

TOPOLOGY, LOGIC AND SET THEORY I

Gary G. Miller

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Abstract. Standard topology is formulated in terms of the adherence of one subspace to another. Equivalently, this can be expressed in terms of the (asymmetric) separation of one subspace from another. A single intuitive axiom suffices. This abstractly characterizes "adherence" as a relational morphism which associates "union" with "or" and "arbitrary union" with "existential quantification." A function turns out to be continuous just in case it preserves adherence.

1. Introduction

"Space" and "continuity" are profound concerns of an extensive literature that crosses disciplines and spans the history of thought. Yet, there is nothing whatsoever in the usual contemporary mathematical formulation of these matters to suggest "spatiality" might have a fundamental role in the scheme of things. Here we show standard set theoretic topology can be formulated so as to display a compellingly simple interrelation between logic, set theory and topology.

One might be wont to say a function $f : X \rightarrow Y$ is continuous just in case it does not "tear" spatial structure. Yet, the informal intent of "continuous map" as "spatial morphism" is not, of course, expressed in usual definitions.

2. Continuity as Preservation of Adherence

Let X be a topological space. Say a subset Y is *separated from* a subset Z and write $Z \mid\langle Y$ just in case there is an open set, disjoint from Z , that contains Y — as the symbol $\mid\langle$ suggests. This directed or asymmetric concept of "separation", which can be equivalently defined by $Z \mid\langle Y \equiv (Y \cap \text{Closure}(Z) = \emptyset)$, has not seen wide use in topology since its introduction by Vaidyanathaswamy (1947). The more familiar *mutual separation* of Mazurkiewicz (1920) is given by $Z \mid\langle Y \equiv ((Z \mid\langle Y) \text{ and } (Y \mid\langle Z))$. For a

development of topology in terms of mutual separation, see [1]. Notice a set U is open just in case it is separated from its complement, $X - U \mid\langle U$. Moreover, a point x adheres to a subset Y just in case $\text{not}(\{x\} \mid\langle Y)$. Thus, we say a subset Z *adheres to* a subset Y , denoted $Z \text{ K } Y$, just in case $\text{not}(Z \mid\langle Y)$.

PROPOSITION 1. A function $f : X \rightarrow Y$, where X and Y are topological spaces, is continuous if and only if

$$Z \text{ K } W \text{ implies } f[Z] \text{ K } f[W]$$

for all Z, W, X , i.e., f is a K -morphism.

Proof. If f is continuous and U is an open set containing $f[W]$ but not intersecting $f[Z]$, then $f^{-1}[U]$ is an open set that contains W while not intersecting Z . Thus, if f is continuous, it is a K -morphism. Suppose, then, f is a K -morphism. Let U be any open subset of $\text{image}(f)$ and $W = (\text{image}(f)) - U$. Then $W \mid\langle U$ in $\text{image}(f)$ and, therefore, in Y . By hypothesis, $f^{-1}[W] \mid\langle f^{-1}[U]$. But, $f^{-1}[W] = X - f^{-1}[U]$. Thus $f^{-1}[U]$ is open, and, therefore, f is continuous. ■

At this stage we have a suggestion that adherence, K , would be the choice for the formalization of spatial structure. However, there are known alternatives to standard topology in which the analogues to "continuity" are morphisms, e.g., for nearness spaces and for uniform continuity in the case of proximity spaces [2]. The deciding factor remains to be the simplicity of some set of characterizing properties for an abstract relation of adherence.

3. The Adherence Axioms

In the spirit of the Kuratowski Closure Axioms, call a binary relation K , with domain the power set 2^X of any specified nonempty set X , an *abstract adherence relation for X* just in case it satisfies the three *adherence axioms*:

A_0 : $Y K Z$ holds only for nonempty Y and Z

A_1 : $(Y \cup Z) K W \equiv (Y K W) \vee (Z K W)$

A_2 : $W K \left[\bigcup_{\alpha \in \beta} Y_\alpha \right] \equiv \bigvee_{\alpha \in \beta} (W K Y_\alpha)$

for all subsets Y, Z and W of X and all collections $\{Y_\alpha : \alpha \in \beta\}$ of subsets. Here the logical signs \vee (or), \bigvee (there exists) will be used to accentuate the "morphism" nature of A_1 and A_2 .

Call a subset U of X K -open just in case $(X-U)K U$ does not hold. As before, write $\text{not}(K)$ as $|\langle$.

Axiom A_0 immediately yields the desired result: X and \emptyset are K -open sets. To see this, take one of Y and Z to be X and the other \emptyset . (Since one can treat X and \emptyset as exceptional cases of open sets, axiom A_0 is but an expositional convenience.)

If desired, axioms A_1 and A_2 can be combined in a single axiom:

$$A : \left[\bigcup_{i \in \eta} Z_i \right] K \left[\bigcup_{\alpha \in \beta} Y_\alpha \right] \equiv \bigvee_{i \in \eta} \bigvee_{\alpha \in \beta} (Z_i K Y_\alpha)$$

where Z_i and Y_α are subsets of X , η and β are index sets and η is finite.

To simplify matters, let us introduce some terminology. Let $()R()$ be any binary relation. Distinguish the "slots" by $(pre)R(post)$. A unary relation $Q() = ()R(Z)$ determined by a fixed choice of Z is called a pre - R relation; similarly, $S() = (Z)R()$, for fixed Z , is called a $post$ - R relation. By a pre -property of R , we mean a property that holds for every pre - R relation. Similarly, each choice of Z yields a $post$ - R relation, and a property that holds for all these is called a $post$ -property of R . A bi -property of R is a property that is both a pre -property and a $post$ -property of R . The following are some unary relational properties of interest.

Let $Q()$ be a unary relation on 2^X . Then, Q satisfies the property stated provided the corresponding formula holds for all indicated subsets and collections of subsets of X :

Monotone: if $Y \subseteq Z$, then, $Q(Y)$ implies $Q(Z)$.

Dissection:
$$Q\left[\bigcup_{\alpha \in \beta} Y_\alpha\right] \equiv \bigvee_{\alpha \in \beta} Q(Y_\alpha)$$

or, equivalently, $Q(Y) \equiv Q(\{y\})$ for some $y \in Y$.

Finite Dissection:
$$Q(Y_1 \cup Y_2) \equiv Q(Y_1) \vee Q(Y_2)$$

Superposition:
$$Q\left[\bigcup_{\alpha \in \beta} Y_\alpha\right] \equiv \bigwedge_{\alpha \in \beta} Q(Y_\alpha).$$

Finite Superposition:
$$Q(Y_1 \cap Y_2) \equiv Q(Y_1) \wedge Q(Y_2).$$

Axiom A₁, then, states *finite dissection is a pre-property of adherence* while *Axiom A₂* states (arbitrary) *dissection is a post-property of K*. Of course, by logically restating *A₁* and *A₂* with equivalent negations of the stated equivalences, we have statements about separation instead of adherence. Then, *Axiom A₁* states *finite superposition is a pre-property of separation*, and *Axiom A₂* states (arbitrary) *superposition is a post-property of |<*.

PROPOSITION 2. Adherence K is bi-monotone, and separation $|<$ is bi-hereditary.

Proof. Suppose $Y \subseteq Z$ and $Y K W$ for some W . Since, logically, $(Y K W \text{ or } Z K W)$ follows from $Y K W$, we conclude from the finite dissection property that $(Y \cup Z) K W$. But, $Y \cup Z = Z$. Therefore, $Z K W$, and K is pre-monotone. The argument for post-monotonicity, of course, is the same with pre and post interchanged and with the use of the post (arbitrary) dissection property of K . Thus, K is bi-monotone. Next, suppose $Z \subseteq Y$ and $Y |< W$ for some W . Then, since $Y = (Y-Z) \cup Z$, by pre-finite superposition, we have $Z |< W$. Thus, separation is pre-hereditary. The post-hereditary case is similar. Thus, separation is bi-hereditary. ■

In other words, *subsets of separated sets are separated* and *supersets of adhering sets adhere*. Let $\mathcal{S}(K)$ be the collection of all K -open sets, sets that are separated from their complements.

PROPOSITION 3. $\mathcal{T}(K)$ is a topology. Moreover, if K is the adherence relation of some topology \mathcal{T} , then $\mathcal{T} = \mathcal{T}(K)$.

Proof. Assume Y_1 and Y_2 are K -open. Then, $(X - Y_1) \mid\langle Y_1$ and $(X - Y_2) \mid\langle Y_2$. By post-heredity, $(X - Y_1) \mid\langle (Y_1 \cap Y_2)$ and $(X - Y_2) \mid\langle (Y_1 \cap Y_2)$. By pre-superposition, $((X - Y_1) \cup (X - Y_2)) \mid\langle (Y_1 \cap Y_2)$. By de Morgan, $(X - (Y_1 \cap Y_2)) \mid\langle (Y_1 \cap Y_2)$. Thus $Y_1 \cap Y_2$ is K -open. Assume $\{Y_\alpha : \alpha \in \beta\}$ is a collection of K -open sets. Then, $(X - Y_\alpha) \mid\langle Y_\alpha$ for $\alpha \in \beta$. By post-heredity, for each $\gamma \in \beta$, $(X - Y_\gamma) \mid\langle Y$ where $Y = \bigcap_{\alpha \in \beta} Y_\alpha$. By pre-superposition, $(\bigcup_{\gamma \in \beta} (X - Y_\gamma)) \mid\langle Y$. By generalized de Morgan, $(X - Y) \mid\langle Y$. Thus, $\bigcap_{\alpha \in \beta} Y_\alpha$ is K -open. Therefore $\mathcal{T}(K)$ is a topology since \emptyset and X are K -open. Now, assume K is the adherence relation of some topology on X . If $Y \in \mathcal{T}$, then $(X - Y) K Y$ so $Y \in \mathcal{T}(K)$, and if $Y \in \mathcal{T}(K)$, then $(X - Y) K Y$ so $Y \in \mathcal{T}$. ■

4. Standard Topology as the Theory of Separation/Adherence

To summarize, a topological space can be regarded as a nonempty set together with a binary relation K on the nonempty subsets of X such that

$$A : \left[\bigcup_{i \in \eta} Z_i \right] K \left[\bigcup_{\alpha \in \beta} Y_\alpha \right] \equiv \bigvee_{i \in \eta} \bigvee_{\alpha \in \beta} (Z_i K Y_\alpha)$$

where Z_i and Y_α are nonempty subsets of X , η and β are index sets and η is finite. This displays the "association" one might desire between the set theoretic \cup , \bigcup and the logical \vee , \bigvee respectively. Moreover, $f : X \rightarrow Y$ is continuous just in case f preserves K . In this way, topology is displayed in terms of "structure and morphism" and is evidently capable of being placed more toward the foundations of mathematics.

REFERENCES

- [1] Szymanski, P. (1941). La notion des ensembles separees comme terme primitif de la topologie, *Mathematica*, 17:65–84.
- [2] Thron, W.J. (1966). *Topological Structures*, Holt, Rinehart and Winston, New York.
- [3] Vaidyanathaswamy, R. (1947). *Treatise on Set Topology, I*. Madras [2nd Ed.], New York, 1968.

Department of Mathematics
University of Victoria
P.O. Box 1700
Victoria, B.C. V8W 2Y2
CANADA