On the Depression of Graphs

by

Mark Schurch
B.Sc., Simon Fraser University, 1998
M.Sc., University of Victoria, 2005

A Dissertation Submitted in Partial Fulfillment of the
Requirements for the Degree of

DOCTOR OF PHILOSOPHY

in the Department of Mathematics and Statistics

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Dr. Christina Mynhardt, Supervisor
(Department of Mathematics and Statistics)

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ABSTRACT

An *edge ordering* of a graph $G = (V, E)$ is an injection $f : E \to \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. A path in G for which the edge ordering f increases along its edge sequence is called an *f-ascent*; an f-ascent is *maximal* if it is not contained in a longer f-ascent. The *depression* of G is the smallest integer k such that any edge ordering f has a maximal f-ascent of length at most k. In this dissertation we discuss various results relating to the depression of a graph. We determine a formula for the depression of the class of trees known as double spiders. A k-kernel of a graph G is a set of vertices $U \subseteq V(G)$ such that for any edge ordering f of G there exists a maximal f-ascent of length at most k which neither starts nor ends in U. We study the concept of k-kernels and discuss related depression results, including an improved upper bound for the depression of trees. We include a characterization of the class of graphs with depression three and without adjacent vertices of degree three or higher, and also construct a large class of graphs with depression three which contains graphs with adjacent vertices of high degree. Lastly, we apply the concept of ascents to edge colourings using possibly fewer than $|E(G)|$ colours (integers). We consider the problem of determining the minimum number of colours for which there exists an edge colouring such that the length of a shortest maximal path of edges with increasing colors has a given length.

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ACKNOWLEDGEMENTS

First and foremost I would like to thank my supervisor Kieka Mynhardt for her guidance and support, without which this dissertation would not have been possible. I am very grateful for her valuable and timely feedback on my error-laden work, and for her flexibility and understanding when I became a distance student. I would like to thank Gary MacGillivray for being a part of my supervisory committee and for his support during my many years at the University of Victoria. I would also like to thank Frank Ruskey for being a part of my supervisory committee, and for introducing me to tatami tilings and the world of combinatorial algorithms. I would like to express my gratitude to the faculty, staff, and graduate students of the Mathematics and Statistics department for providing a welcoming and supportive environment. I would especially like to acknowledge Jill Simmons, Charlie Burton, Ed Moore, Kelly Choo, Amanda Malloch, Maryam Namazi, and Elaine Cumming.

I would like to thank my parents for their unconditional love and support over the years. And finally, I would like to thank Halimah and Emerson for their love and encouragement throughout this process, and for providing such a wonderful distraction.

Chapter 1

Preliminaries

1.1 Introduction

Suppose the edges of a finite graph are totally ordered. The flatness of such an edge ordering is the length of a shortest maximal path along which the edges increase with respect to the ordering. The depression of a graph is the maximum value of the flatness over all edge orderings. In this dissertation we discuss various topics relating to the depression of a graph.

In Chapter 2 we provide a formula for the depression of the class of trees known as double spiders. In Chapter 3 we discuss a method of constructing graphs for which the depression is bounded from above. We establish various results relating to this method, which include an improvement on an existing upper bound for the depression of trees. In Chapter 4 we characterize the class of graphs with depression three and no adjacent vertices of degree three or more. We also construct a large class of graphs with depression three, which includes graphs with adjacent vertices of high degree. In Chapter 5 we allow partial edge orderings and consider the problem of determining the minimum number of labels such that there exists a partial edge ordering with given flatness. Our results include general bounds, classes of graphs which attain some of these bounds, an upper bound for a particular class of trees, and an upper bound for complete graphs. In Chapter 6 we summarize our results and discuss some open problems and further directions of research. In Appendix A we outline an algorithm which was used to verify the depression of many of the example graphs in this dissertation.

In the rest of this chapter we define concepts and introduce notation which is relevant to the main body of work. In general we follow the notation and terminology of [5], which the reader may refer to for background on graph theory topics not defined here. We also discuss some of the motivating results for this research.

1.2 Edge orderings and ascents

An *edge ordering* of a graph $G = (V, E)$ is an injection $f : E \to \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Denote the set of all edge orderings of G by $\mathcal{F}(G)$. A path λ in G for which $f \in \mathcal{F}(G)$ increases along its edge sequence is called an *f-ascent* (or simply *ascent* if the ordering is clear), and if the path λ has length k, it is also called a *(k, f)-ascent*. If the path λ with vertex sequence $v_0, v_1, ..., v_k$ or edge sequence $e_1, e_2, ..., e_k$ forms an f-ascent, we denote this fact by writing λ as $v_0 v_1 \cdots v_k$ or $e_1 e_2 \cdots e_k$. An f-ascent is *maximal* if it is not contained in a longer f-ascent. The *height* of an edge ordering f, denoted by $H(f)$, is the length of a longest f-ascent of G. The *flatness* of an edge ordering f, denoted by $h(f)$, is the length of a shortest maximal f-ascent of G.

1.3 The altitude of a graph

In [3], the *altitude* of G was defined as

$$\alpha(G) = \min_{f \in \mathcal{F}(G)} \{H(f)\}.$$

The interpretation of the altitude of a graph G is that any edge ordering $f \in \mathcal{F}(G)$ has an f-ascent of length at least $\alpha(G)$, and $\alpha(G)$ is the largest integer for which this statement is true. Note that $\alpha(G) \geq 2$ if G has a vertex of degree at least two, and if H is a subgraph of G, then $\alpha(H) \leq \alpha(G)$.

The study of lengths of increasing paths was initiated by Chvátal and Komlós [6], who posed the problem of determining the altitude of the complete graph. This is a difficult problem and $\alpha(K_n)$ is known only for $1 \leq n \leq 8$ (see [3, 6]). In particular, $\alpha(K_6) = 4$ and $\alpha(K_7) = \alpha(K_8) = 5$. Some bounds for the altitude of complete and complete bipartite graphs are given in [3] and some general bounds of $\alpha(G)$ were obtained by Graham and Kleitman in [13]. In [9], Cockayne and Mynhardt gave exact values for $\alpha(K_{3,n})$ and a lower bound for $\alpha(K_{m,n})$. The problem of determining the height of a given edge ordering was shown to be NP-hard by Katrenič and Semanišin [15]. Other work on altitude can be found in e.g. [2, 4, 7, 17, 20, 23].

1.4 The depression of a graph

In [8], the *depression* of G was defined as

$$\varepsilon(G) = \max_{f \in \mathcal{F}(G)} \{h(f)\}.$$

The interpretation of the depression of a graph G is that any edge ordering $f \in \mathcal{F}(G)$ has a maximal f-ascent of length at most $\varepsilon(G)$, and $\varepsilon(G)$ is the smallest integer for which this statement is true.

Clearly, $\varepsilon(G) = 1$ if and only if K_2 is a component of G. Let $\tau(G)$ denote the length of a longest path in G, called the *detour length* of G. If we assume that G is connected and of size at least two, then

$$2 \leq \varepsilon(G), \alpha(G) \leq \tau(G).$$

By taking the edge ordering f for the path P_n, where $n \geq 3$, to increase along its edge sequence we see that $\varepsilon(P_n) = \tau(P_n) = n - 1$. On the other hand, by taking the edge ordering g for the path P_n, $n \geq 3$, as $1, n-1, 2, n-2, ..., \left\lceil \frac{n}{2} \right\rceil$ along its edge sequence, we see that $\alpha(P_n) = 2$.

1.5 Known results on depression

1.5.1 Graphs with depression two

Clearly, if a connected graph G has a vertex which is adjacent to two pendant vertices or to two adjacent vertices of degree two, then $\varepsilon(G) = 2$. In [8] it was shown that the converse of this statement is also true, which gives the following characterization of graphs with depression two.

Theorem 1.5.1. [8] *If G is connected, then $\varepsilon(G) = 2$ if and only if G has a vertex adjacent to two pendant vertices or to two adjacent vertices of degree two.*

The graph shown in Figure 1.1(a) is an example of a graph with a vertex adjacent to two pendant vertices while Figure 1.1(b) is an example of a graph with a vertex adjacent to two adjacent vertices of degree two. In each figure $v_1v_2v_3$ is a maximal ascent for the depicted edge ordering.

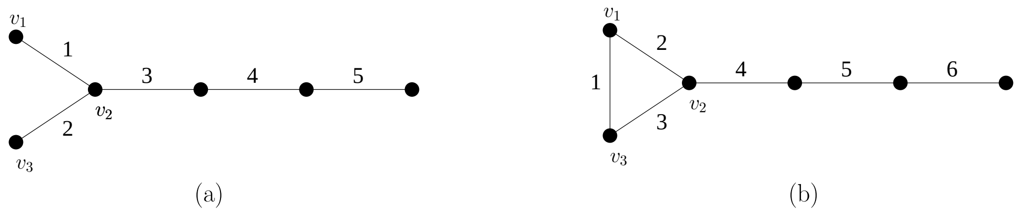

Figure 1.1: Two examples of graphs with depression two: (a) a graph with a vertex adjacent to two pendant vertices, (b) a graph with a vertex adjacent to two adjacent vertices of degree two.

Theorem 1.5.1 shows that there is no forbidden subgraph characterisation of graphs with depression two, because if any vertex of an arbitrary graph G is joined to two new vertices, the resulting graph has depression two. It also follows from Theorem 1.5.1 that if G is a 2-connected graph of order at least four, then $\varepsilon(G) \geq 3$.

1.5.2 Diameter of line graphs

When considering the problem of determining the depression of a graph it is natural to consider the line graph of G, which we denote by $L(G)$, and specifically, how $\varepsilon(G)$ is related to $\operatorname{diam}(L(G))$, the diameter of the line graph of G. The following proposition from [8] shows that there exists an infinite class of graphs G with $\varepsilon(G) \leq \operatorname{diam}(L(G)) + 1$.

Proposition 1.5.2. [8]

(a) *If* $\operatorname{diam}(L(G)) = 1$, *then* $\varepsilon(G) = 2$.

(b) *If* $\operatorname{diam}(L(G)) = 2$, *then* $\varepsilon(G) \leq 3$.

On the other hand, it is not an easy task to find graphs H with $\varepsilon(H) > \operatorname{diam}(L(H)) + 1$. An example of one such graph was given in [8] and is shown in Figure 1.2. In [11], the graph shown in Figure 1.2 was used to construct an infinite class of graphs for which $\varepsilon(H) > \operatorname{diam}(L(H)) + 1$. Furthermore, in [11], this class of graphs is also used

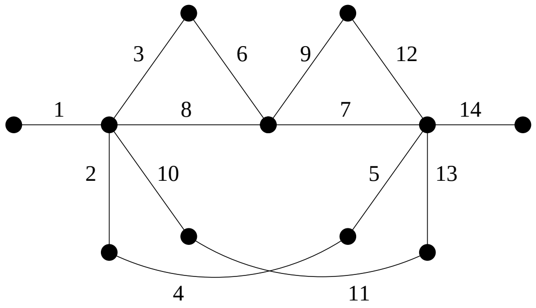

Figure 1.2: A graph H_1 for which $\varepsilon(H_1) \geq 5 > \text{diam}L(H_1) + 1 = 4$.

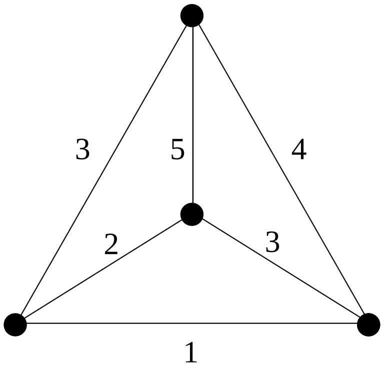

Figure 1.3: An edge ordering of K_4 with flatness three.

to show that the difference $\varepsilon(G) - \text{diam}(L(G))$ may arbitrarily large, as stated in the following result.

Theorem 1.5.3. [11] *For every integer* $n \geq 1$ *there is a graph* G *for which* $\varepsilon(G) - \text{diam}(L(G)) > n$.

1.5.3 Complete graphs and multipartite complete graphs

The depressions of complete graphs and multipartite graphs are corollaries of Theorem 1.5.1 and Proposition 1.5.2.

Corollary 1.5.4. [8] $\varepsilon(K_n) = 3$ *for all* $n \geq 4$.

See Figure 1.3 for an edge ordering of K_4 with flatness three.

Since $\alpha(K_{n-1}) \leq \alpha(K_n)$ and $\alpha(K_6) = 4$, it follows that the complete graphs K_n for $n \geq 6$ constitute an infinite class of graphs G for which $\alpha(G) > \varepsilon(G)$.

Corollary 1.5.5. [8] *$\varepsilon(K_{m,n}) = 3$ for all $2 \leq m \leq n$.*

Corollary 1.5.6. [8] *$\varepsilon(K_{m_1,m_2,\ldots,m_k}) = 3$ for all $m_1 \leq \cdots \leq m_k$ and $k \geq 3$, except $k = 3$ and $m_3 = 1$.*

1.5.4 Paths and cycles

The depressions of paths and cycles were determined in [8].

Proposition 1.5.7. [8] *$\varepsilon(P_n) = n - 1$ for all $n \geq 2$.*

Proposition 1.5.8. [8] *$\varepsilon(C_n) = \left\lceil \frac{n+1}{2} \right\rceil$ for all $n \geq 3$.*

It is a simple matter to verify that the altitude of a cycle of length n is given by

$$\alpha(C_n) = \begin{cases} 2, & \text{if } n = 3 \text{ or } n \text{ is even} \\ 3, & \text{otherwise.} \end{cases}$$

Thus, the cycles C_n for $n \geq 6$ constitute an infinite class of graphs G for which $\alpha(G) < \varepsilon(G)$.

1.5.5 Trees

Theorem 1.5.1 gives the following characterization of trees with depression 2.

Corollary 1.5.9. [8] *If T is a tree, then $\varepsilon(T) = 2$ if and only if some vertex of T is adjacent to at least two leaves.*

A *branch vertex* of a tree is a vertex with degree at least three and a *support vertex* is a vertex adjacent to a leaf. Let $B(T)$ and $L(T)$ respectively denote the sets

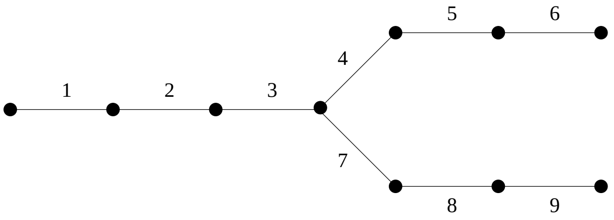

Figure 1.4: An edge ordering of $S(3,3,3)$ for which the maximal ascents are 1 2 3 4 5 6, 1 2 3 7 8 9 , and 4 7 8 9.

of all branch vertices and all leaves of the tree, and $\ell(T)$ the minimum length of a path P between two leaves of T such that no two consecutive vertices of P are branch vertices.

Theorem 1.5.10. [8] For any tree T, $\varepsilon(T) \leq \ell(T)$.

For any tree T and $v \in V(T)$, a $v-L$ *path* is a path from v to some leaf of T. For $v \in V(T)$ and $l \in L(T)$, a (v,l)-*endpath*, or v-*endpath* if l is unimportant, or *endpath* if neither v nor l is important, is a path P from v to l such that each internal vertex of P has degree two in T. A *spider* $S(a_1, a_2, \ldots, a_r)$ is a tree with exactly one branch vertex v and v-endpaths of lengths $a_1, a_2, \cdots, a_r$, where $1 \leq a_1 \leq a_2 \leq \cdots \leq a_r$ and $r = \deg v$. The depression of spiders is given in [8].

Proposition 1.5.11. [8] $\varepsilon(S(a_1, a_2, \ldots, a_r)) = \min\{a_1 + a_2, a_3 + 1\}$.

See Figure 1.4 for an edge ordering f of $S(3,3,3)$ with $h(f) = 4 = \varepsilon(S(3,3,3))$.

An upper bound for the depression of trees related to the above result for spiders was determined in [8]. Those spiders obtained by removing all edges of the tree that are not edges of endpaths are called *hanging spiders* of T. Let $\mathcal{H}(T)$ denote the set of all hanging spiders $H = S(a_1, \ldots, a_r)$, where $r \geq 3$, of T and define

$$s(T) = \min_{H \in \mathcal{H}(T)} \{a_3 + 1\}.$$

Theorem 1.5.12. [8] *For any tree T, $\varepsilon(T) \leq \min\{\ell(T), s(T)\}$.*

The bound does not necessarily give the exact value of the depression of trees, even in the case where $B(T)$ is independent. An improvement on this bound is given in Theorem 3.3.2, which does give the exact value of $\varepsilon(T)$ in the case where $B(T)$ is independent.

A lower bound for the depression of trees was given in [10] and it was shown that this bound gives the exact value of $\varepsilon(T)$ in the case where $B(T)$ is independent. The bound requires the following definition. For $\alpha \in B(T)$ with $\deg \alpha = r$, let $e_1(\alpha), e_2(\alpha), \ldots, e_r(\alpha)$ be an arrangement of the edges incident with α and $\ell_i(\alpha)$ the length of a shortest $\alpha - L$ path that contains $e_i(\alpha)$. We abbreviate $e_i(\alpha)$ and $\ell_i(\alpha)$ to e_i and ℓ_i, if the vertex α is clear from the context. An arrangement $e_1, \ldots, e_r$ is called *suitable* if $\ell_i \leq \ell_j$ whenever $i < j$. From a suitable arrangement $e_1, \ldots, e_r$ of the edges incident with α, define

$$\rho(\alpha) = \min\{\ell_1(\alpha) + \ell_2(\alpha), \ell_3(\alpha) + 1\}.$$

Theorem 1.5.13. [10] *For any tree T, $\varepsilon(T) \geq \min_{\alpha \in B(T)}\{\rho(\alpha)\}$. Moreover, if $B(T)$ is independent, then $\varepsilon(T) = \min_{\alpha \in B(T)}\{\rho(\alpha)\}$.*

1.5.6 Trees with depression three

The following characterization of trees with depression three was given in [16].

Let $\mathcal{S}_k$ be the class of trees S_k, $k \geq 1$, that can be constructed recursively as follows. Let $S_0 = K_2$ with $V(S_0) = \{x_0, y_0\}$. Define $U_0 = \varnothing$ and $Y_0 = \{y_0\}$. Once S_i has been constructed, construct S_{i+1} by performing one of the following two operations.

O1: For any $y \in Y_i$, join y to the vertex u of a new edge ux; let $U_{i+1} = U_i \cup \{u\}$ and $Y_{i+1} = Y_i$.

O2: For any $y \in Y_i$, join y to the central vertex w of a new $P_5 : s, r, w, t, z$; let $U_{i+1} = U_i \cup \{w\}$ and $Y_{i+1} = Y_i \cup \{r, t\}$.

Let $\mathcal{S} = \bigcup_{k \geq 1} \mathcal{S}_k$. Note that $S_0 = K_2$ is not in $\mathcal{S}$. For a tree $S \in \mathcal{S}$, define $U_S = U_k$. Let $\mathcal{G}$ be the class of all graphs G_S constructed as follows.

O3: Add any set $A = A(G_S)$ of new vertices to a tree $S \in \mathcal{S}$ and arbitrary edges between vertices in $A \cup U_S$.

Note that $G_S \in \mathcal{G}$ is a tree if and only if U_S is independent, $\langle A \rangle$ is acyclic, and there is exactly one edge between each component of $\langle A \rangle$ and U_s.

Let $\mathcal{T} = \{T \in \mathcal{G} : T \text{ is a tree}\}$.

Theorem 1.5.14. [16] *For any tree T, $\varepsilon(T) = 3$ if and only if $T \in \mathcal{T}$ and no vertex of T is adjacent to two leaves.*

In the proof of Theorem 1.5.14 the author shows that for each $G \in \mathcal{G}$, $\varepsilon(G) \leq 3$. Therefore,

$$\mathcal{H} = \{H \in \mathcal{G} : H \text{ is not a tree}\}$$

defines a family of graphs which are not trees and have depression at most three. This result is generalized in Section 4.2.

Chapter 2

Double Spiders

The lower bound in Theorem 1.5.13 for the depression of a tree is not exact if the tree contains adjacent branch vertices. In this chapter we determine the depression of a class of trees, called double spiders, which contain adjacent branch vertices.

A *double spider* is a tree with exactly two branch vertices, these two vertices being adjacent. More precisely, the double spider $S(a_1, \ldots, a_k : b_1, \ldots, b_{k'})$ consists of two adjacent vertices u and v with $\deg u = k + 1$, $\deg v = k' + 1$, together with $k \geq 2$ u-endpaths R_i of lengths a_i, $i = 1, \ldots, k$, and $k' \geq k$ v-endpaths R'_j of lengths b_j, $j = 1, \ldots, k'$, where $a_1 \leq \cdots \leq a_k$ and $b_1 \leq \cdots \leq b_{k'}$. Figure 2.1 depicts the double spider $S(2, 3 : 2, 3, 4)$.

2.1 A lower bound

The following lower bound for the depression of double spiders was determined in [10].

Proposition 2.1.1. [10] *Let* $T = S(a_1, \ldots, a_k : b_1, \ldots, b_{k'})$*. Then*

$$\varepsilon(T) \geq \min\{a_1 + a_2, a_3 + 1, b_1 + b_2, b_3 + 1, a_1 + b_2 + 1, a_2 + b_1 + 1\},$$

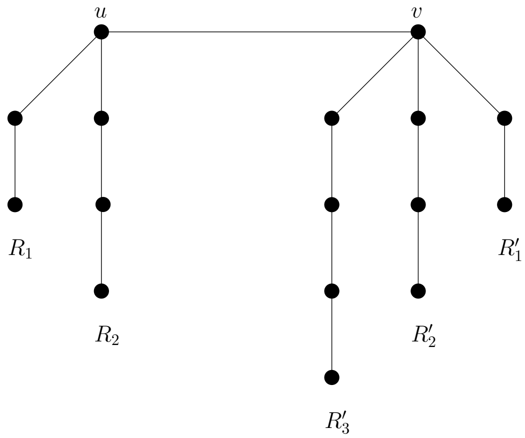

Figure 2.1: The double spider $S(2,3:2,3,4)$.

where we ignore the term a_3+1 *if* $k=2$*, and the term* b_3+1 *if* $k'=2$*.*

Note that the bound in Proposition 2.1.1 improves the bound in Theorem 1.5.13 for double spiders. For example, let $T=S(1,6:3,4)$ with branch vertices u (the support vertex) and v. Using the notation defined in Section 1.5.5, we see that $\ell_1(u)=1$, $\ell_2(u)=4$, $\ell_3(u)=6$, $\ell_1(v)=2$, $\ell_2(v)=3$ and $\ell_3(v)=4$. Therefore $\min_{x\in B(T)}\{\rho(x)\}=5$. However, by Proposition 2.1.1, $\varepsilon(T)\geq\min\{7,7,6,10\}=6$.

The bound in Proposition 2.1.1 is not best possible in all cases. A formula for the depression of double spiders may not be of particular interest in itself, but it may provide ideas for determining the depression of trees in general, or perhaps for special classes of trees with adjacent branch vertices. In Section 2.2 we prove that the depression of double spiders is given by the following formula.

Theorem 2.1.2. *Let* $T=S(a_1,\ldots,a_k:b_1,\ldots,b_{k'})$*. Then*

$$\varepsilon(T)=\begin{cases}\min\{a_1+a_2,a_1+b_2+1,a_2+b_1+1,b_1+b_2\} & \text{if } k=k'=2\\ \min\{a_1+a_2,b_1+b_2,b_3+1,\max\{a_2+2,a_1+b_2+1\}\} & \text{if } k=2<k'\\ \min\{a_1+a_2,a_3+1,b_1+b_2,b_3+1\} & \text{if } k,k'\geq 3.\end{cases} \tag{2.1}$$

The statement of the theorem and the relative complexity of its proof illustrate some of problems one encounters when attempting to determine the depression of graphs with adjacent branch vertices, even for trees with exactly two branch vertices, these vertices being adjacent.

2.2 Proof of the formula for the depression of double spiders

We begin with a lemma.

Lemma 2.2.1. *For any edge ordering f' of a tree T there is an edge ordering f of T with $h(f) \geq h(f')$ such that for each $\alpha \in B(T)$, each α-endpath is a proper subpath of a maximal f-ascent.*

Proof. Let v be a branch vertex of T and $Q_1, \ldots, Q_r$ the v-endpaths of lengths $q_1, \ldots, q_r$, respectively. We first construct an edge ordering f_v that satisfies the requirements for v. Let Q_1 be the endpath $v = v_0, v_1, \ldots, v_{q_1}$. The result is obvious for any endpath of length 1, so we assume $q_1 \geq 2$.

Without loss of generality let $f'(vv_1) < f'(v_1v_2)$. If Q_1 is not an f'-ascent, then there is some smallest index $i \geq 2$ such that $f'(v_{i-2}v_{i-1}) < f'(v_{i-1}v_i) > f'(v_iv_{i+1})$. Relabel the edges $v_iv_{i+1}, \ldots, v_{q_1-1}v_{q_1}$ so that Q_1 is an ascent increasing from v to v_{q_1} (using large enough labels to ensure a total ordering). Relabel the edges of $Q_2, \ldots, Q_r$ similarly if necessary; if f' decreases along the first two edges of Q_i and hence small labels are required to form an ascent from the leaf to v, increase the labels of the other edges by some constant where necessary to ensure a total ordering. The resulting edge ordering g has height no less than that of f'.

If no Q_i is a maximal g-ascent, let $f_v = g$ and we are done. Hence assume without loss of generality that Q_1 is a maximal g-ascent. Since Q_1 is an ascent from v to the leaf, $g(vv_1) = \min_{u \in N(v)}\{g(uv)\}$. It follows that for each $u \in N(v) - \{v_1\}$, v_1vu is a g-ascent which is contained in at least one maximal g-ascent R_u that starts at v_1. For u fixed, denote the set of all these maximal g-ascents R_u by $\mathcal{R}_u$. Define g' by

$$g'(e) = \begin{cases} g(e) + q_1 & \text{if } e \in E(T) - E(Q_1) \\ i & \text{if } e = v_{q_1-i+1}v_{q_1-i},\ i = 1, \ldots, q_1. \end{cases}$$

Then $g'(vv_1) = \min_{u \in N(v)}\{g'(uv)\}$. The maximal g'-ascents in $T - E(Q_1)$ are exactly the same as the maximal g-ascents contained in $T - E(Q_1)$. The other maximal g'-ascents are $Q_1 \cup R_u$ for each $R_u \in \mathcal{R}_u$ and each $u \in N(v) - \{v_1\}$. Therefore $g'(f) \geq h(f')$.

If no Q_i is a maximal g'-ascent, let $f_v = g'$ and we are done. Hence assume without loss of generality that Q_r is a maximal g'-ascent; say Q_r is the path $v = w_0, w_1, \ldots, w_{q_r}$. The maximality of Q_r and the fact $g'(vv_1) = \min_{u \in N(v)}\{g'(uv)\}$ imply that Q_r is an ascent from the leaf w_{q_r} to v, i.e. $g(vw_1) = \max_{u \in N(v)}\{g'(uv)\}$ and $g'(vw_1) > g'(w_1w_2)$. It follows that for each $u \in N(v) - \{w_1\}$, uvw_1 is a g'-ascent which is contained in a maximal g'-ascent R'_u that ends at w_1. In particular, each Q_i, $i \neq r$, is a proper subpath of a maximal g'-ascent. For u fixed, denote the set of all these maximal g'-ascents R'_u by $\mathcal{R}'_u$. By definition of g', each R'_u is also a maximal g-ascent. Let $m = \max_{e \in E(T) - E(Q_r)}\{g'(e)\}$ and define f_v by

$$f_v(e) = \begin{cases} g'(e) & \text{if } e \in E(T) - E(Q_r) \\ m + i & \text{if } e = w_{i-1}w_i,\ i = 1, \ldots, q_r. \end{cases}$$

The maximal f_v-ascents in $T - E(Q_r)$ are exactly the same as the maximal g'-ascents contained in $T - E(Q_r)$. The other maximal f_v-ascents are $Q_r \cup R'_u$ for each $R'_u \in \mathcal{R}'_u$ and each $u \in N(v) - \{w_1\} \neq \phi$. Therefore each Q_i is a proper subpath of a maximal f_v-ascent and $h(f_v) \geq h(g') \geq h(f')$.

By the definition of f_v, each f'-ascent contained entirely in $T - \bigcup_{i=1}^{r} E(Q_i)$ is a subpath of a maximal f_v-ascent, i.e. the relabelling of the paths Q_i did not change the f'-ascents in $T - \bigcup_{i=1}^{r} E(Q_i)$. Hence we may repeat the above process for each branch vertex separately until we obtain the required edge ordering f. □

Proof of Theorem 2.1.2. Let u_i be the neighbour of u on R_i and v_j the neighbour of v on R'_j. Define the edge ordering f as follows.

- From the leaf to u, label the edges of R_1 with the integers $1, \ldots, a_1$.
- From the leaf to v, label the edges of R'_1 with $a_1 + 1, \ldots, a_1 + b_1$.
- For $i = 2, \ldots, k-1$, and in this order, label the edges of R_i with consecutive integers $a_1 + b_1 + 1, \ldots, \sum_{i=1}^{k-1} a_i + b_1$ to form ascents from u to the leaf.
- For $j = 2, \ldots, k'-1$, and in this order, label the edges of R'_j with consecutive integers $\sum_{i=1}^{k-1} a_i + b_1 + 1, \ldots, \sum_{i=1}^{k-1} a_i + \sum_{j=1}^{k'-1} b_j$ to form ascents from v to the leaf.
- Let $f(uv) = \sum_{i=1}^{k-1} a_i + \sum_{j=1}^{k'-1} b_j + 1$.
- From u to the leaf, label the edges of R_k with $\sum_{i=1}^{k-1} a_i + \sum_{j=1}^{k'-1} b_j + 2, \ldots, \sum_{i=1}^{k} a_i + \sum_{j=1}^{k'-1} b_j + 1$.
- From from v to the leaf, label the edges of $R'_{k'}$ with the remaining integers.

See Figures 2.2 and 2.3(a) for the edge ordering f of the double spiders $S(2,3,4 : 2,3,4)$ and $S(2,3 : 2,3,4)$ respectively.

The maximal f-ascents are

- $R_1 \cup R_i$ for $i = 2, \ldots, k$ and $R'_1 \cup R'_j$ for $j = 2, \ldots, k'$,
- $u_i u \cup R_j$ for $2 \le i < j \le k$ and $v_i v \cup R'_j$ for $2 \le i < j \le k'$,
- $R_1 \cup uv \cup R'_{k'}$ and $R'_1 \cup vu \cup R_k$,

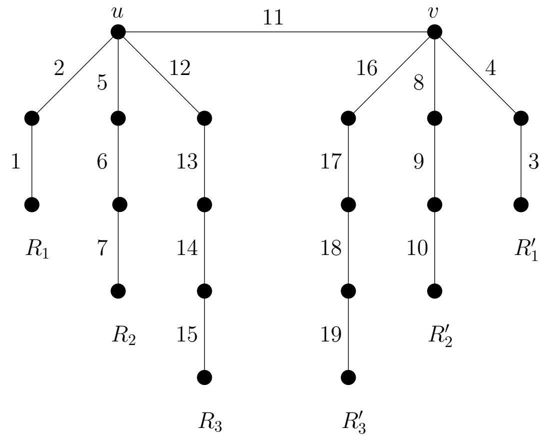

Figure 2.2: The edge ordering f of the double spider $S(2,3,4:2,3,4)$.

- $u_i uv \cup R'_{k'}$ for $i = 2, \ldots, k-1$ and $v_j vu \cup R_k$ for $j = 2, \ldots, k'-1$.

Therefore

$$\begin{aligned}
\varepsilon(T) &\geq h(f) \\
&= \min\{a_1+a_2, b_1+b_2, a_3+1, b_3+1, a_1+1+b_{k'}, b_1+1+a_k, b_{k'}+2, a_k+2\} \\
&= \begin{cases} \min\{a_1+a_2, a_1+b_2+1, a_2+b_1+1, b_1+b_2\} & \text{if} \quad k=k'=2 \\ \min\{a_1+a_2, a_2+2, b_1+b_2, b_3+1\} & \text{if} \quad k=2<k' \\ \min\{a_1+a_2, a_3+1, b_1+b_2, b_3+1\} & \text{if} \quad k,k' \geq 3. \end{cases} \qquad (2.2)
\end{aligned}$$

If $k = 2$, we also define another edge ordering g as follows.

- From the leaf to u, label the edges of R_1 with the integers $1, \ldots, a_1$.
- From the leaf to v, label the edges of R'_1 with $a_1+1, \ldots, a_1+b_1$.
- Let $g(uv) = a_1+b_1+1$.
- Label the edges of R_2 with consecutive integers $a_1+b_1+2, \ldots, a_1+a_2+b_1+1$ to form an ascent from u to the leaf.

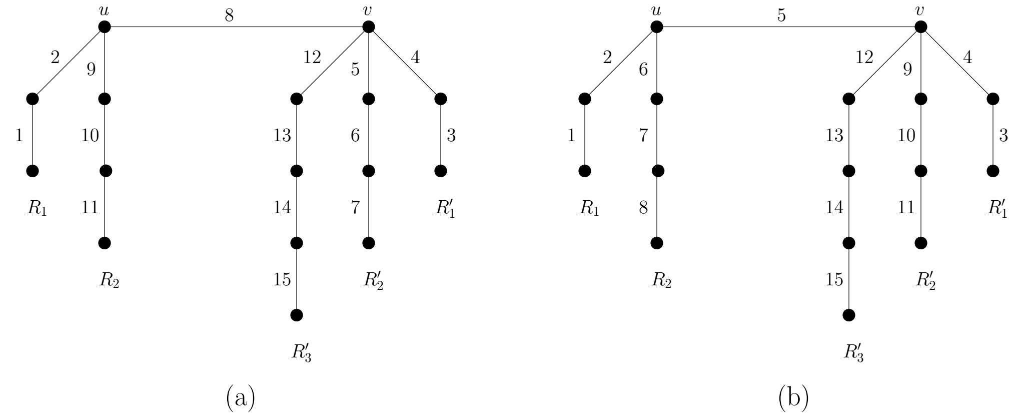

Figure 2.3: (a) The edge ordering f of the double spider $S(2,3:2,3,4)$. (b) The edge ordering g of the double spider $S(2,3:2,3,4)$.

- For $j = 2, \ldots, k'-1$, and in this order, label the edges of R'_j with consecutive integers $a_1 + a_2 + b_1 + 2, \ldots, a_1 + a_2 + \sum_{j=1}^{k'} b_j + 1$ to form ascents from v to the leaf.

See Figure 2.3(b) for the edge ordering g of the double spider $S(2,3:2,3,4)$. The maximal g-ascents are

- $R_1 \cup R_2$ and $R'_1 \cup R'_j$ for $j = 2, \ldots, k'$,
- $v_i v \cup R'_j$ for $2 \le i < j \le k'$,
- $R_1 \cup uv \cup R'_j$ for $2 \le j \le k'$ and $R'_1 \cup vu \cup R_2$.

Therefore

$$\begin{aligned} \varepsilon(T) \ge h(g) &= \min\{a_1 + a_2, b_1 + b_2, b_3 + 1, a_1 + b_2 + 1, a_2 + b_1 + 1\} \\ &= \begin{cases} \min\{a_1 + a_2, a_1 + b_2 + 1, a_2 + b_1 + 1, b_1 + b_2\} & \text{if } \; k = k' = 2 \\ \min\{a_1 + a_2, b_1 + b_2, b_3 + 1, a_1 + b_2 + 1, a_2 + b_1 + 1\} & \text{if } \; k = 2 < k'. \end{cases} \end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3) for the case $k = 2 < k'$, we get

$$\varepsilon(T) \geq \min\{a_1 + a_2, b_1 + b_2, b_3 + 1, \max\{a_2 + 2, a_1 + b_2 + 1\}\}$$

as required.

For the upper bound we prove that each edge ordering of T has height at most the stated minimum. By Theorem 1.5.12, for any edge ordering f of T,

$$h(f) \leq \min\{a_1 + a_2, a_3 + 1, b_1 + b_2, b_3 + 1\}.$$

Thus (2.1) holds for $k, k' \geq 3$. Therefore we assume without loss of generality that $k = 2$. We must show that

$$\varepsilon(T) \leq \begin{cases} \min\{a_1 + b_2 + 1, a_2 + b_1 + 1\} & \text{if} \quad k = k' = 2 \\ \max\{a_2 + 2, a_1 + b_2 + 1\} & \text{if} \quad k = 2 < k' \end{cases}. \tag{2.4}$$

If $a_2 = 1$ or $b_2 = 1$, then u or v is adjacent to two leaves, $\varepsilon(T) = 2$ by Corollary 1.5.9 and we are done, hence we assume that $a_2, b_2 \geq 2$ so that $\varepsilon(T) \geq 3$. Since the values in (2.4) are at least four, we may assume that $\varepsilon(T) \geq 5$, otherwise we are done. By Lemma 2.2.1 we only need to consider edge orderings f with $h(f) = \varepsilon(T)$ such that all endpaths R_i and R'_j are proper subpaths of maximal f-ascents.

Let f be any edge ordering with this property. If λ is a maximal f-ascent that contains any edge of R_i (respectively R'_j) other than uu_i (respectively vv_j), then by the choice of f, $R_i \subseteq \lambda$ (respectively $R'_j \subseteq \lambda$). Therefore, if no endpath of length greater than one is entirely contained in λ, then λ contains at most one edge of any endpath, and edges from at most two endpaths, hence $h(f) \leq 3$, contrary to the choice of f. We therefore assume that

$$\begin{array}{l}\text{any maximal f-ascent contains at least}\\ \text{one endpath of length greater than one.}\end{array} \tag{2.5}$$

Without loss of generality say R_1 is an f-ascent from the leaf to u. Since (by the choice of f) R_1 is not a maximal f-ascent,

$$f(u_1u) < \max\{f(uu_2), f(uv)\}. \tag{2.6}$$

If R_2 is an ascent from the leaf to u and $f(uu_2) > f(uu_1)$, then $R_1 \cup uu_2$ is a maximal f-ascent of length $a_1 + 1$ and (2.4) holds, so we assume this is not the case. If R_2 is an ascent from u to the leaf and $f(uu_2) < f(uu_1)$, then u_2uu_1 is a maximal f-ascent, contradicting (2.5). Hence either

A1. R_2 is an ascent from u to the leaf and $f(uu_2) > f(uu_1)$, so that $R_1 \cup R_2$ is a maximal f-ascent of length $a_1 + a_2$, or

A2. R_2 is an ascent from the leaf to u and $f(uu_2) < f(uu_1)$, so that $R_2 \cup uu_1$ is a maximal f-ascent of length $a_2 + 1$. In this case (2.4) holds if $k = 2 < k'$ and we only have to show that it also holds if $k = k' = 2$.

Suppose $f(uu_1) > f(uv)$. By (2.6), $f(uu_1) < f(uu_2)$, hence A1 holds and $R_1 \cup R_2$ is an f-ascent. If $f(vv_1) < f(uv)$, then by (2.5), R_1' is an ascent (of length greater than one) from the leaf to v, $R_1' \cup vuu_1$ is a maximal f-ascent of length $b_1 + 2$ and (2.4) is satisfied. On the other hand, if $f(vv_1) > f(uv)$, then by (2.5), R_1' is an ascent from v to the leaf, so that $uv \cup R_1'$ is a maximal f-ascent of length $b_1 + 1$ and we are done.

Therefore we assume that $f(uu_1) < f(uv)$. If $f(uv) = \max_{v' \in N(v)}\{f(vv')\}$, then $R_1 \cup uv$ is a maximal f-ascent of length $a_1 + 1$ and (2.4) holds, so we assume this is not the case.

Suppose $f(uv) < f(vv_1)$. If R_1' is an ascent from the leaf to v, then $R_1 \cup uvv_1$ is a maximal f-ascent of length $a_1 + 2$, satisfying (2.4), and if R_1' is an ascent from v to the leaf, then $R_1 \cup uv \cup R_1'$ is a maximal f-ascent of length $a_1 + b_1 + 1$, which also satisfies (2.4).

Assume therefore that $f(uv) > f(vv_1)$. If $f(uv) > f(uu_2)$, then $R_1' \cup vu$ is a maximal f-ascent of length $b_1 + 1$ (because by the choice of f, v_1vu is not a maximal f-ascent), and we are done. Hence we assume that $f(uv) < f(uu_2)$. Then $f(uu_1) < f(uv) < f(uu_2)$ and A1 holds. By (2.5), v_1vuu_2 is not a maximal f-ascent, so either $v_1vu \cup R_2$ is a maximal f-ascent of length $a_2 + 2$, or $R_1' \cup vu \cup R_2$ is a maximal f-ascent of length $a_2 + b_1 + 1$.

Case 1: $v_1vu \cup R_2$ is a maximal f-ascent. Then R_1' is an ascent from v to the leaf and (2.4) is satisfied for $k = 2 < k'$, so we assume $k = k' = 2$. If R_2' is an ascent from v to the leaf, then $R_1' \cup vv_2$ or $v_1v \cup R_2'$ is a maximal f-ascent of length at most $b_2 + 1$, so (2.4) holds. If R_2' is an ascent from the leaf to v, then (2.5) implies that $f(v_2v) < f(vv_1)$ and so $f(uv) = \max_{v' \in N(v)}\{f(vv')\}$, contrary to assumption.

Case 2: $R_1' \cup vu \cup R_2$ is a maximal f-ascent. Then R_1' is an ascent from the leaf to v. If $f(uv) < f(vv_2)$, then either $R_1 \cup uvv_2$ or $R_1 \cup uv \cup R_2'$ is a maximal f-ascent of length $a_1 + 2$ or $a_1 + b_2 + 1$. This together with the maximal ascent $R_1' \cup vu \cup R_2$ satisfies (2.4). Assume that $f(uv) > f(vv_2)$. If R_2' is an ascent from the leaf to v, then $R_1' \cup R_2'$ contains a maximal f-ascent of length at most $b_2 + 1$, and this together with the maximal ascent $R_1' \cup vu \cup R_2$ ensures that (2.4) holds. Thus assume R_2' is an ascent from v to the leaf. By (2.5), $R_1' \cup R_2'$ is an ascent. Also, $v_2vu \cup R_2$ is a maximal

f-ascent of length $a_2 + 2$, hence (2.4) holds if $k = 2 < k'$. Moreover, if $k = k' = 2$, then $R_1 \cup uv$ is a maximal f-ascent of length $a_1 + 1$ and (2.4) holds in this case too. $\square$

Chapter 3

Kernels

Consider two disjoint graphs G_1 and G_2, and vertices $v_i \in V(G_i)$. The *vertex-coalescence of* G_1 *and* G_2 *via* v_1 *and* v_2 is the graph obtained by identifying v_1 and v_2 to form a new vertex v, and is denoted $(G_1 \cdot G_2)(v_1, v_2 : v)$. In forming $G = (G_1 \cdot G_2)(v_1, v_2 : v)$, if v_2 is unimportant we also say we *attach* G_1 *to* G_2 *at* v_1, and if G is the resulting graph, we say that G contains G_1 *as an attachment at* v_1. From Theorem 1.5.1 we see that if v is the central vertex of P_3 or any vertex of K_3, and G is any connected graph containing P_3 or K_3 as an attachment at v, then $\varepsilon(G) = 2$.

From this result an interesting question arises.

Question 1. *What properties should* H *and* $v \in V(H)$ *satisfy so that if we attach* H *to an arbitrary graph at* v*, the resulting graph has depression at most* k*?*

In this chapter we answer this question and extend our consideration to subsets of $V(H)$. That is, for a graph H, we determine a necessary condition for a set $U \subset V(H)$, called a *k-kernel* of H, so that if we add a set of vertices A to H and arbitrary edges between $A \cup U$, the resulting graph has depression at most k. This investigation leads us to an improved upper bound for the depression of trees. The work in this chapter has been accepted for publication in [19].

In Section 3.1 we define k-kernels and ε-kernels of graphs. In Section 3.2 we characterize ε-kernels of paths and k-kernels of cycles. We provide a sufficient condition in Section 3.3 for a set of vertices to be an ε-kernel of a spider and use this result to improve the upper bound for the depression of trees given in Theorem 1.5.12. Lastly, in Section 3.4 we describe a sufficient condition for a vertex of a graph G with $\text{diam}(L(G)) = 2$ to be a k-kernel of G for $k \in \{2, 3\}$. The work in this chapter has been submitted for publication.

3.1 Definition

To help answer Question 1 and to aid our study of graphs with depression three in Chapter 4, we include the following definition which is similar to the one introduced in [16].

We define a *kernel* of a graph G as a set $U \subseteq V(G)$ such that for any edge ordering f of G there exists a maximal f-ascent that neither starts nor ends at a vertex in U. Specifically, we say U is a *k-kernel* if for any edge ordering f of G there exists a maximal (l, f)-ascent, where $l \leq k$, that neither starts nor ends at a vertex in U, and k is smallest integer for which this is true. If $k = \varepsilon(G)$ we say that U is an *ε-kernel* of G. For an illustration of this concept see Figure 3.1, which shows that a vertex v of P_4 with degree two is a 3-kernel of P_4. Furthermore, since $\varepsilon(P_4) = 3$, we also say that v forms an ε-kernel of P_4.

Note that the definition of a kernel given in [16] is equivalent to an ε-kernel as defined here. We have expanded this definition to include cases where $k > \varepsilon(G)$. As illustration of a case where we may be interested in a value of $k > \varepsilon(G)$, consider the graph G shown in Figure 3.2. By Theorem 1.5.1, $\varepsilon(G) = 2$ and the labelling f in the

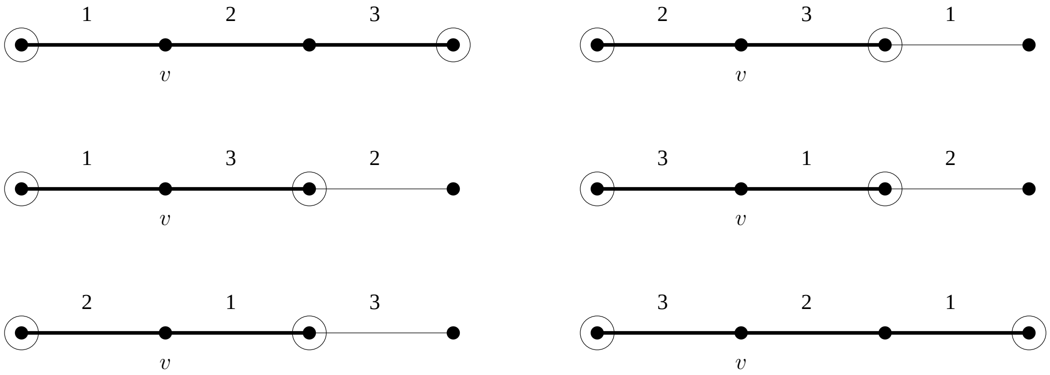

Figure 3.1: The vertex v forms an ε-kernel of P_4.

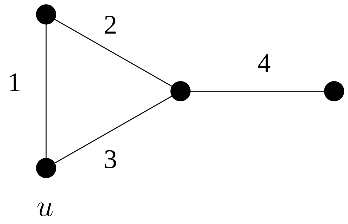

Figure 3.2: The vertex u is not an ε-kernel but is a 3-kernel.

figure shows that the vertex u is not an ε-kernel of G since the only maximal f-ascent of length two (2 3) ends at u. On the other hand, for any labelling f there exists an f-ascent that does not start or end at u and since the longest possible path in G has length three, we conclude that u is a 3-kernel of G.

For any two adjacent edges of a graph G, say e_1 and e_2, and an edge ordering f of G, either e_1e_2 or e_2e_1 is an f-ascent of G that is contained in a maximal f-ascent of length at most $\tau(G)$, the length of a longest path in G. Thus, for the vertex v incident with e_1 and e_2, and any edge ordering f of G, there exists an f-ascent that neither starts nor ends at v. We state the following remark based on this observation.

Remark 3.1.1. *Any vertex* $v \in V(G)$ *with* $\deg(v) \geq 2$ *forms a a* k*-kernel of* G *for some* $\varepsilon(G) \leq k \leq \tau(G)$.

The following theorem relates the concept of kernels to Question 1.

Theorem 3.1.2. [16] *Let U be a k-kernel of a graph H. Form a new graph G by adding any set A of new vertices and arbitrary edges joining vertices in $U \cup A$. Then $\varepsilon(G) \leq k$.*

Therefore, if we are able to identify a k-kernel of a graph G, Theorem 3.1.2 provides us with a method of forming a family of graphs with depression at most k for some $k \geq \varepsilon(G)$. For example, if v is a vertex of P_4 with degree 2 and G is a graph that contains P_4 as an attachment at v, then by Theorem 3.1.2, $\varepsilon(G) \leq \varepsilon(P_4) = 3$.

To aid us in our discussion of kernels we introduce the following terminology.

Let f be an edge ordering of a graph G. If an f-ascent λ neither starts nor ends in a set $A \subset V(G)$, we say that λ is an A-*avoiding* (*maximal*) f-ascent or an a-*avoiding* (*maximal*) f-ascent if A contains a single vertex a (and λ is not contained in a longer f-ascent).

In order to identify a set $U \subseteq V(G)$ as a k-kernel of a graph G, we must show that for every edge ordering $f \in \mathcal{F}(G)$ there exists a U-avoiding maximal ascent of length at most k.

In the following sections we identify k-kernels for various classes of graphs.

3.2 Paths and cycles

In this section we identify k-kernels of paths and cycles. Since $\varepsilon(P_n) = \tau(P_n)$, it follows that any k-kernel of P_n is necessarily an ε-kernel.

Proposition 3.2.1. *Let $U \subseteq V(P_n)$, where $n \geq 3$. Then U forms an ε-kernel of P_n if and only if U is an independent set and for each $u \in U$, $\deg(u) = 2$.*

Proof. Suppose to the contrary that U is an independent set, for each $u \in U$, $\deg(u) = 2$, and U is not an ε-kernel of $P_n = v_1, v_2, ..., v_n$. Then there exists an edge ordering of P_n, say f, such that every maximal f-ascent of length at most $\varepsilon(P_n)$ either starts or ends in U. Since $\varepsilon(P_n) = n-1$, this means that every maximal f-ascent of P_n starts or ends at a vertex in U. Necessarily, for some $3 \leq k \leq n$, either $v_1v_2\cdots v_k$ or $v_kv_{k-1}\cdots v_1$ is a maximal f-ascent. Without loss of generality we assume the former. Since $v_1 \notin U$, it follows that $v_k \in U$ and $k < n$. Since $v_1v_2\cdots v_k$ is a maximal f-ascent, $f(v_{k-1}v_k) > f(v_kv_{k+1})$, which means that $v_{k+1}v_kv_{k-1}$ is an f-ascent that ends at v_{k-1}. Since U is an independent set, v_{k-1} and v_{k+1} are not in U. This implies $v_{k+1}v_kv_{k-1}$ is contained in a longer f-ascent λ. Since λ starts or ends in U, the initial vertex, say $v_{k'}$, is in U and $k' > k+1$. By a similar argument, $v_{k'-1}v_{k'}v_{k'+1}$ is an f-ascent contained in a longer f-ascent, say λ', and the end vertex k'' of λ' is in U, where $k'' > k'-1$. Since P_n is of finite length, eventually we obtain a maximal f-ascent which neither starts nor ends in U, a contradiction.

Conversely, suppose U forms an ε-kernel of P_n. Then every edge ordering f contains a U-avoiding maximal f-ascent. Suppose that U is not an independent set. Let $v_i, v_{i+1} \in U$ for some $2 \leq i \leq n-1$. Let f be the edge ordering defined as $f(v_iv_{i+1}) = 1$, $f(v_jv_{j+1}) = j+1-i$ for each $j > i$, and $f(v_jv_{j+1}) = i-j+f(v_{n-1}v_n)$ for each $j < i$. Thus any maximal f-ascent of P_n starts at either v_i or v_{i+1}, a contradiction. Suppose that U contains an end vertex of P_n. Consider the edge ordering f defined by $f(v_iv_{i+1}) = i$ for all $1 \leq i \leq n-1$. Clearly, $v_1v_2\cdots v_n$ is the only maximal f-ascent and by our assumption it starts or ends in U, a contradiction.

□

Proposition 3.2.2. *Let $U \subseteq V(C_n)$, where $n \geq 3$. If U forms a k-kernel of C_n, then $k = n-1$. Furthermore, U is an $(n-1)$-kernel of C_n if and only if $|U| = 1$.*

Proof. By Remark 3.1.1 any single vertex v forms a k-kernel of C_n for some $\varepsilon(C_n) \leq k \leq n-1$. Consider a cycle $C_n = v_1, v_2, ..., v_n$ and the edge labelling f given by $f(v_iv_{i+1}) = i$ for $1 \leq i \leq n-1$ and $f(v_nv_1) = n$. Then for $v = v_2$ the only v-avoiding maximal f-ascent has length $n-1$. Hence v is an $(n-1)$-kernel of C_n. Since C_n is vertex transitive, any $U \subseteq V(C_n)$ with $|U| = 1$ is an $(n-1)$-kernel of C_n.

Suppose that $U \subseteq V(G)$ and $|U| \geq 2$. Let $u, v \in U$, and say $u = v_1$ and $v = v_k$ where $2 \leq k \leq n$. Let $f : E(C_n) \to \{1, ..., n\}$ such that $f(v_1v_2) = 1$, $f(v_kv_{k+1}) = n$ (or $f(v_kv_1) = n$ if $k = n$), and the remaining edges are labelled so that there are exactly two maximal f-ascents in G, both of which start with the edge labelled 1 and end with the edge labelled n (one in each direction around the cycle). One of the ascents starts at u and the other ends at v, which implies that U is not a kernel of C_n. Therefore, if U forms a k-kernel of C_n, then $|U| = 1$ and $k = n-1$. □

3.3 Spiders

In this section we identify sets which are ε-kernels of a spider $S(a_1, a_2, ..., a_r)$ and use this result to determine a new upper bound for the depression of trees. Recall that for a tree T and a vertex $\alpha \in B(T)$ with $\deg \alpha = r$, an arrangement of the edges $e_1, ..., e_r$ incident with α is called suitable if $\ell_i(\alpha) \leq \ell_j(\alpha)$ whenever $i < j$, where $\ell_i(\alpha)$ is the length of shortest α-endpath containing e_i.

Proposition 3.3.1. *Let $T = S(a_1, a_2, ..., a_r)$. If $U \subseteq V(T) - L(T)$ and $U \cup B(T)$ is independent, then U forms an ε-kernel of T.*

Proof. Let $B(T) = \{v\}$ and $U \subseteq V(T) - L(T)$ such that $U \cup \{v\}$ is independent. By Theorem 1.5.11, $\varepsilon(S(a_1, a_2, ..., a_r)) = \min\{a_1 + a_2, a_3 + 1\}$. Hence, to prove the result we must show that for any edge ordering f of T there exists a U-avoiding maximal f-ascent of length at most $\min\{a_1 + a_2, a_3 + 1\}$.

Let $e_1, e_2, ..., e_r$ be a suitable arrangement of the edges incident with the branch vertex v. For $1 \leq i < j \leq 3$, let $P_{i,j}$ be the path of length $a_i + a_j$ which contains e_i and e_j. From Proposition 3.2.1, for $G = P_{1,2}$, any independent set of $V(G)$ forms an ε-kernel of G, where $\varepsilon(G) = a_1 + a_2$. This implies that for any edge ordering f of T, there exists a $(U \cup \{v\})$-avoiding maximal f-ascent of length at most $a_1 + a_2$ which is contained in $P_{1,2}$. Similarly, there exist $(U \cup \{v\})$-avoiding maximal f-ascents of lengths at most a_1+a_3 and a_2+a_3 contained in $P_{1,3}$ and $P_{2,3}$ respectively. Let $\lambda_{i,j}$ be a $(U \cup \{v\})$-avoiding maximal f-ascent contained in the path $P_{i,j}$ where $1 \leq i < j \leq 3$.

Let f be an edge ordering of T. If $a_1 + a_2 \leq a_3 + 1$, then we are done. Hence we assume $a_1 + a_2 > a_3 + 1$. Suppose to the contrary that there does not exist a U-avoiding maximal f-ascent of length at most a_3+1 in T. Then each $\lambda_{i,j}$ has length at least $a_3 + 2 \geq a_2 + 2 \geq a_1 + 2$, which implies the edges e_i and e_j are contained in $\lambda_{i,j}$.

Without loss of generality assume that $f(e_1) < f(e_2)$. For $1 \leq i \leq 3$, let e'_i be the edge adjacent to e_i and not incident with v. Then, since the length of $\lambda_{1,2}$ is at least $a_2 + 2$, $f(e'_1) < f(e_1)$ and $f(e_2) < f(e'_2)$. This implies that $f(e'_3) < f(e_3) < f(e_2)$ or else the length of $\lambda_{2,3}$ is at most $a_3 + 1$, which is a contradiction. But then either $\lambda_{1,3}$ has length at most $a_1 + 1$ (if $f(e_3) > f(e_1)$), or $\lambda_{1,3}$ has length at most $a_3 + 1$ (if $f(e_3) < f(e_1)$), which again is a contradiction. □

We use Proposition 3.3.1 to establish an upper bound for the depression of a tree. The bound requires the following definition.

An *embedded spider* of a tree T is a subgraph $H = S(a_1, a_2, ..., a_r)$ of T such that

(i) H is a spider,

(ii) no endpath of H contains consecutive vertices in $B(T)$, and

(iii) leaves of H are also leaves of T.

Let $\mathcal{H}_{es}(T)$ denote the set of all embedded spiders $H = S(a_1, a_2, ..., a_r)$ of T, where $r \geq 3$. If $\mathcal{H}_{es}(T) \neq \varnothing$, define

$$\sigma(T) = \min_{H \in \mathcal{H}_{es}(T)} \{a_3 + 1\}.$$

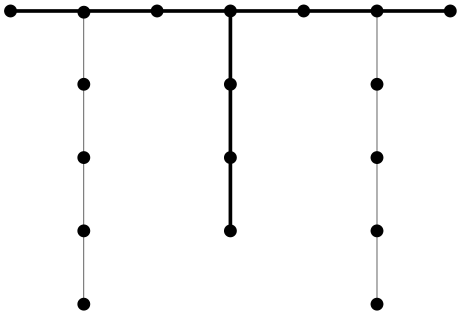

Figure 3.3: A tree T with $\varepsilon(T) = 4$.

If $\mathcal{H}_{es}(T) = \varnothing$, then define $\sigma(T) = \infty$.

Note that $\sigma(T) \leq s(T)$, where $s(T) = \min_{H \in \mathcal{H}(T)}\{a_3 + 1\}$ and $\mathcal{H}(T)$ is the set of all hanging spiders of T with at least three leaves.

Recall that $\ell(T)$ is the minimum length of a path P between two leaves of T such that P contains no two consecutive branch vertices.

Theorem 3.3.2. *For any tree T, $\varepsilon(T) \leq \min\{\ell(T), \sigma(T)\}$.*

Proof. If $\min\{\ell(T), \sigma(T)\} = \ell(T)$, then the result follows from Theorem 1.5.10. Suppose then that $\ell(T) > \sigma(T)$. Let $H = S(a_1, a_2, ..., a_r\}$ be an embedded spider of T such that $a_3 + 1 = \sigma(T)$. Let U be the set of vertices of H that are adjacent to vertices of $T - H$. Since H is an embedded spider, $U \cup B(H)$ is indepdt. By Proposition 3.3.1, U forms an ε-kernel of H. By Theorem 3.1.2, $\varepsilon(T) \leq \min\{a_1 + a_2, a_3 + 1\}$, and since $\ell(T) > \sigma(T)$, $a_1 + a_2 > a_3 + 1$ and the bound is established. ☐

The bound in Theorem 3.3.2 is an improvement on the bound in Theorem 1.5.12. For example, consider the tree T shown in Figure 3.3. We note that $\ell(T) = 5$ and that T does not contain any hanging spiders with at least three leaves, thus from Theorem 1.5.12 it follows that $\varepsilon(T) \leq 5$. On the other hand, for the embedded spider $S(3, 3, 3)$ indicated by the emphasized edges, $a_3 + 1 = 4$, which implies $\sigma(T) \leq 4$. Hence by Theorem 3.3.2, $\varepsilon(T) \leq 4$.

For the tree shown in Figure 3.3 the lower bound in Theorem 1.5.13 gives $\varepsilon(T) \geq 4$ and it was shown in [10] that this bound is tight for trees with no adjacent branch vertices. Hence, for this example, the bound from Theorem 3.3.2 is best possible. Next we show that in general the bound in Theorem 3.3.2 gives the exact value of $\varepsilon(T)$ whenever $B(T)$ is independent.

Recall that for $\alpha \in B(T)$ with $\deg(\alpha) = r$, from a suitable arrangement $e_1, ..., e_r$ of the edges incident with α,

$$\rho(\alpha) = \min\{\ell_1(\alpha) + \ell_2(\alpha), \ell_3(\alpha) + 1\}.$$

Theorem 3.3.3. *If $B(T)$ is independent, then $\varepsilon(T) = \min\{\ell(T), \sigma(T)\}$.*

Proof. If T is a path, then the result is obvious. We consider then only trees for which $B(T) \neq \varnothing$. To prove the result we show that if $B(T)$ is independent, then the lower bound in Theorem 1.5.13 is equivalent to the upper bound in Theorem 3.3.2, that is, $\min\{\ell(T), \sigma(T)\} = \min_{\alpha \in B(T)}\{\rho(\alpha)\}$. Since for any tree T, $\min_{\alpha \in B(T)}\{\rho(\alpha)\} \leq \varepsilon(T) \leq \min\{\ell(T), \sigma(T)\}$, it is enough to show that $\min\{\ell(T), \sigma(T)\} \leq \min_{\alpha \in B(T)}\{\rho(\alpha)\}$.

Let T be a tree with $B(T)$ independent, and v a vertex in $B(T)$ such that $\rho(v) = \min_{\alpha \in B(T)}\{\rho(\alpha)\} = k$. Necessarily, v is the branch vertex of an embedded spider of T, say $S(a_1, a_2, ..., a_r)$ where $r \geq 3$. By definition $\rho(v) = \min\{a_1 + a_2, a_3 + 1\}$. Moreover, $\ell(T) \leq a_1 + a_2$, and $\sigma(T) \leq a_3 + 1$. Hence, $\min\{\ell(T), \sigma(T)\} \leq \rho(v)$ and the result follows. $\square$

The bound in Theorem 3.3.2 is not always exact for trees with adjacent branch vertices. For example, consider the double spider $S(1, 5 : 3, 3)$ shown in Figure 3.4. From Theorem 3.3.2 we have $\varepsilon(S(1, 5 : 3, 3)) \leq 6$, yet $\varepsilon(S(1, 5 : 3, 3)) = 5$ by Theorem 2.1.2.

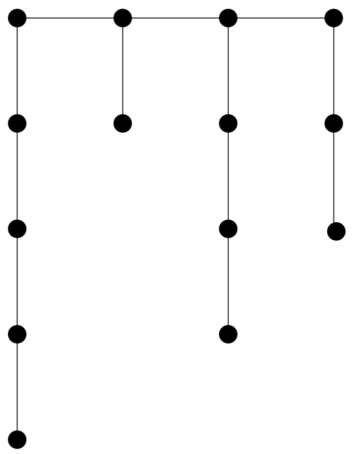

Figure 3.4: The double spider $S(1, 5 : 3, 3)$.

3.4 Graphs whose line graph has diameter two

Recall that if $\mathrm{diam}(L(G)) = 2$, then $\varepsilon(G) \leq 3$. In this section we describe a sufficient condition for a vertex of a graph G with $\mathrm{diam}(L(G)) = 2$ to be a k-kernel of G for $k \in \{2, 3\}$.

We introduce the following notation which we utilize in this section. For a graph G and sets $A, B \subseteq V(G)$, define $E(A, B)$ as the set of all edges $\{a, b\} \in E(G)$ such that $a \in A$ and $b \in B$.

Theorem 3.4.1. *Let G be a graph with* $\mathrm{diam}(L(G)) = 2$*. If v is a vertex such that $N[v]$ is a vertex cover of G, then v forms a k-kernel of G, where $k \in \{2, 3\}$.*

Proof. Let $v \in V(G)$ be a vertex such that $N[v]$ is a vertex cover of G. It suffices to show that for any edge ordering f there exists a v-avoiding maximal f-ascent of length at most three. Suppose $|E(G)| = n$ and let $f : E(G) \to \{1, ..., n\}$ be an edge ordering of G. Let uw and xy be the edges with $f(uw) = 1$ and $f(xy) = n$. Since $\mathrm{diam}(L(G)) = 2$, uw and xy lie on a common P_4. If $v \in \{u, w\} \cap \{x, y\}$, say $v = w = y$, then uvx is a v-avoiding maximal f-ascent of length at most three. Similarly, if (say) $w = y$ and $v \notin \{u, w, x, y\}$, then uwx is a v-avoiding maximal ascent. If $\{u,w\} \cap \{x,y\} = \varnothing$ and $v \notin \{u,w,x,y\}$, then $E(\{u,w\},\{x,y\}) \neq \varnothing$ since uw and xy lie on a common P_4. Any $e \in E(\{u,w\},\{x,y\})$ has label k with $1 < k < n$, thus one of $uwxy$, $uwyx$, $wuxy$ and $wuyx$ is a maximal v-avoiding ascent. We may therefore assume that $v \in \{u,w,x,y\}$ and $v \notin \{u,w\} \cap \{x,y\}$.

Without loss of generality suppose $v = y$. We consider two cases.

Case 1 $\{u,w\} \cap \{x,v\} = \varnothing$. Since $N[v]$ is a vertex cover, v is joined to u or w with an edge labelled k, where $1 < k < n$. In the former case $wuvx$ is a maximal v-avoiding f-ascent, and in the latter case $uwvx$ is such an ascent.

Case 2 $\{u,w\} \cap \{x,v\} \neq \varnothing$. By our assumption $v \notin \{u,w\}$ and we may assume without loss of generality that $x = w$. Let $f(zr) = n-1$ and suppose $v \notin \{z,r\}$. If $r \in \{u,w\}$, then uwz or wuz is a maximal v-avoiding ascent. Hence we may assume $r \notin \{u,w\}$ and similarly $z \notin \{u,w\}$. But zr and uw lie on a common P_4, hence there exists an edge $e \in E(\{z,r\},\{u,w\})$ and this edge has label k with $1 < k < n-1$, thus forming a v-avoiding maximal $(3,f)$-ascent. Therefore $v \in \{z,r\}$; say $v = r$. (Note that possibly $z = u$.)

Let $f(u_1,w_1) = 2$ and suppose $\{u_1,w_1\} \cap \{u,v,w,z\} = \varnothing$. Since $N[v]$ is a vertex cover, v is adjacent to u_1 or w_1 and this edge has label k with $2 < k < n-1$. Since $u_1 \notin \{u,w\}$, u_1w_1 is not adjacent to an edge with a smaller label. Thus u_1w_1vw or w_1u_1vw is a v-avoiding maximal ascent. It follows that $\{u_1,w_1\} \cap \{u,v,w,z\} \neq \varnothing$. We show that the edge labelled 2 is incident with w.

Now suppose that $|\{u_1,w_1\} \cap \{u,v,w,z\}| = 1$ and without loss of generality $w_1 \in \{u,v,w,z\}$. If $w_1 = z$ then u_1zvw is a maximal ascent and if $w_1 = v$ then u_1vw is a maximal ascent. Suppose then that $w_1 = u$. Then u_1 is not incident with an edge with a smaller label. Since $N[v]$ is a vertex cover, v is joined to u_1 or u by an edge with label k, $2 < k < n$ (possibly $k = n-1$ if $u = z$), so u_1uvw or uu_1vw is a v-avoiding maximal ascent. Hence $w_1 = w$ (see Figure 3.5(a) and 3.5(b)). If

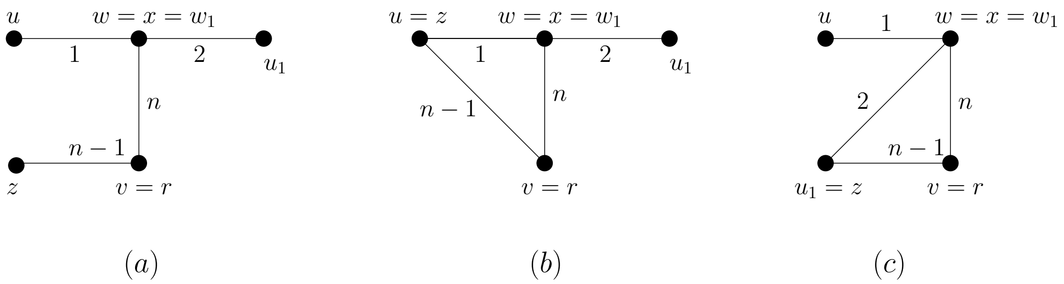

Figure 3.5: In each case the edge labelled 2 is incident with w.

$|E(G)| = 4$, then zvw is a v-avoiding maximal f-ascent.

Suppose next that $|\{u_1, w_1\} \cap \{u, v, w, z\}| = 2$. If $u_1w_1 = zu$, then $z \neq u$ and $uzvw$ is a v-avoiding maximal f-ascent. If $u_1w_1 = vu$, then $z \neq u$ and uvw is such an ascent. Hence $u_1w_1 = zw$; say $u_1 = z$ and $w_1 = w$ and note that $z \neq u$. Thus we see that in each case the edge labelled 2 is incident with w (see Figure 3.5(c)). If $|E(G)| = 4$, then zvw is a v-avoiding maximal f-ascent.

Assume $|E(G)| \geq 5$ and let $f(u_2w_2) = 3$. Suppose $u_2w_2 = uu_1$. If $z \notin \{u, u_1\}$, then since $N[v]$ is a vertex cover, v is joined to u or u_1 by an edge with label k, $3 \leq k \leq n-1$. Suppose $uv \in E(G)$ and consider the ascent u_1uvw. Since u_1 is not incident with the edge labelled 1, and the addition of the edge u_1w with label 2 forms a 4-cycle, u_1uvw is a v-avoiding maximal f-ascent. Similarly, if $u_1v \in E(H)$, then uu_1vw is a v-avoiding maximal f-ascent. If $z = u$, u_1uvw is a v-avoiding maximal ascent, and if $z = u_1$, then uu_1vw is such an ascent. For all other possibilities similar arguments as for u_1w_1 show that $u_2w_2 = zw$ (if $f(zw) \notin \{1, 2\}$) or, without loss of generality, $w_2 = w$ and no edge incident with u_2 has label $1, 2, n-1, n$. Let $u_0 = u$. By repeating the above argument we see that each edge u_iw_i with $f(u_iw_i) = i+1$, $i = 0, 1, ..., n-3$, is incident with w, say $w_i = w$, and possibly $u_i = z$ for one $i = 0, 1, ..., n-3$. Therefore, the graph H is either the graph H_1 or H_2 in Figure 3.6. But in either graph the ascent zvw is a v-avoiding maximal f-ascent and the proof of Case 2 is complete. □

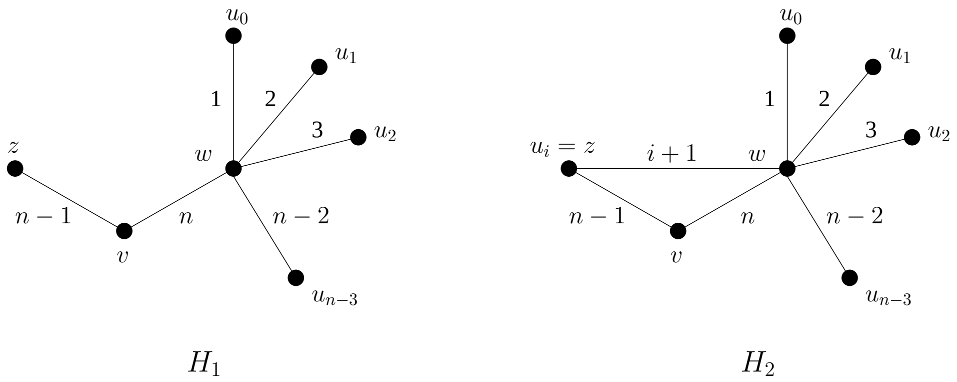

Figure 3.6: The ascent zvw in H_1 and H_2 is a v-avoiding maximal ascent.

Corollary 3.4.2. *Let G be a graph with* $\operatorname{diam}(L(G)) = 2$ *and* $\varepsilon(G) = 3$. *If v is a vertex such that $N[v]$ is a vertex cover of G, then v forms an ε-kernel of G.*

To illustrate the above corollary, note that for $n \geq 4$, $\operatorname{diam}(L(K_n)) = 2$, $\varepsilon(K_n) = 3$, and for any vertex $v \in V(K_n)$, $N[v]$ is a vertex cover of K_n. Therefore, by Corollary 3.4.2 we see that for any $v \in V(K_n)$, v forms an ε-kernel of K_n, $n \geq 4$.

Theorem 3.1.2 and Theorem 3.4.1 allow us to identify a large class of graphs with depression at most three. We state this result in the following corollary.

Corollary 3.4.3. *Let G be a graph with an end-block B such that* $\operatorname{diam}(L(B)) \leq 2$, *and v the cut vertex of G contained in B. If $N[v]$ is a vertex cover of B, then $\varepsilon(G) \leq 3$.*

Next we show that the converse of Theorem 3.4.1 is false. As a counterexample, consider the vertex v of the graph G shown in Figure 3.7. Clearly, $\operatorname{diam}(L(G)) = 2$ and $N[v]$ is not a vertex cover of G. In order to show that v forms a k-kernel, where $k \in \{2, 3\}$, we must show that for every edge ordering f of G there exists a v-avoiding maximal f-ascent of length at most three.

Suppose to the contrary that $f : E(G) \to \{1, 2, ..., 8\}$ is an edge ordering of G for which there does not exist a v-avoiding maximal f-ascent of length at most

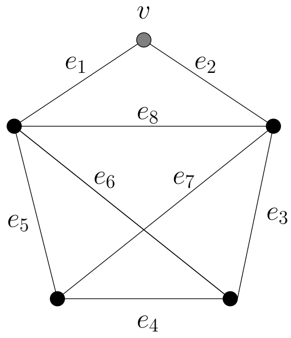

Figure 3.7: A graph G with $\operatorname{diam}(L(G)) = 2$ and a vertex v such that $N[v]$ is not a vertex cover of G.

three. If $\{f^{-1}(1), f^{-1}(8)\} \subseteq \{e_3, e_4, e_5, e_6, e_7, e_8\}$, then there clearly exists a v-avoiding maximal ascent of length at most three. Thus for some $e \in \{e_1, e_2\}$, $f(e) \in \{1, 8\}$, say $f(e_1) = 1$. Then e_1e_2 is contained in a v-avoiding maximal f-ascent λ, and by our assumption, λ has length four. Thus either $\lambda = e_1e_2e_3e_4$ or $\lambda = e_1e_2e_7e_4$ and without loss of generality we assume the former.

Let $k = \min(\{f(e_3), f(e_5), f(e_6), f(e_7)\})$. If $f(e_3) = k$, then either e_3e_7 or $e_3e_7e_5$ is a v-avoiding maximal f-ascent. Similarly, if $f(e_7) = k$, then e_7e_3 or $e_7e_3e_6$ is a v-avoiding maximal f-ascent, and if $f(e_5) = k$, then e_5e_6 or $e_5e_6e_3$ is a v-avoiding maximal f-ascent.

We assume then that $f(e_6) = \min(\{f(e_3), f(e_5), f(e_6), f(e_7)\})$. Then e_6e_5 are the first two edges of a maximal ascent, and since G does not contain a v-avoiding maximal f-ascent of length at most three, it follows that $f(e_5) < f(e_7) < f(e_2)$. Now if $f(e_8) > f(e_2)$, then e_6e_8 is a v-avoiding maximal f-ascent. Assume then that $f(e_8) < f(e_2)$. Since $e_1e_2e_3e_4$ is an f-ascent, it follows that $f(e_8) < f(e_3)$. If $f(e_8) < f(e_5)$, then $e_8e_5e_4$ is a v-avoiding maximal f-ascent. Finally, if $f(e_8) > f(e_5)$, then $e_5e_8e_3$ is a v-avoiding maximal f-ascent.

This covers all cases and establishes the proof of the counterexample to the converse of Theorem 3.4.1.

Chapter 4

Graphs With Depression Three

Theorem 1.5.1 provides a simple characterization of graphs with depression two, which leads us to consider the problem of characterizing graphs with depression three. Although this remains an unsolved problem, a characterization of trees with depression three is given in Theorem 1.5.14 of [16]. In Section 4.1 of this chapter we characterize graphs with depression three and the added property that the graph contains no two adjacent vertices of degree three or more. The work in this section has been accepted for publication in [18]. In Section 4.2 we define a large class of graphs with depression three that do contain adjacent vertices of degree three or more. The work in this section has been submitted for publication.

4.1 Graphs with depression three and no adjacent vertices of degree three or more

Let $\mathcal{H}$ be the set of graphs consisting of $K_{2,m}$ for $m \geq 1$, K_3, P_4, and the spider $S(2,2,2)$ – see Figure 4.1. An ε-kernel of a graph $H \in \mathcal{H}$ is not necessarily unique, but if an ε-kernel consists of a single vertex of H, then we call such a vertex an

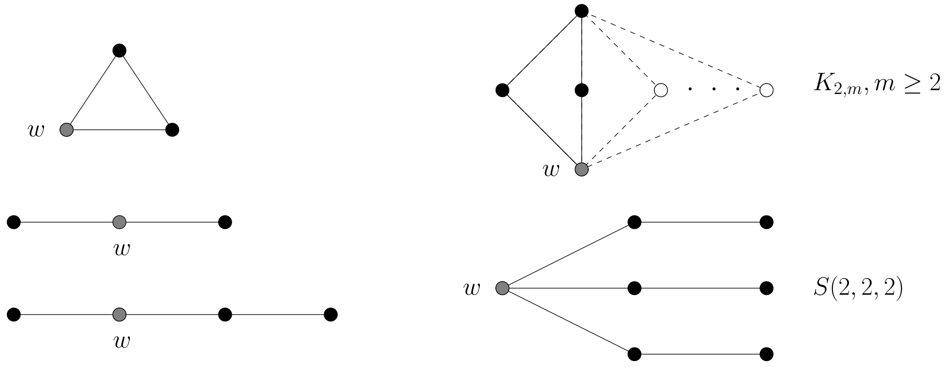

Figure 4.1: The set of graphs $\mathcal{H}$.

ε-kernel vertex.

Our aim in this section is to prove the following theorem.

Theorem 4.1.1. *Let G be a connected graph with $\operatorname{diam}(L(G)) \geq 3$ and no adjacent vertices of degree three or more. Then $\varepsilon(G) \leq 3$ if and only if $G = S(2,2,2)$, or for some $H \in \mathcal{H}$, G contains H as an attachment at an ε-kernel vertex of H.*

A vertex $v \in V(G)$ is called an *end-vertex* if $\deg v = 1$, a *link vertex* if $\deg v = 2$ and a *hub vertex* if $\deg v \geq 3$. A *u-v* path in G in which the internal vertices are link vertices is called a *u-v direct path* or simply a *direct path* if u and v are unimportant. For $U \subseteq V(G)$ and $u \in U$, a v-u direct path is also called a *v-U direct path.* As for trees, a u-v path P in which v is an end-vertex and the internal vertices are link vertices is called a *u-v endpath* or a *u-endpath* if v is unimportant.

We first prove three lemmas.

Lemma 4.1.2. *Let H be a graph in $\mathcal{H}$ and w a vertex in $V(H)$ such that* $\deg(w) = \Delta(H)$*. Then w is an ε-kernel vertex of H.*

Proof. Suppose firstly that $H = K_3$. Then $\varepsilon(H) = 2$ and $\tau(H) = 2$. Hence the result follows from Remark 3.1.1. If $H = K_{2,1}$ or $H = P_4$, then the result follows from Proposition 3.2.1. Suppose then that $H = K_{2,m}$ where $m \geq 2$. Then by Theorem

1.5.1 and Proposition 1.5.2, $\varepsilon(H) = 3$. Moreover, for any vertex $v \in V(K_{2,m})$ where $m \geq 2$, $N[v]$ is a vertex cover of $K_{2,m}$. Thus the result follows from Corollary 3.4.2. Lastly, suppose $H = S(2,2,2)$. Then the result follows from Proposition 3.3.1. □

From Lemma 4.1.2 we see that for each $H \in \mathcal{H}$, the vertex labelled w in Figure 4.1 is an ε-kernel vertex of H. Furthermore, if a graph G has no adjacent vertices of degree three or more, $\text{diam}(L(G)) > 3$, and G contains a graph $H \in \mathcal{H}$ as an attachment at an ε-kernel vertex of H, then G contains H as an attachment at w.

Lemma 4.1.3. *For any graph G, if an edge of G incident with a link vertex is subdivided to form a graph H, then $\varepsilon(H) \geq \varepsilon(G)$.*

Proof. Let G be a graph with a link vertex v and let $e_1 = uv$ and e_2 be the edges incident with v. For any edge ordering $f : E(G) \to \mathbb{Z}$ of G, either e_1e_2 or e_2e_1 is an f-ascent. Without loss of generality assume e_1e_2 is an f-ascent and let H be the graph formed from G by removing e_1 and adding the vertex w and the edges uw and wv. Define the edge ordering f' of H by $f'(uw) = f(e_1)$, $f'(wv) = f(e_1) + 0.5$, and $f'(e) = f(e)$ whenever $e \in E(G)$. Thus for any maximal f-ascent λ of G containing e_1 and e_2 there exists a corresponding f'-ascent λ' in H which contains uw, wv, e_2, thus containing one more edge than λ. Furthermore, for any maximal f-ascent λ of G containing e_1 and not e_2, there exists a corresponding f'-ascent of the same length in H which contains the edges of λ except with e_1 replaced by uw. Lastly, any maximal f-ascent λ in G not containing e_1 is an f'-ascent in H. Therefore for each edge ordering f of G, there exists an edge ordering f' of H such that $h(f') \geq h(f)$. □

Lemma 4.1.4. *Let G be a graph with $\varepsilon(G) = k \geq 4$. If H is formed by attaching P_n to G at an end-vertex of P_n, where $n \geq k$, then $\varepsilon(H) \geq k$.*

Proof. Let $f : E(G) \to \{1, 2, ..., |E(G)|\}$ such that $h(f) = k$. Also, let $G' = v_2, v_3, ..., v_{n+1}$ be a path of order $n \geq k$. Let v_1 be an arbitrary vertex of G and define $H = (G \cdot G')(v_1, v_2 : v)$. Define $f' : E(H) \to \{1, 2, ..., |E(H)|\}$ such that $f'(e) = f(e)$ whenever $e \in E(G)$ or $e = uv$ where $u \in V(G)$, and label the edges on the v-v_{n+1} direct path λ so that its edges form a v-v_{n+1} f'-ascent. Then, for any edge a on λ and $b \in E(G)$, $f'(a) > f'(b)$. Thus for all edges e incident with v that are not on λ, $f'(e) < f(vv_3)$. This implies that any maximal f'-ascent that contains edges of λ has length at least k. Therefore $h(f') \geq k$ and $\varepsilon(H) \geq k$. □

We define the following properties for a graph G.

- **P1:** A graph has property **P1** if and only if it contains a graph $H \in \mathcal{H}$ as an attachment at an ε-kernel vertex of H.
- **P2:** A graph has property **P2** if and only if it contains adjacent vertices of degree three or more.

Proof of Theorem 4.1.1. Suppose that $G = S(2, 2, 2)$ or G contains $H \in \mathcal{H}$ as an attachment at an ε-kernel vertex of H. By Proposition 1.5.2 or Proposition 1.5.11, $\varepsilon(H) \leq 3$, and by Theorem 3.1.2, $\varepsilon(G) \leq \varepsilon(H) \leq 3$.

Conversely, suppose that $G \neq S(2, 2, 2)$ and for all $H \in \mathcal{H}$, G does not contain H as an attachment at an ε-kernel vertex of H. That is, G has neither property **P1** nor property **P2**. It follows from Lemma 4.1.2 that G does not contain a graph $H \in \mathcal{H}$ as an attachment at w, where w is the vertex labelled as such for each graph in Figure 4.1.

We show that $\varepsilon(G) \geq 4$.

First suppose $\Delta(G) = 2$. Since $\operatorname{diam}(L(G)) \geq 3$, $G = C_n$, where $n \geq 6$, or $G = P_n$, where $n \geq 5$. In the former case, $\varepsilon(G) = \lceil \frac{n+1}{2} \rceil \geq 4$ by Proposition 1.5.11, and in the latter case $\varepsilon(G) = \varepsilon(P_n) = n - 1 \geq 4$.

Therefore we assume henceforth that $\Delta(G) \geq 3$. Denote the set of all end-vertices of G by Z, the set of all link vertices by Y and the set of all hub vertices by X'. By Lemma 4.1.4 we may limit our consideration to graphs with endpaths of length at most two, thus, we may assume $d(z, X') \leq 2$ for each $z \in Z$. Define the partition Z_1, Z_2 of Z by $z \in Z_i$ if and only if $z \in Z$ and $d(z, X') = i$.

Let $B_1, B_2, ..., B_k$ be the end-blocks of G that are cycles. By Lemma 4.1.3 and since G does not have property **P1**, we may assume that $B_i \cong C_5$ for each i. For each B_i, $1 \leq i \leq k$, choose a vertex v_i such that $d(v_i, X') = 2$, and let $X'' = \{v_1, ..., v_k\}$ and $X = X' \cup X''$.

Suppose $|X| = 1$. Since $\Delta(G) \geq 3$, $|X'| \geq 1$. Hence $X'' = \varnothing$ and G has exactly one vertex of degree three or more. Thus G is a spider. Since $G \operatorname{diam}(L(G)) \geq 3$ and $G \neq S(2, 2, 2)$, Proposition 1.5.11 shows that $\varepsilon(G) \geq 4$. Hence we assume that $|X| \geq 2$.

Lemma 4.1.1.1 *If* $|X \cup Z| = 2$*, then* $\varepsilon(G) \geq 4$*.*

Proof of Lemma 4.1.1.1. As assumed above, $|X| \geq 2$. Now if $|X \cup Z| = 2$, then $Z = \varnothing$. If $|X'| = 1$, say $X' = \{x\}$, then x lies on two or more cycles, none of which has a cut-vertex other than x, so that $|X''| \geq 2$ and $|X \cup Z| \geq 3$, which is not the case. Thus $|X'| = 2$ and $|X''| = |Z| = 0$. Let $X' = \{u, v\}$. Since $\Delta(G) \geq 3$ there exist at least three u-v direct paths in G. Furthermore, since $\operatorname{diam}(L(G)) \geq 3$, at least two u-v direct paths contain three or more edges, or at least one u-v direct path contains four or more edges.

Suppose G contains exactly two u-v direct paths of length three, e_1, e_2, e_3 and e'_1, e'_2, e'_3, where e_1 and e'_1 are incident with u, and $k \geq 1$ u-v direct paths of length 2. We now describe an edge-ordering $f \in \mathcal{F}(G)$ with flatness four. Let $f(e_2) = 1$ and $f(e'_2) = m$, where $m = |E(G)|$. Also, let $f(e_1) = 2$, $f(e_3) = m - 2$, $f(e'_1) = 3$, $f(e'_3) = m - 1$. Label the edges of the u-v direct paths of length two so that each

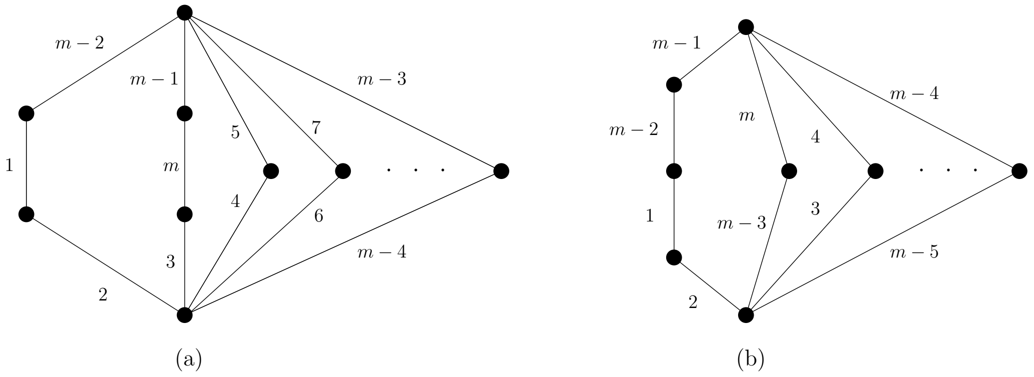

Figure 4.2: Edge orderings with flatness 4 for graphs with $|X \cup Z| = 2$ and $\Delta \geq 3$.

path is labelled with distinct values from the set $\{4, 5, ..., m-4, m-3\}$ so that each direct path forms a u-v direct f-ascent. The paths $e_2e_1e_1'e_2'$ and $e_2e_3e_3'e_2'$ are both maximal $(4, f)$- ascents. Any maximal f-ascent that contains a u-v direct path of length two has length at least 4. Thus f has flatness four – see Figure 4.2(a). By Lemma 4.1.3, if any of the edges on the u-v direct paths in G are subdivided, the depression does not decrease.

Suppose therefore that G contains one u-v direct path Q of length four and $k \geq 2$ u-v direct paths of length two. Let e_1, e_2, e_3, e_4 be the edges of the u-v direct path of length four and let $f(e_2) = 1,\ f(e_1) = 2,\ f(e_3) = m-2$ and $f(e_4) = m-1$. Label the edges of the u-v direct paths of length two with distinct values from the set $\{3, 4, ..., m-3, m\}$ so that each u-v direct path is a u-v direct f-ascent. This labelling is shown in Figure 4.2(b) and also has flatness four. Again, by Lemma 4.1.3, if any of the edges on the u-v direct paths in G are subdivided, the depression does not decrease. $\diamondsuit$

Assume henceforth that $|X \cup Z| \geq 3$. Let G' be the graph with vertex set $X \cup Z$ such that $uv \in E(G')$ if and only if $u, v \in X \cup Z$ and there exits a u-v direct path in G. If $Z \neq \varnothing$, let $x_0 \in Z$, otherwise let x_0 be a vertex that is not a cut vertex of G'. Let $\mu = \operatorname{diam} G'$ and let $V_0, V_1, ..., V_\mu$ be a partition of $V(G')$ such that $V_0 = \{x_0\}$

and for each $i \geq 1$, $v \in V_i$ whenever $d_{G'}(x_0, v) = i$. Denote the set of end-vertices of G' that are not in Z by Ω; note that $X'' \subseteq \Omega$ and that possibly $X' \cap \Omega \neq \varnothing$.

We use G' to aid us in defining a vertex ordering on $X \cup Z$ which we then use to define an edge ordering of $E(G)$ with flatness at least 4. This will show that $\varepsilon(G) \geq 4$. By Lemma 4.1.3 we need only consider graphs in which all direct paths between vertices $u, v \in X$ have length 2, unless u or v is in Ω, in which case, since G does not have property **P1**, one of the u-v direct paths has length 3 and all others have length 2.

We define a vertex ordering $g_k : V(G') \to \mathbb{R}$ of G' recursively as described below.

G1. Define the ordering $g_0 : X \cup \{x_0\} \to \{1, 2, ..., |X \cup \{x_0\}|\}$ so that for $u \in V_i$ and $v \in V_j$, $g_0(u) < g_0(v)$ whenever $0 \leq i < j \leq k$, and within each V_i, the vertices in Ω receive the largest labels, while the vertices in $X - \Omega$ that are not adjacent to a vertex in V_{i+1} receive the smallest labels.

Let X_0 be the set of vertices $v \in X - \Omega$ such that v is not adjacent to a vertex in Z, $g_0^{-1}(g_0(v) - 1) \notin N_{G'}(v)$, and for each vertex $u \in N_{G'}(v)$, $g_0(v) > g_0(u)$. From an ordering g_i on $V(G') - Z$ and its associated set $X_i \subseteq X - \Omega$ we now define the ordering $g_{i+1} : X \to \{1, 2, ..., |X \cup \{x_0\}|\}$ and the set X_{i+1}.

G2. Let v be the vertex such that $g_i(v) = \max_{x \in X_i}\{g_i(x)\}$. Let u be the vertex with the maximum value assigned under g_i over all vertices in $N_{G'}(v)$. Define $g_{i+1}(u) = g_i(v) - 1$ and $g_{i+1}(v) = g_i(v)$.

Now let N_u be the set of vertices t such that $t \in N_{G'}(u)$ and $g_i(u) < g_i(t) < g_i(v)$.

G3. Under g_{i+1}, label the vertices in N_u with the values from the set $\{g_i(v)-2, g_i(v)-3, ..., g_i(v) - 1 - |N_u|\}$ such that $g_{i+1}(t) < g_{i+1}(t')$ whenever $g_i(t) < g_i(t')$ for all $t, t' \in N_u$. For all vertices $x \in X - N_u - \{u, v\}$ let $g_{i+1}(x) \in \{1, 2, ..., |X \cup$

$\{x_0\}|\} - \{g_i(v), g_i(v)-1, ..., g_i(v)-1-|N_u|\}$ such for all $x, x' \in X - N_u - \{u, v\}$, $g_{i+1}(x) < g_{i+1}(x')$ whenever $g_i(x) < g_i(x')$.

Let $X_{i+1} \subseteq X - \Omega$ be the set of vertices v such that $g_{i+1}^{-1}(g_{i+1}(v) - 1) \notin N_{G'}(v)$ and for each vertex $u \in N_{G'}(v)$, $g_{i+1}(v) > g_{i+1}(u)$.

Figure 4.3(b) shows a labelling g_0 of $X \cup \{x_o\} \subseteq V(G')$ as defined in G1 where G' is derived from the graph G shown in Figure 4.3(a). Note that $z, z' \in Z$, $y \in \Omega$, $X_0 = \{v\}$, and u is the neighbour of v assigned the largest label under g_0. The labelling g_1 shown in Figure 4.3(c) is obtained by applying G2 and G3. Note that $X_1 = \varnothing$.

We make two remarks concerning the labels of x_0 and its neighbours for future reference. Recall that x_0 is not a cut vertex. Thus, in G2, if $v \in V_1$, then v is adjacent to some vertex $u' \in V_1 \cup V_2$ such that $g_0(u') > 1 = g_0(x_0)$.

R1. Hence $u \neq x_0$ and $g_{i+1}(x_0) = 1$ for each i.

Also, since $X_i \subseteq X - \Omega$, v is adjacent to at least two vertices of G'; hence if $v \in V_j$ for $j \geq 2$, then $g_i(u) > 2$ for each i.

R2. Therefore the vertex w such that $g_{i+1}(w) = 2$ is adjacent to x_0 for all i.

Lemma 4.1.1.2 $X_{i+1} \subsetneqq X_i$.

Proof of Lemma 4.1.1.2. Let $x \in X - \Omega - X_i$. Then (i) x is adjacent to a vertex y with a larger label under g_i or (ii) x is adjacent to a vertex y' such that $g_i(y') = g_i(x) - 1$. In either case, if $x \in N_u \cup \{u\}$, then by G2 or G3, x is adjacent to a vertex with a larger label under g_{i+1}, namely v or u; hence $x \notin X_{i+1}$. Assume therefore that $x \notin N_u \cup \{u\}$ and suppose first that (i) holds. If $y \in N_u \cup \{u\}$, then $g_i(x) < g_i(y) < g_i(v)$ and by G3, $g_{i+1}(y) \in \{g_i(v) - 1, g_i(v) - 2, ..., g_i(v) - 1 - |N_u|\}$ while $g_{i+1}(x) < g_i(v) - 1 - |N_u|$,

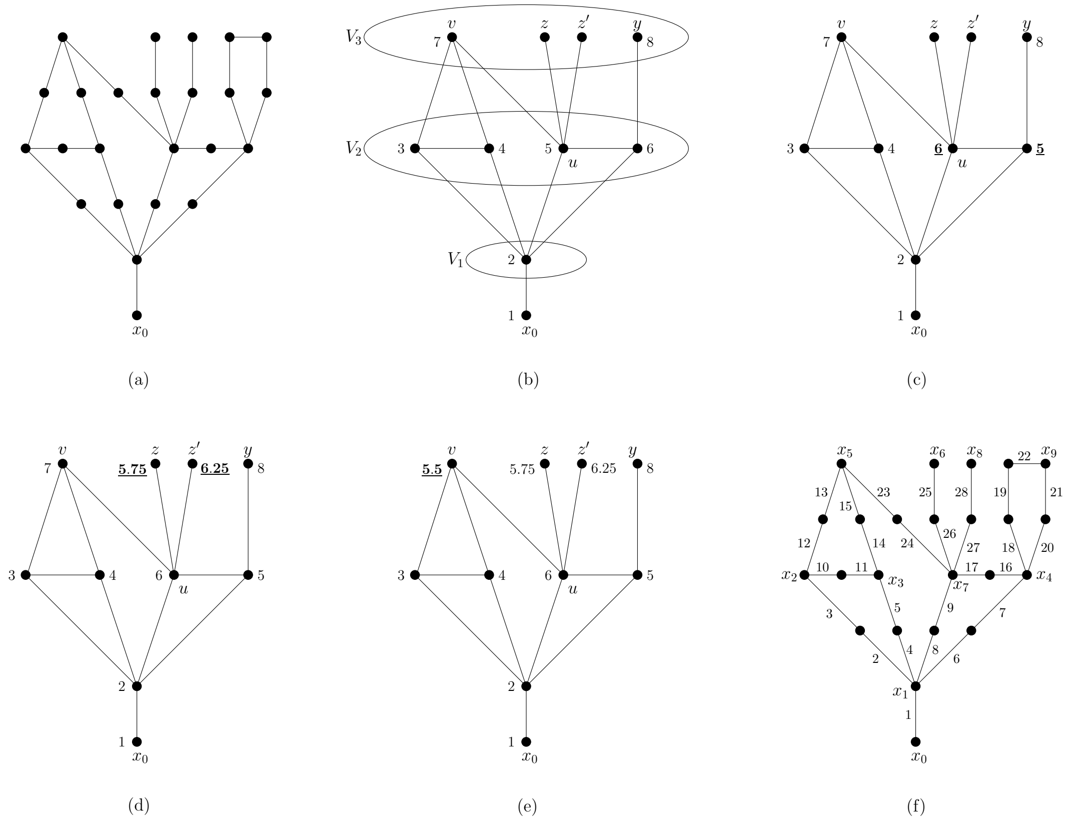

Figure 4.3: Examples of orderings defined by G1-G5 and F1-F3.

hence $g_{i+1}(x) < g_{i+1}(y)$. If $y \notin N_u \cup \{u\}$ then $g_{i+1}(x) < g_{i+1}(y)$ by G3. Hence $x \notin X_{i+1}$.

Suppose that (ii) holds. If $y' \in N_u \cup \{u\}$, then $g_i(y') < g_i(v)$ and thus $g_i(x) < g(v)$. Therefore $y' \neq u$ and by G3, $g_{i+1}(y') \in \{g_i(v) - 2, g_i(v) - 3, ..., g_i(v) - 1 - |N_u|\}$ while $g_{i+1}(x) < g_i(v) - 1 - |N_u|$, so that $g_{i+1}(x) < g_{i+1}(y')$. If $y \notin N_u \cup \{u\}$ then $g_{i+1}(y') = g_{i+1}(x) - 1$ by G3. In either case $x \notin X_{i+1}$.

Therefore $X_{i+1} \subseteq X_i$ and the result follows from the fact that $v \notin X_{i+1}$. $\Diamond$

Beginning with $i = 0$, repeatedly refine g_i to g_{i+1} using the procedure described above until an ordering g_k such that $X_k = \varnothing$ is obtained.

Next extend the labelling g_k to the vertices in Z as follows. For each $x \in X'$, let $Z(x) = Z \cap N_{G'}(x) - \{x_0\}$. Since G does not have property **P1**, $|Z(x)| \leq 2$ for all $x \in X'$. Furthermore, if $|Z(x)| = 2$, then the two vertices in $Z(x)$ are at distance 2 from x in G.

G4. If $Z(x)$ contains a single vertex z, let $g_k(z) = g_k(x) + 0.25$. If $Z(x)$ contains two vertices, say z and z', let $g_k(z) = g_k(x) - 0.25$ and $g_k(z') = g_k(x) + 0.25$.

Figure 4.3(d) contains an example of an extension of a labelling g_k to $Z - \{x_0\}$ as defined in G4.

Now we make one last set of refinements to form the labelling g'_k from g_k. Let $W \subseteq X - \Omega$ be the set of vertices v such that $g_k(v) = \max\{g_k(u) : u \in N_{G'}[v]\}$ and for $u \in N_{G'}(v)$ with $g_k(u) = g(v) - 1$, $Z(u) \neq \varnothing$. Note that if $v \in W$, then $Z(v) = \varnothing$.

G5. For each $v \in W$, let $g'_k(v) = g_k(u) - 0.5$. For all other vertices $v \in V(G')$, let $g'_k(v) = g_k(v)$. This ensures that the label of v is less than the labels of u and the vertices in $Z(u)$, but still greater than those of the vertices in $N_{G'}(v) - \{u\}$.

In Figure 4.3(d) we see that $v \in W$ since v is assigned the largest value over all its neighbours in G', and for the vertex u which is assigned the label $g_1(v) - 1$,

$Z(u) \neq \varnothing$. Furthermore, v is the only vertex in W, thus, the ordering g'_1 as shown in Figure 4.3(e) is obtained by the refinement defined in G5.

Let $S = \{0, 1, 2, ..., |V(G')| - 1\}$ and $\rho : V(G') \to S$ such that $\rho(u) < \rho(v)$ if and only if $g'_k(u) < g'_k(v)$. By R1, $\rho(x_0) = 0$. Let $V(G') = \{x_i : i \in S\}$, where $\rho(x_i) = i$ for each $i \in S$.

We next define the edge ordering $f' : E(G) \to \{1, 2, ..., m\}$, where $m = |E(G)|$, as follows.

F1. For each pair of vertices $x_i, x_j \in V(G')$, if $i < j$ then each x_i-x_j direct path is an x_i-x_j direct f'-ascent.

F2. For $x_i, x_j, x_r, x_s \in V(G')$, where $i < j$ and $r < s$, if (i, j) precedes (r, s) in a lexicographic ordering of $S \times S$, then each label defined by f' on the edges of the x_i-x_j direct paths is less than each label defined by f' on the edges of the x_r-x_s direct paths.

F3. For $x_i, x_j \in V(G')$, if there exist multiple x_i-x_j direct paths, then a longest x_i-x_j direct path contains the smallest and largest labels under f' over all edges contained on x_i-x_j direct paths.

Figure 4.3(f) and Figure 4.4(a) each show an edge ordering which satisfies the constraints outlined in F1-F3. Note that the edge ordering in Figure 4.4(a) has flatness three as indicated by the emphasized maximal ascents.

We now make a set of refinements to f' to form the edge ordering $f : E(G) \to \mathbb{R}$. Let $\deg_{G'} x_0 = r$ and $N_{G'}(x_0) = \{x_{a_1}, x_{a_2}, ..., x_{a_r}\}$, where $a_1 < a_2 < \cdots < a_r$. By R2 and the definition of ρ, $a_1 = 1$.

F4. If $r > 1$, let e be the edge with the smallest label under f' over all edges on x_0-x_{a_2} direct paths and let $f(e) = 1.5$.

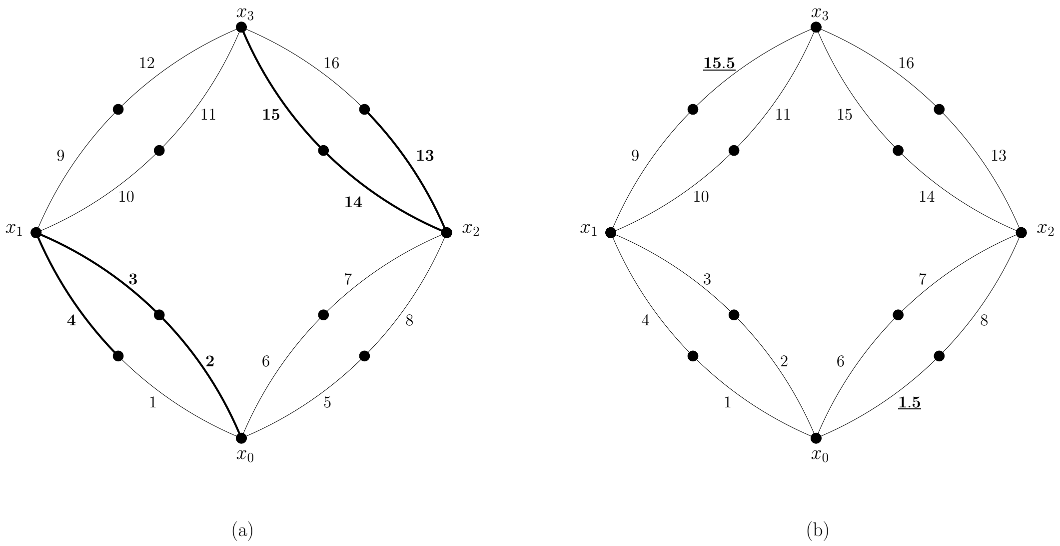

Figure 4.4: An example illustrating the refinements F4-F6.

F5. Let I be the set of indices such that $i \in I$ if and only if $\deg_{G'} x_i > 1$ and $i > j$ for all $x_j \in N_{G'}(x_i)$. For each $i \in I$, let e_i be the edge incident with x_i with the largest label under f' and let $x_{i'}$ be the vertex with the second largest index over all vertices in $N_{G'}(x_i)$. Let $e_{i'}$ be the edge with the largest label over all $x_{i'}$-x_i-direct paths and define $f(e_{'i}) = f'(e_i) - 0.5$.

F6. For all edges $e \in E(G)$ not yet labelled under f, let $f(e) = f'(e)$.

Figure 4.4(b) shows the edge ordering f obtained by applying F4-F6 to the edge ordering f' shown in Figure 4.4(a). Note that the ordering shown in Figure 4.3(f) will remain unchanged after the refinements F4-F6.

Lemma 4.1.1.3 $h(f) \geq 4$.

Proof of Lemma 4.1.1.3. For each vertex $v \in V(G')$, let $E(v)$ be the set of edges of G incident with v, $E^-(v)$ the set of edges of G incident with v that are on a u-v direct f-ascent of length 2 or more for some $u \in V(G)$, and $E^+(v) = E(v) - E^-(v)$. By F1 and F2, for each vertex $v \in V(G')$ and edges $e_1 \in E^-(v)$ and $e_2 \in E^+(v)$,

$f(e_2) \geq f(e_1)$. Also, for all $z \in Z_1$, $E^-(z) = \varnothing$ (since z does not lie on a u-z direct path of length at least two), and for all $v \in X - \{x_0\}$, $|E^-(v)| \geq 1$.

Let λ be any maximal f-ascent in G. Then λ has length at least 2. Let v_0,v_1 and v_2 be the first three vertices of λ, and let $e_1 = v_0v_1$ and $e_2 = v_1v_2$. There are two possibilities for v_1: $v_1 \in Y$ or $v_1 \in X'$.

Case 1 $v_1 \in Y$. Since λ is a maximal f-ascent and $|E^-(v)| \geq 1$ for all $v \in X - \{x_0\}$, $v_0 \in Z$ or $v_0 = x_0$.

First assume $v_0 \in Z - \{x_0\}$. Since $v_1 \in Y$, it follows that $v_0 \in Z_2$ and $v_2 \in X'$. By G4 and the definition of ρ, $\rho(v_2) = \rho(v_0) + 1$. Then by F2, $f(e_2) = \max_{e \in E^-(v_2)}\{f(e)\}$ and thus the ascent e_1e_2 cannot be extended along an edge of $E^-(v_2)$. Furthermore, since $\rho(v_2) = \rho(v_0) + 1$, G4 also implies that there exists a vertex $x_i \in Z_2$ that is adjacent to v_2 in G' and $i = \rho(v_2) + 1$. Thus $E^+(v_2) \neq \varnothing$, and since G does not have properties **P1** or **P2**, each edge in $E^+(v_2)$ is contained in a v_2-x_j direct f-ascent of length at least 2 for some j. Therefore λ has length at least 4.

Now we consider the case where $v_0 = x_0$. Since $v_1 \in Y$, $v_0 \in X \cup Z_2$. Recall that $a_1 = 1$ (see R2). Let P be the x_0-x_1 direct path whose initial edge is the edge e' such that $f(e') = 1$ and let P' be any other x_0-x_1 direct path (if it exists). Let e'' be the terminal edge of P. By F3, P' followed by e'' is an f-ascent; call it ℓ. However, if $x_0 \in \Omega$, then e' followed by ℓ is an f-ascent of length 4 (by F3 and since G does not have property **P1**), and if $x_0 \in X - \Omega$, then by F4, the edge e with the smallest label under f over all edges on x_0-x_{a_2} direct paths followed by ℓ is an f-ascent of length 4. Thus F1 – F4 and the maximality of λ imply that $f(e_1) = 1$ and that v_1 is on an x_0-x_1 direct f-ascent λ' that is a subpath of λ, and the edge e^* of λ' incident with x_1 is in $E^-(x_1)$.

- First suppose $x_0 \in \Omega$. Since G does not have property **P1**, λ' has length 3. Since $|V(G')| \geq 3$, $x_1 \in X'$ and $x_j \in N_{G'}(x_1)$ for some $j > 1$. Thus $E^+(x_1) \neq \varnothing$, which implies that λ has length at least 4.

- Suppose next that $x_0 \in X - \Omega$. Then by the choice of x_0, $Z = \varnothing$. Since x_0 is not a cut vertex, $x_1 \in X - \Omega$ and again $E^+(x_1) \neq \varnothing$. Since $Z = \varnothing$, each edge in $E^+(x_1)$ is contained in a x_1-x_i direct f-ascent of length at least 2 for some $i > 1$. Hence any extension of λ' along an edge in $E^+(x_1)$ has length at least 4. By F3, $f(e^*) = \max_{e \in E^-(x_1)}\{f(e)\}$. Therefore λ' cannot be extended along an edge in $E^-(x_1)$ and thus λ has length at least 4.

- Lastly, suppose $x_0 \in Z_2$. Then $\deg_G x_1 \geq 3$, thus $E^+(x_1) \neq \varnothing$ while $E^-(x_1) = \{e_2\}$. Since G does not have property **P1** (in particular, G does not have an end-vertex at distance 1 from x_1), each edge in $E^+(x_1)$ is contained in a x_1-x_i direct f-ascent of length at least 2 for some $i > 1$. Again λ has length at least 4.

Case 2 $v_1 \in X'$. By the definition of the edge ordering f and since G does not have property **P2**, $v_0 \in Z_1$ or $v_0 \in Y$.

Subcase 2.1 $v_0 \in Z_1$. Then by G4 and F2, e_2 is on a v_1-x_i direct path for some $i > \rho(v_0) = \rho(v_1) + 1$. Since v_0 is an end-vertex and G does not have property **P1** (in particular, G does not have an end-vertex at distance 1 or 2 from v_1), $x_i \in X$.

- If $x_i \in \Omega$, then by F3 the edge incident with x_i with the largest label under f is on the v_1-x_i direct path Q of length 3. Regardless of whether e_2 is on Q or not, each extension of e_1e_2 to a $(3, f)$-ascent can be extended to a $(4, f)$-ascent. Hence λ has length at least 4.

- Suppose that $x_i \in X - \Omega$. Then the v_1-x_i direct path containing e_2 has length 2. Let $e_3 = v_2x_i$. Since $\rho(v_1) < \rho(x_i)$, the definition of g'_k in G5 implies that either $g_k(x_i) \neq \max\{g_k(u) : u \in N_{G'}[x_i]\}$, or $g_k(x_i) = \max\{g_k(u) : u \in N_{G'}[x_i]\}$ and

$g_k(v_1) \neq \max\{g_k(u) : u \in N_{G'}(x_i)\}$. In the former case it is clear that $f(e_3) \neq \max_{e \in E(x_i)}\{f(e)\}$. In the latter case there exists a vertex $u' \in N_{G'}(x_i) - \{v_1\}$ such that $\rho(u') > \rho(v_1)$, and again it follows that $f(e_3) \neq \max_{e \in E(x_i)}\{f(e)\}$. Therefore $v_0v_1v_2x_i$ can be extended to a $(4, f)$-ascent and λ has length at least 4.

Subcase 2.2 $v_0 \in Y$. Since λ is a maximal f-ascent, e_1 is not on a w-v_1 direct $(2, f)$-ascent for any $w \in V(G)$; that is, v_0v_1 cannot be extended to an f-ascent wv_0v_1. Since all direct paths between vertices in X have length 2 or 3, either $v_1 \in X$, or $v_1 \in Y$ and $v_2 \in X$. But by F1 and the maximality of λ the latter case is impossible. Therefore $v_1 \in X$ and by definition of $E^-(v_1)$, $e_1 \notin E^-(v_1)$; hence $e_1 \in E^+(v_1)$. Thus e_1 is on a v_1-x_i direct f-ascent λ_1 (containing v_0) of length at least 2 for some $i > \rho(v_1)$. Since $f(e_2) > f(e_1)$, F1 and F2 now imply that e_2 is on a v_1-x_j direct f-ascent λ_2 for some $j \geq i > \rho(v_1)$. If $j = i$, then clearly $x_j \notin Z$, and if $j > i$, then $j > \rho(v_1) + 1$ and thus, by G4, $x_j \notin Z$. Hence $x_j \in X$ and λ_2 has length at least 2. However, we may assume that λ_2 has length 2, otherwise λ has length at least 4. Let $e_3 = v_2x_j$.

- Suppose $i < j$. We assume that e_3 is the edge incident with x_j with the largest label under f, otherwise λ has length at least 4. Then F3 and the facts that λ_2 has length 2, G does not have property **P1** and $x_j \notin Z$ imply that $x_j \in X - \Omega$. The assumption that e_3 has the largest label amongst the edges incident with x_j now implies two facts: $j = \max\{\rho(u) : u \in N_{G'}[x_j]\}$ and $\rho(v_1) = \max\{\rho(u) : u \in N_{G'}(x_j)\}$. But $\rho(v_1) < j - 1$, hence we have a contradiction to the fact that $X_k = \varnothing$.

- Hence $i = j$. Let e_4 be the edge on λ_1 incident with x_i. Let $e^* = \max\{f(e) : e$ is incident with $x_i\}$. By F3 and the fact that $f(e_1) < f(e_2)$, $e_3 \neq e^*$. If $e_4 \neq e^*$, then $\lambda = e_1e_2e_3...e^*$ has length at least 4, hence assume $e_4 = e^*$. Now if λ_1 has length 3, then $\lambda = e_1e_2e_3e_4$ is a $(4, f)$-ascent, therefore we assume that λ_1 has length 2, and so by F3 all v_1-x_j direct paths have length 2. Since G does not have property **P1**, $N_{G'}(x_i) \neq \{v_1\}$. If $i < \rho(u)$ for any $u \in N_{G'}(x_i)$ then λ has length at least 4. Assume then that $i > \rho(u)$ for all $u \in N_{G'}(x_i)$. Since $X_k = \varnothing$, $\rho(u) = i - 1$ for some $u \in N_{G'}(x_i)$, that is, $x_{i-1} \in N_{G'}(x_i)$. By F2, $e^* = e_4$ is on an x_{i-1}-x_i direct path, hence $v_1 = x_{i-1}$. Let $x_{i'}$ be the vertex with the second largest index over all vertices in $N_{G'}(x_i)$ and let α be the edge with the largest label over all $x_{i'}$-x_i-direct paths. By F5, $f(\alpha) = f(e_4) - 0.5 > f(e_3)$ and therefore λ has length at least 4.

Since the above cases exhaust all possibilities we conclude that $h(f) \geq 4$. $\diamondsuit$

Therefore $\varepsilon(G) \geq 4$ and the result follows. $\square$

The next corollary follows immediately from Theorems 4.1.1, 1.5.2 and 1.5.1.

Corollary 4.1.5. *If G is a connected graph of order at least three, then $\varepsilon(G) = 3$ if and only if no vertex of G is adjacent to two end-vertices or to two adjacent vertices of degree two, and*

(*i*) $\operatorname{diam}(L(G)) = 2$, *or*

(*ii*) $\operatorname{diam}(L(G)) \geq 3$ *and*

  (a) *G has a vertex v and two end-vertices u_1 and u_2 such that $d(v, u_i) = i$, or*

  (b) *G has a vertex v and three end-vertices u_1, u_2, u_3 such that $d(v, u_i) = 2$ for each i, or*

  (c) *G has an end-block $B = K_{2,m}$, $m \geq 2$, whose cut vertex is a vertex of B of degree m.*

4.2 A class of graphs with depression three

Trees with depression three were characterized in [16] and the previous section characterizes graphs with depression three and no adjacent vertices of degree three or more. Furthermore, in Section 3.4 we showed that $\varepsilon(G) \leq 3$ whenever G contains a graph H as an attachment at a vertex v such that v is a vertex cover of H and $\text{diam}(L(H)) \leq 2$. In this section we construct a large class of graphs with depression at most three which contains graphs with cycles and adjacent vertices of degree three or more. The construction is a generalization of the construction used in [16] to characterize trees with depression three.

Let $\mathcal{S}'_k$ be the class of graphs S_k, $k \geq 1$, that can be constructed recursively in k steps as follows. Let $S_0 = K_2$ with $V(S_0) = \{x_0, y_0\}$. Define $U_0 = \varnothing$ and $Y_0 = \{y_0\}$. Once S_i has been constructed, construct S_{i+1} by performing one of the following five operations.

O1: For any $y \in Y_i$, join y to the vertex u_1 of a new edge u_1x_1; let $U_{i+1} = U_i \cup \{u_1\}$ and $Y_{i+1} = Y_i$.

O2: For any $y \in Y_i$, join y to the central vertex u_2 of a new $P_5 : x_2, y_2, u_2, y'_2, x'_2$; let $U_{i+1} = U_i \cup \{u_2\}$ and $Y_{i+1} = Y_i \cup \{y_2, y'_2\}$.

O3: For any $y \in Y_i$, join y to the vertices u_3 and v_3 of a new edge u_3v_3; let $U_{i+1} = U_i \cup \{u_3\}$ and $Y_{i+1} = Y_i$.

O4: For any $y \in Y_i$, join y to the central vertex y_4 and an end vertex u_4 of a new $P_3 : u_4, y_4, x_4$; let $U_{i+1} = U_i \cup \{u_4\}$ and $Y_{i+1} = Y_i$.

O5: For any $y \in Y_i$, join y to the vertex v_5 of the graph $G_5 = (\{x_5, x'_5, v_5, v'_5, v''_5, u_5, y_5\}, \{v_5y_5, y_5x_5, v_5v'_5, v_5v''_5, v'_5v''_5, v'_5u_5, u_5x'_5\})$; let $U_{i+1} = U_i \cup \{u_5\}$ and $Y_{i+1} = Y_i \cup \{y_5\}$.

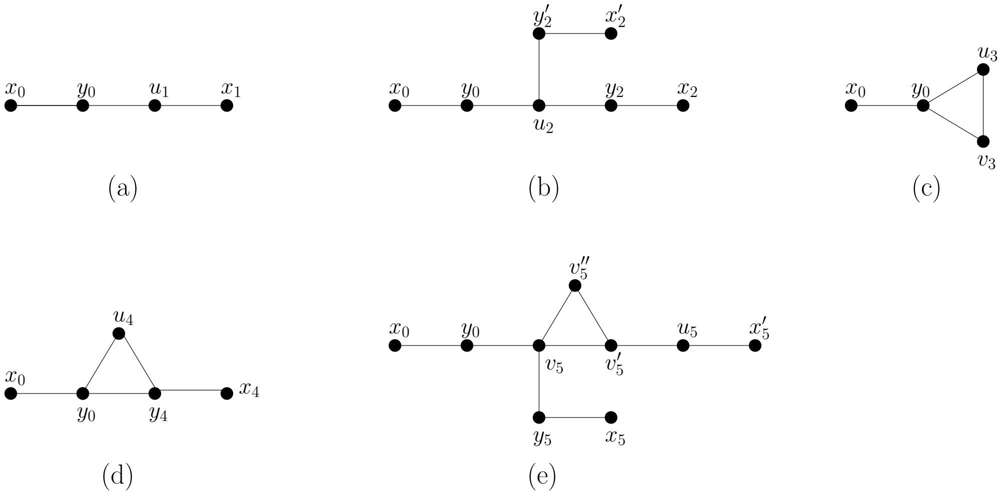

Figure 4.5: S_1 for each of the five operations **O1-O5**.

The operations **O1-O5** performed on S_0 are illustrated in Figure 4.5.

Let $\mathcal{S}_k$ be the family of graphs such that $S_k \in \mathcal{S}_k$ whenever $S_k \in \mathcal{S}_k'$ and in the construction of S_k, any vertex $y \in Y_k$ is involved in **O3** at most once. Define $\mathcal{S} = \bigcup_{k\geq 1} \mathcal{S}_k$. Note that $S_0 = K_2$ is not in $\mathcal{S}$. For a graph $S = S_k \in \mathcal{S}$, define $U_S = U_k$ and $Y_S = Y_k$. Let $\mathcal{G}$ be the class of all graphs G_S formed by performing the following two operations.

O6: Add any set $A = A(G_S)$ of new vertices to a graph $S \in \mathcal{S}$ and arbitrary edges between vertices in $A \cup U_S$.

O7: Add any arbitrary edges between vertices in Y_S.

Remark 4.2.1. *Let $S \in \mathcal{S}$. The operations **O1-O5** show that if $y \in Y_S$, then y is adjacent to exactly one vertex of degree one.*

We define the following property for a graph G.

P3: A graph G has property **P3** with respect to an edge ordering f and sets $U_G, Y_G \subseteq V(G)$, if for each $y \in Y_G$ for which a U_G-avoiding maximal $(2, f)$- or $(3, f)$-ascent ends (starts) at y, there exists a U_G-avoiding maximal $(2, f)$- or $(3, f)$-ascent such that its last (first) edge is assigned the largest (smallest) value under f over all edges incident with y.

Lemma 4.2.2. *If $S \in \mathcal{S}$ and f is an edge ordering of S for which there exists a U_S-avoiding maximal f-ascent of length at most three and all such ascents start or end in Y_S, then S has property **P3** with respect to f, U_S and Y_S.*

Proof. Let $y \in Y_S$ be a vertex for which a U_S-avoiding maximal $(2, f)$- or $(3, f)$-ascent ends at y, A_y be the set of all such f-ascents, and $\lambda = aby$ or $\lambda = acby$, where λ is the maximal f-ascent such that its last edge by is assigned the largest value over all edges of ascents in A_y. Let x be the end vertex adjacent to y. Clearly, $f(by) > f(yx)$.

Suppose to the contrary that $f(by) \neq \max_{v \in N(y)}\{f(vy)\}$. Then there exists an edge $wy \in E(S)$ such that $w \neq b$ and $f(wy) = \max_{v \in N(y)}\{f(vy)\}$. Since λ is a maximal f-ascent, w is a vertex of λ. By the construction of graphs in $\mathcal{S}$, all cycles of S have length three and we may assume that wby is a 3-cycle. If the cycle was introduced by **O3**, then $\lambda = wby$, $b \in U_S$, $w \notin U_S \cup Y_S$, and both w and b have degree 2. But since $f(yw) > f(wb)$ and $\deg(w) = 2$, xyw is a $U_S \cup Y_S$-avoiding maximal f-ascent, a contradiction.

Suppose then that the cycle wby was introduced by **O4**. Then $w \in Y_S$ and there exists an end vertex x' adjacent to w. If $f(x'w) < f(wy)$, then $x'wy$ is a maximal f-ascent, which contradicts our choice of λ. Now if $f(x'w) > f(wy)$, then $xywx'$ is a maximal f-ascent which is also a contradiction.

A similar argument may be used to show that if a U_S-avoiding maximal f-ascent of length at most three starts at y, then there exists a U_S-avoiding maximal $(2, f)$- or $(3, f)$-ascent λ such that for the initial edge yb of λ, $f(yb) = \min_{v \in N(y)}\{f(yv)\}$. $\square$

Theorem 4.2.3. *For each $S \in \mathcal{S}$, $\varepsilon(S) \leq 3$ and U_S is a k-kernel of S for some $k \in \{2, 3\}$.*

Proof. The proof is by induction on k, the number of steps used to construct $S = S_k$ from $K_2 = S_0$. To prove the result we must show that for any edge ordering f of S there exists a U_S-avoiding maximal $(2, f)$- or $(3, f)$-ascent.

If $k = 1$, then S was constructed by performing one of the operations **O1-O5** on $K_2 = S_0$

Case 1 O1 is performed. Then $S = P_4$ and $U_S = \{u_1\}$. Since $\operatorname{diam}(L(S)) = 2$ and $N[u_1]$ is a vertex cover of S, the result follows from Theorem 3.4.1.

Case 2 O2 is performed. Then $S = S(2, 2, 2)$ and $U_S = \{u_2\}$. Consider any edge ordering f of S. Without loss of generality we may assume $f(x_0y_0) < f(y_0u_2)$. If $f(y_0u_2) > y(u_2y_2)$, then either $x_2y_2u_2y_0$ (if $f(x_2y_2) < f(y_2u_2)$) or $y_2u_2y_0$ (if $f(x_2y_2) > f(y_2u_2)$) are u_2-avoiding maximal f-ascents of S with length at most three. The same argument applies if $f(y_0u_2) > f(u'_2y'_2)$. Suppose then that $f(y_0u_2) < f(u_2y_2)$ and $f(y_0u_2) < f(u'_2y'_2)$. To avoid a u_2-avoiding maximal f-ascents of length at most three, both $x_0y_0u_2x_2y_2$ and $x_0y_0u_2x'_2y'_2$ are maximal $(4, f)$-ascents of S. This implies either $y_2u_2y'_2x'_2$ (if $f(y_2u_2) < f(u_2y'_2)$) or $y'_2u_2y_2x_2$ (if $f(y_2u_2) > f(u_2y'_2)$) is a u_2-avoiding maximal f-ascent of the required length.

Case 3 O3 is performed. Then $U_S = \{u_3\}$. Since $\operatorname{diam}(L(S)) = 2$ and $N[u_3]$ is a vertex cover of S, the result follows from Theorem 3.4.1.

Case 4 O4 is performed. Then $U_S = \{u_4\}$. Since $\operatorname{diam}(L(S)) = 2$ and $N[u_4]$ is a vertex cover of S, once again, the result follows from Theorem 3.4.1.

Case 5 O5 is performed. Then $U_S = \{u_5\}$. Suppose to the contrary that u_5 is not a 3-kernel of S. Let f be an edge ordering f of S for which all maximal $(2, f)$- and $(3, f)$-ascents either start or end at u_5. Necessarily, either $x_0y_0v_5y_5x_5$ or its reverse is a $(4, f)$-ascent of S, and without loss of generality we assume the former. Furthermore,

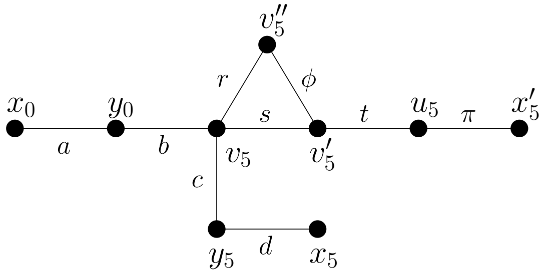

Figure 4.6: Operation O5 is performed, and the paths $abcd$ and rst are f-ascents of S.

by our assumption, neither $v_5''v_5v_5'$ nor its reverse is a maximal $(2, f)$-ascent of S, which implies either $v_5''v_5v_5'u_5$, $v_5''v_5v_5'u_5x_5'$, or the reverse of one of these paths is a maximal f-ascent. We need only consider the former two of these cases since for any f-ascent present in an edge ordering extended from these cases, its reverse will be present in one of the latter cases–with the roles of x_0 and y_0 switched with x_5 and y_5 respectively. These cases are shown in Figure 4.6 where the paths labelled $abcd$ and rst are f-ascents of S. Moving forward we will refer to the labels in this figure to simplify notation.

Firstly, suppose rst is a maximal f-ascent. Then $t > \pi$ and, since u_5 is not a 3-kernel of S, $\pi t\phi r$ is a $(4, f)$-ascent. But then $t < \phi < r < s < t$, which is a contradiction.

Secondly, suppose that $rst\pi$ is an f-ascent of S. If $r < b$, then since $t > r$, either rb (if $\phi > r$) or ϕrb (if $\phi < r$) is a maximal f-ascent, which in either case is a contradiction. Therefore we may assume $r > b$. We may also assume that $\phi > r$, or else abr is a u_5-avoiding maximal f-ascent. Furthermore, if $c > r$, then rcd is a maximal f-ascent, so we may assume $c < r$. Now if $\phi < s$, then ϕs is a u_5-avoiding maximal f-ascent, which is a contradiction. Thus we may assume $\phi > s$. Since $r < s$ by assumption, we now have $c < r < s < \phi$, which implies that $cs\phi$ is a maximal f-ascent, and again we have a contradiction.

This case completes the basis step of the proof.

Assume the result to be true for graphs in $\mathcal{S}$ constructed from K_2 in fewer than $k \geq 2$ steps. Consider any graph $S = S_k$ constructed from K_2 in k steps, and any edge ordering f of S.

Suppose that in the construction of S one of **O1**, **O2** or **O5** was performed at least once. Then S contains $y \in Y_S$ such that y was joined to a new vertex in step $i \geq 2$ and such that y is incident with at least two bridges. Let $y \in Y_S$ be incident to at least two bridges, and x be the vertex of degree one adjacent to y. Note that one of the bridges incident with y is xy. Let $G_1, G_2, ..., G_m$ be the components of $S - y$ which consist of at least two vertices. For each $1 \leq i \leq m$, let G'_i be the subgraph induced by $\{x, y\} \cup V(G_i)$. Then each $G'_i \in \mathcal{S}_j$ for some $1 \leq j < k$. If $G'_i \cong S_j \in \mathcal{S}_j$, then let $U_{G'_i} = U_j$ and f'_i be the edge ordering of G'_i induced by f.

Since y is incident with a bridge other than xy, there exists an i, say $i = 1$, such that $\deg_{G'_1}(y) = 2$. Let $H = S - G_1$ and f_H be the edge ordering of H induced by f. Then $H \cong S_j \in \mathcal{S}_j$ for some $1 \leq j < k$. Let $U_H = U_j$. By the induction hypothesis there exists at least one U_H-avoiding maximal $(2, f_H)$- or $(3, f_H)$-ascent and we may assume that all such maximal f_H-ascents start or end at y, or else there exists a U_S-avoiding maximal f-ascent of length at most three in S and we are done. Without loss of generality assume that there exists a U_H-avoiding maximal f_H-ascent of length at most three which ends at y. Then by Lemma 4.2.2 there exists a maximal f_H-ascent $\lambda = aby$ or $\lambda = acby$ such that $f_H(by) = \max_{v \in N(y)}\{f_H(vy)\}$ and $a \in V(H) - U_H$.

Let b_1 be the neighbour of y in G_1. By the induction hypothesis, there exists at least one $U_{G'_1}$-avoiding maximal $(2, f'_1)$- or $(3, f'_1)$-ascent and we may assume that all such maximal f'_1-ascents start or end at y, or else we are done. Thus either b_1y is the initial or final edge of a $U_{G'_1}$-avoiding maximal f'_1-ascent α of length at most three. If α starts at y, then $f'_1(b_1y) < f(xy) < f(by)$ and λ is a U_S-avoiding maximal f-ascent

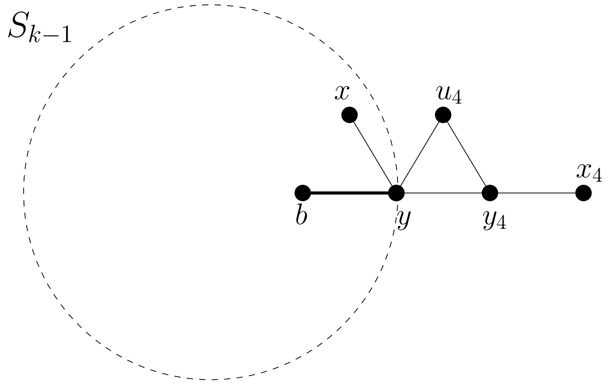

Figure 4.7: S is constructed from S_{k-1} by joining y to y_4 and u_4 of a new $P_3 : u_4, y_4, x_4$.

of length at most three. If α ends at y, then in S either α (if $f_1'(b_1y) > f_H(by)$) or λ (if $f_1'(b_1y) < f_H(by)$) is a U_S-avoiding maximal f-ascent of length at most three.

Suppose then that only **O3** and **O4** are used in the construction of S.

Firstly, suppose that S is constructed from S_{k-1} by joining y to y_4 and u_4 of a new $P_3 : u_4, y_4, x_4$ (see Figure 4.7). Then $U_S = U_{k-1} \cup \{u_4\}$. Let f' be the edge ordering of S_{k-1} induced by f, and x the end vertex adjacent to y. By the induction hypothesis, in S_{k-1} there exists a U_{k-1}-avoiding maximal f'-ascent of length at most three. We may assume that all such f'-ascents start or end at y or else we are done. Without loss of generality assume that there exists a U_{k-1}-avoiding maximal f'-ascent of length at most three which ends at y. By Lemma 4.2.2 there exists a maximal f'-ascent $\lambda = aby$ or $\lambda = acby$ such that $f'(by) = \max_{v \in N(y)}\{f'(vy)\}$ and $a \in V(S_{k-1}) - U_{k-1}$. If λ is a maximal f-ascent, then we are done so we may assume that either

$$f(yu_4) > f(by) \text{ or } f(yy_4) > f(by). \tag{4.1}$$

- Suppose $f(yu_4) > f(by)$. Then $f(yu_4) = \max_{v \in N(y) - y_4}\{f(vy)\}$.
  - If $f(y_4u_4) < f(u_4y)$, then either y_4u_4y or $x_4y_4u_4y$ is a U_S-avoiding maximal f-ascent.

- ○ Suppose $f(y_4u_4) > f(u_4y)$. Then $f(x_4y_4) > f(y_4u_4)$, or else $x_4y_4u_4$ is a U_S-avoiding maximal f-ascent.
    - If $f(yy_4) > f(y_4x_4)$, then $f(yy_4) = \max_{v \in N(y)}\{f(vy)\}$ and x_4y_4y is a U_S-avoiding a maximal f-ascent.
    - If $f(yy_4) < f(y_4x_4)$, then either xyy_4x_4 (if $f(xy) < f(yy_4)$) or y_4yx (if $f(xy) > f(yy_4)$) is a U_S-avoiding maximal f-ascent.

- Suppose then that $f(yu_4) < f(by)$. Then by (4.1), $f(yy_4) > f(by)$ and $f(yy_4) = \max_{v \in N(y)}\{f(vy)\}$. This implies either xyy_4x_4 (if $f(yy_4) < f(y_4x_4)$) or x_4y_4y (if $f(yy_4) > f(y_4x_4)$) is a maximal f-ascent, neither of which starts or ends in U_S.

Secondly, suppose that S is constructed from S_{k-1} by joining $y \in Y_{k-1}$ to the vertices v_3 and u_3 of a new edge u_3v_3. Then $U_S = U_{k-1} \cup \{u_3\}$. Let S' be the subgraph of S induced by $\{x, y, v_3, u_3\}$, f' the edge ordering of S' induced by f, and f'' the edge ordering of S_{k-1} induced by f. Note that $S' \cong S_1 \in \mathcal{S}_1$. Let $U_{S'} = \{u_3\}$. By the basis step, there exists a u_3-avoiding maximal f'-ascent α of length at most three. We may assume that α either starts or ends at y, or else we are done. Without loss of generality assume that α starts at y. Necessarily, $\alpha = yu_3v_3$ and $f'(yx) > f'(yu_3)$. Furthermore, we may assume that $f'(yv_3) > f'(yu_3)$, or else $f'(yv_3) < f'(yu_3) < f'(yx)$ and v_3yx is a U_S-avoiding maximal f-ascent of length two and we are done. Thus $f'(yu_3) = \min_{v \in N(y)}\{f'(vy)\}$.

By the induction hypothesis, there exists a U_{k-1}-avoiding maximal f''-ascent λ of length at most three in S_{k-1}. We may assume that λ starts or ends at y or else we are done. If λ starts at y, then by Lemma 4.2.2 there exists a maximal f''-ascent $\lambda' = aby$ or $\lambda' = acby$ such that $f''(by) = \min_{v \in N(y)}\{f''(vy)\}$ and $a \in V(S_{k-1}) - U_{k-1}$. This implies either λ' or α is a U_S-avoiding maximal f-ascent of length at most three. Assume then that λ ends at y, and furthermore, that all U_{k-1}-avoiding maximal f''-

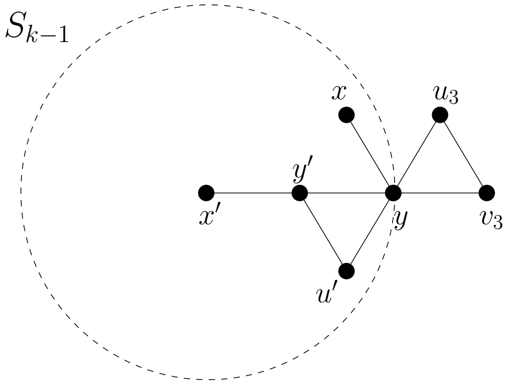

Figure 4.8: S is constructed from S_{k-1} by joining y to u_3 and v_3 of a new edge $\{u_3, v_3\}$.

ascents of length at most three end at y. Then there exists an edge $vy \in E(S_{k-1})$ such that $f''(vy) < f'(yu_3)$ otherwise α is a U_S-avoiding maximal f-ascent of length two and we are done. Let wy be the edge in S_{k-1} such that $f''(wy) = \min_{v \in N(y)}\{f''(vy)\}$. Then $f''(wy) < f'(yu_3) < f'(yv_3)$ which implies $f(wy) = \min_{v \in N(y)}\{f(vy)\}$. Recall that we have assumed S is constructed using only **O3** and **O4**, and that for any graph in $\mathcal{S}$, each vertex in $y \in Y_S$ is involved in **O3** at most once. Thus the edge wy was introduced by **O4**, which implies either $w = u' \in U_{k-1}$ and is adjacent to a vertex $y' \in Y_{k-1}$, or $w = y' \in Y_{k-1}$ and is adjacent to a vertex $u' \in U_{k-1}$. In either case, let x' be the vertex of degree one adjacent to y' – see Figure 4.8.

Suppose $w = y'$. If $f(x'y') < f(y'y)$, then, since $f(y'y) < f(xy)$, $x'y'yx$ is a U_S-avoiding maximal f-ascent of length three. If $f(x'y') > f(y'y)$, then, since $f(y'y) = \min_{v \in N(y)}\{f(vy)\}$, $yy'x'$ is a U_S-avoiding maximal f-ascent of length two.

Suppose then that $w = u'$. Let G_1 be the component of $S - y$ containing w, and G_1' the subgraph of S induced by $V(G_1) \cup \{y, x\}$. Then $G_1' \cong S_j \in \mathcal{S}_j$ for some $1 \le j < k$. Let $U_{G_1'} = U_{S_j}$ and f_1' be the edge ordering of G_1' induced by f. By the induction hypothesis, there exists a $U_{G_1'}$-avoiding maximal f_1' ascent of length at most three in G_1'. Necessarily all $U_{G_1'}$-avoiding maximal f_1' ascent of length at most three start or end at y or else we are done. Suppose there exists such an ascent which starts at y. By Lemma 4.2.2 there exists a $U_{G_1'}$-avoiding maximal f_1' ascent λ of length at most three whose initial edge is $yw = yu'$. But since $f(yu') = \min_{v \in N(y)}\{f(yv)\}$, λ is also a U_S-avoiding maximal f-ascent which is a contradiction. Hence we may assume that there exists a $U_{G_1'}$-avoiding maximal f_1'-ascent λ of length at most three which ends at y. Since $f_1'(u'y) = \min_{v \in N(y)}\{f_1'(vy)\}$, $f_1'(u'y) > f_1'(xy)$ and the last edge of λ is $y'y$. This implies $f_1'(yy') > f_1'(xy)$ or equivalently, $f(y'y) > f(yx)$. Necessarily, $f(x'y') < f(y'y)$, or else $xyy'x'$ is a U_S-avoiding maximal f-ascent of length at most three. Now we look at three cases for the value of $f(y'u')$. In these cases we assume that $\deg_S(y') > 3$ or else either xyy' (if $f(y'u') < f(yy')$) or $yu'y'$ (if $f(y'u') > f(yy')$) is a U_S-avoiding maximal f-ascent.

Case 1 $f(yu') < f(y'u') < f(x'y')$. Then $yu'y'x'$ is a U_S-avoiding maximal f-ascent.

We define the following to aid us in the next two cases. Let H_1 be the component of $S_{k-1} - y'$ containing w, H_1' the the subgraph of S_{k-1} induced by $V(H_1) \cup \{y', x'\}$, and H_2' the subgraph of S_{k-1} induced by $V(S_{k-1}) - V(H_1)$. Then each $H_i \in \mathcal{S}_\ell$ for some $1 \leq \ell < k$. If $H_i' \cong S_\ell \in \mathcal{S}_\ell$, then let $U_{H_i'} = U_\ell$ and f_i be the edge ordering of H_i' induced by f.

Case 2 $f(y'u') < f(x'y')$ and $f(y'u') < f(u'y)$. Then, in H_1', $y'u'yx$ is a $U_{H_1'}$-avoiding maximal f_1-ascent starting at y' and xyy' is a $U_{H_1'}$-avoiding maximal f_1-ascent ending at y. By the induction hypothesis, in H_2, there exists a $U_{H_2'}$-avoiding maximal f_2-ascent of length at most three. We may assume that all such f_2-ascents start or end at y'. Without loss of generality suppose there exists a $U_{H_2'}$-avoiding maximal f_2-ascent of length at most three that ends at y'. By Lemma 4.2.2, there exists a $U_{H_2'}$-avoiding maximal f_2-ascent $\lambda = aby'$ or $\lambda = acby'$ such that $f_2(by') = \max_{v \in N(y')}\{f_2(vy')\}$. Thus, in S, either λ or xyy' is a U_S-avoiding maximal f-ascent of length at most three.

Case 3 $f(y'u') > f(x'y')$. Then either xyy' (if $f(y'u') < f(yy')$) or $yu'y'$ (if $f(y'u') >$

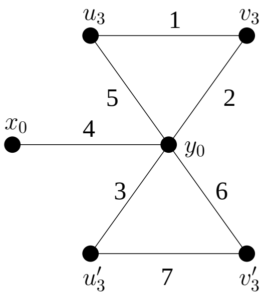

Figure 4.9: A graph G constructed from S_0 by performing **O3** twice at y_0, and an edge labelling f of G for which every maximal f-ascent of length at most three starts or ends in $U_G = \{u_3, u_3'\}$.

$f(yy'))$ is a $U_{H_1'}$-avoiding maximal f_1-ascent which ends at y'. Again, by the induction hypothesis, in H_2, there exists a $U_{H_2'}$-avoiding maximal f_2-ascent of length at most three and we assume that all such f_2-ascents start or end at y'. Suppose there exists a $U_{H_2'}$-avoiding maximal f_2-ascent of length at most three that ends at y'. By Lemma 4.2.2, there exists a $U_{H_2'}$-avoiding maximal f_2-ascent $\lambda = aby'$ or $\lambda = acby'$ such that $f_2(by') = \max_{v \in N(y')}\{f_2(vy')\}$. Therefore, in S, either λ, xyy', or $xu'y'$ is a U_S-avoiding maximal f-ascent of length at most three. Suppose then that there exists a $U_{H_2'}$-avoiding maximal f_2-ascent of length at most three that starts at y'. By Lemma 4.2.2, there exists a $U_{H_2'}$-avoiding maximal f_2-ascent $\lambda = aby'$ or $\lambda = acby'$ such that $f_2(by') = \min_{v \in N(y')}\{f_2(vy')\}$. Necessarily, $f(by') < f(y'x')$, and since $f(y'y) > f(y'x')$ and $f(y'u') > f(y'x')$, λ is a U_S-avoiding maximal f-ascent of length at most three. $\square$

In the construction of $S_k \in \mathcal{S}_k$, any vertex $y \in Y_k$ is involved in **O3** at most once. If not, then U_k is no longer a 3-kernel of S_k. Consider the graph G shown in Figure 4.9, which is constructed from S_0 by performing **O3** twice at y_0. Let $U_G = \{u_3, u_3'\}$. For the edge labelling f of G shown in the figure, any maximal f-ascent of length at most three starts or ends in U_G.

Recall that the graphs $G_S \in \mathcal{G}$ are obtained from a graph $S \in \mathcal{S}$ by performing operations **O6** and **O7**. We now show that these graphs also have depression at most three.

Theorem 4.2.4. *For each* $G_S \in \mathcal{G}$, $\varepsilon(G) \leq 3$.

Proof. Let G'_S be constructed from $S \in \mathcal{S}$ by adding $n \geq 0$ edges between vertices in $Y_{G'_S} = Y_S$ and let $U_{G'_S} = U_S$. If $n = 0$, then $G'_S \in \mathcal{S}$ and by Theorem 4.2.3, $\varepsilon(G'(S)) \leq 3$ and $U_{G'_S}$ is a k-kernel of G'_S, where $k \in \{2, 3\}$.

Suppose that $n \geq 1$. Let f be an edge ordering of G'_S, and f' the edge ordering of S induced by f. If there exists a $(U_S \cup Y_S)$-avoiding maximal f'-ascent of length at most three, then $h(f) \leq 3$. Suppose then that there does not exist a $(U_S \cup Y_S)$-avoiding f'-ascent of length at most three. By Theorem 4.2.3 there exists a U_S-avoiding maximal f'-ascent of length at most three in S, thus all maximal U_S-avoiding $(2, f')$- or $(3, f')$-ascents start or end in Y_S.

Without loss of generality we assume there exists a maximal U_S-avoiding ascent of length at most three which ends in Y_S. By Lemma 4.2.2, S has property **P3**, which implies that there exists a maximal f'-ascent $\lambda = aby_1$ or $\lambda = acby_1$ such that $y_1 \in Y_S$ and $f'(by_1) = \max_{v \in N_S(y_1)}\{f'(vy_1)\}$. Suppose that in G'_S there exists an edge y_1w such that $f(y_1w) = \max_{v \in N_{G'_S}(y_1)}\{f(vy_1)\} > f(by_1)$ and w is not a vertex of λ. Necessarily, $y_1w \notin E(S)$ which implies $w \in Y_S$. Let $w = y_2$, and x_1 and x_2 be the vertices of degree one adjacent to y_1 and y_2 respectively. Since λ is a maximal f'-ascent in S, it follows that $f(y_1x_1) < f(by_1) < f(y_1y_2)$. Therefore, either $x_1y_1y_2x_2$ (if $f(y_2x_2) > f(y_1y_2)$) or $x_2y_2y_1$ (if $f(y_2x_2) < f(y_1y_2)$) is a $U_{G'_S}$-avoiding maximal f-ascent. Hence $U_{G'_S}$ is a k-kernel of G'_S, where $k \in \{2, 3\}$.

Let $G_S \in \mathcal{G}$ be constructed from G'_S by adding any set $A = A(G_S)$ of new vertices to G'_S and arbitrary edges between vertices in $A \cup U_{G'_S}$. Then by Theorem 3.1.2, $\varepsilon(G_S) \leq 3$. □

Note that $\kappa(G_S) = 1$ for each $G_S \in \mathcal{G}_S$. We also note that for each graph G in

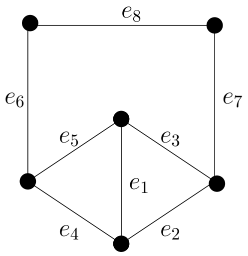

Figure 4.10: A graph H with $\kappa(H) > 1$, $\mathrm{diam}(L(H)) > 2$, and $\varepsilon(H) = 3$.

the classes of graphs with depression three defined in Sections 1.5, 3.4, and 4.1, either $\mathrm{diam}(L(G)) = 2$ or $\kappa(G) = 1$. The graph H shown in Figure 4.10 is an example of a graph with $\kappa(H) > 1$, $\mathrm{diam}(L(H)) > 2$, and $\varepsilon(H) = 3$. We provide the following argument to support the claim that $\varepsilon(H) = 3$. Suppose to the contrary that $\varepsilon(H) > 3$. Let $f : E(H) \to \{1, 2, \ldots, 8\}$ be an edge ordering of H for which every maximal f-ascent has length at least 4. Since e_1 and e_8 are the only edges in H which are at distance three in $L(H)$, it follows that $\{f(e_1), f(e_8)\} = \{1, 8\}$.

Without loss of generality we may assume that $f(e_1) = 1$ and $f(e_8) = 8$. Without loss of generality we may also assume that $f(e_5) = \max\{f(e_2), f(e_3), f(e_4), f(e_5)\}$. Then, since $h(f) > 3$ and $f(e_4) < f(e_5)$, it follows that $e_7e_2e_4e_5$ is a maximal f-ascent. However, this implies e_1e_2 is a maximal f-ascent, a contradiction.

Chapter 5

Edge Colourings

Up to now the edge orderings we have considered were complete orderings; no two edges had the same label. Edge orderings by definition are proper edge colourings since adjacent edges receive different labels/colours. However, in some cases it is also possible to use proper edge colourings in fewer than $|E(G)|$ colours as partial orderings with flatness equal to $\varepsilon(G)$. For example, colouring the edges of C_4 in sequence $1, 2, 3, 2$ gives a partial ordering with flatness $3 = \varepsilon(C_4)$. In this chapter we consider the following question.

What is the smallest integer r such that there exists a proper edge colouring $c : E(G) \to \{1, 2, \ldots, r\}$ for which a shortest maximal ascent has given length k?

Path lengths and colourings have previously been linked. There is an elementary relationship between edge colourings and the altitude of a graph, $\alpha(G)$. Upper bounds on α are established using edge colourings in the following method, also used in [3, 4, 7, 9, 13, 20, 23]. We colour the edges of the graph (not necessarily obtaining a proper colouring) in $t \geq 2$ colours. We then obtain an edge ordering f by first labelling all the edges of one colour with consecutive integers, and then the edges of the next colour, etc. In any f-ascent, once we use edges of one colour, we cannot use edges of a previous colour since all such edges have smaller labels. In [23], this method is used to show that $\alpha(G) \leq \chi_1(G)$, where $\chi_1(G)$ denotes the edge chromatic number of G.

Also, Gallai [12], Hasse [14], Roy [21] and Vitaver [22] independently give a relation between the longest directed path for an orientation D of a graph G, which we denote by $\ell_D(G)$, and the chromatic number of G, $\chi(G)$.

Theorem 5.0.5. [12, 14, 21, 22] *For any orientation D of a graph G, $\ell_D(G) + 1 \geq \chi(G)$. Moreover, there exists an orientation D of G such that $\ell_D(G) + 1 = \chi(G)$.*

Since an edge ordering of G is a vertex ordering of $L(G)$, from Theorem 5.0.5 we again arrive at $\alpha(G) \leq \chi_1(G)$.

In Section 5.1 we define the ε-ascent chromatic index of a graph, denoted $\chi_\varepsilon(G)$. We also state some trivial bounds and characterize the class of graphs G for which $\chi_\varepsilon(G) = \chi_1(G)$, the edge chromatic index of G. In Section 5.2 we provide a lower bound for $\chi_\varepsilon(G)$ in terms of $\varepsilon(G)$ and $\chi_1(G)$. In Section 5.3 we determine the ε-ascent chromatic index for paths and cycles, and show that $\chi_\varepsilon(G) = \varepsilon(G)$ if and only if G is a path or an even cycle. In Section 5.4 we characterize trees for which $\chi_\varepsilon(T) = |E(T)|$. In this section we also define the k-ascent chromatic index of a graph, denoted $\chi_{(k)}(G)$, and determine an upper bound for $\chi_{(3)}(T)$, which is in turn used to establish a bound for the ε-chromatic index of trees with depression three. Finally, in Section 5.5 we provide an upper bound for ε-chromatic index of complete graphs and determine $\chi_\varepsilon(K_n)$ for $2 \leq n \leq 5$ and bounds for $\chi_\varepsilon(K_6)$.

5.1 The ε-ascent chromatic index of a graph

For an edge colouring we assume the colours are integers and the flatness of a colouring c, denoted by $h(c)$, is analogous to the flatness of an edge ordering. We define the ε-

ascent chromatic index of a graph, denoted $\chi_\varepsilon(G)$, as the minimum number of colours of a proper edge colouring c with $h(c) = \varepsilon(G)$.

Therefore, to prove that $\chi_\varepsilon(G) = k$, we must show that

(a) there exists a proper k-edge colouring c such that $h(c) = \varepsilon(G)$, i.e. $\chi_\varepsilon(G) \leq k$, and

(b) for all proper $(k-1)$-edge colourings c, $h(c) < \varepsilon(G)$, i.e. $\chi_\varepsilon(G) \geq k$.

Denote the chromatic index (the minimum number of colours in a proper edge colouring) of G by $\chi_1(G)$.

Remark 5.1.1. *For any graph G,*

(a) $\varepsilon(G) \leq \chi_\varepsilon(G)$*,*

(b) $\chi_1(G) \leq \chi_\varepsilon(G)$*, and*

(c) $\chi_\varepsilon(G) \leq |E(G)|$*.*

As mentioned in Remark 5.1.1, $\chi_\varepsilon(G) \geq \chi_1(G)$. We next characterize graphs G for which $\chi_\varepsilon(G) = \chi_1(G)$.

Proposition 5.1.2. *For any graph G, $\chi_\varepsilon(G) = \chi_1(G)$ if and only if $\varepsilon(G) \leq 2$.*

Proof. If $\varepsilon(G) = 1$, then G has exactly one edge and thus $\chi_\varepsilon(G) = \chi_1(G) = 1$. If $\varepsilon(G) = 2$, then G has at least two adjacent edges. Any proper edge colouring of G has the property that its maximal ascents have length at least two, thus $\chi_\varepsilon(G) = \chi_1(G)$. Now suppose $\varepsilon(G) \geq 3$ and $\chi_\varepsilon(G) = k$, and let $c : E(G) \to \{1, 2, \ldots, k\}$ be a proper edge colouring of G in $\chi_\varepsilon(G)$ colours whose maximal ascents have lengths at least three. Since G has no maximal $(2, c)$-ascents, the colours 1 and k do not occur at the same vertex, for if $c(uv) = 1$ and $c(vw) = k$, then uvw is a maximal $(2, c)$-ascent. Therefore each edge e such that $c(e) = k$ can be recoloured with colour 1 to give a proper $(k-1)$-edge colouring of G. Thus $\chi_1(G) < \chi_\varepsilon(G)$. $\square$

5.2 A lower bound

As mentioned in Remark 5.1.1, $\chi_\varepsilon(G) \geq \chi_1(G)$, and by Proposition 5.1.2, equality holds if and only if $\varepsilon(G) = 2$. We now improve this bound for $\varepsilon(G) \geq 3$.

Proposition 5.2.1. *If $\varepsilon(G) \geq 2$, then $\chi_\varepsilon(G) \geq \chi_1(G) + \varepsilon(G) - 2$.*

Proof. If $\varepsilon(G) = 2$, the result follows from Remark 5.1.1. Let G be a graph with $\varepsilon(G) = k \geq 3$. Suppose, to the contrary, that $\chi_\varepsilon(G) = m \leq \chi_1(G) + k - 3$ and consider a proper m-edge colouring c of G for which $h(c) = \varepsilon(G)$. Suppose that an edge e for which $c(e) = 1$ is adjacent to an edge e' with $c(e') = \chi_1(G)$. Then any maximal ascent containing ee' has length at most $k-1$, a contradiction. In general, if an edge e for which $c(e) = i$ is adjacent to an edge e' for which $c(e') = \chi_1(G) + i - 1$, then $h(c) \leq i + 1 + (\chi_1(G) + k - 3) - (\chi_1(G) + i - 1) = k - 1$. Thus, for $1 \leq i \leq k-2$, an edge assigned colour i is not adjacent to an edge assigned colour $\chi_1(G) + i - 1$. Therefore we may reassign the edges coloured i with colour $\chi_1(G) + i - 1$ for each i such that $1 \leq i \leq k - 2$. Since $k \geq 3$, this gives us a proper edge colouring in $m - (k-2) \leq \chi_1(G) - 1$ colours, a contradiction. □

Corollary 5.2.2. *If $\varepsilon(G) \geq 2$, then $\chi_\varepsilon(G) \geq \Delta(G) + \varepsilon(G) - 2$.*

For each $k \geq 3$, there exists a graph G with $\varepsilon(G) = k$ for which $\chi_\varepsilon(G)$ realizes the bound in Proposition 5.2.1. For example, consider the spider $S(1, t, t)$, where $t \geq 2$. By Theorem 1.5.11, $\varepsilon(S(1, t, t)) = t + 1$. Furthermore, $\chi_1(S(1, t, t)) = 3$, and by Proposition 5.2.1, $\chi_\varepsilon(S(1, t, t)) \geq t + 2$. To establish $\chi_\varepsilon(S(1, t, t)) \leq t + 2$ we consider the edge colouring c of $S(1, t, t)$ shown in Figure 5.1, for which it is easy to verify that $h(c) = t + 1$. Thus for all $t \geq 2$, $\chi_\varepsilon(S(1, t, t)) = t + 2$.

The difference $\chi_\varepsilon(G) - (\chi_1(G) + \varepsilon(G) - 2)$ can also be arbitrarily large. Consider the spider $S(t, t, t)$. By Theorem 1.5.11, $\varepsilon(S(t, t, t)) = t + 1$. Let v be the vertex of $S(t, t, t)$ with degree three, $e_{1,1}, e_{2,1}, e_{3,1}$ the edges incident with v, and $\lambda_i = e_{i,1} e_{i,2} \cdots e_{i,t}$ the

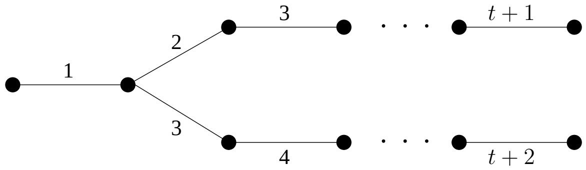

Figure 5.1: An edge colouring of $S(1, t, t)$ with flatness $t + 1$.

v-endpath containing $e_{i,1}$. Let c be a proper r-edge colouring with $h(c) = t + 1$. Without loss of generality we may assume that $c(e_{1,1}) < c(e_{2,1}) < c(e_{3,1})$. Necessarily, $c(e_{3,1}) < c(e_{3,2}) < \cdots < c(e_{3,t})$, otherwise λ_3 contains a maximal (k, c)-ascent, where $k \le t$, which is a contradiction. By a similar argument, $c(e_{1,1}) > c(e_{1,2}) > \cdots > c(e_{1,t})$. Hence $r \ge 2t + 1$. If we let $c(e_{2,i}) = c(e_{3,i})$ for $2 \le i \le t$, then $r \le 2t + 1$ and the resulting edge colouring has the required flatness. Thus $\chi_\varepsilon(S(t, t, t)) = 2t + 1$, and $\chi_\varepsilon(S(t, t, t)) - (\chi_1(S(t, t, t)) + \varepsilon(S(t, t, t)) - 2) = t - 1$.

5.3 Paths and cycles

In this section we determine the ε-ascent chromatic index for paths and cycles. We also show that the only graphs for which $\chi_\varepsilon(G) = \varepsilon(G)$ are paths and even cycles.

Proposition 5.3.1. $\chi_\varepsilon(P_n) = n - 1$ *for all* $n \ge 2$.

Proof. Since $\varepsilon(P_n) = n-1 = |E(P_n)|$, it follows from Remark 5.1.1 that $\chi_\varepsilon(T) = n-1$. □

Proposition 5.3.2. $\chi_\varepsilon(C_n) = \left\lceil \frac{n}{2} \right\rceil + 1$ *for all* $n \ge 3$.

Proof. By Proposition 1.5.8 and Remark 5.1.1, $\chi_\varepsilon(C_n) \ge \varepsilon(C_n) = \left\lceil \frac{n+1}{2} \right\rceil$. Let $C_n = e_1 e_2 \cdots e_n$.

Case 1: n is even. Define the edge ordering c_e of C_n by

$$c_e(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq \frac{n}{2}+1 \\ n-i+2 & \text{if } \frac{n}{2}+1 < i \leq n. \end{cases}$$

It is easy to verify that c_e is a proper $(\frac{n}{2}+1)$-edge colouring with $h(c_e) = \varepsilon(C_n)$. Since n is even, $\frac{n}{2}+1 = \lceil \frac{n+1}{2} \rceil = \varepsilon(C_n)$. Hence, by Remark 5.1.1, $\chi_\varepsilon(C_n) = \frac{n}{2}+1$, and the result holds.

Case 2: n is odd. Define the edge colouring c_o of C_n by

$$c_o(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq \lceil \frac{n}{2} \rceil + 1 \\ n-i+2 & \text{if } \lceil \frac{n}{2} \rceil + 1 < i \leq n. \end{cases}$$

It is easy to verify that c_o is a proper $(\lceil \frac{n}{2} \rceil + 1)$-edge colouring with $h(c_o) = \varepsilon(C_n)$. Since n is odd, $\lceil \frac{n}{2} \rceil + 1 = \lceil \frac{n+1}{2} \rceil + 1 = \varepsilon(C_n) + 1$. To prove the result we must show that any proper $(\frac{n+1}{2})$-edge colouring of C_n has flatness at most $\frac{n-1}{2}$. Suppose, to the contrary, that there exists a proper $(\frac{n+1}{2})$-edge colouring c_o' of C_n with flatness $\varepsilon(c_o) = \frac{n+1}{2}$. Let $c_o'(e_1) = 1$. Since any maximal c_o'-ascent in C_n has length at least $\frac{n+1}{2}$, it follows that

$$c_o'(e_i) = \begin{cases} i & \text{if } 2 \leq i \leq \frac{n+1}{2} \\ n-i+2 & \text{if } \frac{n+1}{2} < i \leq n. \end{cases}$$

But then $c_o'(e_k) = c_o'(e_{k+1})$ for $k = \frac{n+1}{2}$, and c_o' is not a proper edge colouring, a contradiction. □

As mentioned in Remark 5.1.1, $\chi_\varepsilon(G) \geq \varepsilon(G)$. We next characterize graphs G for which $\chi_\varepsilon(G) = \varepsilon(G)$.

Proposition 5.3.3. *If G is connected and $\chi_\varepsilon(G) = \varepsilon(G)$, then $G = C_{2n}$ or $G = P_n$*

for $n \geq 2$.

Proof. Let G be a graph such that $\chi_\varepsilon(G) = \varepsilon(G)$. By Corollary 5.2.2, $\Delta(G) \leq 2$. If $G = P_n$, then $\varepsilon(G) = n - 1$. Hence, from Proposition 5.3.1, $\chi_\varepsilon(G) = n - 1 = \varepsilon(G)$. From Proposition 1.5.8, $\varepsilon(C_n) = \left\lceil \frac{n+1}{2} \right\rceil$. If $G = C_{2k+1}$, then $\varepsilon(G) = k + 1$ and by Proposition 5.3.2, $\chi_\varepsilon(G) = k + 2$. If $G = C_{2k}$, then $\varepsilon(G) = k + 1$ and by Proposition 5.3.2, $\chi_\varepsilon(G) = k + 1$. □

5.4 Trees

Remark 5.1.1 states that $\chi_\varepsilon(G) \leq |E(G)|$. In this section we characterize trees T for which $\chi_\varepsilon(T) = |E(T)|$. We also bound $\chi_\varepsilon(T)$ for trees with $\varepsilon(T) = 3$ and in doing so introduce a variation on the parameter $\chi_\varepsilon(G)$.

Theorem 5.4.1. *Let T be a tree. Then $\chi_\varepsilon(T) = |E(T)|$ if and only if $T = P_n$ or $T = K_{1,n}$, $n \geq 2$.*

Proof. Suppose $T = P_n$, where $n \geq 2$. Then $\varepsilon(T) = n - 1 = |E(T)|$, and by Remark 5.1.1, $\chi_\varepsilon(T) = |E(T)|$. Suppose $T = K_{1,n}$ where $n \geq 2$. Then $\chi_1(T) = n = |E(T)|$, and again by Remark 5.1.1, $\chi_\varepsilon(T) = |E(T)|$.

Conversely, suppose that $T \neq P_n$ and $T \neq K_{1,n}$ for $n \geq 2$. If T contains a vertex which is adjacent to two leaves, then $\varepsilon(T) = 2$ and by Proposition 5.1.2, $\chi_\varepsilon(T) = \chi_1(T)$. Furthermore, since T is not a star, $\text{diam}(L(T)) \geq 2$ which implies $\chi_1(T) < |E(T)|$.

Assume then that no vertex of T is adjacent to two leaves, which by Corollary 1.5.9, implies that $\varepsilon(T) \geq 3$. Let f be an edge ordering of T with $h(f) = \varepsilon(T)$. Suppose also that T has at least three endpaths of length two or more. Let e_1, e_2 and e_3 be pendant edges on three such endpaths. Necessarily each e_i is either the initial or final edge of a maximal f-ascent in T. Without loss of generality we may assume that e_1 and e_2 are both initial edges of a maximal f-ascent in T. Let c be an edge colouring of T such that $c(e_1) = c(e_2) = \min_{e \in E(T)}\{f(e)\}$ and for all other edges $e \notin \{e_1, e_2\}$, $c(e) = f(e)$. Then $h(c) = h(f) = \varepsilon(T)$, which implies $\chi_\varepsilon(T) < |E(T)|$.

Suppose then that T contains at most two endpaths of length two or more. Consider the case where $B(T) = \{v\}$. Since v is not adjacent to two leaves and v has exactly two endpaths of length two or more, $T = S(1, k_1, k_2)$, where $2 \leq k_1 \leq k_2$. By Theorem 1.5.11, $\varepsilon(S(1, k_1, k_2)) = 1 + k_1$. Now we describe an edge colouring c using fewer than $|E(T)|$ labels such that $h(c) = 1 + k_1$. Assign the three edges incident with v the three smallest labels under f where the pendant edge receives the smallest of these labels. For the edges not incident with v, the values of f increase along each of the endpaths and the edges that are the same distance from v are assigned the same values. Thus at least two edges will be assigned the same value under the colouring c and it is easily verified that $h(c) = 1 + k_1 = \varepsilon(T)$.

Suppose then that $|B(T)| = k \geq 2$. Denote the number of leaves of T, which is also the number of endpaths of T, by l. Then $l \geq 2 + k$ and if any vertex in $B(T)$ has degree four or more, $l > 2 + k$ (see Theorem 3.7 in [5]). Necessarily, there exist at least two vertices in $B(T)$, say v_1 and v_2, such that each v_i is incident with exactly one edge that does not lie on a v_i-endpath. Then each v_i has at least two v_i-endpaths. Since no vertex in T is adjacent to two leaves and T has at most two endpaths of length two or more, each v_i has an endpath which is a pendant edge and another which has length two or more, and the degree of v_1 and v_2 is three. Let λ_i be the v_i-endpath of length two or more, y_i the pendant edge of λ_i, w_i the pendant edge incident with v_i, and f an edge ordering of T with $h(f) = \varepsilon(T)$. Then each y_i is either the initial or final edge in a maximal f-ascent in T. If both are initial or both are final edges of a maximal f-ascent in T, then, as before, there exists an edge colouring c of G using fewer than $|E(G)|$ labels such that $h(c) = h(f) = \varepsilon(T)$ and we are done. We assume then, without loss of generality, that y_1 is the initial edge of a maximal f-ascent of T and y_2 is the final edge of a maximal f-ascent of T.

We focus our attention on the vertex v_1. Let the endpath λ_1 be denoted by the edges $e_1 e_2 \ldots e_t$ where $t \geq 2$ and $e_t = y_1$, and e' be the edge incident with v_1 not on a v_1-endpath. We may assume λ_1 is an f-ascent of T of which e_t is the initial edge; if not, then for some i, $2 \leq i \leq t-1$, $f(e_{i+1}) < f(e_i) > f(e_{i-1})$, and there exists an edge colouring c with $c(e_{i+1}) = c(e_{i-1})$ and $h(c) = h(f)$. We also may assume that $f(w_1) > f(e_1)$, otherwise $w_1 e_1$ is a maximal $(2, f)$-ascent and $\varepsilon(T) = 2$, a contradiction. If $f(e') < f(e_1)$, let j be the largest index such that $f(e_j) > f(e')$. If $j < t$, relabel e_{j+1} with $f(e')$, otherwise, relabel e_t with $f(e')$. In either case this new colouring has flatness $\varepsilon(T)$ using fewer than $|E(G)|$ labels. If $f(w_1) > f(e')$, then we define the edge colouring c of T by $c(w_1) = c(y_2) = \max_{e \in E(T)}\{f(e)\}$ and for all other edges e, $c(e) = f(e)$. Since $h(c) = h(f) = \varepsilon(T)$, $\chi_\varepsilon(T) < |E(T)|$.

We assume then that $f(w_1) < f(e')$ and $f(e_1) < f(e')$. Let α be a shortest maximal f-ascent containing the path $w_1 e'$. Necessarily, the length of α is at least $\varepsilon(T)$. Define the edge colouring c as follows: $c(w_1) = f(e_1)$, $c(e_1) = f(w_1)$, for $2 \leq i \leq t$, $c(e_i) = f(e') + i - 2$, and for $e \notin \{e_1, e_2, \ldots, e_t, w_1, e'\}$, $c(e) = f(e)$. Note that the maximal f-ascent containing λ_1 and w_1 is now a maximal c-ascent with its direction reversed. Furthermore, the maximal f-ascent α is also a maximal c-ascent. Also, note that any shortest maximal c-ascent containing $e_1 e'$ has the same length as α, whose length is at least $\varepsilon(T)$. Thus $h(c) = h(f) = \varepsilon(T)$. Moreover, at least two edges receive the same label, hence $\chi_\varepsilon(T) < |E(T)|$. □

Next we discuss a bound of $\chi_\varepsilon(T)$ for trees T with $\varepsilon(T) = 3$. We begin with a slightly more general result.

Theorem 5.4.2. *Let T be a tree with $\varepsilon(T) \geq 3$. Then there exists a proper edge colouring c of T using at most $\chi_1(T) + 2$ colours such that $h(c) \geq 3$.*

Proof. Since $\varepsilon(T) \geq 3$ it follows that $|E(T)| \geq 3$, and by Theorem 1.5.1, no vertex of T is adjacent to two leaves. Note that for any tree T, $\chi_1(T) = \Delta(T)$. If $|E(T)| = 3$, then $T = P_4$, and since $\chi_1(P_4) = 2$ and $\varepsilon(P_4) = 3$, the result follows. Suppose then that $|E(T)| = 4$. The only tree T with four edges and $\varepsilon(T) \geq 3$ is $T = P_5$ and again, since $\chi_1(P_5) = 2$ and $\varepsilon(P_5) = 4$, the result follows.

Suppose the result is true for all trees T with $3 \leq |E(T)| < t$ for some $t \geq 5$ and consider a tree T' with $|E(T')| = t$ and $\varepsilon(T') \geq 3$.

Suppose T' is a path, say $T' = v_0v_1 \ldots v_t$. Let $T = T' - \{v_{t-1}, v_t\}$. By the induction hypothesis there exists a proper edge colouring c of T using at most 4 colours such that $h(c) \geq 3$. If $c(v_{t-3}v_{t-2}) \geq 3$, then let $c(v_tv_{t-1}) = 1$ and $c(v_{t-1}v_{t-2}) = 2$, otherwise let $c(v_tv_{t-1}) = 4$ and $c(v_{t-1}v_{t-2}) = 3$. In either case $v_tv_{t-1}v_{t-2}v_{t-3}$ is a $(3, c)$-ascent in T' which implies that we may extend the colouring c to T' so that it has flatness at least three using at most $\chi_1(T') + 2$ colours.

Suppose now that T' is not a path. Then there exist at least three endpaths in T'. Recall that $B(T')$ is the set of branch vertices of T'.

Case 1 There exists a v-endpath of length two for some $v \in B(T')$. Let vxy be a v-endpath of length two and $T = T' - \{x, y\}$. Since T' does not contain a vertex adjacent to two leaves, it follows that T also does not contain a vertex adjacent to two leaves. Hence $\varepsilon(T) \geq 3$ and by the induction hypothesis T has a proper edge colouring c using at most $\chi_1(T) + 2$ colours such that $h(c) \geq 3$. Let $\deg_T(v) = k$ and C_v be the set of colours assigned to edges incident with v. Note that $\chi_1(T) \geq k$, hence $\chi_1(T') \geq k + 1$, so there are at least $k + 3$ colours available to colour T'.

Suppose that either $\{1, 2\} \cap C_v = \varnothing$ or $\{k+2, k+3\} \cap C_v = \varnothing$. In the former case we extend the edge colouring c to T' by $c(vx) = 2$ and $c(xy) = 1$ and in the latter case by $c(vx) = k + 2$ and $c(xy) = k + 3$. In either case, in T', $h(c) \geq 3$, and since $\chi_1(T') \geq k + 1$, the result holds.

Suppose then that $\{1,2\}\cap C_v \neq \varnothing$ and $\{k+2,k+3\}\cap C_v \neq \varnothing$. Then there exists $j \notin C_v$ such that $3 \leq j \leq k+1$. Suppose there exists $i \in C_v$ such that $i < j$ and for the edge uv assigned i, any edge e adjacent to uv such that $c(e) < c(uv)$ is incident with v. Then, since $h(c) \geq 3$, for any edge vw with $c(vw) > i$, there exists an edge e' incident with w but not with v such that $c(e') > c(vw)$. Thus, if we extend the edge colouring c to T' by $c(vx) = j$ and $c(xy) = j+1$, then the resulting edge colouring of T' has flatness at least three. Therefore, we may assume that for any edge uv with $c(uv) < j$, there exists an edge uw such that $c(uw) < c(uv)$. Hence, if we extend the edge colouring c to T' by $c(vx) = j$ and $c(xy) = j-1$, the resulting edge colouring of T' has flatness at least three.

Case 2 For all $v \in B(T')$, there does not exist a v-endpath of length two. Since $\varepsilon(T') \geq 3$, there exists an endpath of length three or more. Let v_0 be a branch vertex incident with at most one edge not on an endpath, $\deg(v_0) = k$, and $\lambda = v_0v_1, ..v_j$, $j \geq 3$, be a v_0-endpath of maximum length. Let $T = T' - \{v_j, v_{j-1}\}$.

- Suppose $\varepsilon(T) \geq 3$. By the induction hypothesis there exists a proper edge colouring c of T in at most $\chi_1(T) + 2$ colours with flatness at least three. If $c(v_{j-3}v_{j-2}) \geq 3$, then let $c(v_jv_{j-1}) = 1$ and $c(v_{j-1}v_{j-2}) = 2$, otherwise let $c(v_jv_{j-1}) = 4$ and $c(v_{j-1}v_{j-2}) = 3$. In either case $v_jv_{j-1}v_{j-2}v_{j-3}$ is a $(3,c)$-ascent in T', which implies that we may extend the colouring c to T' so that it has flatness at least three.

- Suppose $\varepsilon(T) = 2$. Then in T', v_0 is incident with a pendant edge $e_0 = v_0w$. If T' is a spider, then $T' \cong S(1,3,\ldots,3)$ and Figure 5.2 shows an edge colouring in $\chi_1(T') + 2$ colours with flatness four. Hence assume T' has at least two branch vertices and there are $k-2 \geq 1$ v_0-endpaths of length three which we denote by λ_1,λ_2,. . . ,λ_{k-2}. Let e_i be the edge incident with v_0 which lies on endpath λ_i and e' the edge incident with v_0 that does not lie on an endpath. Let T_1

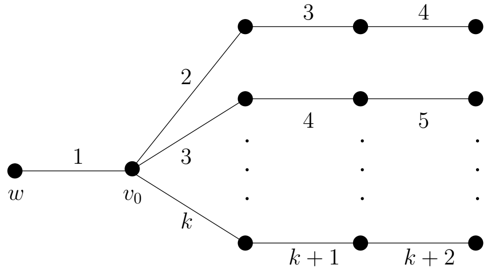

Figure 5.2: A $\chi_1(T')+2$-edge colouring of $T' = S(1,3,\ldots,3)$ with flatness four.

be the component of $T - \{e_1, e_2, \ldots, e_{k-2}\}$ which contains e_0, and note that $\deg_{T_1}(v_0) = 2$. Also, no vertex of T_1 is adjacent to two leaves, hence $\varepsilon(T_1) \geq 3$.

By the induction hypothesis there exists a proper edge colouring c of T_1 in at most $\chi_1(T_1) + 2$ colours such that $h(c) \geq 3$, and without loss of generality we may assume that $c(e') > c(e_0)$. Then we may also assume without loss of generality that $c(e_0) = 1$.

Let $c(e') = m$. We colour the edges of λ_i with $i+1, i+2, i+3$, for $1 \leq i \leq k-2$ and $i \neq m-1$, and the edges of λ_{m-1} (if it exists, in which case $m < k$) with k, $k+1$, $k+2$ to obtain a proper edge colouring c' of T'. Since $k \leq \chi_1(T')$ and $\chi_1(T_1) \leq \chi_1(T')$, it follows that c' uses at most $\chi_1(T') + 2$ colours.

Clearly, the maximal ascents contained in the union of the v_0-endpaths, v_0w, λ_1, λ_2,…,λ_{k-2} all have length at least three (four in fact). If $c(e') < c(e_i)$, then e' followed by λ_i is a $(4, c')$-ascent. Suppose $c(e') > c(e_i)$. Then since $h(c) \geq 3$, there exists an edge e'' adjacent to e' such that $e_0e'e''$ is a $(3, c')$-ascent. Then $e_ie'e''$ is also a $(3, c')$-ascent. Hence $h(c') \geq 3$ and we are done. □

We combine the lower bound from Proposition 5.2.1 with the result from Theorem 5.4.2 in the following corollary.

Corollary 5.4.3. *For any tree T with $\varepsilon(T) = 3$, $\chi_1(T) + 1 \leq \chi_\varepsilon(T) \leq \chi_1(T) + 2$.*

Theorem 5.4.2 does not provide an upper bound for $\chi_\varepsilon(T)$ when $\varepsilon(T) \geq 4$ since we are only guaranteed an edge colouring with flatness at least three. This motivates a generalization of the parameter $\chi_\varepsilon(G)$.

We define the *k-ascent chromatic index* of a graph G, $\chi_{(k)}(G)$, as the minimum number of colours so that there exists a proper edge colouring with flatness k, where $2 \leq k \leq \varepsilon(G)$. Note that $\chi_{(2)}(G) = \chi_1(G)$.

We now restate Theorem 5.4.2 using the parameter $\chi_{(k)}(G)$.

Theorem 5.4.4. *Let T be a tree with $\varepsilon(T) \geq 3$. Then $\chi_{(3)}(T) \leq \chi_1(T) + 2$.*

The bound in Theorem 5.4.4 does not hold for all graphs in general. For example, consider the graph G shown in Figure 5.3. Since $c(e_1) = c(e_4) = 1$, $c(e_2) = c(e_6) = 2$, $c(e_3) = c(e_5) = 3$ is a proper 3-edge colouring of G, we conclude that $\chi_1(G) = 3$, and from Theorem 1.5.1 and Proposition 1.5.2, it follows that $\varepsilon(G) = 3$. Suppose $\chi_{(3)}(G) = \chi_\varepsilon(G) \leq \chi_1(G) + 2 = 5$. Then there exists a proper 5-edge colouring c of G with $h(c) = 3$. Necessarily, the edge coloured 1 is not adjacent to the edge coloured 5. Moreover, $\{c(e_2), c(e_3), c(e_4)\} \cap \{1, 5\} = \varnothing$, otherwise at least one of e_1e_2, e_1e_3, e_4e_5 or their reverse is a maximal $(2, c)$-ascent. Therefore, without loss of generality we may assume $c(e_1) = 1$ and $c(e_5) = 5$. Since $|E(G)| = 6$, two edges are assigned the same colour, and since no other edge is assigned 1 or 5, $c(e_2) = c(e_6) = i$ where $i \in \{2, 3, 4\}$. If $i = 2$, then e_6e_5 is a maximal $(2, c)$-ascent, and if $i = 4$, then $c(e_4) \in \{2, 3\}$ and e_4e_6 is a maximal $(2, c)$-ascent. Suppose then that $i = 3$. If $c(e_4) = 2$, then e_4e_6 is a maximal $(2, c)$-ascent, and if $c(e_4) = 4$, then e_6e_4 is a maximal $(2, c)$-ascent. This completes all cases and we conclude that there does not exist a proper 5-edge colouring of G with flatness three. Hence, $\chi_{(3)}(G) = \chi_\varepsilon(G) = 6 > \chi_1(G) + 2$.

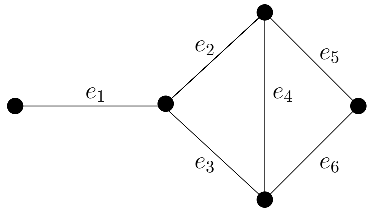

Figure 5.3: A graph G with $\chi_{(3)}(G) = \chi_\varepsilon(G) = 6 > \chi_1(G) + 2$.

5.5 Complete graphs

In this section we consider the problem of determining $\chi_\varepsilon(K_n)$. Trivially, we note that $\chi_\varepsilon(K_2) = 1$, and since $\chi_1(K_3) = |E(K_3)| = 3$, it follows that $\chi_\varepsilon(K_3) = 3$. From Proposition 1.5.4, $\varepsilon(K_n) = 3$ for all $n \geq 4$, hence, for $n \geq 4$ the problem involves determining the minimum number of colours required for there to exist a proper edge colouring of K_n with flatness three.

We first determine an upper bound for $\chi_\varepsilon(K_n)$.

Theorem 5.5.1. *$\chi_\varepsilon(K_n) \leq 2n - 3$ for all $n \geq 2$.*

Proof. As noted previously, $\chi_\varepsilon(K_2) = 1$ and $\chi_\varepsilon(K_3) = 3$, thus the result holds for $n = 2$ and $n = 3$. Consider K_n where $n \geq 4$. Let $V(K_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and define the edge colouring c by $c(v_iv_j) = i + j$. Clearly c is a proper edge colouring of K_n, and furthermore, c uses $2n - 3$ colours. To complete the proof we need only show that every $(2, c)$-ascent is contained in a longer c-ascent.

Suppose $\lambda = v_av_bv_c$ is a $(2, c)$-ascent of K_n. Note that $a \neq n - 1$ since $v_{n-1}v_b$ is assigned the largest value over all edges incident with v_b and hence $v_{n-1}v_bv_c$ is not a c-ascent. We now consider the following cases.

Case 1 $c = n - 1$. If $b \leq n - 3$ and $a \leq n - 3$, or $b \leq n - 4$ and $a = n - 2$, then there exists an edge $v_{n-1}v_k$ such that $k \neq a$ and $c(v_{n-1}v_k) > b + n - 1$, which implies λ is not a maximal c-ascent. If $b = n - 3$ and $a = n - 2$, then there exists an edge v_av_k such that $c(v_av_k) < c(v_av_b)$, and λ is not a maximal c-ascent. If $b = n - 2$, then

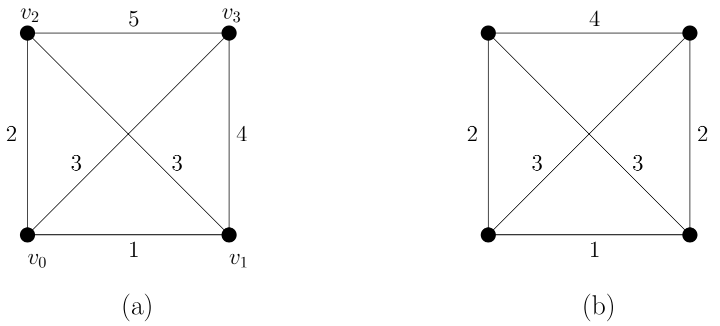

Figure 5.4: (a) A 5-edge colouring of K_4 with flatness 3. (b) A 4-edge colouring of K_4 with flatness 2.

there exists an edge $v_a v_k$ such that $k < n-2$, which implies $c(v_a v_k) < c(v_a v_b)$, and again, λ is not a maximal c-ascent.

Case 2 $b = n-1$. By the definition of the edge colouring c, it follows that $v_a v_{n-1}$ is assigned the largest value over all edges incident with v_a. Thus, since $n \geq 4$, there exists an edge $v_a v_k$ such that $k \neq c$ and $c(v_a v_k) < c(v_a v_{n-1})$, which implies that λ is not a maximal c-ascent.

Case 3 $n-1 \notin \{a,b,c\}$. By the definition of the edge colouring c, $c(v_b v_c) < c(v_c v_{n-1})$, which implies that λ is not a maximal c-ascent.

We have now considered all possible cases for a $(2,c)$-ascent λ and shown that in each case λ is not a maximal c-ascent. □

Figures 5.4(a) and 5.5 depict the edge colouring c for K_4 and K_5 respectively as defined in the proof of Theorem 5.5.1. Next we determine $\chi_\varepsilon(K_4)$ and $\chi_\varepsilon(K_5)$.

Proposition 5.5.2. $\chi_\varepsilon(K_4) = 5$.

Proof. By Theorem 5.5.1, $\chi_\varepsilon(K_4) \leq 5$.

To complete the proof we now show that $\chi_\varepsilon(K_4) \geq 5$. Suppose, to the contrary, that there exists a colouring $c : E(K_4) \to \{1,2,3,4\}$ of K_4 with flatness three. Necessarily, an edge assigned colour 1 is not incident with an edge coloured 4. This

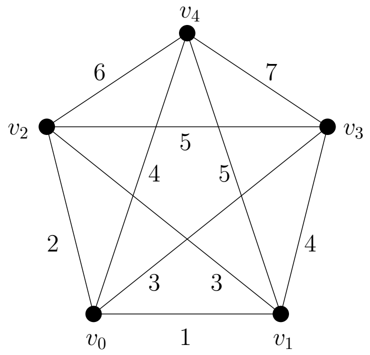

Figure 5.5: A 7-edge colouring c of K_5 with $h(c) = \varepsilon(K_5) = 3$.

implies the colours 1 and 4 are used once each and furthermore, the colours 2 and 3 are each used twice. Any proper colouring with this configuration is equivalent to the one shown in Figure 5.4(b) and both of the ascents 2 3 are maximal ascents. Thus $\chi_\varepsilon(K_4) \geq 5$. $\square$

Proposition 5.5.3. $\chi_\varepsilon(K_5) = 7$.

Proof. By Theorem 5.5.1, $\chi_\varepsilon(K_5) \leq 7$.

We now show that $\chi_\varepsilon(K_5) \geq 7$. Suppose, to the contrary, that there exists a proper 6-edge colouring c of K_5 with $h(c) = 3$. Let $V(K_5) = \{v_0, v_1, v_2, v_3, v_4\}$. Without loss of generality let $c(v_0v_1) = 1$. If $c(v_iv_j) = 6$, then $i, j \geq 2$, otherwise $h(c) = 2$. Without loss of generality let $c(v_2v_3) = 6$. To avoid a maximal $(2, c)$-ascent, $c(e) \in \{2, 3, 4, 5\}$ for each $e \in E(K_5) - \{v_0v_1, v_2v_3\}$. Then, since the maximum size of an independent edge set of K_5 is two and there are 8 edges in $E(K_5) - \{v_0v_1, v_2v_3\}$, $|c^{-1}(i)| = 2$ for each $i \in \{2, 3, 4, 5\}$. Additionally, for each colour $i \in \{2, 3, 4, 5\}$, there exists $x \in \{0, 1, 2, 3\}$ such that $c(v_xv_4) = i$. Note that $c(v_0v_4) \neq 5$ or else $v_1v_0v_4$ is a maximal $(2, c)$-ascent, a contradiction. By similar arguments, $c(v_1v_4) \neq 5$, $c(v_2v_4) \neq 2$, and $c(v_3v_4) \neq 2$. Thus, either $c(v_2v_4) = 5$ or $c(v_3v_4) = 5$, and we assume without loss of generality that $c(v_2v_4) = 5$. Since another edge must also be assigned the colour 5, either $c(v_0v_3) = 5$ or $c(v_1v_3) = 5$ and without loss of generality we may

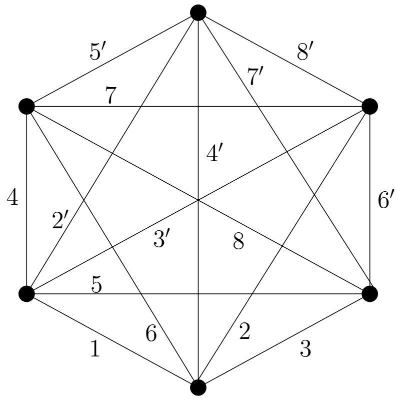

Figure 5.6: An 8-edge colouring c of K_6 with $h(c) = \varepsilon(K_6) = 3$.

assume that $c(v_0v_3) = 5$. If $c(v_1v_3) = 2$, then $v_1v_3v_0$ is a maximal $(2, c)$-ascent. Hence $3 \leq c(v_1v_3) \leq 4$. Since $c(v_2v_3) = 6$ and $c(v_0v_3) = 5$, if $c(v_3v_4) = 3$, then $c(v_1v_3) = 4$ and $v_3v_4v_2$ is a maximal $(2, c)$-ascent. This implies $c(v_3v_4) = 4$ and $c(v_1v_3) = 3$. Now $c(v_0v_4), c(v_1v_4) \in \{2, 3\}$ and since $c(v_1v_3) = 3$, $c(v_1v_4) = 2$ and $c(v_0v_4) = 3$. Moreover, since $c(v_1v_2), c(v_0v_2) \in \{2, 4\}$ and $c(v_1v_4) = 2$, $c(v_1v_2) = 4$ and $c(v_0v_2) = 2$. Then $v_0v_2v_1$ is a maximal $(2, c)$-ascent and the result follows. □

For $2 \leq n \leq 5$, the bound provided in Theorem 5.5.1 is best possible. However, this is not always the case as we illustrate next with K_6.

Proposition 5.5.4. *$7 \leq \chi_\varepsilon(K_6) \leq 8$.*

Proof. Suppose $\chi_\varepsilon(K_6) \leq 6$. Then there exists a colouring $c : E(K_6) \to \{1, 2, 3, 4, 5, 6\}$ with $h(c) = 3$. Since the maximum size of an independent edge set of K_6 is three, any colour can be used at most three times. Moreover, an edge coloured 1 is not adjacent to an edge coloured 6, otherwise $h(c) = 2$. Suppose the colours 1 and 6 are each used exactly once. Since $|E(K_6)| = 15$, there are 13 edges to be coloured with colours from the set $\{2, 3, 4, 5\}$, and by the pigeonhole principle one of these colours is used at least four times, a contradiction. Hence, three edges are coloured 1 or 6.

Without loss of generality we assume two edges are coloured 1 and one is coloured 6; necessarily these edges form an independent edge set. Since there are 12 edges of K_6 to be coloured with colours from the set $\{2, 3, 4, 5\}$, each colour is used exactly three times. Consider the three edges coloured 5. Since only one edge is coloured 6, there exists at least one edge, say e, coloured 5 which is not adjacent to the edge coloured 6. Furthermore, the edge e is adjacent to an edge e' which is coloured 1. Thus $e'e$ is a maximal $(2, c)$-ascent, which is a contradiction. Therefore $\chi_\varepsilon(K_6) \geq 7$.

To show that $\chi_\varepsilon(K_6) \leq 8$, we show that the 8-edge colouring of K_6 in Figure 5.6 has flatness three. Note that in the figure the labels k and k' are assumed to be the same label for each $k \in \{2, 3, \ldots, 8\}$ and the notation is used to differentiate between edges which are assigned the same colour. We lexicographically consider all $(2, c)$-ascents which are the first two edges of a maximal c-ascent and include in brackets the colour of an edge which extends the ascent to a $(3, c)$-ascent. The $(2, c)$-ascents we need to consider are 1 2 ($6'$), 1 $2'$ ($8'$), 1 3 ($6'$), 1 $3'$ ($6'$), 1 4 ($5'$), 1 $4'$ ($8'$), 1 5 ($6'$), 1 6 (7), 2 3 ($6'$), 2 $3'$ (4), 2 $4'$ ($5'$), 2 6 (8), $2'$ $3'$ ($6'$), $2'$ 4 (7), $2'$ $4'$ (6), $2'$ 5 ($6'$), 3 $4'$ ($8'$), 3 6 (7), 4 5 ($6'$), $4'$ 6 (7). Hence $h(c) = 3$. □

Chapter 6

Summary and Further Directions of Research

In this chapter we provide a summary of the work accomplished in this dissertation. We conclude with a list of several open problems related to this body of work.

6.1 Summary of results

In this dissertation we considered various problems related to the depression of a graph. We determined a formula for the depression of the class of trees known as double spiders. We investigated the concept of k-kernels of a graph G. In particular, we identified k-kernels for paths, cycles, and spiders. From the k-kernel result for spiders we were able to determine an improved upper bound for the depression of trees. We also provided a sufficient condition for a vertex $v \in V(G)$ to be a 3-kernel (or 2-kernel) of a graph G with $\operatorname{diam}(L(G)) = 2$.

In this dissertation we also investigated graphs with depression three. We provided a characterization of graphs with depression three and the added property that no two adjacent vertices both have degree three or more. We also constructed a large class of graphs with depression three which includes cyclic graphs and graphs with adjacent vertices of degree three or more.

We considered the problem of determining the smallest integer r such that there exists a proper edge colouring $c : E(G) \to \{1, 2, \ldots, r\}$ for which a shortest maximal ascent has given length k, denoted by $\chi_{(k)}(G)$, or $\chi_\varepsilon(G)$ if $k = \varepsilon(G)$. We established various general bounds for $\chi_\varepsilon(G)$ and in some cases identified classes of graphs which achieve these bounds. We determined $\chi_\varepsilon(G)$ for paths and cycles, and showed that $\chi_{(3)}(T) \leq \chi_1(T) + 2$ for any tree T. We also determined $\chi_\varepsilon(K_n)$ for $2 \leq k \leq 5$, and showed that $\chi_\varepsilon(K_n) \leq 2n - 3$ for all $n \geq 2$.

We developed an algorithm, which is included in Appendix A, to determine the depression of a given graph G. This algorithm was used to verify the depression of many of the example graphs in this dissertation.

6.2 Open problems

A characterization of trees with depression three is given in [16] and graphs with depression three and no adjacent vertices of degree three or more are characterized in Section 4.1 (Theorem 4.1.1), however, the characterization of graphs with depression three remains an unsolved problem.

Problem 6.2.1. *Characterize the class of graphs with depression three.*

Problem 6.2.2. *Does there exist a characterization similar to Theorem 4.1.1 for graphs with depression $k \geq 4$ and no adjacent vertices of degree three or more?*

In Section 4.2 we described 7 operations, **O1** − **O7**, which are used to construct a large class of graphs with depression three.

Problem 6.2.3. *Does there exist a finite number of operations of the type **O1-O7** that would yield all graphs with depression three?*

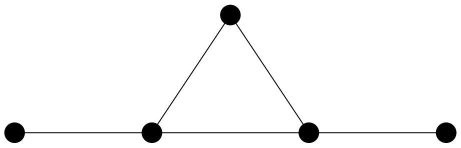

Figure 6.1: A cyclic graph G for which $\chi_\varepsilon(G) = E(G)$.

In Chapter 2 we describe a formula for the depression of the family of trees known as double spiders. As mentioned previously, a characterization of trees with depression three was established in [16]. Further research into the depression of trees with the end goal of determining a formula for the depression of trees in general may include: a characterization of trees with depression four, a formula for the depression of other families of trees, an improved algorithm for determining the depression of a given tree, and improved bounds on the depression of trees, perhaps by using the depression of double spiders.

Problem 6.2.4. *Determine a formula for the depression of trees.*

In Section 5.2 we proved that $\chi_\varepsilon(G) \geq \chi_1(G) + \varepsilon(G) - 2$ and we also showed that the difference $\chi_\varepsilon(G) - [\chi_1(G) + \varepsilon(G) - 2]$ can be arbitrarily large. Prove or disprove:

Problem 6.2.5. *The ratio* $\chi_\varepsilon(G)/\left[\chi_1(G) + \varepsilon(G)\right]$ *is bounded.*

Theorem 5.4.1 states that the only trees T for which $\chi_\varepsilon(T) = |E(T)|$ are stars and paths. The graph shown in Figure 6.1 is an example of a cyclic graph G for which $\chi_\varepsilon(G) = |E(G)|$ (we omit verification of this claim). It remains an open problem to identify all such graphs.

Problem 6.2.6. *Characterize the class of graphs for which* $\chi_\varepsilon(G) = |E(G)|$.

In Section 5.5 we investigated the problem of determining the ε-ascent chromatic index for complete graphs. We established an upper bound for $\chi_\varepsilon(K_n)$ when $n \geq 2$, and determined the exact value for $\chi_\varepsilon(K_n)$ when $2 \leq n \leq 5$. With further research into this problem we aspire to determine a general formula of $\chi_\varepsilon(K_n)$. Failing this we seek to improve the bounds for $\chi_\varepsilon(K_n)$.

Problem 6.2.7. *Determine* $\chi_\varepsilon(K_n)$ *for* $n \geq 6$.

In Appendix A we outline an algorithm for determining the depression of a graph.

Problem 6.2.8. *Develop an improved algorithm for determining the depression of a graph.*

In [15], the problem of determining the height of a given edge ordering was shown to be NP-hard. The complexity of determining the flatness of a given edge ordering is not known.

Problem 6.2.9. *Determine the complexity of the problem of evaluating the flatness of a given edge ordering.*

Appendix A

Depression Algorithm

Here we outline an algorithm for determining the depression of a graph G.

One task in determining the depression of a graph G is to determine the length of the shortest maximal f-ascent for a given edge ordering $f : E(G) \to \{1, 2, \ldots, |E(G)|\}$. Let e_i be the edge in G for which $f(e_i) = i$ for $i = 1, \ldots, m$, where $m = |E(G)|$. Now consider the line graph of G, denoted by $L(G)$, where $V(L(G)) = \{e_1, e_2, \ldots, e_m\}$. We aim to construct our maximal f-ascents of G in $L(G)$. Note that not all paths in $L(G)$ correspond to a path in G. Specifically, a path $P = v_1, v_2, \ldots, v_k$ in $L(G)$ corresponds to a path in G if and only if, for all $0 \leq i < j \leq k$, if $\{v_i, v_j\} \in E(L(G))$, then $j = i + 1$. That is, a path P in $L(G)$ corresponds to a path in G if and only if P is an induced path in $L(G)$. Furthermore, in order for a path $P = v_1, v_2, \ldots, v_k$ to correspond to an f-ascent in G, it follows that for each edge $\{v_i = e_j, v_{i+1} = e_k\}$ of P, $k > j$.

The following procedure *EXTEND*(i) determines all paths P in $L(G)$ which correspond to a maximal f-ascent in G that begins with the vertex e_i. These maximal ascents are stored in the set X, hence, if we run the procedure *EXTEND*(i) for each i from 1 to $n - 1$, sequentially and initializing $P := e_i$ each time, then the flatness of

f is the length of the shortest path in X.

```
proc EXTEND($i$)
  $m := i + 1$;
  while $m \leq n$ do
    if $\{e_i, e_m\} \in E\,(L(G))$ and $N(P \backslash e_i) \cap \{e_m\} = \varnothing$ then
      add $e_m$ to $P$; EXTEND($m$);
      remove $e_m$ from $P$; $m := m + 1$;
    else
      $m := m + 1$;
    end if
  end do
  if $P$ is not a subpath of a path already in $X$ then
    $X := X \cup \{P\}$;
  end if
end proc
```

Naively, we could repeat this procedure over all $m!$ edge orderings of G, but many of these orientations are equivalent. It is easy to see that a vertex ordering f of a graph G can be used to define an acyclic orientation D of the edges of G by orienting the edge $\{a, b\}$ as (a, b) whenever $f(a) < f(b)$. It is also easy to see that more than one vertex ordering may correspond to the same acyclic orientation. For example, consider $P_3 = v_1, v_2, v_3$. The vertex orderings $f = \{(v_1, 2), (v_2, 1), (v_3, 3)\}$ and $g = \{(v_1, 3), (v_2, 1), (v_3, 2)\}$ determine the same acyclic orientation of P_3 – see figure A.1. Moreover, if two edge orderings of G, say h and f, correspond to the same acyclic orientation of $L(G)$, then $h(f) = h(g)$. Thus, we need only consider a minimal set of edge orderings of G which determines the set of all acyclic orientations

Figure A.1: Two vertex orderings which define equivalent orientations of P_3.

of $L(G)$.

To construct the set of all acyclic orientations of $L(G)$ we use the algorithm outlined in [1] and for each of these orientations we obtain a total ordering of the vertices of $L(G)$ (or an edge ordering of G) using a topological sort. We may further reduce our set of orderings by a factor of two by noting that two acyclic orientations d and $\overline{d}$ which are complementary, that is, $(a, b) \in d$ if and only if $(b, a) \in \overline{d}$, will correspond to orderings which yield the same flatness. Thus we generate all acyclic orientations of $L(G)$ with the orientation of one edge fixed. The depression of G is then the maximum flatness over all such edge orderings.

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