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RESEARCH

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# Inclusion relations for certain families of integral operators associated with conic regions

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## Abstract

In this work, we introduce certain subclasses of analytic functions involving the integral operators that generalize the class of uniformly starlike, convex, and close-to-convex functions with respect to symmetric points. We then establish various inclusion relations for these newly defined classes.

**MSC:** 30C45; 30C50

**Keywords:** Sakaguchi functions; Schwarz function; Subordination; Functions with positive real parts; Analytic functions; Conic domain; Uniformly starlike; Integral operators; Symmetrical points

## 1 Introduction

Let  $\mathcal{A}$  be the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

analytic in the open unit disc  $\mathfrak{A} = \{z \in \mathbb{C} : |z| < 1\}$ , and let  $\mathcal{S}$  be the class of functions in  $\mathcal{A}$  that are univalent in  $\mathfrak{A}$ . Also let  $\mathcal{S}^*$ ,  $\mathcal{C}$ ,  $\mathcal{K}$ , and  $\mathcal{C}^*$  be the subclasses of  $\mathcal{A}$  consisting of all functions that are starlike, convex, close-to-convex, and quasiconvex, respectively; for details, see [1].

Let  $f$  and  $g$  be analytic in  $\mathfrak{A}$ . We say that  $f$  is subordinate to  $g$ , written as  $f(z) \prec g(z)$ , if there exists a Schwarz function  $w$  that is analytic in  $\mathfrak{A}$  with  $w(0) = 0$  and  $|w(z)| < 1$  ( $z \in \mathfrak{A}$ ) and such that  $f(z) = g(w(z))$ . In particular, when  $g$  is univalent, then such a subordination is equivalent to  $f(0) = g(0)$  and  $f(\mathfrak{A}) \subseteq g(\mathfrak{A})$ ; see [1].

Two points  $A$  and  $A'$  are said to be symmetrical with respect to  $M$  if  $M$  is the midpoint of the line segment  $AA'$ . Sakaguchi [2] introduced and studied the class  $\mathcal{S}_s^*$  of starlike functions with respect to symmetrical points  $z$  and  $-z$  belonging to the open unit disc  $\mathfrak{A}$ . The class  $\mathcal{S}_s^*$  includes the classes of convex and odd starlike functions with respect to the origin. It was shown [2] that a necessary and sufficient condition for  $f(z) \in \mathcal{S}_s^*$  to be univalent and starlike with respect to symmetrical points in  $\mathfrak{A}$  is that

$$\frac{2zf'(z)}{f(z) - f(-z)} \in \mathcal{P}, \quad z \in \mathfrak{A}.$$

Das and Singh [3] defined the classes  $\mathcal{C}_s$  of convex functions with respect to symmetrical points and showed that a necessary and sufficient condition for  $f(z) \in \mathcal{C}_s$  is that

$$\frac{2(zf'(z))'}{(f(z) - f(-z))'} \in \mathcal{P}, \quad z \in \mathfrak{A}.$$

It is also well known [3] that  $f(z) \in \mathcal{C}_s$  if and only if  $zf(z) \in \mathcal{S}_s^*$ .

The classes  $k - \mathcal{CV}$  and  $k - \mathcal{ST}$  with  $k \geq 0$  denote the famous classes of  $k$ -uniformly convex and  $k$ -starlike functions, respectively, introduced by Kanas and Wisniowska, respectively. For some details see [4–7].

Consider the domain

$$\Omega_k = \{u + iv; u > k\sqrt{(u - 1)^2 + v^2}\}. \tag{1.2}$$

For fixed  $k$ ,  $\Omega_k$  represents the conic region bounded successively by the imaginary axis ( $k = 0$ ), the right branch of a hyperbola ( $0 < k < 1$ ), a parabola ( $k = 1$ ), and an ellipse ( $k > 1$ ). This domain was studied by Kanas [4–6]. The function  $p_k$  with  $p_k(0) = 1$  and  $p'_k(0) > 0$  plays the role of extremal and is given by

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k = 0, \\ 1 + \frac{2}{\pi^2} (\log \frac{1+\sqrt{z}}{1-\sqrt{z}})^2, & k = 1, \\ 1 + \frac{2}{1-k^2} \sinh^2[(\frac{2}{\pi} \arccos k) \operatorname{arctanh} \sqrt{z}], & 0 < k < 1, \\ 1 + \frac{1}{k^2-1} \sin[\frac{\pi}{2R(t)} \int_0^{\sqrt{t}} \frac{u(z)}{\sqrt{1-x^2}\sqrt{1-(tx)^2}} dx] + \frac{1}{k^2-1}, & k > 1, \end{cases} \tag{1.3}$$

with  $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$ ,  $t \in (0, 1)$ ,  $z \in E$ , and  $t$  chosen such that  $k = \cosh(\frac{\pi R'(t)}{4R(t)})$ , where  $R(t)$  is Legendre’s complete elliptic integral of the first kind, and  $R'(t)$  is the complementary integral of  $R(t)$  (see [5, 6]). Let  $\mathcal{P}_{p_k}$  denote the class of all functions  $p(z)$  that are analytic in  $E$  with  $p(0) = 1$  and  $p(z) < p_k(z)$  for  $z \in E$ . Clearly, we can see that  $\mathcal{P}_{p_k} \subset \mathcal{P}$ , where  $\mathcal{P}$  is the class of functions with positive real parts (see [1]). More precisely,

$$\mathcal{P}_{p_k} \subset \mathcal{P}\left(\frac{k}{1+k}\right) \subset \mathcal{P}.$$

For more detail regarding conic domains and related classes, see [4–6, 8–11].

Recently, Noor [12] defined the classes  $k - \mathcal{ST}_s$ ,  $k - \mathcal{UCV}_s$ , and  $k - \mathcal{UK}_s$  of  $k$ -uniformly starlike, convex, and close to convex functions with respect to symmetrical points and studied various interesting properties for these classes.

We consider the following one-parameter families of integral operators:

$$\mathcal{I}_\beta^\alpha f(z) = \frac{(\beta + 1)^\alpha}{\Gamma(\alpha)z^\beta} \int_0^z t^{\beta-1} \left(\log \frac{z}{t}\right)^{\alpha-1} f(t) dt, \tag{1.4}$$

$$\mathcal{L}_\beta^\alpha f(z) = \binom{\alpha + \beta}{\beta} \frac{\alpha}{z^\beta} \int_0^z t^{\beta-1} \left(1 - \frac{t}{z}\right)^{\alpha-1} f(t) dt, \tag{1.5}$$

and

$$\mathfrak{J}_\beta f(z) = \frac{\beta + 1}{z^\beta} \int_0^z t^{\beta-1} f(t) dt, \tag{1.6}$$

where  $\alpha \geq 0, \beta > -1$ , and  $\Gamma$  is the familiar gamma function. We note that  $\mathfrak{J}_\beta : \mathcal{A} \rightarrow \mathcal{A}$  defined by (1.6) is the generalized Bernardi operator introduced in [13] for  $\beta = 1, 2, 3, \dots$ , and for any real number  $\beta > -1$ , this operator was studied by Owa and Srivastava [14, 15]. For the operators  $\mathfrak{L}_\beta^\alpha$  and  $\mathcal{I}_\beta^\alpha$ , we refer to [16, 17]. Also, for  $\alpha = 1$ , we see that

$$\mathfrak{J}_\beta f(z) = \mathfrak{L}_\beta^1 f(z) = \mathcal{I}_\beta^1 f(z).$$

We can represent these operators as follows:

$$\begin{aligned} \mathcal{I}_\beta^\alpha f(z) &= z + \sum_{n=2}^\infty \left(\frac{\beta + 1}{\beta + n}\right)^\alpha a_n z^n \\ &= \left(z + \sum_{n=2}^\infty \left(\frac{\beta + 1}{\beta + n}\right)^\alpha z^n\right) * f(z), \end{aligned} \tag{1.7}$$

$$\begin{aligned} \mathfrak{L}_\beta^\alpha f(z) &= z + \sum_{n=2}^\infty \frac{\Gamma(\beta + n)\Gamma(\alpha + \beta + 1)}{\Gamma(\alpha + \beta + n)\Gamma(\beta + 1)} a_n z^n \\ &= \left(\frac{\alpha + \beta}{\beta}\right) {}_2F_1(1, \beta; \alpha + \beta; z) * f(z), \end{aligned} \tag{1.8}$$

and

$$\mathfrak{J}_\beta f(z) = z + \sum_{n=2}^\infty \left(\frac{\beta + 1}{\beta + n}\right) a_n z^n, \tag{1.9}$$

where  ${}_2F_1$  denotes the Gaussian hypergeometric function, and the symbol  $*$  stands for the convolution (Hadamard product).

By (1.7) and (1.8) we can easily derive the identities

$$z(\mathcal{I}_\beta^\alpha f(z))' = (\beta + 1)\mathcal{I}_\beta^{\alpha-1} f(z) - \beta \mathcal{I}_\beta^\alpha f(z) \tag{1.10}$$

and

$$z(\mathfrak{L}_\beta^\alpha f(z))' = (\alpha + \beta)\mathfrak{L}_\beta^{\alpha-1} f(z) - (\alpha + \beta - 1)\mathfrak{L}_\beta^\alpha f(z), \tag{1.11}$$

where  $\alpha \geq 1$  and  $\beta > -1$ . From (1.10) we have

$$\left[ \frac{1}{1 + \beta} p(z) + \frac{\beta}{1 + \beta} \right] = \frac{\mathcal{I}_\beta^{\alpha-1} f(z)}{\mathcal{I}_\beta^\alpha f(z)}$$

with

$$p(z) = \frac{z(\mathcal{I}_\beta^\alpha f(z))'}{\mathcal{I}_\beta^\alpha f(z)}.$$

With the help of these integral operators, we now define the following classes.

**Definition 1.1** Let  $f(z) \in \mathcal{A}$ . Then  $f(z) \in k-ST_s(\alpha, \beta)$ ,  $\alpha \geq 0, \beta > -1$ , if  $\mathcal{I}_\beta^\alpha f(z) \in k-ST_s$  in  $\mathfrak{A}$ .

**Definition 1.2** Let  $f(z) \in \mathcal{A}$ . Then  $f(z) \in k - \mathcal{ST}_s^*(\alpha, \beta)$ ,  $\alpha \geq 0, \beta > -1$ , if  $\mathcal{I}_\beta^\alpha f(z) \in k - \mathcal{ST}_s$  in  $\mathfrak{A}$ .

**Definition 1.3** Let  $f(z) \in \mathcal{A}$ . Then  $f(z) \in k - \mathcal{UK}_s(\alpha, \beta)$ ,  $\alpha \geq 0, \beta > -1$ , if  $\mathcal{I}_\beta^\alpha f(z) \in k - \mathcal{UK}_s$  in  $\mathfrak{A}$ .

**Definition 1.4** Let  $f(z) \in \mathcal{A}$ . Then  $f(z) \in k - \mathcal{UK}_s^*(\alpha, \beta)$ ,  $\alpha \geq 0, \beta > -1$ , if  $\mathcal{I}_\beta^\alpha f(z) \in k - \mathcal{UK}_s$  in  $\mathfrak{A}$ .

### 2 A set of lemmas

In this section, we give the following lemmas, which will be used in our investigation.

**Lemma 2.1** ([4]) Let  $k \geq 0$ , and let  $\beta_1, \gamma \in \mathbb{C}$  be such that  $\beta_1 \neq 0$  and  $\Re\{\frac{\beta_1 k}{k+1} + \gamma\} > 0$ . Suppose that  $p(z)$  is analytic in  $\mathfrak{A}$  with  $p(0) = 1$  and satisfies

$$\left( p(z) + \frac{zp'(z)}{\beta_1 p(z) + \gamma} \right) \prec p_k(z) \tag{2.1}$$

and that  $q(z)$  is an analytic function satisfying

$$q(z) + \frac{zq'(z)}{\beta_1 q(z) + \gamma} = p_k(z). \tag{2.2}$$

Then  $q(z)$  is univalent,  $p(z) \prec q(z) \prec p_k(z)$ , and  $q(z)$  is the best dominant of (2.1) given as

$$q(z) = \left[ \beta_1 \int_0^1 \left( t^{\beta_1 + \gamma - 1} \exp \int_z^{tz} \frac{p_k(u) - 1}{u} du \right) dt \right]^{-1} - \frac{\gamma}{\beta_1}. \tag{2.3}$$

**Lemma 2.2** ([18]) Let  $\lambda, \rho \in \mathbb{C}$  be such that  $\lambda \neq 0$ , and let  $\phi(z) \in \mathcal{A}$  be convex and univalent in  $\mathbb{U}$  with  $\Re\{\lambda\phi(z) + \rho\} > 0$  ( $z \in \mathbb{U}$ ). Also, let  $q(z) \in \mathcal{A}$  and  $q(z) \prec \phi(z)$ . If  $p(z)$  is analytic in  $\mathbb{U}$  with  $p(0) = 1$  and satisfies

$$\left( p(z) + \frac{zp'(z)}{\lambda q(z) + \rho} \right) \prec \phi(z), \tag{2.4}$$

then  $p(z) \prec \phi(z)$ .

### 3 The main results and their consequences

Our first main result is stated as the following:

**Theorem 3.1** Let  $f(z) \in k - \mathcal{ST}_s(\alpha, \beta)$ . Then the odd function

$$\psi(z) = \frac{1}{2}[f(z) - f(-z)] \in k - \mathcal{ST}(\alpha, \beta).$$

*Proof* Note that

$$\mathcal{I}_\beta^\alpha \psi(z) = \frac{1}{2}[\mathcal{I}_\beta^\alpha f(z) - \mathcal{I}_\beta^\alpha f(-z)].$$

We want to show that  $\mathcal{I}_\beta^\alpha \psi(z) \in k - \mathcal{ST}$ . Now, for  $f(z) \in k - \mathcal{ST}_s(\alpha, \beta)$ , this implies that  $\mathcal{I}_\beta^\alpha f(z) \in k - \mathcal{ST}_s$ . Then, for  $z \in \mathfrak{A}$ ,

$$\begin{aligned} \frac{z(\mathcal{I}_\beta^\alpha \psi(z))'}{\mathcal{I}_\beta^\alpha \psi(z)} &= \frac{1}{2} \left[ \frac{2z(\mathcal{I}_\beta^\alpha f(z))'}{\mathcal{I}_\beta^\alpha f(z) - \mathcal{I}_\beta^\alpha f(-z)} + \frac{2(-z)(\mathcal{I}_\beta^\alpha f(-z))'}{\mathcal{I}_\beta^\alpha f(-z) - \mathcal{I}_\beta^\alpha f(z)} \right] \\ &= \frac{1}{2} [h_1(z) + h_2(z)] \\ &= h(z). \end{aligned}$$

and  $h_i(z) \prec p_k(z)$ ,  $i = 1, 2$ . This implies that  $h(z) \prec p_k(z)$  in  $\mathfrak{A}$ , and therefore  $\mathcal{I}_\beta^\alpha \psi(z) \in k - \mathcal{ST}$ . Consequently,  $\psi(z) \in k - \mathcal{ST}(\alpha, \beta)$  in  $\mathfrak{A}$ . □

Similarly, we can prove that if  $f(z) \in k - \mathcal{ST}_s^*(\alpha, \beta)$ , then

$$\phi(z) = \frac{1}{2} [f(z) - f(-z)] \in k - \mathcal{ST}^*(\alpha, \beta).$$

Taking  $\alpha = 0$ , we obtain the following result proved by Noor [12].

**Corollary 3.2** *Let  $f(z) \in k - \mathcal{ST}_s$ . Then the odd function*

$$\psi(z) = \frac{1}{2} [f(z) - f(-z)] \in k - \mathcal{ST}.$$

Note that, for  $k = \alpha = 0$ , the function  $\psi(z) = \frac{1}{2} [f(z) - f(-z)]$  is a starlike function in  $\mathfrak{A}$ ; see [2].

**Theorem 3.3** *Let  $\alpha \geq 2$  and  $\beta > -1$ . Then  $k - \mathcal{ST}(\alpha - 1, \beta) \subset k - \mathcal{ST}(\alpha, \beta)$ .*

*Proof* Let  $f(z) \in k - \mathcal{ST}(\alpha - 1, \beta)$  and set

$$p(z) = \frac{z(\mathcal{I}_\beta^\alpha f(z))'}{\mathcal{I}_\beta^\alpha f(z)}. \tag{3.1}$$

Note that  $p(z)$  is analytic in  $\mathfrak{A}$  with  $p(0) = 1$ .

From (3.1) and identity (1.10) we have

$$\frac{\mathcal{I}_\beta^{\alpha-1} f(z)}{\mathcal{I}_\beta^\alpha f(z)} = (1 - \gamma)p(z) + \gamma \tag{3.2}$$

with

$$\gamma = \frac{\beta}{\beta + 1}. \tag{3.3}$$

Logarithmic differentiation of (3.2) yields

$$\frac{z(\mathcal{I}_\beta^{\alpha-1} f(z))'}{\mathcal{I}_\beta^{\alpha-1} f(z)} = \left\{ p(z) + \frac{(1 - \gamma)zp'(z)}{(1 - \gamma)zp(z) + \gamma} \right\},$$

and thus it follows that

$$\left( p(z) + \frac{zp'(z)}{zp(z) + \beta} \right) \prec p_k(z).$$

Using Lemma 2.1, we have

$$p(z) \prec q(z) \prec p_k(z)$$

with

$$q(z) = \left[ \int_0^1 \left( t^\beta \exp \int_z^{tz} \frac{p_k(u) - 1}{u} du \right) dt \right]^{-1} - \beta.$$

This proves that  $f(z) \in k - \mathcal{ST}(\alpha, \beta)$  in  $\mathfrak{A}$ , and the proof is complete. □

**Theorem 3.4** *Let  $\alpha \geq 2$  and  $\beta > -1$ . Then  $k - \mathcal{ST}^*(\alpha - 1, \beta) \subset k - \mathcal{ST}^*(\alpha, \beta)$ .*

*Proof* Let

$$\frac{z(\mathfrak{L}_\beta^\alpha f(z))'}{\mathfrak{L}_\beta^\alpha f(z)} = h(z), \tag{3.4}$$

where  $h(z)$  is analytic in  $\mathfrak{A}$  with  $h(0) = 1$ .

From (3.4) and identity (1.11) we get

$$\frac{1}{\alpha + \beta} \frac{z(\mathfrak{L}_\beta^\alpha f(z))'}{\mathfrak{L}_\beta^\alpha f(z)} + \left( 1 - \frac{1}{\alpha + \beta} \right) = \frac{\mathfrak{L}_\beta^{\alpha-1} f(z)}{\mathfrak{L}_\beta^\alpha f(z)}. \tag{3.5}$$

Logarithmic differentiation of (3.5), together with (3.4), gives us

$$\begin{aligned} \frac{z(\mathfrak{L}_\beta^{\alpha-1} f(z))'}{\mathfrak{L}_\beta^{\alpha-1} f(z)} &= h(z) + \frac{\frac{1}{\alpha+\beta}zh'(z)}{\frac{1}{\alpha+\beta}h(z) + \frac{\alpha+\beta-1}{\alpha+\beta}} \\ &= h(z) + \frac{zh'(z)}{h(z) + \alpha + \beta - 1}. \end{aligned}$$

Since  $f(z) \in k - \mathcal{ST}^*(\alpha - 1, \beta)$ , it follows that

$$h(z) + \frac{zh'(z)}{h(z) + \alpha + \beta - 1} \prec p_k(z).$$

Applying Lemma 2.1, we have

$$h(z) \prec p_k(z).$$

This proves our result. □

**Theorem 3.5** *Let  $\alpha \geq 2$  and  $\beta > -1$ . Then  $k - \mathcal{ST}_s(\alpha - 1, \beta) \subset k - \mathcal{ST}_s(\alpha, \beta)$ .*

*Proof* Let  $f(z) \in k - \mathcal{ST}_s(\alpha - 1, \beta)$ . Then, using Theorems 3.1 and 3.3, we have

$$\psi(z) = \frac{f(z) - f(-z)}{2} \in k - \mathcal{ST}(\alpha - 1, \beta) \subset k - \mathcal{ST}(\alpha, \beta).$$

From this it easily follows that  $f(z) \in k - \mathcal{ST}_s(\alpha, \beta)$ , and this completes the proof. □

A similar result for the class  $k - \mathcal{ST}_s^*(\alpha, \beta)$  can be easily proved.

**Theorem 3.6** *Let  $\alpha \geq 1$  and  $\beta > 0$ . Then  $k - \mathcal{UK}_s(\alpha - 1, \beta) \subset k - \mathcal{UK}_s(\alpha, \beta)$ .*

*Proof* Let  $f(z) \in k - \mathcal{UK}_s(\alpha - 1, \beta)$ . Then there exists  $g(z) \in k - \mathcal{ST}_s(\alpha - 1, \beta)$  such that

$$\frac{2z(\mathcal{I}_\beta^{\alpha-1}f(z))'}{\mathcal{I}_\beta^{\alpha-1}g(z) - \mathcal{I}_\beta^{\alpha-1}g(-z)} = \frac{z(\mathcal{I}_\beta^{\alpha-1}f(z))'}{\mathcal{I}_\beta^{\alpha-1}\psi(z)} \in P,$$

where  $\psi(z) = \frac{\mathcal{I}_\beta^{\alpha-1}g(z) - \mathcal{I}_\beta^{\alpha-1}g(-z)}{2} \in k - \mathcal{ST}(\alpha - 1, \beta) \subset k - \mathcal{ST}(\alpha, \beta)$  in  $\mathfrak{A}$ .

Let us set

$$\frac{z(\mathcal{I}_\beta^\alpha f(z))'}{\mathcal{I}_\beta^\alpha \psi(z)} = p(z), \tag{3.6}$$

where  $p(z)$  is analytic in  $\mathfrak{A}$  with  $p(0) = 1$ . Then by (3.6) and identity (1.10) we get

$$\frac{\mathcal{I}_\beta^{\alpha-1}\psi(z)}{\mathcal{I}_\beta^\alpha\psi(z)} = (1 - \gamma)p_0(z) + \gamma,$$

where  $p_0(z) = \frac{z(\mathcal{I}_\beta^\alpha\psi(z))'}{\mathcal{I}_\beta^\alpha\psi(z)}$ , and  $\gamma$  is given by (3.3). Now by simple computations we obtain

$$\begin{aligned} \frac{z(\mathcal{I}_\beta^{\alpha-1}f(z))'}{z\mathcal{I}_\beta^{\alpha-1}\psi(z)} &= \frac{z(\mathcal{I}_\beta^{\alpha-1}f(z))'}{\mathcal{I}_\beta^\alpha\psi(z)[(1 - \gamma)p_0(z) + \gamma]} \\ &= \frac{z[(z(\mathcal{I}_\beta^\alpha f(z)))'] + \beta z(\mathcal{I}_\beta^\alpha f(z))'}{(\beta + 1)\mathcal{I}_\beta^\alpha\psi(z)[(1 - \gamma)p_0(z) + \gamma]} \\ &= \frac{\beta p(z) + p(z)p_0(z) + zp'(z)}{(\beta + 1)[(1 - \frac{\beta}{1+\beta})p_0(z) + \frac{\beta}{1+\beta}]} \\ &= \frac{\beta p(z) + p(z)p_0(z) + zp'(z)}{p_0(z) + \beta} \\ &= p(z) + \frac{zp'(z)}{p_0(z) + \beta}. \end{aligned}$$

Since  $f(z) \in k - \mathcal{UK}_s(\alpha - 1, \beta)$ , it follows that

$$p(z) + \frac{zp'(z)}{p_0(z) + \beta} \in \mathcal{P} \quad \text{in } \mathfrak{A}.$$

Applying Lemma.2.2, we have  $p(z) \in \mathcal{P}$  in  $\mathfrak{A}$ . This proves  $f(z) \in k - \mathcal{UK}_s(\alpha, \beta)$  in  $\mathfrak{A}$ . □

By a similar argument we can easily prove the following inclusion result.

**Theorem 3.7** *Let  $\alpha \geq 1$  and  $\beta > 0$ . Then  $k - \mathcal{UK}^*(\alpha - 1, \beta) \subset k - \mathcal{UK}^*(\alpha, \beta)$ .*

**Theorem 3.8** *Let  $f(z) \in k - \mathcal{ST}_s(\alpha, \beta)$  in  $\mathfrak{A}$ . Then*

$$\Re \left\{ \frac{z(\mathcal{I}_\beta^{\alpha-1} f(z))'}{\mathcal{I}_\beta^{\alpha-1} \varphi(z)} \right\} > 0$$

for  $|z| < R(\beta, \gamma_0)$ , where

$$R(\beta, \gamma_0) = \frac{(1 + \beta)}{(2 - \gamma_0) + \sqrt{(2 - \gamma_0)^2 + (1 + \beta)(\beta + 2\gamma_0 - 1)}}$$

with

$$\gamma_0 = \frac{k}{k + 1}. \tag{3.7}$$

*Proof* Let  $f(z) \in k - \mathcal{ST}_s(\alpha, \beta)$ . Then

$$\varphi(z) = \frac{f(z) - f(-z)}{2} \in k - \mathcal{ST}(\alpha, \beta),$$

and hence

$$\frac{z(\mathcal{I}_\beta^\alpha f(z))'}{\mathcal{I}_\beta^\alpha \varphi(z)} \in \mathcal{P}(p_k) \subset \mathcal{P}(\gamma_0),$$

where  $\gamma_0$  is given by (3.7). Let

$$\begin{aligned} \frac{z(\mathcal{I}_\beta^\alpha f(z))'}{\mathcal{I}_\beta^\alpha \varphi(z)} &= h(z), \quad h(z) \in \mathcal{P}(\gamma_0), \\ &= (1 - \gamma_0)h_0(z) + \gamma_0, \quad h_0(z) \in \mathcal{P}. \end{aligned} \tag{3.8}$$

Then, proceeding as in Theorem 3.5, we have

$$\frac{z(\mathcal{I}_\beta^{\alpha-1} f(z))'}{\mathcal{I}_\beta^{\alpha-1} \varphi(z)} = h(z) + \frac{zh'(z)}{p(z) + \beta}, \tag{3.9}$$

where  $p(z) = \frac{z(\mathcal{I}_\beta^\alpha \varphi(z))'}{\mathcal{I}_\beta^\alpha \varphi(z)} \in \mathcal{P}(\gamma)$ . Using (3.8) and  $p(z) = (1 - \gamma_0)p_0(z) + \gamma_0$  in (3.9), we have

$$\frac{z(\mathcal{I}_\beta^{\alpha-1} f(z))'}{\mathcal{I}_\beta^{\alpha-1} \varphi(z)} = (1 - \gamma_0)h_0(z) + \gamma_0 + \frac{(1 - \gamma_0)zh'_0(z)}{(1 - \gamma_0)p_0(z) + \gamma_0 + \beta}$$

with  $h_0(z) \in \mathcal{P}$ ,  $p_0(z) \in \mathcal{P}$ , that is,

$$\frac{1}{1 - \gamma_0} \left[ \frac{z(\mathcal{I}_\beta^{\alpha-1} f(z))'}{\mathcal{I}_\beta^{\alpha-1} \varphi(z)} - \gamma_0 \right] = h_0(z) + \frac{zh'_0(z)}{(1 - \gamma_0)p_0(z) + \gamma_0 + \beta}.$$

Using the distortion result for the class  $\mathcal{P}$ , we obtain

$$\begin{aligned} & \Re \left[ \frac{1}{1-\gamma_0} \left\{ \frac{z(\mathcal{I}_\beta^{\alpha-1} f(z))'}{\mathcal{I}_\beta^{\alpha-1} \varphi(z)} - \gamma_0 \right\} \right] \\ & \geq \Re h_0(z) \left\{ 1 - \frac{\frac{2r}{1-r^2}}{(1-\gamma_0)\frac{1-r}{1+r} + (\gamma_0 + \beta)} \right\} \\ & = \Re h_0(z) \left\{ 1 - \frac{2r}{(1-\gamma_0)(1+r)^2 + (1-r^2)(\gamma_0 + \beta)} \right\}. \end{aligned} \tag{3.10}$$

Right-hand side of (3.10) is greater than or equal to zero for  $|z| < R(\beta, \gamma_0)$ , where  $R(\beta, \gamma_0)$  is the least positive root of the equation

$$T(r) := (1 - \beta - 2\gamma_0)r^2 - 2(2 - \gamma_0)r + (1 + \beta) = 0,$$

that is,

$$\begin{aligned} R(\beta, \gamma_0) &= \frac{2(2 - \gamma_0) - \sqrt{4(2 - \gamma_0)^2 + 4(1 + \beta)(\beta + 2\gamma_0 - 1)}}{2(1 - \beta - 2\gamma_0)} \\ &= \frac{(1 + \beta)}{(2 - \gamma_0) + \sqrt{(2 - \gamma_0)^2 + (1 + \beta)(\beta + 2\gamma_0 - 1)}}. \end{aligned}$$

The proof is completed. □

*Particular Cases*

(i) For  $\beta = 0$  and  $\gamma_0 = \frac{k}{k+1} = 0$  (i.e.,  $k = 0$ ), we have  $f(z) \in \mathcal{S}_s^*(\alpha, 0)$  ( $\psi \in \mathcal{S}^*(\alpha, 0)$ ) and

$$R(0, 0) = \frac{1}{2 + \sqrt{3}}.$$

(ii) For  $k = 1$  and  $\beta = 0$ ,

$$R\left(0, \frac{1}{2}\right) = \frac{1}{3}.$$

(iii) For  $k = 1$  and  $\beta = 1$ ,

$$R\left(1, \frac{1}{2}\right) = \frac{4}{4 + \sqrt{17}}.$$

**Theorem 3.9** *Let  $\mathcal{L}_\beta^\alpha f(z) \in k - \mathcal{ST}$ . Then*

$$\mathcal{L}_\beta^{\alpha-1} f(z) \in \mathcal{S}^*(\gamma_0), \quad \gamma_0 = \frac{k}{k+1}$$

for  $|z| < R_1$ , where

$$R_1(\alpha, \beta, \gamma_0) = \frac{\alpha + \beta}{2 - \gamma_0 + \sqrt{(2 - \gamma_0)^2 + (\alpha + \beta)(2\gamma_0 + \alpha + \beta - 2)}}.$$

*Proof* Since  $\mathcal{L}_{\beta}^{\alpha} f(z) \in k - \mathcal{ST}$ , we have

$$\frac{z(\mathcal{L}_{\beta}^{\alpha} f(z))'}{\mathcal{L}_{\beta}^{\alpha} f(z)} = h(z), \quad h(z) \prec p_k(z)$$

in  $\mathfrak{A}$ . With a similar argument as in Theorem 3.5, we have

$$\frac{z(\mathcal{L}_{\beta}^{\alpha-1} f(z))'}{\mathcal{L}_{\beta}^{\alpha-1} f(z)} = h(z) + \frac{zh'(z)}{h(z) + \alpha + \beta - 1},$$

that is,

$$\begin{aligned} & \Re \left[ \frac{1}{1 - \gamma_0} \left\{ \frac{z(\mathcal{L}_{\beta}^{\alpha-1} f(z))'}{\mathcal{L}_{\beta}^{\alpha-1} f(z)} - \gamma_0 \right\} \right] \\ &= \Re \left[ h_0(z) + \frac{zh_0'(z)}{(1 - \gamma_0)h_0(z) + (\gamma_0 + \alpha + \beta - 1)} \right] \\ &\geq \Re h_0(z) \left[ 1 - \frac{\frac{2r}{1-r^2}}{(1 - \gamma_0)\frac{1-r}{1+r} + (\gamma_0 + \alpha + \beta - 1)} \right], \end{aligned} \tag{3.11}$$

where

$$h(z) = (1 - \gamma_0)h_0(z) + \gamma_0, \quad h_0 \in \mathcal{P}, \gamma_0 = \frac{k}{k + 1}.$$

The right-hand side of (3.11) is greater than or equal to zero for  $|z| < R_1$ , where  $R_1$  is the least positive root of the equation

$$T(r) := (2 - 2\gamma_0 - \alpha - \beta)r^2 - 2(2 - \gamma_0)r + \alpha + \beta = 0,$$

that is,

$$\begin{aligned} R_1(\alpha, \beta, \gamma_0) &= \frac{2 - \gamma_0 - \sqrt{(2 - \gamma_0)^2 + (\alpha + \beta)(2\gamma_0 + \alpha + \beta - 2)}}{2(2 - \alpha - \beta - 2\gamma_0)} \\ &= \frac{\alpha + \beta}{2 - \gamma_0 + \sqrt{(2 - \gamma_0)^2 + (\alpha + \beta)(2\gamma_0 + \alpha + \beta - 2)}}. \end{aligned}$$

This completes the proof. □

### 4 Conclusion

In this paper, we have defined some new classes of analytic functions involving integral operators. We have shown that these classes generalize the well-known classes, and already existing results can be obtained as a particular cases of our results. Inclusion relations of these classes are also a significant part of our work. We believe that the work presented in this paper will give researchers a new direction and will motivate them to explore more interesting facts on similar lines.

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**Authors' contributions**

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