspondence, there is a continuous surjection $\gamma : \beta X \mapsto \hat{D}\cup\{\infty\}$ such that $\gamma^* = \eta$. Let g be in $C(\hat{D}\cup\{\infty\})$ and write $g = \Gamma_{\tilde{D}}(f)$. For x in X, we have $g(\gamma(i_X(x))) = \eta(\Gamma_{\tilde{D}}(f))(i_X(x)) = \Gamma_{C_b(X)}(i(f))(i_X(x)) = \delta_x(f) = f(x)$. Therefore, for a $g = \Gamma_{\tilde{D}}(f)$ in $C(\hat{D} \cup \{\infty\})$ and x in X, $g(\gamma \circ i_X(x)) = 0$ if and only if $f(x) = 0$. It follows that, for every a in $\mathcal{A}$, $\|a\|_X = \|a\|_{\hat{D}\cup\infty} \circ \gamma \circ i_X$. $\|a\|_{\hat{D}\cup\{\infty\}}$ is upper semi-continuous and $\gamma \circ i_X$ is continuous, so $\|a\|_X$ is upper semi-continuous.

Now, if $\mathcal{A}$ is continuous over $\hat{D}$, then $\|a\|_{\hat{D}\cup\{\infty\}}$ is also continuous, because if $\{g_\lambda\}_{\lambda\in\Lambda}$ is an approximate unit for $C_0(Y)$, by non-degeneracy of α, $\lim_\lambda(1 - g_\lambda)a = 0$, so $\|a\|_{\hat{D}}$ is in $C_0(\hat{D})$, and therefore extends continuously to $\hat{D} \cup \{\infty\}$. Therefore, $\|a\|_X = \|a\|_{\hat{D}\cup\{\infty\}} \circ \gamma \circ i_X$ is continuous. For injectivity, suppose $\|a\|_X = 0$. $i_X(X)$ is dense in βX and γ is surjective, so $\gamma(i_X(X))$ is dense in $\hat{D} \cup \{\infty\}$. Since $\|a\|_{\hat{D}}|_{\gamma(i_X(X))} = 0$, by continuity of $\|a\|_{\hat{D}}$ and density of $\gamma(i_X(X))$, it follows that $\|a\|_{\hat{D}} = 0$. So, $a = 0$ by injectivity of $\mathcal{A}$ over $\hat{D}$. ☐

Remark 2.1.7. *The injectivity statement in the above corollary can fail if a D-algebra $\mathcal{A}$ is not continuous over the Gelfand spectrum of D. This is because, unlike continuous functions, upper semi-continuous functions are not necessarily determined by their restriction to dense subspaces. We provide an example: Let $\mathcal{A}$, be the C*-algebra consisting of bounded functions on* $[0, 1]$ *which are continuous on* $[0, 1/2) \cup (1/2, 1]$. *Then, $\mathcal{A}$ has a natural $D =$* $C[0, 1]$ *structure gotten by point-wise multiplication of functions.* $D \subseteq C_b([0, 1/2) \cup (1/2, 1])$, *but $\mathcal{A}$ is not injective over* $[0, 1/2) \cup (1/2, 1]$ *(the function δ which is zero everywhere on* $[0, 1]$ *except equal to* 1 *at* $1/2$ *is in $\mathcal{A}$). Notice also that $\mathcal{A}$ is continuous over* $[0, 1/2) \cup (1/2, 1]$, *so continuity over some space X doesn't necessarily imply continuity over the Gelfand spectrum.*

When renormalizing a bundle of C*-algebras, a transformation of the parameter occurs. At the level of the section algebra, this transformation is a pullback of the D-algebra structure by a continuous map on the spectrum $\hat{D}$ when D is represented as $D = C_0(\hat{D})$ (when D is not represented as such, this is not always case; see remark 2.1.18). Such a construction was first considered in Raeburn and Williams [42]. The version presented here appears in Raeburn and Williams pg. 155 [41]. For this definition, we will need the concept of a C*-tensor product. A good reference for C*-tensor products is Murphy chapter 6.3 [33].

Definition 2.1.8. *Let $(\mathcal{A}, \alpha)$ be a $C_0(Y)$-algebra, and $f : X \mapsto Y$ a continuous (not necessarily proper) map between locally compact Hausdorff spaces X, Y. We define the* ***pullback of*** $(\mathcal{A}, \alpha)$ ***by*** *f, denoted $(f^*\mathcal{A}, f^*\alpha)$, as follows: consider the C*-tensor* *product $C_0(X) \otimes \mathcal{A}$, and the closed ideal I_f which is densely spanned by the elements*

$$\{h \otimes ga - h(g \circ f) \otimes a : h \in C_0(X),\ g \in C_0(Y)\ a \in \mathcal{A}\}.$$

*Define $f^*\mathcal{A}$ to be the quotient $C_0(X) \otimes \mathcal{A}/I_f$. For h in $C_0(X)$ and a in $\mathcal{A}$, denote $h \otimes_f a$ to be the image of the basic tensor $h \otimes a$ under the quotient map π_f. Define a non-degenerate *-homomorphism $f^*\alpha : C_0(X) \mapsto Z(\mathcal{M}(f^*\mathcal{A}))$ on basic tensors by*

$$f^*\alpha(k)(h \otimes_f a) = kh \otimes_f a,\ k,\ h \in C_0(X),\ a \in \mathcal{A}.$$

We characterize when $(f^*\mathcal{A}, f^*\alpha)$ is a $C_0(X)$-algebra. The first statement of this proposition is essentially Raeburn and Williams proposition 1.3 [42].

Proposition 2.1.9. *Let $(\mathcal{A}, \alpha)$ be a $C_0(Y)$-algebra and $f : X \mapsto Y$ a continuous map between locally compact Hausdorff spaces X, Y.*

*(i) For every point x in X, $(f^*\mathcal{A})_x \simeq \mathcal{A}_{f(x)}$ via the *-isomorphism sending $(h \otimes_f a)(x)$ to $h(x)a(f(x))$ for all h in $C_0(X)$ and a in $\mathcal{A}$.*

(ii) $f^*\alpha$ is injective if and only if for every open set $U \subseteq X$, there is a point u in U such that $\mathcal{A}_{f(u)} \neq 0$.

Proof. First, we prove (i): let $ev_x : C_0(X) \mapsto \mathbb{C}$ and $ev_{f(x)} : \mathcal{A} \mapsto \mathcal{A}_{f(x)}$ be the evaluation maps. Using the isomorphism $C_0(X) \otimes \mathcal{A} \simeq C_0(X, \mathcal{A})$, it is easy to see that $\ker(id_{C_0(X)} \otimes ev_{f(x)}) = C_0(X) \otimes \mathcal{A}^{f(x)}$ and $\ker(ev_x \otimes id_{\mathcal{A}_{f(x)}}) = C_0(X)^x \otimes \mathcal{A}_{f(x)}$. Therefore, $\mu_x := (ev_x \otimes id_{\mathcal{A}}) \circ (id_{C_0(X)} \otimes ev_{f(x)})$ has kernel $C_0(X) \otimes \mathcal{A}^{f(x)} + C_0(X)^x \otimes \mathcal{A}$. More explicitly, μ_x is defined on basic tensors $h \otimes a$, h in $C_0(X)$, a in $\mathcal{A}$, as $\mu_x(h \otimes a) = h(x)a(f(x))$. Since μ_x annihilates the generators of I_f, it passes down to a *-homorphism $\pi_x : f^*\mathcal{A} \mapsto \mathcal{A}_{f(x)}$ with kernel $C_0(X) \otimes_f \mathcal{A}^{f(x)} + C_0(X)^x \otimes_f \mathcal{A}$. Since $h \otimes_f ga = h(g \circ f) \otimes_f a$ (in particular for g in $C_0(Y)$ vanishing at $f(x)$) it follows that $C_0(X) \otimes_f \mathcal{A}^{f(x)} \subseteq C_0(X)^x \otimes_f \mathcal{A}$. Therefore, $\ker(\pi_x) = C_0(X)^x \otimes_f \mathcal{A} = f^*\mathcal{A}^x$, proving (i).

(ii): For h in $C_0(X)$, let U be an open set in X for which $h(v) \neq 0$ for all points v in U. By assumption we can choose a point u in U and element a in $\mathcal{A}$ such that $a(f(u)) \neq 0$. Then, $(h(h \otimes_\varphi a))(u) = \pi_u(h(h \otimes_\varphi a)) = (h(u))^2 a(f(u)) \neq 0$. Therefore, $\varphi_*\alpha(h) \neq 0$. For the converse, let U be open in X, and h a function in $C_0(X)$ supported on U. Then, by injectivity of $f^*\alpha$, there is b in $\varphi_*\mathcal{A}$ such that $hb \neq 0$. By injectivity of $f^*\mathcal{A}$ over X (proposition 2.1.5), it follows that there is a point u in U such that $0 \neq hb(u) \in \mathcal{A}_{f(u)}$. □

Corollary 2.1.10. *Let X and Y be locally compact spaces. If $(\mathcal{A}, \alpha)$ is a $C_0(Y)$-algebra such that $\mathcal{A}_y \neq 0$ for all y in Y, then for any continuous map $f : X \mapsto Y$, the pullback $f^*(\mathcal{A}, \alpha)$ is a $C_0(X)$-algebra such that $(f^*\mathcal{A})_x \neq 0$ for all x in X.*

Let $(\mathcal{A}, \alpha)$ be a D-algebra with $D \subseteq C_b(X)$. Denote $S^1(\mathcal{A})$ to be the state space of $\mathcal{A}$ (positive, norm one linear functionals on $\mathcal{A}$), and for x in X, denote $S^1_x(\mathcal{A})$ be states that vanish on the ideal $\mathcal{A}^x$, i.e, states that are pullbacks of states on $\mathcal{A}_x$ by the evaluation map. Similarly, denote $T^1_x(\mathcal{A})$ to be set of states τ in $S^1_x(\mathcal{A})$ that satisfy the trace condition; i.e., $\tau(ab) = \tau(ba)$ for all a, b in $\mathcal{A}$. Starting from a trace τ in $T^1_x(\mathcal{A})$, a renormalization procedure will allow us to produce, via iteration of an induced operator on the trace space, a sequence of traces τ_n in $T^1_{x_n}(\mathcal{A})$, where the sequence $\{x_n\}_{n\in\mathbb{N}}$ is obtained by iterating the renormalization dynamics on X. Knowing the boundedness of $\{\tau_n\}_{n\in\mathbb{N}}$, and properties of its accumulation points will be useful when we demonstrate in corollary 4.1.6 that properties of $\mathcal{A}_x$ can be inferred from properties of the renormalization procedure and limiting properties of the algebras $\mathcal{A}_{x_n}$. In particular, we will need the following facts:

Proposition 2.1.11. *Let $(\mathcal{A}, \alpha)$ be a D-algebra with $D \subseteq C_b(X)$. Suppose $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X converging to x_∞, and $\{s_n\}_{n\in\mathbb{N}}$ is a sequence of states, with s_n in $S^1_{x_n}(\mathcal{A})$ for all n in $\mathbb{N}$, that has a weak* limit s in $S^1(\mathcal{A})$. Then, s is a state in $S^1_{x_\infty}(\mathcal{A})$. If $\mathcal{A}$ is unital, or if $\mathcal{A}_x$ is unital for all x in X and $\mathcal{A}$ is continuous over X, then any sequence of states $\{s_n\}_{n\in\mathbb{N}}$ with s_n in $S_{x_n}(\mathcal{A})$ has an accumulation point.*

Proof. Suppose a is in $\mathcal{A}^{x_\infty}$, i.e. $a(x_\infty) = 0$. By upper semi-continuity over X (proposition 2.1.6), for every $\epsilon > 0$ there is an open set U_{x_∞} containing x_∞ such that for all points x in U_{x_∞}, $\|a(x)\| < \epsilon$. Since $\lim_{n\to\infty} x_n = x_\infty$, there is some N such that for all $n > N$, x_n is in U_x. Therefore, for $n > N$, $|s_n(a)| = |s'_n(a(x_n))| < \epsilon$ where s'_n denotes the descent of s_n to the state on $\mathcal{A}_{x_n}$. By weak* convergence of s_n, we have $|s(a)| \leq \epsilon$. As ϵ was arbitrary, $s(a) = 0$.

If $\mathcal{A}$ is unital, then $S^1(\mathcal{A})$ is weak* compact, so any sequence of states s_n has an accumulation point. Suppose $\mathcal{A}$ is continuous over X with unital fibres. The space $S^{\leq 1}(\mathcal{A})$ of positive linear functionals with norm bounded by 1 is weak* compact, so let s be an accumulation point in $S^{\leq 1}(\mathcal{A})$ of the sequence s_n of states in $S^1_{x_n}(\mathcal{A})$. It suffices to show $\|s\| = 1$. By a similar argument to the above, s is in $S^{\leq 1}_{x_\infty}(\mathcal{A})$. Let a be a positive element in $\mathcal{A}$ such that $a(x_\infty) = 1$. Since s is in $S_{x_\infty}(\mathcal{A})$, it follows that $\|s\| = s(a) = \lim_{n\to\infty} s'_n(a(x_n))$, where s'_n denotes the descent of s_n to a normed one state on $\mathcal{A}_{x_n}$. By continuity of $\mathcal{A}$ over X, for every $\epsilon > 0$ there is an open neighbourhood U_{x_∞} of x_∞ in X such that $1 - \epsilon \leq a(x) \leq 1 + \epsilon$ for all x in U_{x_∞}. Since x_n converges to x_∞, there is an N in $\mathbb{N}$ such that x_n is in U_{x_∞} for all $n > N$. Therefore, $1 - \epsilon \leq s'_n(a(x_n)) \leq 1 + \epsilon$ for all $n > N$, so $1 - \epsilon \leq \|s\| \leq 1 + \epsilon$. As ϵ was arbitrary, it follows that $\|s\| = 1$. $\square$

Remark 2.1.12. *The last claim in the above proposition is not true if continuity of $\mathcal{A}$ over X is replaced with upper semi-continuity. For instance, take $\mathcal{A}$ to be the functions f on $[0, 1]$ that are continuous on $(0, 1]$ and $\lim_{t\to 0} f(t) = 0$, with the $C[0, 1]$-algebra* *structure given by point-wise multiplication of functions. Let t_n be a sequence in $[0,1]$ converging to 0 and let s_n be the state defined by evaluation of functions in $\mathcal{A}$ at t_n. Then, the only weak* limit point of s_n is 0.*

2.1.1 AF Algebras

Recall that an approximately finite (AF) algebra is a C^*-algebra which is the inductive limit of finite dimensional C^*-algebras; see Rordam, Larsen and Lausten chapter 6 [48] for a good reference on inductive limits and their universal properties. In this section, we construct the section algebra of a field of unital AF algebras over a rather large space X. X is large enough that every isomorphism class of AF algebras is represented (many times) as a fibre in this field (corollary 2.1.16). In proposition 2.1.17, we prove that this section algebra is a continuous field over X. X is also equipped with a natural continuous map $\sigma : X \mapsto X$, and we briefly compute the pullback of $\mathcal{A}$ by the maps σ^k, for k in $\mathbb{N}$, on X.

First, let us introduce the space X for which we will paramaterize our field by. This space appears in Treviño [51] and its bi-infinite analogue is used as a moduli space for flat surfaces built out of bi-infinite Bratteli diagrams. Let $Mat_{k,n}(\mathbb{Z}_{\geq 0})$ denote the set of $k \times n$ matrices with non-negative integer entries, and let $\mathcal{M}$ be the subset of matrices in $\bigcup_{k,n\in\mathbb{N}} Mat_{k,n}(\mathbb{Z}_{\geq 0})$ for which every row and column is non-zero. For k, n in $\mathbb{N}$ and A in $\mathcal{M} \cap Mat_{k,n}$, denote $s(A) = k, r(A) = n$ to be the source of A and the range of A, respectively. Therefore, we think of $\mathcal{M}$ as the directed edges of a graph with vertices the natural numbers $\mathbb{N}$. For a natural number n, define the space of finite paths

$$P^n = \{(A_1, A_2, ...A_n) \in \mathcal{M}^n : r(A_i) = s(A_{i+1})\ \forall i\}.$$

We will make the convention that P^0 refers to the vertices of the above graph, i.e., $P^0 = \mathbb{N}$. Similarly, define the space of infinite paths

$$X = \{(A_1, A_2, ...) \in \mathcal{M}^{\mathbb{N}} : r(A_i) = s(A_{i+1})\ \forall i \in \mathbb{N}\}.$$

X is a closed subspace of $\mathcal{M}^{\mathbb{N}}$ relative to the product topology, with discrete topology on $\mathcal{M}$. This turns X into a totally disconnected second countable paracompact Hausdorff space. To a finite path $z = (A_1, A_2..., A_n)$ in P^n, we associate a directed graph $B(z) = (E(z), V(z), r, s)$ as follows: let $V_0(z) = \{1, ..., s(A_1)\}$, and for $1 \leq i \leq n$, $V_i(z) = \{1, ..., r(A_i)\}$. Let $V(z) = \bigsqcup_{i=0}^n V_i(z)$ be the vertex set. Let

$$E_i(z) = \{e^j_{v,w} := (j, v, w) : v \in V_{i-1}, w \in V_i, (A_i)_{v,w} \neq 0,\ 1 \leq j \leq (A_i)_{v,w}\}$$

and $E(z) = \bigsqcup_{i=1}^n E_i$ be the edge set with $s(e^j_{v,w}) = v$ and $r(e^j_{v,w}) = w$. Let $P^n(z)$ denote the set of paths in $B(z)$ of length n. Note that these paths must start at vertices in $V_0(z)$ and end at vertices in $V_n(z)$. If x is an infinite path, we do the same process indefinitely to produce a graph $B(x) = (\bigsqcup_{n=1}^{\infty} E_i(x), \bigsqcup_{n=0}^{\infty} V_i(x), r, s)$ which is a Bratteli diagram; see Davidson Ch. 3 [6] for a good reference on Bratteli diagrams and their context as combinatorical models for AF algebras. By construction, the adjacency matrix of this Bratteli diagram from the $n-1$ level of vertices $(V_{n-1}(x))$ to the n^{th} level $(V_n(x))$ is equal to the matrix x_n. Therefore, we can think of X as parameterizing a space of Bratteli Diagrams. For finite paths z in P^n and w in P^m such that $r(z) = s(w)$, Let $z * w$ denote the concatenation of z and w, which is a finite path in P^{n+m}. Since $r(z) = s(w)$, we have $V_n(z) = V_0(w)$, so for a length n path p in $P^n(z)$ and a length m path q in $P^m(w)$ such that $r(p) = s(q)$, the concatenation $p * q$ into a length $n+m$ path in $P^{n+m}(z*w)$ also makes sense.

Since X parameterizes a space of Bratteli diagrams $\{B(x)\}_{x \in X}$, it must also parameterize a space of AF algebras $\{A_x\}_{x\in X}$. This is because, given a Bratteli diagram $B(x)$, one can construct an AF algbera A_x that is unique up to isomorphism (Davidson Ch.3 [6]). We present this construction in a way that allows us to simultaneously construct the section algebra $\mathcal{A}$ of the field $\{A_x\}_{x\in X}$.

We begin the construction of $\mathcal{A}$. For a finite path z in P^n, n in $\mathbb{N} \cup \{0\}$, and vertex v in $V_n(z)$, let $H(z), H^v(z)$ be the finite dimensional Hilbert spaces with orthonormal basis $\{\delta_p : p \in P^n(z)\}$, $\{\delta_p : p \in P^n(z), r(p) = v\}$, respectively. Let $P_z^v : H(z) \mapsto H(z)$ denote the projection of $H(z)$ onto $H^v(z)$. Define

$$A(z) = \{a \in B(H(z)) : [a, P_z^v] = 0, \forall v \in V_n(z)\},$$

where $[a,b] = ab - ba$ is the commutator. More concretely, $A(z)$ is the direct sum of the full matrix algebras $B(H^v(z))$, ranging through all vertices v in $V_n(z)$. Let e be a matrix in $\mathcal{M}$ such that $r(z) = s(e)$. For edges f in $E(e)$, let $R_f(z) : H(z) \mapsto H(z*e)$ be the partial isometry defined on basis elements δ_p in $H(z)$ as

$$R_f(z)(\delta_p) = \begin{cases} 0 & \text{if } r(p) \neq s(f) \\ \delta_{p*f} & \text{if } r(p) = s(f). \end{cases}$$

Let $\varphi_{z,z*e} : A(z) \mapsto A(z*e)$ be the map defined for a in $A(z)$ as $\varphi_{z,z*e}(a) = \sum_{f\in E(e)} R_f(z) a (R_f(z))^*$. We will clarify the relation between $\varphi_{z,z*e}$ and the edge set $E_{n+1}(z*e) = E(e)$ in $B(z*e)$ momentarily (proposition 2.1.13). First, we need a combinatorical description of *-homomorphisms between finite dimensional C^*-algebras.

Suppose $\varphi : A \mapsto B$ is a *-homomorphism between finite dimensional C^*-algebras $A = \oplus_{i=1}^{k} M_{l_i}(\mathbb{C})$ and $B = \oplus_{j=1}^{n} M_{m_j}(\mathbb{C})$. For $i \leq k$ and $j \leq n$ let $\varphi_{i,j} : M_{l_i}(\mathbb{C}) \mapsto M_{m_j}(\mathbb{C})$ the induced $*$-homomorphism between the i^{th} and j^{th} factors of A, B, respectively. $\varphi_{i,j}$ is just the simultaneous restriction of the domain to the i^{th} factor, and compression of the range of φ by the projection in B onto the j^{th} factor. By Davidson theorem 1.10.7 [6], and thinking of $\varphi_{i,j}$ as a representation on the Hilbert space $\mathbb{C}^{m_j}$, $\varphi_{i,j}$ is unitarily equivalent to the direct sum of $a_{i,j}$ diagonal copies of the identity representation for $M_{l_i}(\mathbb{C}) = B(\mathbb{C}^{l_i})$. The matrix a in $Mat_{k,n}(\mathbb{N} \cup \{0\})$ with entries $a_{i,j}$ for $i \leq k$ and $j \leq n$ is called the multiplicity matrix of φ. This matrix determines φ uniquely, up to conjugation by unitaries in B (see Davidson corollary 3.2.2 [6]).

Proposition 2.1.13. *$\varphi_{z,z*e} : A(z) \mapsto A(z*e)$ is a unital and injective $*$-homomorphism with multiplicity matrix e.*

Proof. Each path p in $P^{n+1}(z * e)$ has a unique decomposition $p = q * f$, where q is a path in $P^n(z)$ and f is an edge in $E(e)$, so $\varphi_{z,z*e}(1) = \sum_{f \in E(e)} R_f(z)(R_f(z))^* = 1$, and by uniqueness of the decomposition, $(R_f(z))^* R_{f'}(z) = 0$ for distinct edges f, f' in $E(e)$. Therefore, for elements a, b in $A(z)$, we have

$$\varphi_{z,z*e}(a)\varphi_{z,z*e}(b) = \sum_{(f,f') \in E(e)^2} R_f(z) a (R_f(z))^* R_{f'}(z) b (R_{f'}(z))^* =$$

$$= \sum_{f \in E(e)} R_f(z) ab (R_f(z))^* = \varphi_{z,z*e}(ab).$$

So, $\varphi_{z,z*e}$ is multiplicative. It is immediate from the formula for $\varphi_{z,z*e}$ that it is linear and $*$-preserving. Hence, $\varphi_{z,z*e}$ is a $*$-homomorphism. The support projection of $R_f(z)$ is $P_z^{s(f)}$, so for vertices v and w in $V_n(z)$ and $V_{n+1}(z * e)$ respectively, the $*$-homomorphism $P^w_{z*e}\varphi_{z,z*e}P^w_{z*e}$ sends an element a in the full matrix algebra $P^v_z A(z) P^v_z = B(H^v(z))$ into the full matrix algebra $P^w_{z*e} A(z) P^w_{z*e} = B(H^w(z * e))$ as the orthogonal sum $\sum_{f:s(f)=v,r(f)=w} R_f(z) a (R_f(z))^*$. Therefore, $P^w_{z*e}\varphi_{z,z*e}P^w_{z*e} : B(H^v(z)) \mapsto B(H^w(z * e))$ is unitarily equivalent (as a representation) to the direct sum of $|E(e)_{v,w}| = e_{v,w}$ diagonal copies of $id_{B(H^v(z))}$. Hence, the connecting matrix of $\varphi_{z,z*e}$ is e. Since $\sum_w e_{v,w} > 0$ for each v in $V_n(z)$, $\varphi_{z,z*e}$ is injective. □

For $n \geq 1$, let $A_n = \prod_{w \in P^n} A(w)$ (note that P^n is an infinite set, so the distinction between the direct product Π and direct sum $\bigoplus$ over P^n is important). For a in A_n, let $a(z)$ denote the factor of a in $A(z)$. Define a connecting homomorphism $\varphi_n : A_n \mapsto A_{n+1}$ for a in A_n factor-wise by $\varphi_n(a)(z * e) = \varphi_{z,z*e}(a(z))$, for all z in P^n and e in $\mathcal{M}$ such that $r(z) = s(e)$. We now find an inductive limit $(\mathcal{A}, \mu_n)$ for this inductive system $\{\varphi_n\}$.

For x in X and $n \geq 1$, we will denote the restriction of x to the first n edges to be $x_{[1,n]}$. For $n = 0$, we will denote $s(x)$ as $x_{[1,0]}$. For $n \geq 0$, let $\varphi_{n,x} = \varphi_{x_{[1,n]},x_{[1,n+1]}}$. Choose an inductive limit $\{A_x, \mu_{n,x}\}_{n \geq 0}$ of the system $\{\varphi_{n,x}\}_{n \geq 0}$. For x' in X, let $\pi_{x'} : \Pi_{x \in X} A_x \mapsto A_{x'}$ be the projection onto factor indexed by x'. Define $\mu_n : A_n \mapsto \prod_{x \in X} A_x$ factor-wise by $\pi_x(\mu_n(a)) = \mu_{n,x}(a(x_{[1,n]}))$, for all x in X and a in A_n. For a in A_n and

x in X, we have

$$\pi_x(\mu_{n+1}(\varphi_n(a))) = \mu_{n+1,x}(\varphi_n(a)(x_{[1,n+1]})) = \mu_{n+1,x}(\pi_x(\varphi_n(a))) =$$
$$\mu_{n+1,x}\varphi_{n,x}(a(x_{[1,n]})) = \mu_{n,x}(a(x_{[1,n]})) = \pi_x(\mu_n(a)),$$

and so $\mu_{n+1} \circ \varphi_n = \mu_n$. By proposition 2.1.13, $\varphi_{n,x}$ is unit preserving and injective for all x in X. Therefore, $\mu_{n,x}$ is also unit preserving and injective for all x in X. As μ_n pointwise is just $\mu_{n,x}$, it follows that μ_n is unit preserving and injective, so the C*-algebra $\overline{\bigcup_{n\in\mathbb{N}} \mu_n(A_n)} = \mathcal{A}$ is unital, and together with $\mu_n : A_n \mapsto \mathcal{A}$ is a unit-preserving inductive limit of $\{\varphi_n\}$.

We will now describe the D-algebra structure of $\mathcal{A}$ and gather a couple facts about it. For z in P^n, let $U(z) = \{x \in X : x_{[1,n]} = z\}$. Consider

$$C_u(X) = \{f \in C_b(X) : \forall\epsilon > 0,\ \exists n \in \mathbb{N} : \forall z \in P^n, \forall x, y \in U(z), |f(x) - f(y)| < \epsilon\}.$$

We can think of these functions as the uniformly continuous functions on X. It is easy to verify $C_u(X)$ is a C*-sub-algebra of $C_b(X)$.

Proposition 2.1.14. *$\mathcal{A}$ is a $C_u(X)$-algebra with product $f \cdot a$, for f in $C_u(X)$, a in $\mathcal{A}$, defined point-wise as $\pi_x(f \cdot a) = f(x)\pi_x(a)$, for all x in X. Moreover, $C_u(X)$ unitally embeds into the centre $Z(\mathcal{A})$ of $\mathcal{A}$.*

Proof. Consider $C_u^n(X) = \{f \in C_b(X) : f \text{ is constant on } U(z)\ \forall z \in P^n\}$. Then $\bigcup_{n=1}^{\infty} C_u^n(X)$ is a dense *-sub-algebra of $C_u(X)$ and each $C_u^n(X)$ can be identified with the sub-algebras $\Pi_{z\in P^n}\mathbb{C}(1(z))$ in the centre of A_n. Under these identifications, the inclusion $C_u^n(X) \mapsto C_u^{n+1}(X)$ becomes the φ_n restricted to $\Pi_{z\in P^n}\mathbb{C}((1(z))$. Taking the inductive limit, we then obtain our unital embedding $C_u(X) \mapsto Z(\mathcal{A})$. For f in $C_u^n(X)$ and x in X, $\pi_x(f) = \mu_{n,x}(f(x_{[1,n]})1(x_{[1,n]})) = f(x)\pi_x(1)$. By density of $\bigcup_{n=1}^{\infty} C_u^n(X)$ in $C_u(X)$, $\pi_x(f) = f(x)\pi_x(1)$ for all f in $C_u(X)$. □

For z in P^n, denote χ_z to be the characteristic function on the cylinder set $U(z)$, which is in $C_u^n(X)$.

Proposition 2.1.15. *The kernel of $\pi_x : \mathcal{A} \mapsto A_x$ is the ideal $\mathcal{A}^x$. Also,*

$$\mathcal{A}^x = \{a \in \mathcal{A} : a\chi_{x_{[1,n]}} \to 0 \text{ as } n \to \infty\}$$

Proof. It is easy to check that $J = \{a \in \mathcal{A} : a\chi_{x_{[1,n]}} \to 0 \text{ as } n \to \infty\}$ is a closed two-sided ideal. For f in $C_u(X)$ such that $f(x) = 0$, $f\chi_{x_{[1,n]}} \to 0$ as $n \to \infty$, so, for any a in $\mathcal{A}$, $f \cdot a\chi_{x_{[1,n]}} \to 0$ as $n \to \infty$. Therefore, the generators of the ideal $\mathcal{A}^x$ are contained in the ideal J, so $\mathcal{A}^x \subseteq J$. If a is in J, then $(1 - \chi_{x_{[1,n]}})a \to a$ as $n \to \infty$, and $(1 - \chi_{x_{[1,n]}})a \in \mathcal{A}^x$, for all $n \geq 1$, so by closure, a is in $\mathcal{A}^x$. Therefore $\mathcal{A}^x = J$. Let I be the kernel of $\pi_x : \mathcal{A} \mapsto A_x$. Since for a in J, $(1-\chi_{x_{[1,n]}})a \to a$ as $n \to \infty$ and $(1-\chi_{x_{[1,n]}})a$

is in I for $n \geq 1$, we have $J \subseteq I$. It suffices to show $I_n := I \cap \mu_n(A_n) \subseteq J \cap \mu_n(A_n) := J_n$ for $n \geq 1$. Suppose a is in A_n and $0 = \pi_x(\mu_n(a)) = \mu_{n,x}(a(x_{[1,n]}))$. By injectivity of $\mu_{n,x}$, we conclude $a(x_{[1,n]}) = 0$. Therefore $\mu_n(a)\chi_{x_{[1,k]}} = 0$ for $k \geq n$, so $\mu_n(a)$ is in J_n. Therefore, $I_n \subseteq J_n$. □

Corollary 2.1.16. *For each x in X, $\mathcal{A}_x$ is isomorphic to the AF algebra A_x. For every unital AF algebra A, there is x' in X such that $\mathcal{A}_{x'}$ is isomorphic to A*

Proof. The first conclusion immediately follows from proposition 2.1.15. As for the second conclusion, let A be a unital AF algebra. A can be written as the closure of an increasing union of finite dimensional sub-algebras A_n that contain the unit and $A_0 = \mathbb{C}1$. Therefore, A is the inductive limit algebra of the sequence of inclusions $i_n : A_n \mapsto A_{n+1}$. For $n \geq 0$, let x'_{n+1} be the adjacency matrix of i_n. Since i_n is unital and injective, the rows and columns of x'_{n+1} are non-zero. Hence, $x' = (x'_1, x'_2, ...)$ is in X. Since $A_0 = A(s(x'))$ and both $\varphi_{n,x'}$ and i_n are unital with the same adjacency matrix for all $n \geq 0$, it follows that there are *-isomorphisms $\psi_n : A_n \mapsto A(x_{[1,n]})$ such that $\psi_{n+1} \circ i_n = \varphi_{n,x'} \circ \psi_n$, so $\mathcal{A}_{x'} \simeq A$. □

Proposition 2.1.17. *$\mathcal{A}$ is a continuous over X. Moreover, for every a in $\mathcal{A}$, $\|a\|_X$ is in $C_u(X)$.*

Proof. It suffices to show the proposition for all a in $\mu_n(A_n)$ and $n \geq 1$. Write $a = \mu_n(a')$ for a' in A_n. By proposition 2.1.15, $\|a(x)\| = \|\pi_x(a)\|$ for all x in X. Therefore, $\|a(x)\| = \|\mu_{n,x}(a'(x_{[1,n]}))\|$ for all x in X. Since the $\mu_{n,x}$ are injective, $\|\mu_{n,x}(a'(x_{[1,n]}))\| = \|a'(x_{[1,n]})\|$ for all x in X, so the function $\|a\|_X$ is in $C_u^n(X)$. □

We will describe the pullback of the $C_u(X)$-algebra $\mathcal{A}$ by the k^{th} shift $\sigma^k : X \mapsto X$ given by $\sigma(A_1, A_2, ...,) = (A_{k+1}, A_{k+2}, ...)$ for $(A_1, A_2, ...)$ in X. For $n \geq k$, we will also denote $\sigma^k : P^n \mapsto P^{n-k}$ to be the shift on finite paths. This shouldn't cause confusion because the length of the path will always be specified. For a natural number $n \geq k$, define $\sigma^{k*}A_n := \Pi_{z \in P^n} A(\sigma^k z)$, and $\varphi_{n,\sigma^k} : \sigma^{k*}A_n \mapsto \sigma^{k*}A_{n+1}$ be the *-homomorphism defined factor-wise by $\varphi_{n,\sigma^k}(a)(z * e) = \varphi_{\sigma^k z, \sigma^k z * e}(a(z))$ for all a in $\sigma^{k*}A_n$, z in P^n, and e in A such that $r(z) = s(e)$. Define $\mu_{n,\sigma^k} : \sigma^{k*}A_n \mapsto \Pi_{x \in X} A_{\sigma^k x}$ factor-wise by $\pi_x(\mu_n(a)) = \mu_{n-k,\sigma^k x}(a(\sigma^k(x_{[1,n]})))$ for all x in X and a in $\sigma^* A_n$. Then, by the same computation done above, $\mu_{n+1,\sigma^k} \circ \varphi_{n,\sigma^k} = \mu_{n,\sigma^k}$. Since the μ_{n,σ^k}, for $n \geq k$, are injective, $\sigma^{k*}\mathcal{A} := \bigcup_{n \in \mathbb{N}} \mu_{n,\sigma^k}(\sigma^{k*}A_n)$ along with $\mu_{n,\sigma^k} : \sigma^{k*}A_n \mapsto \sigma^{k*}\mathcal{A}$, $n \geq k$, is an inductive limit of $\{\varphi_{n,\sigma^k}\}_{n \geq k}$. The $C_u(X)$ structure is given point-wise by $\pi_x(fa) = f(x)\pi_x(a)$ for all f in $C_u(X)$, a in $\sigma^{k*}\mathcal{A}$, and x in X, and so the fibre of $\sigma^{k*}\mathcal{A}$ at x is isomorphic to $A_{\sigma^k(x)}$.

Remark 2.1.18. *The $C_u(X)$-algebra $C_u(X) \otimes_{\sigma^k} \mathcal{A}$, defined in a similar way to the pullbacks in definition 2.1.8, and $\sigma^{k*}\mathcal{A}$ are not isomorphic as $C_u(X)$-algebras. Essentially, this is because if $\Pi_{i \in \mathbb{N}} A_i$ and $\Pi_{j \in \mathbb{N}} B_j$ are two direct products of sequences of* *non-zero C*-algebras* $\{A_i\}_{i\in\mathbb{N}}$ *and* $\{B_j\}_{j\in\mathbb{N}}$*, then* $(\Pi_i A_i) \otimes (\Pi_j B_j)$ *and* $\Pi_{i,j}(A_i \otimes B_j)$ *are not isomorphic. This situation is an outlier in our thesis, because all other section algebras presented will be coming from groupoid bundles over locally compact Hausdorff spaces, whose induced section algebras behave well with the definition of pullback given in section 2.1; see proposition 2.2.11 for a precise statement.*

2.2 Groupoid Bundles

Groupoid C*-algebras, introduced by Renault [44], have provided interesting models for well-known C*-algebras, as well as completely new C*-algebras. Here, a topological groupoid is a non-empty locally compact Hausdorff space G, together with a closed set $G \circ G \subseteq G \times G$ (called the composable pairs), a continuous map $p : G \circ G \mapsto G$ (called the product map), and a continuous involution ${}^{-1} : G \mapsto G$ (called the inverse map). For (g_1, g_2) in $G \circ G$, denote $p(g_1, g_2)$ by $g_1 g_2$. $G \circ G$, p, and ${}^{-1}$ must satisfy the following axioms:

(i) the product is associative, meaning that if g_1, g_2, g_3 are in G and (g_1, g_2), (g_2, g_3) are in $G \circ G$, then $(g_1 g_2, g_3)$, $(g_1, g_2 g_3)$ are in $G \circ G$, and $(g_1 g_2)g_3 = g_1(g_2 g_3)$.

(ii) For all g in G, (g, g^{-1}), (g, g^{-1}) are in $G \circ G$, and $g^1 g$, gg^{-1} serve as partial identities, meaning whenever e and f are in G and (e, g), (g, f) are in $G \circ G$, we have $egg^{-1} = e$ and $g^{-1}gf = f$.

The axioms imply identites like $(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$ whenever (g_1, g_2) is in $G \circ G$. See Putnam lemma 3.1.3 [40] for a number of these identities. The elements of the form $g^{-1}g$ for g in G will be called units, and the set of units will be denoted G^0. The mappings $r : G \mapsto G^0$ and $s : G \mapsto G^0$ defined for g in G as $r(g) = gg^{-1}$ and $s(g) = g^{-1}g$ are called the range and source maps, respectively. It follows from the axioms that g_1 and g_2 in G are composable if and only if $s(g_1) = r(g_2)$. In our thesis, we will only consider étale groupoids, which are topological groupoids for which the range (or equivalently the source) map is a local homeomorphism from G to G^0. When this is true, the groupoid C*-algebra of G can be constructed solely from the groupoid operations without any technical measure theory criterion. This construction starts by endowing $C_c(G)$, the continuous compactly supported $\mathbb{C}$-valued functions on G, with a product, defined for a and b in $C_c(G)$ point-wise as

$$ab(g) = \sum_{h\in G: r(h)=r(g)} a(h)b(h^{-1}g), \ g \in G$$

and an involution, defined for a in $C_c(G)$ point-wise as

$$a^*(g) = \overline{a}(g^{-1}), \ g \in G.$$

The facts that the above sum for the product ab involves only finitely many terms, and that ab is a continuous compactly support function are consequences of the étale condition. This product and involution turn $C_c(G)$ into a $*$-algebra. Now, to equip $C_c(G)$ with a pre-C^*-norm, one must embed $C_c(G)$ *-homomorphically into the bounded operators on some Hilbert space. This can be done by associating to each unit u in G^0 the Hilbert space $\ell^2(s^{-1}\{u\})$ of square-summable sequences, indexed by the discrete set $s^{-1}\{u\}$, and a *-homomorphism $\pi^u : C_c(G) \mapsto B(\ell^2(s^{-1}\{u\})$ into the bounded operators on $\ell^2(s^{-1}\{u\})$, defined for each a in $C_c(G)$ and ψ in $\ell^2(s^{-1}\{u\})$ as

$$\pi^u(a)(\psi)(g) = \sum_{h \in G : r(h)=r(g)} a(h)\psi(h^{-1}g),\ g \in s^{-1}\{u\}.$$

Then, the *-homomorphism $\pi = \oplus_{u \in G^0}\pi^u$ is shown to be faithful on $C_c(G)$. The C*-algebra $\overline{\pi(C_c(G))}$, (equipped with the operator norm) denoted by $C_r^*(G)$, is called the reduced C^*-algebra of G. For the details of its construction, see Putnam section 3.3 [40].

There is also a “maximal” completion of $C_c(G)$. One considers an auxilliary norm $\|\cdot\|_I$ on $C_c(G)$, called the I-norm (Renault pg. 50 [44]), and defines a pre-C*-norm $\|\cdot\|$ on $C_c(G)$ by letting, for a in $C_c(G)$, $\|a\|$ be the supremum of all the operator norms $\|\mu(a)\|$, where $\mu : C_c(G) \mapsto B(H)$ is a *-homomorphism with $\|\mu\| \leq 1$ relative to the I-norm on $C_c(G)$ and the operator norm on $B(H)$. The completion of $C_c(G)$ in this pre-C^* norm is denoted $C^*(G)$, and is called the full groupoid C^*-algebra of G. For details, see Renault Chapter 2 [44].

There is a canonical surjective $*$-homomorphism $C^*(G) \mapsto C_r^*(G)$ which is the identity on $C_c(G)$. For most groupoids encountered, this homomorphism is a *-isomorphism. In general, one needs an amenability hypothesis (for which there are several such notions for groupoids). The strong connection between the groupoid G and the $*$-algebra $C_c(G)$ means that it is often the case that a groupoid property will have a corresponding C^*-property, and vice-versa. This is the case for $C_0(X)$-algebras (proposition 2.2.8), and later on will be the case for renormalization procedures. The corresponding notion of a $C_0(X)$-algebra, a groupoid bundle over X, will be introduced in this section (definition 2.2.3), and some aspects of the correspondence will be established, namely the fibres of a groupoid bundle X will correspond to the fibres of the induced $C_0(X)$-algebra (proposition 2.2.9), and the pullback of a groupoid bundle over X (definition 2.2.10) will correspond to the pullback of the induced $C_0(X)$-algebra (proposition 2.2.11). Examples of groupoid bundles will be considered in sections 2.2.1, 2.2.2, and 2.2.3. The author is not aware of any pre-existing literature containing the results on groupoid bundles appearing in this section, though the reader will see they are elementary enough to have likely been considered before. Renault defines groupoid bundles in definition 3.3 [45]. His results are focused on amenability properties of such groupoids (see Renault theorem 3.5 [45]). We will be working with the full groupoid C*-algebra, but the analogous constructions and results hold for the reduced C^*-algebras.

The following definitions are standard in the theory of étale groupoids; see for instance Putnam definition 3.2.8 [40] for the first definition and Putnam definition 3.4.6 [40] for the second one.

Definition 2.2.1. *A G-**set** U of a groupoid G is an open set for which $r(U)$ and $s(U)$ are open in G^0 and $r : U \mapsto r(U)$, $s : U \mapsto s(U)$ are homeomorphisms. We will call $r \circ (s|_U)^{-1}$ the G-**map of** U, and it shall be denoted γ_U.*

Notice that G being étale is equivalent to saying that G has a basis for its topology consisting of G-sets.

Definition 2.2.2. *Let G be an étale groupoid. A set S of G^0 is said to be G-**invariant** if for any g in G such that $s(g)$ is in S, we have that $r(g)$ is also in S. G is said to be **minimal** if there are no non-trivial closed G invariant sets. G being minimal is easily seen to be equivalent the orbits $\mathcal{O}_u = \{v \in G^0 : \exists g \in G : s(g) = u \text{ and } r(g) = v\}$ being dense for every u in G^0 (the closure of an orbit is G-invariant, and conversely every closed invariant set must contain a whole orbit).*

It is an easy check that if S is G-invariant. then $r^{-1}(S) = s^{-1}(S)$ is a groupoid, which we shall denote G_S, and will be called the reduction of G to S. Roughly speaking, a topological groupoid bundle structure on a groupoid G is a continuous parameterization of a collection of (pairwise disjoint) groupoid reductions to invariant sets. This is made precise below.

Definition 2.2.3. *Let X be a locally compact Hausdorff space. A topological **groupoid bundle over** X, denoted (G, π), is a topological groupoid G together with a continuous surjection $\pi : G \mapsto X$ such that, for every g in G, $\pi(g) = \pi(r(g)) = \pi(s(g))$. If G is étale, we will call (G, π) an **étale groupoid bundle**.*

*For a point x in X, denote $G^x = \pi^{-1}(X \setminus \{x\})$, which is open in G, and denote $G_x = \pi^{-1}(x) = G \setminus G^x$, which is closed in G. Notice that G^x and G_x are the reductions to the invariant sets of units $\pi^{-1}(X \setminus \{x\}) \cap G^0$, $\pi^{-1}(\{x\}) \cap G^0$, respectively. We will call the closed sub-groupoid G_x the **fibre of** G **at** x.*

The only difference between our definition of groupoid bundle and Renault's (definition 3.3 [45]) is that we do not assume $\pi : G \mapsto X$ to be an open map.

Proposition 2.2.4. *If (G, π) is an étale groupoid bundle over X, then for every point x in X, G_x is étale.*

Proof. Let U be a G-set, and $\gamma = r \circ (s|_U)^{-1}$. Then, $\gamma(U \cap \pi^{-1}(x)) = \gamma(U) \cap \pi^{-1}(x)$, so $U_x := U \cap \pi^{-1}(x)$ is a G_x-set. □

We will now see, in the context of minimality, how properties of a fibre groupoid in a groupoid bundle affect properties of neighbouring fibre groupoids. First, we need two definitions.

Definition 2.2.5. *Let G be an étale groupoid such that its unit space G^0 is a metric space with metric d. For $\epsilon > 0$, we say G is **ϵ-minimal** if for every u and v in G^0, there is a g in G for which $s(g) = u$ and $d(r(g), v) < \epsilon$.*

Note that if G is an étale groupoid such that its unit space is a metric space, then G being minimal is equivalent to G being ϵ-minimal for all $\epsilon > 0$. We now define what it means for a family of metric spaces to vary continuously.

Definition 2.2.6. *let $f : X \mapsto Y$ be a continuous surjection, and assume $f^{-1}(y)$ is a metric space, with metric d_y, for every y in Y. Denote the set $\Delta_f X = \{(x_1, x_2) \in X \times X : f(x) = f(y)\}$; this is just the set $(f \times f)^{-1}(\Delta_{id_Y} Y)$. We say $(d_y)_{y \in Y}$ **varies continuously over** Y if the induced function $d : \Delta_f X \mapsto \mathbb{R}$, defined for (x_1, x_2) in $\Delta_f X$ as $d(x_1, x_2) = d_{f(x_i)}(x_1, x_2)$, is continuous.*

Recall that a function $f : X \mapsto Y$ is proper if for every compact set K of Y, $f^{-1}(K)$ is compact. The following proposition will be useful when we consider groupoid renormalization procedures.

Proposition 2.2.7. *Let (G, π) be an étale groupoid bundle over X, such that each unit space G^0_x is a metric space with d_x, for x in X, and $(d_x)_{x \in X}$ varies continuously over X. Assume also that $\pi : G^0 \mapsto X$ is a proper map. Then, whenever x in X is such that G_x is minimal, for every $\epsilon > 0$ there is an open neighbourhood V of x such that for every $\tilde{x}$ in V, $G_{\tilde{x}}$ is ϵ-minimal.*

Proof. First, we show if $f : X \mapsto Y$ is a proper map with Y locally compact and Hausdorff, and U is an open set in X containing a pre-image $f^{-1}(y) \subseteq U$ for some y in Y, then there is an open neighbourhood V of y for which $f^{-1}(V) \subseteq U$. If this were not true then we can find (by the axiom of choice) a net x_V, indexed by the set of all pre-compact neighbourhoods V of y, in X such that x_V is in $f^{-1}(V)$ but not in U. Then, by fixing a pre-compact neighbourhood V_0 of y, we see from properness that $f^{-1}(\overline{V_0})$ is compact, so the sub-net consisiting of x_V for $V \subseteq V_0$ has a convergent sub-net $(x_\lambda)_{\lambda \in \Lambda}$, converging to a point x. Since Y is locally compact and Hausdorff, the intersection of all pre-compact neighbourhoods of y is just y, so $\lim_V f(x_V) = y$, and therefore $f(x) = \lim_\lambda(f(x_\lambda)) = y$. Hence x is in $f^{-1}(y)$. However, $f^{-1}(y) \subseteq U$, so x_λ, by its convergence, must eventually lie inside U, a contradiction. Therefore, there must exist an open neighbourhood V of y such that $f^{-1}(V) \subseteq U$.

We will denote $\pi : G \mapsto X$ restricted to G^0 as π_0. Let $\epsilon > 0$. By assumption, the induced function $d : \Delta_{\pi_0} G^0 \mapsto \mathbb{R}$ is continuous, so for every u in G^0_x, we can find an open set W_u of G_0 containing u, and with the property that $d(\Delta_{\pi_0} W_u) < \epsilon$, where

$\Delta_{\pi_0} W_u = \Delta_{\pi_0} G^0 \cap (W_u \times W_u)$. By compactness of G^0_x, we can find a finite sub-cover $\{W_{u_i}\}_{i=1}^M$ of G^0_x. Since G_x is minimal, and each W_{u_i} intersects G^0_x non-trivially, we can find open sets $U_{i,j}$ in G, for $1 \leq i \leq M$, $1 \leq j \leq l_i$ such that $\bigcup_{j=1}^{l_i} s(U_{i,j})$ covers G^0_x and $r(U_{i,j}) \subseteq W_{u_i}$. Now, since π_0 is proper, by the above argument we can find and open set V of x such that $\pi_0^{-1}(V) \subseteq \bigcup_{j=1}^{l_i} s(U_{i,j})$ for all $1 \leq i \leq M$ and $\pi_0^{-1}(V) \subseteq \bigcup_{i=1}^{M} W_{u_i}$. Therefore, for u and v in $\pi_0^{-1}(\tilde{x})$, where $\tilde{x}$ is in V, there is an $i \leq M$ and a $j \leq l_i$ such that v is in W_{u_i} and u is in $s(U_{i,j})$. Hence, there is a g in $U_{i,j}$ such that $s(g) = u$ and $r(g)$ is in W_{u_i}. By the assumption on W_{u_i}, we then have $d_{\tilde{x}}(r(g), v) = d(r(g), v) < \epsilon$. Therefore, V satisfies the conclusion of the proposition. □

We now show if (G, π) is an étale groupoid bundle over X, then $C^*(G)$ is naturally a $C_0(X)$-algebra.

Proposition 2.2.8. *Let (G, π) be an étale groupoid bundle over X. Then $\pi : G \mapsto X$ induces an injective and non-degenerate *-homomorphism $\pi^* : C_0(X) \mapsto Z(\mathcal{M}(C^*(G)))$ which, for f in $C_0(X)$ and h in $C_c(G)$, $\pi^*(f)h$ is the function in $C_c(G)$ given point-wise by $(\pi^*(f)h)(g) = f(\pi(g))h(g)$, for g in G. Hence, $(C^*(G), \pi^*)$ is a $C_0(X)$-algebra (definition 2.1.1).*

Proof. $\pi|_{G^0} = \pi_0 : G^0 \mapsto X$ is continuous and, since $\pi(g) = \pi(r(g))$ for all g in G, it is also a surjection. Therefore, the induced *-homomorphism $\pi_0^* : C_0(X) \mapsto \mathcal{M}(C_0(G^0))$ is injective and non-degnerate. Let $\lambda : C_0(G^0) \mapsto \mathcal{M}(C^*(G))$ be the *-homomorphism where, for k in $C_0(G^0)$ and h in $C_c(G)$, $\lambda(k)h$ is the function in $C_c(G)$ given point-wise by

$$(\lambda(k)h)(g) = k(r(g))h(g), \ g \in G.$$

λ is injective and non-degenerate (Renault Ch. 2 proposition 1.14 [44]), so it extends to the multiplier $\tilde{\lambda} : \mathcal{M}(C_0(G^0)) \mapsto \mathcal{M}(C^*(G))$. Therefore, $\pi^* := \tilde{\lambda} \circ \pi_0^*$ is an injective and non-degenerate *-homomorphism from $C_0(X)$ to $\mathcal{M}(C^*(G))$. For f in $C_0(X)$ and h in $C_c(G)$, let k a function in $C_c(G^0)$ such that $k = 1$ on $r(\text{supp}(h))$. By the definition of multiplier extensions,

$$\pi^*(f)h = \tilde{\lambda}(\pi_0^*(f))(\lambda(k)h) = \lambda(\pi_0^*(f)k)(h).$$

Therefore $\pi^*(f)h$ is in $C_c(G)$. Moreover, the support of $\pi^*(f)h$ is contained in the support of h, so for g in G,

$$(\pi^*(f)h)(g) = f(\pi(r(g)))k(r(g))h(g) = f(\pi(r(g)))h(g).$$

Since $\pi_0 \circ r = \pi$, it follows that $(\pi^*(f)h)(g) = f(\pi(g))h(g)$. Now, to show π^* maps into the centre of $\mathcal{M}(C^*(G))$, by strong density of $C^*(G)$ inside of $\mathcal{M}(C^*(G))$, it suffices to check that for any f in $C_0(X)$ and a, b in $C_c(G)$, $\pi^*(f)ab = a(\pi^*(f)b)$. This is automatic from the above formula for $\pi^*(f)h$ and the fact that $\pi = \pi_0 \circ r = \pi_0 \circ s$. □

For x in X, G^x is an open sub-groupoid that is the reduction to an open G-invariant set, so we can (and do) identify $C^*(G^x)$ with the ideal in $C^*(G)$ that has $C_c(G^x)$ as a dense $*$-sub-algebra. By Putnam Theorem 3.4.8 [40] the restriction map $C_c(G) \mapsto C_c(G_x)$ extends to a surjective $*$-homomorphism $\rho_x : C^*(G) \mapsto C^*(G_x)$ with $C^*(G^x)$ its kernel. Therefore, we can identify the quotient of $C^*(G)$ by $C^*(G^x)$ with $C^*(G_x)$. Using these identifications, we have a correspondence between our groupoid bundle notion of fibre and the $C_0(X)$-algebra notion of fibre (definition 2.1.3):

Proposition 2.2.9. *Let (G, π) be an étale groupoid bundle over X. Then, for any point x in X, $C^*(G)^x = C^*(G^x)$, and therefore $C^*(G)_x \simeq C^*(G_x)$ using the identification mentioned above.*

Proof. It is easy to see that $\pi^*(C_c(X \setminus \{x\}))C_c(G) \subseteq C_c(G^x)$, so $C^*(G)^x \subseteq C^*(G^x)$. For h in $C_c(G^x)$, choose a function ϕ in $C_c(X \setminus \{x\})$ such that $\phi = 1$ on the compact set $\pi(\text{supp}(h)) \subseteq X \setminus \{x\}$. Then, $\pi^*(\phi)h = h$. Therefore, h is in $\pi^*(C_c(X \setminus \{x\}))C_c(G)$, so the inclusion $C^*(G^x) \subseteq C^*(G)^x$ also holds. The second conclusion follows immediately from the first. $\square$

Definition 2.2.10. *Let (G, π) be an étale groupoid bundle over Y and $f : X \mapsto Y$ a continuous (not necessarily proper) function between locally compact Hausdorff spaces X and Y. We define the **pullback of** (G, π) **by** f (denoted $(f^*G, f^*\pi)$) as follows:*

Let $f^*G := \{(x, g) \in X \times G : f(x) = \pi(g)\}$ be the fibred product of X and G by f and π. This is a locally compact Hausdorff topological étale groupoid with (x, g) and (x', g') in f^*G composable if and only if $x = x'$ and $s(g) = r(g')$, in which case the product $(x, g)(x', g') = (x, gg')$. Involution is defined for pairs (x, g) in f^*G as $(x, g)^{-1} = (x, g^{-1})$. $f^*\pi : f^*G \mapsto X$ is defined for (x, g) in f^*G as $f^*\pi(x, g) = x$, giving f^*G the structure of a groupoid bundle over X. For any open set U in X and open G-set V, $U \times_f V = \{(x, g) \in U \times V : f(x) = \pi(g)\}$ is an open f^*G-set, so $(f^*G, f^*\pi)$ is an étale groupoid bundle over X.

We now show that the groupoid and C*-algbera versions of pullback are in correspondence.

Proposition 2.2.11. *Let (G, π) be a groupoid bundle over Y and $f : X \mapsto Y$ a continuous map between locally compact Hausdorff spaces X and Y. Then, $(f^*C^*(G), f^*\pi^*)$ (definition 2.1.8) and $(C^*(f^*G), (f^*\pi)^*)$ are isomorphic as $C_0(X)$-algebras (definition 2.1.2).*

Proof. Consider the maps $\Phi_1 : C_c(X) \mapsto \mathcal{M}(C_c(X \times G))$ and $\Phi_2 : C_c(G) \mapsto \mathcal{M}(C_c(X \times G))$ defined for compactly supported functions h, a, and b in X, G, and $X \times G$ by

$$\Phi_1(h)b(x, g) = h(x)b(x, g), \; (x, g) \text{ in } X \times G$$

and

$$\Phi_2(a)b(x,g) = \sum_{g' \in G : r(g') = r(g)} a(g')b(x, g'^{-1}g), \ (x,g) \text{ in } X \times G.$$

It is routine to check that these are $*$-homomorphisms between *-algebras. Now, the images of Φ_1 and Φ_2 commute, $\Phi_1(C_c(X))\Phi_2(C_c(G))$ is contained in $C_c(X \times G)$ and spans a dense *-sub-algebra, so by the universal property of the algebraic tensor product $\hat{\otimes}$, there is a *-homomorphism $\Phi_1 \hat{\otimes} \Phi_2 : C_c(X) \hat{\otimes} C_c(G) \mapsto C_c(X \times G)$ with dense image defined on basic tensors $h \otimes a$ in $C_c(X) \hat{\otimes} C_c(G)$ by

$$\Phi_1 \hat{\otimes} \Phi_2(h \otimes a)(x,g) = h(x)a(g), \ (x,g) \text{ in } X \times G.$$

f^*G is the reduction of $X \times G$ to the closed $X \times G$-invariant set of units $\{(x,u) \in X \times G^0 : f(x) = \pi(u)\}$, so the map $r : C_c(X \times G) \mapsto C^*(f^*G)$ which sends a function b in $C_c(X \times G)$ to its restriction $b|_{f^*G} = r(b)$ is a homomorphism of *-algebras and has dense image. Let $ev_x^1 : C^*(f^*G) \mapsto C^*(f^*G)_x = C^*(G_{f(x)})$ be the evaluation map at x in X, and let $ev_x \otimes ev_{f(x)} = \mu_x : C_0(X) \otimes C^*(G) \mapsto C^*(G)_{f(x)} = C^*(G_{f(x)})$ be the mapping defined in the proof of proposition 2.1.9. By proposition 2.2.9, the evaluation maps $ev_{f(x)} : C^*(G) \mapsto C^*(G_{f(x)})$ and $ev_x^1 : C^*(f^*G) \mapsto C^*(G_{f(x)})$, when restricted to compactly supported functions on G and f^*G, respectively, are just the restrictions of the domains to $G_{f(x)}$, $(f^*G)_x = G_{f(x)}$, respectively. So, for $h \otimes a$ in $C_c(X) \hat{\otimes} C_c(G)$, $ev_x^1(r \circ \Phi_1 \hat{\otimes} \Phi_2(h \otimes a)) = ha|_{(f^*G)_x} = h(x)a|_{G_{f(x)}} = \mu_x(h \otimes a)$. By linearity, it follows that $ev_x^1 r(\Phi_1 \hat{\otimes} \Phi_2)(b) = \mu_x(b)$ for all points x in X and elements b in $C_c(X) \hat{\otimes} C_c(G)$. By injectivity of $C^*(f^*G)$ and $f^*C^*(G)$ over X and the preceding equality, for any b in $C_c(X) \hat{\otimes} C_c(G)$ we have

$$\|r(\Phi_1 \hat{\otimes} \Phi_2(b))\| = \sup_{x \in X} \|ev_x^1 r(\Phi_1 \hat{\otimes} \Phi_2)(b)\| = \sup_{x \in X} \|\mu_x(b)\| = \|\pi_f(b)\|$$

where $\pi_f : C_0(X) \otimes C^*(G) \mapsto f^*C^*(G)$ is the quotient map. Therefore, $r \circ \Phi_1 \hat{\otimes} \Phi_2$ extends to a *-homomorphism $\tilde{\Phi}$ from $C_0(X) \otimes C^*(G)$ to $C^*(f^*G)$ with dense range, and kernel equal to the kernel of π_f, such that $ev_x^1 \circ \tilde{\Phi} = \mu_x$ for all points x in X. Hence, $\tilde{\Phi}$ descends to an isomorphism of $C_0(X)$-algebras $f^*C^*(G)$ and $C^*(f^*G)$. $\square$

2.2.1 AF Groupoid Bundles

The space for which the field of AF algebras constructed in section 2.1.1 fibres over is too large to visualize, and its section algebra displays pathological behaviour (remark 2.1.18), so it is desirable to consider fields of AF algebras that fibre over smaller locally compact spaces. In this section, we construct groupoid bundles over the Cantor set whose fibres are tail equivalence groupoids. One advantage of having a compact parameter space is that, given a fixed parameter and a renormalization procedure for the groupoid bundle, the infinite sequence of successive renormalizations of the groupoid (or algebra) at this parameter will always have an accumulation point. This is what Treviño calls "bounded geometry" in [51]. This fact makes the unique trace condition in corollary 4.1.6 more applicable, for instance. First, it will be useful to have an alternative description of surjective graph homomorphisms between finite directed graphs.

Definition 2.2.12. *Let $G = (E, V, r, s)$ be a finite directed graph. A* ***coding*** *of G is a pair of functions (n, B) on the vertex set, edge set, respectively, such that*

- *a vertex v in V maps to a finite set $n(v)$*
- *an edge e in E maps to a collection of finite sets $B_{i,j}(e)$, one for each index (i, j) in $n(s(e)) \times n(r(e))$*
- *For each $e \in E$ there is an (i, j) in $n(s(e)) \times n(r(e))$ for which $B_{i,j}(e)$ is non-empty.*

We will say the coding is ***non-degenerate*** *if G has no sources or sinks and for each e in E, (i, j) in $n(s(e)) \times n(r(e))$, there is (i', j') in $n(s(e)) \times n(r(e))$ such that $B_{i',j}(e)$ and $B_{i,j'}(e)$ are non-empty.*

A coding (n, B) on G determines a larger directed graph $G(B) = (E(B), V(B), r_B, s_B)$ and a surjective graph homomorphism $\pi : G(B) \mapsto G$, defined as follows:

- the edge set is the disjoint union $E(B) = \bigsqcup_{e \in E} \bigsqcup_{(i,j) \in n(s(e)) \times n(r(e))} B_{i,j}(e)$
- the vertex set is the disjoint union $V(B) = \bigsqcup_{v \in V} n(v)$

If f is an edge in $B_{i,j}(e)$, then

- $\pi(f) = e$, $\pi(j) = r(e)$, $\pi(i) = s(e)$
- $r_B(e) = j$, $s_B(e) = i$.

Conversely, given any surjective directed graph homomorphism $\varphi : H \mapsto G$ between finite graphs, there is a coding (B, n) on G and a directed graph isomorphism $\psi : H \mapsto G(B)$ such that $\varphi \circ \psi^{-1} = \pi$. The coding is $n(v) = \varphi^{-1}(v)$ and $B_{i,j}(e) = \varphi^{-1}(e) \cap H_i \cap H^j$, where H_i, H^j are the edges of H that begin, end, at the vertices i, j, respectively. We will say a surjective graph homomorphism $\pi : H \mapsto G$ is non-degenerate if G has no sources or sinks, and if for every edge e in G and vertices v, w in H such that $\pi(v) = s(e)$, $\pi(w) = r(e)$, there are edges f and f' such that $\pi(f) = \pi(f') = e$ and $s(f) = v$, $r(f') = w$. Obviously non-degeneracy of the graph homomorphism implies non-degeneracy of the coding, and vice-versa. Note that if $\pi : H \mapsto G$ is non-degenerate, then H has no sources or sinks. We will assume non-degeneracy for the remainder of this section.

Let P_G^n denote the paths of length n in G and X_G denote the (one-sided) infinite path space for G. To a coded graph G with coding (B, n), and each path x in X_G, we define a Bratteli diagram $\mathcal{B}_x$ by letting $V_{x,n} = n(s(x_{n+1}))$ for $n \geq 0$, and $E_{x,n} = \bigsqcup_{(i,j)\in n(s(x_n))\times n(r(x_n))} B_{i,j}(x_n)$ with range and source maps defined on $E_{x,n}$ by $r_x(f) = r_B(f) \in V_{x,n}$ and $s_x(f) = s_B(f) \in V_{x,n-1}$.

Denote $G(B) = H$, and $\pi : H \mapsto G$ the surjective graph homomorphism defined above. Let σ_H, σ_G denote the shifts on X_H, X_G, respectively. Let

$$R_H = \{(x, y) \in X_H \times X_H : \exists n \in \mathbb{N} : \sigma_H^n x = \sigma_H^n y\},$$

which is the tail equivalence relation on X_H. Given a finite path z in P_H^n denote $U(z) = \{x \in X_H : x|_{[1,n]} = z\}$, which is the cylinder set at z. For z, w in P_H^n such that $r(z) = r(w)$, define

$$U(z, w) = \{(x, y) \in X_H \times X_H : x|_{(n,\infty)} = y|_{(n,\infty)} \text{ and } x|_{[1,n]} = z, y|_{[1,n]} = w\}.$$

The sets $\{U(z, w)\}_{n,z,w\in P_H^n : r(z)=r(w)}$ form a basis for a topology on R_H making it an étale groupoid. For a more detailed description of these groupoids, see Putnam chapter 3.5 [40]. Now, consider

$$R_\pi := \{(x, y) \in R_H : \pi(x) = \pi(y)\}.$$

R_π is an open sub-groupoid of R_H with basis consisting of the $U(z, w)$ such that $\pi(z) = \pi(w)$. Therefore, R_π is an étale groupoid, with groupoid bundle structure $\hat{\pi} : R_\pi \mapsto X_G$ via $\hat{\pi}(x, y) = \pi(x) = \pi(y)$. Notice that $\hat{\pi}^{-1}(x)$ is isomorphic to the groupoid of tail equivalence on the Bratteli diagram $\mathcal{B}_x$. Therefore, $(C^*(R_\pi), \pi^*)$ is a $C(X_G)$ algebra with fibres $C^*(R_\pi)_x$ isomorphic to an AF algebra associated to the Bratteli diagram $\mathcal{B}_x$. Here is another way to describe $(C^*(R_\pi), \hat{\pi}^*)$: Let A_H be the stationary AF algebra of the adjacency matrix for H. A_H has a canonical diagonal $C(X_H) \subseteq A_H$. $C(X_G)$ embeds into A_H via $\pi^* : C(X_G) \mapsto C(X_H) \subseteq A_H$. The commutant of $\pi^*(C(X_G))$ in A_H is equal to $C^*(R_\pi)$, and $\hat{\pi}^*$ is π^* with range restricted to $\hat{\pi}^*(C(X_G))' = C^*(R_\pi)$.

2.2.2 Rotation Groupoid Bundle

This next example is rather simple to describe. Let $\mathbb{R}/\mathbb{Z}$ be the quotient of the additive group $\mathbb{R}$ by the sub-group $\mathbb{Z}$. For t in $\mathbb{R}$, we will denote $\bar{t}$ to be the image of t under the quotient map. Equip $\mathbb{R}/\mathbb{Z}$ with the quotient topology. Let

$$\alpha : (0, 1) \times \mathbb{R}/\mathbb{Z} \mapsto (0, 1) \times \mathbb{R}/\mathbb{Z}$$

be the homeomorphism $\alpha(\theta, x) = (\theta, x + \overline{\theta})$. Notice that if we identify $\mathbb{R}/\mathbb{Z}$ with the circle S^1 via the exponential map $\overline{t} \mapsto e^{2\pi i t}$, then under this identification $\alpha(\theta, \cdot)$ is rotation of the circle by the angle $2\pi\theta$. We prefer to work with $\mathbb{R}/\mathbb{Z}$. Let

$$H = (0,1) \times \mathbb{R}/\mathbb{Z} \times_{\alpha} \mathbb{Z}$$

be the corresponding transformation groupoid of α (see Renault example 1.2.a [44]). Our conventions are as follows: points (θ, x, n) and (ψ, y, m) in H are composable if and only if $\theta = \psi$ and $y = x - n\overline{\theta}$, in which case the product $(\theta, x, n)(y, m) = (\theta, x, n+m)$. Involution of a point (θ, x, n) in H is given by $(\theta, x, n)^{-1} = (\theta, x - n\overline{\theta}, -n)$. The bundle structure

$$\pi : H \mapsto (0,1)$$

is given by the projection onto the first co-ordinate $\pi(\theta, x, n) = \theta$. If H is given the product topology as the space $(0,1) \times \mathbb{R}/\mathbb{Z} \times \mathbb{Z}$, then these operations turn H into an étale groupoid bundle (H, π). For θ in $(0,1)$, $C^*(H_\theta) = A_\theta$ is what is known as a rotation algebra. A detailed analysis of rotation algebras can be found in Davidson chapter 4 [6].

2.2.3 Iterated Function System Groupoids and Bundles

Let $\{\gamma_1, ..., \gamma_v\}$ be an iterated function system (IFS) on $\mathbb{R}^d$ of the form $\gamma_i(z) = Az + w_i$, for $i = 1, ..., v$ and A an invertible matrix with norm $\|A\| < 1$. Recently, Korfanty [20] constructed a groupoid out of such a single matrix system. We will now briefly review the construction.

Let $\Sigma_n = \{1, ..., v\}^n$. For $\eta = (\eta_1, \eta_2, ..., \eta_n)$ in Σ_n, define $\gamma_\eta = \gamma_{\eta_1} \circ \gamma_{\eta_2} \circ ... \circ \gamma_{\eta_n}$. We denote the attractor for the IFS to be K, which is the unique non-empty compact set of $\mathbb{R}^d$ such that $\bigcup_{i=1}^{v} \gamma_i(K) = K$. The proof of its existence and uniqueness can be found in Falconer theorem 9.1 [11]. For x in K, define $\mathcal{F}^{-n}(x) = \bigcup_{\eta \in \Sigma_n} \gamma_\eta^{-1}(x) \cap K$. The equivalence relation

$$R = \{(x, y) \in K^2 : \ \exists n \in \mathbb{N} : \mathcal{F}^{-n}(x) = \mathcal{F}^{-n}(y)\}$$

can be given an étale topology, and is the object of study in Korfanty's thesis [20].

We will now show that R can be enveloped into a "Cuntz-Pimsner" étale groupoid as an open subgroupoid, with subspace topology the same as the topology considered by Korfanty. This enveloping groupoid will be used to construct the renormalization procedure for the groupoid R. After this construction, we show how to construct groupoid bundles starting from the groupoid R. If we interpret the parameter space as a probability space, then the fibre groupoids can be thought of as random iterated function systems. All of our constructions (except the groupoid bundles) and arguments in this section are generalizations of the ones found in Korfany Ch. 4 [20].

First, for $(n, m), (k, l)$ in $(\mathbb{N} \cup \{0\})^2$, define the product

$$(n, m) * (k, l) = \begin{cases} (n, m - k + l) & \text{if } m \geq k \\ (n + k - m, l) & \text{if } m \leq k \end{cases}$$

and involution $(n, m)^{-1} = (m, n)$. It is an easy computation to show $*$ is associative and $^{-1}$ is involutive. The map $\pi : ((\mathbb{N} \cup \{0\})^2, *) \mapsto (\mathbb{Z}, +)$ via $\pi(n, m) = n - m$ is multiplicative and inverse preserving.

Now, define

$$T(R) = \{(x, n, m, y) \in K \times (\mathbb{N} \cup \{0\})^2 \times K : \mathcal{F}^{-n}(x) = \mathcal{F}^{-m}(y)\}.$$

For (x, n, m, y) in $T(R)$ let $(x, n, m, y)^{-1} = (y, m, n, x)$ be the involution, which is in $T(R)$. Given pairs $(x, n, m, y), (y, k, l, z)$ in $T(R)$, it can be shown the product $(x, n, m, y) * (y, k, l, z) = (x, (n, m) * (k, l), z)$ is in $T(R)$. We show this for the case $m \geq k$:

$$\mathcal{F}^{-(m-k+l)}(z) = \mathcal{F}^{-(m-k)}\mathcal{F}^{-l}(z) = \mathcal{F}^{-(m-k)}\mathcal{F}^{-k}(y) = \mathcal{F}^{-m}(y) = \mathcal{F}^{-n}(x).$$

An easy computation shows the product $*$ is associative and $^{-1}$ is involutive. Define a groupoid

$$O(R) = \{(x, \pi(n, m), y) : (x, n, m, y) \in T(R)\}.$$

The involution of a point (x, k, y) in $O(R)$ is $(x, k, y)^{-1} = (y, -k, x)$, and points (x, k, y), (y', l, z) in $O(R)$ are composable if and only if $y = y'$, in which case the product $(x, k, y)(y, l, z) = (x, k + l, z)$. $O(R)$ is closed under this inverse and product because the natural surjection $\tilde{\pi} : T(R) \mapsto O(R)$ is multiplicative and inverse preserving. Notice R is isomorphic to the sub-groupoid $\{(x, 0, y) : (x, 0, y) \in O(R)\}$ of $O(R)$ in an obvious way. $O(R)$ will be referred to as the **enveloping Cuntz-Pimsner groupoid** of R and $T(R)$ the **Toeplitz extension of** $O(R)$.

We now topologize $O(R)$ into an étale groupoid containing the isomorphic copy of R as an open sub-groupoid. For (x, k, y) in $O(R)$, denote $\gamma_{(x,k,y)} : \mathbb{R}^d \mapsto \mathbb{R}^d$ to be the map sending z in $\mathbb{R}^d$ $\gamma_{(x,k,y)}(z) = A^k z + x - A^k y$. Observe $\gamma_{(x,k,y)}$ is affine with linear factor A^k and sends y to x. It is uniquely determined by these properties.

Proposition 2.2.13. *The map sending (x, k, y) in $O(R)$ to $\gamma_{(x,k,y)}$ is a groupoid homomorphism from $O(R)$ to the group of affine transformations of the form $A^n(-) + w$ for some n in $\mathbb{Z}$ and w in $\mathbb{R}^d$.*

Proof. For (x, k, y) and (y, l, z) in $O(R)$, $\gamma_{(x,k,y)}\gamma_{(y,l,z)}$ and $\gamma_{(x,k+l,z)}$ are both affine with linear factor A^{k+l} and send z to x, and are therefore equal. A similar observation shows $\gamma^{-1}_{(x,k,y)} = \gamma_{(y,-k,x)}$. □

Proposition 2.2.14. *For (x,n,m,y) in $T(R)$, let η, ε be words in Σ_n, Σ_m, respectively such that x is in $\gamma_\eta(K)$, y is in $\gamma_\varepsilon(K)$ and $\gamma_\eta^{-1}(x) = \gamma_\varepsilon^{-1}(y)$. Then, $\gamma_\eta\gamma_\varepsilon^{-1} = \gamma_{(x,n-m,y)}$.*

Proof. Both maps are affine with the same linear factor, and send y to x. □

For a point x in K, let $U_n(x)$ be the open neighbourhood of x as in Korfanty Definition 4.2.6 [20]. i.e., $U_n(x) = K \setminus \bigcup_{\eta\in\Sigma_n : x\notin\gamma_\eta(K)} \gamma_\eta(K)$. Let (x,n,m,y) be in $T(R)$. Since $U_m(y) \subseteq \bigcup_{\varepsilon\in\Sigma_m : y\in\gamma_\varepsilon(K)} \gamma_\varepsilon(K)$, by proposition 2.2.14,

$$\gamma_{(x,n-m,y)}(U_n(y)) \subseteq \gamma_{(x,n-m,y)}(\bigcup_{\varepsilon : y\in\gamma_\varepsilon(K)} \gamma_\varepsilon(K)) = \bigcup_{\eta : x\in\eta(K)} \eta(K) \subseteq K,$$

so $\gamma_{(x,n-m,y)}$ restricts to a continuous function from $U_m(y)$ to K, and hence $\gamma_{(x,n-m,y)}^{-1}(U_n(x))$ is open.

Definition 2.2.15. *For (x,n,m,y) in $T(R)$, define $U(x,n,m,y) = U_m(y) \cap \gamma_{(x,n-m,y)}^{-1}(U_n(x))$, which, by the above discussion, is open in K. Since $\gamma_{(x,n-m,y)}^{-1} = \gamma_{(y,m-n,x)}$, we have $\gamma_{(x,n-m,y)}(U(x,n,m,y)) = U(y,m,n,x)$. Define $\gamma_{(x,n,m,y)}$ to be the homeomorphism $\gamma_{(x,n-m,y)} : U(x,n,m,y) \mapsto U(y,m,n,x)$.*

Proposition 2.2.16. *If $\tilde{y}$ is a point in $U(x,n,m,y)$, then $(\gamma_{(x,n,m,y)}(\tilde{y}), n, m, \tilde{y})$ is in $T(R)$.*

Proof. Let $\gamma_{(x,n,m,y)}(\tilde{y}) = \tilde{x}$. By symmetry ($\gamma_{(x,n,m,y)}^{-1} = \gamma_{(y,m,n,x)}$) it suffices to prove $\mathcal{F}^{-m}(\tilde{y}) \subseteq \mathcal{F}^{-n}(\tilde{x})$. Let ε be a word in Σ_m such that $\tilde{y}$ is in $\gamma_\varepsilon(K)$. Then, y is also in $\gamma_\varepsilon(K)$, otherwise $\tilde{y}$ is in $U_m(y) \cap \gamma_\varepsilon(K) = \emptyset$. Therefore, $\gamma_\varepsilon^{-1}(y)$ is in $\mathcal{F}^{-m}(y)$. Since (x,n,m,y) is in $T(R)$, there is a word η in Σ_n such that x is in $\gamma_\eta(K)$ and $\gamma_\eta^{-1}(x) = \gamma_\varepsilon^{-1}(y)$. By proposition 2.2.14, $\gamma_\eta\gamma_\varepsilon^{-1}(\tilde{y}) = \gamma_{(x,n,m,y)}(\tilde{y}) = \tilde{x}$. Therefore, $\gamma_\eta^{-1}(\tilde{x}) = \gamma_\varepsilon^{-1}(\tilde{y})$, proving $\mathcal{F}^{-m}(\tilde{y}) \subseteq \mathcal{F}^{-n}(\tilde{x})$. □

Definition 2.2.17. *Let (x,n,m,y) be a point in $T(R)$. For U an open subset of $U(x,n,m,y)$, define $\gamma_{(x,n,m,y)}^U$ to be the restriction $\gamma_{(x,n,m,y)} : U \mapsto \gamma_{(x,n,m,y)}(U)$. Denote*

$$\Gamma = \{\gamma_{(x,n,m,y)}^U : (x,n,m,y) \in T(R),\ U \text{ open } \subseteq U(x,n,m,y)\}.$$

For γ in Γ, let $D(\gamma), R(\gamma)$ denote its domain and range, respectively, and $c(\gamma) = c(\gamma_{(x,n,m,y)}^U) = n - m$ the co-cycle of γ. For γ_1 and γ_2 in Γ, define $D(\gamma_1 \circ \gamma_2) = \gamma_2^{-1}(D(\gamma_1) \cap R(\gamma_2))$.

We show the system Γ of homeomorphisms generates an étale topology for $O(R)$:

Proposition 2.2.18. *The following hold:*

(i) $\{U \text{ open in } K : id_U \in \Gamma\}$ is a basis for the relative topology on K.

(ii) If γ is in Γ, then γ^{-1} is in Γ.

(iii) For any γ_1, γ_2 in Γ and point $\tilde{y}$ in $D(\gamma_1 \circ \gamma_2)$, there is a γ_3 in Γ such that $\tilde{y}$ is in $D(\gamma_3) \subseteq D(\gamma_1 \circ \gamma_2)$ and $c(\gamma_3) = c(\gamma_1) + c(\gamma_2)$. Moreover, $\gamma_3|_{D(\gamma_3)} = \gamma_1 \circ \gamma_2|_{D(\gamma_3)}$.

(iv) Let $graph(\gamma) = \{(x, y) : \gamma(y) = x\}$. Suppose (x, y) is in $graph(\gamma_1) \cap graph(\gamma_2)$ and $c(\gamma_1) = c(\gamma_2)$. Then, there is a γ_3 in Γ with $c(\gamma_3) = c(\gamma_2) = c(\gamma_1)$ such that (x, y) is in $graph(\gamma_3) \subseteq graph(\gamma_1) \cap graph(\gamma_2)$. Moreover, $\gamma_3 = \gamma_2|_{D(\gamma_3)} = \gamma_1|_{D(\gamma_3)}$.

Proof. *i*) : For an open set U of K, $id_U = \gamma^U_{(x,0,0,x)}$ is in Γ.

ii) For $\gamma^U_{(x,n,m,y)}$ in Γ, $(\gamma^U_{(x,n,m,y)})^{-1} = \gamma^{\gamma_{(x,n,m,y)}(U)}_{(y,m,n,x)}$ is in Γ.

iii): Write $\gamma_1 = \gamma^{U_1}_{(x_1,n_1,m_1,y_1)}$ and $\gamma_2 = \gamma^{U_2}_{(x_2,n_2,m_2,y_2)}$. By proposition 2.2.16, for $\tilde{w} = \gamma_2(\tilde{y})$ and $\tilde{x} = \gamma_1(\tilde{w})$, the points $(\tilde{x}, n_1, m_1, \tilde{w})$ and $(\tilde{w}, n_2, m_2, \tilde{y})$ are in $T(R)$. Hence, $\tilde{y}$ is in $U(\tilde{x}, (n_1, m_1) * (n_2, m_2), \tilde{y}) =: U$. Let $\gamma_3 := \gamma^{U \cap D(\gamma_1 \circ \gamma_2)}_{(\tilde{x},(n_1,m_1)*(n_2,m_2),\tilde{y})}$. Then, $\tilde{y}$ is in $D(\gamma_3) = D(\gamma_1 \circ \gamma_2) \cap U \subseteq D(\gamma_1 \circ \gamma_2)$ and $c(\gamma_3) = c(\gamma_1) + c(\gamma_2)$. γ_3 and $\gamma_1 \circ \gamma_2$ are both affine maps with matrix $A^{c(\gamma_3)} = A^{c(\gamma_1)+c(\gamma_2)}$ and they both send $\tilde{y}$ to $\tilde{x}$. Hence, $\gamma_3 = \gamma_1 \circ \gamma_2|_{D(\gamma_3)}$.

iv): By proposition 2.2.16, (x, n_1, m_1, y) and (x, n_2, m_2, y) are in $T(R)$, so consider $\gamma_3 = \gamma^{U_1 \cap U_2 \cap U(x,n_1,m_1,y)}_{(x,n_1,m_1,y)}$. By a similar argument as in *iii*), $\gamma_3 = \gamma_2|_{D(\gamma_3)} = \gamma_1|_{D(\gamma_3)}$. ☐

Now, for γ in Γ, define $U_\gamma = \{(\gamma(y), c(\gamma), y) : y \in D(\gamma)\} \subseteq O(R)$. For α in $T(R)$, $\tilde{\pi}(\alpha)$ is in U_{γ_α}, so these sets cover $O(R)$. By proposition 2.2.18 *iv*), if β is a point in $U_{\gamma_1} \cap U_{\gamma_2}$, there is $\gamma_3 \in \Gamma$ such that β is in $U_{\gamma_3} \subseteq U_{\gamma_1} \cap U_{\gamma_2}$, so the $U'_\gamma s$ form a basis for a topology on $O(R)$. Proposition 2.2.18 *ii*) implies the involution $\alpha \mapsto \alpha^{-1}$ is a homeomorphism. *iii*) implies the product is continuous. The groupoid is étale because the r, s restricted to a U_γ are homoeomorphisms (U_γ is the graph of γ, so r,s can be identified homeomorphically with γ, γ^{-1}). Since K is compact, Hausdorff and r, s are local homeomorphisms from $O(R)$ to K, $O(R)$ is locally compact. $O(R)$ is Hausdorff because the clopen set $\{(x, k, y) \in O(R)\}$ for fixed k embeds into K continuously via $(x, k, y) \mapsto (r(x, k, y), s(x, k, y)) = (x, y)$.

Remark 2.2.19. *If we replace the co-cycle of $\gamma = \gamma^U_{(x,n,m,y)}$ (definition 2.2.17) with $d(\gamma) = (n, m)$, and the addition of $c(\gamma_1) + c(\gamma_2)$ with the product $d(\gamma_1) * d(\gamma_2)$ in proposition 2.2.18 and beyond, we get a proof that the involution and product on $T(R)$ are continuous, as well as the "range" and "source" maps $r(\alpha) = \alpha * \alpha^{-1}$, $s(\alpha) = \alpha^{-1} * \alpha$ are local homeomorphisms onto $(\mathbb{N} \cup \{0\}) \times K$. We can then define a normed topological *-algebra $C_c(T(R))$, with formulas for involution and product analogous to that for $C_c(O(R))$. Its completion $\mathcal{T}_R$ is a C^*-algebra which is an extension of $\mathcal{O}_R := C^*(O(R))$, aptly called the Toeplitz extension of $\mathcal{O}_R$.*

The open sets U_γ for γ in Γ such that $c(\gamma) = 0$ form a basis for a topology on the copy of R in $O(R)$ which is the same as the one constructed in Korfanty Ch. 4 [20] , so R is an open subgroupoid of $O(R)$, the "diagonal" of $O(R)$.

Now, we show how to construct a bundle of IFS-like groupoids that parallels the construction of a bundle of AF groupoids from finite graphs in section 2.2.1. Let $\{\gamma_1, ..., \gamma_v\}$ be a single (contractive) matrix affine system on $\mathbb{R}^d$, and let K be its attractor. Choose any subsets $\alpha_j \subseteq \{1, ..., v\}$ for $1 \leq j \leq n$, and let $\mathcal{A} = \{\alpha_j\}_{j=1}^n$. Consider the new single matrix system $\{\beta_{i,j}\}_{i \in \alpha_j, j \leq n}$ on $\mathbb{R}^{d+1}$ where $\beta_{i,j} = \gamma_i \times \beta_j$, $\beta_j(t) = \frac{1}{2n-1}t + \frac{2(j-1)}{2n-1}$. Notice that $\beta_j([0,1]) \subseteq [0,1]$ and the system $\{\beta_j\}$ restricted to its attractor is topologically conjugate to the standard system on the Cantor set $\{1, ...n\}^{\mathbb{N}}$, where, for a point x in $\{1, ...n\}^{\mathbb{N}}$, and $1 \leq j \leq v$, $\beta_j(x) = jx$. Now, let $K_0 = K \times [0,1]$ and for a word $j_1 j_2 ... j_m$ in $\{1, .., n\}^m$, let $K_{j_1...j_m} = \bigcup_{i_1...i_m \in \alpha_{j_1} \times ... \times \alpha_{j_m}} \gamma_{i_1} ... \gamma_{i_m}(K) \times \beta_{j_1,...,j_m}([0,1])$. Now take the union $K_m := \bigcup_{j_1...j_m \in \{1,...,n\}^m} K_{j_1...j_m}$. Notice that this union is disjoint, because $\beta_j([0,1]) \cap \beta_i([0,1]) = \emptyset$ for $i \neq j$. By construction, $K_{m+1} = \bigcup_{i,j} \beta_{i,j}(K_m)$, and $K_{m+1} \subseteq K_m$. Therefore, $K_{\mathcal{A}} := \bigcap_{m \in \mathbb{N}} K_m$ is the attractor for $\{\beta_{i,j}\}$. For a point x in $\{1, ..., n\}^{\mathbb{N}}$, notice that $K_{x_1...x_m} \subseteq K_{x_1...x_{m+1}}$, so $K_x := \bigcap_{m \in \mathbb{N}} K_m$ is a non-empty compact space. Let $\pi : \mathbb{R}^{d+1} \mapsto \mathbb{R}$ be the projection onto the last factor, and restrict the domain to $K_{\mathcal{A}}$. Then, for x in $\{1, ..., n\}^{\mathbb{N}}$ (identified as a point in the attractor for $\{\beta_j\}$), $\pi^{-1}(x) = K_x$, so $\pi : K_{\mathcal{A}} \mapsto \{1, .., n\}^{\mathbb{N}}$ is a continuous surjective factor map from the system $\{\beta_{i,j}\}$ on $K_{\mathcal{A}}$ to the system $\{\beta_j\}$ on $\{1, ..., n\}^{\mathbb{N}}$. Let $R_{\mathcal{A}}$ denote the étale equivalence relation associated to $\{K_{\mathcal{A}}, \beta_{i,j}\}$, and let $R_\pi = \{(x, y) \in R_{\mathcal{A}} : \pi(x) = \pi(y)\}$. Suppose (x, n, n, y) is a point in $T(R_{\mathcal{A}})$ such that $\pi(x) = \pi(y)$. Let $\gamma = \gamma_{(x,n,n,y)}$. Since π is a factor map, $\gamma_{(x,n,n,y)}$ descends to $\gamma_{(\pi(x),n,n,\pi(y))} = id$ under π. Therefore, $U_\gamma \subseteq R_\pi$, so R_π is an open subgroupoid of $R_{\mathcal{A}}$, and is therefore étale with a groupoid bundle structure $\hat{\pi} : R_\pi \mapsto \{1, ..., n\}^{\mathbb{N}}$ via $\hat{\pi}(x, y) = \pi(x) = \pi(y)$. For z in $\{1, ..., n\}^{\mathbb{N}}$, and $x \in K_z$, define

$$\mathcal{F}_z^{-m}(x) := \bigcup_{i_1...i_m \in \alpha_{z_1} \times ... \times \alpha_{z_m}} \gamma_{i_1...i_m}^{-1}(x) \cap K_{\sigma^m z}.$$

The fibres $\hat{\pi}^{-1}(z)$ are the étale equivalence relations

$$R_z = \{(x, y) \in K_z \times K_z : \exists m \in \mathbb{N} : \mathcal{F}_z^{-m}(x) = \mathcal{F}_z^{-m}(y)\}.$$

Notice also that $C^*(R_\pi)$ is the commutant of $\pi^*(C(\{1, ..., n\}^{\mathbb{N}}))$ in $C^*(R_{\mathcal{A}})$.

Section 3

Equivalence

The concept of strong Morita equivalence, defined by Rieffel in [47] (and nowadays just called Morita equivalence) has been important to the theory of C^*-algebras ever since its discovery. Morita equivalent C^*-algebras have isomorphic representation theory, K-theory, ideal structure, and are stably isomorphic when the algebras are σ-unital (see Raeburn and Williams chapters 3.3 and 5.5 for proofs of the first two claims, and the last claim, respectively). Rieffel deduced Mackey's imprimitivity theorem from a specially constructed Morita equivalence [47]. In Alain Connes' theory of non-commutative geometry, two Morita equivalent C^*-algebras are said to have the same underlying non-commutative space [4], making Morita equivalence the natural notion of isomorphism in this setting. Morita equivalent pairs of C^*-algebras are in abundance and appear quite naturally, compared to *-isomorphic pairs. For instance, the C^*-algebra of a foliation is Morita equivalent to the C^*-algebra of any transversal to the foliation. Muhly, Renault and Williams showed these and many more Morita equivalent pairs can be realized first by an equivalence of the corresponding groupoid models [31].

In section 3.3 we present this concept of groupoid equivalence, as well as in section 3.4 the corresponding notion of groupoid bundle equivalence. As an example of the above discussion, as shown by Rieffel in [46], two rotation algebras A_θ and A_φ, defined in section 2.2.2, with irrational parameters θ and ψ in $(0, 1/2]$, are never *-isomorphic, but are Morita equivalent whenever they are in the same $SL(2, \mathbb{Z})$ orbit (acting on $\mathbb{R}$ by Mobius transformations). In section 3.4.2 we present a groupoid equivalence model for these Morita equivalences of Rieffel. The abundance of Morita equivalent pairs makes it a desirable formalism for renormalization, because, given a field of C^*-algebras $(\mathcal{A}_x)_{x \in X}$, an abundance of Morita equivalent pairs $\mathcal{A}_{x'}$, $\mathcal{A}_x$ makes the existence of a dynamical system $\sigma : X_1 \mapsto X$ defined on a large subset of parameters $X_1 \subseteq X$ along with Morita equivalences between $\mathcal{A}_{\sigma(x)}$ and $\mathcal{A}_x$ plausible. Moreover, it seems to be the correct formalism, for if two unital C^*-algebras A and B are Morita equivalent, as in Raeburn and Williams [41] theorem 3.19 there is a larger C^*-algebra C constructed out of the Morita equivalence and full complementary projections P_A, P_B in C such that the cut-downs $P_A C P_A$ and $P_B C P_B$ are $*$-isomorphic to A and B, respectively. Therefore, we can think of two Morita equivalent C^*-algebras as differing only by scale, and the Morita equivalence being a scaling transformation. This section is devoted to presenting Morita equivalence and groupoid equivalence, the analogous notions for D-algebras and groupoid bundles, and the associated constructions that will be useful in the later sections. We will provide examples of Morita equivalence and groupoid equivalence for the D-algebras and groupoid bundles constructed in Section 2.

3.1 Morita Equivalence

A Morita equivalence between two C^*-algebras is formulated in terms of Hilbert C^*-modules (definition [3.1.1]), which are generalizations of Hilbert spaces in the sense that the $\mathbb{C}$-module structure is replaced with an action of a C^*-algebra A and the $\mathbb{C}$-valued inner product replaced with an A valued one. Such objects were first considered by Kaplansky [16] when A is a commutative C^*-algebra. It was not until much later in the work of Paschke [35] and in Rieffel's first paper on Morita equivalence [47] that Hilbert C^*-modules were considered in full generality. They also appear in the formulation of Kasparov's KK-theory in a fundamental way [19].

In definition 3.1.2 we define Morita equivalence in terms of Hilbert C^*-modules. In proposition 3.1.6 we show that inductive limits of inductive sequences of Morita equivalences exist, which will be used to construct examples of Morita equivalent pairs of AF-algebras in section 3.2.1. One can think of a Morita equivalence as a generalized isomorphism, and just like isomorphisms, Morita equivalences can be composed. We briefly present how this is done in the discussion after proposition 3.1.6. We present the concept of a basis of a Hilbert C^*-module in definition 3.1.7, which generalizes the concept of a basis of a Hilbert space, and remark on its usefulness to renormalization (remark 3.1.8).

A Morita equivalence induces many relations between the pair of algebras, but the one we will be most interested in is the induced operator between the trace spaces, defined in proposition 3.1.10, as it will be used in our arguments in section 4. We then present the corresponding notion of Morita equivalence for D-algebras (definition 3.2.1) and provide such an example in 3.2.1 of an equivalence between the section algebra of the field of unital AF-algebras defined in section 2.1.1 and its pull-back by the shift.

For our purposes, if F is a complex vector space and A is a C^*-algebra, a right A-module structure on F is a bi-linear map $F \times A \mapsto F$, with image of (f, a) in $F \times A$ denoted fa, that is associative in the sense that if f is in F and a_1, a_2 are in A, then $(fa_1)a_2 = f(a_1a_2)$. A left A-module structure on F is defined analogously.

Definition 3.1.1. *Let B be a C^*-algebra. A* ***right (resp. left) Hilbert B-module*** *$F \leftarrow B$ (resp. $B \rightarrow F$) is a complex vector space F with a right (resp. left) B-module* *structure together with a map* $\langle \cdot, \cdot \rangle : F \times F \mapsto B$ *that is*

- *(i) right (resp. left) B-linear:* $\langle f_1, f_2 b\rangle = \langle f_1, f_2\rangle b$, *for all* f_1, f_2 *in* F *and* b *in* B,
- *(ii) Hermitian:* $\langle f_1, f_2\rangle^* = \langle f_2, f_1\rangle$, *for all* f_1, f_2 *in* F,
- *(iii) positive:* $\langle f, f\rangle \geq 0$, *for all* f *in* F,
- *(iv) faithful:* $\langle f, f\rangle \neq 0$, *for all non-zero* f *in* F,
- *(v) and* F *is complete with respect to the norm* $\|f\| := \|\sqrt{\langle f, f\rangle}\|$ *(see Raeburn and Williams corollary 2.7* [41] *for the proof that* $\|\cdot\|$ *is a norm).*

We will call $\langle \cdot, \cdot \rangle$ *a* ***B-inner product*** *if it satisfies these conditions. We say a Hilbert* B*-module is* ***full*** *if the image of the* B*-inner product linearly spans a dense sub-algebra in* B*. In general, the closed linear span of* $\langle \cdot, \cdot \rangle$ *is an ideal in* B*.*

Now, we present the definition of Morita equivalence, first considered by Rieffel [47].

Definition 3.1.2. *Let* A *and* B *be* C^**-algebras. An* A*-*B ***imprimitivity bi-module*** *is a complex vector space* F *together with a full left Hilbert* A*-module structure, and a full right Hilbert* B*-module structure satisfying the following two compatibility conditions:*

- *(i) For every* a *in* A, b *in* B, *and* e, f *in* F, *we have* $\langle be, f\rangle_* = \langle e, b^* f\rangle_*$ *and* ${}_*\langle ea, f\rangle = {}_*\langle e, fa^*\rangle$
- *(ii)* ${}_*\langle f_1, f_2\rangle f_3 = f_1\langle f_2, f_3\rangle_*$ *for all* f_1, f_2, f_3 *in* F, *where* ${}_*\langle \cdot, \cdot\rangle, \langle \cdot, \cdot\rangle_*$ *are the* A, B *inner products, respectively.*

We will frequently represent an A*-*B *imprimitivity bi-module* F *schematically as* $A \rightarrow F \leftarrow B$. A *and* B *are said to be* ***Morita equivalent*** *if there exists an* A*-*B *imprimitivity bi-module.*

Definition 3.1.3. *Let* A *and* B *be* C^**-algebras. An* ***isomorphism*** *of* A*-*B *imprimitivity bi-modules* F_1, F_2 *is a surjective linear map* $U : F_1 \mapsto F_2$ *such that* ${}_*\langle U(f_1), U(f_2)\rangle = {}_*\langle f_1, f_2\rangle$ *and* $\langle U(f_1), U(f_2)\rangle_* = \langle f_1, f_2\rangle_*$ *for all* f_1, f_2 *in* F*. These conditions automatically imply* U *is injective, left* A*-linear, and right* B*-linear. We say* F_1 *and* F_2 *are* ***isomorphic*** *if there exists such an isomorphism.*

We now provide a simple example of Morita equivalence. Recall that for an $n \times m$ matrix a with i, j entries, for $1 \leq i \leq n$, $1 \leq j \leq m$, $a_{i,j}$ in $\mathbb{C}$, the transpose a^t is the $m \times n$ matrix with entries $a^t_{j,i} = a_{i,j}$, for $1 \leq i \leq n$, $1 \leq j \leq m$. The conjugate $\overline{a}$ of a is the matrix with the complex conjugate entries of a.

Example 3.1.4. *Let $F = \mathbb{C}^n$, where x in F is viewed as a column vector $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$. Then, the standard $\mathbb{C}$-vector space structure on F is a right $\mathbb{C}$-module structure for F and the inner product $\langle \cdot, \cdot \rangle_* : F \times F \mapsto \mathbb{C}$, defined for x, y in F as $\langle x, y \rangle_* = \overline{x}^t y = \sum_{i=1}^n \overline{x_i} y_i$, is a $\mathbb{C}$-inner product in the sense above. Hence, F is naturally a full Hilbert $\mathbb{C}$-module. A matrix in $M_n(\mathbb{C})$ acts on the left of a column vector by matrix multiplication, and this action defines a left $M_n(\mathbb{C})$-module structure on F that commutes with the above scalar multiplication of $\mathbb{C}$.*

F can also be equipped with a bilinear form ${}_*\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \mapsto M_n(\mathbb{C})$ defined for x, y in $M_n(\mathbb{C})$ as ${}_*\langle x, y \rangle = x\overline{y}^t$, which is the matrix whose i, j entry, for $1 \leq i \leq n$, $1 \leq j \leq n$, is $x_i \overline{y_j}$. By associativity of matrix multiplication, for a in $M_n(\mathbb{C})$, x, y, z in F, we have $(ax)\overline{y}^t = a(x\overline{y}^t)$ and $(x\overline{y}^t)z = x(\overline{y}^t z)$. Therefore, ${}_*\langle \cdot, \cdot \rangle$ is $M_n(\mathbb{C})$ linear in the left-most entry, and is compatible with $\langle \cdot, \cdot \rangle_*$ in the sense above. Thinking of $M_n(\mathbb{C})$ as operators on $\mathbb{C}^n$, and using the Hilbert space characterization of positive operators, we see that, for x in F, $x\overline{x}^t$ is positive and non-zero if x is non-zero. By elementary linear algebra, $\overline{(x\overline{y}^t)}^t = y\overline{x}^t$ for any x any y in F. Since F is a finite dimensional vector space, the norm $\| \sqrt{{}_*\langle x, x \rangle} \|$ is complete. By these above properties, $M_n(\mathbb{C}) \to F$ is a left Hilbert $M_n(\mathbb{C})$-module that is compatible with the right full Hilbert $\mathbb{C}$-module structure $F \leftarrow \mathbb{C}$. It is full, because for a matrix a in $M_n(\mathbb{C})$, $a = \sum_{i \leq n, j \leq n} a_{i,j} e^{i,j} = \sum_{i \leq n, j \leq n} a_{i,j} {}_*\langle e_i, e_j \rangle$, where $e^{i,j}$ is the $n \times n$ matrix whose i, j entry is 1 with all other entries zero, and where e_i is the vector in F whose i^{th} entry is 1 with all other entries zero. Therefore, F is an $M_n(\mathbb{C})$-$\mathbb{C}$ imprimitivity bi-module. In a similar way, an infinite dimensional Hilbert space H can be equipped with a $\mathcal{K}(H)$-$\mathbb{C}$ imprimitivity bi-module structure, where $\mathcal{K}(H)$ are the compact operators on H.

When constructing an imprimivity bi-module, it is often more natural to first define it on a dense subspace F_0 of the vector space F with actions and inner products defined on dense $*$-sub-algebras of C^*-algebras. The following definition of a pre-imprimitivity bi-module appears in Raeburn and William definition 3.9 [41].

Definition 3.1.5. *Given dense $*$-sub-algebras A_0 and B_0, of C^*-algebras A, B, respectively, an A_0-B_0 **pre-imprimitivity bi-module** is a vector space F_0 with left A_0-module and right B_0-module structures, along with bi-linear forms ${}_*\langle \cdot, \cdot \rangle : F_0 \times F_0 \mapsto A_0$, $\langle \cdot, \cdot \rangle_* : F_0 \times F_0 \mapsto B_0$ that satisfy (i)-(iv) in definition 3.1.1, (i)-(ii) in definition 3.1.2, and whose images linearly span a dense set in A_0, B_0, respectively, with the additional assumption that for all a in A_0, b in B_0 and f in F_0, we have ${}_*\langle fb, fb \rangle \leq \|b\|^2 {}_*\langle f, f \rangle$ and $\langle af, af \rangle_* \leq \|a\|^2 \langle f, f \rangle_*$.*

By Raeburn and Williams proposition 3.12 [41], an A_0-B_0 pre-imprimitivity bi-module F_0 completes to an A-B imprimitivity bi-module F, Where A_0 and B_0 are contained the C^*-algebras A, B, respectively, as dense $*$-sub-algebras.

We now show that inductive limits of imprimitivity bi-modules exists. If $\gamma_n : C_n \mapsto C_{n+1}$ for n in $\mathbb{N}$ is a sequence of morphisms in some category, and k, m are in $\mathbb{N}$ such that $k > m$, let $\gamma_{k,m} := \gamma_{k-1} \circ \gamma_{k-2} \circ \ldots \circ \gamma_m$.

Proposition 3.1.6. *Let A_n, B_n, n in $\mathbb{N}$, be a sequence of C^*-algebras, and let $A_n \to F_n \leftarrow B_n$ be a sequence of A_n-B_n imprimitivity bi-modules. Suppose we have *-homomorphisms $\varphi_n : A_n \mapsto A_{n+1}$, $\psi_n : B_n \mapsto B_{n+1}$ and linear maps $\phi_n : F_n \mapsto F_{n+1}$, for n in $\mathbb{N}$, such that*

(i) $\phi_n(af) = \varphi_n(a)\phi_n(f)$ *and* $\phi_n(fb) = \phi_n(f)\psi_n(b)$ *for all a in A_n, b in B_n and f in F_n*

(ii) $\langle \phi_n(f_1), \phi_n(f_2)\rangle_* = \psi_n(\langle f_1, f_2\rangle_*)$ *and* ${}_*\langle \phi_n(f_1), \phi_n(f_2)\rangle = \varphi_n({}_*\langle f_1, f_2\rangle)$ *for all f_1, f_2 in F_n.*

Then, there is a Banach space F and a sequence of linear maps $\theta_n : F_n \mapsto F$ such that $\theta_{n+1} \circ \phi_n = \theta_n$ and $\|\theta_n(f)\| = \lim_{k\to\infty} \|\phi_{k,n}(f)\|$, for all n in $\mathbb{N}$, for all f in F_n. We will call such a sequence $\{\theta_n, F\}_{n\in\mathbb{N}}$ a Banach space inductive limit of $\{\phi_n\}_{n\in\mathbb{N}}$. If $\{\theta'_n, F'\}_{n\in\mathbb{N}}$ is another Banach space inductive limit of $\{\phi_n\}_{n\in\mathbb{N}}$, then there is an isometric isomorphism $\gamma : F \mapsto F'$ such that $\gamma \circ \theta_n = \theta'_n$ for all n in $\mathbb{N}$.

Given any inductive limits $\{\mu_n, A\}_{n\in\mathbb{N}}$, $\{\theta_n, F\}_{n\in\mathbb{N}}$, $\{\eta_n, B\}_{n\in\mathbb{N}}$ of the respective sequences $\{\varphi_n\}_{n\in\mathbb{N}}$, $\{\phi_n\}_{n\in\mathbb{N}}$, $\{\psi_n\}_{n\in\mathbb{N}}$, there is an induced A-B bi-module structure on F defined in the following way: For any n, m in $\mathbb{N}$, a in A_n, f in F_n, g in F_m and b in B_m, let

- $\theta_n(f)\eta_m(b) := \theta_{n+m}(\phi_{n+m,n}(f)\psi_{n+m,m}(b))$
- $\mu_n(a)\theta_m(g) := \theta_{n+m}(\varphi_{n+m,n}(a)\phi_{n+m,m}(g))$
- $\langle \theta_n(f), \theta_m(g)\rangle_* := \eta_{n+m}(\langle \phi_{n+m,n}(f), \phi_{n+m,m}(g)\rangle_*)$
- ${}_*\langle \theta_n(f), \theta_m(g)\rangle := \mu_{n+m}({}_*\langle \phi_{n+m,n}(f), \phi_{n+m,m}(g)\rangle)$

Then, the above left, right actions and inner products are well-defined and extend to an A-B imprimitivity bi-module structure on F. Also, the inductive limit norm on F is the same as the norm(s) coming from the inner-product(s).

Proof. For a proof of the first part of the proposition claiming existence and uniqueness of a Banach space inductive limit $\{\theta_n, F\}_{n\in\mathbb{N}}$ of $\{\phi_n\}_{n\in\mathbb{N}}$, see Rordam, Larsen and Lausten proposition 6.2.4 [48]. Their proof shows the existence of inductive limits for C*-algebras, but if the objects are replaced with Banach spaces and the connecting *-homomorphisms replaced with norm-decreasing linear maps $\phi_n : F_n \mapsto F_{n+1}$, then the proof carries through. To see that the ϕ_n are norm decreasing, let f be in F_n. Then,

$\|\phi_n(f)\| := \|\sqrt{\langle \phi_n(f), \phi_n(f)\rangle_*}\| = \|\psi_n(\sqrt{\langle f, f\rangle_*})\|$. Since ψ_n is a *-homomorphism, it is norm-decreasing. Therefore, $\|\psi_n(\sqrt{\langle f, f\rangle_*})\| \leq \|\sqrt{\langle f, f\rangle}\| = \|f\|$. Hence, $\|\phi_n\| \leq 1$.

We first show the right $B_\infty := \bigcup_{n\in\mathbb{N}} \eta_n(B_n)$ action on $F_\infty := \bigcup_{n\in\mathbb{N}} \theta_n(F_n)$ is well-defined. For b in B_m, f in F_n and $m' \geq m$, $n' \geq n$,

$$\theta_{n'}(\phi_{n',n}(f))\eta_{m'}(\psi_{m',m}(b)) = \theta_{n'+m'}(\phi_{n'+m',n'}(\phi_{n',n}(f))\psi_{n'+m',m'}(\psi_{m',m}(b))).$$

By hypothesis,

$$\phi_{n'+m',n'}(\phi_{n',n}(f))\psi_{n'+m',m'}(\psi_{m',m}(b)) = \phi_{n'+m',n+m}(\phi_{n+m,n}(f)\psi_{n+m,m}(b)).$$

Since $\theta_{n'+m'} \circ \phi_{n'+m',n+m} = \theta_{n+m}$, it follows that

$$\theta_{n'+m'}(\phi_{n'+m',n'}(\phi_{n',n}(f))\psi_{n'+m',m'}(\psi_{m',m}(b))) = \theta_{n+m}(\phi_{n+m,n}(f)\psi_{n+m,m}(b)).$$

Therefore, $\theta_{n'}(\phi_{n',n}(f))\eta_{m'}(\psi_{m',m}(b)) = \theta_n(f)\eta_m(b)$, so the action is well-defined.

The axioms of a B_∞-module structure on F_∞ and (i)-(iii) in definition 3.1.1 hold for the B_∞-valued bi-linear form on F_∞ because it and the B_∞-module structure are built out of the B_n-inner products on F_n. To verify any of these axioms, one just views the elements under consideration in B_n and F_n for fixed n, and uses the fact that F_n is a Hilbert B_n-module.

To see that the induced pre-norm on F_∞ from $\langle\cdot,\cdot\rangle_*$ is faithful, let $f = \theta_n(f')$ for f' in F_n, n in $\mathbb{N}$. Then,

$$\begin{aligned}\|f\| &:= \lim_{k\to\infty} \|\sqrt{\langle \phi_{k,n}(f'), \phi_{k,n}(f')\rangle_*}\| = \lim_{k\to\infty} \|\psi_{k,n}(\sqrt{\langle f', f'\rangle_*})\| = \\ &= \|\eta_n(\sqrt{\langle f', f'\rangle_*})\| = \|\sqrt{\langle f, f\rangle_*}\|.\end{aligned}$$

Therefore, the induced norm on F_∞ from $\langle\cdot,\cdot\rangle_*$ is equal to the norm on the Banach space F, and is therefore faithful.

Similarly, it can be shown the left A_∞ action on F_∞ and the bi-linear form ${}_*\langle\cdot,\cdot\rangle$ satisfy (i)-(iv) in definition 3.1.1 and that the induced norm from ${}_*\langle\cdot,\cdot\rangle$ is equal to the Banach space norm on F_∞.

We show (i) holds in definition 3.1.2 for the above pair of actions. The verification of (ii) follows by a similar calculation. Let $a = \mu_n(a')$, $f = \theta_m(f')$, $b = \eta_k(b')$ for a' in A_n, f' in F_m, and b' in B_k. Then,

$$\begin{aligned}&(af)b = \theta_{n+m+k}(\phi_{n+m+k,n+m}(\varphi_{n+m,n}(a')\phi_{n+m,m}(f'))\psi_{n+m+k,k}(b')) = \\ &\theta_{n+m+k}(\varphi_{n+m+k}(a')\phi_{n+m+k,n+m}(\phi_{n+m,m}(f'))\psi_{n+m+k,k}(b')) = \\ &\theta_{n+m+k}(\varphi_{n+m+k}(a')\phi_{n+m+k,m}(f')\psi_{n+m+k,k}(b')).\end{aligned}$$

Similarily,

$$a(fb) = \theta_{n+m+k}(\varphi_{n+m+k,n}(a')\phi_{n+m+k,m+k}(\phi_{m+k,m}(f')\psi_{m+k,k}(b'))) =$$
$$\theta_{n+m+k}(\varphi_{n+m+k,n}(a')\phi_{n+m+k,m+k}(\phi_{m+k,m}(f'))\psi_{n+m+k,k}(b')) =$$
$$\theta_{n+m+k}(\varphi_{n+m+k,n}(a')\phi_{n+m+k,m}(f')\psi_{n+m+k,k}(b')).$$

Therefore, $(af)b = a(fb)$.

Since the norms induced from ${}_*\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_*$ are equal, it follows that for any a in A_∞, b in B_∞ and f in F_∞, we have ${}_*\langle fb, fb\rangle \leq \|b\|^2{}_*\langle f, f\rangle$ and $\langle af, af\rangle_* \leq \|a\|^2\langle f, f\rangle_*$. Therefore, F_∞ is a A_∞-B_∞ pre-imprimitivity bi-module. By Raeburn and Williams proposition 3.12 [41], F_∞ completes to an A-B imprimitivity bi-module F'. Since the induced norms are equal to the Banach space norm on F restricted to F_∞, F' is isometrically isomorphic to F, so we may assume $F = F'$. □

Given two imprimitivity bi-modules $A_1 \to F_1 \leftarrow B$, $B \to F_2 \leftarrow A_2$, we outline how to compose them into an imprimitivity bi-module $A_1 \to F_1 \otimes_B F_2 \leftarrow A_2$: start with the $\mathbb{C}$-module tensor product $F_1 \odot F_2$ (see Dummite and Foote Section 10.4 [9] for the definition of tensor products of modules) and let N be the closed A_1-A_2 sub-module generated by the set

$$\{f_1 b \odot f_2 - f_1 \odot b f_2 : \ f_1 \in F_1, \ f_2 \in F_2, \ b \in B\}.$$

Let $F_1 \odot_B F_2 := F_1 \odot F_2/N$ denote the quotient module, and denote $f_1 \otimes_B f_2$ to be the image of the basic tensor $f_1 \odot f_2$ under the quotient map. $F_1 \odot_B F_2$ has a left A_1-module structure and right A_2-module structure inherited from the left, right module structures on F_1, F_2, respectively. $F_1 \odot_B F_2$ can be equipped with left A_1 and right A_2 inner products, defined on basic tensors as

- ${}_{12}\langle f_1 \otimes_B f_2, g_1 \otimes_B g_2\rangle = {}_*\langle f_1{}_*\langle f_2, g_2\rangle, g_1\rangle$ for all f_1, g_1 in F_1, f_2, g_2 in F_2
- $\langle f_1 \otimes_B f_2, g_1 \otimes_B g_2\rangle_{12} = \langle f_2, \langle f_1, g_1\rangle_* g_2\rangle_*$ for all f_1, g_1 in F_1, f_2, g_2 in F_2.

By Raeburn and Williams proposition 3.16 [41] these inner-products give $F_1 \odot_B F_2$ the structure of an A_1-A_2 pre-imprimitivity bi-module (definition 3.9 [41]) and can therefore be completed (proposition 3.12 [41]) to an A_1-A_2 imprimitivity bi-module $A_1 \to F_1 \otimes_B F_2 := \overline{F_1 \odot_B F_2} \leftarrow A_2$. $F_1 \otimes_B F_2$ is called the **balanced tensor product** of F_1 and F_2. We present the concept of a basis for a Hilbert bi-module.

Bases for Hilbert C^*-modules are central to Pimsner and Popa's work on Jones' index theory for subfactors [38], and the index theory for C^*-algebras developed by Kajiwara and Watatani [15]. In their considerations, a basis is what we regard as a finite basis, and the situation when one might need an infinite basis, for instance when algebras defining the bi-module structure are not unital, is not considered. The extension to infinite basis is natural enough.

Definition 3.1.7. *Let $A \to F \leftarrow B$ be an A-B imprimitivity bi-module (definition 3.1.2). A* ***left basis for*** *F, denoted $\mathcal{B}$ is a collection of vectors $\varepsilon_{i,j}$ in F, indexed by i in $\mathbb{N}$ and $j \leq n_i$, j, n_i in $\mathbb{N}$, such that the sequence $e_i = \sum_{j \leq n_i} {}_*\langle \varepsilon_{i,j}, \varepsilon_{i,j} \rangle$ satisfies $e_i \leq 1_A$ for all i in $\mathbb{N}$ and forms an approximate unit for A. We call a finite collection of vectors $\mathcal{B} = \{\varepsilon_j\}_{j=1}^n$ a* ***finite left basis*** *for F if $\sum_{j=1}^n {}_*\langle \varepsilon_j, \varepsilon_j \rangle = 1$.*

Remark 3.1.8. *Assume $A \to F \leftarrow B$ is an A-B imprimitivity bi-module, where A and B are unital, and $\{\varepsilon_i\}_{i=1}^n$ is a finite left basis for F. Then, as in Rieffel proposition 2.1* [46]*, we can consider the element z in $M_n(B)$ defined entry-wise as $z_{i,j} = \langle \varepsilon_i, \varepsilon_j \rangle_*$ for $i \leq n$, $j \leq n$ and the map $\phi_z : A \mapsto M_n(B)$ where, for a in A, $(\phi_z(a))_{i,j} = \langle \varepsilon_i, a\varepsilon_j \rangle_*$, $i \leq n$, $j \leq n$. The proof of Rieffel proposition 2.1* [46] *shows that z is a full projection and ϕ_z is an injective $*$-homomorphism of A onto $zM_n(B)z$. This projection is the analog of the characteristic function χ_Y of a first return domain $Y \subseteq X$ for a dynamical system $T : X \mapsto X$. We will elaborate more on this analogy in remark 3.3.15 after we present groupoid equivalences and their induced imprimitivity bi-modules. Clearly though, the concept of a basis will be an important one to us.*

It is often useful to think of an A-B imprimitivity bi-module F as a generalized isomorphism from A to B. Moreover, the above construction shows how, by choosing a basis, F can be interpreted as a $*$-isomorphism $\phi_z : A \mapsto zM_n(B)z$, with the full projection z representing the "scale" of A relative to $M_n(B)$. The next definition is an attempt to qualitatively describe the scaling properties of F.

Definition 3.1.9. *We say an A-B imprimitivity bi-module $A \to F \leftarrow B$ is* ***bounded*** *if there is a number $M > 0$ and a left basis $\mathcal{B} = \{\varepsilon_{i,j} \in F : i \in \mathbb{N},\ j \leq n_i\}$ for F such that for all i in $\mathbb{N}$, $\sum_{j \leq n_i} \langle \varepsilon_{i,j}, \varepsilon_{i,j} \rangle_* \leq M 1_B$. We will call F a* ***contraction*** *if $M \leq 1$, and a* ***strict contraction*** *if there is a left basis $\mathcal{B}$ and a non-zero positive element b in B such that for all i in $\mathbb{N}$, $\sum_{j \leq n_i} \langle \varepsilon_{i,j}, \varepsilon_{i,j} \rangle_* \leq 1_B - b$*

Given an A-B imprimitivity bi-module, the following construction of an operator between trace spaces will be a very useful tool for us in section 4. The construction of this operator when A and B are unital is done in Rieffel [46]. The non-unital case is considered by Combes and Zettl [3]. In the case where the algebra A only acts on the Hilbert B-module F by bounded B-linear and adjointable operators is done in Laca and Neshveyev [22]. We will follow the treatment of this operator there.

First, recall from the discussion preceding proposition 2.1.11 that $T^1(A)$ denotes the set of normed one traces. Similarly, denote $T^{<\infty}(A)$ to be the set of finite traces, i.e., positive linear functionals on A satisfying the trace condition. Second, if $\mathcal{B} = \{\varepsilon_{i,j} \in F : i \in \mathbb{N},\ j \leq n_i\}$ is a left basis for $A \to F \leftarrow B$, for i in $\mathbb{N}$, let $\Delta_i(\mathcal{B}) : A \mapsto B$ be the linear map defined, for a in A, as $\Delta_i(\mathcal{B})(a) = \sum_{j \leq n_i} \langle \varepsilon_{i,j}, a\varepsilon_{i,j} \rangle_*$. If $\mathcal{B}$ is a finite left basis, define $\Delta(\mathcal{B}) : A \mapsto B$, for a in A, as $\Delta(\mathcal{B})(a) = \sum_{j=1}^n \langle \varepsilon_j, a\varepsilon_j \rangle_*$.

Proposition 3.1.10. *Let $A \to F \leftarrow B$ be a bounded A-B imprimitivity bi-module. Suppose τ is in $T^{<\infty}(A)$ and $\mathcal{B}$ is a left basis for F (definition 3.1.7). For a in A, define $S\tau(a) := \lim_{i\to\infty} \tau(\Delta_i(\mathcal{B})(a))$ (which exists by Laca and Neshveyev theorem 1.1 (ii)* [22]*). Then,*

- *The limit $S\tau(a)$ exists and is independent of the choice of $\mathcal{B}$ (theorem 1.1 (iii)* [22]*).*
- *$S\tau$ is in $T^{<\infty}(A)$. (automatic from theorem 1.1 (iii)* [22] *and the fact that F is bounded).*

We will call S the ***trace operator*** *of F and it will be sometimes denoted S_F in case we need to keep track of F. In the absence of a basis, the trace operator can be defined by a more general formula. See Laca and Neshveyev* [22] *theorem 1.1 for this general formula.*

We cannot stress enough the importance of the first bullet point above. This fact will be used to show in section 4.1.1 that if by succesively iterating a renormalization procedure, starting at a parameter algebra A_x, we can find arbitarily small "non-commutative first return domains" (see remark 3.3.15) that are uniformly spaced out and cover the whole algebra, then A_x will have at most one trace. In our arguments we will never have to compute the trace operator, and will only use the properties listed above. For clarity though, we provide the example of how the trace on $M_n(\mathbb{C})$ arises from the trace operator coming from the $M_n(\mathbb{C})$-$\mathbb{C}$ imprimitivity bi-module $F = \mathbb{C}^n$ of example 3.1.4.

Example 3.1.11. *Let $M_n(\mathbb{C}) \to F \leftarrow \mathbb{C}$ be as in example 3.1.4. The standard orthonormal basis $\{e_i\}_{i=1}^n$ basis for $\mathbb{C}^n$ is a left basis in the sense of definition 3.1.7 because $1_{M_n(\mathbb{C})} = \sum_{i=1}^n e^{i,i} = \sum_{i=1}^n {}_*\langle e_i, e_i\rangle$. Notice that $id_{\mathbb{C}} : \mathbb{C} \mapsto \mathbb{C}$ is a trace for $\mathbb{C}$. Denote Tr to be the trace $S_F(id_{\mathbb{C}})$ on $M_n(\mathbb{C})$. For a in $M_n(\mathbb{C})$, we have $Tr(a) = \sum_{j=1}^n \langle e_i, ae_i\rangle_* = \sum_{j=1}^n a_{i,i}$, so Tr is the standard trace on $M_n(\mathbb{C})$.*

3.2 $C_0(X)$-equivalence

The renormalization procedures considered in this thesis will transform a D-algebra to its pull-back by its renormalization dynamics. To make sense of this, we will need a concept of Morita equivalence between D-algebras. Such a concept is considered in Raeburn and Williams definition 5.6 [41], and is defined as follows.

First, suppose F is a right Hilbert B-module, and (B, β) is a D-algebra. Then, F inherits an action of D for f in F, g in D by the formula

$$fg := \lim_{\lambda} f(e_\lambda g), \ \{e_\lambda\}_{\lambda\in\Lambda} \text{ an approximate unit in } B.$$

A similar formula holds if F is a left Hilbert B-module.

Definition 3.2.1. *Let (A,α), (B,β) be D-algebras. An (A,α)-(B,β)* ***imprimitivity bi-module****, denoted $(A,\alpha) \to F \leftarrow (B,\beta)$, is an A-B imprimitivity bi-module $A \to F \leftarrow B$ (definition 3.1.2) such that $gf = fg$ for all f in F, g in D. (A,α) and (B,β) are called D-**equivalent** if there exists an (A,α)-(B,β) imprimitivity bi-module.*

We record a fact about D-equivalences:

Proposition 3.2.2. *Let $(A,\alpha) \to F \leftarrow (B,\beta)$ be an (A,α)-(B,β) imprimitivity bi-module, and assume $D \subseteq C_b(X)$. For x in X, define F^x to be the closed A-B sub-module generated by the set $\{fg \in F : f \in F,\ g \in D,\ g(x) = 0\}$. For f_1, f_2 in F, define ${}_x\langle f_1, f_2\rangle := {}_*\langle f_1, f_2\rangle(x)$ and $\langle f_1, f_2\rangle_x := \langle f_1, f_2\rangle_*(x)$. Then, $F^x = \{f : {}_x\langle f, f\rangle = 0\} = \{f : \langle f, f\rangle_x = 0\}$. Let $F_x := F/F^x$ and for f in F, denote $f(x)$ to be the image of f under the quotient map. F_x has the structure of an A_x-B_x imprimitivity bi-module with products $a(x)f(x) := af(x)$, $f(x)b(x) := fb(x)$ and inner products ${}_x\langle\cdot,\cdot\rangle$, $\langle\cdot,\cdot\rangle_x$.*

Proof. The proof is straightforward; see Raeburn and Williams lemma 5.12 [41] for the details. ☐

3.2.1 Equivalence for AF Algebras

Fix k in $\mathbb{N} \cup \{0\}$, let X be as in section 2.1.1, $\sigma^k : X \mapsto X$ the k^{th} shift on X, and $(\sigma^{*k}\mathcal{A}, \sigma^{*k}\alpha)$ the $C_u(X)$-algebras constructed in the same section, with fibres $(\sigma^{*k}\mathcal{A})_x = A_{\sigma^k x}$, for all x in X. In this section, we construct a $(\sigma^{*k}\mathcal{A}, \sigma^{*k}\alpha)$- $(\mathcal{A}, \alpha)$ imprimitivity bi-module E^k. First, we construct the finite stages of the fibres E^k_x, for any x in X. Then, we take their inductive limits, and see how the limits glue together to form E^k.

For a finite path $z = z_1 z_2 ... z_n$ in P^n, and $n \geq k$, recall that we denote $z_{k+1}z_{k+2}...z_n$ by $\sigma^k z$, and this will not cause confusion as the length of path will always be specified. Also, denote $B(H,K)$ to be the bounded operators from a Hilbert space H to another Hilbert space K, and recall the definitions of the Hilbert spaces $H(z)$ in section 2.1.1. For $n \geq k$ and a finite path z in P^n, let

$$E^k(z) := \{T \in B(H(z), H(\sigma^k z)) : P^v_{\sigma^k z} T - T P^v_z = 0 \ \forall v \in V_n(z)\}.$$

More concretely, $E^k(z) = \bigoplus_{v \in V_n(z)} B(H^v(z), H^v(\sigma^k z))$. For a in $A(z)$ and T in $E^k(z)$ let Ta denote the composition as operators $a : H(z) \mapsto H(z)$ and $T : H(z) \mapsto H(\sigma^k z)$. Since $P^v_z a = a P^v_z$ for all v in $V_n(z)$, $0 = (P^v_{\sigma^k z} T - T P^v_z)a = P^v_{\sigma^k z} T a - T a P^v_z$ for all v in $V_n(z)$. Therefore, Ta is in $E^k(z)$, so the operator composite defines a right $A(z)$-module structure on $E^k(z)$. Now, for operators S, T in $E^k(z)$, let $\langle S, T\rangle_* = S^*T$ be the composite of T with the operator adjoint of S. For all vertices v in $V_n(z)$, $S^*TP^v_z = S^*P^v_{\sigma^k z}T = (P^v_{\sigma^k z}S)^*T = (SP^v_z)^*T = P^v_z S^*T$. Therefore, $\langle S, T\rangle_*$ is in $A(z)$.

We check that $\langle\cdot,\cdot\rangle_*$ defines an $A(z)$-valued inner product: for a in $A(z)$ and S, T in $E^k(z)$,

(i) $(S^*T)a = S^*(Ta)$ by associativity of operator composition

(ii) $(S^*T)^* = T^*S$ since for all vectors ψ and φ in $H(z)$, $\langle S^*T\psi,\varphi\rangle = \langle T\psi, S\varphi\rangle = \langle\psi, T^*S\varphi\rangle$

(iii) $T^*T \geq 0$ because for all vectors ψ in $H(z)$, $\langle T^*T\psi,\psi\rangle = \langle T\psi, T\psi\rangle \geq 0$

(iv) $T^*T \neq 0$ if $T \neq 0$ because if $T\psi \neq 0$ for some vector ψ in $H(z)$, then $\langle T^*T\psi,\psi\rangle = \langle T\psi, T\psi\rangle \neq 0$

(v) $E^k(z)$ is complete with respect to the norm $\|\sqrt{\langle T,T\rangle_*}\|$ because $E^k(z)$ is a finite dimensional vector space.

Therefore, $E^k(z)$ equipped with the above right $A(z)$-module action and bi-linear form $\langle\cdot,\cdot\rangle$ is a right Hilbert $A(z)$-module. For paths p and q in $P^n(z)$ such that $r(p) = r(q)$, let Let $S_{\sigma^k p,q}$ be the operator in $E^k(z)$ defined on a basis vector δ_r, r in $P^n(z)$ by

$$S_{\sigma^k p,q}(\delta_r) = \begin{cases} 0 & \text{if } r \neq q \\ \delta_{\sigma^k p} & \text{if } r = q \end{cases}$$

Then, for paths p, q in $P^n(z)$ such that $r(p) = r(q)$, $\langle S_{\sigma^k(p),q}, S_{\sigma^k(p),p}\rangle_*$ is the rank one operator sending δ_p to δ_q. Such operators (ranging over p, q) linearly span $A(z)$, so it follows that $E^k(z)$ is a full Hilbert $A(z)$-module. Similarly, for a in $A(\sigma^k z)$ and T in $E^k(z)$, the operator composite aT defines a left $A(\sigma^k z)$-module. For S and T in $E^k(z)$, let ${}_*\langle S,T\rangle = ST^*$. By similar observations as above, this defines an inner product on $E(z)$ which is full and compatible with the left $A(\sigma^k z)$-module structure. $A(\sigma^k z) \to E^k(z)$ and $E^k(z) \leftarrow A(z)$ are compatible because of associativity of operator composites: for operators a in $A(\sigma^k z)$, b in $A(z)$ and S, T, U in $E^k(z)$, we have

(i) $(aT)b = a(Tb)$, and

(ii) $(ST^*)U = S(T^*U)$.

Now, let e be an edge in $\mathcal{M}$ such that $r(z) = s(e)$. Let $\phi_{z,z*e} : E^k(z) \mapsto E^k(z*e)$ be the linear map defined for T in $E^k(z)$ as $\phi_{z,z*e}(T) = \sum_{f\in E(e)} R_f(\sigma^k z)T(R_f(z))^*$. Since the partial isometries $R_f(Z)$, f in $E(e)$ have pair-wise orthogonal ranges, for T in $E^k(z)$ and b in $A(z)$ we have

$$\phi_{z,z*e}(T)\varphi_{z,z*e}(b) = (\sum_{f\in E(e)} R_f(\sigma^k z)T(R_f(z))^*)(\sum_{f'\in E(e)} R_{f'}(z)b(R_{f'}(z))^*) =$$

$$= \sum_{f\in E(e)} R_f(\sigma^k z)TP_z^{s(f)}b(R_f(z))^*.$$

Since b is in $A(z)$, for f in $E(e)$, $P_z^{s(f)} b(R_f(z))^* = bP_z^{s(f)}(R_f(z))^* = b(R_f(z))^*$. Therefore, $\sum_{f \in E(e)} R_f(\sigma^k z) T P_z^{s(f)} b(R_f(z))^* = \phi_{z,z*e}(Tb)$. Hence, $\phi_{z,z*e}(T)\varphi_{z,z*e}(b) = \phi_{z,z*e}(Tb)$. Again, Using the pair-wise orthogonality of the ranges of $R_f(\sigma^k z)$, f in $E(e)$, For S and T in $E^k(z)$, we have

$$\langle \phi_{z,z*e}(S), \phi_{z,z*e}(T) \rangle_* = \sum_{(f,f') \in E(e)^2} R_f(z) S^* (R_f(\sigma^k z))^* R_{f'}(\sigma^k z) T (R_f(z))^* =$$
$$\sum_{f \in E(e)} R_f(z) S^* P_{\sigma^k z}^{s(f)} T (R_f(z))^*.$$

Since $P_{\sigma^k z}^{s(f)} T = T P_z^{s(f)}$ for f in $E(e)$ and $P_z^{s(f)}(R_f(z))^* = (R_f(z))^*$, it follows that

$$\langle \phi_{z,z*e}(S), \phi_{z,z*e}(T) \rangle_* = \sum_{f \in E(e)} R_f(z) S^* T (R_f(z))^* = \varphi_{z,z*e}(\langle T, S \rangle_*).$$

Therefore, $\langle \phi_{z,z*e}(S), \phi_{z,z*e}(T) \rangle_* = \varphi_{z,z*e}(\langle S, T \rangle_*)$. Similarly, for a in $A(\sigma^k z)$ and S, T in $E^k(z)$, $\phi_{z,z*e}(aT) = \varphi_{z,z*e}(a)\phi_{z,z*e}(T)$ and ${}_*\langle \phi_{z,z*e}(S), \phi_{z,z*e}(T) \rangle = \varphi_{\sigma^k z, \sigma^k z*e}({}_*\langle S, T \rangle)$.

Now, for a point x in X and $n \geq k$ let $\phi_{n,x} = \phi_{x_{[1,n]}, x_{[1,n+1]}}$. Then, by the above calculations, the inductive sequences $\{\varphi_{n-k,\sigma^k x}\}_{n \geq k}$, $\{\phi_{n,x}\}_{n \geq k}$, $\{\varphi_{n,x}\}_{n \geq k}$ form an inductive sequence of the $E^k(x_{[1,n]})$ as imprimitivity bi-modules. Let $\{\mu_{n-k,\sigma^k x}, A_{\sigma^k x}\}_{n \geq k}$, $\{\theta_{n,x}, E_x^k\}_{n \geq k}$, $\{\mu_{n,x}, A_x\}_{n \geq k}$ be the inductive limits of the respective sequences, and equip E_x^k with the $A_{\sigma^k x}$ - A_x imprimitivity bi-module structure as in proposition 3.1.6.

Equip $E_n^k := \Pi_{z \in P^n} E^k(z)$ with the $\sigma^{k*}A_n$-A_n bi-module structure defined factor-wise, for a in $\sigma^{k*}A_n$, S, T in E_n^k, b in A_n, and z in P^n as

- $aT(z) = a(z)T(z)$, $Tb(z) = T(z)b(z)$,
- $\langle S, T \rangle_*(z) = \langle S(z), T(z) \rangle_*$, ${}_*\langle S, T \rangle_*(z) = {}_*\langle S(z), T(z) \rangle$.

Let $\varphi_{n,\sigma^k} : \sigma^{k*}A_n \mapsto \sigma^{k*}A_{n+1}$ and $\varphi_n : A_n \mapsto A_{n+1}$ be as in section 2.1.1 and define $\phi_n : E_n^k \mapsto E_{n+1}^k$ pointwise, for T in E_n^k, by

$$\phi_n(T)(z * e) = \varphi_{z,z*e}(T(z)) \text{ for all } z \text{ in } P^n,\ e \text{ in } \mathcal{M} \text{ such that } r(z) = s(e).$$

Then from the above (factor-wise) calculations, $\{\varphi_{n,\sigma^k}\}_{n \geq k}$, $\{\phi_n\}_{n \geq k}$, $\{\varphi_n\}_{n \geq k}$ is an inductive sequence of imprimitivity bi-modules. Consider the map $\theta_n : E_n^k \mapsto \Pi_{x \in X} E_x^k$, defined factor-wise, for all T in E_n^k and x in X, by $\pi_x(\theta_n(T)) = \theta_{n,x}(T(x_{[1,n]}))$. Then, the θ_n are injective and $\theta_{n+1} \circ \phi_n = \theta_n$, so $E^k := \overline{\bigcup_{n \geq 1}(\theta_n(E_n))}$ along with $\{\theta_n : E_n^k \mapsto E^k\}_{n \geq 1}$ is an inductive limit of $\{\phi_n\}_{n \geq 1}$. Equip E^k with the $\sigma^{k*}\mathcal{A}$ - $\mathcal{A}$ imprimitivity bi-module structure as in proposition 3.1.6. Then, it is easy to show, for all a in $\sigma^{k*}\mathcal{A}$, S, T in E^k, and b in $\mathcal{A}$, that $\pi_x(aT) = \pi_x(a)\pi_x(T)$, $\pi_x(Tb) = \pi_x(T)\pi_x(b)$, $\pi_x(\langle S, T \rangle_*) = \langle \pi_x(S), \pi_x(T) \rangle_*$, and $\pi_x({}_*\langle S, T \rangle) = \langle \pi_x(S), \pi_x(T) \rangle_*$ for all x in X, so the $\sigma^{k*}\mathcal{A}$-$\mathcal{A}$ bi-module structure on E^k is given point-wise by the Hilbert $A_{\sigma^k x}$-A_x bi-module structure on E^k_x, for all x in X. Since the $C_u(X)$ structures on $\sigma^{k*}A$ and A are also point-wise multiplication, it follows that E^k is also a $(\sigma^{k*}\mathcal{A}, \sigma^{k*}\alpha)$-$(\mathcal{A}, \alpha)$ imprimitivity bi-module.

3.3 Groupoid Equivalence

In Muhly, Renault and Williams [31] a notion of Morita equivalence between groupoids was developed, and was shown that many known examples of Morita equivalent pairs of C^*-algebras have such a groupoid model. Here, we present this notion of groupoid equivalence between étale groupoids, and prove that the composition of étale groupoid equivalences (definition 3.3.4, and the construction after proposition 3.3.5) is an étale groupoid equivalence. We then present how an imprimitivity bi-module arises from a groupoid equivalence (theorem 3.3.14), and prove the composition of groupoid equivalences is a groupoid model for the balanced tensor product defined in section 3.1. We define the notion of an equivalence between groupoid bundles over a space X in definition 3.4.1 and show that the induced Morita equivalence are $C_0(X)$-equivalences. The latter sections are devoted to examples of groupoid bundle equivalences between the groupoid bundles introduced in Section 2. We assume all topological groupoids mentioned are locally compact Hausdorff and étale with paracompact unit spaces. The following two definitions are due to Muhly, Renault and Williams [31].

Definition 3.3.1. *Let G be a topological groupoid and E a topological space. A **left (resp. right)** G **action on** E, denoted $G \to E$ ($G \leftarrow E$) consists of a continuous open surjection $\rho : E \mapsto G^0$ (resp. σ), and a continuous product map $p : G \circ E \mapsto E$ ($p : E \circ G \mapsto E$), where $G \circ E = \{(g, e) \in G \times E : s(g) = \rho(e)\}$ ($E \circ G = \{(e, g) : r(g) = \sigma(e)\}$). The image $p(g, e)$ is denoted ge (resp. $p(e, g) = eg$) and is required to be*

(i) associative: if (g_1, g_2) is in $G \circ G$ and (g_2, e) in $G \circ E$, then (g_1g_2, e), (g_1, g_2e) are in $G \circ E$, and $g_1(g_2e) = (g_1g_2)e$

(ii) invertible: if u is in G^0, and $\rho(e) = u$, then $ue = e$
(a right action must also satisfy these two conditions). We say the action is

(iii) free: if $ge = e$ implies g is in G^0

(iv) proper: if the graph of the action $P : G \circ E \mapsto E^2$, defined for (g, e) in $G \circ E$ as $P(g, e) = (ge, e)$, is a proper map.

*If the action satisfies $(i) - (iv)$, then $G \to E$ is called a **principal** G**-action***

Definition 3.3.2. *Let G, H be topological groupoids with paracompact unit spaces. A G-H **equivalence** $G \to E \leftarrow H$ consists of a principal left G action, as well as a* *principal right H action on a second countable locally compact space E together with the two compatibility conditions:*

(i) the actions commute: if (g, e) is in $G \circ E$ and (e, h) in $E \circ H$, then (g, eh), (ge, h) are in $G \circ E$, $E \circ H$, respectively, and $(ge)h = g(eh)$.

(ii) ρ, σ are injective modulo H, G, respectively: if e_1, e_2 are in E such that $\rho(e_1) = \rho(e_2)$, there is h in H for which $e_1 h = e_2$. If e_1, e_2 are in E such that $\sigma(e_1) = \sigma(e_2)$, there is g in G for which $ge_1 = e_2$.

The following proposition will be useful to us when we construct the product of groupoid equivalences.

Proposition 3.3.3. *Let $G \to E$ be a groupoid action of an étale groupoid G. Then, the product map $p : G \circ E \mapsto E$ is open.*

Proof. For g in G, by the étale property, there is an open set $U_g \subseteq G$ such that $r, s|_{U_g}$ are homeomorphisms, i.e., U_g is a G-set. Let $U_s := s(U_g)$ and $U_r := r(U_g)$. Consider the continuous maps $\beta_r : \rho^{-1}(U_s) \mapsto \rho^{-1}(U_r)$, defined for e in $\rho^{-1}(U_s)$ as $\beta_r(e) = (s|_{U_s})^{-1}(\rho(e))e$, and $\beta_s : \rho^{-1}(U_r) \mapsto \rho^{-1}(U_s)$, defined for f in $\rho^{-1}(U_r)$ as $\beta_s(f) = ((r|_{U_r})^{-1}(\rho(f))^{-1})f$. It is obvious, by invertibility of the action, that $\beta_s \circ \beta_r = id_{\rho^{-1}(U_s)}$ and $\beta_r \circ \beta_s = id_{\rho^{-1}(U_r)}$. Therefore, β_r is a homeomorphism. For V open in E, we have

$$p(U_g \circ V) = p(U_g \circ \rho^{-1}(U_s \cap V)) = \beta_r(\rho^{-1}(U_s) \cap V).$$

Therefore, $p(U_g \circ V)$ is open. Since G-sets generate the topology on G and sets $W \circ V$, W open in G, V open in E, generate the topology on $G \circ E$, it follows that p is an open map. □

We define the analogous notion of a G-set (definition 2.2.1) for a groupoid equivalence.

Definition 3.3.4. *Let $G \to E \leftarrow H$ be a G-H equivalence between étale groupoids. We call an open set $U \subseteq E$ an **E-set** if $\sigma|_U, \rho|_U$ are homeomorphisms. This is equivalent to σ and ρ being injective on U, since these maps are already open and continuous. We say E is **étale** if every point e in E has a neighbourhood which is an E-set, i.e., ρ, σ are local homeomorphisms. If U is an E-set, we will call $\sigma \circ (\rho|_U)^{-1}$ the **E-map of** U, and we will denote it γ_U.*

We now show that when a groupoid equivalence is étale, the groupoid actions are automatically proper.

Proposition 3.3.5. *Suppose two groupoid actions $G \to E \leftarrow H$ for étale groupoids G, H satisfy $(i) - (iii)$ in definition 3.3.1, and $(i) - (ii)$ in definition 3.3.2. Suppose ρ and σ are local homeomorphisms. Then, the actions are proper, and therefore define an étale G-H equivalence.*

Proof. We prove this for the left action. Let $E^\sigma = \{(x,y) \in E^2 : \sigma(x) = \sigma(y)\}$, which is closed in E^2. By injectivity of σ modulo G, the graph of the action $P : G \circ E \mapsto E^\sigma$ is a surjection. P is also continuous and injective because the action of G is continuous and free. To show P is proper, it suffices to show P is a homeomorphism. By proposition 3.3.3, the product map $p : G \circ E \mapsto E$ is open. Also, the projection $\pi_2 : G \circ E \mapsto E$ is open because for U, V open in G, E, respectively, $\pi_2(U \circ V) = \rho^{-1}(s(U)) \cap V$. Let U be an open set in G and V an open set in E such that $\sigma|_V$ is injective. We prove $P(U \circ V) = (p(U \circ V) \times \pi_2(U \circ V)) \cap E^\sigma$. " $\subseteq$ " is automatic. Suppose (u,v) is in $(p(U \circ V) \times \pi_2(U \circ V)) \cap E^\sigma$. Therefore, v is in V and there is g in U and v' in V such that $gv' = u$. Since $\sigma(v') = \sigma(gv') = \sigma(v)$, it follows by injectivity of σ on V that $v = v'$. Hence, $P(U \circ V) = (p(U \circ V) \times \pi_2(U \circ V)) \cap E^\sigma$, so $P(U \circ V)$ is open. since E-sets V generate the topology on E and sets $U \circ V$ generate the topology on $G \circ E$, it follows that P is an open map. □

We now show how to compose two étale groupoid equivalences $G_1 \to E_1 \leftarrow H$, $H \to E_2 \leftarrow G_2$ between étale groupoids into an étale groupoid equivalence $G_1 \to E_1 \times_H E_2 \leftarrow G_2$. The fact that groupoid equivalences can be composed is alluded to in Muhly, Renault and Williams [31]. The construction of the product of groupoid equivalences is done in full generality by Holkar [14]. As the proof in general is quite technical we prefer to simplify things by just considering the étale case.

Consider the fibred product $E_1 \circ E_2 := \{(e_1,e_2) \in E_1 \times E_2 : \sigma_1(e_1) = \rho_2(e_2)\}$ and the relation $\sim$ on $E_1 \circ E_2$ defined by $(e_1,e_2) \sim (f_1,f_2)$ if there is h in H such that $\sigma_1(e_1) = r(h)$ and $(e_1h, h^{-1}e_2) = (f_1,f_2)$. $\sim$ is an equivalence relation:

- (reflexivity) for (e_1,e_2) in $E_1 \circ E_2$, $(e_1,e_2) \sim (e_1\sigma_1(e_1), \sigma_1(e_2)e_2) = (e_1,e_2)$,
- (symmetry) for h in H and (e_1,e_2) in $E_1 \circ E_2$ such that $\sigma_1(e_1) = r(h)$, $(e_1,e_2) \sim (e_1h, h^{-1}e_2)$ implies $(e_1h, h^{-1}e_2) \sim ((e_1h)h^{-1}, h(h^{-1}e_2)) = (e_1,e_2)$,
- (transitivity) for h_1, h_2 in H, such that $s(h_1) = r(h_2)$ and (e_1,e_2) in $E_1 \circ E_2$ such that $\sigma_1(e_1) = r(h_1)$, $(e_1,e_2) \sim (e_1h_1, h_1^{-1}e_2)$ and $(e_1h_1, h_1^{-1}e_2) \sim ((e_1h_1)h_2, h_2^{-1}(h_1^{-1}e_2))$ implies $(e_1,e_2) \sim (e_1(h_1h_2), (h_1h_2)^{-1}e_2)$.

Let $q : E_1 \circ E_2 \mapsto E_1 \circ E_2/\sim \; =: E_1 \times_H E_2$ be the quotient map, and give $E_1 \times_H E_2$ the quotient topology. For (e_1,e_2) in $E_1 \circ E_2$, we will denote $q((e_1,e_2))$ as $[e_1,e_2]$.

Proposition 3.3.6. *$q : E_1 \circ E_2 \mapsto E_1 \times_H E_2$ is an open map.*

Proof. Let (e_1,e_2) be in $E_1 \circ E_2$, and U_1, U_2 be open neighbourhoods of e_1, e_2, respectively. We prove the proposition for the open set $U_1 \circ U_2 = \{(u_1,u_2) \in U_1 \times U_2 : \sigma_1(u_1) = \rho_2(u_2)\}$. This amounts to showing the saturation $U_1 \circ_H U_2 := \{(f_1h, h^{-1}f_2) : (f_1,f_2) \in U_1 \circ U_2,\ h \in H,\ \sigma_1(f_1) = r(h)\}$ is open. Let $(f_1h, h^{-1}f_2)$ be in $U_1 \circ_H U_2$, where (f_1,f_2) is in $U_1 \circ U_2$ and h is in H such that $\sigma_1(f_1) = r(h)$. Let U_h be a G-set about h. By proposition 3.3.3, the sets $p_1(U_1 \circ U_h)$, $p_2(U_h^{-1} \circ U_2)$ are open in E_1, E_2, where p_1, p_2 are the product maps. Therefore, $W = p_1(U_1 \circ U_h) \circ p_2(U_h^{-1} \circ U_2)$ is an open neighbourhood of $(f_1h, h^{-1}f_2)$. Let $w = (u_1h_1, h_2^{-1}u_2)$ be in W, where u_1, u_2 are in U_1, U_2, respectively, and h_1, h_2 are in U_h such that $\sigma_1(u_1) = r(h_1)$ and $r(h_2) = \rho_2(u_2)$. Then $s(h_1) = s(h_2)$, and so $h_1 = h_2 = h'$. Therefore, $\sigma_1(u_1) = r(h') = \rho_2(u_2)$, so (u_1, u_2) is in $U_1 \circ U_2$ and $w = (u_1h', h'^{-1}u_2)$ is in $U_1 \circ_H U_2$. Hence, $W \subseteq U_1 \circ_H U_2$, proving that the latter is open. ☐

Corollary 3.3.7. *$E_1 \times_H E_2$ is locally compact and second countable.*

Proof. $E_1 \circ E_2$ is locally compact and second countable because it is a closed subspace of $E_1 \times E_2$. $q : E_1 \circ E_2 \mapsto E_1 \times_H E_2$ is a continuous open surjection, and so local compactness and second countability pass on to $E_1 \times_H E_2$. ☐

We now define the left and right actions of G_1, G_2, respectively, on $E_1 \times_H E_2$. First, we will define the structure maps ρ_{12}, σ_{12}. Let $\pi_1 : E_1 \times E_2 \mapsto E_1$, $\pi_2 : E_1 \times E_2 \mapsto E_2$ be the projections and $\tilde{\rho_{12}} := \rho_1 \circ \pi_1|_{E_1 \circ E_2}$, $\tilde{\sigma_{12}} := \sigma_2 \circ \pi_2|_{E_1 \circ E_2}$. For (e_1, e_2) in $E_1 \circ E_2$ and h in H such that $\sigma_1(e_1) = r(h)$, $\tilde{\rho_{12}}(e_1, e_2) = \rho_1(e_1) = \rho_1(e_1h) = \tilde{\rho_{12}}(e_1h, h^{-1}e_2)$, so $\tilde{\rho_{12}}$ descends to a continuous map $\rho_{12} : E_1 \times_H E_2 \mapsto G_1^0$. ρ_{12} is a surjection because for every e in E_1, $\emptyset \neq (e, \rho_2^{-1}\{\sigma(e)\}) \subseteq E_1 \circ E_2$ and ρ_1 surjects E_1 onto G_1^0. ρ_{12} is an open map because $\rho_{12}(q(V_1 \circ V_2)) = \rho_1(V_1 \cap \sigma_1^{-1}(\rho_2(V_2)))$ and ρ_1, ρ_2 are open maps. $\tilde{\sigma_{12}}$ also descends to a continuous open surjection $\sigma_{12} : E_1 \times_H E_2 \mapsto G_2^0$ defined by $\sigma_{12}[e_1, e_2] = \sigma_2(e_2)$. The proof is similar.

Let $p_{G_1} : G_1 \circ E_1 \mapsto E_1$ and $p_{G_2} : E_2 \circ G_2 \mapsto E_2$ be the product maps. σ_1, ρ_2 are invariant under the the left action of G_1, right action of G_2, respectively. Therefore, $(p_{G_1} \times id_{E_2})^{-1}(E_1 \circ E_2) = G_1 \circ (E_1 \circ E_2)$ and $(id_{E_1} \times p_{G_2})^{-1}(E_1 \circ E_2) = (E_1 \circ E_2) \circ G_2$. Hence,

$\tilde{p_1} := p_{G_1} \times id_{E_2} : G_1 \circ (E_1 \circ E_2) \mapsto E_1 \circ E_2$ and $\tilde{p_2} := id_{E_1} \times p_{G_2} : (E_1 \circ E_2) \circ G_2 \mapsto E_1 \circ E_2$ define continuous and proper actions with structure maps $\tilde{\rho_{12}}$, $\tilde{\sigma_{12}}$, respectively. For g_1 in G_1, g_2 in G_2, h in H, and (e_1, e_2) in $E_1 \circ E_2$ such that $s(g_1) = \rho_1(e_1)$, $\sigma_1(e_1) = r(h)$, and $\sigma_2(e_2) = r(g_2)$, $(e_1, e_2) \sim (e_1h, h^{-1}e_2)$ if and only if $(g_1e_1, e_2) \sim (g_1e_1h, h^{-1}e_2)$ if and only if $(e_1, e_2g_2) \sim (e_1h, h^{-1}e_2g_2)$. Therefore, the actions on $E_1 \circ E_2$ descend to actions on $E_1 \times_H E_2$ (with structure maps ρ_{12}, σ_{12}) making the following diagrams commute:

$$\begin{tikzcd}
G_1 \circ (E_1 \circ E_2) \arrow[r, "\tilde{p_1}"] \arrow[d, "id_{G_1} \times q"] & E_1 \circ E_2 \arrow[d, "q"] \\
G_1 \circ (E_1 \times_H E_2) \arrow[r, "p_1"] & E_1 \times_H E_2
\end{tikzcd}$$

$$\begin{tikzcd}
(E_1 \circ E_2) \circ G_2 \arrow[r, "\tilde{p_2}"] \arrow[d, "q \times id_{G_2}"] & E_1 \circ E_2 \arrow[d, "q"] \\
(E_1 \times_H E_2) \circ G_2 \arrow[r, "p_2"] & E_1 \times_H E_2
\end{tikzcd}$$

By proposition 3.3.6, $id_{G_1} \times q : G_1 \times (E_1 \circ E_2) \mapsto G_1 \times (E_1 \times_H E_2)$ is an open map. Since $G_1 \circ (E_1 \circ E_2)$ is a closed subspace of $G_1 \times (E_1 \circ E_2)$ saturated under $id_{G_1} \times q$ and $(id_{G_1} \times q)(G_1 \circ (E_1 \circ E_2)) = G_1 \circ (E_1 \times_H E_2)$, the restriction $id_{G_1} \times q : G_1 \circ (E_1 \circ E_2) \mapsto G_1 \circ E_1 \times_H E_2$ is a quotient map. So, p_1 is the descent of $\tilde{p_1}$ by quotient maps, and is therefore continuous. Similarly, p_2 is continuous. The actions on $E_1 \times_H E_2$ are associative and commute with each other because they do have these properties at the level of $E_1 \circ E_2$.

We show that ρ_{12}, σ_{12} are injective modulo the right G_2 action, left G_1 action, respectively: Suppose $[e_1, e_2]$ and $[f_1, f_2]$ are in $E_1 \times_H E_2$ and $\rho_{12}[e_1, e_2] = \rho_{12}[f_1, f_2]$. Then, by definition of ρ_{12}, $\rho_1(e_1) = \rho(e_2)$. By injectivity of ρ modulo H, there is h in H such that $e_1 h = f_1$. Therefore, $[e_1, e_2] = [f_1, h^{-1}e_2]$. Since $\rho_2(h^{-1}e_2) = s(h) = \sigma_1(f_1) = \rho_2(f_2)$, there is g in G_2 such that $h^{-1}e_2 g = f_2$. Therefore, $[e_1, e_2]g = [f_1, h^{-1}e_2 g] = [f_1, f_2]$. By a similar argument, σ_{12} is injective modulo G_1.

Now, we show that the actions are free: for g in G and $[e_1, e_2]$ in $E_1 \times_H E_2$ such that $s(g) = \rho_{12}[e_1, e_2]$, $g[e_1, e_2] = [e_1, e_2]$ if and only if there is h in H: $(ge_1, e_2) = (e_1 h, h^{-1}e_2)$. Therefore, $e_2 = h^{-1}e_2$, so h is in H^0. Hence, $ge_1 = e_1$, so g is in G_1^0. Similarly, the G_2 action is free. Therefore, the actions $G_1 \to E_1 \times_H E_2 \leftarrow G_2$ satisfy (i)-(iii) in definition 3.3.1 and (i)-(ii) in definition 3.3.2, so by proposition 3.3.5 to show $G_1 \to E_1 \times_H E_2 \leftarrow G_2$ is an étale equivalence, it suffices to show ρ_{12} and σ_{12} are local homeomorphisms.

Proposition 3.3.8. *Let $G_1 \to E_1 \leftarrow H$ and $H \to E_2 \leftarrow G_2$ be groupoid equivalences. If U is an E_1-set (definition 3.3.4) and V is an E_2 set, then $q(U \circ V)$ is an $E_1 \times_H E_2$-set. When $\sigma(U) \subseteq \rho(V)$, for its $E_1 \times_H E_2$-map (definition 3.3.4), we have $\gamma_V \circ \gamma_U = \gamma_{q(U \circ V)}$.*

Proof. We show $\rho_{12}|_{q(U \circ V)}$ is injective. The proof for σ_{12} is similar. Suppose (u_1, v_1), (u_2, v_2) are in $U \circ V$ and $\rho_{12}[u_1, v_1] = \rho_{12}[u_2, v_2]$. Then, by injectivity of ρ_{12} modulo G_2, there is a g in G_2 and h in H such that $(u_1, v_1) = (u_2 h, h^{-1}v_2 g)$. Since U is an E_1-set $u_1 = u_2 h$ implies h is a unit. Therefore, $v_1 = v_2 g$. Since V is an E_2-set, $v_1 = v_2 g$ implies g is a unit. Therefore, $(u_1, v_1) = (u_2, v_2)$. The conclusion for the E-maps is obvious. □

Corollary 3.3.9. *The product of étale groupoid equivalences is an étale groupoid equivalence.*

Proposition 3.3.10. *Let $G_1 \to E_1 \leftarrow H$ and $H \to E_2 \leftarrow G_2$ be groupoid equivalences. Suppose either* *(i) E_1 is Hausdorff and $\sigma_2 : E_2 \mapsto G_2^0$ is a local homeomorphism or*

(ii) E_2 is Hausdorff and $\rho_1 : E_1 \mapsto G_1^0$ is a local homeomorphism

Then $E_1 \times_H E_2$ is Hausdorff.

Proof. We prove (i). The proof of (ii) is similar. Since $E_1 \times_H E_2$ continuously surjects onto Hausdorff spaces G_1^0, G_2^0 via ρ_{12}, σ_{12}, it suffices to prove the Hausdorff property for pairs of distinct points $[e_1, e_2]$, $[f_1, f_2]$ in $E_1 \times_H E_2$ such that $\rho_{12}[e_1, e_2] = \rho_{12}[f_1, f_2]$ and $\sigma_{12}[e_1, e_2] = \sigma_{12}[f_1, f_2]$. In this case, there are h_1, h_2 in H such that $e_1 = f_1 h_1$ and $e_2 = h_2 f_2$. Hence, $[e_1, e_2] = [f_1 h_1 h_2^{-1}, f_2]$. Since $[e_1, e_2]$ and $[f_1, f_2]$ are distinct, $h := h_1 h_2^{-1}$ is not a unit. By the Hausdorff condition on E_1, we may choose disjoint neighbourhoods U about $f_1 h$ and V about f_1. Let W be a neighbourhood of f_2 such that $\sigma_2|_W$ is injective. Let $Z_1 = q(U \circ W), Z_2 = q(V \circ W)$. Z_1, Z_2 are open neighbourhoods about $[f_1 h, f_2] = [e_1, e_2]$ and $[f_1, f_2]$, respectively. Suppose z is in $Z_1 \cap Z_2$. Then, there are $(x_1, x_2) \in U \circ W$, $(y_1, y_2) \in V \circ W$ and an h in H such that $x_1 h = y_1$, $h^{-1} x_2 = y_2$. Therefore, $\sigma_2(x_2) = \sigma_2(y_2)$, so $x_2 = y_2$ by injectivity of σ_2 on W, implying that h is a unit. Therefore, $x_1 = y_1$, which is impossible since U, V were chosen to be disjoint. Therefore, $Z_1 \cap Z_2 = \emptyset$, proving the proposition. ☐

Corollary 3.3.11. *The product of Hausdorff étale groupoid equivalences is a Hausdorff étale groupoid equivalence.*

Definition 3.3.12. *Let $G \to E_1 \leftarrow H$ and $G \to E_2 \leftarrow H$ be groupoid equivalences. We say E_1 **is topologically conjugate** to E_2 if there is a homeomorphism $\varphi : E_1 \mapsto E_2$ such that $\rho_2 \circ \varphi = \rho_1$, $\sigma_2 \circ \varphi = \sigma_1$ and for all $g \in G$, $e \in E_1$, $h \in H$ such that $s(g) = \rho_1(e)$, $r(h) = \sigma_1(e)$, we have $\varphi(geh) = g\varphi(e)h$. We call such a map φ a topological conjugacy.*

Proposition 3.3.13. *Suppose $G_1 \to E_1 \leftarrow H$ and $G_1 \to F_1 \leftarrow H$ are topologically conjugate étale equivalences, via a conjugacy $\varphi_1 : E_1 \mapsto F_1$ and $H \to E_2 \leftarrow G_2$, $H \to F_2 \leftarrow G_2$ are topologically conjugate via a conjugacy $\varphi_2 : E_2 \mapsto F_2$. Then, $E_1 \times_H E_2$ is topologically conjugate to $F_1 \times_H F_2$ via the conjugacy $\varphi_1 \times_H \varphi_2 : E_1 \times_H E_2 \mapsto F_1 \times_H F_2$, defined for $[e_1, e_2]$ in $E_1 \times_H E_2$ as $\varphi_1 \times_H \varphi_2[e_1, e_2] = [\varphi_1(e_1), \varphi_2(e_2)]$.*

Proof. $\varphi_1 \times_H \varphi_2$ is well-defined because, for (e_1, e_2) in $E_1 \circ E_2$ and h in H such that $\sigma_1(e_1) = r(h)$, $\varphi_1 \times \varphi_2(e_1 h, h^{-1} e_2) = (\varphi_1(e_1)h, h^{-1}\varphi_2(e_2))$ and is continuous because it is the descent of a continuous map by quotient maps. Similarly, $\varphi_1^{-1} \times_H \varphi_2^{-1}$ is well defined, continuous, and is the inverse of $\varphi_1 \times_H \varphi_2$. Therefore $\varphi_1 \times_H \varphi_2$ is a homeomorphism, and it commutes with the left actions of G_1, right actions of G_2 because φ_1, φ_2 do. ☐

We present how Muhly, Renault and Williams produce a $C^*(G)$-$C^*(H)$ imprimitivity bi-module from a (G, H) equivalence in [31].

Theorem 3.3.14. *Let $G \to E \leftarrow H$ be an étale groupoid equivalence, and let $F_0 = C_c(E)$, $A_0 = C_c(G)$, $B_0 = C_c(H)$, where A_0, B_0 are equipped with their structure as dense *-sub-algebras of $C^*(G), C^*(H)$. Then, the following left, right actions and pre-inner products ${}_*\langle \cdot, \cdot \rangle$, $\langle \cdot, \cdot \rangle_*$ equip F_0 with the structure of a A_0- B_0 pre-imprimitivity bi-module (see definition 3.1.5) $A_0 \to F_0 \leftarrow B_0$:*

- $af(e) = \sum_{g \in G : r(g) = \rho(e)} a(g) f(g^{-1}e)$, $a \in A_0$, $f \in F_0$, $e \in E$
- $fb(e) = \sum_{h \in H : r(h) = \sigma(e)} f(eh) b(h^{-1})$, $f \in F_0$, $b \in B_0$, $e \in E$
- ${}_*\langle f_1, f_2 \rangle(g) = \sum_{e \in E : \rho(e) = r(g)} f_1(g^{-1}e)\overline{f_2}(e)$, $f_1, f_2 \in F_0$, $g \in G$
- $\langle f_1, f_2 \rangle_*(h) = \sum_{e \in E : \sigma(e) = r(h)} \overline{f_1}(e) f_2(eh)$, $f_1, f_2 \in F_0$, $h \in H$.

By Raeburn and Williams proposition 3.12 [41], *$A_0 \to F_0 \leftarrow B_0$ completes to a Morita equivalence $C^*(G) \to F_E \leftarrow C^*(H)$ (definition 3.1.2).*

Remark 3.3.15. *Recall from remark 3.1.8 that a left basis $\mathcal{B} = \{\varepsilon_i\}_{i=1}^n$ basis of an A-B imprimitivity bi-module F determined a projection z in $M_n(B)$ that we said was the analog to the characteristic function χ_Y of a first return domain $Y \subseteq X$ for a dynamical system $T : X \mapsto X$. To elaborate more on this analogy, let's assume $T : X \mapsto X$ is a homeomorphism of a compact Hausdorff space X, $Y \subseteq X$ is clopen and compact, and that every orbit of T eventually intersects Y. Such a situation can occur when T is a homeomorphism of the Cantor set. Denote G_T to be the transformation groupoid of T. Then, the pair of actions $G_T|_Y \to G_T^Y \leftarrow G_T$, considered in the introduction, is a groupoid equivalence in the sense of definition 3.3.2. The induced $C^*(G_T|_Y)$-$C^*(G_T)$ imprimitivity bi-module, $C^*(G_T^Y)$, will then contain χ_Y, and the single element set $\{\chi_Y\}$ will be a left $C^*(G_T^Y)$ basis. To see the case of a basis with n elements in this special setting, let $\{\phi_i\}_{i=1}^n$ be a partition of unity in $C(Y) \subseteq C^*(G_T^Y)$. Then, $\mathcal{B} = \{\sqrt{\phi_i}\}_{i=1}^n$ will be a left basis for $C^*(G_T^Y)$. The projection z in $M_n(C^*(G_T))$ it determines has the property that the sum of its diagonal entries, which is $\Delta(\mathcal{B})(1)$, is equal to χ_Y. Therefore, we can think of choosing a basis $\mathcal{B} = \{\varepsilon_i\}_{i=1}^n$ for an A-B imprimitivity bi-module as realizing the bi-module F as a non-commutative first return procedure. The diagonal entries $\{\langle \varepsilon_i, \varepsilon_i \rangle_*\}_{i=1}^n$ of the projection z in $M_n(B)$ determined by a left basis $\mathcal{B}$ is a cover of the non-commutative first return domain. Although $\Delta(\mathcal{B})(1) = \sum_{i=1}^n \langle \varepsilon_i, \varepsilon_i \rangle_*$ is in general not a projection, we will think of it as a* ***non-commutative first return domain****.*

Now, we prove the following functorality result. This is proven in Holkar [14] in the general case, but we prefer to provide a simpler proof for the étale case. Recall the definition of the composition of imprimitivity bi-modules in the discussion after proposition 3.1.6.

Proposition 3.3.16. *Let $G_1 \to E_1 \leftarrow H$, $H \to E_2 \leftarrow G_2$ be étale groupoid equivalences. Then, the composition $F_{E_1} \otimes_{C^*(H)} F_{E_2}$ and $F_{E_1 \times_H E_2}$ are isomorphic as $C^*(G_1)-C^*(G_2)$ imprimitivity bi-modules (definition 3.1.3).*

Proof. Consider the bi-linear map $\tilde{U}$ from $C_c(E_1) \times C_c(E_2)$ to functions on $E_1 \times_H E_2$ defined for (f_1, f_2) in $C_c(E_1) \times C_c(E_2)$ by $\tilde{U}(f_1, f_2)([e_1, e_2]) =$ $\sum_{h \in H : \sigma_1(e_1) = r(h)} f_1(e_1 h) f_2(h^{-1} e_2)$, where $[e_1, e_2]$ is in $E_1 \times_H E_2$. Since the supports of f_1 and f_2 can be covered by finitely many E-sets, for any $[e_1, e_2]$ in $E_1 \times_H E_2$there are at most finitely many h in H for which $e_1 h$ is in the support of f_1 and $h^{-1} e_2$ is in the support of f_2. Therefore, the sum in $U(f_1, f_2)[e_1, e_2]$ has finitely many non-zero terms. First, we prove $\tilde{U}(f_1, f_2)$ is in $C_c(E_1 \times_H E_2)$. By bi-linearity, and the étale property of E_1, E_2 it suffices to prove this for f_1, f_2 supported in E-sets U_1, U_2, respectively. Let $\text{supp}(f_1) = K_1$, $\text{supp}(f_2) = K_2$, and $K := q(K_1 \circ K_2)$. For (e_1, e_2) in $K_1 \circ K_2$, it follows from the E-set property of U_1, U_2 that $\tilde{U}(f_1, f_2)[e_1, e_2] = f_1(e_1) f_2(e_2) = f_1 \otimes f_2(e_1, e_2)$. For $[e_1, e_2]$ not in K, we have $\tilde{U}(f_1, f_2)([e_1, e_2]) = 0$ since none of the summands will land in $K_1 \circ K_2$. Therefore, we can write $\tilde{U}(f_1, f_2) = f_1 \otimes f_2 \circ (q|_{U_1 \circ U_2})^{-1}$,and so $\tilde{U}(f_1, f_2)$ is in $C_c(E_1 \times_H E_2)$. Since the functions of the form $f_1 \otimes f_2|_{E_1 \circ E_2}$ densely span $C_c(E_1 \circ E_2)$ and $q : E_1 \circ E_2 \mapsto E_1 \times_H E_2$ is a local homeomorphism, it follows that functions $f_1 \otimes f_2 \circ (q|_{U_1 \circ U_2})^{-1}$ densely span $C_c(E_1 \times E_2)$, and therefore the linear map $U : C_c(E_1) \otimes C_c(E_2) \mapsto C_c(E_1 \times_H E_2)$ defined on basic tensors by $U(f_1 \otimes f_2) = \tilde{U}(f_1, f_2)$ has dense image. Let ${}_{12}\langle \cdot, \cdot \rangle$, $\langle \cdot, \cdot \rangle_{12}$ be the pre-inner products defined in section 3.1 below proposition 3.1.6. We now show for all basic tensors $\varphi_1 \otimes \varphi_2$, $\psi_1 \otimes \psi_2$ in $C_c(E_1) \otimes C_c(E_2)$,

$$ {}_*\langle U(\varphi_1 \otimes \varphi_2), U(\psi_1 \otimes \psi_2) \rangle = {}_{12}\langle \varphi_1 \otimes \varphi_2, \psi_1 \otimes \psi_2 \rangle \text{ and} $$

$$ \langle U(\varphi_1 \otimes \varphi_2), U(\psi_1 \otimes \psi_2) \rangle_* = \langle \varphi_1 \otimes \varphi_2, \psi_1 \otimes \psi_2 \rangle_{12}. $$

We show the equality for the right inner products. The calculation for the left is similar. For g in G, we have

$$ \begin{aligned} &\langle U(\varphi_1 \otimes \varphi_2, U(\psi_1 \otimes \psi_2) \rangle_*(g) = \\ &\sum_{[x,y] : \sigma_2(y) = r(g)} \overline{U(\varphi_1 \otimes \varphi_2)}[x, y] U(\psi_1 \otimes \psi_2)[x, yg] = \\ &\sum_{[x,y] : \sigma_2(y) = r(g)} \sum_{h_1, h_2 \in H : r(h_i) = \sigma_1(x)} \overline{\varphi_1}(x h_1)) \overline{\varphi_2}(h_1^{-1} y) \psi_1(x h_2) \psi_2(h_2^{-1} y g) = (*) \end{aligned} $$

By injectivity of σ_{12} modulo G_1, we can replace the $[x, y] : \sigma_2(y) = r(g)$ summand by $g \in G_1 : \sigma_{12}([g_1 x_0, y_0]) = r(g)$, for any fixed $[x_0, y_0] \in E_1 \times_H E_2$, $\sigma_2(y_0) = r(g)$. Therefore,

$$
\begin{aligned}
&(*) = \\
&\sum_{h_1,h_2:r(h_i)=\rho_2(y_0)} \overline{\varphi_2}(h_1^{-1}y_0)\psi_2(h_2^{-1}y_0g) \sum_{g_1:s(g_1)=\rho(x_0)} \overline{\varphi_1}(g_1x_0h_1)\psi_1(g_1x_0h_1h_1^{-1}h_2) \\
&= \sum_{h_1,h_2:r(h_i)=\rho_2(y_0)} \overline{\varphi_2}(h_1^{-1}y_0)\langle\varphi_1,\psi_1\rangle_*(h_1^{-1}h_2)\psi_2(h_2^{-1}y_0g) = \\
&\sum_{h_1:r(h_1)=\rho_2(y_0)} \overline{\varphi_2}(h_1^{-1}y_0) \sum_{h_2:r(h_2)=r(h_1)} \langle\varphi_1,\psi_1\rangle_*(h_1^{-1}h_2)\psi_2((h_1^{-1}h_2)^{-1}h_1^{-1}y_0g) = \\
&\sum_{h_1:r(h_1)=\rho_2(y_0)} \overline{\varphi_2}(h_1^{-1}y_0)(\langle\varphi_1,\psi_1\rangle_*\psi_2(h_1^{-1}y_0g)) = \langle\varphi_2, \langle\varphi_1,\psi_1\rangle_*\psi_2\rangle_* = \\
&\langle\varphi_1\otimes\varphi_2, \psi_1\otimes\psi_2\rangle_{12}.
\end{aligned}
$$

Therefore, U descends to a linear map $\hat{U} : C_c(E_1) \otimes_{C^*(H)} C_c(E_2) \mapsto C_c(E_1 \times_H E_2)$ with dense range preserving the pre-inner products, so $\hat{U}$ extends to a $C^*(G_1)$-$C^*(G_2)$ Hilbert bi-module isomorphism. ☐

The next proposition will be useful to us when we consider groupoid renormalization.

Proposition 3.3.17. *Suppose $G \to E \leftarrow H$ is an étale groupoid equivalence. Let W be an E-set, and let ϕ, ψ be in in $C_c(W) \subseteq C_c(E)$. Then, ${}_*\langle\phi,\psi\rangle = \phi\overline{\psi} \circ \rho|_W^{-1}$ and $\langle\phi,\psi\rangle_* = \overline{\phi}\psi \circ \sigma|_W^{-1}$.*

Proof. We prove the first equation. The second is similar. For g in G, ${}_*\langle\phi,\psi\rangle(g) = \sum_{e\in E:\rho(e)=r(g)} \phi(g^{-1}e)\overline{\psi}(e)$. If $g^{-1}e \neq e$, then the fact that σ is a homeomorphism on $\operatorname{supp}(\phi) \cup \operatorname{supp}(\psi)$ and $\sigma(g^{-1}e) = \sigma(e)$ implies $\phi(g^{-1}e)\overline{\psi}(e) = 0$. Since the action of G is free, $g^{-1}e \neq e$ holds if g is not a unit, so if g is not in G^0, then ${}_*\langle\phi,\psi\rangle(g) = 0$. Now, suppose g is not in $\rho(W)$. Then, for any e in E such that $\rho(e) = r(g) = g$, e is not in W, so $\phi(g^{-1}e)\overline{\psi}(e) = 0$. Hence, ${}_*\langle\phi,\psi\rangle$ is in $C_c(\rho(W))$. For $w \in W$, since ρ is a homeomorphism on W, if e in E is such that $\rho(e) = \rho(w)$ and $e \neq w$, then e is not in W, so $\phi(e)\overline{\psi}(e) = 0$. Therefore, ${}_*\langle\phi,\psi\rangle(\rho(w)) = \phi\overline{\psi}(w)$. ☐

3.4 Groupoid Bundle Equivalence

We provide a groupoid bundle analog to $C_0(X)$-equivalences.

Definition 3.4.1. *Let (G,π), (H,μ) be groupoid bundles over a locally compact Hausdorff space X. A* ***groupoid bundle equivalence*** *$(G,\pi) \to E \leftarrow (H,\mu)$, or (G,π)-(H,μ) equivalence for short, is a groupoid equivalence $G \to E \leftarrow H$ (definition 3.3.2) such that $\pi\circ\rho = \mu\circ\sigma$. Denote $E^x := \rho^{-1}(\pi^{-1}(X \setminus \{x\})) = \sigma^{-1}(\mu^{-1}(X \setminus \{x\}))$ and*

$E_x := \rho^{-1}(\pi^{-1}\{x\}) = \sigma^{-1}(\mu^{-1}\{x\})$. $\pi \circ \rho = \mu \circ \sigma$ implies $G_x \to E_x \leftarrow H_x$ is a groupoid equivalence for all x in X. We will call E_x the ***fibre of** E* ***at** x. $(G, \pi) \to E \leftarrow (H, \mu)$ is said to be an* ***étale groupoid bundle equivalence*** *if it is an étale G-H equivalence*

Proposition 3.4.2. *Let (G, π), (H, μ) be étale groupoid bundles over X, and let $(G, \pi) \to E \leftarrow (H, \mu)$ be an étale groupoid bundle equivalence. The induced Morita equivalence $C^*(G) \to F_E \leftarrow C^*(H)$ is then a $C_0(X)$-equivalence (definition 3.2.1). Also, for every x in X, $(F_E)^x = F_{E^x}$, and so $(F_E)_x \simeq F_{(E_x)}$ as Hilbert $C^*(G_x)$-$C^*(H_x)$ bi-modules (definition 3.1.3).*

Proof. Let a be in $C_c(G)$, f in $C_c(E)$, b in $C_c(H)$, and k in $C_0(X)$. Then, for e in E,

$$(ka)fb(e) = \sum_{g \in G : r(g) = \rho(e)} k(\pi(g))a(g)fb(g^{-1}e) = k(\pi(\rho(e)))afb(e),$$

and

$$af(bk)(e) = \sum_{h \in H : r(h) = \sigma(e)} af(eh)b(h^{-1})k(\mu(h^{-1})) = k(\mu(\sigma(e)))afb(e).$$

Since $\pi \circ \rho = \mu \circ \sigma$, it follows that $(ka)fb = af(bk)$. Since a was arbitary in a dense sub-algebra, it follows from non-degeneracy of π^*, μ^*, that $kf = fk$. Therefore, F_E is a $C_0(X)$-equivalence. The proofs of $(F_E)^x = F_{E^x}$ for all x in X and of the corollary that $(F_E)_x \simeq F_{E_x}$ are similar to the proof of proposition 2.2.9 □

3.4.1 AF Groupoid Equivalence

Let $\pi : H \mapsto G$ be a non-degenerate surjective graph homomorphism, and let $(R_\pi, \hat{\pi})$ be the groupoid bundle over X_G as in section 2.2.1. Since the groupoid bundle $(R_\pi, \hat{\pi})$ of AF groupoids is over a much nicer space than the field of AF algebras in section 2.1.1, we can provide a groupoid model for the equivalences between AF algebras presented in section 3.2.1 that forms a groupoid bundle equivalence between $(\sigma_G^* R_\pi, \sigma_G^* \hat{\pi})$ and $(R_\pi, \hat{\pi})$ over X_G, where σ_G is the shift map. We provide such a model in this section.

Let $F = \{(x, y) \in X_H \times X_H : (x, \sigma_H y) \in R_\pi\}$. We can equip F with left $\sigma_G^* R_\pi$, right R_π actions as follows: Let $\rho : F \mapsto \sigma_G^* X_H$ and $\sigma : F \mapsto X_H$ be the structure maps defined for (x, y) in F as $\rho(x, y) = (x, \pi(y))$ and $\sigma(x, y) = y$. Then, $((x_1, x_2), z)$ in $\sigma_G^* R_\pi$ is composable with (x, y) if and only if $x_2 = x$ and $\pi(y) = z$, in which case $((x_1, x_2), z)(x, y) = (x_1, y)$. Similarily, (y_1, y_2) in R_π is composable with (x, y) if and only if $y = y_1$, in which case $(x, y)(y_1, y_2) = (x, y_2)$. F is closed under the left product because $(x_1, \sigma_H y) = (x_1, x_2)(x_2, \sigma_H y)$. The factors on the right are in R_π, so by closure of R_π, (x_1, y) is in F. Similarily, (x, y_2) is in F because $(x, \sigma_H y_2) = (x, \sigma_H y_1)(\sigma_H y_1, \sigma_H y_2)$, and both factors on the right are in R_π. Associativity of either action is trivial. For (x_1, x_2) in $\sigma_G^* R_\pi$ and (x_2, y) in F, $(x_1, x_2)(x_2, y) = (x_2, y)$ if and only if $x_1 = x_2$. Therefore, the left action is invertible and free. Similarly for the right action. The actions are compatible with each other because left multiplication only changes the left-most co-ordinate x for a point (x, y) in F, and right multiplication only changes the right-most co-ordinate y of (x, y), but keeps $\pi(y)$ the same.

We show ρ, σ are bijections modulo the right, left actions, respectively: σ is surjective because for every y in X_H, $(\sigma_H y, y)$ is in F. If (x_1, y) and (x_2, y) are in F, then $g = ((x_1, x_2), \pi(y)) = ((x_1, \sigma_H y), \pi(y))((x_2, \sigma_H y), \pi(y))$ is in $\sigma_G^* R_\pi$, and $g(x_2, y) = (x_1, y)$. Therefore, σ is bijective modulo the left $\sigma_G^* R_\pi$ action. Now, let (x, z) be a point in $\sigma_G^* X_H$. By the non-degenerate assumption on the surjection $\pi : H \mapsto G$, there is an edge f in H such that $\pi(f) = z_1$ and $r(f) = s(x)$. Therefore, $(x, f * x)$ is a point in F such that $\rho(x, f * x) = (x, z)$. Hence, ρ is surjective. If (x, y_1) and (x, y_2) are in F and $\pi(y_1) = \pi(y_2)$, then $(\sigma_H y_1, \sigma_H y_2) = (x, \sigma_H y_1)^{-1}(x, \sigma_H y_2)$. Therefore, $h = (y_1, y_2)$ is in R_π and $(x, y_1)h = (x, y_2)$, so ρ is bijective modulo the right R_π action.

As for a topology on F, for n in $\mathbb{N}$, let z in P_H^n and w in P_H^{n+1} be paths in H such that $r(z) = r(w)$ and $\pi(z) = \sigma_G \pi(w)$. Define $U(z, w) = \{(x, y) \in X_H \times X_H : x_{[n+1,\infty)} = y_{[n+2,\infty)} \text{ and } x_{[1,n]} = z,\ y_{[1,n+1]} = w\}$. Then, $U(z, w)$ is contained in F, and such sets form a basis for a topology on F. For such a pair z, w, σ maps $U(z, w)$ bijectively onto $U(w)$ and ρ maps $U(z, w)$ onto $U_{w_1}(z) := \{(x, w_1 * \pi(x)) \in X_H \times X_G : x_{[1,n]} = z\}$. Continuity is also easily checked. Therefore, σ and ρ are local homeomorphisms. Then, by proposition 3.3.5, the actions are proper. Hence, $\sigma_G^* R_\pi \to F \leftarrow R_\pi$ is an étale groupoid equivalence. Since $\sigma_G^* \hat{\pi}(\rho(x, y)) = \pi(y) = \pi(\sigma(x, y))$ for all (x, y) in F, it follows that F is an étale groupoid bundle equivalence $(\sigma_G^* R_\pi, \sigma_G^* \hat{\pi}) \to F \leftarrow (R_\pi, \hat{\pi})$.

3.4.2 Rotation Groupoid Equivalence

Let $X_0 = (0, 1)$ and $X_1 = \bigcup_{n \in \mathbb{N}} (\frac{1}{n+1}, \frac{1}{n})$. The Gauss map is the continuous surjection $G : X_1 \mapsto X_0$ defined for θ in X_1 as $G(\theta) = \frac{1}{\theta} - \lfloor \frac{1}{\theta} \rfloor$, where $\lfloor x \rfloor$ for x in $\mathbb{R}_+$ is the largest non-negative integer bounded from above by x. Let $i : X_1 \mapsto X_0$ be the inclusion. Let H be the groupoid bundle over $(0, 1)$ as in section 2.2.2. We identify $G^* H$ with the transformation groupoid

$$X_1 \times \mathbb{R}/\mathbb{Z} \times_{G^*\alpha} \mathbb{Z},$$

where $G^*\alpha : X_1 \times \mathbb{R}/\mathbb{Z} \mapsto X_1 \times \mathbb{R}/\mathbb{Z}$ is the homeomorphism defined for (θ, x) in $X_1 \times \mathbb{R}/\mathbb{Z}$ as $G^*\alpha(\theta, x) = (\theta, x + \overline{G(\theta)})$. Under this identification, $G^*\pi$ becomes the projection of $X_1 \times \mathbb{R}/\mathbb{Z} \times_{G^*\alpha} \mathbb{Z}$ onto the first co-ordinate. $i^* H$ will be identified with the transformation groupoid of α restricted to $X_1 \times \mathbb{R}/\mathbb{Z}$, which is just $\pi^{-1}(X_1)$. Then, $i^*\pi$ is identified with π restricted to $\pi^{-1}(X_1)$. We will show $E = X_1 \times \mathbb{R}$ equipped with structure maps $\rho, \sigma : E \mapsto X_1$, defined for (θ, t) in E as $\rho(\theta, t) = (\theta, \bar{t})$, $\sigma(\theta, t) = (\theta, \overline{\theta t})$, and proposed left, right actions of $G^* H$, $i^* H$, defined for $((\theta, x, n), (\psi, t))$ in $G^* H \circ E$, $((\psi, t), (\theta, y, m))$ in $E \circ i^* H$ as

$$(\theta, x, n)(\psi, t) = (\theta, t + \frac{n}{\theta}),\ (\psi, t)(\theta, y, m) = (\theta, t - m),$$

is a groupoid bundle equivalence $(G^*H, G^*\pi) \to E \leftarrow (i^*H, i^*\pi)$. First, we show the formula above and ρ define a free and proper left action of G^*H.

Assume (θ, x, n) in G^*H and (ψ, t) in E are such that $s(\theta, x, n) = \rho(\psi, t)$. Then,

$$\rho((\theta, x, n)(\psi, t)) = \rho(\theta, t + \frac{n}{\theta}) = (\theta, \bar{t} + \overline{\frac{n}{\theta}}) = (\theta, x - n\overline{G(\theta)} + n\overline{G(\theta)}) = r(\theta, x, n).$$

So, for $((\theta, y, m), (\theta, x, n))$ in $G^*H \circ G^*H$, both $((\theta, y, m), [(\theta, x, n)(\theta, t)])$ and $([(\theta, y, m)(\theta, x, n)], (\theta, t)]$ are in $G^*H \circ E$. We have

$$\begin{aligned}&(\theta, y, m)[(\theta, x, n)(\theta, t)] = (\theta, y, m)[(\theta, t + \frac{n}{\theta})] = (\theta, t + \frac{n}{\theta} + \frac{m}{\theta}) = \\ &= (\theta, y, m + n)(\theta, t) = [(\theta, y, m)(\theta, x, n)](\theta, t),\end{aligned}$$

so the action is associative. $(\theta, t + \frac{n}{\theta}) = (\theta, t)$ if and only if $n = 0$, so the left action is free and invertible.

$$G^*H \circ E = \{(\theta, \bar{t} + nG(\theta), n, \theta, t) : t \in \mathbb{R},\ n \in \mathbb{Z}, \text{ and } \theta \in X_1\},$$

so the map $G^*H \circ E \mapsto X_1 \times \mathbb{Z} \times \mathbb{R}$ defined by forgetting the second and fourth co-ordinate is a homeomorphism. Under this homeomorphism, the product map sends (θ, n, t) in $X_1 \times \mathbb{Z} \times \mathbb{R}$ to $(\theta, t + \frac{n}{\theta})$ and is therefore continuous. Similar calculations will now be done to show σ and the formula above define a free and proper right action of i^*H.

Assume (θ, y, m) in i^*H and (ψ, t) in E are such that $r(\theta, y, m) = \sigma(\psi, t)$. Then,

$$\sigma((\psi, t)(\theta, y, m)) = \sigma(\theta, t - m) = (\theta, \overline{\theta t - \theta m}) = (\theta, \sigma(t) - m\bar{\theta}) = s(\theta, y, m).$$

So, for $((\theta, y, m), (\theta, x, n))$ in $i^*H \circ i^*H$, both $([(\theta, t)(\theta, y, m)], (\theta, x, n))$ and $((\theta, t), [(\theta, y, m)(\theta, x, n)])$ are in $E \circ i^*H$. We have

$$\begin{aligned}&[(\theta, t)(\theta, y, m)](\theta, x, n) = [(\theta, t - m)](\theta, x, n) = (\theta, t - m - n) = \\ &= (\theta, t)(\theta, y, m + n) = (\theta, t)[(\theta, y, m)(\theta, x, n)],\end{aligned}$$

so the action is associative. $(\theta, t - m) = (\theta, t)$ if and only if $m = 0$, so the right action is free and invertible.

$$E \circ i^*H = \{(\theta, t, \theta, \overline{\theta t}, m) : \theta \in X_1,\ t \in \mathbb{R} \text{ and } m \in \mathbb{Z}\},$$

so the map $E \circ i^*H \mapsto X_1 \times \mathbb{R} \times \mathbb{Z}$ defined by forgetting the third and fourth co-ordinate is a homeomorphism. Under this homeomorphism, the product map sends (θ, t, m) in $X_1 \times \mathbb{R} \times \mathbb{Z}$ to $(\theta, t - m)$ and is therefore continuous. We now show the actions commute.

Assume (θ, x, n) in G^*H, (θ, t) in E, and (θ, y, m) in i^*H are such that $s(\theta, x, n) =$

$\rho(\theta, t)$ and $\sigma(\theta, t) = r(\theta, y, m)$. Then, we have

$$\rho((\theta, t)(\theta, y, m)) = (\theta, \overline{t - m}) = (\theta, \bar{t}) = (\theta, \rho(t))$$

and

$$\sigma((\theta, x, n)(\theta, t)) = (\theta, \overline{\theta(t + \frac{n}{\theta})}) = (\theta, \overline{\theta t}) = \sigma(\theta, t).$$

So, $([(\theta, x, n)(\theta, t)], (\theta, y, m))$ is in $E \circ i^*H$ and $((\theta, x, n), [(\theta, t)(\theta, y, m)])$ is in $G^*H \circ E$. We have

$$[(\theta, x, n)(\theta, t)](\theta, y, m) = (\theta, t + \frac{n}{\theta} - m) = (\theta, t - m + \frac{n}{\theta}) = (\theta, x, n)[(\theta, t)(\theta, y, m)].$$

Therefore, the actions commute. Both ρ, σ are surjective and open (moreover, they are local homeomorphisms), so all that remains to show is that they are injective modulo the right, left actions.

Assume (θ, s) and (ψ, t) are such that $\sigma((\theta, s)) = \sigma((\psi, t))$. Then, $\theta = \psi$ and $\theta(s - t) = n$ is in $\mathbb{Z}$. Hence, $((\psi, \bar{t} + nG(\psi), n), (\psi, t))$ is in G^*H and $(\psi, \bar{t} + nG(\psi), n)(\psi, t) = (\theta, t + \frac{n}{\theta}) = (\theta, s)$. Therefore, σ is injective modulo the left G^*H action.

Assume (θ, s) and (ψ, t) are such that $\rho((\theta, s)) = \rho((\psi, t))$. Then, $\theta = \psi$ and $s - t = m$ is in $\mathbb{Z}$ Hence, $((\theta, s), (\theta, \overline{\theta s}, m))$ is in $E \circ i^*H$ and $(\theta, s)(\theta, \overline{\theta s}, m) = (\theta, s - m) = (\psi, t)$. Therefore, ρ is injective modulo the right action. Since ρ and σ are local homeomorphisms, it follows from the above properties of the actions and proposition 3.3.5 that the actions are proper. We have shown

$$G^*H \to E \leftarrow i^*H$$

is an étale groupoid equivalence. For (θ, t) in E, $G^*\pi(\rho(\theta, t)) = \theta = i^*\pi(\sigma(\theta, t))$, so $(G^*H, G^*\pi) \to E \leftarrow (i^*H, i^*\pi)$ is an étale groupoid bundle equivalence. The fibre-wise groupoid equivalences can be shown to be groupoid models for Rieffel's bi-modules between rotation algebras in [46].

3.4.3 IFS Groupoid Equivalence

Let R and $O(R)$ be the groupoids associated to a single matrix affine IFS $\{\gamma_1, ..., \gamma_v\}$ as in section 2.2.3, with attractor denoted as K. For k in $\mathbb{Z}$, let

$$E^{-k} := \{(x, -k, y) : (x, -k, y) \in O(R)\}.$$

Proposition 3.4.3. *If r, s surject onto K when restricted to E^{-k}, then $R \to E^{-k} \leftarrow R$ is an étale groupoid equivalence, where the left, right actions are the groupoid product, and the topology of E^{-k} the subspace topology coming from $O(R)$.*

Proof. The left, right actions commute and are associative because they are coming from the groupoid product. The structure maps r, s surject onto the unit space K of R by assumption, and are bijections modulo the right, left actions because if $r(g) = r(h)$, for g and h in E^{-k}, then $h^{-1}g$ is in R, and the analogous statement holds for s. By the construction of the topology on $O(R)$, E^{-k} is a clopen set, and therefore $r, s|_{E^{-k}}$ are still local homeomorphisms, so the actions are étale. The actions are also continuous because the groupoid product is. Therefore, by proposition 3.3.5, the actions are proper and define an étale groupoid equivalence. $\square$

There is an analogous groupoid bundle equivalence between the field of groupoids constructed at the end of 2.2.3 and its pull back by the shift. As in the above proposition, we must make an assumption on r and s for this to be true.

Section 4
Renormalization

In this section, we will formally define renormalization procedures in the C^*-algebra setting (definition 4.1.1) as well as in the groupoid setting (definition 4.2.7), which have only been alluded to so far. In the introduction, we mentioned various applications of renormalization procedures for dynamical systems which demonstrated that the long term behaviour of such a procedure can be used to infer properties of the dynamical systems. We will demonstrate that this is true also in the C^*-algebra and groupoid setting by showing an analogous statement to Masur's criterion (see Masur theorem 3 [27], or our introduction) holds, given some rather complicated conditions on the iterates which we explain in section 4.1. These conditions will be simplified greatly in the groupoid setting, and we will even consider "iterable" conditions in which one only needs to check the properties of the first iteration of the renormalization procedure at a parameter to obtain the necessary conditions for all iterates. As a consequence, we will provide criterion for when a parameter groupoid is simple (theorem 4.2.10) and when it has at most one invariant probability measure on its unit space (theorem 4.2.13) in terms of properties of the renormalization dynamics. We will check these conditions for the field of AF algebras in section 4.1.2, the rotation groupoid bundle in section 4.2.1, as well as the IFS groupoids in section 4.2.2.

4.1 C^*-algebra renormalization

Recall from the introduction the definition of a renormalization procedure for a dynamical system, and its origins going back to Feigenbaum [12] and Coullet and Tresser [5]. In this section, we formulate the analogous definition of a renormalization procedure for C^*-algebras, and then make some comments about it. Then, we introduce the finite type condition (definition 4.1.2) for such a procedure that, when assumed, will imply an analog of Masur's criterion holds for the field of C^*-algebras the renormalization procedure acts on (see theorem 4.1.5). Then, in section 4.1.2 we will present the example of a renormalization procedure for AF algebras using the information from section 2.1.1 and section 3.2.1, and consider the unique trace conditions in this setting. We will, in the process, recover the unique trace criterion appearing in Treviño [51]. Recall the definition of pullback of a D-algebra by a continuous function (2.1.8), the definition of a D-equivalence (3.2.1), and the definition of evaluation of a D-equivalence (3.2.2).

Definition 4.1.1. *Let X be a Hausdorff space, and $(\mathcal{A}, \alpha)$ be a D-algebra, with $D \subseteq C_b(X)$. A* ***renormalization procedure*** *consists of*

- *a subspace X_1 of X and a surjection $\sigma : X_1 \mapsto X$. We denote $i : X_1 \mapsto X$ to be the inclusion.*
- *for each x in X_1, a bounded $\mathcal{A}_{\sigma(x)}$-$\mathcal{A}_x$ imprimitivity bi-module (see definition 3.1.9) $\mathcal{A}_{\sigma(x)} \to F_x \leftarrow A_x$.*

We will denote the above data by the triple $(\mathcal{A}, F, \sigma)$. For $n = 1, 2, ..., \infty$, we call $X_n := \bigcap_{k=1}^{n} \sigma^{-k}(X)$ the ***n^{th} renormalizable parameters*** *when $n < \infty$ and X_∞ the* ***infinitely renormalizable paramaters****.*

We say $(\mathcal{A}, F, \sigma)$ is ***upper semi-continuous*** *if $D = C_0(X)$, X_1 and X are locally compact, $\sigma : X_1 \mapsto X$ is continuous, and each bi-module F_x is the point-wise evaluation of a $C_0(X_1)$-equivalence $(\sigma^*\mathcal{A}, \sigma^*\alpha) \to F \leftarrow (i^*\mathcal{A}, i^*\alpha)$. Similarily, $(\mathcal{A}, F, \sigma)$ is* ***continuous*** *if it is upper semi-continuous and $\mathcal{A}$ is a continuous field over X.*

We make three comments on the definition now. We are primarily interested in the infinitely renormalizable parameters, but it is often the case they do not form a nice space, and so it is useful to "complete" X_∞ to include parameters that are only finitely renormalizable.

Since the D-algebra elements of $\mathcal{A}$ can be thought of as sections of the bundle $\{\mathcal{A}_x\}_{x \in X}$ (definition 2.1.3), the D-algebra structure can be thought of as providing the bundle $\{\mathcal{A}_x\}_{x \in X}$ a topology (by specification of the allowed sections of the bundle). As mentioned in our introduction, Masur's criterion links the topology of the orbit of a flat surface under the renormalization dynamics with properties of the flat surface. In order to make similar connections in our setting, it will be necessary to have a topology for $\{\mathcal{A}_x\}_{x \in X}$. Also, with a D-algebra and an upper semi-continuous renormalization procedure for it, we can build a C^*-algebra, via a Cuntz-Pimsner algebra construction, which we expect has interesting links with the renormalization dynamics. We briefly present this construction in section 6.

We consider imprimitivity bi-modules for the reasons mentioned in the introduction of section 3.1. The strictly contractive condition is supposed to replicate the fact that the first return domains are smaller than the original space the dynamics was acting on, so the scale of the renormalized system is strictly smaller than the original one. The author is not convinced this the right condition on the imprimitivity bi-modules, but it seems better than nothing, as it rules out trivial examples like the identity bi-module, which is obviously not an example of renormalization. We come up with a better notion to replicate this in the groupoid setting, which we believe is close to the correct one. This notion is not readily transferable to the C^*-algebra setting as it is formulated in terms of metric properties of the groupoids (see definition 4.2.7).

Recall from the discussion after proposition 3.1.6 the definition of the composition of imprimitivity bi-modules. For x in X_k, we will denote F_x^k to be the composition

$$F_x^k := F_{\sigma^{k-1}x} \otimes_{\mathcal{A}_{\sigma^{k-1}x}} F_{\sigma^{k-2}x} \otimes_{\mathcal{A}_{\sigma^{k-2}x}} \ldots \otimes_{\mathcal{A}_{\sigma x}} F_x.$$

If $(\mathcal{A}, F, \sigma)$ is upper semi-continuous, then the composition above can be first done at the level of the $C_0(X_1)$-equivalence F, but we won't need this until section 6 so we will postpone it for now.

The trace operator $S_x^k := S_{F_x^k}$ (definition 3.1.10) maps finite traces to finite traces, so by the functorality result in Laca and Neshveyev (proposition 1.2 [22]), $S_x^{k_1+k_2} = S^{k_2}_{\sigma^{k_1}x} \circ S_x^{k_1}$ for all k_1, k_2 in $\mathbb{N}$, and x in X_∞. Therefore, we can regard the function sending (x,k) in $X_\infty \times \mathbb{N}$ to S_x^k as a co-cycle.

If A is a unital C^* algebra and a is in A, we will denote $NC(a)$ to be subset of A consisting of the elements $\sum_{k=1}^n x_k^* a x_k$, for any n in $\mathbb{N}$ and any family $\{x_k\}_{k=1}^n$ such that $\sum_{k=1}^n x_k x_k^* = 1$. Note that for b in $NC(a)$, and for any trace τ in $T^{<\infty}(A)$, $\tau(a) = \tau(b)$.

Recall from the paragraph above definition 3.1.10 that given an A-B imprimitivity bi-module F, A, B unital, and a left basis $\mathcal{B} = \{\varepsilon_i\}_{i=1}^n$ for F, $\Delta(\mathcal{B}) : A \mapsto B$ is the operator defined for a in A as $\Delta(\mathcal{B})(a) = \sum_{i=1}^n \langle \varepsilon_i, a\varepsilon_i\rangle_*$.

Definition 4.1.2. *Let $(\mathcal{A}, \alpha)$ be a D-algebra, $D \subseteq C_b(X)$, such that $\mathcal{A}_x$ is unital for every x in X, and let $(\mathcal{A}, F, \sigma)$ be a renormalization procedure. For an infinitely renormalizable parameter x in X and an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$, we say x* ***is finite type along*** *$\sigma^{n_k}x$ if there are finitely many positive elements $a_1, \ldots, a_m$ of $\mathcal{A}$ for which the following holds:*
For every positive element a in $\mathcal{A}_x$ and $\epsilon > 0$ there is a K in $\mathbb{N}$ such that for all $k \geq K$, there are (finite) left-F_x^k basis $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_N$ for which the positive cone spanned by $\{\Delta(\mathcal{B}_n)(a_m(\sigma^{n_k}x))\}_{n\leq N, m\leq M}$ intersects the ϵ neighbourhood of $NC(a)$. The $a_1, \ldots, a_M$ are called ***the generators for*** *x* ***along*** *$\sigma^{n_k}x$.*

We make one remark on the definition. The generators $a_1, \ldots, a_M$ should be thought of as some partition of a unit and the elements $\Delta(\mathcal{B}(a_m(\sigma^{n_k}x)))$ as a slice of the non-commutative renormalization domain determined by the basis $\mathcal{B}$ (see remark 3.3.15). We will then interpret the finite type condition along a sub-sequence $\sigma^{n_k}x$ as saying for each n_k, we can find evenly spread out slices of non-commutative renormalization domains for the procedure $\mathcal{A}_{\sigma^{n_k}x} \to F_x^{n_k} \leftarrow \mathcal{A}_x$, and we are only allowed to slice such domains using a finite partition $a_1, .., a_M$ that varies upper semi-continuously. Our argument for Masur's criterion in general requires this condition (or something slightly weaker), but it is desirable to have easier to check conditions that imply this. As we believe groupoid renormalization shows more promise, we will only find these conditions in the groupoid setting. Any conditions we put on our groupoid renormalization procedure that imply Masur's criterion will first imply that it is finite type, so it is important to introduce the above definition. If the reader would now like an example of a renormalization procedure satisfying the finite type condition, see proposition 4.1.7, and the discussion above it.

We would like to motivate the next definition. The renormalization procedure introduced in Douady and Hubbard [8] takes a quadratic polynomial $f_c(z) = z^2 + c$ and finds a domain U such that some iterate f_c^n sends U to an open set V, containing U, in a 2-to-1 fashion. The quasi-conformal theory in [8] then asserts that $f_c^n : U \mapsto V$ is quasi-conformally conjugate to another quadratic f_{c_1} in a neighbourhood of its filled Julia set (see also Milnor [30] for a nice description of this procedure). If f_c is infinitely renormalizable, then the procedure produces a sequence of domains $U_n \subseteq V_n$ such that $V_n \setminus U_n$ is conformally equivalent to an annulus with outer radius 1 and inner radius μ_n. f_c is said to be **a-priori bounded** if there is a positive number $\mu > 0$ such that $\mu_n \geq \mu$ for every n in $\mathbb{N}$. When f_c is a-priori bounded, the renormalization procedure can be used to prove that its Julia set is locally connected (Lyubich 9.2 theorem 6 [25]). The reason why this condition is called "a-priori", is that it is almost impossible to directly show such bounds, so some additional property of the quadratic must also be used. Hence, the bounds are a-priori knowledge. For our purposes, in the following definition, the renormalization domains $U \subseteq V$ in the discussion above are replaced with the non-commutative first return domain $\Delta(\mathcal{B})(1)$ associated to a finite basis $\mathcal{B}$ (see remark 3.1.8) and a slice $\Delta(\mathcal{B})(a) \leq \Delta(\mathcal{B})(1)$ of it by a positive element a in $\mathcal{A}$ with $a \leq 1$. The conformal modulus of the annulus $V \setminus U$ is replaced with the ratio $\tau(\Delta(\mathcal{B})(a)))/\tau(\Delta(\mathcal{B})(1))$ of measures under tracial states τ.

Definition 4.1.3. *Suppose $(\mathcal{A}, F, \sigma)$ is a renormalization procedure. For x in X_∞ and $\{n_k\}_{k=1}^\infty$ a sequence of positive integers, we say a positive element a in $\mathcal{A}$ is* ***a-priori bounded along*** *$\{\sigma^{n_k}x\}_{k\in\mathbb{N}}$ if for every trace τ in $T^1(A_x)$, we have* $\liminf_k \frac{S_x^{n_k}\tau(a(\sigma^{n_k}x))}{\|S_x^{n_k}\tau\|} \neq 0$.

As in the above discussion, this condition would be impossible to determine without some additional information, such as that considered in the next proposition.

Proposition 4.1.4. *Suppose $(\mathcal{A}, F, \sigma)$ is a renormalization procedure and $(\mathcal{A}, \alpha)$ is a D-algebra, $D \subseteq C_b(X)$, that is continuous over X with unital fibres. If x is an infinitely renormalizable parameter with a limit point $x_* = \lim_{k\to\infty} \sigma^{n_k}x$, where $\{n_k\}$ is an increasing sequence in $\mathbb{N}$, for which the algebra $\mathcal{A}_{x_*}$ is simple, then a positive element a in $\mathcal{A}$ is a-priori bounded along $\{\sigma^{n_k}x\}_{k\in\mathbb{N}}$ if and only if $a(x_*) \neq 0$.*

Proof. Let m_k be any increasing subsequence of n_k and τ any trace in $T^1(\mathcal{A})$. Since $\mathcal{A}$ is continuous over X, has unital fibres, and $\lim_{k\to\infty} \sigma^{m_k}x = x_*$, by proposition 2.1.11

there is a sub-sequence m'_k of m_k for which the sequence $\frac{S_x^{m'_k}\tau}{\|S_x^{m'_k}\tau\|}$ (considered as a trace in $T^1_{\sigma^{m'_k}x}(\mathcal{A})$) has a weak* limit τ_* in $T^1_{x_*}(\mathcal{A})$. Since A_{x_*} is simple, the trace τ_* is faithful when considered as a trace in $T^1(\mathcal{A}_{x_*})$ (Murphy remark 6.2.3 [33]). Therefore, $a(x_*) \neq 0$ if and only if $0 \neq \tau_*(a) = \lim_{k\to\infty} \frac{S_x^{n_k}\tau(a(\sigma^{m'_k}x))}{\|S_x^{m'_k}\tau\|}$. Since the trace τ and the increasing subsequence $\{m_k\}$ of $\{n_k\}$ were chosen arbitrarily, it follows that $a(x_*) \neq 0$ if and only if a is a-priori bounded along $\{\sigma^{n_k}x\}_{k\in\mathbb{N}}$. $\square$

4.1.1 Masur's Criterion

Recall from the introduction that there is a continuous time renormalization procedure, known as Teichmüller flow, for the space of flat surfaces of a fixed finite genus that are identified up to conformal isotopy, known as Teichmüller space. We think of a flat surface as a dynamical system by equipping it with its translation flow in the vertical direction. Masur's criterion says that a finite genus flat surface M whose vertical translation flow is minimal is uniquely ergodic if the orbit of M in Teichmüller space under the Teichmüller flow has an accumulation point (Masur theorem 3 [27]).

The first indication that there might be a Masur's criterion for C^*-algebras came from the paper of Treviño [51], where unique ergodicity of an infinite genus flat surfaces constructed out of a bi-infinite Bratteli diagram is established when the renormalization dynamics has an accumulation point that is a minimal flat surface of the same type. Since the ergodic measures of these flat surfaces are the same as the tracial states of the AF algebra associated to the positive part of the bi-infinite Bratteli diagram, one can translate his result into a unique trace criterion for AF algebras (the resulting criterion is corollary 4.1.10).

We show this result and a stronger one can be proven for AF algebras when one works with the renormalization procedure on the AF algebra rather than the one on the flat surface. First, we will give the general Masur's criterion argument in theorem 4.1.5. Our techniques of proof differ from Treviño's. His primary tool is Sobolev theory, and ours is the induced operator on traces, the "renormalization operator" and its properties listed in proposition 3.1.10. We now prove Masur's criterion.

Theorem 4.1.5. *Suppose $(\mathcal{A}, E, \sigma)$ is a unital renormalization procedure, and x is an infinitely renormalizable parameter. If x is finite type along $\{\sigma^{n_k}x\}_{k\in\mathbb{N}}$ with a-priori bounded generators $a_1, ..., a_M$, then $T^1(A_x)$ contains at most one trace.*

Proof. Suppose, for the sake of contradiction, that $T^1_x(\mathcal{A})$ contains more than one trace. Since $\mathcal{A}_x$ is unital, by Alaoglu's theorem (Pedersen theorem 2.5.2 [36]), $T^1_x(\mathcal{A})$ is weak* compact, and so, by the Krein-Milman theorem (Pedersen theorem 2.5.4), $T^1_x(\mathcal{A})$ is the weak* closed convex span of its extreme points. Therefore, there are at least two distinct extremal traces τ_0 and τ_1 in $T^1(\mathcal{A}_x)$. For n in $\mathbb{N}\cup\{0\}$ and λ in $\mathbb{C}$, define

$\tau^n_\lambda := \lambda(S^n_x\tau_1 - S^n_x\tau_0) + S^n_x\tau_0$ and $\mu^n := \frac{\|\tau_1^n\|}{\|\tau_0^n\|}$ for n in $\mathbb{N}$. Notice that if we swap τ_0 with τ_1, the new μ^n is the reciprocal of the old. Therefore, we may assume, by re-indexing the traces if necessary, there is a sub-sequence $\{j_k\}_{k\in\mathbb{N}}$ of $\{n_k\}_{k\in\mathbb{N}}$ and a positive number L in $\mathbb{R}$ such that $\mu^{j_k} \leq L$ for all k in $\mathbb{N}$. For $m \leq M$ and k in $\mathbb{N}$, the affine linear map l^k_m, defined for λ in $\mathbb{R}$ as $l^k_m(\lambda) = \tau^{j_k}_\lambda(a_m)$ has $\lambda^k_m = \frac{-\tau_0^{j_k}(a_m)}{\tau_1^{j_k}(a_m)-\tau_0^{j_k}(a_m)}$ as its zero. Write $\lambda^k_m = \frac{1}{1-\mu^k_m}$, where $\mu^k_m = \frac{\tau_1^{j_k}(a_m)}{\tau_0^{j_k}(a_m)}$. Since a_m for $m \leq M$ is a-priori bounded, there is a $D > 0$ and K in $\mathbb{N}$ such that $0 < \frac{\tau_1^{j_k}(a_m)}{\|\tau_1^{j_k}\|}\frac{\|\tau_0^{j_k}\|}{\tau_0^{j_k}(a_m)} < D$ for all $m \leq M$ and $k \geq K$. Therefore, $0 < \mu^k_m = \frac{\tau_1^{j_k}(a_m)}{\|\tau_1^{j_k}\|}\frac{\|\tau_0^{j_k}\|}{\tau_0^{j_k}(a_m)}\mu^{j_k} < DL$ for all $m \leq M$ and $k \geq K$. It follows that $|\lambda^k_m| > \delta := \min\{\frac{1}{|1-DL|}, 1\}$ for all $m \leq M$ and $k \geq K$. Hence, for any $m \leq M$ and $k \geq K$, the zero of the line l^k_m does not intersect $[-\delta, \delta]$. Therefore, the intermediate value theorem implies that for every $m \leq M$ and $k \geq K$, $l^k_m(\lambda)$ is either positive for all λ in $[-\delta, \delta]$ or negative for all λ in $[-\delta, \delta]$. Since $l^k_m(0) = \tau_0^{j_k}(a_m) \geq 0$, the former holds for every $m \leq M$ and $k \geq K$.

Now, let a be a positive element in $\mathcal{A}_x$ and $\epsilon > 0$. Since $a_1, ..., a_M$ are generators, there is a $j = j_{k_0}$ for some $k_0 \geq K$ in $\mathbb{N}$, left F^j_x basis $\mathcal{B}_1, ..., \mathcal{B}_N$, an element b in the positive linear span of $\{\Delta(\mathcal{B}_n)(a_m(\sigma^j x))\}_{n\leq N, m\leq M}$, and an element c in $NC(a)$ such that $\|b - c\| \leq \epsilon$. By the definition of τ^j_λ and the above argument, we have $\tau^0_\lambda(\Delta(\mathcal{B}_n)(a_m(\sigma^j x))) = \tau^j_\lambda(a_m) > 0$ for all $m \leq M$, $n \leq N$, and λ in $[-\delta, \delta]$. Since b is a positive linear combination of elements in $\{\Delta(\mathcal{B}_n)(a_m(\sigma^j x))\}_{n\leq N, m\leq M}$, it follows that $\tau^0_\lambda(b) \geq 0$ for all λ in $[-\delta, \delta]$. Therefore,

$$\tau^0_\lambda(a) = \tau^0_\lambda(c) \geq \tau^0_\lambda(b) - |\tau^0_\lambda(b - c)|$$

for all λ in $[-\delta, \delta]$. Since $\|\tau^0_\lambda\| \leq 2\delta + 1$ for all λ in $[-\delta, \delta]$, and $\|b - c\| \leq \epsilon$ it follows that

$$\tau^0_\lambda(b) - |\tau^0_\lambda(b - c)| \geq \tau^0_\lambda(b) - (2\delta + 1)\epsilon \geq -(2\delta + 1)\epsilon.$$

Therefore, $\tau^0_\lambda(a) \geq -(2\delta + 1)\epsilon$ for all λ in $[-\delta, \delta]$. Since ϵ was arbitrary, it follows that $\tau^0_\lambda(a) \geq 0$ for all positive elements a in $\mathcal{A}_x$ and all λ in $[-\delta, \delta]$. Also, $\tau^0_\lambda(1) = 1$, so τ^0_λ is in $T^1(\mathcal{A}_x)$ for all λ in $[-\delta, \delta]$. τ_0 and τ_1 were chosen to be distinct, so $\lambda \mapsto \tau^0_\lambda$ for λ in $[-\delta, \delta]$ is a non-trivial line of traces in $T^1(\mathcal{A}_x)$ such that $\tau^0_0 = \tau_0$. Therefore, τ_0 is not an extreme point, which is a contradiction.

□

Corollary 4.1.6. *Let* $(\mathcal{A}, \alpha)$ *be a D-algebra,* $D \subseteq C_b(X)$, *that is continuous over* X *and has unital fibres* $\mathcal{A}_x$ *for all* x *in* X. *Suppose* $(\mathcal{A}, E, \sigma)$ *is a renormalization procedure, and a point* x *in* X *is finite type along a sequence* $\{\sigma^{n_k}x\}_{k\in\mathbb{N}}$ *that converges to a point* x_∞, *and has generators* $a_1, ..., a_M$ *that are non-zero at* x_∞. *Then, if* $\mathcal{A}_{x_\infty}$ *is simple,* $\mathcal{A}_x$ *has at most one trace.*

4.1.2 AF Algebra Renormalization

Let X be as in section 2.1.1, and let $(\mathcal{A}, \alpha)$ be the section algebra of the field of AF algebras $\{\mathcal{A}_x\}_{x \in X}$ defined in the same section. the E_x^k defined in section 3.2.1 is isomorphic (as $\mathcal{A}_{\sigma^k x}$-$\mathcal{A}_x$ Hilbert bi-modules) to the composite $\tilde{E}_x^k := E_{\sigma^{k-1}x} \otimes_{\mathcal{A}_{\sigma^{k-1}x}} E_{\sigma^{k-2}x} \otimes_{\mathcal{A}_{\sigma^{k-2}x}} ... \otimes_{A_{\sigma x}} E_x$. This is easily seen first at the finite stages. For $n \geq k$, consider the $A(\sigma^k(x_{[1,n]}))$-$A(x_{[1,n]})$ bi-module $\tilde{E}^k(x_{[1,n]}) := E(\sigma^{k-1}(x_{[1,n]})) \otimes_{A(\sigma^{k-1}(x_{[1,n]})} E(\sigma^{k-2}(x_{[1,n]})) \otimes_{A(\sigma^{k-2}(x_{[1,n]}))} \otimes ... \otimes_{A_{\sigma(x_{[1,n]})}} E(x_{[1,n]})$. The map $\Phi_{n,k} : \tilde{E}^k(x_{[1,n]}) \mapsto E^k(x_{[1,n]})$ defined on basic tensors $T_1 \otimes T_2 \otimes ... \otimes T_k$ in $\tilde{E}^k(x_{[1,n]})$ as the operator composite $T_1 \circ T_2 \circ ... \circ T_k = \Phi_{n,k}(T_1 \otimes ... \otimes T_k)$ is easily seen to be a Hilbert bi-module isomorphism commuting with the inductive sequences $\tilde{E}^k(x_{[1,k]}) \to \tilde{E}^k(x_{[1,k+1]}) \to \tilde{E}^k(x_{[1,k+2]}) \to ....$ and $E^k(x_{[1,k]}) \to E^k(x_{[1,k+1]}) \to E^k(x_{[1,k+2]}) \to ....$ Therefore, the limit map $\Phi_k := \lim_{n \to \infty} \Phi_{n,k} : \tilde{E}_x^k \mapsto E_x^k$ (which exists, due to the universal property of inductive limits) is an isomorphism of Hilbert $A_{\sigma^k x}$-A_x bi-modules. We will work with E_x^k from now on.

Proposition 4.1.7. *Let $x = (x_1, x_2, ...)$ be a sequence of matrices in X, and suppose x_{n_k} is a sub-sequence for which $r(x_{n_k}) = r(x_{n_{k'}}) = M$ for all k, k' in $\mathbb{N}$. For a natural number m, $m \leq M$, let a'_m be the element in A_0, defined pointwise for N in $\mathbb{N}$, $n \leq N$ as*

$$a'_m(N)(\delta_n) = 0 \text{ if } N \neq M \text{ or } n \neq m, \text{ and } a'_m(M)(\delta_m) = \delta_m.$$

Let $a_m = \mu_0(a'_m)$. Then, x is finite type along $\{\sigma^{n_k} x\}_{k \in \mathbb{N}}$ with generators $a_1, ..., a_M$ defined above.

Proof. For p in $P^{n_k}(x)$ Let χ_p be the element in $A(x_{[1,n_k]})$ defined for δ_q, q in $P^{n_k}(x)$ as $\chi_p(\delta_q) = 0$ if $q \neq p$ and $\chi_p(\delta_p) = \delta_p$. Consider the equivalence

$$\mathbb{C}^M = A(\sigma^{n_k} x_{[1,n_k]}) \to E^{n_k}(x_{[1,n_k]}) \leftarrow A(x_{[1,n_k]}).$$

For each $m \leq M$ choose a path q_m in $P^{n_k}(x)$ with $r(q_m) = m$. This can be done because the matrices x_n have every column non-zero. for $m = r(p)$, let $q_{r(p)} = p$. Then, the operators $T_{m,q_m} : H(x_{[1,n_k]}) \mapsto H(M)$ defined for q in $P^{n_k}(x)$ as

$$T_{m,q_m}(\delta_q) = 0 \text{ if } q \neq q_m \text{ and } T_{m,q_m}(\delta_{q_m}) = \delta_m$$

are in $E^{n_k}(x_{[1,n_k]})$. Since $T_{m,q_m} T^*_{m,q_m}$ is the projection onto the m^{th} co-ordinate of $\mathbb{C}^M$, it follows that $\sum_{m=1}^{M} {}_*\langle T_{m,q_m}, T_{m,q_m} \rangle = 1$. Therefore, $\mathcal{B}_p = \{T_{m,q_m}\}_{m \leq M}$ is a left basis for $E^{n_k}(x_{[1,n_k]})$. Since $a'_{r(p)}(M)$ is also the projection onto the $r(p)^{th}$ co-ordinate of $\mathbb{C}^M$, we have $\Delta(\mathcal{B}_p)(a'_{r(p)}(M)) = T^*_{r(p),p} T_{r(p),p} = \chi_p$. For any positive element a in $A(x_{[1,n_k]})$, $NC(a)$ contains its diagonal, which is an element of the form $b = \sum_{p \in P^{n_k}(x)} t_p \chi_p$, where t_p are non-negative numbers. Therefore, $\sum_{p \in P^{n_k}(x)} t_p \Delta(\mathcal{B}_p)(a'_{r(p)}(M))$ is in $NC(a)$.

Hence, for every $k' \geq k$ there are left basis $\mathcal{B}'_1, ..., \mathcal{B}'_N$ for $E^{n_{k'}}(x_{[1,n_{k'}]})$ for which the positive cone spanned by $\{\Delta(\mathcal{B}'_n)(a'_m(M))\}_{n \leq N, m \leq M}$ intersects $NC(\varphi_{x_{[1,n_{k'}]}, x_{[1,n_k]}}(a))$. Since $a_m(\sigma^{n_{k'}}x) = \mu_{0,\sigma^{n_{k'}}x}(a'_m(M))$, by applying the respective inductive limits to the elements a'_m, $\varphi_{x_{[1,n_{k'}]}, x_{[1,n_k]}}(a)$ and basis elements of $\mathcal{B}'_n$, we have that for all $k' > k$, there are left $E_x^{n_{k'}}$ basis $\mathcal{B}_1$,...,$\mathcal{B}_N$ for which the positive cone spanned by $\{\Delta(\mathcal{B}_n)(a_m(\sigma^{n_{k'}}x))\}_{n \leq N, m \leq M}$ intersects $NC(\mu_{n_k,x}(a))$. Therefore, we have proven the proposition for positive elements in $\bigcup_{k=1}^{\infty} \mu_{n_k,x}(A(x_{[1,n_k]}))$. Since every positive element in $\mathcal{A}_x$ can be approximated arbitrarily by positive elements in $\bigcup_{k=1}^{\infty} \mu_{n_k,x}(A(x_{[1,n_k]}))$, the proposition follows. □

Definition 4.1.8. *Let $z = (z_1, z_2, .., z_n)$ be a path of length n in $(\mathcal{M}, \mathbb{N}, r, s)$. We say z is simple if the matrix $z_1 z_2 ... z_n$ has all non-zero entries. This is equivalent to saying, for the directed graph $B(z) = (E(z), V(z), r, s)$, there is a path connecting any pair of vertices v in $V_0(z)$ and w in $V_n(z)$.*

Proposition 4.1.9. *Suppose $x = (x_1, x_2, ...)$ is a sequence of matrices in X that has a finite simple path z in P^n occuring infinitely often; i.e., there is an increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ such that $x_{[n_k+1, n_k+n]} = z$ for all k in $\mathbb{N}$. Then, $M := r(x_{n_k}) = r(x_{n_{k'}})$, for k, k' in $\mathbb{N}$, so x is finite type along $\sigma^{n_k}x$ with generators $a_1, ..., a_M$ as in proposition 4.1.7. Moreover, $a_1, .., a_M$ are a-priori bounded. Consequently, $\mathcal{A}_x$ has a unique tracial state.*

Proof. Let τ be a trace in $T^1(A_x)$ and $\{m_k\}_{k \in \mathbb{N}}$ an increasing sub-sequence of $\{n_k\}_{k \in \mathbb{N}}$. For each k in $\mathbb{N}$ the sub-algebras $\mu_{n,\sigma^{m_k}}(A(\sigma^{m_k}(x_{[m_k+1, m_k+n]}))) = \mu_{n,\sigma^{m_k}x}(A(z))$ of $A_{\sigma^{m_k}x}$ contain $a_m(\sigma^{m_k}x)$ for $m \leq M$. Moreover, $a_m(\sigma^{m_k}x) = \mu_{n,\sigma^{m_k}x}(\chi_m)$, where χ_m is the operator in $A(z)$ defined for δ_p, p in $P^n(z)$ as

$$\chi_m(\delta_p) = 0 \text{ if } s(p) \neq m \text{ and } \chi_m(\delta_p) = \delta_p \text{ if } s(p) = m.$$

Consider the sequence traces $\tau_k = \frac{S_x^{m_k}\tau \circ \mu_{n,\sigma^{m_k}x}}{\|S_x^{m_k}\tau\|}$ in $T^1(A(z))$. And let τ_* be a weak* accumulation point of $\{\tau_k\}_{k \in \mathbb{N}}$ in $T^1(A(z))$ (such a trace exists because $A(z)$ is unital). For p in $P^n(z)$, let χ_p be operator in $A(z)$ which is the projection onto the 1-dimensional sub-space spanned by δ_p. Since $\sum_{p \in P^n(z)} \chi_p = 1$, there is a path q in $P^n(z)$ for which $\tau_*(\chi_q) \neq 0$. For each $m \leq M$, let q_m be a path from the vertex m to the vertex $r(q)$. Such a path exists because z is simple. Let T_{q,q_m} be the operator in $A(z)$ defined for δ_p, p in $P^n(z)$ as

$$T_{q,q_m}(p) = 0 \text{ if } p \neq q_m \text{ and } T_{q,q_m}(q_m) = q.$$

Then, $T^*_{q,q_m} T_{q,q_m} = \chi_{q_m}$ and $T_{q,q_m} T^*_{q,q_m} = \chi_q$. Therefore, $\tau_*(\chi_{q_m}) = \tau_*(\chi_q) \neq 0$. Since $\chi_{q_m} \leq \chi_m$, it follows that $\tau_*(\chi_m) \neq 0$. Hence, for any $m \leq M$, the sequence $\frac{S_x^{m_k}\tau \circ \mu_{n,\sigma^{m_k}x}(\chi_m)}{\|S_x^{m_k}\tau\|} = \frac{S_x^{m_k}\tau(a_m(\sigma^{m_k}x)}{\|S_x^{m_k}\tau\|}$ does not converge to zero. Since the trace τ and the increasing sub-sequence $\{m_k\}$ of $\{n_k\}$ were arbitrary, it follows that $a_1, ..., a_m$ are a-priori bounded. Therefore, by proposition 4.1.7 and theorem 4.1.5, $\mathcal{A}_x$ has at most one tracial state. Every finite dimensional C^*-algebra has a tracial state. Since a unital AF algebra A is the inductive limit of an inductive sequence $\{\varphi_n : A_n \mapsto A_{n+1}\}_{n\in\mathbb{N}}$ of finite dimensional C^*-algebras A_n, with φ_n unital, its tracial state space is the inverse limit of the sequence $\{(\varphi_n)^* : T^1(A_{n+1}) \mapsto T^1(A_n)\}_{n\in\mathbb{N}}$, where $(\varphi_n)^*$ sends τ in $T^1(A_{n+1})$ to $\tau \circ \varphi_n$. Therefore, $T^1(A)$, being an inverse limit of non-empty compact spaces, is non-empty. Hence, $\mathcal{A}_x$ has a unique tracial state. □

From an interval exchange mapping T, Veech produces, using the renormalizations of it, a sequence of square matrices $A_1, A_2, ...$ with $\det(A_i) = \pm 1$, for i in $\mathbb{N}$, and proved that if the sequence had a finite simple path occuring infinitely often, then T is uniquely ergodic (Veech proposition 3.30 [53]). It can be shown that the tracial state space of the AF algebra associated to $A_1, A_2, ...$ is isomorphic to the space of ergodic probability measures for T, so Veech's criterion is a corollary of ours. We also recover Theorem 1 of Treviño [51], but recasted in terms of AF algebras:

Corollary 4.1.10. *If x is a sequence of matrices in X such that $\sigma^{n_x}x$ has an accumulation point x_* for which the AF algebra $\mathcal{A}_{x_*}$ is simple, then $\mathcal{A}_x$ has a unique tracial state.*

Remark 4.1.11. *It would be interesting to put our a-priori bounded condition, relative to the generators constructed in proposition 4.1.7, in the context of the work of Bezuglyi, Kwiatkowski, Medynets and Solomyak (BKMS)* [2]*, which focuses on determining unique trace criterion for AF algebras in terms of Bratteli diagrams. We expect there is significant overlap with our a-priori bounded condition and their notion of "exact finite rank" (BKMS definition 3.5* [2]*). Their methods of proof are not related to ours and specific to Bratteli diagrams. Proposition 4.1.9 above is implied by BKMS proposition 4.12* [2] *in the special case when $x = (x_1, x_2, ...)$ in X satisfies, in addition to the hypothesis in proposition 4.1.9, $r(x_i) = r(x_j)$ for all i, j in $\mathbb{N}$ (which can be achieved from a telescoping) and each x_i has a positive power with all non-zero entries.*

4.2 Groupoid Renormalization

We now find groupoid analogs to the ideas presented in section 4.1. We will only consider the finite type condition (recall definition 4.1.2) with one generator $a_1 = 1$. When this happens, the generators are automatically a-priori bounded (recall definition 4.1.3), so having the finite type condition will automatically imply the Masur's criterion.

When we first consider this condition, we will frame it in terms of a sequence of groupoid equivalences $G_n \to E_n \leftarrow G_0$, where G_0 should be thought of as a fibre groupoid in a groupoid bundle, and the E_n as the n^{th} iteration of a renormalization procedure. Then, we will prove in theorem 4.2.5 that, whenever a groupoid has such a sequence of equivalences, there can be at most one G_0-invariant probability measure on its unit space G_0^0. This will essentially be the argument presented for Masur's criterion in the C^*-algebra setting (theorem 4.1.5) with some simplifications.

We will then introduce renormalization procedures for groupoid bundles in definition 4.2.7. We will consider a condition on the first iteration of a procedure that will give us a "Masur's criterion" for when a fibre in the groupoid bundle is simple, and prove that when a certain basis generated from the renormalization procedure contains arbitrarily fine covers of (uniformly) bounded dimension, the fibre groupoid will have at most one invariant probability measure on its unit space (theorem 4.2.13).

We will specialize these criterion to the case of groupoid bundles coming from continuous families of isometries acting on uniformly finite dimensional compact metric spaces (see definition 4.2.14). The results in this section will imply, for instance the well-known result that rotation of the circle by an irrational angle is a minimal and uniquely ergodic dynamical system. We then consider unique invariant probability measure criterion for the IFS groupoids presented in section 2.2.3. Both groupoid equivalences presented in sections 3.4.2, 3.4.3 will be shown to be examples of renormalization procedures and will be used to prove the unique measure results above. It should be mentioned the groupoid equivalences presented for AF groupoids in section 3.4.1 are also examples of renormalizations procedure for groupoid bundles, however we will not have an occasion to show this because we already established a unique trace criterion for AF algebras in section 4.1.2.

We start by introducing the finite type condition with one generator $a_1 = 1$ in the language of groupoid equiavelences. First, it will be convenient throughout this section to have a notion of dimension for a cover.

Definition 4.2.1. *Let X be a topological space and $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of X indexed by a set $\mathcal{A}$. We say $\mathcal{U}$ is **finite dimensional** if there is a d in $\mathbb{N}$ such that for every α in $\mathcal{A}$, the number of β in $\mathcal{A}$ for which $U_\beta \cap U_\alpha \neq \emptyset$ is bounded by d. We will let $d(\mathcal{U})$ be the minimal such d, and call this number the **dimension** of $\mathcal{U}$.*

The dimension of a covering is also known as its "order" (see Munkres page 305 [32]). A topological space X has finite covering dimension if there is a d in $\mathbb{N}$ for which every open cover has an open refinement of dimension bounded by d (see also Munkres page 305).

Definition 4.2.2. *Let $G_0 = G$, and $\{G_n\}_{n\geq 1}$ be étale groupoids, with G^0 a metric space, along with étale groupoid equivalences $G_n \to E_n \leftarrow G$ (definition 3.3.4). We say $\{E_n\}_{n\geq 1}$ satisifies the **strong finite type condition** if there are numbers D and M in $\mathbb{N}$ such that the following holds:*

For every $\epsilon > 0$ there is N in $\mathbb{N}$ such that for all $n \geq N$, there are open sets $\{U_{i,j}\}_{i \leq \tilde{M}, j \leq l}$, $\tilde{M} \leq M$, of E_n with the properties:

- *$\rho, \sigma|_{U_{i,j}}$ are homeomorphisms (i.e., $U_{i,j}$ is an E_n-set) for all $i \leq \tilde{M}$, $j \leq l$,*
- *for fixed j, $\{\rho(U_{i,j})\}_{i \leq \tilde{M}}$ covers G_n^0, and*
- *$\{\bigcup_{i \leq \tilde{M}} \sigma(U_{i,j})\}_{j \leq l}$ is an at most D-dimensional cover of G_0 by sets of diameter less than ϵ.*

We now translate the above definition into a statement about the existence of certain functions in the corresponding bi-module. Given an open subset U of a locally compact space X, we will say a collection of open sets $\{V_m\}_{m \in \mathbb{N}}$ is an **exhaustion of** U if for each m in $\mathbb{N}$, $\overline{V_m}$ is compact and $\overline{V_m} \subseteq V_{m+1} \subseteq U$, with $\bigcup_{m \in \mathbb{N}} V_m = U$.

Proposition 4.2.3. *For n in $\mathbb{N}$, let $G_n \to E_n \leftarrow G$ be an étale groupoid equivalence, and assume $\{E_n\}_{n \geq 1}$ satisfies the strong finite type condition. Then, there are constants M, D in $\mathbb{N}$ for which following holds: for every compact set $K \subseteq G^0$ and $\epsilon > 0$, there is N in $\mathbb{N}$ and positive functions $\{\psi_{i,j}^m\}_{i \leq M, j \leq l}^{m \in \mathbb{N}} \subseteq C_c(E_n)$ for each $n \geq N$ such that, for $j \leq l$, and m in $\mathbb{N}$,*

(i) $\psi_j^m := \sum_{i=1}^M \langle \sqrt{\psi_{i,j}^m}, \sqrt{\psi_{i,j}^m} \rangle_*$ satisfy $\sum_{j=1}^l \psi_j^m = 1$ on a neighbourhood of K and $\sum_{j=1}^l \psi_j^m \leq 1$ on G^0,

(ii) the diameter of the support of ψ_j^m is less than ϵ, and

(iii) $\eta_j^m := \sum_{i=1}^M {}_*\langle \sqrt{\psi_{i,j}^m}, \sqrt{\psi_{i,j}^m} \rangle$ satisfy $\frac{e_j^m}{MD} \leq \eta_j^m \leq M$ for some approximate unit e_j^m in $C_c(G_n^0)$.

Proof. Fix a compact set K and $\epsilon > 0$. Then, let $\{U_{i,j}\}_{i \leq M, j \leq l}$ be open sets in E_n as in definition 4.2.2 for the fixed ϵ and $n \geq N$. For each $i \leq M$ and $j \leq l$, choose an exhaustion $\{V_{i,j}^m\}_{m \in \mathbb{N}}$ of $U_{i,j}$ (as in the above paragraph) such that, for any $j \leq l$, the collection of open sets $\{W_j^m := \bigcup_{i \leq M} \rho(V_{i,j}^m)\}_{m \in \mathbb{N}}$ is an exhaustion of G_n^0, and, for any m in $\mathbb{N}$, the collection of open sets $\{V_j^m := \bigcup_{i \leq M} \sigma(V_{i,j}^m)\}_{j \leq l}$ is a D-dimensional cover of K. Now, for $i \leq M$ and $j \leq l$, let $\phi_{i,j}^m$, for m in $\mathbb{N}$, be a positive function in $C_c(V_{i,j}^{m+1})$ such that

$$0 \leq \phi_{i,j}^m \leq 1 \text{ and } \phi_{i,j}^m = 1 \text{ on } \overline{V_{i,j}^m}. \tag{*}$$

By proposition 3.3.17, $\phi_j^m := \sum_{i=1}^M \langle \sqrt{\phi_{i,j}^m}, \sqrt{\phi_{i,j}^m} \rangle_* = \sum_{i=1}^M \phi_{i,j}^m \circ \sigma^{-1}$. Since $\{V_j^m\}_{j \leq l}$ is a D-dimensional cover of K, by (*), we have $\sum_{j=1}^l \phi_j^m = \sum_{j=1}^l \sum_{i=1}^M \phi_{i,j}^m \circ \sigma^{-1} \geq 1$ on an open neighbourhood of K, and $\sum_{j=1}^l \phi_j^m \leq MD$ on G^0. Therefore, for each m in $\mathbb{N}$, we can find a bounded, positive and invertible function ϕ^m in $C(G_0)$ such that $1 \leq \phi^m \leq MD$, $\sum_{j=1}^l \frac{\phi_j^m}{\phi^m} = 1$ on an open neighbourhood of K and bounded by 1 on all of G_0. Let $\psi_j^m := \frac{\phi_j^m}{\phi^m}$ and $\psi_{i,j}^m := \frac{\phi_{i,j}^m}{\phi^m \circ \sigma}$. Again, using proposition 3.3.17, we have $\sum_{i=1}^{M}\langle\sqrt{\psi^m_{i,j}},\sqrt{\psi^m_{i,j}}\rangle_* = \psi^m_j$. Now, we consider the functions in the left inner product. Let $\varphi^m_j := \sum_{i=1}^{M} {}_*\langle\sqrt{\phi^m_{i,j}},\sqrt{\phi^m_{i,j}}\rangle = \sum_{i=1}^{M}\phi^m_{i,j}\circ\rho^{-1}$. Since W^m_j is an exhaustion of of G^0_n, by (*), there is a approximate unit $\{e^m_j\}_{m\in\mathbb{N}}$ for $C_0(G^0_n)$ such that e^m_j is in $C_c(W^m_j)$ and $\varphi^m_k = \sum_{i=1}^{M}\phi^m_{i,j}\circ\rho^{-1} \geq e^m_j$. Also, $\varphi^m_j \leq M$. Since $1 \leq \phi^m\circ\sigma \leq MD$, $\frac{e^m_j}{MD} \leq \eta^m_j := \sum_{i=1}^{M} {}_*\langle\sqrt{\psi^m_{i,j}},\sqrt{\psi^m_{i,j}}\rangle \leq M$. ☐

Definition 4.2.4. *Let G be an étale groupoid, and μ a Radon measure on G^0 (see Pedersen definition 6.3.1* [36]*). μ is **G-invariant** if for every open G-set U (definition 2.2.1), we have $\mu(r(U)) = \mu(s(U))$. Recall from Putnam lemma 3.4.2 that this is equivalent to the property that for every G-map $\gamma : U \mapsto V$ and f in $C_c(V)$, we have $\int f\circ\gamma\ d\mu = \int f\ d\mu$*

For a finite Radon measure μ on G^0, we will let $\omega_\mu : C_0(G^0) \mapsto \mathbb{C}$ denote the positive linear functional obtained by integration with respect to μ. Recall that the Radon-Nikodyn theorem (Pedersen 6.5.6 [36]) states there is a 1-1 correspondence between finite Radon measures and positive linear functionals on $C_0(G^0)$ via the map $\mu \mapsto \omega_\mu$, and that $\|\omega_\mu\| = \mu(G^0)$. The set of G-invariant measures μ such that $\mu(G^0) \leq 1$ is a weak* closed set (when viewed as positive linear functionals): suppose $\{\mu_\lambda\}_{\lambda\in\Lambda}$ is a net of G-invariant measures such that $\lim_\lambda \mu_\lambda = \mu$. Then, for a G-map $\gamma : U \mapsto V$ and an f in $C_c(V)$, by the definition of the weak* topology (see Pedersen 2.4.8 [36]), we have

$$\int f\circ\gamma\ d\mu = \lim_\lambda \int f\circ\gamma\ d\mu_\lambda = \lim_\lambda \int f\ d\mu_\lambda = \int f\ d\mu.$$

Therefore, μ is G-invariant. The fact that $\|\omega_\mu\| \leq 1$ follows from Alaoglu's theorem (Pedersen 2.5.2 [36]).

Let $E : C^*(G) \mapsto C_0(G^0)$ be the conditional expectation defined in Putnam section 3.4.1 [40]. Then, by Putnam theorem 3.4.4 [40], for a finite G-invariant Radon measure μ, the positive linear functional $\tau_\mu = \omega_\mu\circ E$ is a trace on $C^*(G)$. Obviously $\|\tau_\mu\| = \|\omega_\mu\|$, and the map $\omega_\mu \mapsto \tau_\mu$ is weak* continuous, and injective. We shall denote $T^{\leq 1}_G$ to be set of traces of the form τ_μ for some G-invariant measure μ with $\mu(G^0) \leq 1$. Also, denote T^1_G to be set of all traces induced from G-invariant probability measures. T^1_G happens to be all the tracial states on $C^*(G)$ when G is an equivalence relation (Putnam theorem 3.4.5 [40]), but this need not be true for a general étale groupoid.

Theorem 4.2.5. *If $\{G_n \to E_n \leftarrow G\}_{n\geq 1}$ is a sequence of étale groupoid equivalences that satisifies the strong finite type condition (definition 4.2.2), then there is at most one G-invariant probability measure for G^0.*

Proof. As shown above, $T^{\leq 1}_G$ is a weak* compact convex set, so by the Krein-Milman theorem (Pedersen 2.5.4), $T^{\leq 1}_G$ is the (weak*) closure of the convex span of its extreme points. Now, it follows that if $T^{\leq 1}_G$ has at most one non-zero extreme point, then either $T^{\leq 1}_G$ is isometric (as a convex set) to the interval $[0, 1]$ or 0. So, to prove there is at most one G-invariant probability measure, it suffices to show there is at most one non-zero extremal trace of $T_G^{\leq 1}$. We will suppose, to the contrary, that there are two non-zero extremal traces τ_0, τ_1 of $T_G^{\leq 1}$. Since they are non-zero and extremal, it follows that $\|\tau_0\| = \|\tau_1\| = 1$. Let $S_n = S_{C^*(E_n)}$ be the trace operator (definition 3.1.10) induced from the $C^*(G_n)$-$C^*(G)$ imprimitivity bi-module $C^*(E)$. It maps finite traces to finite traces because, by proposition 4.2.3 (*iii*), we can choose a left basis (definition 3.1.7) for $C^*(E_n)$ with at most M-summands in each term. So, $\tau_i^n := S_n\tau_i$ is in $T^{<\infty}(C^*(G_n)$ for $i = 0, 1$. Define $\tau_\lambda^n = \lambda(\tau_1^n - \tau_0^n) + \tau_0^n$ for λ in $\mathbb{R}$, and n in $\mathbb{N} \cup \{0\}$. By taking a sub-sequence of E_n and possibly swapping τ_0 with τ_1, we may assume without loss of generality that the sequence $\lambda_n = \frac{\|\tau_0^n\|}{\|\tau_0^n\| - \|\tau_1^n\|}$ does not intersect $[-1/4, 1/4]$. Let f be a positive function in $C_c(G^0)$. By uniform continuity of f, for every $\delta > 0$, there is $\epsilon > 0$ such that $d(x, y) < \epsilon$ for x, y in G, then $\|f(x) - f(y)\| < \delta$. Let $\mathrm{supp}(f) = K$. Now, for the fixed ϵ and K above, by proposition 4.2.3, there is N in $\mathbb{N}$ and functions (depending on K and ϵ) $\{\psi_{i,j}^m\}_{i \leq M, j \leq l}$ in $C_c(E_n)$ for $n \geq N$ and m in $\mathbb{N}$ with the properties as in proposition 4.2.3. Then, for any x_j in $\mathrm{supp}(\psi_j^m)$, $j \leq l$, and m in $\mathbb{N}$, we have

$$|f - \sum_{j=1}^{l} f(x_j)\psi_j^m| = |\sum_{j=1}^{l}(f - f(x_j))\psi_j^m| \leq \delta(\sum_{j=1}^{l} \psi_j^m) \leq \delta.$$

Therefore, for λ in $[-1/4, 1/4]$, using the above inequality, $\|\tau_\lambda\| \leq 2|\lambda| + 1$ and $\tau_\lambda(\psi_j^m) = \tau_\lambda^n(\eta_j^m)$, we have

$$\tau_\lambda(f) \geq (\sum_{j=1}^{l} f(x_j)\tau_\lambda(\psi_j^m)) - (2|\lambda| + 1)\delta \geq \sum_{j=1}^{l} f(x_j)\tau_\lambda^n(\eta_j^m) - 2\delta.$$

We now show there is $\mu > 0$ independent of K, ϵ, and $n \geq N$ such that there is large enough m for which $\tau_\lambda^n(\eta_j^m)$ is positive for λ in $[-\mu, \mu]$ and $j \leq l$. This will prove the theorem, as follows: by the last string of inequalities above and positivity of $\tau_\lambda^n(\eta_j^m)$ for λ in $[-\mu, \mu]$, $\mu < 1/4$, we have $\tau_\lambda(f) \geq -2\delta$ for $\lambda \in [-\mu, \mu]$. As δ was arbitrary and μ independent of f, δ, it follows, by density of $C_c(G^0)$ in $C_0(G^0)$, that for all positive functions f in $C_0(G^0)$, $\tau_\lambda(f) \geq 0$ for λ in $[-\mu, \mu]$. Since $\tau_\lambda = \omega_\lambda \circ E$ for a linear functional ω_λ in $C_0(G^0)$, positivity of τ_λ is determined by checking positivity for all positive f in $C_0(G^0)$. Therefore, τ_λ is a positive normed one trace on $C^*(G)$ for all λ in $[-\mu, \mu]$. Since τ_0, τ_1 were assumed distinct, it follows that τ_0 is a non-trivial convex combination of traces $\tau_{-\mu}, \tau_\mu$ in $T_G^{\leq 1}$, contradicting extremality of τ_0 and thus proving the theorem. We now show there is such a μ as described above.

Recall that a sequence $\lambda_n = \frac{a_n}{a_n - b_n}$, $a_n > 0$, $b_n \geq 0$ does not intersect $[-\mu, \mu]$, $\mu < 1$ if and only if $\frac{b_n}{a_n} \leq 1 + \frac{1}{\mu}$ for all n in $\mathbb{N}$. By proposition 4.2.3, for $j \leq l$ and m in $\mathbb{N}$, $\frac{e_j^m}{MD} \leq \eta_j^m \leq M$ for some approximate unit $\{e_j^m\}_{m \in \mathbb{N}}$ and constants M, D in

$\mathbb{N}$ (independent of K and ϵ). Therefore, $\frac{\tau_1^n(\eta_j^m)}{\tau_0^n(\eta_j^m)} \leq M^2 D \frac{\|\tau_1^n\|}{\tau_0^n(e_j^m)}$ for all m in $\mathbb{N}$. Since $\frac{\|\tau_0^n\|}{\|\tau_0^n\|-\|\tau_1^n\|}$ doesn't intersect $[-1/4, 1/4]$ and $\tau_0^n(e_k^m) \to \|\tau_0^n\|$ as $m \to \infty$, it follows that there is an m_n in $\mathbb{N}$ large enough so that $\frac{\|\tau_1^n\|}{\tau_0^n(e_j^{m_n})} \leq 10$. So, for $\mu = \frac{1}{10M^2D-1}$, the ratio $\mu_{n,j,\epsilon} = \frac{\tau_0^n(\eta_j^{m_n})}{\tau_0^n(\eta_j^{m_n})-\tau_1^n(\eta_j^{m_n})}$ doesn't intersect $[-\mu, \mu]$. $\mu_{n,j,\epsilon}$ is the zero for the line $\lambda \mapsto \tau_\lambda^n(\eta_k^{m_n})$, λ in $\mathbb{R}$. Since it doesn't intersect $[-\mu, \mu]$, by the intermediate value theorem, this line is either positive on $[-\mu, \mu]$ or negative. Since $\tau_0^n(\eta_k^{m_n}) \geq 0$, we have the former. $\square$

We now introduce renormalization procedures for étale groupoid bundles. We will first recall the definitions of contractions, dilations, local contractions, and local dilations of metric spaces.

Definition 4.2.6. *Let (X, d_X) and (Y, d_Y) be metric spaces, and let λ be a positive number strictly less than 1. We say a function $f : X \mapsto Y$ is a **λ-contraction** (or **λ-dilation**) if for every x_1, x_2 in X, we have $d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2)$ (or $d_Y(f(x_1), f(x_2)) = \lambda d_X(x_1, x_2)$).*

*We say $f : X \mapsto Y$ is **locally a λ-contraction** (**locally a λ-dilation**) if every x in X has an open neighbourhood U_x such that $f|_{U_x}$ is a λ-contraction (λ-dilation).*

Recall the definition of pullback of a groupoid bundle by a continuous function (2.2.10) and the definition of a groupoid bundle equivalence (3.4.1)

Definition 4.2.7. *Let (G, π) be an étale groupoid bundle over X, and assume (G_x^0, d_x) is a metric space for every x in X. A **renormalization procedure for** (G, π) consists of the following data:*

- *an open subspace X_1 of X and a surjective local homeomorphism $f : X_1 \mapsto X$. Let $i : X_1 \mapsto X$ denote the inclusion,*
- *a groupoid bundle equivalence $f^*(G, \pi) \to E \leftarrow i^*(G, \pi)$,*
- *and for every x in X_1, there is a $\lambda_x < 1$ such that every E_x-map (definition 3.3.4) is locally a λ_x-contraction. We will say E_x is **locally a λ_x-contraction**. We will also assume there is an $\epsilon_X > 0$ such that for every x in X_1 and u in G_x^0, there is a v in G_{fx}^0 and an E_x-map $\gamma : B_{\epsilon_X}(v) \mapsto U$ that is a λ_x-contraction and $\gamma(v) = u$. In short, we say E is **ϵ_X-iterable**.*

  *If for all x in X_1, every E_x-map is locally a λ_x-dilation and for every u in G_x^0 there is a v in G_{fx}^0 and an E_x-map $\gamma : B_{\epsilon_X}(v) \mapsto B_{\epsilon_X \lambda_x}(u)$ that is a λ_x-dilation and $\gamma(v) = u$, we will say E is an **ϵ_X-iterable dilation***

*We will denote the above data by (G, E, f). f will be called the **renormalization dynamics** and $X_n := \bigcap_{k=1}^n f^{-k}X$ will be called the n-**renormalizable parameters**.*

*When $n = \infty$, X_∞ is called the **infinitely renormalizable parameters**. for x in X_n, we will denote $\lambda_x^n = \Pi_{k=0}^{n-1}\lambda_{f^k x}$.*

If (G, E, f) is a renormalization procedure of an étale groupoid bundle (G, π) over X, for x in X_n, the product $E_{f^{n-1}x} \times_{G_{f^{n-1}x}} E_{f^{n-2}x} \times .. \times_{G_{fx}} E_x$, constructed after proposition 3.3.5, will be denoted as E_x^n.

Proposition 4.2.8. *Let (G, E, f) be a renormalization procedure for an étale groupoid bundle (G, π) over X, and let x be an infinitely renormalizable parameter. Then, for any $\epsilon \leq \epsilon_X$, each n in $\mathbb{N}$ and any v in G_x^0 there is an open neighbourhood V of v with diameter bounded by $\lambda_x^n 2\epsilon$ which is the image of an E_x^n-map $\gamma : B_\epsilon(u) \mapsto V$, where u is in $G_{f^n x}^0$, that is a λ_x^n-contraction and $\gamma(u) = v$.*

Proof. Since E is ϵ-iterable, we can find u_i in $E_{f^{i-1}x}$ for $1 \leq i \leq n$, $E_{f^{i-1}x}$-maps γ_i with domain containing $B_\epsilon(u_i)$, and $\gamma_i(u_i) = u_{i-1}$ for $1 \leq i \leq n$, where we denote $u_0 = v$. Since the contraction factor of γ_i is $\lambda_{f^{i-1}x}$, it follows that $\gamma_i(B_\epsilon(u_i)) \subseteq B_{\epsilon\lambda_{f^{i-1}x}}(u_{i-1})$, so we can let $u_n = u$, and consider the composite $\gamma := \gamma_1 \circ \gamma_2 \circ ... \circ \gamma_n : B_\epsilon(u) \mapsto V$, where $V = \gamma_1 \circ \gamma_2 \circ ... \circ \gamma_n(B_\epsilon(u))$. Therefore $\gamma(u) = v$. By proposition 3.3.8, the composite of $E_{f^{i-1}x}$ maps, $1 \leq i \leq n$, is an E_x^n-map. V has diameter at most $\lambda_x^n 2\epsilon$ because the contraction factor of a composite of contractions is bounded by the product of each individual factor. Hence, γ is also a λ_x^n-contraction. □

Recall the definition of ϵ-minimality in 2.2.5 for an étale groupoid with a metric d on its unit space. The next proposition could be quite practical in proving minimality of a fibre groupoid G_x at an infinitely renormalizable parameter when the iterability constant ϵ_X is large.

Proposition 4.2.9. *Suppose (G, E, f) is a renormalization procedure for a groupoid bundle (G, π) with metrics d_x on the unit spaces G_x^0, for x in X. Let x be an infinitely renormalizable parameter. Let n_k be an increasing sequence in $\mathbb{N}$. If $G_{f^{n_k}x}$ is ϵ_k-minimal for $\epsilon_k \leq \epsilon_X$, for all k in $\mathbb{N}$, and $\lim_{n\to\infty} \lambda_x^n = 0$, then G_x is minimal.*

Proof. It suffices to show, for each u in G_x^0, the orbit $\mathcal{O}_u = \{v \in G_x^0 : \ \exists g \in G : r(g) = v,\ s(g) = u\}$ is dense in G_x^0. By proposition 4.2.8, for each w in G_x^0 and k in $\mathbb{N}$, we can find an open neighbourhood W of w with diameter at most $\lambda_x^{n_k} 2\epsilon_X$ that is the image of a $E_x^{n_k}$-map $\gamma : B_{\epsilon_X}(\tilde{w}) \mapsto W$ such that $\gamma(\tilde{w}) = w$. Let e be in $E_x^{n_x}$ such that $\sigma(e) = u$, and denote $\rho(e) = \tilde{u}$. By ϵ_k-minimality there is h in $G_{f^{n_k}x}$ such that $s(h) = \tilde{u}$ and that $r(h)$ is in $B_\epsilon(\tilde{w})$. Let U_γ be an $E_x^{n_k}$-set defining γ ($\gamma = \gamma_{U_\gamma}$). Let $\tilde{e}$ be the unique element in U_γ such that $\rho(\tilde{e}) = r(h)$. Then e and $e' = h^{-1}\tilde{e}$ both satisfy $\rho(e) = \rho(e')$, so there is a g in G_x^0 such that $r(g) = \sigma(e) = u$ and $eg = e'$. Therefore, $s(g) = \sigma(e') = \sigma(\tilde{e}) = \gamma(\rho(\tilde{e}))$, so that $s(g)$ is in W. Since W contains w and has diameter bounded by $\lambda_x^{n_k} 2\epsilon_X$, it follows that $d_x(s(g), w) < \lambda_x^{n_k} 2\epsilon_X$. By definition of the orbit, $s(g)$ is in $\mathcal{O}_u$, so we have shown for all k in $\mathbb{N}$ and u, w in G_x^0 that $\mathcal{O}_u \cap B_{\lambda_x^{n_k} 2\epsilon_X}(w) \neq \emptyset$. Since $\lim_{k\to\infty} \lambda_x^{n_k} = 0$, it follows that G_x is minimal. □

As an application of this result, we prove an interesting relation between the renormalization dynamics and the properties of the groupoid bundle.

Theorem 4.2.10. *Assume (G, E, f) is a renomalization procedure for a groupoid bundle (G, π) over X for which the metrics d_x on G_x^0 vary continuously over X (see definition 2.2.6), and for which $\pi : G^0 \mapsto X$ is proper.*

If x is an infinitely renormalizable parameter such that $\lim_{n\to\infty} \lambda_x^n = 0$ and the orbit $\{f^n x\}_{n\in\mathbb{N}}$ contains an accumulation point x_* for which G_{x_*} is minimal, then G_x is minimal.

Proof. Let $\epsilon < \epsilon_X$, and let n_k be an increasing sequence in $\mathbb{N}$ for which $\lim_{k\to\infty} f^{n_k} x = x_*$. Since $\pi_0 = \pi : G^0 \mapsto X$ is proper, the metrics $(d_x)_{x\in X}$ vary continuously, and G_{x_*} is minimal, by proposition 2.2.7, for any $\epsilon \le \epsilon_X$, there is an open neighbourhood U of x_* for which $G_{\tilde{x}}$ is ϵ-minimal for all $\tilde{x}$ in U. Fix $\epsilon \le \epsilon_X$ and an open set U as above. Since $\{f^{n_k} x\}_{k\in\mathbb{N}}$ converges to x_*, there is a K in $\mathbb{N}$ for which $f^{n_k} x$ is in U, for all $k \ge K$. Therefore, $G_{f^{n_k} x}$ is ϵ-minimal along the sub-sequence for all $k \ge K$, and by assumption $\lim_{n\to\infty} \lambda_x^n = 0$, so by proposition 4.2.9, G_x is minimal. □

Definition 4.2.11. *Let (G, E, f) be a renormalization procedure for an étale groupoid bundle (G, π) over X, and let x be an infinitely renormalizable parameter. For n in $\mathbb{N}$ and $0 < \epsilon \le \delta$, we will denote $\mathcal{E}_x(n, \epsilon, \delta)$ to be the collection of open sets V of G_x^0 which are images of E_x^n-maps $\gamma : U \mapsto V$ that are λ_x^n-contractions such that $B_\epsilon(u_1) \subseteq U \subseteq B_\delta(u_2)$ for some u_1, u_2 in $G^0_{f^n x}$. We will denote the collection $\bigcup_{n\in\mathbb{N}} \mathcal{E}_x(n, \epsilon, \delta)$ by $\mathcal{E}_x(\epsilon, \delta)$, and call this the ϵ-δ-**renormalization basis** of G_x^0.*

Regarding the above definition, it follows from proposition 4.2.8 that for any ϵ, δ such that $\epsilon \le \epsilon_X \le \delta$ and n in $\mathbb{N}$, $\mathcal{E}_x(n, \epsilon, \delta)$ contains an open cover of G_x^0 by sets with diameter less than $\epsilon\lambda_x^n$, and that any open set in $\mathcal{E}_x(n, \epsilon, \delta)$ has diameter bounded by $\lambda_x^n 2\delta$. Hence, if $\lim_{n\to\infty} \lambda_x^n = 0$, $\mathcal{E}_x(\epsilon, \delta)$ is a basis for the topology on G_x^0.

Now, we will show that if the hypothesis of theorem 4.2.10 holds at a parameter x, along with the extra assumption that the renormalization basis contains arbitrarily small D-dimensional covers, for a fixed D, then G_x^0 has at most one G_x-invariant probability measure. First, we prove a lemma.

Lemma 4.2.12. *Let (G, E, f) be a renormalization procedure for a groupoid bundle (G, π) over X, and let x be an infinitely renormalizable parameter such that $\lim_{n\to\infty} \lambda_x^n = 0$, and let $\{n_k\}_{k\in\mathbb{N}}$ an increasing sequence in $\mathbb{N}$. Suppose there are numbers $\epsilon \le \epsilon_X \le \delta$, and M, D in $\mathbb{N}$ for which the following hold:*

- *for each k in $\mathbb{N}$, $\mathcal{E}_x(n_k, \epsilon, \delta)$ contains a D-dimensional cover,*
- *for each k in $\mathbb{N}$ there is a collection of $G_{f^{n_k} x}$-sets $U_{i,j}$, $i \le M$, $j \le l_i$ such that $\bigcup_{i=1}^M s(U_{i,j}) = G^0_{f^{n_k} x}$, each $\bigcup_{i=1}^M r(U_{i,j})$ has diameter at most $\epsilon/2$, and every point in $G^0_{f^{n_k} x}$ is within $\epsilon/2$ of $\bigcup_{i\le M, j\le l_i} r(U_{i,j})$.*

Then, G_x^0 has at most one G_x invariant probability measure.

Proof. For each k in $\mathbb{N}$, let $\{V_m\}_{m=1}^s$ be a D-dimensional cover consisiting of elements of $\mathcal{E}_x(n_k, \epsilon, \delta)$. Therefore, by the definition of such sets, there are $E_x^{n_k}$-maps $\gamma_m : W_m \mapsto V_m$ for $1 \leq m \leq s$ such that W_m contains an ϵ-ball $B_\epsilon(u_m)$. Since there is a j_m such that u_m is within $\epsilon/2$ of $\bigcup_{i \leq M} r(U_{i,j_m})$, and this set has diameter at most $\epsilon/2$, it follows that $\bigcup_{i \leq M} r(U_{i,j_m}) \subseteq B_\epsilon(u_m) \subseteq W_m$. Denote $\beta_{i,m} : s(U_{i,j_m}) \mapsto r(U_{i,j_m})$ to be the corresponding $G_{f^{n_k}x}$-map of the $G_{f^{n_k}x}$-set U_{i,j_m}. For $1 \leq i \leq M$ and $1 \leq m \leq s$, let $\gamma_{i,m} = \gamma_m \circ \beta_{i,m}$, $\gamma_{m,0} = \gamma_m$, and let $V_{i,m}$ be an $E_x^{n_k}$-set corresponding to the $E_x^{n_k}$-map $\gamma_{i,m}$. Then,

- for each $m \leq s$, $\{\rho(V_{i,m})\}_{i=0}^M$ is a cover of $G_{f^{n_k}x}^0$,
- for each $m \leq s$, $\bigcup_{i=0}^M \sigma(V_{i,m}) = V_m$ has diameter at most $\lambda_x^{n_k} 2\delta$, and
- $\{V_m\}_{m=1}^s$ is an at most D-dimensional cover of G_x^0.

Since $\lim_{k\to\infty} \lambda_x^{n_k} 2\delta = 0$, it follows that $G_{f^{n_k}x} \to E_x^{n_k} \leftarrow G_x$ satisfies the strong finite type condition in definition 4.2.2, and so by proposition 4.2.5, G_x^0 has at most one G_x-invariant probability measure. □

Theorem 4.2.13. *Assume (G, E, f) is a renomalization procedure for a groupoid bundle (G, π) over X for which the metrics d_x on G_x^0 vary continuously over X (definition 2.2.6), and for which $\pi : G^0 \mapsto X$ is proper.*

Let x be an infinitely renormalizable parameter such that $\lim_{n\to\infty} \lambda_x^n = 0$, and assume $\{n_k\}_{k\in\mathbb{N}}$ is an increasing sequence in $\mathbb{N}$ such that $f^{n_k}x$ converges to a parameter x_* for which G_{x_*} is minimal. Then, if there is a D in $\mathbb{N}$ and $\epsilon \leq \epsilon_X \leq \delta$ for which $\mathcal{E}_x(n_k, \epsilon, \delta)$ contains a D-dimensional cover, for every k in $\mathbb{N}$, then G_x^0 has at most one G_x-invariant probability measure.

Proof. Notice that the proof of proposition 2.2.7 gives $G_{\tilde{x}}$-sets satisfying bullet point two in lemma 4.2.12 for $\tilde{x}$ in a neighbourhood U of x, and the fact that $f^{n_k}x$ is eventually in U, along with the assumption that the $\mathcal{E}_x(n_k, \epsilon, \delta)$ contain covers of at most D-dimension imply the hypothesis of lemma 4.2.12 applies. Therefore, G_x^0 has at most one G_x-invariant probability measure. □

We specialize our analysis to certain types of renormalization procedures for isometries on compact metric spaces which are finite dimensional in the following sense.

Definition 4.2.14. *Let (X, d) be a metric space. We say (X, d) is **uniformly finite dimensional** if there is a D in $\mathbb{N}$ and ϵ' such that for every $\epsilon < \epsilon'$, there is an open cover $\mathcal{U}_\epsilon = \{B_\epsilon(x_\alpha)\}_{\alpha\in\mathcal{A}}$ by ϵ-balls with dimension bounded by D. We will let $D(X, d)$ be the minimal such D and call this the **uniform dimension** of (X, d).*

$\mathbb{R}^n$ is uniformly finite dimensional with the euclidean metric. It can be shown that every compact Riemannian manifold M is uniformly finite dimensional (choose a finite cover by metric balls in which the metric is bi-Lipschitz equivalent, via the exponential map, to the euclidean metric, and use the fact euclidean balls are uniformly finite dimensional with the euclidean metric, then bootstrap from local to global).

Definition 4.2.15. *Let (X, d) be a compact metric space and G a discrete group. A G-**action by isometries**, denoted α, is a collection of homeomorphisms $\alpha^g : X \mapsto X$ for g in G such that $d(\alpha^g(x_1), \alpha^g(x_2)) = d(x_1, x_2)$ for every x_1, x_2 in X, $\alpha^{g_1}\alpha^{g_2} = \alpha^{g_1 g_2}$ for all g_1, g_2 in G, and $\alpha^1 = id_X$, where 1 is the identity element in G.*

*Let Y be a locally compact Hausdorff space. By a **continuous family of** G-**actions by isometries of** X **over** Y, we shall mean a family $\{\alpha_y\}_{y \in Y}$ of G-actions by isometries of X such that the combined map $\alpha^g : X \times Y \mapsto X \times Y$ defined, for (x, y) in $X \times Y$, as $\alpha^g(x, y) = (\alpha^g_y(x), y)$ is a homeomorphism, for each g in G.*

Given a continuous family of G-actions by isometries of (X, d), the transformation groupoid $H = G \times_\alpha (X \times Y)$ has the natural bundle structure given by the projection π onto the Y co-ordinate of $G \times X \times Y$. $\pi_0 : H^0 \mapsto Y = \pi : X \times Y \mapsto Y$ is proper since X is compact, and the family of metrics $d_y = d$ on $\pi_0^{-1}(\{y\}) = X \times \{y\}$ is obviously continuous over Y.

Proposition 4.2.16. *Let $\{\alpha_y\}_{y \in Y}$ be a continuous family of G-actions by isometries of a uniformly finite dimensional compact metric space (X, d). Denote (H, π) to be the induced étale groupoid bundle over Y. Suppose there is a renormalization procedure $(H, E, f : Y_1 \mapsto Y)$ for $G \times_\alpha X \times Y$ that is an ϵ_Y-iterable dilation. Then, if y is an infinitely renormalizable parameter such that $H_y = G \times_{\alpha_y} X$ is minimal and $\lim_{n \to \infty} \lambda^n_y = 0$, there is at most one H_y invariant probability measure for X.*

Proof. We will show the hypothesis in lemma 4.2.12 applies in this setting. Fix $\epsilon > 0$ smaller than ϵ_Y and ϵ'. By the uniformly finite dimensional property of (X, d) there is a D in $\mathbb{N}$ such that, for every n in $\mathbb{N}$, there exists a cover $\{B_{\epsilon\lambda^n_y}(u_i)\}_{i=1}^l$ with dimension bounded by D. Fix n in $\mathbb{N}$. Since E is an ϵ_X-iterable dilation, for each $i \leq l$ and $1 \leq k \leq n$, there are u^k_i in X and $E_{f^{k-1}y}$-maps $\gamma^k_i : B_\epsilon(u^k_i) \mapsto B_{\epsilon\lambda_{f^{k-1}y}}(u^{k-1}_i)$ that are $\lambda_{f^{k-1}y}$-dilations and $\gamma^k_i(u^k_i) = u^{k-1}_i$, where we denote $u^0_i = u_i$. We can compose the γ^k_i, and the resulting map $\gamma_i := \gamma_1 \circ \gamma_2 \circ \ldots \circ \gamma_n : B_\epsilon(u^n_i) \mapsto B_{\epsilon\lambda^n_y}(u_i)$ is an E^n_y-map. Therefore, the cover $\{B_{\epsilon\lambda^n_y}(u_i)\}_{i=1}^l$ is contained in $\mathcal{E}_y(n, \epsilon)$. So, the first bullet point holds for the the hypothesis in lemma 4.2.12.

Now, let's fix a cover $\{B_{\epsilon/8}(v_i)\}_{i=1}^M$ of X. Since the groupoid H_y is minimal and E^n_y is a $(H_{f^n y}, H_y)$ equivalence, $H_{f^n y}$ is also minimal. Hence, for $1 \leq i \leq M$ and $1 \leq j \leq M$, we can find an element $g_{i,j}$ in G such that $d(\alpha^{g_{i,j}}_{f^n y} v_j, v_i) < \epsilon/8$. Therefore, $\alpha^{g_{i,j}}_{f^n y}(B_{\epsilon/8}(v_j)) = B_{\epsilon/8}(\alpha^{g_{i,j}}_{f^n y} v_j) \subseteq B_{\epsilon/4}(v_i)$. Hence, the collection of $H_{f^n y}$-sets $U_{i,j} = (g_{i,j}, \alpha^{g_{i,j}}_{f^n y}(B_{\epsilon/4}(v_j)))$ for $1 \leq i \leq l$ and $1 \leq j \leq M$ has the properties that $\bigcup_{j=1}^M r(U_{i,j}) \subseteq$

$B_{\epsilon/4}(v_i)$ and $\bigcup_{j=1}^{M} s(U_{i,j}) = X$. The above containment implies any x in X is within $\epsilon/2$ of $\bigcup_{i,j}^{M} r(U_{i,j})$, since $\{B_{\epsilon/4}(v_i)\}_{i=1}^{M}$ is a cover. We have shown the second bullet point holds for the hypothesis in lemma 4.2.12. Hence, the hypothesis applies to show there is at most one H_y invariant probability measure for X. $\square$

Remark 4.2.17. *It is not much more difficult to extend our groupoid renormalization theory to the case of the general finite type condition with* $a_1, ..., a_M$ *generators. Basically, these generators will correspond to open sets* $U_1, ..., U_M$ *in* G^0 *for a groupoid bundle* (G, π). *The third bullet point in definition 4.2.2 is then relaxed to allow the elements of D-dimensionsional covers to come from covers of* $U_m \cap G^0_{f^n x}$, *for any* $m \leq M$, *in the renormalized groupoid's unit space* $(G_{f^n x})^0$. *The effect this more general finite type condition will have on the hypothesis of Masur's criterion (theorem 4.2.13) is, essentially, to replace the assumption of a minimal accumulation point of* $\{f^n x\}_{n\in\mathbb{N}}$ *with the assumption that* $U_1, ..., U_M$ *are a-priori bounded along a sub-sequence of* $\{f^n x\}_{n\in\mathbb{N}}$, *and that the renormalization basis is generated from covers of* $U_1, .., U_m$ *in an appropriate sense, while keeping the same assumption of finite dimensionality. The general finite type condition is essential if we want to give the unique trace criterion for AF algebras a groupoid proof, or for proving unique groupoid invariant probability measure for generalizations of Korfanty's IFS groupoids to groupoids associated to graph iterated function systems.*

4.2.1 Rotation Groupoid Renormalization

In this section, we show the groupoid bundle (H, π) over $X = (0, 1)$ constructed in section 2.2.2, along with the groupoid bundle equivalence constructed in section 3.4.2, is a renormalization procedure and that the results proved in the last section are applicable to this setting. We can equip $\mathbb{R}/\mathbb{Z}$ the metric d, where for z and w in $\mathbb{R}/\mathbb{Z}$, $d(z, w)$ is defined to be the smallest positive number t for which either $z - w = \overline{t}$ or $w - z = \overline{t}$. One readily checks that this defines a metric on $\mathbb{R}/\mathbb{Z}$, which, for t in $\mathbb{R}$ and $\epsilon \leq 1$, the ϵ-ball for $\overline{t}$ is just $\overline{(t - \epsilon, t + \epsilon)}$. Moreover, the quotient map $\rho : \mathbb{R} \mapsto \mathbb{R}/\mathbb{Z}$, whenever restricted to an interval of length less than 1/2, is an isometry.

We show $(\mathbb{R}/\mathbb{Z}, d)$ is uniformly finite dimensional: let $\epsilon \leq 1/4$, and let l in $\mathbb{N}$ be the first number such that $l\epsilon > 1$. Let I_n be the interval $((n - 1/2)\epsilon, (n + 1)\epsilon)$. Then, $\{\overline{I_n}\}_{0\leq n\leq l-1}$ is a cover of $\mathbb{R}/\mathbb{Z}$. Each interval is of length $\frac{3}{2}\epsilon$ and is the translation by ϵ of the preceeding one, so for $0 \leq n \leq l-1$, at most two intervals (I_{n-1}, I_{n+1}) intersect I_n. By our choice of l, the only extra overlaps that are possible after projecting $I_n \mapsto \overline{I_n}$ are between the pairs $(\overline{I_{l-1}}, \overline{I_0})$, $(\overline{I_{l-1}}, \overline{I_1})$, and $(\overline{I_{l-2}}, \overline{I_0})$. Therefore, the cover $\{\overline{I_n}\}_{0\leq n\leq l-1}$ is at-most 3-dimensional. Each I_n is the $\frac{3}{4}\epsilon$ ball centred at a point in $\mathbb{R}/\mathbb{Z}$, so by making a change in variables $\delta = \frac{3}{4}\epsilon$, we can find 3-dimensional covers by δ balls for all $\delta < \frac{3}{16}$.

Since each $\mathbb{Z}$ action induced from the homeomorphism $\alpha_\theta(z) = z + \overline{\theta}$, z in $\mathbb{R}/\mathbb{Z}$, θ in $(0, 1)$, is an isometry with respect to the metric d and the $(\alpha_\theta)_{\theta\in(0,1)}$ vary continuously over $(0,1)$, it follows that (H,π) is the transformation groupoid of a continuous family of $\mathbb{Z}$ actions by isometries on a uniformly finite dimensional compact space.

Recall from section 3.4.2 the definition of the Gauss map $G : X_1 \mapsto X$, along with the groupoid bundle equivalence $(G^*H, G^*\pi) \to E \leftarrow (i^*H, i^*\pi)$. We will now prove (H,E,G) is a renormalization procedure for (H,π). Let t be in $\mathbb{R}$ and θ be in $X_1 = \bigcup_{n=1}^{\infty}(\frac{1}{n+1},\frac{1}{n})$. Denote ρ and σ to be the structure maps for E_θ, and let $M_\theta : \mathbb{R} \mapsto \mathbb{R}$ be defined, for t in $\mathbb{R}$, as $M_\theta(t) = \theta t$. Then, $\sigma = \rho \circ M_\theta$. Recall from above that $\rho : (t-1/4, t+1/4) \mapsto \overline{(t-1/4,t+1/4)}$ is an isometry. Since $\theta < 1$, $M_\theta((t-1/4,t+1/4)))$ is another interval of length smaller than $\frac{1}{4}$, so $\rho : M_\theta((t-1/4,t+1/4))) \mapsto \overline{M_\theta((t-1/4,t+1/4)))}$ is also an isometry. Hence, $\gamma = \rho \circ M_\theta \circ (\rho|_{(t-1/4,t+1/4)})^{-1}$ is an E_θ-map that is isometrically equivalent to $M_\theta : (t-1/4, t+1/4) \mapsto M_\theta((t-1/4,t+1/4))) = (\theta t - \theta/4, \theta t + \theta/4)$, which is a θ-dilation. The intervals $\{(t-\frac{1}{4}, t+\frac{1}{4})\}_{t\in\mathbb{R}}$ cover all of $\mathbb{R}$ and consist of, when projected down to $\mathbb{R}/\mathbb{Z}$, all the $\frac{1}{4}$-balls in $(\mathbb{R}/\mathbb{Z}, d)$. It follows that for every θ in X_1, every E_θ-map is locally a θ-dilation. The E_θ-map associated to the E_θ-set $(t-1/4,t+1/4)$ is a θ-dilation sending the ball $B_{1/4}(\overline{t})$ to the ball $B_{\theta/4}(\overline{\theta t})$ and the map $t \mapsto \overline{\theta t}$ is surjective, so E is a $\frac{1}{4}$-iterable dilation. We have shown (H,E,G) is a renormalization procedure for (H,π).

We will now compute the constants λ_θ^j for θ irrational (infinitely renormalizable) and j in $\mathbb{N}$. Relevant to our computation are the denominators $k_j(\theta)$ of the convergents for θ, defined as follows. First, for j in $\mathbb{N}$, denote $a_j(\theta) = \lfloor \frac{1}{G^{j-1}(\theta)} \rfloor$, which is the largest natural number bounded from above by $\frac{1}{G^{j-1}(\theta)}$. Let $k_{-1}(\theta) = 0$, $k_0(\theta) = 1$, and define recursively $k_j(\theta) = a_j(\theta)k_{j-1}(\theta) + k_{j-2}(\theta)$ for $j \geq 1$.

Lemma 4.2.18. *Let θ be an irrational number in $(0,1)$, and $\lambda_\theta^j = \Pi_{k=0}^{j-1}\lambda_{G^k\theta} = \Pi_{k=0}^{j-1}G^k\theta$. Then for any j in $\mathbb{N}$, we have $\lambda_\theta^j = \frac{G^{j-1}(\theta)}{k_{j-1}(\theta)+G^{j-1}(\theta)k_{j-2}(\theta)}$.*

Proof. We prove by induction. For $j=1$, $k_{-1}(\theta) = 0$ and $k_0(\theta) = 1$, so $\theta = \lambda_\theta^1 = G^0(\theta) = \theta$. Suppose the lemma is true for $j-1 \geq 1$. Then, by induction we have $\lambda_\theta^j = G^{j-1}(\theta)\lambda_\theta^{j-1} = G^{j-1}(\theta)\frac{G^{j-2}(\theta)}{k_{j-2}(\theta)+G^{j-2}(\theta)k_{j-3}(\theta)}$. Write $G^{j-2}(\theta)$ as $\frac{1}{a_{j-1}(\theta)+G^{j-1}(\theta)}$. Then, we have

$$\frac{G^{j-2}(\theta)}{k_{j-2}(\theta)+G^{j-2}(\theta)k_{j-3}(\theta)} = \frac{1}{(a_{j-1}(\theta)+G^{j-1}(\theta))(k_{j-2}(\theta)+\frac{k_{j-3}(\theta)}{a_{j-1}(\theta)+G^{j-1}(\theta)})}.$$

By distributing the product in the denominator, we see that the above term in the denominator is equal to $a_{j-1}(\theta)k_{j-2}(\theta) + k_{j-3}(\theta) + G^{j-1}(\theta)k_{j-2}(\theta)$. Using that $a_{j-1}(\theta)k_{j-2}(\theta) + k_{j-3}(\theta) = k_{j-1}(\theta)$, we have

$$\frac{1}{(a_{j-1}(\theta)+G^{j-1}(\theta))(k_{j-2}(\theta)+\frac{k_{j-3}(\theta)}{a_{j-1}(\theta)+G^{j-1}(\theta)})} = \frac{1}{k_{j-1}(\theta)+G^{j-1}(\theta)k_{j-2}(\theta)}.$$

Multiplying the above last number by $G^{j-1}(\theta)$ we see that $\lambda_\theta^j = \frac{G^{j-1}(\theta)}{k_{j-1}(\theta)+G^{j-1}(\theta)k_{j-2}(\theta)}$, proving the induction hypothesis. $\square$

Corollary 4.2.19. *If θ is an irrational number in $(0,1)$, then $\lim_{j\to\infty} \lambda_\theta^j = 0$.*

Proof. By lemma 4.2.18, $\lambda_\theta^j = \frac{G^{j-1}(\theta)}{k_{j-1}(\theta)+G^{j-1}(\theta)k_{j-2}(\theta)} \leq \frac{1}{k_{j-1}(\theta)}$. Since θ is irrational, $a_j(\theta) \geq 1$ for all j, so $\frac{1}{k_{j-1}(\theta)} \leq \frac{1}{j-1}$. $\square$

Now, we apply our results from the previous section.

Lemma 4.2.20. *For any irrational θ in $(0,1)$, H_θ is $1/4$-minimal.*

Proof. For any θ in $(0,1/4)$, H_θ is $1/4$-minimal, because the distance between consecutive points in an orbit is constant and less than $1/4$ apart. Whenever θ is an angle in $(0,1)$ such that $0 < n\theta - m < 1/4$ for some n and m in $\mathbb{Z}$, H_θ is $1/4$-minimal, because $\alpha_\theta^n = \alpha_{n\theta-m}$, so α_θ contains the orbits of a $1/4$-minimal parameter, and is therefore $1/4$-minimal. Clearly this is true for irrational θ ($\{\overline{n\theta}\}_{n\in\mathbb{N}}$ is infinite and therefore has a limit point, so there are n_1, n_2 and m in $\mathbb{Z}$ such that $0 < n_1\theta - n_2\theta + m < 1/4$). $\square$

Corollary 4.2.21. *For irrational θ in $(0,1)$, H_θ is minimal.*

Proof. An irrational number θ is infinitely renormalizable, and, by lemma 4.2.20, is $1/4$-minimal. Since the renormalization procedure is $1/4$-iterable, and $\lim_{n\to\infty} \lambda_\theta^n = 0$ (corollary 4.2.19), it follows from proposition 4.2.9 that H_θ is minimal. $\square$

Corollary 4.2.22. *For irrational θ in $(0,1)$, $\mathbb{R}/\mathbb{Z}$ has a unique H_θ invariant probability measure.*

Proof. The pushforward of Lebesgue on $[0,1)$ via the quotient map is an H_θ invariant probability measure. Since H_θ is minimal and $\lim_{n\to\infty} \lambda_\theta^n = 0$, by theorem 4.2.16, This measure is the unique H_θ invariant probability measure. $\square$

4.2.2 IFS Groupoid Renormalization

Let R and $O(R)$ be the groupoids associated to a single matrix affine IFS $\{\gamma_1, ..., \gamma_v\}$ as in section 2.2.3. By proposition 3.4.3, $R \to E^{-k} \leftarrow R$ is a groupoid equivalence whenever r and s surject onto K when restricted to E^{-k} (this will an assumption throughout most of this section; see definition 4.2.27). Equip $R^0 = K$ with the Euclidean metric restricted to K. Since every E^{-k} map is locally of the form $A^k(-) + v$, v in $\mathbb{R}^d$, every E^{-k}-map is locally a $\|A\|^k$-contraction. Since K is compact, there is an ϵ_* for which E^{-k} is ϵ_*-iterable. Therefore, $(R, E^{-k}, *)$ is a renormalization procedure with trivial renormalization dynamics $* \mapsto *$. So, to show K has at most one R-invariant probability measure when R is minimal, it suffices to show, by theorem 4.2.13, that for some $\epsilon \leq \epsilon_* \leq \delta$, the ϵ-δ-renormalization basis contains D-dimensional covers in $\mathcal{E}(n, \epsilon, \delta)$ for every n in $\mathbb{N}$. We will show this in proposition 4.2.32. It is often the case that R is not minimal, and so we would like to consider a more general case where there is a large open invariant set $K \setminus S$ such that $R|_{K\setminus S}$ is minimal. We will put assumptions on the "singularity set" (definition 4.2.29) S that are natural enough to include the examples of the Sierpinski gasket's and the Sierpinski carpet's singularities. Since $K \setminus S$ is no longer compact, we cannot appeal to 4.2.13, but instead we will apply lemma 4.2.12. We will then prove existence of the unique invariant probability measure in proposition 4.2.33, establishing existence and uniqueness for a large class of these groupoids modulo their singularities (we believe all the examples appearing in Korfanty [20] satisfy our assumptions, but we have not rigorously checked this yet). First, we make a standing assumption that our IFS satisfies the following property. This property also appears in Korfanty lemma 5.2.1 [20].

Definition 4.2.23. *Let $\Upsilon = \{\gamma_1, .., \gamma_v\}$ be an iterated function system, and let K be its attractor. We say Υ satisfies the **strong attractor open set condition** if there is an open set $W \neq \emptyset$ in K such that $\gamma_i(W) \subseteq W$ and $\gamma_i(K) \cap \gamma_j(W) = \emptyset$ for all $1 \leq i \neq j \leq v$.*

By uniqueness of the attractor, W is an open and dense set of K. Definition 4.2.23 is equivalent to assuming that the above W is an open set in K such that $\gamma_i(W) \subseteq W$, $\gamma_i(W)$ open in K, and $\gamma_i(W) \cap \gamma_j(W) = \emptyset$ for $1 \leq i \neq j \leq v$.

The above definition is implied by the "**strong open set condition**" considered by Lalley (page 700 [23]), which assumes there is an open set V in $\mathbb{R}^d$ (rather than in K) for which $\gamma_i(V) \subseteq V$, $\gamma_i(V) \cap \gamma_j(V) = \emptyset$ for all $1 \leq i \neq j \leq v$, and $V \cap K \neq \emptyset$. It is easy to see from these properties that $W = V \cap K$ satisfies the strong attractor open set condition; since γ_i are affine maps, $\gamma_i(V)$ is open in $\mathbb{R}^d$ for all $i \leq v$. Hence, $\gamma_i(V) \cap \gamma_j(V) = \emptyset$ implies $\overline{\gamma_i(V)} \cap \gamma_j(V) = \emptyset$. Uniqueness of the attractor implies $K \subseteq \overline{V}$, so that $\gamma_i(K) \cap \gamma_j(W) \subseteq \overline{\gamma_i(V)} \cap \gamma_j(V) = \emptyset$ for all $1 \leq i \neq j \leq v$.

The "**open set condition**" drops the assumption that $V \cap K \neq \emptyset$. In Schief theorem 2.2 [49] it is shown that the strong open set condition and the open set condition are equivalent when the iterated function system elements are metric dilations.

One could define the "**attractor open set condition**", by dropping the assumption that $\gamma_i(W)$ is open in K for all $1 \leq i \leq v$, but it seems difficult to prove anything with such limited assumptions. We will assume the strong attractor open set condition to prove useful alternative descriptions to canonical open sets in K as suggested in Korfanty remark 4.2.12 [20]. To do this, let's denote $T_m(x) = \{\varepsilon \in \Sigma_m : \exists\, x_\varepsilon \in K : \gamma_\varepsilon(x_\varepsilon) = x\}$, and define

- $K_m(x) := \bigcup_{\varepsilon \in T_m(x)} \gamma_\varepsilon(K)$
- $V_m(x) := \operatorname{int}(K_m(x))$
- $W_m(x) := \bigcup_{\varepsilon \in T_m(x)} \gamma_\varepsilon(W)$

Recall that $U_m(x) = K \setminus \bigcup_{\eta \notin T_m(x)} \gamma_\eta(K)$.

Proposition 4.2.24. $U_m(x) = V_m(x)$

Proof. Since $U_m(x) \subseteq K_m(x)$, it follows that $U_m(x) \subseteq \text{int}(K_m(x)) = V_m(x)$. Suppose y is in $V_m(x) \setminus U_m(x)$ and let U_y be an open neighbourhood of y contained in $V_m(x)$. Since $W_m(x)$ is open and dense in $V_m(x)$, $W_m(x) \cap U_y$ is open and dense in U_y. y is not in $U_m(x)$, so there is some η not in $T_m(x)$ such that y is in $\gamma_\eta(K)$. Therefore, $U_y \cap \gamma_\eta(W) \neq \emptyset$. This means, by density of $W_m(x) \cap U_y$ and open-ness of $\gamma_\eta(W)$, that $W_m(x) \cap \gamma_\eta(W) \neq \emptyset$, which is a contradiction. $\square$

For $\gamma = \gamma^U_{(x,m,n,y)}$ in Γ (definition 2.2.17), recall the notations $c(\gamma) = m - n$ and $d(\gamma) = (m, n)$.

Proposition 4.2.25. *Let γ be in Γ with $c(\gamma) = k$ and $d(\gamma) = (M + k, M)$. Then, for every x in $D(\gamma)$ and $m \geq M$, there is a bijection $b : T_{m+k}(\gamma(x)) \mapsto T_m(x)$ such that $\gamma_\varepsilon^{-1}(\gamma(x)) = \gamma_{b\varepsilon}^{-1}(x)$ for all ε in $T_{m+k}(\gamma(x))$.*

Proof. Note that if ε, η are two distinct elements in $T_m(x)$, then $\gamma_\varepsilon^{-1}(x) \neq \gamma_\eta^{-1}(x)$, otherwise $\gamma_\varepsilon \gamma_\eta^{-1}(x) = x$. But then $\gamma_\eta = \gamma_\varepsilon$, which is a contradiction to the strong attractor open set condition. Therefore, $T_m(x)$ is in bijection with $\mathcal{F}^{-m}(x)$ via the map sending ε in $T_m(x)$ to $\gamma_\varepsilon^{-1}(x)$. By proposition 2.2.16, for x in $D(\gamma)$ and $m \geq M$, we have $\mathcal{F}^{-m}(x) = \mathcal{F}^{-(m+k)}(\gamma(x))$. Therefore, the function $b : T_{m+k}(\gamma(x)) \mapsto T_m(x)$ defined by letting $b\varepsilon$, for ε in $T_{m+k}(\gamma(x))$, be the unique word in $T_m(x)$ such that $\gamma_\varepsilon^{-1}(\gamma(x)) = \gamma_{b\varepsilon}^{-1}(x)$ is a well defined bijection. $\square$

Proposition 4.2.26. *Let γ be in Γ with $c(\gamma) = k$ and $d(\gamma) = (M + k, M)$. Suppose $K_m(x) \subseteq D(\gamma)$ for some $m \geq M$. Then, $\gamma(U_m(x)) = U_{m+k}(\gamma(x))$.*

Proof. Since γ is a homeomorphism from $D(\gamma)$ onto an open set of K and $K_m(x) \subseteq D(\gamma)$, $\gamma(\text{int}(K_m(x))) = \text{int}(\gamma(K_m(x)))$. By proposition 4.2.25 and proposition 2.2.14,

$$\gamma(K_m(x)) = \bigcup_{\eta \in T_m(x)} \gamma(\gamma_\eta(K)) = \bigcup_{\eta \in T_m(x)} \gamma_{b^{-1}\eta}(K) = K_{m+k}(\gamma(x)).$$

By proposition 4.2.24, $\text{int}(K_n(y)) = U_n(y)$ for any n in $\mathbb{N}$,and y in K. Combining these equalities,

$$\gamma(U_m(x)) = \gamma(\text{int}(K_m(x))) = \text{int}(\gamma(K_m(x))) = \text{int}(K_{m+k}(\gamma(x))) = U_{m+k}(\gamma(x)).$$

$\square$

Definition 4.2.27. *We say E is* ***primitive*** *if there is a k in $\mathbb{N}$ such that the range and source maps r, s of $O(R)$ surject onto K when restricted to E^{-k}. We will call the smallest such k the* ***primitivity constant*** *of E.*

Definition 4.2.28. *We say sets $C, D \subseteq K$ are R-**equivalent** if there is an R-map $\gamma : U \mapsto V$ such that $C \subseteq U$ and $\gamma(C) = D$.*

Definition 4.2.29. *Suppose E is primitive with primitivity constant k. We say a closed set $S \subseteq K$ is **singular** if it is fixed by E^{-k}, i.e., for g in E^{-k}, $r(g)$ is in S if and only if $s(g)$ is in S, and there is an open set U in K containing S and a collection of open sets $\{U_i\}_{i=1}^n$ in K such that $\bigcup_{i=1}^n U_i = U \setminus S$ and each $U_i \setminus S$ is R-equivalent to some V_i not intersecting U.*

Note that since $r, s|_{E^{-k}} \mapsto K$ are surjective, S being fixed by E^{-k} implies that S is an R-invariant set. If S is singular, we denote the groupoid R restricted to units in $K \setminus S$ to be $R|_{K \setminus S}$.

Theorem 4.2.30. *Suppose E is primitive, and S is a singular set such that the equivalence classes of points in $K \setminus S$ are dense (in K). Then, $C^*(R|_{K\setminus S})$ has at most one trace.*

Before we prove this, we will establish a couple lemmas.

Lemma 4.2.31. *Assume the hypothesis in theorem 4.2.30. Let k be the primitivity constant for E. Then, there are numbers m, j in $\mathbb{N}$ and an open set U containing S with a cover $\{U_i\}_{i=1}^n$ of $U \setminus S$ as in definition 4.2.29 such that*

(i) the range and source maps surject onto K when restricted to the set

$$\begin{aligned} E_m^{-k} := \{(x, -k, y) : \exists\ \gamma \in \Gamma : d(\gamma) = \\ = (M + k, M)),\ m \geq M,\ K_m(x) \subseteq D(\gamma) \text{ and } \gamma(x) = y\} \end{aligned}$$

(ii) for every y in K and x in $K \setminus U$, the sets $U_{m+j}(x)$, $U_i \setminus S$, $1 \leq i \leq n$, are all R-equivalent to subsets of $U_m(y)$

Proof. We start by proving *i*). Given x in K, by surjectivity of r on E^{-k}, choose γ_x in Γ with $d(\gamma_x) = (M_x + k, M_x)$ and x in $D(\gamma_x) =: U_x$. Given y in K, by surjectivity of s on E^{-k}, choose γ'_y in Γ with $d(\gamma'_y) = (M_y + k, M_y)$ and x_y in $D(\gamma'_y) =: V_y$, such that $\gamma'_y(x_y) = y$. Denote $R(\gamma'_y) = \gamma'_y(V_y)$ by W_y. Choose finite subcovers $\{U_{x_i}\}_{i=1}^q$ and $\{W_{y_j}\}_{j=1}^r$ of K, and let N be the maximum of the M_{x_i}, M_{y_j}, $i \leq q$, $j \leq r$. Since $\sup_{x\in K} \text{diameter}(K_m(x)) \to 0$ as $m \to \infty$, by the Lebesgue number lemma (Munkres lemma 27.5 [32]) applied to the covers $\{U_{x_i}\}$, $\{W_{y_j}\}$ of K, there is M in $\mathbb{N}$, $M \geq N$, such that for every $m \geq M$ and z in K, $K_m(z)$ is contained in one of the cover elements of $\{U_{x_i}\}_{i=1}^q$ and one of $\{W_{y_j}\}_{j=1}^r$. This proves the surjectivity of r on E_m^{-k} for $m \geq M$. By proposition 4.2.26, applied to $\gamma'^{-1}_{y_j}$ in Γ, where y_j is such that $V_{y_j} \supset K_{m+k}(z)$, we have $U_m(\gamma'^{-1}_{y_j}(z)) = \gamma'^{-1}_{y_j}(U_{m+k}(z))$. Therefore, $U_m(\gamma'^{-1}_{y_j}(z)) \subseteq D(\gamma_{y_j})$ and $\gamma'_{y_j}(\gamma'^{-1}_{y_j}(z)) = z$, proving surjectivity of s on E_m^{-k} for $m \geq M$.

We prove ii). Fix $m \geq M$. By density of the equivalence classes for points in x in $K \setminus S$, and the fact that diameter$(U_{m+j}(x)) \to 0$ as $j \to \infty$, there is some $j_x \in \mathbb{N}$ such that $U_{m+j_x}(x)$ is R-equivalent to a subset of $U_m(y)$. Since there are only a finite number of possible different sets $U_m(y)$, we may take j_x to work for all y in K. The sets $\{U_{m+j_x}\}_{x \in K \setminus S}$ cover the compact set $K \setminus U$. Choose a finite sub-cover $\{U_{m+j_{x_i}}(x_i)\}_{i=1}^{l}$, and let $j = \sup_{i=1,..,l} j_{x_i}$. Then, for x in $K \setminus U$ there is i such that x is in $U_{m+j_{x_i}}(x_i)$. Since $j \geq j_{x_i}$, we have $U_{m+j}(x) \subseteq U_{m+j_{x_i}}(x)$ (this follows from $\gamma_\beta(K) = \bigcup_{\alpha \in \Sigma_{j-j_{x_i}}} \gamma_{\beta\alpha}(K)$ for β in $\Sigma_{j_{x_i}}$). By Korfanty lemma 4.2.8, $U_{m+j_{x_i}}(x) \subseteq U_{m+j_{x_i}}(x_i)$, so for every y in K, the sets $\{U_{m+j}(x)\}_{x \in K \setminus U}$ are all R-equivalent to subsets of $U_m(y)$. The j chosen above only depends on U and not on the covering elements $\{U_i\}_{i=1}^{n}$, so by chopping these sets into small enough pieces (while preserving the union), we may assume that the R-equivalent set V_i of $U_i \setminus S$ has diameter small enough such that V_i is contained in one of the $U_{m+j}(x)$, for some x in $K \setminus U$ (here we are applying the Lebesgue number lemma to the cover $\{U_{m+j}(x)\}_{x \in K \setminus U}$ of the compact set $K \setminus U$). By transitivity of R-equivalence, it follows that for all y in K, $U_i \setminus S$ is R-equivalent to a subset of $U_m(y)$, proving the lemma. □

Lemma 4.2.32. *Assume that s surjects onto K when restricted to E_m^{-k} (defined in lemma 4.2.31). Then, there is D in $\mathbb{N}$ such that for every r in $\mathbb{N}$, the cover consisting of the distinct sets of the form $U_{m+rk}(y)$ is at most D-dimensional (definition 4.2.1)*

Proof. Choose $y_1, ..., y_l$ such that the $U_{m+rk}(y_i)$ consist of all the distinct sets of the form $U_{m+rk}(y)$. By surjectivity of s on E_m^{-k} and proposition 4.2.26 (iterated r times), there are, for $1 \leq i \leq l$, x_i in K and E^{-rk}-maps $\gamma_i : U_m(x_i) \mapsto U_{m+rk}(y_i)$ with $\gamma_i(x_i) = y_i$ that are, by the proof of proposition 2.2.18 locally maps in Γ such that, locally, $d(\gamma_i) = (M + rk, M)$ for some $M \leq m$. Let $b_i : T_{m+rk}(y_i) \mapsto T_m(x_i)$ be the bijections as in proposition 4.2.25. Fix i and ε in $T_{m+rk}(y_i)$. Consider all $1 \leq j \leq l$ such that ε is in $T_{m+rk}(y_j)$ and $b_j\varepsilon = \eta$, for a fixed η in Σ_m. We show that for two distinct j, j' such that $b_j\varepsilon = \eta = b_{j'}\varepsilon$, we have $U_m(x_j) \neq U_m(x_{j'})$. Suppose this is not true, i.e, $U_m(x_j) = U_m(x_{j'})$. Since $\gamma_j, \gamma_{j'}$ each are the composites of r maps in Γ, and maps in Γ are restrictions of affine maps, we have that γ_j and $\gamma_{j'}$ are restrictions of affine maps. Since γ_j, $\gamma_{(y_j,rk,x_j)}|_{U_m(x_j)}$ and $\gamma_{j'}$, $\gamma_{(y_{j'},rk,x_{j'})}|_{U_m(x_{j'})}$ send x_j to y_j and $x_{j'}$ to $y_{j'}$, respectively, and have the same linear factor A^{rk}, it follows that $\gamma_j = \gamma_{(y_j,rk,x_j)}|_{U_m(x_j)}$ and $\gamma_{j'} = \gamma_{(y_{j'},rk,x_{j'})}|_{U_m(x_{j'})}$. By proposition 2.2.14, we have $\gamma_{(y_{j'},rk,x_{j'})} = \gamma_\varepsilon \circ \gamma_\eta^{-1} = \gamma_{(y_j,rk,x_j)}$. So, $U_m(x_j) = U_m(x_{j'})$ implies $\gamma_j = \gamma_{j'}$. Therefore, $U_{m+rk}(y_j) = \gamma_j(U_m(x_j)) = \gamma_{j'}(U_m(x_{j'})) = U_{m+rk}(y_{j'})$, which is a contradiction. Therefore, the number of j such that ε is in $T_{m+rk}(y_j)$ and $b_j\varepsilon = \eta$ is bounded by the number of distinct $U_m(x_j)$, which is bounded by 2^{v^m}. As there are only v^m different η in Σ_m, it follows that the number of j such that ε is in $T_{m+rk}(y_j)$ is bounded by $2^{v^m}v^m$. Since b_i is a bijection, there can only be at-most v^m different ε in $T_{m+rk}(y_i)$. Therefore, the number of $T_{m+rk}(y_j)$ intersecting $T_{m+rk}(y_i)$ is bounded by $D = 2^{v^m}v^mv^m$. Since $U_{m+rk}(y_i) \cap U_{m+rk}(y_j) \neq \emptyset$

implies $T_{m+rk}(y_i) \cap T_{m+rk}(y_j) \neq \emptyset$, it follows that $\{U_{m+rk}(y_i)\}_{i=1}^{l}$ is an at most D-dimensional cover. $\square$

We now prove theorem 4.2.30. Notice that lemma 4.2.32 is bullet point one in proposition 4.2.12 and lemma 4.2.31 (*ii*) is bullet two in 4.2.12. The only exception is the ϵ-balls have been replaced with the open sets $U_m(x)$. We could prove that the hypothesis in Lemma 4.2.32 applies by using the Lebesgue number lemma and some tweaking of the conditions in lemma 4.2.31. It is more natural, however to work with what we got, so instead we will replicate the proof of lemma 4.2.12 tailored to this setting.

Proof. Let m, k be as in lemma 4.2.31, and let D be as in lemma 4.2.32. For $\epsilon > 0$, let N be large enough so that $\sup_{x\in K} \text{diameter}(U_{m+rk}(x)) < \epsilon$ for $r \geq N$. Let $\{U_{m+rk}(y_i)\}_{i=1}^{l}$ be the at most D-dimensional cover of K as in lemma 4.2.32. By lemma 4.2.31 combined with proposition 4.2.25, there are, for $1 \leq 1 \leq l$, E^{-rk}-maps γ_i and x_i in K such that $c(\gamma_i) = rk$, $\gamma_{\eta_i}(x_i) = y_i$, and $\gamma_i(U_m(x_i)) = U_{m+rk}(y_i)$. By lemma 4.2.31 there is a finite cover (independent of ϵ, r) $\{W_i\}_{i=1}^{M}$ of $K \setminus S$ such that each $W_i \setminus S$ is R-equivalent to a subset $V_{i,j}$ of $U_m(x_j)$ for all $1 \leq i \leq M$, $1 \leq j \leq l$, by an R-map $\gamma_{i,j} : W_i \setminus S \mapsto V_{i,j}$. Define $V_{0,j} = U_m(x_j) \setminus S$, $\gamma_{0,j} = id_{V_{0,j}}$, and $\beta_{i,j} = \gamma_j \circ \gamma_{i,j}$ for $0 \leq i \leq M$, $1 \leq j \leq l$. Since the $\beta_{i,j}$ are (locally) in Γ, the sets $U_{i,j} = \{(x, -rk, \beta_{i,j}(x)) : x \in W_i \setminus S\}$ for $i > 0$ and $U_{0,j} = \{(x, -rk, \beta_{0,j}(x)) : x \in V_{0,j}\}$ are open in E^{-rk}. By construction, these sets have the following properties:

- $r, s|_{U_{i,j}}$ are homeomorphisms
- For fixed j, $\{r(U_{i,j}\}_{i=0}^{M}$ is an open cover of $K \setminus S$ and $\bigcup_{i=0}^{M} s(U_{i,j}) = U_{m+rk}(y_j) \setminus S$.
- $\{\bigcup_{i=0}^{m} s(U_{i,j}) = U_{m+rk}(y_j) \setminus S\}_{j=1}^{l}$ is a D-dimensional cover of $K \setminus S$ by open sets with diameter less than ϵ.

These covering properties, along with the fact that E^{-rk} fixes S, imply the groupoid equivalences $R|_{K\setminus S} \to E^{-rk}|_{K\setminus S} \leftarrow R|_{K\setminus S}$ for r in $\mathbb{N}$, satisfy the strong finite type condition with respect to $K \setminus S$ (definition 4.2.2), so by theorem 4.2.5, $C^*(R|_{K\setminus S})$ has at most one tracial state coming from a $R|_{K\setminus S}$ probability measure. Since $R|_{K\setminus S}$ is principal, by Putnam theorem 3.4.5 [40] it follows that these are the only types of tracial states on $C^*(R|_{K\setminus S})$. $\square$

Let $\phi : \{1, ..., v\}^{\mathbb{N}} \mapsto K$ be the coding of K as in Korfanty theorem 2.1.11 [20], which is a continuous surjection defined for $x = x_1x_2...$ in $\{1, ..., v\}^{\mathbb{N}}$ by $\phi(x) = \lim_{n\to\infty} \gamma_{x_1}\gamma_{x_2}...\gamma_{x_n}(K)$. Let μ be the unique tail invariant probability measure for $\{1, ..., v\}^{\mathbb{N}}$ and let $\omega := \phi_*\mu$ be the pushforward to K. Since μ is a Radon measure and ϕ is continuous between compact spaces, ω is a Radon measure, and therefore defines a normed-one linear functional τ_ω on $C(K)$ by integration. We can extend τ_ω to a state on $C^*(R)$ by pre-composing with the conditional expectation $E : C^*(R) \mapsto C(K)$ defined in Putnam section 3.4.1 [40].

Proposition 4.2.33. *τ_ω is a trace on $C^*(R)$.*

Proof. By Korfanty proposition 4.3.2, $C^*(R)$ and E are the the inductive limit of $C^*(R_n)$ and its canonical conditional expectation E_n, so it suffices to show $\tau_\omega \circ E_n$ is a trace on $C^*(R_n)$ for all n. By Putnam theorem 3.4.4 [40], this is equivalent to showing ω is an R_n-invariant measure, i.e., for every open set $U \subseteq R_n$ such that $r|_U, s|_U$ are homeomorphisms, $\omega(r(U)) = \omega(s(U))$. Let $U_r := r(U)$, $U_s := s(U)$, and $\gamma := r \circ s|_{U_s}^{-1}$. The preceeding equality can be re-written as $\omega(\gamma(U_s)) = \omega(U_s)$. Recall from proposition 2.2.18 or Korfanty lemma 4.2.16 that the open sets $U_\gamma = \{(\gamma(z), z) : z \in U\}$ for $\gamma = \gamma^U_{(x,n,n,y)}$, (x, n, n, y) in $T(R)$, and $U \subseteq U(x, n, n, y)$ form a basis for the topology on R_n. Moreover, U_γ is an R_n-set with $r(U_\gamma) = \gamma_{(x,n,n,y)}(U) \subseteq U(y, n, n, x)$, $s(U_\gamma) = U$, and $r \circ s|_{U_s}^{-1} = \gamma_{(x,n,n,y)}|_U$. We first show R_n-invariance for the G-sets U_γ, i.e. $\omega(\gamma_{(x,n,n,y)}(U)) = \omega(U)$: Since $U \subseteq U_n(y)$ and $\gamma(U) \subseteq U_n(x)$, we have $U \cap \gamma_\alpha(K) = \emptyset$ and $\gamma(U) \cap \gamma_\beta(K) = \emptyset$ for α not in $T_n(y)$ and β not in $T_n(x)$. Therefore, $\phi^{-1}(U) \cap U(\alpha) = \emptyset$ and $\phi^{-1}(\gamma(U)) \cap U(\beta) = \emptyset$, where, for η in Σ_n, $U(\eta) := \{x \in \{1, .., v\}^{\mathbb{N}} : x|_{[1,n]} = \eta\}$. Hence, $\omega(U) = \sum_{\varepsilon \in T_n(y)} \mu(\phi^{-1}(U) \cap U(\varepsilon))$ and $\omega(\gamma(U)) = \sum_{\eta \in T_n(x)} \mu(\phi^{-1}(\gamma(U) \cap U(\eta))$. Let $b : T_n(y) \mapsto T_n(x)$ be the bijection, defined in proposition 4.2.25, such that, for ε in $T_n(y)$, $\gamma_\varepsilon^{-1}(y) = \gamma_{b\varepsilon}^{-1}(x)$. Therefore, to show $\omega(U) = \omega(\gamma(U))$, it suffices to show $\mu(\phi^{-1}(U) \cap U(\varepsilon)) = \mu(\phi^{-1}(\gamma(U)) \cap U(b\varepsilon))$ for all ε in $T_n(y)$. Let $b\varepsilon = \eta$. By invariance of μ, $\mu(\phi^{-1}(U) \cap U(\varepsilon)) = \mu(\eta\varepsilon^{-1}(\phi^{-1}(U) \cap U(\varepsilon)))$. Therefore, it suffices to show $\eta\varepsilon^{-1}(\phi^{-1}(U) \cap U(\varepsilon)) = \phi^{-1}(\gamma(U)) \cap U(\eta)$. Since each are subsets of $U(\eta)$, it suffices to show

$$\varepsilon^{-1}(\phi^{-1}(U) \cap U(\varepsilon)) = \eta^{-1}(\phi^{-1}(\gamma(U)) \cap U(\eta)).$$

$\varepsilon^{-1}(\phi^{-1}(U) \cap U(\varepsilon)) = \varepsilon^{-1}(\phi^{-1}(U)) = (\phi \circ \varepsilon)^{-1}(U) = \phi^{-1}\gamma_\varepsilon^{-1}(U)$, and similarly $\eta^{-1}(\phi^{-1}(\gamma(U)) \cap U(\eta)) = \phi^{-1}\gamma_\eta^{-1}(\gamma(U))$. Therefore, it suffices to prove $\gamma_\varepsilon^{-1}(U) = \gamma_\eta^{-1}(\gamma(U))$. Since $\gamma_\varepsilon^{-1}(y) = \gamma_\eta^{-1}(x)$, by proposition 2.2.14, $\gamma_\eta \circ \gamma_\varepsilon^{-1}|_{U \cap \gamma_\varepsilon(K)} = \gamma|_{U \cap \gamma_\varepsilon(K)}$. Therefore, $\gamma_\varepsilon^{-1}(U) \subseteq \gamma_\eta^{-1}(\gamma(U))$. The other containment follows from symmetry $((\gamma^U_{(x,n,n,y)})^{-1} = \gamma^{\gamma(U)}_{(y,n,n,x)})$.

Now, for a general R-set U, let ψ be in $C_c(U_r)$ such that $0 \leq \psi \leq 1$, and let $\{U_{\gamma_i}\}_{i=1}^n \subseteq U$ be basis elements such that $U_i := r(U_{\gamma_i})$ cover $\mathrm{supp}(\psi)$. Let $\{\phi_i\}_{i=1}^n$ be positive functions such that $\phi_i \in C_c(U_i)$ and $\sum_{i=1}^n \phi_i = 1$ on $\mathrm{supp}(\psi)$. Then, by R_n-invariance of ω with respect to the U_{γ_i},

$$\int \psi \circ \gamma \; d\omega = \sum_{i=1}^{n} \int (\psi\phi_i) \circ \gamma \; d\omega = \sum_{i=1}^{n} \int \psi\phi_i \; d\omega = \int \psi.$$

Since ω is a borel measure, we have

$$\omega(U_r) = \sup\{\int \psi \; d\omega = \int \psi \circ \gamma \; d\omega : \; 0 \leq \psi \leq 1, \; \psi \in C_c(U_r)\} = \omega(\gamma^{-1}(U_r)).$$

□

Since $\mu(V) \neq 0$ for all open sets $V \subseteq \{1, ..., v\}^{\mathbb{N}}$, we have $\omega(K \setminus S) \neq 0$. Therefore, $\tau := \frac{1}{\omega(K \setminus S)} \tau_\omega|_{C^*(R|_{K \setminus S})}$ is a normed one trace for $C^*(R|_{K \setminus S})$. By theorem 4.2.30 it is the unique trace when E is primitive, S is singular, and $R|_{K \setminus S}$ is minimal.

Corollary 4.2.34. *If $\Upsilon = \{\gamma_1, ..., \gamma_v\}$ satisifes the strong attractor open set condition and R is minimal, then $C^*(R)$ has a unique tracial state.*

Proof. Let W be the open set in the strong attractor open set condition. Since $\gamma_1(W)$ is open, it follows that $U_1 = \{(w, -1, \gamma_1(w))\}$ is a non-empty open set in E^{-1} such that $s(U_1)$ and $r(U_1)$ are open in K. Since R is minimal, every equivalence class intersects $s(U_1)$ and $r(U_1)$. Therefore, r and s are surjective on E^{-1}, so E is primitive. With $S = \emptyset$, we may apply theorem 4.2.30 and proposition 4.2.33 to conclude $C^*(R)$ has a unique trace. □

Section 5

Renormalization Groupoid

Recall from the previous section that the most important piece of information of a groupoid renomalization procedure (definition 4.2.7) for a groupoid bundle (G, π) over X was a groupoid bundle equivalence $f^*(G, \pi) \to E \leftarrow i^*(G, \pi)$, where $f : X_1 \mapsto X$ is a continuous surjection from an open set X_1 of X, and $i : X_1 \mapsto X$ is the inclusion. X_n was the open set inductively defined as $X_n = f^{-1}(X_{n-1})$ for $n \geq 1$, where $X_0 = X$. In this section, we will be concerned with constructing a groupoid $\mathcal{F}_E$ that contains the information about such a groupoid bundle equivalence, as well as its iterations E^n_x, for n in $\mathbb{N}$ and x in X_n, when $f : X_1 \mapsto X$ is a local homeomorphism. We will then show that any groupoid bundle equivalence $f^*(G, \pi) \to E \leftarrow i^*(G, \pi)$ comes from essentially the same constructions appearing in section 2.2.1 and 3.4.1 for the AF groupoid bundle.

Let us fix an étale groupoid bundle (G, π) over X, a local homeomorphism $f : X_1 \mapsto X$ from an open set X_1 of a second countable locally compact Hausdorff space X onto X, and a groupoid bundle equivalence $f^*(G, \pi) \to E \leftarrow i^*(G, \pi)$. We will first give a description of such information that will make it easier to iterate E.

Since $i : X_1 \mapsto X$ is just the inclusion, we can identify i^*G with the restriction $G_1 := G|_{\pi^{-1}(X_1)}$ and $i^*\pi$ with $\pi_1 := \pi|_{\pi^{-1}(X_1)}$. Similarly, for n in $\mathbb{N}$, denote (G_n, π_n) to be the groupoid bundle gotten by restricting to $\pi^{-1}(X_n)$. Hence, (G_1, π_1) acts on the right of E.

Now, given the left action $f^*(G, \pi) \to E$ we can define a left action $(G, \pi) \to E$ in the following way. Recall that a point h in f^*G is of the form $h = (g, x)$, for some g in G, x in X_1 such that $\pi(g) = f(x)$. Let's denote $\pi_G : f^*G \mapsto G$ to be the projection defined, for (g, x) in f^*G, as $\pi_G(g, x) = g$. Restricted to the unit space $(f^*G)^0$, $\pi_G : (f^*G)^0 \mapsto G^0$ is an open map, since π_G sends the basis element $U \times_f V = \{(g, x) \in U \times V : \pi(g) = f(x)\}$, for U open in G^0 and V open in X_1, to $U \cap \pi^{-1}(f(V))$, and f is an open map by assumption. Similarly, π_G is surjective because f is. Therefore, $\rho_f := \pi_G \circ \rho : E \mapsto G^0$ is an open surjection. We will let $G \circ_f E = \{(g, e) \in G \times E : s(g) = \rho_f(e)\}$ and define $p_f : G \circ_f E \mapsto E$ for (g, e) in $G \circ_f E$ as $p_f(g, e) = p((g, \pi(\sigma(e))), e) = (g, \pi(\sigma(e)))e$, where $p : f^*G \circ E \mapsto E$ is the original product map. This is well defined, because, by definition of a point $\rho(e)$ in $(f^*G)^0$, we have $\pi(g) = \pi(s(g)) = \pi(\pi_G \circ \rho(e)) = f(f^*\pi(\rho(e))) = f(\pi(\sigma(e)))$, so $(g, \pi(\sigma(e)))$ is in f^*G. We will denote $p_f(g, e)$ by ge.

Given g_1, g_2 such that $s(g_1) = r(g_2)$, and an e in E such that $s(g_2) = \rho_f(e)$, both (g_1, g_2e) and (g_1g_2, e) are in $G \circ_f E$ because $\rho_f(g_2e) = \pi_G(\rho((g_2, \pi(\sigma(e)))e)) = \pi_G((r(g_2), \pi(\sigma(e)))) = r(g_2)$. Write $x = \pi(\sigma(e))$. We have $g_1[(g_2, x)e] = (g_1, x)[(g_2, x)e] = (g_1g_2, x)e = (g_1g_2)e$. Therefore, the action is associative. Similarily, the action is invertible and free because the f^*G action is. Notice that the map $\Phi : f^*G \circ E \mapsto G \circ_f E$ defined for $((g, x), e)$ in $f^*G \circ E$ as $\Phi((g, x), e) = (g, e)$ is a bijection since x is uniquely determined by e. Moreover, it is continuous, because it is the restriction of a projection and is open, because it sends the basis element $U \circ V \subseteq f^*G \circ E$, for U open in f^*G and V open in E, to the basis element $\pi_G(U \cap \rho(V)) \circ V \subseteq G \circ E$. Therefore, Φ is a homoeomorphism. By the definition of p_f, we have $p_f \circ \Phi = p$. This intertwining implies that p_f is a continuous, proper action and that it commutes with the right G_1 action (p has these properties). σ remains injective modulo this new left G action because every product under the left f^*G action is represented by a product of the left G action. The relation between p_f and σ is now $\pi \circ \rho_f = f(\pi \circ \sigma)$, and whenever $\rho_f(e_1) = \rho_f(e_2)$ for e_1 and e_2 in E, there is a g in G_1 such that $e_1g = e_2$ only if $\pi(\sigma(e_2)) = \pi(s(g)) = \pi(r(g)) = \pi(\sigma(e_1))$. Conversely, if $\rho_f(e_1) = \rho_f(e_2)$ and $\pi(\sigma(e_2)) = \pi(\sigma(e_1))$, then $\rho(e_1) = \rho(e_2)$, and so the injectivity of ρ modulo the right G_1 action applies to show there is g in G_1 such that $e_1g = e_2$. Since f is a local homeomorphism, π_G is as well. Therefore, $\pi_G \circ \rho = \rho_f$ is a local homeomorphism. We will give a name to such a pair of actions $G \to E \leftarrow G_1$:

Definition 5.0.1. *Let (G, π) be an étale groupoid bundle over X, and let $f : X_1 \mapsto X$ be a local homeomorphism of an open set X_1 of X onto X. An f-**equivalence** is a pair of commuting étale groupoid actions $G \to E \leftarrow G|_{\pi^{-1}(X_1)} = G_1$ such that*

- $f \circ \pi \circ \sigma = \pi \circ \rho_f$
- *the above relation induces a map $\rho : E \mapsto (f^*G)^0$ defined for e in E as $\rho(e) = (\rho_f(e), \pi \circ (e))$. We require this map to be surjective, or equivalently, for any u in G^0 and x in X_1 such that $\pi(u) = f(x)$, there is an e in E such that $\pi(\sigma(e)) = x$ and $\rho_f(e) = u$.*
- *ρ and σ are injective modulo the right G_1 action, left G action, respectively. The condition of injectivity of ρ_f is equivalent to the property that if e_1 and e_2 are in E such that $\rho_f(e_1) = \rho_f(e_2)$ and $\pi(\sigma(e_1)) = \pi(\sigma(e_2))$, then there is a g in G_1 such that $e_1g = e_2$.*

The above conditions for an f-equivalence imply ρ is open, for if U is an open set in E such that $\rho_f|_U, \sigma|_U$ are injective and $\pi(\sigma(U))$ is contained in an open neighbourhood W such that $f : W \mapsto f(W)$ is a homeomorphism, then $\rho(U) = \rho_f(U) \times_f \pi(\sigma(U)) =$

$\rho_f(U) \times_f W$. Since $\rho_f(U)$ and W are open, it follows from the above equation that $\rho(U)$ is open. Therefore, there is an induced left f^*G action on E such that $f^*(G,\pi) \to E \leftarrow i^*(G,\pi)$ is a groupoid bundle equivalence. Clearly such a construction is an inverse to the one starting from a groupoud bundle equivalence as above and constructing an f-equivalence.

For an open set U contained in X, we will denote $\pi^{-1}(U)$ by G_U and $\pi|_{\pi^{-1}(U)}$ by π_U. Also, denote $\sigma^{-1}(\pi^{-1}(U))$ by E_U and $\rho^{-1}(\pi^{-1}(U))$ by ${}_UE$. There is a local characterization of an f-equivalence, which we now state.

Proposition 5.0.2. *Let (G,π) be an étale groupoid bundle and $f : X_1 \mapsto X$ a local homeomorphism from an open set X_1 of X onto X. A pair of commuting groupoid actions $G \to E \leftarrow G_1$ is an f-equivalence if and only if $\pi \circ \rho_f = f \circ \pi \circ \sigma$ and for every open set U of X_1 such that $f : U \mapsto f(U)$ is a homeomorphism, the restricted action $G_{f(U)} \to E_U \leftarrow G_U$ is an étale groupoid equivalence.*

Proof. The proof is straightforward. ☐

We now show how to construct a groupoid out of an f-equivalence. We will inductively define $X_n = f^{-1}(X_{n-1})$ for $n \geq 1$, where $X_0 = X$, and $G_n := G_{X_n}$, $E_n := E_{X_n}$. Then, the restricted actions $G_{n-1} \to E_n \leftarrow G_n$ are $f : X_n \mapsto X_{n-1}$ equivalences, by the local characterization above. We will let E^n denote the product $E_1 \times_{G_1} E_2 \times_{G_2} ... \times_{G_{n-1}} E_n$, and we will equip E^n with its commuting left G_0 and right G_n groupoid actions as in section 3.3. For $(e_1, e_2, ..., e_n)$ in $E_1 \times E_2 \times ... \times E_1$ such that $\sigma(e_i) = \rho(e_{i+1})$ for all $1 \leq i \leq n-1$, we will denote $e_1e_2...e_n$ to be its image in E^n under the quotient map $E_1 \times E_2 \times ... \times E_1 \mapsto E^n$.

Since the E_i are $f : X_i \mapsto X_{i-1}$ equivalences for all $1 \leq i \leq n$ and $e = e_1...e_n$ in E^n, we have $f^n(\pi(\sigma(e_n))) = f^{n-1}(\pi(\rho(e_n))) = f^{n-1}(\pi(\sigma(e_{n-1}))) = ... = \pi(\rho(e_1))$. Let U be an open set in X_n such that $f^n : U \mapsto f^n(U)$ is a homeomorphism. Then, for each $1 \leq i \leq n$, $f : f^{i-1}(U) \mapsto f^i(U)$ is a homeomorphism, and so each action $G_{f^{i-1}(U)} \to E_{f^i(U)} \leftarrow G_{f^i(U)}$ is an étale groupoid equivalence. It is easy to see that

$$E^n_U = E_{f^{n-1}(U)} \times_{G_{f^{n-1}(U)}} E_{f^{n-2}(U)} \times_{G_{f^{n-2}(U)}} ... \times_{G_{f(U)}} E_U.$$

Therefore, E^n_U is the product of étale groupoid equivalence, so by proposition 3.3.9 it is also an étale groupoid equivalence. So, we see from the local characterization in proposition 5.0.2 that $G \to E^n \leftarrow G_n$ is an $f^n : X_n \mapsto X$ equivalence. The dual actions $G_n \to E^n \leftarrow G_0$ have (g,e) in $G_n \circ E^n$ if and only if (e, g^{-1}) is in $E \circ G_n$ with product $ge = eg^{-1}$, and similarly for the dual G_0 action. We will denote E^n, when equipped with the dual actions, to be E^{-n}, and we will denote e in E^n by e^{-1} when regarded as being in E^{-n}. Also, we will denote G by E^0. Now, for (m,n) in $(\mathbb{N} \cup \{0\})^2$, we will let $E^{(m,n)}$ denote $E^{-m} \times_G E^n$. Since the properties second countability, local compactness and Hausdorff-ness are preserved under the product of étale groupoid equivalences (proposition 3.3.9 and 3.3.10), and this product is locally the product of étale equivalence, it follows that $E^{(m,n)}$ is second countable, locally compact, and Hausdorff (remember that X is second countable, locally compact and Hausdorff). For a in E^m and b in E^n such that $\rho_{f^m}(a) = \rho_{f^n}(b)$, we will let $a^{-1}b$ denote the image of (a^{-1}, b) in $E^{-m} \circ E^n$ under the quotient map $q_{(m,n)} : E^{-m} \circ E^n \mapsto E^{(m,n)}$. Since $E^{(m,n)}$ is locally the product of étale groupoid equivalences, by proposition 3.3.6, $q_{(m,n)}$ is locally an open map, and is therefore open. For U open in E^m and V open in E^{-n}, we will let $U^{-1}V$ denote the (open) image of $U^{-1} \circ V = \{(u^{-1}, v) \in U^{-1} \times V : \rho_{f^m}(u) = \rho_{f^n}(v)\}$ under $q_{(m,n)}$. Now, we will let $\mathcal{E} = \bigsqcup_{(m,n)\in(\mathbb{N}\cup\{0\})^2} E^{(m,n)}$, and describe a product and involution on it, making it the "Toeplitz extension" of the groupoid we want to consider.

First, for n in $\mathbb{N} \cup \{0\}$, since $f^{*n}G \to E^n \leftarrow G_n$ is a groupoid equivalence, $E^n \times_{G_n} \times E^{-n}$ is isomorphic to $f^{n*}G$, via the isomorphism $\Phi_n : E^n \times_{G_n} \times E^{-n} \mapsto f^{*n}G$ sending $[a, b^{-1}]$ in $E^n \times_{G_n} \times E^{-n}$, for a and b in E^n such that $\sigma(a) = \sigma(b)$, to the unique groupoid element $(g, \pi(\sigma(a))$ in $f^{*n}G$ such that $a = (g, \pi(\sigma(e)))b = gb$. We will denote g by ab^{-1}, so that $\pi_G \circ \Phi_n([a, b^{-1}]) = ab^{-1}$. Since Φ_n is a homeomorphism and π_G is continuous, it follows that the map $p_{(n,n)} : E^n \circ E^{-n} \mapsto G^0$ $[a, b^{-1}]$ sending (a, b^{-1}) to ab^{-1} is continuous. More generally, if $a = a_1a_2..a_n$ is in E^n and $b = b_1b_2...b_k$ is in E^k, such that $\sigma(a) = \sigma(b)$, we will denote ab^{-1} to be the element in E^{n-k} defined as

$$ab^{-1} = \begin{cases} [a(b_{k-n+1}...b_k)^{-1}](b_1...b_{k-n})^{-1} & \text{if } n \leq k \\ (a_1...a_{n-k})[(a_{n-k+1}...a_n)b^{-1}] & \text{if } n \geq k. \end{cases}$$

The mappings $p_{(n,k)} : E^n \circ E^{-k} \mapsto E^{n-k}$, defined for (a, b^{-1}) in $E^n \circ E^{-k}$ as $p_{(n,k)}(a, b^{-1}) = ab^{-1}$ are continuous, as they are a composite of either $p_{(n,n)}$ (when $n \leq k$) or $p_{(k,k)}$ (when $n \geq k$) with the product maps of the groupoid actions. Recall the product $(\mathbb{N} \cup \{0\})^2 \mapsto \mathbb{N} \cup \{0\}$, defined for (m, n) in $\mathbb{N} \cup \{0\}$ and (k, l) in $\mathbb{N} \cup \{0\}$ as

$$(m, n)(k, l) = \begin{cases} (m + k - n, l) & \text{if } n \leq k \\ (m, n - k + l) & \text{if } n \geq k \end{cases}$$

For $\alpha = a^{-1}b$ in $E^{(m,n)}$ and $\beta = c^{-1}d$ in $E^{(k,l)}$ such that $\sigma(b) = \sigma(c)$, the product $\alpha\beta$ in $E^{(m,n)(k,l)}$ is defined as

$$\alpha\beta = \begin{cases} [a^{-1}(bc^{-1})]d & \text{if } n \leq k \\ a^{-1}[(bc^{-1})d] & \text{if } n \geq k. \end{cases}$$

Since α and β can also be represented as $(a^{-1}g^{-1})(gb)$ and $(c^{-1}h^{-1})(hd)$ for g and h in G such that $s(g) = \rho_{f^n}(b)$ and $s(h) = \rho_{f^l}(d)$, we must check that the product with these representatives gives the same answer. We will do this for the case when $n \geq k$, and the case when $n < k$ is handled similarily. Since the map $p_{(n,k)} : E^n \circ E^{-k} \mapsto E^{n-k}$ is left and right G-equivariant, we have $a^{-1}g^{-1}[(gbc^{-1}h^{-1})hd] = a^{-1}g^{-1}g[(bc^{-1})(h^{-1}hd)] = a^{-1}[(bc^{-1})d]$. Therefore, $\alpha\beta$ is well defined. It will be convenient to denote, for $\alpha = a^{-1}b$ in $E^{(m,n)}$, $\sigma(a)$ by $\rho_*(\alpha)$ and $\sigma(b)$ by $\sigma_*(\alpha)$.

We will let $\mathcal{E} \circ \mathcal{E} = \{(\alpha, \beta) \in \mathcal{E} \times \mathcal{E} : \sigma_*(\alpha) = \rho_*(\beta)\}$ and denote $p_{\mathcal{E}} : \mathcal{E} \circ \mathcal{E} \mapsto \mathcal{E}$ to be the product map, defined for (α, β) in $\mathcal{E} \circ \mathcal{E}$ as $p_{\mathcal{E}}(\alpha, \beta) = \alpha\beta$. It is a straightforward check that this product is associative, and we will omit this. For (m, n), (k, l) in $(\mathbb{N} \cup \{0\})^2$, where $n \geq k$, the product map $p_{\mathcal{E}} : E^{(m,n)} \circ E^{(k,l)} \mapsto E^{(m,n)(k,l)}$ has a lift to a map $\tilde{p_{\mathcal{E}}} : (E^{-m} \circ E^n) \circ (E^{-k} \circ E^l) \mapsto E^{-m} \circ E^{n-k+l}$, defined for (a^{-1}, b, c^{-1}, d) in $E^{-m} \times E^n \times E^{-k} \times E^l$ such that $\rho_{f^m}(a) = \rho_{f^n}(b)$, $\sigma(b) = \sigma(c)$, $\rho_{f^k}(c) = \rho_{f^l}(d)$ as $\tilde{p_{\mathcal{E}}}(a^{-1}, b, c^{-1}, d) = (a^{-1}, p_{(n,k)}(b, c^{-1})h)$. Since $p_{(n,k)}$ is continuous and the composition $(p_{(n,k)}(b, c^{-1}), h) \mapsto p_{(n,k)}(b, c^{-1})h$ is continuous, $\tilde{p_{\mathcal{E}}}$ is continuous. Moreover, locally, this mapping is the composition of étale groupoid equivalences (as in proposition 3.3.6) so, $\tilde{p_{\mathcal{E}}}$ is an open map. The mappings $q_1 = q_{(m,n)} \times q_{(k,l)} : (E^{-m} \circ E^n) \circ (E^{-k} \circ E^l) \mapsto E^{(m,n)} \circ E^{(k,l)}$ and $q_2 = q_{(m,n-k+l)}$ are also seen to be open, by the same reasoning. The diagram

$$\begin{tikzcd}
E^{-m} \circ E^n \circ E^{-k} \circ E^l \arrow[r, "\tilde{p_{\mathcal{E}}}"] \arrow[d, "q_1"] & E^{-m} \circ E^{n-k+l} \arrow[d, "q_2"] \\
E^{(m,n)} \circ E^{(k,l)} \arrow[r, "p_{\mathcal{E}}"] & E^{(m,n-k+l)}
\end{tikzcd}$$

commutes. Therefore, $p_{\mathcal{E}} : E^{(m,n)} \circ E^{(k,l)} \mapsto E^{(m,n-k+l)}$ is a continuous and open mapping. The same is true when $n \leq k$. Therefore, $p_{\mathcal{E}} : \mathcal{E} \circ \mathcal{E} \mapsto \mathcal{E}$ is continuous and open.

Define the involution of $\alpha = a^{-1}b$ in $E^{(m,n)}$ to be $\alpha^{-1} = b^{-1}a$, which is in $E^{(n,m)}$ For an open set U contained in E^m and an open set V contained in E^n, it is easy to see that $(U^{-1}V)^{-1} = V^{-1}U$. Therefore, ${}^{-1} : \mathcal{E} \mapsto \mathcal{E}$ is a homeomorphism. Moreover, for (α, β) in $\mathcal{E} \circ \mathcal{E}$, we have that $(\beta^{-1}, \alpha^{-1})$ is in $\mathcal{E} \circ \mathcal{E}$, and $(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}$.

For n in $\mathbb{N} \cup \{0\}$, let $\Delta_n = \{e^{-1}e : e \in E^n\}$. Since $\sigma : E^n \mapsto G_n^0$ is injective modulo the left G-action, $e^{-1}e$ in Δ_n is completely determined by $\sigma_*(e^{-1}e) = \sigma(e)$. Therefore, $\sigma_* : \Delta_n \mapsto G_n^0$ is a bijection. $\sigma_* : \Delta_n \mapsto G_n^0$ is continuous, because it is the restriction of the left structure map for $E^{(n,n)} = E^{-n} \times_G E^n$. Also, for U open in E^n, $\sigma_*(U^{-1}U \cap \Delta_n) = \sigma(U)$, so $\sigma_* : \Delta_n \mapsto G_n^0$ is also open. Therefore, Δ_n is homeomorphic to G_n^0. Let $\Delta = \bigsqcup_{n=0}^{\infty} \Delta_n$, and define the range and source maps $r_{\mathcal{E}} : \mathcal{E} \mapsto \Delta$ and $s_{\mathcal{E}} : \mathcal{E} \mapsto \Delta$, respectively, for $\alpha = a^{-1}b$ in $E^{(m,n)}$ as $r(\alpha) = \alpha\alpha^{-1} = a^{-1}b(a^{-1}b)^{-1} = a^{-1}(bb^{-1})a = a^{-1}a$, and $s_{\mathcal{E}}(\alpha) = \alpha^{-1}\alpha = b^{-1}aa^{-1}b = b^{-1}b$. $r_{\mathcal{E}}$ and $s_{\mathcal{E}}$ are continuous because the product and the involution are.

Proposition 5.0.3. *$r_{\mathcal{E}}$ and $s_{\mathcal{E}}$ are local homeomorphisms.*

Proof. It suffices to prove $r_{\mathcal{E}} : E^{(m,n)} \mapsto \Delta_m$ is a local homeomorphism, for any (m, n) in $(\mathbb{N} \cup \{0\})^2$. For every a in E^m, by surjectivity of $\rho_{f^n} : E^n \mapsto G^0$, there is b in E^n

such that $\rho_{f^m}(a) = \rho_{f^n}(b)$, so $a^{-1}b$ is in $E^{(m,n)}$, and $r_{\mathcal{E}}(a^{-1}b) = a^{-1}a$. Therefore, $r_{\mathcal{E}}$ surjects $E^{(m,n)}$ onto Δ_m. For any $a^{-1}b$ in $E^{(m,n)}$, let $U_a \subseteq E^m$ and $U_b \subseteq E^n$ be open sets about a, b, respectively, such that $\sigma|_{U_a}$ and $\rho_{f^n}|_{U_b}$ are injective, and $\rho_{f^m}(U_a) = \rho_{f^n}(U_b)$. Then, $r_{\mathcal{E}}(U_a^{-1}U_b) = U_a^{-1}U_a \cap \Delta_m$, so $r_{\mathcal{E}}(U_a^{-1}U_b)$ is open. If $\sigma(e) = \sigma(e')$ for e, e' in U_a, then by injectivity of $\sigma|_{U_a}$, we have $e = e'$. Since $\rho_{f^m}(U_a) = \rho_{f^n}(U_b)$, and $\rho_{f^n}|_{U_b}$ is injective, there is a unique b' in U_b such that $\rho_{f^m}(e) = \rho_{f^n}(b')$. Therefore, $e^{-1}b'$ is the unique element in $U_a^{-1}U_b$ such that $r_{\mathcal{E}}(e^{-1}b') = e^{-1}e$. Hence, $r_{\mathcal{E}}|_{U_a^{-1}U_b}$ is injective. Since open sets of the form $U_a^{-1}U_b$ with the above properties form a basis for $E^{(m,n)}$, it follows that $r_{\mathcal{E}}$ is a local homeomorphism. □

Now, we will take a quotient of $\mathcal{E}$ to get our desired groupoid. First, let's define an equivalence relation $\sim$ on $\mathcal{E}$ where, for α and β in $\mathcal{E}$, $\alpha \sim \beta$ if $\sigma_*(\alpha) = \sigma_*(\beta)$ and $\alpha\beta^{-1}$ is in Δ. since $\Delta^{-1} = \Delta$, $\alpha\beta^{-1}$ being in Δ is equivalent to $\beta\alpha^{-1} = (\alpha\beta^{-1})^{-1}$ being in Δ. Therefore, $\alpha \sim \beta$ implies $\beta \sim \alpha$. By definition of Δ, $\alpha \sim \alpha$ for any α in $\mathcal{E}$. Before we show transitivity of $\sim$, we would like to prove some other results first. These results depend on two observations:

- For any α in $\mathcal{E}$ and ω in Δ such that $\sigma_*(\omega) = \rho_*(\alpha)$, we have that $\alpha^{-1}\omega\alpha$ is in Δ.
- If ω_1 is in Δ_{k_1} and ω_2 is in Δ_{k_2} such that $k_1 \leq k_2$ and $\sigma_*(\omega_1) = \rho_*(\omega_2)$, then $\omega_1\omega_2 = \omega_2$.

These observations are straightforward to verify. For instance, to show $\alpha^{-1}\omega\alpha$ is in Δ, we can write $\omega = e^{-1}e$. Then, by associativity of the product, $\alpha^{-1}(e^{-1}e)\alpha = (e\alpha)^{-1}(e\alpha)$, which is in Δ.

Proposition 5.0.4. *If $\alpha \sim \beta$ for α, β in $\mathcal{E}$, then $\alpha^{-1} \sim \beta^{-1}$.*

Proof. From the first observation and the hypothesis, we know that both $\alpha^{-1}(\alpha\beta^{-1})\alpha = (\alpha^{-1}\alpha)\beta^{-1}\alpha$ and $\beta^{-1}(\alpha\beta^{-1})\beta = \beta^{-1}\alpha(\beta^{-1}\beta)$ are in Δ. It is easy to see that $\beta^{-1}\beta\beta^{-1} = \beta^{-1}$ and $\alpha\alpha^{-1}\alpha = \alpha$. Since $\beta^{-1}\alpha$ is in Δ, $\alpha^{-1}\alpha$ and $\beta^{-1}\beta$ are composable. By the second observation, either $\alpha^{-1}\alpha\beta^{-1}\beta = \beta^{-1}\beta$ or $\alpha^{-1}\alpha\beta^{-1}\beta = \alpha^{-1}\alpha$. Therefore, either $\alpha^{-1}\alpha(\beta^{-1}\alpha) = (\alpha^{-1}\alpha)(\beta^{-1}\beta)\beta^{-1}\alpha = (\beta^{-1}\beta)\beta^{-1}\alpha = \beta^{-1}\alpha$, or $(\beta^{-1}\alpha)(\beta^{-1}\beta) = (\beta^{-1}\alpha)(\alpha^{-1}\alpha)(\beta^{-1}\beta) = \beta^{-1}\alpha(\alpha^{-1}\alpha) = \beta^{-1}\alpha$, In either case, we have shown $\beta^{-1}\alpha$ is in Δ. Therefore $\beta^{-1} \sim \alpha^{-1}$, so that $\alpha^{-1} \sim \beta^{-1}$. □

Proposition 5.0.5. *Suppose α_1, α_2, β_1, β_2 are elements of $\mathcal{E}$ such that $\alpha_1 \sim \alpha_2$, $\beta_1 \sim \beta_2$, and $\sigma_*(\alpha_1) = \rho_*(\beta_1)$. Then $\sigma_*(\alpha_2) = \rho_*(\beta_2)$, and $\alpha_1\beta_1 \sim \alpha_2\beta_2$.*

Proof. $\alpha_1\alpha_2^{-1}$ and $\beta_1\beta_2^{-1}$ are in Δ, so $\sigma_*(\alpha_1) = \sigma_*(\alpha_2)$ and $\rho_*(\beta_1) = \rho_*(\beta_2)$. By assumption, $\sigma_*(\alpha_1) = \rho_*(\beta_1)$, so we have $\sigma_*(\alpha_2) = \sigma_*(\alpha_1) = \rho_*(\beta_1) = \rho_*(\beta_2)$. Now, let $\alpha_1^{-1}\alpha_2 = \omega_1$, which is in Δ by proposition 5.0.4, and let $\omega_2 = \beta_1\beta_2^{-1}$. From above, ω_1 and ω_2 are composable, so by the second observation, either $\omega_1\omega_2 = \omega_2$ or $\omega_1\omega_2 = \omega_1$. Assume first that $\omega_1\omega_2 = \omega_2$. Then, $\alpha_1\beta_1(\alpha_2\beta_2)^{-1} = \alpha_1\omega_2\alpha_2^{-1} = \alpha_1\omega_1\omega_2\alpha_2^{-1} = (\alpha_1\alpha_1^{-1})(\alpha_2\omega_2\alpha_2^{-1})$. Since Δ is closed under multiplication, and $\alpha_2\omega_2\alpha_2^{-1}$ is in Δ by the first oberservation, it follows that $\alpha_1\beta_1(\alpha_2\beta_2)^{-1}$ is in Δ. Now, assume $\omega_1\omega_2 = \omega_1$. If α_1 is in $E^{(m,n)}$ and α_2 is in $E^{(k,l)}$, then either $\alpha_1^{-1}\alpha_2$ is in $E^{(n,m-k+l)}$ $(m \geq k)$ or $\alpha_1^{-1}\alpha_2$ is in $E^{(n+k-m,l)}$ $(k \geq m)$. Since $\alpha_1^{-1}\alpha_2$ is in Δ, we have $n = m - k + l$, so $\alpha_1^{-1}\alpha_2$ is either in Δ_n $(m \geq k)$ or it is in Δ_l $(k \geq m)$. This equality, along with $\alpha_1^{-1}\alpha_2\alpha_2^{-1}\alpha_2 = \alpha_1^{-1}\alpha_2$ and $\alpha_1^{-1}\alpha_1\alpha_1^{-1}\alpha_2 = \alpha_1^{-1}\alpha_2$ implies from the second observation that either $\alpha_1^{-1}\alpha_2 = \alpha_1^{-1}\alpha_1$ $(m \geq k)$ or $\alpha_1^{-1}\alpha_2 = \alpha_2^{-1}\alpha_2$ $(k \geq m)$. Therefore, either $\alpha_1^{-1}\alpha_1\omega_2 = \alpha_1^{-1}\alpha_1$, or $\omega_2\alpha_2^{-1}\alpha_2 = \alpha_2^{-1}\alpha_2$. Hence, either $\alpha_1\omega_2\alpha_2^{-1} = \alpha_1\alpha_1^{-1}\alpha_1\omega_2\alpha_2^{-1} = \alpha_1\alpha_1^{-1}\alpha_1\alpha_2 = \alpha_1\alpha_2^{-1}$, or $\alpha_1\omega_2\alpha_2^{-1} = \alpha_1\omega_2\alpha_2^{-1}\alpha_2\alpha_2^{-1} = \alpha_1\alpha_2^{-1}\alpha_2\alpha_2^{-1} = \alpha_1\alpha_2^{-1}$ In either case, we have that $\alpha_1\beta_1(\alpha_2\beta_2)^{-1}$ is in Δ. □

We now show $\sim$ is transitive.

Proposition 5.0.6. *For α, β, and γ in $\mathcal{E}$, if $\alpha \sim \beta$ and $\beta \sim \gamma$, then $\alpha \sim \gamma$.*

Proof. $\alpha\beta^{-1}$ and $\beta\gamma^{-1}$ are in Δ, so by closure of Δ under the product, $\alpha\beta^{-1}\beta\gamma^{-1}$ is in Δ. Since $\beta^{-1}\beta \sim \alpha^{-1}\alpha$, it follows from proposition 5.0.5 that $\alpha\beta^{-1}\beta\gamma^{-1} \sim \alpha\alpha^{-1}\alpha\gamma^{-1} = \alpha\gamma^{-1}$. Now, to prove the proposition, it suffices to show that if $z = \alpha\gamma^{-1}$ is equivalent to an element $u = \alpha\beta^{-1}\beta\gamma^{-1}$ in Δ, then z is in Δ. Assume z is in $E^{(n,n)}$ and u is in Δ_k. Since $z \sim u$, we have that zu is in Δ. If $n \geq k$, then $zu = zz^{-1}zu = z(z^{-1}z)u = z(z^{-1}z) = z$, so in this case z is in Δ. Now, let's assume $n \leq k$. Since $\sigma_*(zu) = \sigma_*(u)$ and zu is in Δ_k, it follows that $zu = u$. Let's write $u = b^{-1}b$ for $b = b_1b_2...b_k$ in E^k, $z = e^{-1}a$ for $e = e_1e_2...e_n$ in E^n and $a = a_1a_2...a_n$ in E^n. By the definition of the product, $(e^{-1}a)(b^{-1}b) = [e^{-1}(ab^{-1})]b = [(ba^{-1})e]^{-1}b$. Since we have $[(ba^{-1})e]^{-1}b = b^{-1}b$, it follows that $b = (ba^{-1})e$. Expanding this, we have

$$b_1...b_k = b_1...b_{k-n}[(b_{k-n+1}...b_k)(a_1...a_n)^{-1}]e_1...e_n.$$

Let $g = (b_{k-n+1}...b_k)(a_1...a_n)^{-1}$. From the above expansion, we have that $g(e_1...e_n) = b_{k-n+1}...b_k$. By definition of g, we have $g(a_1...a_n) = b_{k-n+1}...b_k$. Therefore, $g(e_1...e_n) = g(a_1...a_n)$. Since the action is invertible, we have $(e_1...e_n) = (a_1...a_n)$. Hence, $z = e^{-1}a$ is in Δ. □

Now, let $\mathcal{F}$ be the quotient space $\mathcal{E}/\sim$, which we equip with the quotient topology. For α in $\mathcal{E}$, we will denote $\overline{\alpha}$ to be its image under the quotient map $q : \mathcal{E} \mapsto \mathcal{F}$. We define the involution $\overline{\alpha}^{-1}$ to be $\overline{\alpha^{-1}}$, which is well defined by proposition 5.0.4, and continuous because it is the descent of the continuous involution at the level of $\mathcal{E}$ by a quotient map. For $\overline{\alpha}$, $\overline{\beta}$ in $\mathcal{F}$ such that $\sigma_*(\alpha) = \rho_*(\beta)$, define $\overline{\alpha}\overline{\beta} := \overline{\alpha\beta}$ to be the product, which is well defined, by proposition 5.0.5. We will show the product map $p_{\mathcal{F}} : \mathcal{F} \circ \mathcal{F} \mapsto \mathcal{F}$ is continuous a bit later. Notice that for α, β such that $\sigma_*(\alpha) = \rho_*(\beta)$, we have $\alpha\beta\beta^{-1} \sim \alpha\alpha^{-1}\alpha = \alpha$ and $\alpha^{-1}\alpha\beta \sim \beta\beta^{-1}\beta = \beta$. Therefore,

$\overline{\alpha}\overline{\beta}\overline{\beta}^{-1} = \overline{\alpha}$ and $\overline{\alpha}^{-1}\overline{\alpha}\overline{\beta} = \overline{\beta}$. These were the only identities missing from $\mathcal{E}$ for it to be a groupoid. Hence, $\mathcal{F}$ is a groupoid. We show that it is a second countable locally compact Hausdorff étale groupoid with unit space homeomorphic to G^0. First, we prove a lemma.

Lemma 5.0.7. *If α and β are elements in $E^{(m,n)}$ such that $\alpha \sim \beta$, then $\alpha = \beta$.*

Proof. Since $\alpha\beta^{-1}$ is in Δ, we know that $\sigma_*(\alpha^{-1}\alpha) = \sigma_*(\beta^{-1}\beta)$, and $\alpha^{-1}\alpha$, $\beta^{-1}\beta$ are in Δ_n. Therfore, by injectivity of $\sigma_* : \Delta_n \mapsto G^0_n$, we have $\alpha^{-1}\alpha = \beta^{-1}\beta$. Similarily, $\alpha\alpha^{-1} = \beta\beta^{-1} = \alpha\beta^{-1}$. Therefore, we have $\alpha = \alpha\alpha^{-1}\alpha = (\alpha\beta^{-1})\beta = (\beta\beta^{-1})\beta = \beta$. $\square$

Proposition 5.0.8. *$q : \mathcal{E} \mapsto \mathcal{F}$ is a local homeomorphism. Moreover, for every (m, n) in $(\mathbb{N} \cup \{0\})^2$, $q : E^{(m,n)} \mapsto \mathcal{F}$ is a homeomorphism onto its (open) image.*

Proof. By lemma 5.0.7, we have that the restriction $q : E^{(m,n)} \mapsto \mathcal{F}$, for any (m, n) in $\mathbb{N} \cup \{0\}$ is an injection. For $a^{-1}b$ in $E^{(m,n)}$, let $U_a \subseteq E^m$ and $U_b \subseteq E^n$ be open neighbourhoods of a, b, respectively, such that $\rho_{f^m}|_{U_a}$, $\rho_{f^n}|_{U_b}$, $\sigma|_{U_a}$ and $\sigma|_{U_b}$ are all homeomorphisms onto their open images. Let $W = U_a^{-1}U_b$. We show the saturation $q^{-1}q(W)$ of W is open. For k in $\mathbb{N}$, let $U_{k,a}$, $U_{k,b}$ be the unique open sets in Δ_k such that $\sigma_*(U_{k,a}) = \sigma_*(U_a) \cap G^0_k$ and $\sigma_*(U_{k,b}) = \sigma_*(U_b) \cap G^0_k$. Since $G \to E^k \leftarrow G_k$ is locally étale, it follows easily that Δ_k is open in $E^{(k,k)}$, so $U_{k,a}$ and $U_{k,b}$ are open in $E^{(k,k)}$. For $m' \geq m$ and $n' \geq n$, let $W_{m',n'} = U_{m',a}U_a^{-1}U_bU_{n',b}$. This set is open, because the product map $p_{\mathcal{E}}$ is open. For $\tilde{a}$ in U_a and $\tilde{b}$ in U_b such that $\rho_{f^m}(\tilde{a}) = \rho_{f^n}(\tilde{b})$, $\sigma_*(\tilde{a})$ is in $G^0_{m'}$, and $\sigma_*(\tilde{b})$ is in $G^0_{n'}$, there are unique elements ω_1 in $U_{n',a}$, ω_2 in $U_{m',b}$ such that $\sigma_*(\tilde{b}) = \sigma_*(\omega_1)$ and $\sigma_*(\tilde{a}) = \sigma(\omega_2)$, so that $\omega_2\tilde{a}^{-1}\tilde{b}\omega_1$ is in $W_{m',n'}$ and $\omega_2\tilde{a}^{-1}\tilde{b}\omega_1 \sim \tilde{a}^{-1}\tilde{b}$. By lemma 5.0.7, it follows that $E^{(m',n')} \cap q^{-1}q(W) = W_{m',n'}$. Now, for an arbitrary element α in $E^{(k,l)} \cap q^{-1}q(W)$, we may choose n_1 and n_2 in $\mathbb{N}$ large enough so that $k + n_1 \geq m$ and $l + n_2 \geq m$, and elements ω_2 in Δ_{k+n_1}, ω_1 in Δ_{l+n_2} such that $\sigma_*(\omega_2) = \rho_*(\alpha)$ and $\sigma_*(\alpha) = \rho_*(\omega_1)$. Then, $\omega_2\alpha\omega_1$ is in $W_{k+n_1,l+n_2}$, since $E^{(k+n_1,l+n_2)} \cap q^{-1}q(W) = W_{k+n_1,l+n_2}$. Since $W_{k+n_1,l+n_2}$ is open, it then follows by continuity of the product map $p_{\mathcal{E}}$ that there are open neighbourhoods U_{ω_2}, U_α, U_{ω_1} such that $U_{\omega_2}U_\alpha U_{\omega_1} \subseteq W_{k+n_1,l+n_2}$. Therefore, $q^{-1}q(W)$ contains the open neighbourhood $V_\alpha = \rho_*^{-1}(\sigma_*(U_{\omega_2})) \cap U_\alpha \cap \sigma_*^{-1}(\sigma_*(U_{\omega_1}))$. Hence, $q^{-1}q(W)$ is open, so $q : E^{(m,n)} \mapsto \mathcal{F}$ is a homoemorphism onto its (open) image. $\square$

Since $\mathcal{E}$ is locally compact and 2nd countable, and these properties are preserved by local homeomorphisms, it follows that $\mathcal{F}$ is locally compact and second countable. As for Hausdorff, let $\overline{\alpha}$ and $\overline{\beta}$ be two distinct equivalence classes in $\mathcal{F}$. Since the maps ρ_* and σ_* descend to continuous maps from $\mathcal{F}$ to the Hausdorff space G^0, it suffices to prove the Hausdorff property for distinct $\overline{\alpha}$ and $\overline{\beta}$ such that $\sigma_*(\alpha) = \sigma_*(\beta)$ and $\rho_*(\alpha) = \rho_*(\beta)$. These equalities imply we can find representatives α and β of the respective equivalence classes that are in the same clopen set $E^{(m,n)}$ for some (m, n) in

$\mathbb{N} \cup \{0\}$. Since $E^{(m,n)}$ is Hausdorff, we can find open disjoint neighbourhoods U_α and U_β of α and β in $E^{(m,n)}$, respectively. Since $q : E^{(m,n)} \mapsto \mathcal{F}$ is a homeomorphism onto its open image, it follows that $q(U_\alpha)$ and $q(U_\beta)$ are open disjoint neighbourhoods of $\overline{\alpha}$, $\overline{\beta}$, respectively. Therefore, $\mathcal{F}$ is also Hausdorff.

Since q is an open map, $q \times q$ is also open. $(q \times q)^{-1}(\mathcal{F} \circ \mathcal{F}) = \mathcal{E} \circ \mathcal{E}$ is closed, and $q \times q$ saturated, so the restriction $q \times q : \mathcal{E} \circ \mathcal{E} \mapsto \mathcal{F} \circ \mathcal{F}$ is a quotient map. Therefore, the product map $p_{\mathcal{F}}$ is the descent of the product map $p_{\mathcal{E}}$ by quotient maps, and is therefore continuous. Now, we show $\mathcal{F}$ is étale with unit space isomorphic to G^0. We proved in proposition 5.0.6 that if an element z in $\mathcal{F}$ is equivalent to an element in Δ, then z is in Δ. Therefore, $q^{-1}(\mathcal{F}^0) = \Delta$. Also, for u and v in Δ, $u \sim v$ if and only if $\sigma_*(u) = \sigma_*(v)$, so the local homemorphism $\sigma_* : \Delta \mapsto G^0$ descends to a homeomorphism $\overline{\sigma_*} : \mathcal{F}^0 \mapsto G^0$ such that $\overline{\sigma_*} \circ q = \sigma_*$. Hence, $\mathcal{F}^0$ is homeomorphic to G^0. Since Δ is open in $\mathcal{E}$, we have from proposition 5.0.8 that $q : \Delta \mapsto \mathcal{F}^0$ is a local homeomorphism. Therefore, the range and source maps on $\mathcal{F}$ are the descents of the range and source maps $r_{\mathcal{E}}$, $s_{\mathcal{E}}$, which are local homeomorphisms, by the local homeomorphisms $q : \mathcal{E} \mapsto \mathcal{F}$ and $q : \Delta \mapsto \mathcal{F}^0$. Therefore, the range and source maps on $\mathcal{F}$ are local homeomorphisms, so $\mathcal{F}$ is étale.

We can specialize this construction to the most basic case when the groupoid G is equal to the space X, with bundle structure $\pi = id_X : X \mapsto X$ and groupoid equivalence $X \to E_f \leftarrow X_1$, where $E_f = \{(y, x) \in X \times X_1 : f(x) = y\}$ is equipped with the structure maps defined for (y, x) in E_f as $\sigma(y, x) = x$, $\rho(y, x) = y = f(\sigma(y, x))$. Then, it is easy to see that $\mathcal{E}_{E_f}$ is

$$T_f = \{(x_1, (m, n), x_2) \in X \times (\mathbb{N} \cup \{0\})^2 \times X : x_1 \in X_m, x_2 \in X_n, f^m(x_1) = f^n(x_2)\},$$

with product, defined for pairs $(x_1, (m, n), x_2)$, $(y_1, (k, l), y_2)$ in T_f when $x_2 = y_1$ as $(x_1, (m, n), x_2)(y_1, (k, l), y_2) = (x_1, (m, n)(k, l), y_2)$. The involution is $(x, (m, n), y)^{-1} = (y, (n, m), x)$ for $(x, (m, n), y)$ in T_f. The topology on T_f has basis elements consisting of sets $U \times (m, n) \times V \subseteq E_f^{(m,n)}$ such that $f^m|_U$ and $f^n|_V$ are homeomorphisms, with $f^m(U) = f^n(V)$. The groupoid $\mathcal{F}_{E_f}$ is the groupoid $O_f = \{(x, m-n, y) \in X \times \mathbb{Z} \times X : (x, m, n, y) \in T_f\}$, and the quotient map $q : T_f \mapsto O_f$ sends (x, m, n, y) in T_f to $(x, m-n, y)$ in O_f. The groupoids O_f of local homeomorphisms have been considered by Deaconu and Muhly in [7].

Now, given an f-equivalence E, the Toeplitz extension $\mathcal{E}_E$ and the groupoid $\mathcal{F}_E$ factor down onto T_f, O_f, respectively, in the following way. For (m, n) in $(\mathbb{N} \cup \{0\})^2$, let $\gamma_{(m,n)} : E^{(m,n)} \mapsto E_f^{(m,n)}$ be the map defined, for $a^{-1}b$ in $E^{(m,n)}$ as $\gamma_{(m,n)}(a^{-1}b) = (\pi \circ \sigma(a), m, n, \pi \circ \sigma(b))$. $(\pi \circ \sigma(a), m, n, \pi \circ \sigma(b))$ is in $E_f^{(m,n)}$, because $\pi \circ \rho_{f^m} = f^m \circ \pi \circ \sigma$, so $\rho_{f^m}(a) = \rho_{f^n}(b)$ implies $f^m(\pi \circ \sigma(a)) = f^n(\pi \circ \sigma(b))$. For U open in X_m and V open in X_n, such that $f^m|_U$, $f^n|_V$ are homeomorphisms, and $f^m(U) = f^n(V) = W$, we have $\gamma_{(m,n)}^{-1}(U \times (m, n) \times V) = {}_U E_V^{(m,n)} = E_U^{-m} \times_{G_W} E_V^n$. Therefore, $\gamma_{(m,n)}$ is continuous.

We will let $\gamma = \bigcup_{(m,n)} \gamma_{(m,n)} : \mathcal{E}_E \mapsto T_f$. We show γ is multiplicative and inverse preserving. For $a^{-1}b$ in $E^{(m,n)}$ and $c^{-1}d$ in $E^{(k,l)}$ such that $\sigma(b) = \sigma(c) = u$, we have $\gamma(a^{-1}b) = (\pi \circ \sigma(a), (m,n), \pi(u))$ and $\gamma(c^{-1}d) = (\pi(u), (k,l), \pi \circ \sigma(d))$. Also, $\gamma(a^{-1}bc^{-1}d) = (\pi\circ\sigma(a), (m,n)(k,l), \pi\circ\sigma(d))$, and $(\pi\circ\sigma(a), (m,n), \pi(u))(\pi(u), (k,l), \pi\circ\sigma(d)) = (\pi \circ \sigma(a), (m,n)(k,l), \pi \circ \sigma(d))$, so γ is multiplicative. For $a^{-1}b$ in $E^{(m,n)}$, $\gamma(b^{-1}a) = (\pi \circ \sigma(b), (n,m), \pi \circ \sigma(a)) = (\pi \circ \sigma(a), (m,n), \pi \circ \sigma(b))^{-1}$, so γ is inverse preserving. Hence, $\gamma : \mathcal{E} \mapsto T_f$ is a continuous surjection preserving the involution and product of the Toeplitz extensions. For e in E^n, we have $\gamma(e^{-1}e) = (\pi \circ (e), (n,n), \pi \circ (e))$, so γ descends to a continuous surjective groupoid homomorphism $\pi_E : \mathcal{F}_E \mapsto O_f$ via the quotient maps $q_1 : \mathcal{E}_E \mapsto \mathcal{F}_E$ and $q_2 : T_f \mapsto O_f$. Therefore, the f-equivalence structure leads to a continuous surjective groupoid homomorphism π_E.

We can recover the f-equivalence from the structure $\pi_E : \mathcal{F}_E \mapsto O_f$. First, suppose $a^{-1}b$ is in $E^{(n,n)}$ and $\pi_E(\overline{a^{-1}b}) = (x, 0, x)$. Therefore, $\pi \circ \sigma(a) = \pi \circ \sigma(b)$ and $\rho_{f^n}(a) = \rho_{f^n}(b)$, so by the property of an f^n-equivalence, there is g in G_n such that $b = ag$. Hence, we have $a^{-1}b = (a^{-1}a)g \sim g$, and $\pi_E^{-1}(O_f^0) = q(E^{(0,0)}) \simeq G$. Moreover, the diagram

$$\begin{tikzcd}
G \arrow[r, "q"] \arrow[dr, "\pi"'] & \pi_E^{-1}(O_f^0) \arrow[d, "\pi_E"] \\
& O_f^0 = X
\end{tikzcd}$$

commutes, so $q : (G, \pi) \mapsto (\pi_E^{-1}(O_f^0), \pi_E)$ is an isomorphism of groupoid bundles. We now apply the same argument to recover the f-equivalence E. Consider $O_f^{(0,1)} = \{(y, -1, x) : x \in X_1 \text{ and } f(x) = y\}$. Now, suppose $a = a_1a_2..a_n$ and $b = b_1..b_{n+1}$ are such that $\rho_{f^n}(a) = \rho_{f^{n+1}}(b)$ and $\pi_E(\overline{a^{-1}b}) = (y, -1, x)$, for $(y, -1, x)$ in $O_f^{(0,1)}$. It follows that $\pi \circ \sigma(b_n) = \pi \circ \rho_f(b_{n+1}) = f(\pi \circ \sigma(b_{n+1})) = x$, so that $\pi_E(\overline{a^{-1}b_1b_2...b_n})$ is in O_f^0. Let g be in G_n such that $g = \overline{a^{-1}b_1b_2...b_n}$. Then, $a^{-1}b \sim gb_{n+1}$ and hence $\pi_E^{-1}(O_f^{(0,1)}) = q(E^{(0,1)}) \simeq E$. Since q is multiplicative, the left and right groupoid actions $\pi_E^{-1}(O_f^0) \to \pi_E^{-1}(O_f^{(0,1)}) \leftarrow \pi_E^{-1}(O_f^0 \cap X_1)$ is conjugate to the respective left, right actions

$$G = E^{(0,0)} \to E = E^{(0,1)} \leftarrow G_1 = E^{(0,0)} \cap \gamma^{-1}(X_1)$$

in $\mathcal{E}_E$, which is by definition the mutiplication by the left and right f-equivalence actions. Therefore, we can recover the f-equivalence from π_E.

We can consider this reconstruction in a more abstract setting. Suppose H is a second countable locally compact Hausdorff étale groupoid equipped with a continuous surjective groupoid homomorphism $\pi : H \mapsto O_f$. Since O_f^0 is open, $H_\pi := \pi^{-1}(O_f^0)$ is an open subgroupoid of H, so it is a second countable locally compact Hausdorff étale groupoid as well. Since $\pi : H_\pi \mapsto O_f^0 = X$ is a groupoid homorphism onto the unit space groupoid X, we have $\pi(r(g)) = \pi(g) = \pi(s(g))$ for g in H_π. Hence, (H_π, π) is a groupoid bundle over X. $O_f^{(0,1)}$ is open in O_f since it it is the image of the clopen set $E_f^{(0,1)}$ under q. Hence, $E_\pi := \pi^{-1}(O_f^{(0,1)})$ is open in H, and is therefore second countable, locally compact, Hausdorff with the property that $\rho_f := r : E_\pi \mapsto H^0 = H_\pi^0$ and $\sigma := s : E_\pi \mapsto H^0 = H_\pi^0$ are local homeomorphisms onto their (open) images (which is not necessarily all of H^0 in either case).

By the definition of $O_f^{(0,1)}$, we have $\pi \circ \rho_f = r \circ \pi = f(s \circ \pi) = f(\pi \circ \sigma)$. E_π is closed under groupoid multiplication on the left by H_π and on the right by $(H_\pi)_1$ since $(0,0)(0,1) = (0,1)$ and $(0,1)(0,0) = (0,1)$. Now, for e_1 and e_2 in E such that $\sigma(e_1) = \sigma(e_2)$, we have $s \circ \pi(e_1) = s \circ \pi(e_2) = x$, so that $g = e_1 e_2^{-1}$ is in $(H_\pi)_{f(x)}$ and $ge_2 = e_1$. Therefore, σ is injective modulo the left H_π action. If $\rho(e_1) = \rho(e_2)$ and $\pi \circ \sigma(e_1) = \pi \circ \sigma(e_2) = x$, then $h = e_1^{-1} e_2$ is in $(H_\pi)_x$, so $e_1 h = e_2$. In summary, if we assume that for every x in X_1, the range and source maps restricted to $\pi^{-1}((f(x), -1, x))$ surject onto $\pi^{-1}(f(x))$, $\pi^{-1}(x)$, respectively, then $H_\pi \to E_\pi \leftarrow (H_\pi)_1$ is an f-equivalence. It can then be shown that the corresponding groupoid $\mathcal{F}_{E_\pi}$ is isomorphic to H via the map sending a word in $\mathcal{F}_{E_\pi}$ to the product of all its factors.

If we consider the construction of bundles of AF groupoids $(R_\pi, \hat{\pi})$ in section 2.2.1 from surjective non-degenerate graph homomorphisms $\pi : H \mapsto G$, and the groupoid equivalences $\sigma_G^*(R_\pi, \hat{\pi}) \to E \leftarrow (R_\pi, \hat{\pi})$ as in section 3.4.1, then the groupoid $\mathcal{F}_E$ is isomorphic to the groupoid G_{σ_H}, where $\sigma_H : X_H \mapsto X_H$ is the shift, and E is recoverd from the factor map $\pi : G_{\sigma_H} \mapsto G_{\sigma_G}$ sending (x, k, y) in G_{σ_H} to $(\pi(x), k, \pi(y))$. The groupoids G_{σ_H} are considered by Kumjian, Pask, Raeburn, and Renault in [21].

For an IFS groupoid R as in section 2.2.3 with groupoid equivalence $E = \{(x, -1, y) \in O(R)\}$, $\mathcal{F}_E$ is isomorphic to $O(R)$. The same is true if we do not assume the range and source map surject onto the attractor K. At the end of section 2.2.3 we constructed from R another IFS groupoid $R_{\mathcal{A}}$, which had the groupoid of a standard IFS Υ with $|\mathcal{A}|$ elements on a Cantor set as a factor. This factor map extends to a factor map $\pi : O(R_{\mathcal{A}}) \mapsto O_\sigma$, where $\sigma : \mathcal{A}^{\mathbb{N}} \mapsto \mathcal{A}^{\mathbb{N}}$ is the shift on the symbol set $\mathcal{A}$. It is easy to see that the groupoid bundle $(R_\pi, \hat{\pi})$ constructed at the end of section 2.2.3 is canonically identified with the groupoid bundle $(O(R_{\mathcal{A}}))_\pi, \pi)$. We could then, as above, construct a σ-equivalence E for the groupoid bundle $(R_\pi, \hat{\pi})$ appearing in section when the surjectivity conditions, as above, on r and s are met, which would be a groupoid renormalization procedure (R_π, E, σ).

The iterations E^n of the rotation groupoid bundle equivalence $G^*(H, \pi) \to E \leftarrow i^*(H, \pi)$ defined in section 3.4.2 have interesting connections to the continued fraction algorithm. For instance, for a fixed irrational θ in $(0,1)$, E_θ^n is homeomorphic to $k_n(\theta)$ disjoint copies of $\mathbb{R}$, where $k_n(\theta)$ is the n^{th}-convergent of θ (the denominator in a rational approximation of θ). The groupoid $\mathcal{F}_E$ could then be an interesting one to compute.

The correspondence between f-equivalences and groupoids which factor onto O_f, leads us to a question: what do the properties of a renormalization procedure (G, E, f) correspond to at the level of the groupoid $\mathcal{F}_E$, or $C^*(\mathcal{F}_E)$?

Section 6
Further Work

In this section, we provide future research directions the author (or an interested reader) might follow related to both the renormalization groupoid introduced in the previous section and the analogous construction for a C^*-algebra renormalization procedure.

Anantharaman-Delaroche in [1] (definition 2.1) defined the notion of a locally contracting étale groupoid. Since a renormalization procedure (G, E, f) (definition 4.2.7) for an étale groupoid bundle (G, π) is locally contracting in a sense, it is natural to ask whether its renormalization groupoid $\mathcal{F}_E$ is locally contracting. If we assume

- $\pi : G^0 \mapsto X$ is open, proper, $(G^0_x, d_x)_{x \in X_\infty}$ is continuously varying in the sense of definition 2.2.6, $\lim_{n\to\infty} \sup_{x\in X} \lambda^n_x = 0$,
- $\sigma : X_1 \mapsto X$ has a dense number of periodic points with corresponding fibre groupoids minimal, and
- the existence of a metric d on X and a number $\lambda < 1$ for which every local inverse of f is locally a λ-contraction,

then we claim $\mathcal{F}_E$ is locally contracting. The proof we know of relies on a "local" Lebesgue number lemma for continuously varying metric spaces. Although the conditions above include all the renormalization procedures considered in this thesis, it would be desirable to prove this claim with fewer assumptions, which will be future work. Assuming the above conditions, and that $\mathcal{F}_E$ is essentially free (defined below Anantharaman-Delaroche definition 1.1.2 [1]), Anantharaman-Delaroche proposition 2.4 [1] implies $C^*(\mathcal{F}_E)$ is purely infinite. Simplicity of $C^*(\mathcal{F}_E)$ follows if we also assume O_f is minimal. We do not need all the above assumptions for proving that $\mathcal{F}_E$ is minimal, but we will leave a careful analysis of the properties of $\mathcal{F}_E$ for future work. So, we ask the following question, which is motivated by the idea that their should be a strong link between the renormalization dynamics, and the dynamics of the renormalization procedure.

Question 6.0.1. *Suppose* (G, E, f) *is a groupoid renormalization procedure for* (G, π)

with renormalization dynamics a local homeomorphism. If O_f is locally contracting, when is $\mathcal{F}_E$ locally contracting? Similarily, if O_f is minimal, when is $\mathcal{F}_E$ minimal?

Given an upper semi-continuous C^*-algebra renormalization procedure (definition 4.1.1) $(\mathcal{A}, F, \sigma)$ where $(\mathcal{A}, \sigma)$ is a D-algebra, $D = C_0(X) \subseteq C_b(X)$, and $\sigma : X_1 \mapsto X$ has a transfer operator $\Phi_\sigma : C_0(X_1) \mapsto C_0(X)$, which is a positive, faithful linear map such that $\Phi_\sigma(gf \circ \sigma) = \Phi_\sigma(g)f$ for all g in $C_0(X_1)$, f in $C_0(X)$, we can construct a C^*-algebra in the following way. The map Φ_σ extends to a linear map $\Phi_{\mathcal{A}} : \sigma^*\mathcal{A} \mapsto \mathcal{A}$ that sends a basic tensor $g \otimes_\sigma a$ in $C_0(X_1) \otimes_\sigma \mathcal{A} = \sigma^*\mathcal{A}$ to $\Phi_\sigma(g)a$. One can show that $\Phi_{\mathcal{A}}$ is also positive, faithful and satisfies $\Phi_{\mathcal{A}}(ab) = \Phi_{\mathcal{A}}(a)b$ for all a in $\sigma^*\mathcal{A}$ and b in $\mathcal{A}$, where the right action of $\mathcal{A}$ on $\sigma^*\mathcal{A} = C_0(X) \otimes_\sigma \mathcal{A}$ is the obvious one. Consider the dual Morita equivalence $i^*\mathcal{A} \rightarrow F^* \leftarrow \sigma^*\mathcal{A}$. The non-degeneracy of the right action $\sigma^*\mathcal{A} \leftarrow \mathcal{A}$ allows us to define a right action $F^* \leftarrow \mathcal{A}$. This action, along with $\langle \cdot, \cdot \rangle_{\mathcal{A}} := \Phi_{\mathcal{A}}(\langle \cdot, \cdot \rangle_*)$ defines a right Hilbert $\mathcal{A}$-module structure on F^*, as can be checked. The left action $i^*\mathcal{A} \rightarrow F^*$ can be extended, by non-degeneracy, to a left action $\mathcal{A} \rightarrow F^*$ acting on the right Hilbert $\mathcal{A}$-module F^* by bounded, adjointable right $\mathcal{A}$-linear operators. Such a pair of actions $\mathcal{A} \rightarrow F^* \leftarrow \mathcal{A}$ is called an $\mathcal{A}$-$\mathcal{A}$ correspondence. The left and right actions of $\mathcal{A}$ have the property that for all e in F^* and g in $C_0(X)$, we have $eg = (g \circ \sigma)e$. By Pimsner [37], we can associate two C^*-algebras $\mathcal{O}_{F^*}$, $\mathcal{T}_{F^*}$ the Cuntz-Pimnser algebra of F^*, and its Toeplitz extension, respectively.

If (G, E, f) is a groupoid renormalization procedure for (G, π), denote F_E to be the induced Hilbert $f^*C^*(G)$ - $i^*C^*(G)$ imprimitivity bi-module (definition 3.3.14). Using the universal properties of $\mathcal{O}_{F_E^*}$ and $\mathcal{T}_{F_E^*}$, as well as the universal property of the groupoid C^*-algebra of $\mathcal{F}_E$ and its "Toeplitz extension" $\mathcal{E}_E$, one can show that $C^*(\mathcal{F}_E) \simeq \mathcal{O}_{F_E^*}$ and $C^*(\mathcal{E}_E) \simeq \mathcal{T}_{F_E^*}$ when f is finite-to-one and the transfer operator Φ_f is chosen to be $\Phi_f(g)(y) = \sum_{x \in f^{-1}(y)} g(x)$.

The structure $\pi : \mathcal{F}_E \mapsto O_f$ corresponds to an injective *-homomorphism $\pi : C^*(O_f) \mapsto \mathcal{M}(C^*(\mathcal{F}_E))$ into the multiplier algebra (for the definition, see the beginning of section 2) of $C^*(\mathcal{F}_E)$. More generally, it is suspected that for a C^*-algebra renormalization procedure $(\mathcal{A}, F, \sigma)$ as above, $\mathcal{O}_{F^*}$ has, in a natural way, the structure of an injective $*$-homomorphism $\varphi : \mathcal{O}_{\sigma, \Phi_\sigma} \mapsto \mathcal{M}(\mathcal{O}_{F^*})$, where $\mathcal{O}_{\sigma, \Phi_\sigma}$ is the C^*-algebra of the transfer operator Φ_σ considered by Exel [10], but we haven't checked this rigorously yet. The commutant $\varphi(C_0(X))' \cap \mathcal{O}_{F^*}$ should be equal to $\mathcal{A}$, with α equal to $\varphi : C_0(X) \mapsto \varphi(C_0(X))' \cap \mathcal{O}_{\mathcal{F}^*}$. F^* should be the elements in the "twisted" commutant

$$\varphi(C_0(X))^\sigma \cap \mathcal{O}_{F^*} = \{T \in \mathcal{O}_{F^*} : \forall g \in C_0(X),\ Tg = (g \circ \sigma)T\}.$$

In this way, one should be able to recover $\mathcal{A} \rightarrow F^* \leftarrow \mathcal{A}$. When σ is a local homoemorphism, the original actions $\sigma^*\mathcal{A} \rightarrow F \leftarrow i^*\mathcal{A}$ can be recovered from the $\mathcal{A}$ - $\mathcal{A}$ correspondence.

The C^*-algebra $\mathcal{O}_{F^*}$ encodes the dynamics of the backwards iterations of the renormalization procedure. We can also turn F into an $\mathcal{A}$ - $\mathcal{A}$ correspondence when $X_1 = X$. First, consider the actions $\sigma^*\mathcal{A} \to F \to i^*\mathcal{A} = \mathcal{A}$. $\mathcal{A}$ acts non-degenerately on the left of $\sigma^*\mathcal{A} = C_0(X_1) \otimes_\sigma \mathcal{A}$ in the obvious way. Since the action $\sigma^*\mathcal{A} \to F$ is also non-degenerate, these two actions induce a left action $\mathcal{A} \to F$ by bounded, adjointable, right $\mathcal{A}$-linear operators, so that $\mathcal{A} \to F \leftarrow \mathcal{A}$ is an $\mathcal{A}$ - $\mathcal{A}$ correspondence. Its Cuntz-Pimsner algebra $\mathcal{O}_F$ encodes the dynamics of the forwards iterations of the renormalization procedure.

A Cuntz-Pimsner algebra has a canonical action of the circle, called the gauge action. See Pimsner remark 1.2(2) [37] for the construction. It would be interesting to study the structure of the KMS state spaces relative to both the gauge actions for $\mathcal{O}_F$ and $\mathcal{O}_{F^*}$ (or their Toeplitz extensions). By Laca and Neshveyev theorem 2.1 [22], The structure of these spaces will be related to properties of the trace operators S_x (definition 3.1.10).

Since the groupoid $\mathcal{F}_E$ of a groupoid renormalization procedure (G, E, f) contains the information of (G, E, f), and is generated by the iterations of $E \subseteq \mathcal{F}_E$ we could try and develop our renormalization theory for groupoids using the structure $\pi : \mathcal{F}_E \mapsto O_f$ instead of (G, E, f). Since we have more to work with in $\mathcal{F}_E$, our results may also be stronger. This approach to renormalization theory in terms of the renormalization groupoid would have the theory of étale groupoids at its disposal, possibly making it more easier to develop and would allow us to use pre-existing concepts. For example, we could use a similar notion to Anantharaman-Delaroche's locally contracting property in replace of the contraction properties considered on (G, E, f). This same approach to renormalization theory in the C^*-algebra setting may be desirable. One reason is that there doesn't seem to be a natural notion of contraction for an equivalence $\sigma^*\mathcal{A} \to F \leftarrow i^*\mathcal{A}$, but if we use instead the renormalization C^*-algebra $\mathcal{O}_{F^*}$ and its (conjectured) structure $\mathcal{O}_{\sigma,\Phi_\sigma} \mapsto \mathcal{M}(\mathcal{O}_{F^*})$, then we could possibly develop a notion of contraction around the purely infinite property, which has been well-studied by many.

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