

THE MULTIDIMENSIONAL WEYL FRACTIONAL
INTEGRAL OF CERTAIN CLASSES
OF POLYNOMIALS

by

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Abstract

Recently, H.M. Srivastava, S.P. Goyal, and R.M. Jain [7] introduced and studied some interesting new generalizations of the multidimensional Riemann–Liouville and Weyl fractional integral operators. The object of the present paper is to apply their multidimensional Weyl fractional integral to certain general classes of polynomials in several variables, defined (for example) by Equation (2.9) below. Each of the main theorems proved in this paper provides a unification (and generalization) of the corrected versions of various earlier results given, among others, by R.K. Raina [3].

1. Introduction

In a recent paper Srivastava [4, p. 221, Equation (2.7)] established the following Weyl fractional integral of order μ :

$$(1.1) \quad \{\Gamma(\mu)\}^{-1} \int_x^\infty (t-x)^{\mu-1} \left\{ \exp(-st) S_n^m(t) \right\} dt \\ = s^{-\mu} \exp(-sx) \sum_{j=0}^{[n/m]} (-n)_{mj} \left[-\frac{1}{s} \right]^j A_{n,j} L_j^{(-\mu-j)}(sx), \\ (\min\{\operatorname{Re}(s), \operatorname{Re}(\mu)\} > 0),$$

where $S_n^m(t)$ are the polynomials defined earlier by him by (*cf.* [4, p. 220, Equation (2.4)])

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$$(1.2) \quad S_n^m(x) = \sum_{j=0}^{[n/m]} (-n)_{mj} A_{n,j} \frac{x^j}{j!}.$$

Here $A_{n,j}$ ($n, j \geq 0$) are arbitrary constants and $(a)_n$ denotes the usual Pochhammer symbol defined by

$$(1.3) \quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0, \\ a(a+1)\cdots(a+n-1), & \text{if } n = 1, 2, 3, \dots \end{cases}$$

Subsequently, Raina [3] gave some multidimensional generalizations of (1.1). Many of the results of Raina [3] are in error. The main purposes of this paper are to correct these results of Raina [3] and to extend them by making use of a general class of multiple Weyl fractional integrals introduced and studied recently by Srivastava, Goyal, and Jain [7].

2. Corrected and Modified Forms of Raina's Results

We start with the classical Laplace transform (*cf.* [2, Vol. I, Chapter 4]):

$$(2.1) \quad \mathcal{L}\{f(t):s\} = \int_0^\infty e^{-st} f(t) dt,$$

and its multidimensional extension defined by

$$(2.2) \quad \mathcal{L}\{f(t_1, \dots, t_r) : s_1, \dots, s_r\} \\ = \int_0^\infty \cdots \int_0^\infty \exp(-\sum s_i t_i) f(t_1, \dots, t_r) dt_1 \cdots dt_r,$$

valid for certain suitably constrained spaces of functions $f(t)$ and $f(t_1, \dots, t_r)$. Here, and throughout this paper, we find it to be convenient to write:

$$\sum z_i = z_1 + \cdots + z_r \quad \text{and} \quad \Pi\{\zeta_i\} = \zeta_1 \cdots \zeta_r.$$

Using the multiple Laplace transform (2.2), it is fairly easy to derive (*cf.* [3, p. 285, Equation (2.6)])

$$\begin{aligned}
(2.3) \quad & \mathcal{L} \left\{ \Pi \left\{ t_i^{\mu_i-1} \right\} \left[x + \Sigma t_i^{\alpha_i} \right]^n : s_1, \dots, s_r \right\} \\
&= \int_0^\infty \cdots \int_0^\infty \Pi \left\{ t_i^{\mu_i-1} \right\} \left[x + \Sigma t_i^{\alpha_i} \right]^n \exp(-\Sigma s_i t_i) \Pi \{ dt_i \} \\
&= x^n \Pi \left\{ \frac{\Gamma(\mu_i)}{s_i} \right\} F \begin{matrix} 1:1; \dots; 1 \\ 0:0; \dots; 0 \end{matrix} \left[\begin{matrix} [-n:1] : \\ \text{---} : \\ [\mu_1:\alpha_1]; \dots; [\mu_r:\alpha_r]; \\ \text{---} ; \dots; \text{---} ; \end{matrix} \right. \\
& \qquad \qquad \qquad \left. -\frac{1}{xs_1^{\alpha_1}}, \dots, -\frac{1}{xs_r^{\alpha_r}} \right],
\end{aligned}$$

provided that

$$\operatorname{Re}(\mu_i) > 0, \quad \operatorname{Re}(s_i) > 0, \quad \text{and} \quad \alpha_i > 0 \quad (i = 1, \dots, r).$$

The multivariable hypergeometric function on the right-hand side of (2.3) is the generalized Lauricella function of several variables ([5]; see also [8, p. 37 *et seq.*]).

For $\alpha_i = m_i$ ($i = 1, \dots, r$), where m_i are positive *integers*, (2.3) will yield the following *corrected* form of one of Raina's results [3, p. 285, Equation (2.6)]:

$$\begin{aligned}
(2.4) \quad & \mathcal{L} \left\{ \Pi \left\{ t_i^{\mu_i-1} \right\} \left[x + \Sigma t_i^{m_i} \right]^n : s_1, \dots, s_r \right\} \\
&= \int_0^\infty \cdots \int_0^\infty \Pi \left\{ t_i^{\mu_i-1} \right\} \left[x + \Sigma t_i^{m_i} \right]^n \exp(-\Sigma s_i t_i) \Pi \{ dt_i \} \\
&= x^n \Pi \left\{ \frac{\Gamma(\mu_i)}{s_i} \right\} F \begin{matrix} 1:m_1; \dots, m_r \\ 0:0; \dots; 0 \end{matrix} \left[\begin{matrix} [-n:1] : \\ \text{---} : \\ \Delta(m_1;\mu_1); \dots; \Delta(m_r;\mu_r); \\ \text{---} ; \dots; \text{---} ; \end{matrix} \right. \\
& \qquad \qquad \qquad \left. -\frac{1}{x} \left[\frac{m_1}{s_1} \right]^{m_1}, \dots, -\frac{1}{x} \left[\frac{m_r}{s_r} \right]^{m_r} \right],
\end{aligned}$$

where, for convenience,

$$\Delta(m; \lambda) = \frac{\lambda}{m}, \frac{\lambda+1}{m}, \dots, \frac{\lambda+m-1}{m} \quad (m = 1, 2, 3, \dots),$$

and, for convergence,

$$\min\{\operatorname{Re}(\mu_i), \operatorname{Re}(s_i)\} > 0 \quad (i = 1, \dots, r).$$

For $r = 1$ and $m_1 = 1$, (2.4) would obviously correspond to an earlier result of Srivastava [4, p. 220, Equation (2.2)].

Next we consider the multiple Weyl fractional integral of order μ_1, \dots, μ_r defined by (*cf.*, *e.g.*, [7]; see also [2, Vol. II, Chapter 13])

$$(2.5) \quad \mathcal{W}_{\mu_1, \dots, \mu_r} \{f(x_1, \dots, x_r)\} \\ = \prod\{\Gamma(\mu_i)\}^{-1} \int_{x_1}^{\infty} \dots \int_{x_r}^{\infty} \prod\{(t_i - x_i)^{\mu_i - 1}\} f(t_1, \dots, t_r) dt_1 \dots dt_r,$$

where

$$\operatorname{Re}(\mu_i) > 0 \quad (i = 1, \dots, r).$$

By choosing $f(t_1, \dots, t_r)$ as the product

$$S_n^m \left[\sum x_i^{\alpha_i} + \sum (t_i - x_i)^{\beta_i} \right] \exp(-\sum s_i t_i),$$

and writing $t_i + x_i$ for t_i , we get the following multiple Weyl fractional integral:

$$(2.6) \quad \int_{x_1}^{\infty} \dots \int_{x_r}^{\infty} \prod \left\{ \frac{(t_i - x_i)^{\mu_i - 1}}{\Gamma(\mu_i)} \right\} \exp(-\sum s_i t_i) \\ \cdot S_n^m \left[\sum x_i^{\alpha_i} + \sum (t_i - x_i)^{\beta_i} \right] dt_1 \dots dt_r \\ = \frac{\exp(-\sum s_i x_i)}{\prod \left\{ s_i^{\mu_i} \right\}} \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j!} A_{n,j} \left[\sum x_i^{\alpha_i} \right]^j$$

$$\cdot F_{0:0;\dots;0}^{1:1;\dots;1} \left[\begin{array}{c} [-j:1]: [\mu_1:\beta_1]; \dots; [\mu_r:\beta_r]; \\ \text{---} : \text{---} ; \dots; \text{---} ; \\ \frac{1}{s_1 \Sigma x_i^{\alpha_i}}, \dots, \frac{1}{s_r \Sigma x_i^{\alpha_i}} \end{array} \right],$$

where

$$\operatorname{Re}(\mu_i) > 0, \operatorname{Re}(s_i) > 0, \text{ and } \beta_i > 0 \quad (i = 1, \dots, r).$$

Our result (2.6) provides a mild generalization of the corrected form of another result of Raina [3, p. 285, Equation (3.1)].

Formula (2.6) yields the following two interesting special cases under suitable substitutions. For example, if $\beta_i = m_i$ ($i = 1, \dots, r$), where m_i are positive integers, we obtain

$$(2.7) \quad \int_{x_1}^{\infty} \dots \int_{x_r}^{\infty} \Pi \left\{ \frac{(t_i - x_i)^{\mu_i - 1}}{\Gamma(\mu_i)} \right\} \exp(-\Sigma s_i t_i) S_n^m \left[\Sigma x_i^{\alpha_i} + \Sigma (t_i - x_i)^{m_i} \right] dt_1 \dots dt_r$$

$$= \frac{\exp(-\Sigma s_i x_i)}{\Pi \{s_i^{\mu_i}\}} \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j!} A_{n,j} \left[\Sigma x_i^{\alpha_i} \right]^j$$

$$\cdot F_{0:0;\dots;0}^{1:m_1;\dots;m_r} \left[\begin{array}{c} -j: \Delta(m_1; \mu_1); \dots; \Delta(m_r; \mu_r); \\ \text{---} : \text{---} ; \dots; \text{---} ; \\ \frac{(m_1/s_1)^{m_1}}{\Sigma x_i^{\alpha_i}}, \dots, \frac{(m_r/s_r)^{m_r}}{\Sigma x_i^{\alpha_i}} \end{array} \right],$$

provided that

$$\min\{\operatorname{Re}(\mu_i), \operatorname{Re}(s_i)\} > 0 \quad (i = 1, \dots, r).$$

If, in this last formula (2.7), we further set $\alpha_i = m_i = 1$ ($i = 1, \dots, r$), we arrive at the following corrected form of the aforementioned result of Raina (*cf.* [3, p. 285, Equation (3.1)]):

$$\begin{aligned}
(2.8) \quad & \int_{x_1}^{\infty} \cdots \int_{x_r}^{\infty} \Pi \left\{ \frac{(t_i - x_i)^{\mu_i - 1}}{\Gamma(\mu_i)} \right\} \exp(-\Sigma s_i t_i) S_n^m(\Sigma t_i) dt_1 \cdots dt_r \\
& = \frac{\exp(-\Sigma s_i x_i)}{\Pi \{s_i^{\mu_i}\}} \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j!} A_{n,j}(\Sigma x_i)^j \\
& \quad \cdot F_{0:0;\dots;0}^{1:1;\dots;1} \left[\begin{matrix} -j: \mu_1; \dots; \mu_r; \\ \frac{1}{s_1 \Sigma x_i}, \dots, \frac{1}{s_r \Sigma x_i} \end{matrix} \right],
\end{aligned}$$

provided that

$$\min\{\operatorname{Re}(\mu_i), \operatorname{Re}(s_i)\} > 0 \quad (i = 1, \dots, r).$$

A substantial variation of Formula (2.8) can easily be established by considering the set of polynomials of the type

$$\mathcal{S}_n^{m_1, \dots, m_r} \left[\begin{matrix} \alpha_1 & q_1 \\ x_1 & + t_1 \end{matrix}, \dots, \begin{matrix} \alpha_r & q_r \\ x_r & + t_r \end{matrix} \right],$$

where (*cf.* [6, p. 686, Equation (1.4)])

$$(2.9) \quad \mathcal{S}_n^{m_1, \dots, m_r}(x_1, \dots, x_r) = \sum_{k_1, \dots, k_r=0}^{M \leq n} (-n)_M$$

$$\cdot A(n; k_1, \dots, k_r) \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_r^{k_r}}{k_r!}$$

and

$$M \equiv m_1 k_1 + \cdots + m_r k_r \quad (m_i = 1, 2, 3, \dots; i = 1, \dots, r),$$

the coefficients $A(n; k_1, \dots, k_r)$ being arbitrary constants, real or complex. Thus we have

$$(2.10) \quad \int_{x_1}^{\infty} \cdots \int_{x_r}^{\infty} \Pi \left\{ (t_i - x_i)^{\mu_i - 1} \right\} \exp(-\Sigma s_i t_i)$$

$$\begin{aligned}
& \cdot \mathcal{S}_n^{m_1, \dots, m_r} \left[x_1^{\alpha_1} + (t_1 - x_1)^{q_1}, \dots, x_r^{\alpha_r} + (t_r - x_r)^{q_r} \right] dt_1 \cdots dt_r \\
& = \exp(-\sum_i s_i x_i) \sum_{j_1, \dots, j_r = 0}^{m_1 j_1 + \dots + m_r j_r \leq n} \frac{(-n)_{m_1 j_1 + \dots + m_r j_r}}{j_1! \cdots j_r!} \\
& \cdot A(n; j_1, \dots, j_r) \prod_{i=1}^r \left\{ \frac{\Gamma(\mu_i + q_i j_i)}{s_i^{\mu_i + q_i j_i}} \right. \\
& \cdot \left. {}_1F_{q_i} \left[\begin{matrix} -j_i; \\ \Delta(q_i; 1 - \mu_i - q_i j_i); \end{matrix} \right. \left. -x_i^i \left[-\frac{s_i}{q_i} \right]^{q_i} \right] \right\},
\end{aligned}$$

where

$$\min\{\operatorname{Re}(\mu_i), \operatorname{Re}(s_i)\} > 0; \quad q_i = 1, 2, 3, \dots \quad (i = 1, \dots, r).$$

Setting $\alpha_i = q_i$ ($i = 1, \dots, r$), (2.10) immediately yields

$$\begin{aligned}
(2.11) \quad & \int_{x_1}^{\infty} \cdots \int_{x_r}^{\infty} \prod \left\{ (t_i - x_i)^{\mu_i - 1} \right\} \exp(-\sum_i s_i t_i) \\
& \cdot \mathcal{S}_n^{m_1, \dots, m_r} \left[x_1^{q_1} + (t_1 - x_1)^{q_1}, \dots, x_r^{q_r} + (t_r - x_r)^{q_r} \right] dt_1 \cdots dt_r \\
& = \exp(-\sum_i s_i x_i) \sum_{j_1, \dots, j_r = 0}^{m_1 j_1 + \dots + m_r j_r \leq n} \frac{(-n)_{m_1 j_1 + \dots + m_r j_r}}{j_1! \cdots j_r!} \\
& \cdot A(n; j_1, \dots, j_r) \prod_{i=1}^r \left\{ \frac{\Gamma(\mu_i + q_i j_i)}{s_i^{\mu_i + q_i j_i}} {}_1F_{q_i} \left[\begin{matrix} -j_i; \\ \Delta(q_i; 1 - \mu_i - q_i j_i); \end{matrix} \right. \right. \\
& \left. \left. - \left[-\frac{s_i x_i}{q_i} \right]^{q_i} \right] \right\},
\end{aligned}$$

which holds true under the conditions stated with (2.10).

In its special case when $m_i = q_i$ ($i = 1, \dots, r$), Formula (2.11) would correspond to the corrected form of the main result of Raina [3, p. 286, Equation (3.3)].

Finally, setting $\alpha_i = q_i = 1$ ($i = 1, \dots, r$), (2.11) assumes the elegant form:

$$\begin{aligned}
 (2.12) \quad & \int_{x_1}^{\infty} \cdots \int_{x_r}^{\infty} \Pi \left\{ (t_i - x_i)^{\mu_i - 1} \right\} \exp(-\Sigma s_i t_i) \\
 & \cdot \mathcal{S}_n^{m_1, \dots, m_r}(t_1, \dots, t_r) dt_1 \cdots dt_r \\
 & = \exp(-\Sigma s_i t_i) \sum_{\substack{m_1 j_1 + \cdots + m_r j_r \leq n \\ j_1, \dots, j_r = 0}}^{m_1 j_1 + \cdots + m_r j_r \leq n} (-n)_{m_1 j_1 + \cdots + m_r j_r} \\
 & \cdot A(n; j_1, \dots, j_r) \prod_{i=1}^r \left\{ \frac{(-1)^{j_i} \Gamma(\mu_i)}{s_i^{\mu_i + j_i}} L_n^{(-\mu_i - j_i)}(s_i x_i) \right\},
 \end{aligned}$$

where

$$\min\{\operatorname{Re}(\mu_i), \operatorname{Re}(s_i)\} > 0 \quad (i = 1, \dots, r)$$

and we have made use of the relationship:

$$(2.13) \quad {}_1F_1 \left[\begin{matrix} -n; \\ \alpha + 1; \end{matrix} z \right] = \frac{n!}{(1+\alpha)_n} L_n^{(\alpha)}(z),$$

$L_n^{(\alpha)}(z)$ being [just as in Equation (1.1)] the classical Laguerre polynomial of order α and degree n in z .

Srivastava's result (1.1) above corresponds to the special case of (2.12) when $r = 1$.

3. Further Multidimensional Weyl Fractional Integrals

Making use of an interesting generalization of the multidimensional Weyl fractional integral (2.5), which was introduced and studied recently by Srivastava, Goyal, and Jain [7], we have

$$\begin{aligned}
(3.1) \quad & \int_{x_1}^{\infty} \cdots \int_{x_r}^{\infty} \Pi \left\{ \frac{(t_i - x_i)^{\mu_i - 1}}{\Gamma(\mu_i)} \right\} \exp(-\Sigma s_i t_i) \\
& \cdot S_n^m \left[z \Pi \left\{ \left[1 - \frac{x_i}{t_i} \right]^{\rho_i} \right\} \right] \mathcal{S}_n^{m_1, \dots, m_r} (z_1 t_1, \dots, z_r t_r) dt_1 \cdots dt_r \\
& = \exp(-\Sigma s_i x_i) \int_0^{\infty} \cdots \int_0^{\infty} \Pi \left\{ \frac{t_i^{\mu_i - 1}}{\Gamma(\mu_i)} \right\} \exp(-\Sigma s_i t_i) \\
& \cdot \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j!} A_{n,j} z^j \Pi \left\{ \left[1 - \frac{x_i}{x_i + t_i} \right]^{\rho_i j} \right\} \\
& \cdot \sum_{\substack{m_1 k_1 + \dots + m_r k_r \leq N \\ k_1, \dots, k_r = 0}} \frac{(-N)_{m_1 k_1 + \dots + m_r k_r}}{k_1! \cdots k_r!} A(N; k_1, \dots, k_r) \\
& \cdot \Pi \left\{ z_i (x_i + t_i) \right\}^{k_i} dt_1 \cdots dt_r,
\end{aligned}$$

which holds true under the constraints derivable from Section 2.

Now, evaluating the multiple Laplace transform occurring on the right-hand side of (3.1), we can easily obtain

$$\begin{aligned}
(3.2) \quad & \int_{x_1}^{\infty} \cdots \int_{x_r}^{\infty} \Pi \left\{ \frac{(t_i - x_i)^{\mu_i - 1}}{\Gamma(\mu_i)} \right\} \exp(-\Sigma s_i t_i) \\
& \cdot S_n^m \left[z \Pi \left\{ \left[1 - \frac{x_i}{t_i} \right]^{\rho_i} \right\} \right] \mathcal{S}_N^{m_1, \dots, m_r} (z_1 t_1, \dots, z_r t_r) dt_1 \cdots dt_r \\
& = \exp(-\Sigma s_i x_i) \sum_{j=0}^{[n/m]} \sum_{k_1, \dots, k_r = 0}^{m_1 k_1 + \dots + m_r k_r \leq N} \frac{(-n)_{mj}}{j!}
\end{aligned}$$

$$\cdot \frac{(-N)_{m_1 k_1 + \dots + m_r k_r}}{k_1! \dots k_r!} A_{n,j} A(N; k_1, \dots, k_r) z^j$$

$$\cdot \prod_{i=1}^r \left[\frac{\Gamma(\mu_i + \rho_i j)}{\Gamma(\mu_i)} \right] z_i^{k_i} s_i^{-\mu_i - \rho_i j} x_i^{k_i - \rho_i j} {}_2F_0 \left[\begin{matrix} -k_i + \rho_i j, \mu_i + \rho_i j; \\ \text{---}; \end{matrix} -\frac{1}{s_i x_i} \right],$$

provided that

$$(3.3) \quad \operatorname{Re}(\mu_i + \rho_i j) > 0 \quad (j = 1, \dots, [n/m]); \quad \operatorname{Re}(s_i) > 0; \quad i = 1, \dots, r.$$

Our result (3.2) can be rewritten in terms of the Whittaker function $W_{k,m}(z)$, since (cf. [1, p. 264, Equation (5)])

$$(3.4) \quad W_{k,m}(x) = e^{-x/2} x^k {}_2F_0 \left[\begin{matrix} \frac{1}{2} - k + m, \frac{1}{2} - k - m; \\ \text{---}; \end{matrix} -\frac{1}{x} \right].$$

Further, if

$$(3.5) \quad k_i - \rho_i j = 0, 1, 2, \dots \quad (i = 1, \dots, r; \quad j = 1, \dots, [n/m]),$$

we can rewrite (3.2) in terms of Laguerre polynomials [cf. Equation (2.13)]:

$$(3.6) \quad L_n^{(\alpha)}(z) = \frac{(\alpha+1)_n}{n!} {}_1F_1 \left[\begin{matrix} -n; \\ z \end{matrix} \right]$$

$$= \frac{(-z)^n}{n!} {}_2F_0 \left[\begin{matrix} -n, -\alpha - n; \\ \text{---}; \end{matrix} -\frac{1}{z} \right],$$

or, equivalently,

$$(3.7) \quad {}_2F_0 \left[\begin{matrix} -n, \alpha; \\ \text{---}; \end{matrix} -\frac{1}{z} \right] = n! \left[-\frac{1}{z} \right]^n L_n^{(-\alpha-n)}(z).$$

Our integral formula (3.2) thus yields

$$\begin{aligned}
(3.8) \quad & \int_{x_1}^{\infty} \cdots \int_{x_r}^{\infty} \Pi \left\{ \frac{(t_i - x_i)^{\mu_i - 1}}{\Gamma(\mu_i)} \right\} \exp(-\Sigma s_i t_i) \\
& \cdot S_n^m \left[z \Pi \left\{ \left[1 - \frac{x_i}{t_i} \right]^{\rho_i} \right\} \right] \mathcal{S}_N^{m_1, \dots, m_r} (z_1 t_1, \dots, z_r t_r) dt_1 \cdots dt_r \\
& = \exp(-\Sigma s_i x_i) \sum_{j=0}^{[n/m]} \sum_{k_1, \dots, k_r = 0}^{m_1 k_1 + \dots + m_r k_r \leq N} \frac{(-n)_{mj}}{j!} \frac{(-N)_{m_1 k_1 + \dots + m_r k_r}}{k_1! \cdots k_r!} \\
& \cdot A_{n,j} A(N; k_1, \dots, k_r) z^j \prod_{i=1}^r \left\{ \frac{\Gamma(\mu_i + \rho_i j)}{\Gamma(\mu_i)} z_i^{k_i} x_i^{-\mu_i - k_i} \right. \\
& \quad \left. \cdot (-1)^{k_i - \rho_i j} (k_i - \rho_i j)! L_{k_i - \rho_i j}^{(-\mu_i - k_i)}(s_i x_i) \right\},
\end{aligned}$$

where the conditions in (3.3) are assumed to hold true.

Setting $z = 0$ and $A_{0,0} = 1$ in (3.8), we have the following variation of the multidimensional Weyl fractional integral (2.12):

$$\begin{aligned}
(3.9) \quad & \int_{x_1}^{\infty} \cdots \int_{x_r}^{\infty} \Pi \left\{ (t_i - x_i)^{\mu_i - 1} \right\} \exp(-\Sigma s_i t_i) \mathcal{S}_N^{m_1, \dots, m_r} (z_1 t_1, \dots, z_r t_r) \\
& \quad \cdot dt_1 \cdots dt_r \\
& = \exp(-\Sigma s_i x_i) \sum_{k_1, \dots, k_r = 0}^{m_1 k_1 + \dots + m_r k_r \leq N} (-N)_{m_1 k_1 + \dots + m_r k_r} \\
& \quad \cdot A(N; k_1, \dots, k_r) \prod_{i=1}^r \left\{ \frac{z_i^{k_i} \Gamma(\mu_i)}{s_i^{\mu_i + k_i}} L_{k_i}^{(-\mu_i - k_i)}(s_i x_i) \right\}.
\end{aligned}$$

provided that the conditions in (3.3) hold true.

We conclude by remarking that many more results for Weyl fractional integrals in

one, two, and more dimensions can be derived as consequences of the results presented in this paper.

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