NEIGHBORHOODS OF A CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract

By making use of the familiar concept of neighborhoods of analytic functions, the authors prove several inclusion relations associated with the (n, δ)-neighborhoods of various subclasses of starlike and convex functions of complex order. Special cases of some of these inclusion relations are shown to yield known results.

1. Introduction and Definitions

Let $\mathcal{A}(n)$ denote the class of functions $f(z)$ of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad \left(a_k \geqq 0;\ n \in \mathbb{N} := \{1, 2, 3, \cdots\}\right), \tag{1.1}$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Following Goodman [4] and Ruscheweyh [6], we define the (n, δ)-neighborhood of a function $f(z) \in \mathcal{A}(n)$ by

$$N_{n,\delta}(f) := \left\{ g \in \mathcal{A}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^n \ \text{ and } \right.$$
$$\left. \sum_{k=n+1}^{\infty} k\,|a_k - b_k| \leqq \delta \right\}. \tag{1.2}$$

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In particular, for the identity function

$$e(z)=z, \tag{1.3}$$

we immediately have

$$N_{n,\delta}(e) := \left\{ g \in \mathcal{A}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k\,|b_k| \leqq \delta \right\}. \tag{1.4}$$

The main object of the present paper is to investigate the (n,δ)-neighborhoods of the following subclasses of the class $\mathcal{A}(n)$ of *normalized* analytic functions in $\mathcal{U}$ with negative coefficients.

A function $f(z) \in \mathcal{A}(n)$ is said to be *starlike of complex order* γ $(\gamma \in \mathbb{C} \setminus \{0\})$, that is, $f \in \mathcal{S}_n^*(\gamma)$, if it also satisfies the inequality:

$$\operatorname{Re}\left\{1+\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\right\}>0 \quad (z \in \mathcal{U};\ \gamma \in \mathbb{C}\setminus\{0\}). \tag{1.5}$$

Furthermore, a function $f(z) \in \mathcal{A}(n)$ is said to be *convex of complex order* γ $(\gamma \in \mathbb{C}\setminus\{0\})$, that is, $f \in \mathcal{C}_n(\gamma)$, if it also satisfies the inequality:

$$\operatorname{Re}\left\{1+\frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right\}>0 \quad (z \in \mathcal{U};\gamma \in \mathbb{C}\setminus\{0\}). \tag{1.6}$$

The classes $\mathcal{S}_n^*(\gamma)$ and $\mathcal{C}_n(\gamma)$ stem essentially from the classes of starlike and convex functions of complex order, which were considered earlier by Nasr and Aouf [5] and Wiatrowski [9], respectively (see also [3] and [7]).

Finally, let $\mathcal{S}_n(\gamma,\lambda,\beta)$ denote the subclass of $\mathcal{A}(n)$ consisting of functions $f(z)$ which satisfy the inequality:

$$\left|\frac{1}{\gamma}\left(\frac{zf'(z)+\lambda z^2 f''(z)}{\lambda z f'(z)+(1-\lambda)f(z)}-1\right)\right|<\beta \tag{1.7}$$

$$(z \in \mathcal{U};\ \gamma \in \mathbb{C}\setminus\{0\};\ 0 \leqq \lambda \leqq 1;\ 0 \leqq \beta < 1).$$

Also let $\mathcal{R}_n(\gamma,\lambda,\beta)$ denote the subclass of $\mathcal{A}(n)$ consisting of functions $f(z)$ which satisfy the inequality:

$$\left|\frac{1}{\gamma}\left\{f'(z)+\lambda z f''(z)-1\right\}\right|<\beta \tag{1.8}$$

$$(z \in \mathcal{U};\ \gamma \in \mathbb{C}\setminus\{0\};\ 0 \leqq \lambda \leqq 1;\ 0 \leqq \beta < 1).$$

Various further subclasses of the classes $\mathcal{S}_n(\gamma,\lambda,\beta)$ and $\mathcal{R}_n(\gamma,\lambda,\beta)$ *with* $\gamma=1$ were studied in many earlier works (*cf.*, *e.g.*, Altintaş [1] and Srivastava *et al.* [8]; see also the references cited in these earlier works). Clearly, we have

$$\mathcal{S}_n(\gamma,0,1) \subset \mathcal{S}_n^*(\gamma) \quad \text{and} \quad \mathcal{R}_n(\gamma,0,1) \subset \mathcal{C}_n(\gamma) \tag{1.9}$$

$$(n \in \mathbb{N};\ \gamma \in \mathbb{C}\setminus\{0\}).$$

2. A Set of Inclusion Relations Involving $N_{n,\delta}(e)$

In our investigation of the inclusion relations involving $N_{n,\delta}(e)$, we shall require Lemma 1 and Lemma 2 below.

Lemma 1. *Let the function $f \in \mathcal{A}(n)$ be defined by* (1.1). *Then $f(z)$ is in the class $\mathcal{S}_n(\gamma,\lambda,\beta)$ if and only if*

$$\sum_{k=n+1}^{\infty} \{\lambda(k-1)+1\}\,(k+\beta|\gamma|-1)\,a_k \leqq \beta|\gamma|. \tag{2.1}$$

Proof. We first suppose that $f \in \mathcal{S}_n(\gamma,\lambda,\beta)$. Then, by appealing to the condition (1.7), we readily get

$$\operatorname{Re}\left\{\frac{zf'(z)+\lambda z^2 f''(z)}{\lambda z f'(z)+(1-\lambda)f(z)}-1\right\} > -\beta|\gamma| \quad (z \in \mathcal{U}) \tag{2.2}$$

or, equivalently,

$$\operatorname{Re}\left\{\frac{-\sum\limits_{k=n+1}^{\infty}\{\lambda(k-1)+1\}\,(k-1)a_k z^k}{z-\sum\limits_{k=n+1}^{\infty}\{\lambda(k-1)+1\}\,a_k z^k}\right\} > -\beta|\gamma| \quad (z \in \mathcal{U}), \tag{2.3}$$

where we have made use of the definition (1.1).

Now choose values of z on the real axis and let $z \to 1-$ through real values. Then the inequality (2.3) immediately yields the desired condition (2.1).

Conversely, by applying the hypothesis (2.1) and letting $|z|=1$, we find that

$$\begin{aligned}\left|\frac{zf'(z)+\lambda z^2 f''(z)}{\lambda z f'(z)+(1-\lambda)f(z)}-1\right| &= \left|\frac{\sum\limits_{k=n+1}^{\infty}\{\lambda(k-1)+1\}\,(k-1)a_k z^k}{z-\sum\limits_{k=n+1}^{\infty}\{\lambda(k-1)+1\}\,a_k z^k}\right| \\ &\leqq \frac{\beta|\gamma|\left(1-\sum\limits_{k=n+1}^{\infty}\{\lambda(k-1)+1\}\,a_k\right)}{1-\sum\limits_{k=n+1}^{\infty}\{\lambda(k-1)+1\}\,a_k} \\ &\leqq \beta|\gamma|.\end{aligned} \tag{2.4}$$

Hence, by the maximum modulus theorem, we have

$$f \in \mathcal{S}_n(\gamma,\lambda,\beta),$$

which evidently completes the proof of Lemma 1.

Similarly, we can prove

Lemma 2. *Let the function* $f \in \mathcal{A}(n)$ *be defined by* (1.1). *Then* $f(z)$ *is in the class* $\mathcal{R}_n(\gamma, \lambda, \beta)$ *if and only if*

$$\sum_{k=n+1}^{\infty} k\{\lambda(k-1)+1\} a_k \leqq \beta|\gamma|. \tag{2.5}$$

Remark 1. A special case of Lemma 1 when

$$\gamma = 1 \quad \text{and} \quad \beta = 1-\alpha \quad (0 \leqq \alpha < 1)$$

was given earlier by Altintaş [1, p. 489, Theorem 1].

Our first inclusion relation involving $N_{n,\delta}(e)$ is given by

Theorem 1. *Let*

$$\delta = \frac{(n+1)\beta|\gamma|}{(\lambda n+1)(n+\beta|\gamma|)} \quad (|\gamma| < 1). \tag{2.6}$$

Then

$$\mathcal{S}_n(\gamma, \lambda, \beta) \subset N_{n,\delta}(e). \tag{2.7}$$

Proof. For $f \in \mathcal{S}_n(\gamma, \lambda, \beta)$, Lemma 1 immediately yields

$$(\lambda n+1)(n+\beta|\gamma|) \sum_{k=n+1}^{\infty} a_k \leqq \beta|\gamma|,$$

so that

$$\sum_{k=n+1}^{\infty} a_k \leqq \frac{\beta|\gamma|}{(\lambda n+1)(n+\beta|\gamma|)}. \tag{2.8}$$

On the other hand, we also find from (2.1) and (2.8) that

$$\begin{aligned}
(\lambda n+1) \sum_{k=n+1}^{\infty} k a_k &\leqq \beta|\gamma| + (1-\beta|\gamma|)(\lambda n+1) \sum_{k=n+1}^{\infty} a_k \\
&\leqq \beta|\gamma| + (1-\beta|\gamma|)(\lambda n+1) \frac{\beta|\gamma|}{(\lambda n+1)(n+\beta|\gamma|)} \\
&\leqq \frac{(n+1)\beta|\gamma|}{n+\beta|\gamma|} \quad (|\gamma| < 1),
\end{aligned}$$

that is,

$$\sum_{k=n+1}^{\infty} k a_k \leqq \frac{(n+1)\beta|\gamma|}{(\lambda n+1)(n+\beta|\gamma|)} = \delta, \tag{2.9}$$

which, in view of the definition (1.4), proves Theorem 1.

By similarly applying Lemma 2 instead of Lemma 1, we can prove

Theorem 2. *Let*

$$\delta = \frac{\beta|\gamma|}{\lambda n+1}. \tag{2.10}$$

Then

$$\mathcal{R}_n(\gamma, \lambda, \beta) \subset N_{n,\delta}(e). \tag{2.11}$$

Remark 2. A special case of Theorem 1 when

$$\gamma = 1 - \alpha \quad (0 \leqq \alpha < 1), \quad \lambda = 0, \quad \text{and} \quad \beta = 0 \tag{2.12}$$

was proven recently by Altintaş and Owa [2, p. 798, Theorem 2.1].

3. Neighborhoods for the Classes $\mathcal{S}_n^{(\alpha)}(\gamma, \lambda, \beta)$ and $\mathcal{R}_n^{(\alpha)}(\gamma, \lambda, \beta)$

In this section we determine the neighborhood for each of the classes

$$\mathcal{S}_n^{(\alpha)}(\gamma, \lambda, \beta) \quad \text{and} \quad \mathcal{R}_n^{(\alpha)}(\gamma, \lambda, \beta),$$

which we define as follows. A function $f \in \mathcal{A}(n)$ is said to be in the class $\mathcal{S}_n^{(\alpha)}(\gamma, \lambda, \beta)$ if there exists a function $g \in \mathcal{S}_n(\gamma, \lambda, \beta)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \alpha \quad (z \in \mathcal{U};\, 0 \leqq \alpha < 1). \tag{3.1}$$

Analogously, a function $f \in \mathcal{A}(n)$ is said to be in the class $\mathcal{R}_n^{(\alpha)}(\gamma, \lambda, \beta)$ if there exists a function $g \in \mathcal{R}_n(\gamma, \lambda, \beta)$ such that the inequality (3.1) holds true.

Theorem 3. *If* $g \in \mathcal{S}_n(\gamma, \lambda, \beta)$ *and*

$$\alpha = 1 - \frac{(\lambda n + 1)(n + \beta|\gamma|)\delta}{n(n+1)\left\{\lambda(n + \beta|\gamma|) + 1\right\}}, \tag{3.2}$$

then

$$N_{n,\delta}(g) \subset \mathcal{S}_n^{(\alpha)}(\gamma, \lambda, \beta). \tag{3.3}$$

Proof. Suppose that $f \in N_{n,\delta}(g)$. We then find from the definition (1.2) that

$$\sum_{k=n+1}^{\infty} k\,|a_k - b_k| \leqq \delta, \tag{3.4}$$

which readily implies the coefficient inequality:

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \leqq \frac{\delta}{n+1} \quad (n \in \mathbb{N}). \tag{3.5}$$

Next, since $g \in \mathcal{S}_n(\gamma, \lambda, \beta)$, we have [*cf.* Equation (2.8)]

$$\sum_{k=n+1}^{\infty} b_k \leqq \frac{\beta|\gamma|}{(\lambda n + 1)(n + \beta|\gamma|)}, \tag{3.6}$$

so that

$$\begin{aligned}\left|\frac{f(z)}{g(z)}-1\right| &< \frac{\sum\limits_{k=n+1}^{\infty} |a_k-b_k|}{1-\sum\limits_{k=n+1}^{\infty} b_k} \\ &\leqq \frac{\delta}{n+1}\cdot\frac{(\lambda n+1)(n+\beta|\gamma|)}{n\{\lambda(n+\beta|\gamma|)+1\}} \\ &= 1-\alpha, \end{aligned} \tag{3.7}$$

provided that α is given precisely by (3.2). Thus, by definition, $f \in \mathcal{S}_n^{(\alpha)}(\gamma,\lambda,\beta)$ for α given by (3.2), which evidently completes our proof of Theorem 3.

Our proof of Theorem 4 below is much akin to that of Theorem 3.

Theorem 4. *If* $g \in \mathcal{R}_n(\gamma,\lambda,\beta)$ *and*

$$\alpha = 1-\frac{(\lambda n+1)\delta}{(n+1)(\lambda n+1)-\beta|\gamma|}, \tag{3.8}$$

then

$$N_{n,\delta}(g) \subset \mathcal{R}_n^{(\alpha)}(\gamma,\lambda,\beta). \tag{3.9}$$

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