
Faculty of Science

Faculty Publications

Parameterized Quantum Fractional Integral Inequalities Defined by Using n -Polynomial Convex Functions

Rozana Liko, Hari M. Srivastava, Pshtiwan O. Mohammed, Artion Kashuri, Eman Al-Sarairah, Soubhagya K. Sahoo, and Mohamed S. Soliman

2022

©2022 by the authors. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license.

<http://creativecommons.org/licenses/by/4.0/>

This article was originally published at:

<https://doi.org/10.3390/axioms11120727>

Citation for this paper:

Liko, R., Srivastava, H. M., Mohammed, P. O., Kashuri, A., Al-Sarairah, E., Sahoo, S. K., & Soliman, M. S. (2022). Parameterized Quantum Fractional Integral Inequalities Defined by Using n -Polynomial Convex Functions. *Axioms*, 11(12), 727.
<https://doi.org/10.3390/axioms11120727>

Article

Parameterized Quantum Fractional Integral Inequalities Defined by Using n -Polynomial Convex Functions

Rozana Liko ¹, Hari Mohan Srivastava ^{2,3,4,5}, Pshtiwan Othman Mohammed ^{6,*}, Artion Kashuri ¹, Eman Al-Sarairah ^{7,8}, Soubhagya Kumar Sahoo ⁹ and Mohamed S. Soliman ^{10,*}

- ¹ Department of Mathematics, Faculty of Technical and Natural Sciences, University "Ismail Qemali", 9400 Vlora, Albania
 - ² Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada
 - ³ Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
 - ⁴ Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, Baku AZ1007, Azerbaijan
 - ⁵ Center for Converging Humanities, Kyung Hee University, 26 Kyunghedae-ro, Dongdaemun-gu, Seoul 02447, Republic of Korea
 - ⁶ Department of Mathematics, College of Education, University of Sulaimani, Sulaimani 46001, Iraq
 - ⁷ Department of Mathematics, Khalifa University, Abu Dhabi P.O. Box 127788, United Arab Emirates
 - ⁸ Department of Mathematics, Al-Hussein Bin Talal University, Ma'an P.O. Box 33011, Jordan
 - ⁹ Department of Mathematics, Institute of Technical Education and Research, Siksha 'O' Anusandhan University, Bhubaneswar 751030, India
 - ¹⁰ Department of Electrical Engineering, College of Engineering, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia
- * Correspondence: pshtiwan.muhammad@univsul.edu.iq (P.O.M.); soliman@tu.edu.sa (M.S.S.)



Citation: Liko, R.; Srivastava, H.M.; Mohammed, P.O.; Kashuri, A.; Al-Sarairah, E.; Sahoo, S.K.; Soliman, M.S. Parameterized Quantum Fractional Integral Inequalities Defined by Using n -Polynomial Convex Functions. *Axioms* **2022**, *11*, 727. <https://doi.org/10.3390/axioms11120727>

Academic Editors: Sevtaç Sümer Eker and Juan J. Nieto

Received: 24 November 2022
Accepted: 12 December 2022
Published: 13 December 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

Abstract: Convexity performs the appropriate role in the theoretical study of inequalities according to the nature and behaviour. There is a strong relation between symmetry and convexity. In this article, we consider a new parameterized quantum fractional integral identity. Following that, our main results are established, which consist of some integral inequalities of Ostrowski and midpoint type pertaining to n -polynomial convex functions. From our main results, we discuss in detail several special cases. Finally, an example and an application to special means of positive real numbers are presented to support our theoretical results.

Keywords: Ostrowski inequality; Riemann–Liouville q -fractional integrals; q -Hölder's inequality; q -power mean inequality; n -polynomial convex functions; special means

MSC: 26A51; 26A33; 26D07; 26D10; 26D15; 26E60

1. Introduction

Integral inequalities are very useful tools for finding estimations. They can be applied in different fields of mathematics such as fractional calculus and discrete fractional calculus, etc. (see [1–4]).

Convexity study is crucial regarding theoretical behavior of mathematical inequalities. For some other theoretical studies of inequalities on different types of convex functions, see, e.g., GA-convex [5], MT-convex [6], (α, m) -convex [7], a generalized class of convexity [8], and many other types can be found in [9].

Definition 1 ([9]). A function $\Theta : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if

$$\Theta(v\omega_1 + (1-v)\omega_2) \leq v\Theta(\omega_1) + (1-v)\Theta(\omega_2)$$

holds for all $\omega_1, \omega_2 \in I$ and $v \in [0, 1]$. Likewise, Θ is concave if $-\Theta$ is convex.

Definition 2 ([10,11]). Let $n \in \mathbb{N}$. A nonnegative function $\Theta : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called n -polynomial convex if

$$\Theta(v\omega_1 + (1 - v)\omega_2) \leq \frac{1}{n} \sum_{\ell=1}^n (1 - (1 - v)^\ell) \Theta(\omega_1) + \frac{1}{n} \sum_{\ell=1}^n (1 - v^\ell) \Theta(\omega_2)$$

holds for all $\omega_1, \omega_2 \in I$ and $v \in [0, 1]$.

Remark 1. From [10], every nonnegative convex function is also an n -polynomial convex function. Moreover, we get the convex function by taking $n = 1$ in Definition 2.

Symmetry has a significant role in integral inequality models with convexity. Furthermore, for convex functions and their types, many basic inequalities are found, such as Hermite–Hadamard type [12], Hermite–Hadamard–Fejér type [13], Ostrowski type [14–16], Simpson type [17,18], Hardy type [19], and Olsen type [20].

In this paper, we are focused in Ostrowski and midpoint type inequalities as follows.

Theorem 1 ([21]). (Ostrowski type inequality) Assume that $\Theta : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable function on I and let $\omega_1, \omega_2 \in I^\circ$ (the interior of I) with $\omega_1 < \omega_2$. If $|\Theta'(\xi)| \leq \mathcal{M}$ for all $\xi \in [\omega_1, \omega_2]$, then the following inequality holds true:

$$\left| \Theta(\xi) - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \Theta(v) dv \right| \leq \mathcal{M}(\omega_2 - \omega_1) \left[\frac{1}{4} + \frac{\left(\xi - \frac{\omega_1 + \omega_2}{2}\right)^2}{(\omega_2 - \omega_1)^2} \right]. \tag{1}$$

Choosing $\xi = \frac{\omega_1 + \omega_2}{2}$ in (1), we get the following midpoint type inequality:

$$\left| \Theta\left(\frac{\omega_1 + \omega_2}{2}\right) - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \Theta(v) dv \right| \leq \frac{\mathcal{M}(\omega_2 - \omega_1)}{4}. \tag{2}$$

Let us recall some published papers about above inequalities using quantum calculus that inspired us.

The \dot{q} -analogue of the trapezium’s inequality was discovered by Tariboon and Ntouyas [22] using the concepts of quantum calculus, also known as calculus without limits, on the finite intervals. See [23] for more information on how to get classical calculus by taking $\dot{q} \rightarrow 1^-$. An updated version of the \dot{q} -analogue of the inequality of the trapezium was discovered by Alp et al. [24]. Meanwhile, \dot{q} -analogues of trapezium-like inequalities involving first order \dot{q} -differentiable convex functions were deduced by Sudsutad et al. [25] and Noor et al. [26]. These analogues were created by Liu and Zhuang [27] by using twice \dot{q} -differentiable convex functions. Budak et al. [28] are able to further develop certain quantum Hermite–Hadamard-type inequality. Ali et al. [29] presented some quantum Ostrowski-type inequalities for twice quantum differentiable functions. Convexity was used by Butt et al. [30] to generate some new quantum Simpson–Newton-like estimates in the frame of Mercer type inequalities. Aljinović et al. [31] established Ostrowski inequality for quantum calculus. Wang et al. [32] developed new Ostrowski-type inequalities via \dot{q} -fractional integrals involving s -convex functions. To the best of our knowledge, the recently published paper from Wang et al. [32] is the first one working in this new direction that mixed together the fractional calculus with quantum calculus. Inspired by it, we will try to give some new quantum fractional integral inequalities of Ostrowski and midpoint type.

The article is set up as follows. The purpose of the Section 2 is to review several earlier findings of fractional calculus and \dot{q} -calculus to provide the main interpretation of this article. In Section 3, we look at proving a new parameterized quantum fractional integral identity and demonstrate some integral inequalities of Ostrowski and midpoint type via n -polynomial convex functions. From our main results, we will discuss in detail

several special cases. In Section 4, we offer an example and an application to special means of positive real numbers in order to show the efficiency of our theoretical results. The conclusion and future research will be given in Section 5.

2. Preliminaries

Let us denote, respectively, $\mathcal{L}[\omega_1, \omega_2]$ the set of all Lebesgue integrable functions on $[\omega_1, \omega_2]$ and $\mathcal{C}[\omega_1, \omega_2]$ the set of all differentiable continuous functions on $[\omega_1, \omega_2]$.

2.1. Fractional Calculus

Definition 3. Let $\alpha > 0, 0 \leq \omega_1 < \omega_2$ and $\Theta \in \mathcal{L}[\omega_1, \omega_2]$. Then the Riemann–Liouville fractional integral operators of order α are defined by

$$J_{\omega_1^+}^\alpha \Theta(\xi) = \frac{1}{\Gamma(\alpha)} \int_{\omega_1}^\xi (\xi - v)^{\alpha-1} \Theta(v) dv, \quad \omega_1 < \xi \tag{3}$$

and

$$J_{\omega_2^-}^\alpha \Theta(\xi) = \frac{1}{\Gamma(\alpha)} \int_\xi^{\omega_2} (v - \xi)^{\alpha-1} \Theta(v) dv, \quad \xi < \omega_2,$$

where $\Gamma(\cdot)$ is gamma function, defined by

$$\Gamma(\alpha) = \int_0^\infty v^{\alpha-1} e^{-v} dv, \quad \Gamma(\alpha + 1) = \alpha \Gamma(\alpha).$$

For $\alpha = 1$, we get the classical Riemann integrals.

2.2. Quantum Calculus

Throughout the remaining paper, let us consider $0 < q < 1$ as a constant.

Definition 4 ([23]). For $\Theta \in \mathcal{C}[\omega_1, \omega_2]$, the left q -derivative of Θ at $\xi \in [\omega_1, \omega_2]$ is given by

$${}_{\omega_1} D_q \Theta(\xi) = \frac{\Theta(\xi) - \Theta(q\xi + (1 - q)\omega_1)}{(1 - q)(\xi - \omega_1)}, \quad \xi \neq \omega_1. \tag{4}$$

The function Θ is said to be q -differentiable on $[\omega_1, \omega_2]$ if ${}_{\omega_1} D_q \Theta(\xi)$ exists for all $\xi \in [\omega_1, \omega_2]$. If we choose $\omega_1 = 0$, then we will use the notation ${}_{\omega_1} D_q \Theta(\xi) = D_q \Theta(\xi)$, which is the q -Jackson derivative [23,24,33] for more details.

The q -integer is expressed as follows:

$$[n]_q := \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \dots + q^{n-1}, \quad n \in \mathbb{N}, q \in (0, 1).$$

The following q -integral along with its properties can be studied in [24].

Definition 5. Suppose that $\Theta \in \mathcal{C}[\omega_1, \omega_2]$. Then q -definite integral for $\xi \in [\omega_1, \omega_2]$ is defined as

$$\int_{\omega_1}^\xi \Theta(v) {}_{\omega_1} d_q v = (1 - q)(\xi - \omega_1) \sum_{r=0}^\infty q^r \Theta(q^r \xi + (1 - q^r)\omega_1). \tag{5}$$

Choosing $\omega_1 = 0$ in (5), we have

$$\int_0^\xi \Theta(v) d_q v = (1 - q)\xi \sum_{r=0}^\infty q^r \Theta(q^r \xi),$$

which gives

$$\int_0^1 v^{\alpha+s} d_{\dot{q}}v = \frac{1}{[\alpha + s + 1]_{\dot{q}}}, \quad \int_0^1 v^\alpha (1 - v)^s d_{\dot{q}}v = \frac{\Gamma_{\dot{q}}(\alpha + 1)\Gamma_{\dot{q}}(s + 1)}{\Gamma_{\dot{q}}(\alpha + s + 2)},$$

where the \dot{q} -gamma function for $\xi > 0$ is defined by

$$\Gamma_{\dot{q}}(\xi) = \int_0^\infty v^{\xi-1} \mathcal{E}_{\dot{q}}^{-\dot{q}v} d_{\dot{q}}v, \\ \Gamma_{\dot{q}}(\xi + 1) = [\xi]_{\dot{q}} \Gamma_{\dot{q}}(\xi)$$

and the \dot{q} -exponential function is given as

$$\mathcal{E}_{\dot{q}}^v = \sum_{r=0}^\infty \dot{q}^{\frac{r(r-1)}{2}} \frac{v^r}{[r]_{\dot{q}}!}.$$

The following \dot{q} -fractional integrals can be studied in [34].

Definition 6. Let $\alpha > 0, 0 \leq \omega_1 < \omega_2$ and $\Theta \in \mathcal{L}[\omega_1, \omega_2]$. Then the Riemann–Liouville \dot{q} -fractional integrals of order α are defined by

$$J_{\dot{q}, \omega_1^+}^\alpha \Theta(\xi) = \frac{1}{\Gamma_{\dot{q}}(\alpha)} \int_{\omega_1}^\xi (\xi - \dot{q}v)^{\alpha-1} \Theta(v)_{\omega_1} d_{\dot{q}}v, \quad \omega_1 < \xi \tag{6}$$

and

$$J_{\dot{q}, \omega_2^-}^\alpha \Theta(\xi) = \frac{1}{\Gamma_{\dot{q}}(\alpha)} \int_{\dot{q}\xi}^{\omega_2} (v - \dot{q}\xi)^{\alpha-1} \Theta(v) d_{\dot{q}}v, \quad \xi < \omega_2,$$

where $\Gamma_{\dot{q}}(\cdot)$ is \dot{q} -gamma function. For $\dot{q} \rightarrow 1^-$, we get the Riemann–Liouville fractional integral operators.

Theorem 2 ([22]). (\dot{q} -integration by parts) Let $\Theta_1, \Theta_2 \in \mathcal{C}[\omega_1, \omega_2]$, then for all $\xi \in [\omega_1, \omega_2]$, and we have

$$\int_{\omega_1}^\xi \Theta_1(v)_{\omega_1} D_{\dot{q}} \Theta_2(v)_{\omega_1} d_{\dot{q}}v = \Theta_1(\xi) \Theta_2(\xi) - \Theta_1(\omega_1) \Theta_2(\omega_1) \\ - \int_{\omega_1}^\xi \Theta_2(\dot{q}v + (1 - \dot{q})\omega_1)_{\omega_1} D_{\dot{q}} \Theta_1(v)_{\omega_1} d_{\dot{q}}v. \tag{7}$$

Theorem 3 ([35]). (\dot{q} -Hölder’s inequality) Let Θ_1, Θ_2 be two \dot{q} -integrable functions on $[\omega_1, \omega_2]$ such that $p, \dot{q}_* > 1$, and $\frac{1}{p} + \frac{1}{\dot{q}_*} = 1$, and then we have

$$\int_{\omega_1}^{\omega_2} |\Theta_1(v) \Theta_2(v)|_{\omega_1} d_{\dot{q}}v \leq \left(\int_{\omega_1}^{\omega_2} |\Theta_1(v)|^p_{\omega_1} d_{\dot{q}}v \right)^{\frac{1}{p}} \left(\int_{\omega_1}^{\omega_2} |\Theta_2(v)|^{\dot{q}_*}_{\omega_1} d_{\dot{q}}v \right)^{\frac{1}{\dot{q}_*}}. \tag{8}$$

Theorem 4 ([35]). (\dot{q} -power mean inequality) Let Θ_1, Θ_2 be two \dot{q} -integrable functions on $[\omega_1, \omega_2]$ such that $\dot{q}_* \geq 1$, and then we have

$$\int_{\omega_1}^{\omega_2} |\Theta_1(v) \Theta_2(v)|_{\omega_1} d_{\dot{q}}v \leq \left(\int_{\omega_1}^{\omega_2} |\Theta_1(v)|_{\omega_1} d_{\dot{q}}v \right)^{1 - \frac{1}{\dot{q}_*}} \left(\int_{\omega_1}^{\omega_2} |\Theta_1(v)| |\Theta_2(v)|^{\dot{q}_*}_{\omega_1} d_{\dot{q}}v \right)^{\frac{1}{\dot{q}_*}}. \tag{9}$$

3. Main Results

For the simplicity of notations, let

$$\delta(\xi, \alpha) := \int_0^1 |\xi - v^\alpha| dv, \quad \rho(\xi, p, \alpha) := \int_0^1 |\xi - v^\alpha|^p dv.$$

Let us recall the well-known beta and hypergeometric functions below:

$$\beta(x, y) := \int_0^1 v^{x-1}(1-v)^{y-1}dv, \quad x, y > 0$$

and

$${}_2F_1(\omega_1, \omega_2; \tau; z) := \frac{1}{\beta(\omega_2, \tau - \omega_2)} \int_0^1 v^{\omega_2-1}(1-v)^{\tau-\omega_2-1}(1-zv)^{-\omega_1}dv$$

for $\Re(\tau) > \Re(\omega_2) > 0$, and $|z| \leq 1$.

To establish our main results, we need the following lemmas.

Lemma 1. For $\alpha > 0$ and $0 \leq \xi \leq 1$, we have

$$\delta(\xi, \alpha) := \begin{cases} \frac{1}{\alpha + 1}, & \text{for } \xi = 0; \\ \frac{2\alpha\xi^{1+\frac{1}{\alpha}} + 1}{\alpha + 1} - \xi, & \text{for } 0 < \xi < 1; \\ \frac{\alpha}{\alpha + 1}, & \text{for } \xi = 1. \end{cases}$$

Proof. The proof is evident. \square

Lemma 2. For $\alpha > 0$, $p \geq 1$ and $0 \leq \xi \leq 1$, we have

$$\rho(\xi, p, \alpha) := \begin{cases} \frac{1}{p\alpha + 1}, & \text{for } \xi = 0; \\ \frac{\xi^{p+\frac{1}{\alpha}}}{\alpha} \beta\left(\frac{1}{\alpha}, p + 1\right) + \frac{(1-\xi)^{p+1}}{\alpha(p+1)} {}_2F_1\left(1 - \frac{1}{\alpha}, 1; p + 2; 1 - \xi\right), & \text{for } 0 < \xi < 1; \\ \frac{1}{\alpha} \beta\left(\frac{1}{\alpha}, p + 1\right), & \text{for } \xi = 1. \end{cases}$$

Proof. The proof is a straightforward computations. We omit here their details. \square

Lemma 3. Suppose $\Theta : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a \dot{q} -differentiable mapping, where $0 < \dot{q} < 1$, such that $0 \leq \omega_1 < \omega_2$. If $D_{\dot{q}}\Theta(x) \in \mathcal{L}[\omega_1, \omega_2]$ for all $x \in [\omega_1, \omega_2]$, $\theta, \omega \in \mathbb{R}$ and $\alpha \in \mathbb{N}$, and then we have

$$\begin{aligned} & \frac{\theta(\omega_2 - x)^\alpha (\Theta(\omega_2) - \Theta(x)) - \omega(x - \omega_1)^\alpha (\Theta(x) - \Theta(\omega_1))}{\omega_2 - \omega_1} + \left(\frac{(x - \omega_1)^\alpha + (\omega_2 - x)^\alpha}{\omega_2 - \omega_1} \right) \Theta(x) \\ & + \frac{[\alpha]_{\dot{q}}(1 - \dot{q})}{\dot{q}^\alpha(\omega_2 - \omega_1)} ((x - \omega_1)^\alpha + (\omega_2 - x)^\alpha) \Theta(x) - \frac{\Gamma_{\dot{q}}(\alpha + 1)}{\dot{q}^\alpha(\omega_2 - \omega_1)} \left[J_{\dot{q}, x^+}^\alpha \Theta(\omega_2) + J_{\dot{q}, x^-}^\alpha \Theta(\omega_1) \right] \\ & = \frac{(\omega_2 - x)^{\alpha+1}}{\omega_2 - \omega_1} \int_0^1 (\theta - v^\alpha) D_{\dot{q}}\Theta(vx + (1 - v)\omega_2) d_{\dot{q}}v \\ & - \frac{(x - \omega_1)^{\alpha+1}}{\omega_2 - \omega_1} \int_0^1 (\omega - v^\alpha) D_{\dot{q}}\Theta((1 - v)\omega_1 + vx) d_{\dot{q}}v. \end{aligned} \tag{10}$$

Proof. Let us denote, respectively,

$$I_1 := \int_0^1 (\theta - v^\alpha) D_{\dot{q}}\Theta(vx + (1 - v)\omega_2) d_{\dot{q}}v$$

and

$$I_2 := \int_0^1 (\omega - v^\alpha) D_{\dot{q}}\Theta((1 - v)\omega_1 + vx) d_{\dot{q}}v.$$

With the help of \dot{q} -integration by parts, we have

$$\begin{aligned}
 I_1 &= \theta \int_0^1 D_{\dot{q}}\Theta(vx + (1 - v)\omega_2)d_{\dot{q}}v - \int_0^1 v^\alpha D_{\dot{q}}\Theta(vx + (1 - v)\omega_2)d_{\dot{q}}v \\
 &= \frac{\theta}{\omega_2 - x} \int_x^{\omega_2} {}_x D_{\dot{q}}\Theta(v) {}_x d_{\dot{q}}v - v^\alpha \frac{\Theta(vx + (1 - v)\omega_2)}{x - \omega_2} \Big|_0^1 \\
 &\quad + \frac{[\alpha]_{\dot{q}}}{x - \omega_2} \int_0^1 v^{\alpha-1} \Theta(\dot{q}vx + (1 - \dot{q}v)\omega_2)d_{\dot{q}}v \\
 &= \frac{\theta}{\omega_2 - x} (\Theta(\omega_2) - \Theta(x)) + \frac{\Theta(x)}{\omega_2 - x} + \frac{[\alpha]_{\dot{q}}(1 - \dot{q})}{\dot{q}^\alpha(\omega_2 - x)} \Theta(x) \\
 &\quad - \frac{[\alpha]_{\dot{q}}\Gamma_{\dot{q}}(\alpha)}{\dot{q}^\alpha(\omega_2 - x)^{\alpha+1}} \frac{1}{\Gamma_{\dot{q}}(\alpha)} \int_x^{\omega_2} (\omega_2 - \dot{q}v)^{\alpha-1} \Theta(v)d_{\dot{q}}v \\
 &= \frac{\theta}{\omega_2 - x} (\Theta(\omega_2) - \Theta(x)) + \frac{\Theta(x)}{\omega_2 - x} + \frac{[\alpha]_{\dot{q}}(1 - \dot{q})}{\dot{q}^\alpha(\omega_2 - x)} \Theta(x) \\
 &\quad - \frac{\Gamma_{\dot{q}}(\alpha + 1)}{\dot{q}^\alpha(\omega_2 - x)^{\alpha+1}} J_{\dot{q},x^+}^\alpha \Theta(\omega_2).
 \end{aligned} \tag{11}$$

Similarly, we get

$$\begin{aligned}
 I_2 &= \omega \int_0^1 D_{\dot{q}}\Theta((1 - v)\omega_1 + vx)d_{\dot{q}}v - \int_0^1 v^\alpha D_{\dot{q}}\Theta((1 - v)\omega_1 + vx)d_{\dot{q}}v \\
 &= \frac{\omega}{x - \omega_1} (\Theta(x) - \Theta(\omega_1)) - \frac{\Theta(x)}{x - \omega_1} - \frac{[\alpha]_{\dot{q}}(1 - \dot{q})}{\dot{q}^\alpha(x - \omega_1)} \Theta(x) \\
 &\quad + \frac{\Gamma_{\dot{q}}(\alpha + 1)}{\dot{q}^\alpha(x - \omega_1)^{\alpha+1}} J_{\dot{q},x^-}^\alpha \Theta(\omega_1).
 \end{aligned} \tag{12}$$

By multiplying (11) by $\frac{(\omega_2-x)^{\alpha+1}}{\omega_2-\omega_1}$ and (12) with $\frac{(x-\omega_1)^{\alpha+1}}{\omega_2-\omega_1}$ and subtracting them, we acquire (10). \square

Remark 2. Considering $\dot{q} \rightarrow 1^-$ in Lemma 3, we have

$$\begin{aligned}
 &\frac{\theta(\omega_2 - x)^\alpha (\Theta(\omega_2) - \Theta(x)) - \omega(x - \omega_1)^\alpha (\Theta(x) - \Theta(\omega_1))}{\omega_2 - \omega_1} + \left(\frac{(x - \omega_1)^\alpha + (\omega_2 - x)^\alpha}{\omega_2 - \omega_1} \right) \Theta(x) \\
 &\quad - \frac{\Gamma(\alpha + 1)}{\omega_2 - \omega_1} [J_{x^+}^\alpha \Theta(\omega_2) + J_{x^-}^\alpha \Theta(\omega_1)] \\
 &= \frac{(\omega_2 - x)^{\alpha+1}}{\omega_2 - \omega_1} \int_0^1 (\theta - v^\alpha) \Theta'(vx + (1 - v)\omega_2) dv \\
 &\quad - \frac{(x - \omega_1)^{\alpha+1}}{\omega_2 - \omega_1} \int_0^1 (\omega - v^\alpha) \Theta'((1 - v)\omega_1 + vx) dv.
 \end{aligned} \tag{13}$$

Remark 3. Choosing $\alpha = 1$ in Lemma 3, we get the following \dot{q} -identity:

$$\begin{aligned}
 &\frac{\theta(\omega_2 - x)(\Theta(\omega_2) - \Theta(x)) - \omega(x - \omega_1)(\Theta(x) - \Theta(\omega_1))}{\omega_2 - \omega_1} + \frac{\Theta(x)}{\dot{q}} - \frac{1}{\dot{q}(\omega_2 - \omega_1)} \int_{\dot{q}\omega_1}^{\omega_2} \Theta(v) {}_{\omega_1} d_{\dot{q}}v \\
 &= \frac{(\omega_2 - x)^2}{\omega_2 - \omega_1} \int_0^1 (\theta - v) D_{\dot{q}}\Theta(vx + (1 - v)\omega_2) d_{\dot{q}}v \\
 &\quad - \frac{(x - \omega_1)^2}{\omega_2 - \omega_1} \int_0^1 (\omega - v) D_{\dot{q}}\Theta((1 - v)\omega_1 + vx) d_{\dot{q}}v.
 \end{aligned} \tag{14}$$

By using Lemmas 1–3, we established the following \dot{q} -fractional integral inequalities.

Theorem 5. Let $\Theta : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a \dot{q} -differentiable mapping, where $0 < \dot{q} < 1$, such that $0 \leq \omega_1 < \omega_2$ and $\theta, \varpi \in [0, 1]$. If $D_{\dot{q}}\Theta(x) \in \mathcal{L}[\omega_1, \omega_2]$ for every $x \in [\omega_1, \omega_2]$ and $|D_{\dot{q}}\Theta(x)|^{\dot{q}^*}$ is n -polynomial convex function for all $n \in \mathbb{N}$, such that $p, \dot{q}^* > 1$, and $\frac{1}{p} + \frac{1}{\dot{q}^*} = 1$, and then for $\alpha \in \mathbb{N}$, the following \dot{q} -fractional integral inequality holds true:

$$\begin{aligned} & \left| \frac{\theta(\omega_2 - x)^\alpha (\Theta(\omega_2) - \Theta(x)) - \varpi(x - \omega_1)^\alpha (\Theta(x) - \Theta(\omega_1))}{\omega_2 - \omega_1} + \left(\frac{(x - \omega_1)^\alpha + (\omega_2 - x)^\alpha}{\omega_2 - \omega_1} \right) \Theta(x) \right. \\ & \left. + \frac{[\alpha]_{\dot{q}}(1 - \dot{q})}{\dot{q}^\alpha(\omega_2 - \omega_1)} ((x - \omega_1)^\alpha + (\omega_2 - x)^\alpha) \Theta(x) - \frac{\Gamma_{\dot{q}}(\alpha + 1)}{\dot{q}^\alpha(\omega_2 - \omega_1)} [J_{\dot{q}, x^+}^\alpha \Theta(\omega_2) + J_{\dot{q}, x^-}^\alpha \Theta(\omega_1)] \right| \\ & \leq \frac{1}{\omega_2 - \omega_1} \left(1 - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{[\ell + 1]_{\dot{q}}} \right)^{\frac{1}{\dot{q}^*}} \left\{ (\omega_2 - x)^{\alpha+1} A_{\dot{q}}^{\frac{1}{p}}(\theta, p, \alpha) [|D_{\dot{q}}\Theta(x)|^{\dot{q}^*} + |D_{\dot{q}}\Theta(\omega_2)|^{\dot{q}^*}]^{\frac{1}{\dot{q}^*}} \right. \\ & \left. + (x - \omega_1)^{\alpha+1} A_{\dot{q}}^{\frac{1}{p}}(\varpi, p, \alpha) [|D_{\dot{q}}\Theta(x)|^{\dot{q}^*} + |D_{\dot{q}}\Theta(\omega_1)|^{\dot{q}^*}]^{\frac{1}{\dot{q}^*}} \right\}, \end{aligned} \tag{15}$$

where

$$A_{\dot{q}}(\theta, p, \alpha) := \int_0^1 |\theta - v^\alpha|^p d_{\dot{q}}v.$$

Proof. By using Lemma 3, \dot{q} -Hölder’s inequality, n -polynomial convexity of $|D_{\dot{q}}\Theta(x)|^{\dot{q}^*}$ and properties of modulus, we have

$$\begin{aligned} & \left| \frac{\theta(\omega_2 - x)^\alpha (\Theta(\omega_2) - \Theta(x)) - \varpi(x - \omega_1)^\alpha (\Theta(x) - \Theta(\omega_1))}{\omega_2 - \omega_1} + \left(\frac{(x - \omega_1)^\alpha + (\omega_2 - x)^\alpha}{\omega_2 - \omega_1} \right) \Theta(x) \right. \\ & \left. + \frac{[\alpha]_{\dot{q}}(1 - \dot{q})}{\dot{q}^\alpha(\omega_2 - \omega_1)} ((x - \omega_1)^\alpha + (\omega_2 - x)^\alpha) \Theta(x) - \frac{\Gamma_{\dot{q}}(\alpha + 1)}{\dot{q}^\alpha(\omega_2 - \omega_1)} [J_{\dot{q}, x^+}^\alpha \Theta(\omega_2) + J_{\dot{q}, x^-}^\alpha \Theta(\omega_1)] \right| \\ & \leq \frac{(\omega_2 - x)^{\alpha+1}}{\omega_2 - \omega_1} \int_0^1 |\theta - v^\alpha| |D_{\dot{q}}\Theta(vx + (1 - v)\omega_2)| d_{\dot{q}}v \\ & \quad + \frac{(x - \omega_1)^{\alpha+1}}{\omega_2 - \omega_1} \int_0^1 |\varpi - v^\alpha| |D_{\dot{q}}\Theta((1 - v)\omega_1 + vx)| d_{\dot{q}}v \\ & \leq \frac{(\omega_2 - x)^{\alpha+1}}{\omega_2 - \omega_1} \left(\int_0^1 |\theta - v^\alpha|^p d_{\dot{q}}v \right)^{\frac{1}{p}} \left(\int_0^1 |D_{\dot{q}}\Theta(vx + (1 - v)\omega_2)|^{\dot{q}^*} d_{\dot{q}}v \right)^{\frac{1}{\dot{q}^*}} \\ & \quad + \frac{(x - \omega_1)^{\alpha+1}}{\omega_2 - \omega_1} \left(\int_0^1 |\varpi - v^\alpha|^p d_{\dot{q}}v \right)^{\frac{1}{p}} \left(\int_0^1 |D_{\dot{q}}\Theta((1 - v)\omega_1 + vx)|^{\dot{q}^*} d_{\dot{q}}v \right)^{\frac{1}{\dot{q}^*}} \\ & \leq \frac{(\omega_2 - x)^{\alpha+1}}{\omega_2 - \omega_1} A_{\dot{q}}^{\frac{1}{p}}(\theta, p, \alpha) \left[\int_0^1 \frac{1}{n} \sum_{\ell=1}^n (1 - (1 - v)^\ell) |D_{\dot{q}}\Theta(x)|^{\dot{q}^*} d_{\dot{q}}v + \int_0^1 \frac{1}{n} \sum_{\ell=1}^n (1 - v^\ell) |D_{\dot{q}}\Theta(\omega_2)|^{\dot{q}^*} d_{\dot{q}}v \right]^{\frac{1}{\dot{q}^*}} \\ & \quad + \frac{(x - \omega_1)^{\alpha+1}}{\omega_2 - \omega_1} A_{\dot{q}}^{\frac{1}{p}}(\varpi, p, \alpha) \left[\int_0^1 \frac{1}{n} \sum_{\ell=1}^n (1 - (1 - v)^\ell) |D_{\dot{q}}\Theta(x)|^{\dot{q}^*} d_{\dot{q}}v + \int_0^1 \frac{1}{n} \sum_{\ell=1}^n (1 - v^\ell) |D_{\dot{q}}\Theta(\omega_1)|^{\dot{q}^*} d_{\dot{q}}v \right]^{\frac{1}{\dot{q}^*}} \\ & = \frac{1}{\omega_2 - \omega_1} \left(1 - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{[\ell + 1]_{\dot{q}}} \right)^{\frac{1}{\dot{q}^*}} \left\{ (\omega_2 - x)^{\alpha+1} A_{\dot{q}}^{\frac{1}{p}}(\theta, p, \alpha) [|D_{\dot{q}}\Theta(x)|^{\dot{q}^*} + |D_{\dot{q}}\Theta(\omega_2)|^{\dot{q}^*}]^{\frac{1}{\dot{q}^*}} \right. \\ & \left. + (x - \omega_1)^{\alpha+1} A_{\dot{q}}^{\frac{1}{p}}(\varpi, p, \alpha) [|D_{\dot{q}}\Theta(x)|^{\dot{q}^*} + |D_{\dot{q}}\Theta(\omega_1)|^{\dot{q}^*}]^{\frac{1}{\dot{q}^*}} \right\}. \end{aligned}$$

This concludes the desired proof. \square

Corollary 1. Suppose $\dot{q} \rightarrow 1^-$ in Theorem 5. Then we have

$$\begin{aligned} & \left| \frac{\theta(\omega_2 - x)^\alpha (\Theta(\omega_2) - \Theta(x)) - \varpi(x - \omega_1)^\alpha (\Theta(x) - \Theta(\omega_1))}{\omega_2 - \omega_1} + \left(\frac{(x - \omega_1)^\alpha + (\omega_2 - x)^\alpha}{\omega_2 - \omega_1} \right) \Theta(x) \right. \\ & \left. - \frac{\Gamma(\alpha + 1)}{\omega_2 - \omega_1} [J_{x^+}^\alpha \Theta(\omega_2) + J_{x^-}^\alpha \Theta(\omega_1)] \right| \\ & \leq \frac{1}{\omega_2 - \omega_1} \left(1 - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{\ell + 1} \right)^{\frac{1}{\dot{q}^*}} \left\{ (\omega_2 - x)^{\alpha+1} \rho^{\frac{1}{p}}(\theta, p, \alpha) [|\Theta'(x)|^{\dot{q}^*} + |\Theta'(\omega_2)|^{\dot{q}^*}]^{\frac{1}{\dot{q}^*}} \right. \\ & \left. + (x - \omega_1)^{\alpha+1} \rho^{\frac{1}{p}}(\varpi, p, \alpha) [|\Theta'(x)|^{\dot{q}^*} + |\Theta'(\omega_1)|^{\dot{q}^*}]^{\frac{1}{\dot{q}^*}} \right\}. \end{aligned} \tag{16}$$

Corollary 2. Considering $\alpha = 1$ in Theorem 5, we get

$$\begin{aligned} & \left| \frac{\theta(\omega_2 - x)(\Theta(\omega_2) - \Theta(x)) - \varpi(x - \omega_1)(\Theta(x) - \Theta(\omega_1))}{\omega_2 - \omega_1} + \frac{\Theta(x)}{\dot{q}} - \frac{1}{\dot{q}(\omega_2 - \omega_1)} \int_{\dot{q}\omega_1}^{\omega_2} \Theta(v) \omega_1 d_{\dot{q}} v \right| \\ & \leq \frac{1}{\omega_2 - \omega_1} \left(1 - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{[\ell + 1]_{\dot{q}}} \right)^{\frac{1}{\dot{q}^*}} \left\{ (\omega_2 - x)^2 A_{\dot{q}}^{\frac{1}{p}}(\theta, p) [|\mathbb{D}_{\dot{q}}\Theta(x)|^{\dot{q}^*} + |\mathbb{D}_{\dot{q}}\Theta(\omega_2)|^{\dot{q}^*}]^{\frac{1}{\dot{q}^*}} \right. \\ & \left. + (x - \omega_1)^2 A_{\dot{q}}^{\frac{1}{p}}(\varpi, p) [|\mathbb{D}_{\dot{q}}\Theta(x)|^{\dot{q}^*} + |\mathbb{D}_{\dot{q}}\Theta(\omega_1)|^{\dot{q}^*}]^{\frac{1}{\dot{q}^*}} \right\}, \end{aligned} \tag{17}$$

where

$$A_{\dot{q}}(\theta, p) := \int_0^1 |\theta - v|^p d_{\dot{q}} v.$$

Corollary 3. Taking $|\mathbb{D}_{\dot{q}}\Theta| \leq \mathcal{K}$ in Theorem 5, we obtain

$$\begin{aligned} & \left| \frac{\theta(\omega_2 - x)^\alpha (\Theta(\omega_2) - \Theta(x)) - \varpi(x - \omega_1)^\alpha (\Theta(x) - \Theta(\omega_1))}{\omega_2 - \omega_1} + \left(\frac{(x - \omega_1)^\alpha + (\omega_2 - x)^\alpha}{\omega_2 - \omega_1} \right) \Theta(x) \right. \\ & \left. + \frac{[\alpha]_{\dot{q}}(1 - \dot{q})}{\dot{q}^\alpha(\omega_2 - \omega_1)} ((x - \omega_1)^\alpha + (\omega_2 - x)^\alpha) \Theta(x) - \frac{\Gamma_{\dot{q}}(\alpha + 1)}{\dot{q}^\alpha(\omega_2 - \omega_1)} [J_{\dot{q},x^+}^\alpha \Theta(\omega_2) + J_{\dot{q},x^-}^\alpha \Theta(\omega_1)] \right| \\ & \leq \frac{2^{\frac{1}{\dot{q}^*}} \mathcal{K}}{\omega_2 - \omega_1} \left(1 - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{[\ell + 1]_{\dot{q}}} \right)^{\frac{1}{\dot{q}^*}} \left\{ (\omega_2 - x)^{\alpha+1} A_{\dot{q}}^{\frac{1}{p}}(\theta, p, \alpha) + (x - \omega_1)^{\alpha+1} A_{\dot{q}}^{\frac{1}{p}}(\varpi, p, \alpha) \right\}. \end{aligned} \tag{18}$$

Corollary 4. Considering $n = 1$ in Theorem 5, we have

$$\begin{aligned} & \left| \frac{\theta(\omega_2 - x)^\alpha (\Theta(\omega_2) - \Theta(x)) - \varpi(x - \omega_1)^\alpha (\Theta(x) - \Theta(\omega_1))}{\omega_2 - \omega_1} + \left(\frac{(x - \omega_1)^\alpha + (\omega_2 - x)^\alpha}{\omega_2 - \omega_1} \right) \Theta(x) \right. \\ & \left. + \frac{[\alpha]_{\dot{q}}(1 - \dot{q})}{\dot{q}^\alpha(\omega_2 - \omega_1)} ((x - \omega_1)^\alpha + (\omega_2 - x)^\alpha) \Theta(x) - \frac{\Gamma_{\dot{q}}(\alpha + 1)}{\dot{q}^\alpha(\omega_2 - \omega_1)} [J_{\dot{q},x^+}^\alpha \Theta(\omega_2) + J_{\dot{q},x^-}^\alpha \Theta(\omega_1)] \right| \\ & \leq \frac{1}{\omega_2 - \omega_1} \left(\frac{\dot{q}}{1 + \dot{q}} \right)^{\frac{1}{\dot{q}^*}} \left\{ (\omega_2 - x)^{\alpha+1} A_{\dot{q}}^{\frac{1}{p}}(\theta, p, \alpha) [|\mathbb{D}_{\dot{q}}\Theta(x)|^{\dot{q}^*} + |\mathbb{D}_{\dot{q}}\Theta(\omega_2)|^{\dot{q}^*}]^{\frac{1}{\dot{q}^*}} \right. \\ & \left. + (x - \omega_1)^{\alpha+1} A_{\dot{q}}^{\frac{1}{p}}(\varpi, p, \alpha) [|\mathbb{D}_{\dot{q}}\Theta(x)|^{\dot{q}^*} + |\mathbb{D}_{\dot{q}}\Theta(\omega_1)|^{\dot{q}^*}]^{\frac{1}{\dot{q}^*}} \right\}. \end{aligned} \tag{19}$$

Corollary 5. Taking $\theta = \varpi$ in Theorem 5, we get

$$\begin{aligned}
 & \left| \frac{\theta[(\omega_2 - x)^\alpha(\Theta(\omega_2) - \Theta(x)) - (x - \omega_1)^\alpha(\Theta(x) - \Theta(\omega_1))]}{\omega_2 - \omega_1} + \left(\frac{(x - \omega_1)^\alpha + (\omega_2 - x)^\alpha}{\omega_2 - \omega_1} \right) \Theta(x) \right. \\
 & \left. + \frac{[\alpha]_{\dot{q}}(1 - \dot{q})}{\dot{q}^\alpha(\omega_2 - \omega_1)} ((x - \omega_1)^\alpha + (\omega_2 - x)^\alpha)\Theta(x) - \frac{\Gamma_{\dot{q}}(\alpha + 1)}{\dot{q}^\alpha(\omega_2 - \omega_1)} [J_{\dot{q},x^+}^\alpha \Theta(\omega_2) + J_{\dot{q},x^-}^\alpha \Theta(\omega_1)] \right| \\
 & \leq \frac{1}{\omega_2 - \omega_1} \left(1 - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{[\ell + 1]_{\dot{q}}} \right)^{\frac{1}{\dot{q}^*}} A_{\dot{q}}^{\frac{1}{p}}(\theta, p, \alpha) \left\{ (\omega_2 - x)^{\alpha+1} [|\mathcal{D}_{\dot{q}}\Theta(x)|^{\dot{q}^*} + |\mathcal{D}_{\dot{q}}\Theta(\omega_2)|^{\dot{q}^*}]^{\frac{1}{\dot{q}^*}} \right. \\
 & \left. + (x - \omega_1)^{\alpha+1} [|\mathcal{D}_{\dot{q}}\Theta(x)|^{\dot{q}^*} + |\mathcal{D}_{\dot{q}}\Theta(\omega_1)|^{\dot{q}^*}]^{\frac{1}{\dot{q}^*}} \right\}. \tag{20}
 \end{aligned}$$

Corollary 6. Choosing $\theta = 0$ and $\alpha = 1$ in Corollary 5, we obtain the following \dot{q} -integral inequality of Ostrowski type:

$$\begin{aligned}
 & \left| \Theta(x) - \frac{1}{\omega_2 - \omega_1} \int_{\dot{q}\omega_1}^{\omega_2} \Theta(v) {}_{\omega_1}d_{\dot{q}}v \right| \\
 & \leq \frac{\dot{q}}{\omega_2 - \omega_1} \left(1 - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{[\ell + 1]_{\dot{q}}} \right)^{\frac{1}{\dot{q}^*}} \left(\frac{1}{[p + 1]_{\dot{q}}} \right)^{\frac{1}{p}} \left\{ (\omega_2 - x)^2 [|\mathcal{D}_{\dot{q}}\Theta(x)|^{\dot{q}^*} + |\mathcal{D}_{\dot{q}}\Theta(\omega_2)|^{\dot{q}^*}]^{\frac{1}{\dot{q}^*}} \right. \\
 & \left. + (x - \omega_1)^2 [|\mathcal{D}_{\dot{q}}\Theta(x)|^{\dot{q}^*} + |\mathcal{D}_{\dot{q}}\Theta(\omega_1)|^{\dot{q}^*}]^{\frac{1}{\dot{q}^*}} \right\}. \tag{21}
 \end{aligned}$$

Corollary 7. Taking $x = \frac{\omega_1 + \omega_2}{2}$ in Corollary 6, we have the following \dot{q} -integral inequality of midpoint type:

$$\begin{aligned}
 & \left| \Theta\left(\frac{\omega_1 + \omega_2}{2}\right) - \frac{1}{\omega_2 - \omega_1} \int_{\dot{q}\omega_1}^{\omega_2} \Theta(v) {}_{\omega_1}d_{\dot{q}}v \right| \\
 & \leq \frac{\dot{q}(\omega_2 - \omega_1)}{4} \left(1 - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{[\ell + 1]_{\dot{q}}} \right)^{\frac{1}{\dot{q}^*}} \left(\frac{1}{[p + 1]_{\dot{q}}} \right)^{\frac{1}{p}} \left\{ \left[\left| \mathcal{D}_{\dot{q}}\Theta\left(\frac{\omega_1 + \omega_2}{2}\right) \right|^{\dot{q}^*} + |\mathcal{D}_{\dot{q}}\Theta(\omega_1)|^{\dot{q}^*} \right]^{\frac{1}{\dot{q}^*}} \right. \\
 & \left. + \left[\left| \mathcal{D}_{\dot{q}}\Theta\left(\frac{\omega_1 + \omega_2}{2}\right) \right|^{\dot{q}^*} + |\mathcal{D}_{\dot{q}}\Theta(\omega_2)|^{\dot{q}^*} \right]^{\frac{1}{\dot{q}^*}} \right\}. \tag{22}
 \end{aligned}$$

Theorem 6. Suppose $\Theta : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a \dot{q} -differentiable mapping, where $0 < \dot{q} < 1$, such that $0 \leq \omega_1 < \omega_2$ and $\theta, \varpi \in [0, 1]$. If $\mathcal{D}_{\dot{q}}\Theta(x) \in \mathcal{L}[\omega_1, \omega_2]$ for every $x \in [\omega_1, \omega_2]$ and $|\mathcal{D}_{\dot{q}}\Theta(x)|^{\dot{q}^*}$ is n -polynomial convex function for all $n \in \mathbb{N}$, such that $\dot{q}^* \geq 1$, and then for $\alpha \in \mathbb{N}$, the following \dot{q} -fractional integral inequality holds true:

$$\begin{aligned}
 & \left| \frac{\theta(\omega_2 - x)^\alpha(\Theta(\omega_2) - \Theta(x)) - \varpi(x - \omega_1)^\alpha(\Theta(x) - \Theta(\omega_1))}{\omega_2 - \omega_1} + \left(\frac{(x - \omega_1)^\alpha + (\omega_2 - x)^\alpha}{\omega_2 - \omega_1} \right) \Theta(x) \right. \\
 & \left. + \frac{[\alpha]_{\dot{q}}(1 - \dot{q})}{\dot{q}^\alpha(\omega_2 - \omega_1)} ((x - \omega_1)^\alpha + (\omega_2 - x)^\alpha)\Theta(x) - \frac{\Gamma_{\dot{q}}(\alpha + 1)}{\dot{q}^\alpha(\omega_2 - \omega_1)} [J_{\dot{q},x^+}^\alpha \Theta(\omega_2) + J_{\dot{q},x^-}^\alpha \Theta(\omega_1)] \right| \\
 & \leq \frac{(\omega_2 - x)^{\alpha+1}}{\omega_2 - \omega_1} A_{\dot{q}}^{1 - \frac{1}{\dot{q}^*}}(\theta, \alpha) \left[\left(A_{\dot{q}}(\theta, \alpha) - \frac{1}{n} \sum_{\ell=1}^n B_{\dot{q}}(\theta, \alpha, \ell) \right) |\mathcal{D}_{\dot{q}}\Theta(x)|^{\dot{q}^*} \right. \\
 & \left. + \left(A_{\dot{q}}(\theta, \alpha) - \frac{1}{n} \sum_{\ell=1}^n C_{\dot{q}}(\theta, \alpha, \ell) \right) |\mathcal{D}_{\dot{q}}\Theta(\omega_2)|^{\dot{q}^*} \right]^{\frac{1}{\dot{q}^*}} \\
 & + \frac{(x - \omega_1)^{\alpha+1}}{\omega_2 - \omega_1} A_{\dot{q}}^{1 - \frac{1}{\dot{q}^*}}(\varpi, \alpha) \left[\left(A_{\dot{q}}(\varpi, \alpha) - \frac{1}{n} \sum_{\ell=1}^n B_{\dot{q}}(\varpi, \alpha, \ell) \right) |\mathcal{D}_{\dot{q}}\Theta(x)|^{\dot{q}^*} \right. \\
 & \left. + \left(A_{\dot{q}}(\varpi, \alpha) - \frac{1}{n} \sum_{\ell=1}^n C_{\dot{q}}(\varpi, \alpha, \ell) \right) |\mathcal{D}_{\dot{q}}\Theta(\omega_1)|^{\dot{q}^*} \right]^{\frac{1}{\dot{q}^*}}, \tag{23}
 \end{aligned}$$

where

$$A_{\dot{q}}(\theta, \alpha) := \int_0^1 |\theta - v^\alpha| d_{\dot{q}}v,$$

$$B_{\dot{q}}(\theta, \alpha, \ell) := \int_0^1 (1 - v)^\ell |\theta - v^\alpha| d_{\dot{q}}v$$

and

$$C_{\dot{q}}(\theta, \alpha, \ell) := \int_0^1 v^\ell |\theta - v^\alpha| d_{\dot{q}}v.$$

Proof. By using Lemma 3, \dot{q} -power mean inequality, n -polynomial convexity of $|D_{\dot{q}}\Theta(x)|^{\dot{q}_*}$ and properties of modulus, we have

$$\begin{aligned} & \left| \frac{\theta(\omega_2 - x)^\alpha (\Theta(\omega_2) - \Theta(x)) - \omega(x - \omega_1)^\alpha (\Theta(x) - \Theta(\omega_1))}{\omega_2 - \omega_1} + \left(\frac{(x - \omega_1)^\alpha + (\omega_2 - x)^\alpha}{\omega_2 - \omega_1} \right) \Theta(x) \right. \\ & \left. + \frac{[\alpha]_{\dot{q}}(1 - \dot{q})}{\dot{q}^\alpha(\omega_2 - \omega_1)} ((x - \omega_1)^\alpha + (\omega_2 - x)^\alpha) \Theta(x) - \frac{\Gamma_{\dot{q}}(\alpha + 1)}{\dot{q}^\alpha(\omega_2 - \omega_1)} \left[J_{\dot{q}, x^+}^\alpha \Theta(\omega_2) + J_{\dot{q}, x^-}^\alpha \Theta(\omega_1) \right] \right| \\ & \leq \frac{(\omega_2 - x)^{\alpha+1}}{\omega_2 - \omega_1} \int_0^1 |\theta - v^\alpha| |D_{\dot{q}}\Theta(vx + (1 - v)\omega_2)| d_{\dot{q}}v \\ & \quad + \frac{(x - \omega_1)^{\alpha+1}}{\omega_2 - \omega_1} \int_0^1 |\omega - v^\alpha| |D_{\dot{q}}\Theta((1 - v)\omega_1 + vx)| d_{\dot{q}}v \\ & \leq \frac{(\omega_2 - x)^{\alpha+1}}{\omega_2 - \omega_1} \left(\int_0^1 |\theta - v^\alpha| d_{\dot{q}}v \right)^{1 - \frac{1}{\dot{q}_*}} \left(\int_0^1 |\theta - v^\alpha| |D_{\dot{q}}\Theta(vx + (1 - v)\omega_2)|^{\dot{q}_*} d_{\dot{q}}v \right)^{\frac{1}{\dot{q}_*}} \\ & \quad + \frac{(x - \omega_1)^{\alpha+1}}{\omega_2 - \omega_1} \left(\int_0^1 |\omega - v^\alpha| d_{\dot{q}}v \right)^{1 - \frac{1}{\dot{q}_*}} \left(\int_0^1 |\omega - v^\alpha| |D_{\dot{q}}\Theta((1 - v)\omega_1 + vx)|^{\dot{q}_*} d_{\dot{q}}v \right)^{\frac{1}{\dot{q}_*}} \\ & \leq \frac{(\omega_2 - x)^{\alpha+1}}{\omega_2 - \omega_1} A_{\dot{q}}^{1 - \frac{1}{\dot{q}_*}}(\theta, \alpha) \\ & \quad \times \left[\int_0^1 |\theta - v^\alpha| \left(\frac{1}{n} \sum_{\ell=1}^n (1 - (1 - v)^\ell) |D_{\dot{q}}\Theta(x)|^{\dot{q}_*} + \frac{1}{n} \sum_{\ell=1}^n (1 - v^\ell) |D_{\dot{q}}\Theta(\omega_2)|^{\dot{q}_*} \right) d_{\dot{q}}v \right]^{\frac{1}{\dot{q}_*}} \\ & \quad + \frac{(x - \omega_1)^{\alpha+1}}{\omega_2 - \omega_1} A_{\dot{q}}^{1 - \frac{1}{\dot{q}_*}}(\omega, \alpha) \\ & \quad \times \left[\int_0^1 |\omega - v^\alpha| \left(\frac{1}{n} \sum_{\ell=1}^n (1 - (1 - v)^\ell) |D_{\dot{q}}\Theta(x)|^{\dot{q}_*} + \frac{1}{n} \sum_{\ell=1}^n (1 - v^\ell) |D_{\dot{q}}\Theta(\omega_1)|^{\dot{q}_*} \right) d_{\dot{q}}v \right]^{\frac{1}{\dot{q}_*}} \\ & = \frac{(\omega_2 - x)^{\alpha+1}}{\omega_2 - \omega_1} A_{\dot{q}}^{1 - \frac{1}{\dot{q}_*}}(\theta, \alpha) \\ & \quad \times \left[\left(A_{\dot{q}}(\theta, \alpha) - \frac{1}{n} \sum_{\ell=1}^n B_{\dot{q}}(\theta, \alpha, \ell) \right) |D_{\dot{q}}\Theta(x)|^{\dot{q}_*} + \left(A_{\dot{q}}(\theta, \alpha) - \frac{1}{n} \sum_{\ell=1}^n C_{\dot{q}}(\theta, \alpha, \ell) \right) |D_{\dot{q}}\Theta(\omega_2)|^{\dot{q}_*} \right]^{\frac{1}{\dot{q}_*}} \\ & \quad + \frac{(x - \omega_1)^{\alpha+1}}{\omega_2 - \omega_1} A_{\dot{q}}^{1 - \frac{1}{\dot{q}_*}}(\omega, \alpha) \\ & \quad \times \left[\left(A_{\dot{q}}(\omega, \alpha) - \frac{1}{n} \sum_{\ell=1}^n B_{\dot{q}}(\omega, \alpha, \ell) \right) |D_{\dot{q}}\Theta(x)|^{\dot{q}_*} + \left(A_{\dot{q}}(\omega, \alpha) - \frac{1}{n} \sum_{\ell=1}^n C_{\dot{q}}(\omega, \alpha, \ell) \right) |D_{\dot{q}}\Theta(\omega_1)|^{\dot{q}_*} \right]^{\frac{1}{\dot{q}_*}}. \end{aligned}$$

This completes the proof. \square

Corollary 8. Taking $\dot{q}_* = 1$ in Theorem 6, we have

$$\begin{aligned}
 & \left| \frac{\theta(\omega_2 - x)^\alpha (\Theta(\omega_2) - \Theta(x)) - \varpi(x - \omega_1)^\alpha (\Theta(x) - \Theta(\omega_1))}{\omega_2 - \omega_1} + \left(\frac{(x - \omega_1)^\alpha + (\omega_2 - x)^\alpha}{\omega_2 - \omega_1} \right) \Theta(x) \right. \\
 & \left. + \frac{[\alpha]_{\dot{q}}(1 - \dot{q})}{\dot{q}^\alpha(\omega_2 - \omega_1)} \left((x - \omega_1)^\alpha + (\omega_2 - x)^\alpha \right) \Theta(x) - \frac{\Gamma_{\dot{q}}(\alpha + 1)}{\dot{q}^\alpha(\omega_2 - \omega_1)} \left[J_{\dot{q}, x^+}^\alpha \Theta(\omega_2) + J_{\dot{q}, x^-}^\alpha \Theta(\omega_1) \right] \right| \\
 & \leq \frac{(\omega_2 - x)^{\alpha+1}}{\omega_2 - \omega_1} \left[\left(A_{\dot{q}}(\theta, \alpha) - \frac{1}{n} \sum_{\ell=1}^n B_{\dot{q}}(\theta, \alpha, \ell) \right) |D_{\dot{q}}\Theta(x)| + \left(A_{\dot{q}}(\theta, \alpha) - \frac{1}{n} \sum_{\ell=1}^n C_{\dot{q}}(\theta, \alpha, \ell) \right) |D_{\dot{q}}\Theta(\omega_2)| \right] \quad (24) \\
 & + \frac{(x - \omega_1)^{\alpha+1}}{\omega_2 - \omega_1} \left[\left(A_{\dot{q}}(\varpi, \alpha) - \frac{1}{n} \sum_{\ell=1}^n B_{\dot{q}}(\varpi, \alpha, \ell) \right) |D_{\dot{q}}\Theta(x)| + \left(A_{\dot{q}}(\varpi, \alpha) - \frac{1}{n} \sum_{\ell=1}^n C_{\dot{q}}(\varpi, \alpha, \ell) \right) |D_{\dot{q}}\Theta(\omega_1)| \right].
 \end{aligned}$$

Corollary 9. *If we choose $\dot{q} \rightarrow 1^-$ in Theorem 6, we get*

$$\begin{aligned}
 & \left| \frac{\theta(\omega_2 - x)^\alpha (\Theta(\omega_2) - \Theta(x)) - \varpi(x - \omega_1)^\alpha (\Theta(x) - \Theta(\omega_1))}{\omega_2 - \omega_1} + \left(\frac{(x - \omega_1)^\alpha + (\omega_2 - x)^\alpha}{\omega_2 - \omega_1} \right) \Theta(x) \right. \\
 & \left. - \frac{\Gamma(\alpha + 1)}{\omega_2 - \omega_1} \left[J_{x^+}^\alpha \Theta(\omega_2) + J_{x^-}^\alpha \Theta(\omega_1) \right] \right| \\
 & \leq \frac{(\omega_2 - x)^{\alpha+1}}{\omega_2 - \omega_1} \delta^{1 - \frac{1}{\dot{q}^*}}(\theta, \alpha) \left[\left(\delta(\theta, \alpha) - \frac{1}{n} \sum_{\ell=1}^n B(\theta, \alpha, \ell) \right) |\Theta'(x)|^{\dot{q}^*} + \left(\delta(\theta, \alpha) - \frac{1}{n} \sum_{\ell=1}^n C(\theta, \alpha, \ell) \right) |\Theta'(\omega_2)|^{\dot{q}^*} \right]^{\frac{1}{\dot{q}^*}} \quad (25) \\
 & + \frac{(x - \omega_1)^{\alpha+1}}{\omega_2 - \omega_1} \delta^{1 - \frac{1}{\dot{q}^*}}(\varpi, \alpha) \left[\left(\delta(\varpi, \alpha) - \frac{1}{n} \sum_{\ell=1}^n B(\varpi, \alpha, \ell) \right) |\Theta'(x)|^{\dot{q}^*} + \left(\delta(\varpi, \alpha) - \frac{1}{n} \sum_{\ell=1}^n C(\varpi, \alpha, \ell) \right) |\Theta'(\omega_1)|^{\dot{q}^*} \right]^{\frac{1}{\dot{q}^*}},
 \end{aligned}$$

where

$$B(\theta, \alpha, \ell) := \int_0^1 (1 - v)^\ell |\theta - v^\alpha| dv$$

and

$$C(\theta, \alpha, \ell) := \int_0^1 v^\ell |\theta - v^\alpha| dv.$$

Corollary 10. *Taking $\alpha = 1$ in Theorem 6, we obtain*

$$\begin{aligned}
 & \left| \frac{\theta(\omega_2 - x) (\Theta(\omega_2) - \Theta(x)) - \varpi(x - \omega_1) (\Theta(x) - \Theta(\omega_1))}{\omega_2 - \omega_1} + \frac{\Theta(x)}{\dot{q}} - \frac{1}{\dot{q}(\omega_2 - \omega_1)} \int_{\dot{q}\omega_1}^{\omega_2} \Theta(v) {}_{\omega_1}d_{\dot{q}}v \right| \\
 & \leq \frac{(\omega_2 - x)^2}{\omega_2 - \omega_1} A_{\dot{q}}^{1 - \frac{1}{\dot{q}^*}}(\theta) \left[\left(A_{\dot{q}}(\theta) - \frac{1}{n} \sum_{\ell=1}^n B_{\dot{q}}(\theta, \ell) \right) |D_{\dot{q}}\Theta(x)|^{\dot{q}^*} + \left(A_{\dot{q}}(\theta) - \frac{1}{n} \sum_{\ell=1}^n C_{\dot{q}}(\theta, \ell) \right) |D_{\dot{q}}\Theta(\omega_2)|^{\dot{q}^*} \right]^{\frac{1}{\dot{q}^*}} \quad (26) \\
 & + \frac{(x - \omega_1)^2}{\omega_2 - \omega_1} A_{\dot{q}}^{1 - \frac{1}{\dot{q}^*}}(\varpi) \left[\left(A_{\dot{q}}(\varpi) - \frac{1}{n} \sum_{\ell=1}^n B_{\dot{q}}(\varpi, \ell) \right) |D_{\dot{q}}\Theta(x)|^{\dot{q}^*} + \left(A_{\dot{q}}(\varpi) - \frac{1}{n} \sum_{\ell=1}^n C_{\dot{q}}(\varpi, \ell) \right) |D_{\dot{q}}\Theta(\omega_1)|^{\dot{q}^*} \right]^{\frac{1}{\dot{q}^*}},
 \end{aligned}$$

where

$$A_{\dot{q}}(\theta) := \int_0^1 |\theta - v| d_{\dot{q}}v,$$

$$B_{\dot{q}}(\theta, \ell) := \int_0^1 (1 - v)^\ell |\theta - v| d_{\dot{q}}v$$

and

$$C_{\dot{q}}(\theta, \ell) := \int_0^1 v^\ell |\theta - v| d_{\dot{q}}v.$$

Corollary 11. *Choosing $|D_{\dot{q}}\Theta| \leq \mathcal{K}$ in Theorem 6, we have*

$$\begin{aligned}
 & \left| \frac{\theta(\omega_2 - x)^\alpha (\Theta(\omega_2) - \Theta(x)) - \varpi(x - \omega_1)^\alpha (\Theta(x) - \Theta(\omega_1))}{\omega_2 - \omega_1} + \left(\frac{(x - \omega_1)^\alpha + (\omega_2 - x)^\alpha}{\omega_2 - \omega_1} \right) \Theta(x) \right. \\
 & \left. + \frac{[\alpha]_{\dot{q}}(1 - \dot{q})}{\dot{q}^\alpha(\omega_2 - \omega_1)} ((x - \omega_1)^\alpha + (\omega_2 - x)^\alpha) \Theta(x) - \frac{\Gamma_{\dot{q}}(\alpha + 1)}{\dot{q}^\alpha(\omega_2 - \omega_1)} \left[J_{\dot{q},x^+}^\alpha \Theta(\omega_2) + J_{\dot{q},x^-}^\alpha \Theta(\omega_1) \right] \right| \\
 \leq & \mathcal{K} \frac{(\omega_2 - x)^{\alpha+1}}{\omega_2 - \omega_1} A_{\dot{q}}^{1-\frac{1}{\dot{q}^*}}(\theta, \alpha) \left[2A_{\dot{q}}(\theta, \alpha) - \frac{1}{n} \sum_{\ell=1}^n (B_{\dot{q}}(\theta, \alpha, \ell) + C_{\dot{q}}(\theta, \alpha, \ell)) \right]^{\frac{1}{\dot{q}^*}} \\
 & + \mathcal{K} \frac{(x - \omega_1)^{\alpha+1}}{\omega_2 - \omega_1} A_{\dot{q}}^{1-\frac{1}{\dot{q}^*}}(\varpi, \alpha) \left[2A_{\dot{q}}(\varpi, \alpha) - \frac{1}{n} \sum_{\ell=1}^n (B_{\dot{q}}(\varpi, \alpha, \ell) + C_{\dot{q}}(\varpi, \alpha, \ell)) \right]^{\frac{1}{\dot{q}^*}}. \tag{27}
 \end{aligned}$$

Corollary 12. Taking $n = 1$ in Theorem 6, we get

$$\begin{aligned}
 & \left| \frac{\theta(\omega_2 - x)^\alpha (\Theta(\omega_2) - \Theta(x)) - \varpi(x - \omega_1)^\alpha (\Theta(x) - \Theta(\omega_1))}{\omega_2 - \omega_1} + \left(\frac{(x - \omega_1)^\alpha + (\omega_2 - x)^\alpha}{\omega_2 - \omega_1} \right) \Theta(x) \right. \\
 & \left. + \frac{[\alpha]_{\dot{q}}(1 - \dot{q})}{\dot{q}^\alpha(\omega_2 - \omega_1)} ((x - \omega_1)^\alpha + (\omega_2 - x)^\alpha) \Theta(x) - \frac{\Gamma_{\dot{q}}(\alpha + 1)}{\dot{q}^\alpha(\omega_2 - \omega_1)} \left[J_{\dot{q},x^+}^\alpha \Theta(\omega_2) + J_{\dot{q},x^-}^\alpha \Theta(\omega_1) \right] \right| \\
 \leq & \frac{(\omega_2 - x)^{\alpha+1}}{\omega_2 - \omega_1} A_{\dot{q}}^{1-\frac{1}{\dot{q}^*}}(\theta, \alpha) [C_{\dot{q}}(\theta, \alpha) |D_{\dot{q}}\Theta(x)|^{\dot{q}^*} + (A_{\dot{q}}(\theta, \alpha) - C_{\dot{q}}(\theta, \alpha)) |D_{\dot{q}}\Theta(\omega_2)|^{\dot{q}^*}]^{\frac{1}{\dot{q}^*}} \\
 & + \frac{(x - \omega_1)^{\alpha+1}}{\omega_2 - \omega_1} A_{\dot{q}}^{1-\frac{1}{\dot{q}^*}}(\varpi, \alpha) [C_{\dot{q}}(\varpi, \alpha) |D_{\dot{q}}\Theta(x)|^{\dot{q}^*} + (A_{\dot{q}}(\varpi, \alpha) - C_{\dot{q}}(\varpi, \alpha)) |D_{\dot{q}}\Theta(\omega_1)|^{\dot{q}^*}]^{\frac{1}{\dot{q}^*}}, \tag{28}
 \end{aligned}$$

where

$$C_{\dot{q}}(\theta, \alpha) := \int_0^1 v |\theta - v^\alpha| d_{\dot{q}}v.$$

Corollary 13. Choosing $\theta = \varpi$ in Theorem 6, we obtain

$$\begin{aligned}
 & \left| \frac{\theta[(\omega_2 - x)^\alpha (\Theta(\omega_2) - \Theta(x)) - (x - \omega_1)^\alpha (\Theta(x) - \Theta(\omega_1))]}{\omega_2 - \omega_1} + \left(\frac{(x - \omega_1)^\alpha + (\omega_2 - x)^\alpha}{\omega_2 - \omega_1} \right) \Theta(x) \right. \\
 & \left. + \frac{[\alpha]_{\dot{q}}(1 - \dot{q})}{\dot{q}^\alpha(\omega_2 - \omega_1)} ((x - \omega_1)^\alpha + (\omega_2 - x)^\alpha) \Theta(x) - \frac{\Gamma_{\dot{q}}(\alpha + 1)}{\dot{q}^\alpha(\omega_2 - \omega_1)} \left[J_{\dot{q},x^+}^\alpha \Theta(\omega_2) + J_{\dot{q},x^-}^\alpha \Theta(\omega_1) \right] \right| \\
 \leq & \frac{(\omega_2 - x)^{\alpha+1}}{\omega_2 - \omega_1} A_{\dot{q}}^{1-\frac{1}{\dot{q}^*}}(\theta, \alpha) \\
 & \times \left[\left(A_{\dot{q}}(\theta, \alpha) - \frac{1}{n} \sum_{\ell=1}^n B_{\dot{q}}(\theta, \alpha, \ell) \right) |D_{\dot{q}}\Theta(x)|^{\dot{q}^*} + \left(A_{\dot{q}}(\theta, \alpha) - \frac{1}{n} \sum_{\ell=1}^n C_{\dot{q}}(\theta, \alpha, \ell) \right) |D_{\dot{q}}\Theta(\omega_2)|^{\dot{q}^*} \right]^{\frac{1}{\dot{q}^*}} \\
 & + \frac{(x - \omega_1)^{\alpha+1}}{\omega_2 - \omega_1} A_{\dot{q}}^{1-\frac{1}{\dot{q}^*}}(\theta, \alpha) \\
 & \times \left[\left(A_{\dot{q}}(\theta, \alpha) - \frac{1}{n} \sum_{\ell=1}^n B_{\dot{q}}(\theta, \alpha, \ell) \right) |D_{\dot{q}}\Theta(x)|^{\dot{q}^*} + \left(A_{\dot{q}}(\theta, \alpha) - \frac{1}{n} \sum_{\ell=1}^n C_{\dot{q}}(\theta, \alpha, \ell) \right) |D_{\dot{q}}\Theta(\omega_1)|^{\dot{q}^*} \right]^{\frac{1}{\dot{q}^*}}. \tag{29}
 \end{aligned}$$

Corollary 14. Taking $\theta = 0$ and $\alpha = 1$ in Corollary 13, we have the following \dot{q} -integral inequality of Ostrowski type:

$$\begin{aligned}
 & \left| \Theta(x) - \frac{1}{\omega_2 - \omega_1} \int_{\dot{q}\omega_1}^{\omega_2} \Theta(v) \omega_1 d_{\dot{q}}v \right| \\
 & \leq \frac{\dot{q}(\omega_2 - x)^2}{\omega_2 - \omega_1} \left(\frac{1}{1 + \dot{q}} \right)^{1 - \frac{1}{\dot{q}^*}} \\
 & \quad \times \left[\left(\frac{1}{1 + \dot{q}} - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{[\ell + 1]_{\dot{q}}[\ell + 2]_{\dot{q}}} \right) |D_{\dot{q}}\Theta(x)|^{\dot{q}^*} + \left(\frac{1}{1 + \dot{q}} - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{[\ell + 2]_{\dot{q}}} \right) |D_{\dot{q}}\Theta(\omega_2)|^{\dot{q}^*} \right]^{\frac{1}{\dot{q}^*}} \quad (30) \\
 & \quad + \frac{\dot{q}(x - \omega_1)^2}{\omega_2 - \omega_1} \left(\frac{1}{1 + \dot{q}} \right)^{1 - \frac{1}{\dot{q}^*}} \\
 & \quad \times \left[\left(\frac{1}{1 + \dot{q}} - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{[\ell + 1]_{\dot{q}}[\ell + 2]_{\dot{q}}} \right) |D_{\dot{q}}\Theta(x)|^{\dot{q}^*} + \left(\frac{1}{1 + \dot{q}} - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{[\ell + 2]_{\dot{q}}} \right) |D_{\dot{q}}\Theta(\omega_1)|^{\dot{q}^*} \right]^{\frac{1}{\dot{q}^*}}.
 \end{aligned}$$

Corollary 15. *Choosing $x = \frac{\omega_1 + \omega_2}{2}$ in Corollary 14, we get the following \dot{q} -integral inequality of midpoint type:*

$$\begin{aligned}
 & \left| \Theta\left(\frac{\omega_1 + \omega_2}{2}\right) - \frac{1}{\omega_2 - \omega_1} \int_{\dot{q}\omega_1}^{\omega_2} \Theta(v) \omega_1 d_{\dot{q}}v \right| \\
 & \leq \frac{\dot{q}(\omega_2 - \omega_1)}{4} \left(\frac{1}{1 + \dot{q}} \right)^{1 - \frac{1}{\dot{q}^*}} \\
 & \quad \times \left\{ \left[\left(\frac{1}{1 + \dot{q}} - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{[\ell + 1]_{\dot{q}}[\ell + 2]_{\dot{q}}} \right) |D_{\dot{q}}\Theta\left(\frac{\omega_1 + \omega_2}{2}\right)|^{\dot{q}^*} + \left(\frac{1}{1 + \dot{q}} - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{[\ell + 2]_{\dot{q}}} \right) |D_{\dot{q}}\Theta(\omega_1)|^{\dot{q}^*} \right]^{\frac{1}{\dot{q}^*}} \right. \\
 & \quad \left. + \left[\left(\frac{1}{1 + \dot{q}} - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{[\ell + 1]_{\dot{q}}[\ell + 2]_{\dot{q}}} \right) |D_{\dot{q}}\Theta\left(\frac{\omega_1 + \omega_2}{2}\right)|^{\dot{q}^*} + \left(\frac{1}{1 + \dot{q}} - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{[\ell + 2]_{\dot{q}}} \right) |D_{\dot{q}}\Theta(\omega_2)|^{\dot{q}^*} \right]^{\frac{1}{\dot{q}^*}} \right\}. \quad (31)
 \end{aligned}$$

4. Example and Application

4.1. Example

Let $\Theta(v) = \frac{v^{\frac{r+1}{\dot{q}^*} + 1}}{\left[\frac{r+1}{\dot{q}^*} + 1\right]_{\dot{q}}}$, where $r \in \mathbb{N}$, $\dot{q}^* \geq 1$ and $\dot{q} \in (0, 1)$. After simple calculations, we have $|D_{\dot{q}}\Theta(v)|^{\dot{q}^*} = v^{r+1}$, which shows that $|D_{\dot{q}}\Theta(v)|^{\dot{q}^*}$ is convex function for all $v > 0$ and $r \in \mathbb{N}$. Then, by applying Corollaries 2 and 10 with Remark 1 for specific values $n = 1$, $\omega_1 = 0$, $\omega_2 = 1$ such that $x, \theta, \omega \in [0, 1]$, we deduce the following \dot{q} -inequalities:

$$\begin{aligned}
 & \left| \theta(1 - x) \left(\left[\frac{r+1}{\dot{q}^*} + 1 \right]_{\dot{q}} - x^{\frac{r+1}{\dot{q}^*} + 1} \right) - \omega x^{\frac{r+1}{\dot{q}^*} + 2} + \frac{x^{\frac{r+1}{\dot{q}^*} + 1}}{\dot{q}} - \frac{1}{\dot{q} \left[\frac{r+1}{\dot{q}^*} + 2 \right]_{\dot{q}}} \right| \\
 & \leq \left[\frac{r+1}{\dot{q}^*} + 1 \right]_{\dot{q}} \left(\frac{\dot{q}^*}{1 + \dot{q}^*} \right)^{\frac{1}{\dot{q}^*}} \left\{ (1 - x)^2 (1 + x^{r+1})^{\frac{1}{\dot{q}^*}} A_{\dot{q}}^{\frac{1}{p}}(\theta, p) + x^{\frac{r+1}{\dot{q}^*} + 2} A_{\dot{q}}^{\frac{1}{p}}(\omega, p) \right\}, \quad (32)
 \end{aligned}$$

$$\begin{aligned}
 & \left| \theta(1 - x) \left(\left[\frac{r+1}{\dot{q}^*} + 1 \right]_{\dot{q}} - x^{\frac{r+1}{\dot{q}^*} + 1} \right) - \omega x^{\frac{r+1}{\dot{q}^*} + 2} + \frac{x^{\frac{r+1}{\dot{q}^*} + 1}}{\dot{q}} - \frac{1}{\dot{q} \left[\frac{r+1}{\dot{q}^*} + 2 \right]_{\dot{q}}} \right| \\
 & \leq \left[\frac{r+1}{\dot{q}^*} + 1 \right]_{\dot{q}} \left\{ (1 - x)^2 A_{\dot{q}}^{1 - \frac{1}{\dot{q}^*}}(\theta) \left[(x^{r+1} - 1) C_{\dot{q}}(\theta) + A_{\dot{q}}(\theta) \right]^{\frac{1}{\dot{q}^*}} + x^{\frac{r+1}{\dot{q}^*} + 2} A_{\dot{q}}^{1 - \frac{1}{\dot{q}^*}}(\omega) C_{\dot{q}}^{\frac{1}{\dot{q}^*}}(\omega) \right\}, \quad (33)
 \end{aligned}$$

where

$$C_{\dot{q}}(\theta) := \int_0^1 v|\theta - v|d_{\dot{q}}v.$$

4.2. Application to Special Means

We consider the following arithmetic mean for real numbers ω_1 and ω_2 such that $0 \leq \omega_1 < \omega_2$:

$$\mathcal{A}(\omega_1, \omega_2) = \frac{\omega_1 + \omega_2}{2}.$$

For the simplicity of notation, let

$$\Delta_{\dot{q}}(r, \dot{q}_*; \omega_1, \omega_2) := (1 - \dot{q})(\omega_2 - \dot{q}\omega_1) \sum_{n=0}^{\infty} \dot{q}^n (\dot{q}^n \omega_2 + (1 - \dot{q}^n) \dot{q}\omega_1)^{\frac{r+1}{\dot{q}_*} + 1},$$

where $r \in \mathbb{N}$, $\dot{q}_* \geq 1$, and $\dot{q} \in (0, 1)$.

Proposition 1. Let $n, r \in \mathbb{N}$, $\dot{q} \in (0, 1)$ and $\omega_1, \omega_2 \in \mathbb{R}$, where $0 \leq \omega_1 < \omega_2$. Then for $p, \dot{q}_* > 1$ and $\frac{1}{p} + \frac{1}{\dot{q}_*} = 1$, we have

$$\begin{aligned} & \left| \mathcal{A}^{\frac{r+1}{\dot{q}_*} + 1}(\omega_1, \omega_2) - \frac{\Delta_{\dot{q}}(r, \dot{q}_*; \omega_1, \omega_2)}{\omega_2 - \omega_1} \right| \\ & \leq \frac{2^{\frac{1}{\dot{q}_*}} \dot{q}(\omega_2 - \omega_1)}{4} \left(1 - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{[\ell + 1]_{\dot{q}}} \right)^{\frac{1}{\dot{q}_*}} \\ & \times \left(\frac{1}{[p + 1]_{\dot{q}}} \right)^{\frac{1}{p}} \left[\frac{r + 1}{\dot{q}_*} + 1 \right]_{\dot{q}} \left\{ \mathcal{A}^{\frac{1}{\dot{q}_*}}(\omega_1^{r+1}, \mathcal{A}^{r+1}(\omega_1, \omega_2)) + \mathcal{A}^{\frac{1}{\dot{q}_*}}(\omega_2^{r+1}, \mathcal{A}^{r+1}(\omega_1, \omega_2)) \right\}. \end{aligned} \tag{34}$$

Proof. By applying Corollary 7 and Remark 1 with $\Theta(v) = \frac{v^{\frac{r+1}{\dot{q}_*} + 1}}{[\frac{r+1}{\dot{q}_*} + 1]_{\dot{q}}}$ for all $v \in [\omega_1, \omega_2]$, then we can get the desired result (34). \square

Proposition 2. Let $n, r \in \mathbb{N}$, $\dot{q} \in (0, 1)$ and $\omega_1, \omega_2 \in \mathbb{R}$, where $0 \leq \omega_1 < \omega_2$. Then for $\dot{q}_* \geq 1$, we have

$$\begin{aligned} & \left| \mathcal{A}^{\frac{r+1}{\dot{q}_*} + 1}(\omega_1, \omega_2) - \frac{\Delta_{\dot{q}}(r, \dot{q}_*; \omega_1, \omega_2)}{\omega_2 - \omega_1} \right| \\ & \leq \frac{\dot{q}(\omega_2 - \omega_1)}{4} \left(\frac{1}{1 + \dot{q}} \right)^{1 - \frac{1}{\dot{q}_*}} \left[\frac{r + 1}{\dot{q}_*} + 1 \right]_{\dot{q}} \\ & \times \left\{ \left[\left(\frac{1}{1 + \dot{q}} - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{[\ell + 1]_{\dot{q}}[\ell + 2]_{\dot{q}}} \right) \mathcal{A}^{r+1}(\omega_1, \omega_2) + \left(\frac{1}{1 + \dot{q}} - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{[\ell + 2]_{\dot{q}}} \right) \omega_1^{r+1} \right]^{\frac{1}{\dot{q}_*}} \right. \\ & \left. + \left[\left(\frac{1}{1 + \dot{q}} - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{[\ell + 1]_{\dot{q}}[\ell + 2]_{\dot{q}}} \right) \mathcal{A}^{r+1}(\omega_1, \omega_2) + \left(\frac{1}{1 + \dot{q}} - \frac{1}{n} \sum_{\ell=1}^n \frac{1}{[\ell + 2]_{\dot{q}}} \right) \omega_2^{r+1} \right]^{\frac{1}{\dot{q}_*}} \right\}. \end{aligned} \tag{35}$$

Proof. By using Corollary 15 and Remark 1 with $\Theta(v) = \frac{v^{\frac{r+1}{\dot{q}_*} + 1}}{[\frac{r+1}{\dot{q}_*} + 1]_{<\dot{q}}}$ for all $v \in [\omega_1, \omega_2]$, then we can obtain the desired result (35). \square

5. Conclusions

In this article, we considered a new parameterized quantum fractional integral identity. By using this, we have established some quantum fractional integral inequalities of Ostrowski and midpoint type via n -polynomial convex functions. Many consequences and

several special cases are analyzed and an example, and an application is given. Interested readers can use q -deformed real numbers [36] to extend our results. Furthermore, numerical analysis and comparison with fractional calculus, and quantum calculus, separately, can be done as well. We hope that this novel idea, which mixed together fractional calculus and quantum calculus, opens many avenues for interested researchers working in these fascinating fields and that they can discover further approximations for different kinds of convexity.

Author Contributions: Conceptualization, R.L., A.K., E.A.-S. and M.S.S.; data curation, P.O.M. and A.K.; formal analysis, E.A.-S.; funding acquisition, S.K.S. and E.A.-S.; investigation, R.L., H.M.S., P.O.M., A.K., S.K.S., E.A.-S. and M.S.S.; methodology, H.M.S. and P.O.M.; project administration, H.M.S. and M.S.S.; software, R.L., P.O.M. and S.K.S.; supervision, H.M.S. and M.S.S.; validation, R.L., A.K., S.K.S. and M.S.S.; visualization, S.K.S.; writing—original draft, R.L., P.O.M., A.K. and E.A.-S.; writing—review & editing, H.M.S. and M.S.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Cloud, M.J.; Drachman, B.C.; Lebedev, L. *Inequalities*, 2nd ed.; Springer: Cham, Switzerland, 2014.
2. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006; Volume 204.
3. Nawaz, Y.; Arif, M.S.; Abodayeh, K. A third-order two-stage numerical scheme for fractional Stokes problems: A comparative computational study. *J. Comput. Nonlinear Dyn.* **2022**, *17*, 101004. [[CrossRef](#)]
4. Nawaz, Y.; Arif, M.S.; Abodayeh, K. A numerical scheme for fractional mixed convection flow over flat and oscillatory plates. *J. Comput. Nonlinear Dyn.* **2022**, *17*, 071008. [[CrossRef](#)]
5. Zhang, T.Y.; Ji, A.P.; Qi, F. Some inequalities of Hermite–Hadamard type for GA-convex functions with applications to means. *Le Matematiche* **2013**, *68*, 229–239.
6. Mohammed, P.O. Some new Hermite–Hadamard type inequalities for MT-convex functions on differentiable coordinates. *J. King Saud Univ. Sci.* **2018**, *30*, 258–262. [[CrossRef](#)]
7. Shi, D.P.; Xi, B.Y.; Qi, F. Hermite–Hadamard type inequalities for Riemann–Liouville fractional integrals of (α, m) -convex functions. *Fract. Differ. Calc.* **2014**, *4*, 31–43. [[CrossRef](#)]
8. Mohammed, P.O.; Abdeljawad, T.; Zeng, S.; Kashuri, A. Fractional Hermite–Hadamard integral inequalities for a new class of convex functions. *Symmetry* **2020**, *12*, 1485. [[CrossRef](#)]
9. Dragomir, S.S.; Pearce, C.E.M. *Selected Topics on Hermite–Hadamard Inequalities and Applications*; RGMIA Monographs; Victoria University: Footscray, Australia, 2000.
10. Toplu, T.; Kadakal, M.; İşcan, İ. On n -polynomial convexity and some related inequalities. *AIMS Math.* **2020**, *5*, 1304–1318. [[CrossRef](#)]
11. Park, C.; Chu, Y.M.; Saleem, M.S.; Jahangir, N.; Rehman, N. On n -polynomial p -convex functions and some related inequalities. *Adv. Differ. Equ.* **2020**, *2020*, 666. [[CrossRef](#)]
12. Hadamard, J. Étude sur les propriétés des fonctions entières en particulier d’une fonction considérée par Riemann. *J. Math. Pures Appl.* **1893**, *58*, 171–215.
13. Mohammed, P.O.; Abdeljawad, T.; Kashuri, A. Fractional Hermite–Hadamard–Fejér inequalities for a convex function with respect to an increasing function involving a positive weighted symmetric function. *Symmetry* **2020**, *12*, 1503. [[CrossRef](#)]
14. Gavrea, B.; Gavrea, I. On some Ostrowski type inequalities. *Gen. Math.* **2010**, *18*, 33–44.
15. Budak, H.; Pehlivan, E. Weighted Ostrowski, trapezoid and midpoint type inequalities for Riemann–Liouville fractional integrals. *AIMS Math.* **2020**, *5*, 1960–1984. [[CrossRef](#)]
16. Tariq, M.; Sahoo, S.K.; Nasir, J.; Aydi, H.; Alsamir, H. Some Ostrowski type inequalities via n -polynomial exponentially s -convex functions and their applications. *AIMS Math.* **2021**, *6*, 13272–13290. [[CrossRef](#)]
17. Awan, M.U.; Talib, S.; Kashuri, A.; Noor, M.A.; Chu, Y.M. Estimates of quantum bounds pertaining to new q -integral identity with applications. *Adv. Differ. Equ.* **2020**, *2020*, 424. [[CrossRef](#)]
18. Ujević, N. Sharp inequalities of Simpson type and Ostrowski type. *Comput. Math. Appl.* **2004**, *48*, 145–151. [[CrossRef](#)]

19. Kaijser, S.; Nikolova, L.; Persson, L.E.; Wedestig, A. Hardy type inequalities via convexity. *Math. Inequal. Appl.* **2005**, *8*, 403–417. [[CrossRef](#)]
20. Gunawan, H. Eridani, Fractional integrals and generalized Olsen inequalities. *Kyungpook Math. J.* **2009**, *49*, 31–39. [[CrossRef](#)]
21. Ostrowski, A.M. Über die Absolutabweichung einer differentierbaren Funktion von ihren Integralmittelwert. *Comment. Helv.* **1937**, *10*, 226–227. [[CrossRef](#)]
22. Tariboon, J.; Ntouyas, S.K. Quantum integral inequalities on finite intervals. *J. Inequal. Appl.* **2014**, *2014*, 121. [[CrossRef](#)]
23. Kac, V.; Cheung, P. *Quantum Calculus*; Springer: Cham, Switzerland, 2002.
24. Alp, N.; Sarikaya, M.Z.; Kunt, M.; İşcan, İ. q -Hermite–Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. *J. King Saud Univ. Sci.* **2018**, *30*, 193–203. [[CrossRef](#)]
25. Sudsutad, W.; Ntouyas, S.K.; Tariboon, J. Quantum integral inequalities for convex functions. *J. Math. Inequal.* **2015**, *9*, 781–793. [[CrossRef](#)]
26. Noor, M.A.; Noor, K.I.; Awan, M.U. Some quantum estimates for Hermite–Hadamard inequalities. *Appl. Math. Comput.* **2015**, *251*, 675–679. [[CrossRef](#)]
27. Liu, W.J.; Zhuang, H.F. Some quantum estimates of Hermite–Hadamard inequalities for convex functions. *J. Appl. Anal. Comput.* **2017**, *7*, 501–522.
28. Budak, H.; Khan, S.; Ali, M.A.; Chu, Y.M. Refinements of quantum Hermite–Hadamard-type inequalities. *Open Math.* **2021**, *19*, 724–734. [[CrossRef](#)]
29. Ali, M.A.; Budak, H.; Akkurt, A.; Chu, Y.M. Quantum Ostrowski-type inequalities for twice quantum differentiable functions in quantum calculus. *Open Math.* **2021**, *19*, 440–449. [[CrossRef](#)]
30. Butt, S.I.; Budak, H.; Nonlaopon, K. New quantum Mercer estimates of Simpson–Newton-like inequalities via convexity. *Symmetry* **2022**, *14*, 1935. [[CrossRef](#)]
31. Aljinović, A.A.; Kovačević, D.; Puljiz, M.; Keko, A.Ž. On Ostrowski inequality for quantum calculus. *Appl. Math. Comput.* **2021**, *410*, 126454. [[CrossRef](#)]
32. Wang, X.; Khan, K.A.; Ditta, A.; Nosheen, A.; Awan, K.M.; Mabela, R.M. New developments on Ostrowski type inequalities via q -fractional integrals involving s -convex functions. *J. Funct. Spaces* **2022**, *2022*, 9742133. [[CrossRef](#)]
33. Jackson, F.H. On a q -definite integrals. *Q. J. Pure Appl. Math.* **1910**, *41*, 193–203.
34. Mansour, Z.S.I. On fractional q -Sturm–Liouville problems. *J. Fixed Point Theory Appl.* **2017**, *19*, 1591–1612. [[CrossRef](#)]
35. Shaimardan, S. *Hardy-type inequalities quantum calculus*; Lulea University of Technology, Graphic Production: Lulea, Sweden, 2018.
36. Genoud, S.M.; Ovsienko, V. On q -deformed real numbers. *Exp. Math.* **2022**, *31*, 652–660. [[CrossRef](#)]