IRREDUNDANT RAMSEY NUMBERS

by

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B. Sc., University of Victoria, 1987

A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

in the Department
of
Mathematics

We accept this thesis as conforming
to the required standard

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ISBN 0-315-46516-6

Abstract

Given a graph G=(V, E), a set of vertices S is *irredundant* if for no vertex v in S is the closed neighbourhood of v contained in the union of the closed neighbourhoods of the vertices in S-{v}. The *irredundant Ramsey Number* s(m, n) is the least value of p such that for any p-vertex graph G, either G has an irredundant vertex subset of at least n vertices or its complement $\overline{G}$ has an irredundant vertex subset of at least m vertices. The existence of these numbers is guaranteed by Ramsey's theorem. We prove that s(3, 3) = 6, s(3, 4) = 8, s(3, 5) = 12, and s(3, 6) = 15.

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Acknowledgements

The author wishes to thank the following: My supervisor, E. J. Cockayne, and C. M. Mynhardt for their many helpful discussions. The Natural Science and Engineering Research Council for financial support. Finally Richard Chadwick, at the University of Victoria Micro-Computer Demonstration Facility, for providing help in the word processing of this thesis.

Chapter 1

Introduction

In 1930, Frank Ramsey proved a remarkable result which has been the subject for a great amount of research. A special case of Ramsey's theorem can be stated as follows: "Given two positive integers, m and n, there exists a minimum integer p, such that any p-vertex graph either contains an independent vertex subset of size n, or its complement contains an independent vertex subset of size m." The number p is called a Ramsey number and is denoted by r(m, n). These numbers have proved very difficult to evaluate and only seven nontrivial values have ever been calculated. A generalization of the definition, by Chvátal and Harary[10], has provided a fruitful area of research in which many results have been published. The purpose of this thesis is to present a new generalization of these numbers and calculate four nontrivial values.

We generalize the definition to: "Given two positive integers, m and n, there exists a minimum integer p, such that any p-vertex graph either contains an *irredundant* vertex subset of size n, or its complement contains an *irredundant* vertex subset of size m." We call p the *irredundant Ramsey Number* s(m, n). The existence of these numbers is guaranteed by Ramsey's theorem. The property of irredundance which generalizes independence was first introduced in 1976 by Cockayne, Hedetniemi, and Miller[15]. A set of vertices S is *irredundant* if for each vertex v in S, the closed neighbourhood of v is not contained in the union of the closed neighbourhoods of the vertices in S-{v}. Since irredundance is a more complicated property than independence, we anticipate that these numbers will also be very difficult to evaluate.

In Chapter 2, background material on graph theory, irredundance, and Ramsey theory is given and we survey known results in (i) independence, domination, and irredundance, (ii) Ramsey theory, and (iii) generalized Ramsey theory.

In Chapter 3, we calculate the first three nontrivial irredundant Ramsey numbers. We prove that s(3, 3) = 6, s(3, 4) = 8, and s(3, 5) = 12. The number s(3, 3) = 6 follows directly from Ramsey theory. We prove s(3, 4) = 8 and s(3, 5) = 12 by contradiction. The construction used for these proofs is similar. However, the construction proves too complicated to calculate s(3, 6). In Chapter 4, we prove s(3, 6) = 15 using a new technique.

We feel that irredundant Ramsey numbers can be calculated for all values for which classical Ramsey numbers are known. The numbers calculated so far have proved some what more difficult to calculate than the corresponding Ramsey numbers. Each evaluation appears to be a new and challenging problem.

Chapter 2

Preliminaries

2.1 - Graph Theory

The purpose of this section is to provide an overview of the basic graph theory used in this thesis. If more detail is required, the reader is refered to Bondy and Murty[6] or Harary[26].

A *graph* G is an ordered pair of sets (V(G), E(G)). V(G) is called the set of *vertices*. The elements of E(G) are unordered pairs of distinct vertices and are called *edges*. When the graph in question is clear from context, we will drop the G and simply write (V, E). For the remainder of this section let G and H be graphs.

If u and v are vertices of G, an edge $\{u, v\} \in E$ is said to be *incident* with the vertices u and v. The vertices u and v are called the *ends* of the edge. We will write uv to denote such an edge. We also say u and v are *incident with an edge* e.

Two vertices u, v in V are said to be *adjacent* if there exists an edge with which both vertices are incident. Vertices u and v are *nonadjacent* if no such edge exists. A vertex v is *adjacent (nonadjacent) to a set of vertices T* if every vertex in T is adjacent (nonadjacent) to v.

We let p(G) and q(G) denote the number of vertices and edges in G, respectively. Again we will drop the G and write p, q when the graph is clear from context.

A *subgraph* H of G, written $G \supseteq H$, is a graph whose vertex set is a subset of

V(G), and whose edge set is a subset of the edges of G which have both ends in V(H).

Suppose T is a nonempty subset of V(G). The subgraph of G whose vertex set is T and whose edge set is the set of edges of E(G) with both ends in T, is the called the subgraph of G *induced* by T, denoted by G[T]. We say G[T] is an *induced subgraph* of G.

The *union of G and H*, denoted $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

A *path* of length n is a set of n+1 distinct vertices, say $v_0, v_1, \ldots, v_n$, such that v_i is adjacent to v_{i+1} for $i = 0, 1, 2, \ldots, n-1$. We denote such a path by P_n. We say P_n is a *path from v_0 to v_n*. A *cycle* or *circuit* of length n is a set of n distinct vertices, say $v_1, \ldots, v_n$, such that v_i is adjacent to v_{i+1} for $i = 1, 2, \ldots, n-1$ and v_n is adjacent to v_1. We denote such a cycle as C_n. If n is odd (even) we say C_n is an *odd cycle* (*even cycle*). Given a cycle $v_0, v_1, \ldots, v_n$, a *chord* is any edge from v_i to v_j where $|i-j| \neq 1 \mod n$.

A graph is *connected* if there exists a path from u to v for any two distinct vertices u and v. A connected graph with no cycles is called a *tree*.

A *complete* graph G is a graph in which every two distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n. A *clique* is a subgraph H of G such that H is a complete graph. An *independent set (of vertices)*, T, in G is a set such that G[T] contains no edges. A *independent set of edges* is a set in which no two edges are incident with the same vertex.

G is called *bipartite* if the vertices of G can be partitioned into two sets R and S such that R and S are both independent sets. Notice each edge must have one end in R and one end in S. A *complete bipartite* graph is a bipartite graph where each vertex in R is adjacent to S If |R| = n and |S| = m then we denote the complete bipartite graph by $K_{n,m}$ The following is a well known characterization of bipartite graphs

Theorem 2.1.1 A graph is bipartite if and only if it contains no odd cycles.

Two graphs G and H are called *isomorphic* if there exists a function f: V(G) $\rightarrow$ V(H) such that f is one to one, onto and f(u)f(v) $\in$ E(H) if and only if uv $\in$ E(G). An isomporphism from G to itself is called an *automorphism*. The set of automorphisms of a graph together with function composition forms a group called the *automorphism group of G*. A *cyclic graph* is one which contains the cyclic group $Z_{p(G)}$ as a subgroup of the automorphism group of G.

The *complement* of G, denoted $\overline{G}$, is the graph with vertex set V(G) where uv $\in E(\overline{G})$ if and only if uv $\notin$ E(G). Notice if T is a clique in G then V(T) is an independent set in $\overline{G}$. G is called *selfcomplementary* if G is isomorphic to $\overline{G}$.

The *degree* of a vertex v $\in$ V(G), denoted $deg_G(v)$, is the number of edges in E(G) with which v is incident. We will write deg(v) if the graph in question is clear from context. The following result is well known.

Proposition 2.1.2 Let G be a graph.

$$\sum_{v \in V} deg(v) = 2q$$

Corallary 2.1.3 The sum of the degrees of vertices in G is even.

A graph is *k-regular* if every vertex has degree k. The minimum degree taken over all vertices in G is denoted by $\delta(G)$. The maximum degree is denoted by $\Delta(G)$

2.2 - Independence, Domination, and Irredundance

Let G be a graph. The *open neighbourhood* of the vertex v denoted by N(v) is given by $N(v) = \{ u \in V \mid uv \in E \}$. The *closed neighbourhood of v is* $N[v] = \{v\} \cup N(v)$. More generally we define the *closed neighbourhood of a subset* X, $V \supseteq X$, by $N[X] = \bigcup_{x \in X} N[x]$.

If X, Y are subsets of V, X *dominates* Y if and only if $N[X] \supseteq Y$. If X dominates V, then X is called a *dominating set of G*.

A vertex x is called *redundant in a subset X* if $N[X-\{x\}] \supseteq N[x]$ and a set X is called *irredundant* if it contains no vertex which is redundant in X. Thus, a set X is irredundant if and only if for each $x \in X$, N[x] has a vertex which is not in the union of closed neighbourhoods of the remainder of X. The *private neighbourhood,* I(x, X) of x in X is defined by

$$I(x, X) = N[x] - N[X-\{x\}]$$

We will use the following facts about irredundance which are immediate from the definition:

(i) A set X is irredundant if and only if for each $x \in X$, $I(x, X) \neq \varnothing$.

(ii) Any independent set Y is also irredundant since for each $y \in Y$, $y \in I(y, Y)$.

(iii) Any element x of an irredundant set X of G which is not isolated in G[X], has a private neighbour outside the set X.

These ideas can be illustrated by the following example. Let a graph G represent a radio broadcast network. Let the vertices of G be cities and make two cities adjacent if a radio signal can be sent from one city to the other. Finally, let the set T represent the cities containing a transmitter, all of which broadcast the same signal.

All the members of T, plus any vertex adjacent to a member of T receive the transmitted signal. This is exactly what we called N[T]. T is a dominating set if N[T] = V, in other words, every city in the network receives the transmitted signal.

A redundant element, $t \in T$, is a city containing a transmitter such that when the transmitter in t is turned off, all the cities which previously received the signal continue to receive it. That is, N[T] is not reduced with the removal of t ($N[T - \{t\}] \supseteq N[T]$). T is irredundant if no such transmitter exists.

Consider the network in figure 1. We let the transmitters be located at vertices {1, 4, 5}. This is not a dominating set since {8} does not receive the signal. To test whether the set is irredundant we use the following scheme. Construct a table with one row for each member $v \in T$. On that row, list v together with N(v). The vertices which appear in the table are all the elements of N[T]. Cross out any element which appears more that once. Circle all the elements which appear exactly once. Notice if vertex u appears in the row associated with v, and u is circled, then v is not redundant. Removing v from T will remove u from N[T], since u appears in no other rows. Therefore, T is irredundant if each row has at least one element circled. The circled elements in the row associated with v are precisely the private neighbours of v (I(v, T)). We can see in our example each vertex in T has at least one private neighbour; and hence, T is irredundant. Suppose we were to add {8} to T. In our table we would add 8 together with N(8) = {2, 3}. Now, 3 appears in both N(4) and N(8), thus, 3 must be crossed out. At this point 4 no longer has a private neighbour. Therefore, $T \cup \{8\}$ is not an irredundant set.

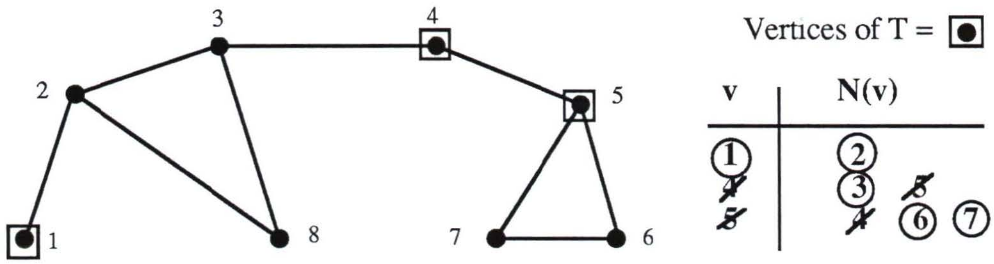

Figure 1 Example of an irredundant set

We notice irredundance is a generalization of independence. If a set T is independent, then each vertex v ∈ T is not adjacent to any other member u ∈ T, i.e. v ∉ N(u). Therefore, if we construct the table of private neighbours as we did in our previous example, each vertex in T will be its own private neighbour. That is, in the row corresponding to v ∈ T, v will be circled since it will not appear in any other row. Thus, each vertex in T has a private neighbour and hence T is irredundant.

Extremal independent, dominating, and irredundant sets are related by the following well-known results.

Proposition 2.2.1 (Berge[3]). If X is maximal independent, then X is minimal dominating.

Proposition 2.2.2 (Berge[4]). X is a maximal independent set if and only if X is independent and dominating.

Proposition 2.2.3 (Cockayne, Hedetniemi and Miller [15]). If X is minimal dominating, then X is maximal irredundant.

Proposition 2.2.4 (Cockayne, Hedetniemi and Miller [15]). A set of vertices is irredundant and dominating if and only if it is a minimal dominating set.

We now define six parameters concerning these types of vertex subsets.

The *lower (upper) independence numbers* $i(G)$ ($\beta(G)$), *domination numbers* $\gamma(G)$ ($\Gamma(G)$) and *irredundance numbers* $ir(G)$ ($IR(G)$) are respectively the smallest (largest) cardinalities of maximal independent, minimal dominating and maximal irredundant sets of vertices of G.

It follows from the above propositions that for any graph G

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G).$$

A great deal of recent research has been done with these six parameters. For an excellent survey see Hedetniemi, Laskar, and Pfaff[27]. We present a sample of well known results.

Bollobás and Cockayne[5] proved the following using the properties of I(x, X).

Theorem 2.2.5 Suppose that a vertex u is not dominated by a maximal irredundant set X. Then for some x in X

(a) I(x, X) is contained in N(u), and

(b) for x_1, x_2 in I(x, X) such that $x_1 \neq x_2$, either x_1 and x_2 are adjacent or for $i = 1, 2$ there exits t_i in X - {x} such that x_i is adjacent to each vertex of $I(t_i, X)$.

They also generalized a result of Allan and Laskar[2].

Theorem 2.2.6 For any graph G, $\gamma \leq 2ir - 1$

The first results involving the parameter IR(G) were published by Cockayne, Favaron, Payan, and Thomason [14].

Theorem 2.2.6 If a graph G = (V, E) has no isolated vertices, then the complement V - X of every irredundant set X is a dominating set.

Corollary 2.2.7 For any graph G with p vertices, $\gamma + IR \leq p$.

Theorem 2.2.8 If a graph G has p vertices, none of which is isolated, and $\gamma + IR = p$, then $\beta = \Gamma = IR$.

Theorem 2.2.9 If a graph G has p vertices and minimum degree $\delta \geq 2$, then $\gamma + IR \leq p - \delta + 2$.

Theorem 2.2.10 For any bipartite graph G, $\beta = \Gamma = IR$.

Theorem 2.2.11 For any graph G, $i + \beta \leq 2p + 2\delta - 2\sqrt{(2p\delta)}$.

They conjectured the following theorem which was later proved by O. Favaron[21].

Theorem 2.2.12 For any graph G, $i + IR \leq 2p + 2\delta - 2\sqrt{(2p\delta)}$.

For a vertex u which is not dominated by the maximal irredundant set X, we define $X_u = \{x \in X \mid N(u) \supseteq I(x,X)\}$.

Theorem 2.2.13 Suppose $IR > \Gamma$. Then for all maximum irredundant sets X which dominate the greatest number of vertices and each u not dominated by X, $|X_u| \geq 2$.

Cockayne, Favaron, Payan, and Thomason also presented a graph with all unequal parameters. ($ir = 2, \gamma = 3, i = 4, \beta = 7, \Gamma = 9$, and $IR = 10$)

Cockayne and Mynhardt[16] showed that for any sequence of 6 integers $m_1, \ldots, m_6$ such that $1 < m_1 \leq m_2 \leq \ldots \leq m_6$ and $m_2 < 2m_1$, there exists a graph such that $ir = m_1, \gamma = m_2, i = m_3, \beta = m_4, \Gamma = m_5$, and $IR = m_6$.

2.3 - Ramsey Theory

One of first problems given in any graph theory course is the following "Given any group of six people, prove either three people are mutual friends or three people are mutual strangers." Let p be a person in a group of six people. Partition the five remaining people into two sets. Let F denote the set of friends of p and let S denote the set of people with whom p is a stranger.

Suppose the set F contains three or more people. If all of these people are mutual strangers, then we have a set of three mutual strangers. On the other hand, if any two members of F are friends, then these two together with p are a set of three mutual friends.

If F does not contain three or more people then S must contain at least three people since $|S| + |F| = 5$. If all the members of S are mutual friends, then a set of at least three mutual friends exists. On the other hand, if any two members of S are strangers, then these two together with p form a set of three mutual strangers.

We can use this result to solve a similar problem: "Given a group of ten people prove either four people are mutual friends or three people are mutual strangers." Let p be a person in a group of ten people. As before, let F be the friends of p and S be the people whom p does not know. Either F contains at least six people or S contains at least four people, since $|F| + |S| = 9$.

Suppose F contains at least six people. From our previous result, we know either three people in F are mutual strangers or three people in F are mutual friends. In the latter case, p together with these three friends forms a set of four mutual friends.

If F does not contain at least six people, then S must contain four. If all the people in S are mutual friends then a set of four friends exists. If two people in S are strangers, then p together with these two forms a set of three mutual strangers.

The following theorem generalizes these two situations.

Theorem 2.3.1 Let $m_1, m_2, \ldots, m_k$ all be at least r, then there exists a smallest integer

$$N = r(m_1, \ldots, m_k, r)$$

so large that if we have any set of N objects, and we partition all the r-subsets of this set arbitrarily into k classes, $C_1, C_2, \ldots, C_k$, then there are m_x objects all of whose r-subsets are in class C_x, for some $x = 1, 2, \ldots, k$.

This theorem was first proved in 1930 by Frank Ramsey[32]. It was rediscovered and popularized by Erdös and Szekeres in 1935[19]. The integer N is called the Ramsey number and denoted by $r(m_1, \ldots, m_k, r)$.

In the above examples, we proved $r(3, 3, 2) \leq 6$ and $r(3, 4, 2) \leq 10$. Consider a set, S, of people. We partition the 2-subsets of S in the following fashion. Let $u, v \in S$. If u and v are strangers, then $\{u, v\}$ is placed in C_1. If u and v are friends then $\{u, v\}$ is placed in C_2. Suppose $|S| \geq N(m_1, m_2, 2)$, Ramsey's theorem says either there exists a subset of S of size at least m_1, all of whose 2-subsets are in C_1, or there exists a subset of S of size at least m_2, all of whose 2-subsets are in C_2. i.e. There are m_1 mutual strangers in S or there are m_2 mutual friends in S.

Increasing the number of classes from 2 to m is equivalent to increasing the types of relationships between people, from two (friends and strangers) to m.

Graph theory is a powerful tool for examining Ramsey's theorem when r is two. If we let the objects of our set, S, be the vertices of the complete graph K_N, then the 2-subsets of S are the edges of the K_N. Partitioning the 2-subsets of S is equivalent to partitioning the edges of K_N into m classes Quite often we call this partitioning an *edge-colouring of K_n*

A *k-colouring* of the edges of a graph G is a function f E(G) → {1, 2, , k} Ramsey's theorem says there exists an integer N such that whenever the edges of K_N are k-coloured arbitrarily, a complete subgraph of m_i vertices, all of whoses edges are colour i, exists for some i When r is two, we will simply write the Ramsey number as $r(m_1, \ldots, m_k)$

In this thesis we will be intrested in the case when k = 2 Our previous examples can be stated as follows "If the edges of K_6 (K_{10}) are coloured red and blue arbitrarily, either a set of three vertices forms a red K_3 or a set of three (four) vertices forms a blue K_3 (K_4) " Hence, $r(3,3) \leq 6$ and $r(3, 4) \leq 10$

A natural correspondence exists between a graph G on n-vertices and a 2-coloured K_n Let the edges of G be the red edges in the K_n and the edges of $\overline{G}$ be the blue edges in K_n Now a red clique in K_n is a clique in G, which is an independent set in $\overline{G}$ Similarily, a blue clique in K_n is an independent set in G i e a clique in $\overline{G}$

We may now define Ramsey numbers in terms of independence The number r(m, n) is the minimum integer p such that any graph on p vertices either has $\beta(\overline{G}) \geq m$ or $\beta(G) \geq n$ To illustrate with our two examples, given a graph G on six (ten) vertices, either G contains an independent set of size at least three (four) or $\overline{G}$ contains an independent set of size at least three

The problem of computing exact values for r(m, n) has proved to be very difficult For a survey of known results and bounds see Chung and Grinstead[8] In 1955, Greenwood and Gleason[25] calculated four nontrivial values (One can easily verify r(1, n) = 1 and r(2, n) = n) Since that time, only three more values have been computed. Table 1 contains all known values together with upper and lower bounds for some other numbers. Trivially, r(m, n) = r(n, m).

Only one three colour Ramsey number has ever been calculated. Namely, r(3, 3, 3) = 17 by Greenwood and Gleason[25].

n \ m	3	4	5	6	7	8	9
3	6	9	14	18	23	28/29	36
4		18	25/28	34/44			
5			42/55				

Table 1 All known 2 colour Ramsey numbers, r(m,n), and some bounds

The problem of computing a Ramsey number, r(m, n) involves two steps. Firstly, a lower bound for the number is obtained by contructing a graph G on p_1 vertices with $\beta(\overline{G}) < m$ and $\beta(G) < n$ and hence $r(m, n) > p_1$. G is called a *(m, n)-graph*. Secondly, one must show that *every* graph on p_2 vertices must have either $\beta(\overline{G}) \geq m$ or $\beta(G) \geq n$. This establishes $r(m, n) \leq p_2$. In the event that $p_1 + 1 = p_2$, a Ramsey number has been calculated.

While no general method exists for constructing graphs to obtain lower bounds, the study of cyclic graphs has been very useful. Cyclic graphs attain the lower bound for r(3,3), r(3,4), r(3,5), r(3,9) and r(4,4). In other cases however, no cyclic (m,n)-graph exists on r(m, n) - 1 vertices. See Chung and Grinstead[8].

The advantage of using cyclic graphs is that the number of graphs which must be examined is greatly reduced, making a computer search feasible. New lower bounds for r(4,7), r(4, 8), r(4, 9), r(5, 7) and r(5, 8) were recently established by Radziszowski and

Kreher[31] using computer searches

The following result due to Erdös and Szekeres[19] and Greenwood and Gleason[25] has been useful for computing upper bounds of r(m, n) when m and n are small It appears that upper bounds too large are produced when m or n gets large

Theorem 2.3.2 For any two integers $m \geq 2$, $n \geq 2$,

$$r(m, n) \leq r(m, n - 1) + r(m - 1, n)$$

Furthermore, if r(m, n - 1) and r(m - 1, n) are both even, then strict inequality applies

Much better upper bounds can be produced when m = 3 using techniques developed by Graver and Yackel[24] and by Kalbfleisch[29]

For $v \in V$, denote the graph induced by the set of vertices adjacent to v by $H_1(v)$ and the graph induced by the set of vertices nonadjacent to v by $H_2(v)$ We say the vertex v is *preferred*

If G is a (m, n)-graph and v is a preferred vertex, then $H_1(v)$ and $H_2(v)$ are (m-1, n) - and (m, n-1) - graphs respectively

Lemma 2.3.4 (Graver and Yackel [24]) If G is a (m, n)-graph on p vertices, then r(m-1, n)-1 is the maximum possible degree for any vertex of G and p - r(m, n-1) is the minimum possible degree for a vertex of G

Given a (3, n)-graph G with preferred vertex v, $H_1(v)$ must be an independent set, hence, the maximum possible degree in G is n - 1 This implies every edge must be in $H_2(v)$ or be incident with exactly one neighbour of v Let Z(v) denote the sum of the degrees of the neighbours of v

This implies in any (3, n) - graph G with preferred vertex v, $q(G) = Z(v) + q(H_2(v))$ If we had a lower bound on the number of edges in (3, n - 1) -graphs (i e

$H_2(v)$), we would have a lower bound on q(G) - Z(v). If we knew something of the degrees of vertices in Z(v), we could maximize q(G) - Z(v). In the event this maximum value is less than the lower bound for $q(H_2(v))$ calculated above, we would have a contradiction to show G is not a (3, n)-graph. See Kalbfleisch[29].

A great deal of work has also been done on asymptotic values for r(m, n). Using the recurrence inequality,

$$r(m, n) \leq r(m - 1, n) + r(m, n - 1)$$

we can conclude,

$$r(m, n) \leq \binom{m + n - 2}{m - 1}.$$

This implies

$$r(m, m) \leq c\, 4^m\, m^{-1/2}$$

for some constant c. Graver and Yackel [24] proved that

$$r(3, n) \leq c\, (n^2 \log \log n) / (\log n).$$

In general,

$$r(m, n) \leq c\, (n^{m-1} \log \log n) / (\log n),$$

where the constant c depends on m. For fixed n > 3 they conjectured that r(m, n) = m^{n-1+c}, asymptotically in m, where c is a constant. Very recently Ajtai, Komlós and Szemerédi[1] have shown

$$R(3, n) < c\, n^2 / \log n.$$

Erdös used an existence argument to construct the following lower bound.

Theorem 2.3.6 (Erdos [17]) If

$$\binom{t}{n} 2^{1-\binom{n}{2}} < 1$$

then r(n, n) > t.

Using Stirling's formula,

$$r(n, n) > n\, 2^{n/2} [1 / (e \sqrt{2}) + o(1)]$$

where o(1) is some term such that $\lim_{n\to\infty} o(1)/n = 0$

The proof of this theorem does not construct an explicit graph, but uses a probabilistic argument to show there exists an (n, n) - graph on t vertices.

The lower bound can be improved by a factor of 2 using the Lovász Local Theorem. (see Spencer [35]).

Theorem 2.3.8 (Lovász Local Theorem [18]) Let G be a finite graph with maximal degree d and vertices 1, ..., m. Let $A_1, \ldots, A_m$ be events in a probability space such that A_i is independent of $\{A_j : \{i,j\} \in E(G)\}$. Assume $P(A_i) \le p$ for $1 \le i \le m$. If 4dp < 1 then

$$P(\overline{A}_1 \cdots \overline{A}_m) > 0.$$

When m and n are distinct, the following result of Spencer [35] generalized Erdos' argument.

Theorem 2.3.7 If for some p, 0 < p < 1,

$$\binom{t}{n} p^{\binom{n}{2}} + \binom{t}{m} (1 - p)^{\binom{m}{2}} < 1$$

then

$$r(m, n) > t$$

Erdős proved using the probabilistic method when $m = 3$

$$r(3, n) \geq c\, k^2 / (\log k)^2$$

Notice this is very close to the upperbound of Ajtai, Komlós, and Szemerédi[1].

2.4 - Generalized Ramsey Theorey

The new results in this thesis are a generalization of the Ramsey numbers above. In this section we will sample the results of another generalization. For an excellent survey of this area see Burr [7].

Let $G_1, \ldots, G_m$ be graphs. The *(generalized) Ramsey number* $r(G_1, \ldots, G_m)$ is the least number n such that when the edges of K_n are coloured arbitrarily with m colours, there exists an i such that the i^{th} coloured subgraph contains a G_i. Although very few Ramsey numbers are known, a remarkable number of generalized Ramsey numbers have been calculated.

The classical Ramsey numbers are produced when each of the G_i's are complete graphs. Since a complete graph on n vertices contains any graph on n vertices as a subgraph, the classical Ramsey numbers are upper bounds for generalized Ramsey numbers. Hence, generalized Ramsey numbers are finite.

The simplest result in generalized Ramsey theory is $r(G, K_2) = n$, where G is any graph on n vertices. Colour the edges of K_n red and blue. If no blue edges exist, then G is a subgraph of the graph induced by the red edges, since all the edges are red and G must be a subgraph of K_n. If a blue edge exists in the colouring, then a blue K_2 exists. To establish the lower bound, colour a K_{n-1} red. Neither a red G nor a blue K_2 exists.

We now present five generalized Ramsey theory results.

Theorem 2.4.1 (Gerencsér, Gyárfás [22]) If $m \geq n$, then

$$r(P_m, P_n) = m + [n/2] - 1.$$

Theorem 2.4.2 (Rosta, Faudree, Schelp [33, 34, 20]) If $m \geq n$, n odd, and $(m, n) \neq (3, 3)$, then

$$r(C_m, C_n) = 2m - 1.$$

If $m \geq n$, m, n even, and $(m, n) \neq (4, 4)$, then

$$r(C_m, C_n) = m + n/2 - 1.$$

If $m > n$, m odd, n even, then

$$r(C_m, C_n) = \max(m + n/2 - 1, 2n - 1).$$

Finally,

$$r(C_3, C_3) = r(C_4, C_4) = 6.$$

Theorem 2.4.3 (Chvátal, Harary [12])

$$r(K_n, 2K_2) = n + 2.$$

If G has n points and is not complete, then

$$r(G, 2K_2) = n + 1.$$

(Here $2K_2$ denotes $K_2 \cup K_2$. This notation generalizes in the obvious way.)

Theorem 2.4.4 (Chvátal [9]) If T is a tree on m points, then

$$r(T, K_n) = (m-1)(n-1) + 1.$$

Theorem 2.4.5 (Cockayne [13]) Let T be a tree on m points which has a point of degree one adjacent to a vertex of degree two. Then

$$r(T, K_{1,n}) = m + n - 2,$$

provided one of the following four conditions holds:

$$m \equiv 0, 2 \pmod{m - 1}$$

$$m \not\equiv 1 \pmod{m - 1} \text{ and } n \geq (m - 3)^2$$

$$m \not\equiv 1 \pmod{m - 1} \text{ and } n \equiv 1 \pmod{m - 2}$$

or

$$n \equiv -1 \pmod{m - 1} \text{ and } n > m - 2.$$

Chapter 3

Irredundant Ramsey Numbers

In this chapter we define the irredundant Ramsey numbers, s(m, n), and prove s(3,3)=6, s(3,4)=8 and s(3,5)=12.

3.1 - Definition of Irredundant Ramsey Numbers

In the previous chapter we presented results in Ramsey theory and generalized Ramsey theory. In both cases, the results always have the same form. If the edges of a sufficiently large complete graph are partitioned into k classes and the subgraph induced by each class is examined, we are guaranteed to find a predetermined structure in one of the subgraphs. In the case of classical Ramsey numbers, the structure is an complete graph (or alternativly an independent set) of predetermined size. In the case of generalized Ramsey numbers, the structure is a predetermined subgraph.

We can define anlogous numbers for irredundance. Informally, if we partition the edges of a sufficiently large complete graph into two classes, and hence produce two subgraphs, an irredundant set of size m must appear in the first subgraph or an irredundant set of size n must appear in the second subgraph, where m and n are positive integers. Again, partitioning the edges of a complete graph on p vertices into two classes is equivalent to selecting an arbitrary graph on p vertices -- one class representing the edges of the graph, the second class representing the edges of the complement of the graph. The problem is to evaluate how large the complete graph must be in order to guarantee the desired structure. We define this number formally. The *irredundant Ramsey number* s(m,n) is the least integer p such that any p-vertex graph G satisfies $IR(\overline{G}) \geq m$ or $IR(G) \geq n$.

Since $IR(G) \geq \beta(G)$ for all G, it follows that $s(m, n) \leq r(m, n)$ i.e., any graph with $\beta(\overline{G}) \geq m$ or $\beta(G) \geq n$ must also have $IR(\overline{G}) \geq m$ or $IR(G) \geq n$. The same recurrence inequality which holds for Ramsey numbers also holds for irredundant Ramsey numbers. The proof is analogous and is omitted.

Proposition 3.1.1 For all positive integers $m \geq 2$, $n \geq 2$,

$$s(m, n) \leq s(m-1, n) + s(m, n-1),$$

with strict inequality when s(m-1, n) and s(m, n-1) are both even.

The inequality for classical Ramsey numbers proved very useful in the calculation of r(3,3), r(3,4), r(3,5) and r(4,4). (See Greenwood and Gleason [25]). However, the upper bounds for s(3,4), s(3, 5), and s(3, 6) given by the inequality are larger than the exact values. We have developed new methods for the calculation of these numbers.

For ease of explanation we will abbreviate IR(G), $IR(\overline{G})$ to IR, $\overline{IR}$, etc. and will refer to the edges of G as red edges and the edges of $\overline{G}$ as blue edges, respectively. Denote by $C(v_1, \ldots, v_n)$ the cycle of G with vertex sequence $v_1, \ldots, v_n$. Extensive use will be made of the following result.

Lemma 3.1.2 The graph $\overline{G}$ has an irredundant set of size 3 if and only if there is a red

K_3 or there is a red 6-cycle $C(v_1, v_2, \ldots, v_6)$ and the edges v_1v_4, v_2v_5, and v_3v_6 are blue.

Proof. Let $X = \{x, y, z\}$ be irredundant in $\overline{G}$. There are three cases to consider.

(i) X is independent in $\overline{G}$ and hence forms a red K_3.

(ii) $\overline{G}[X]$ has exactly one isolate x in $\overline{G}$ and thus xy, xz are red and yz is blue. Since y, z must have private neighbours in $\overline{G}$, say y_1 and z_1 repectively, xy_1 and zy_1 are red, making xzy_1 a red K_3.

(iii) $\overline{G}[X]$ has no isolates. Then in $\overline{G}$, x, y, z have distinct private neighbours x_1, y_1, z_1 in V - X. It follows that xy_1, xz_1, yx_1, yz_1, zx_1, and zy_1 are all red, while xx_1, yy_1, zz_1 are blue. Therefore $C(x, y_1, z, x_1, y, z_1)$ is the required cycle.

The converse in obvious. ■

3.2 - Calculation of s(3, 3), s(3, 4), s(3, 5).

Theorem 3.2.1 $s(3, 3) = 6$.

Proof. Firstly, $s(3,3) \leq r(3,3) = 6$ and secondly, the graph C_5 has neither an induced K_3 nor a 6-cycle. Therefore by lemma 3.1.2, $IR(\overline{C_5}) \leq 2$. Since C_5 is self-complementary, $IR(C_5) \leq 2$ and hence $s(3, 3) > 5$. ■

Using lemmas 3.2.2, 3.2.3, and 3.2.4, we now prove that there is no 8-vertex graph G with $IR < 4$ and $\overline{IR} < 3$. In each of these lemmas, G is an assumed counterexample to this statement. We remark that the non-existence of G could also be extablished by showing that each of the three 8-vertex graphs which have $\beta(\overline{G}) \leq 2$ and $\beta(G) \leq 3$ (see [30], p. 363), have $IR(G) = 4$. However, instead we will illustrate the use of lemma 3 1 2

Lemma 3 2 2 Each vertex of G has degree 2 or 3

Proof If G has a vertex u with degree at least four, consider the set U = {u_1, u_2, u_3, u_4} where N(u) ⊇ U If U is independent in G, then U is irredundant in G and IR ≥ 4, otherwise some $u_i u_j$ is red, $uu_i u_j$ is a red K_3 and by lemma 3 1 2 $\overline{IR} \geq 3$

If G has a vertex u of degree one or less, then there is a set U of six vertices which are not adjacent to u Since r(3,3) = 6, either U contains a red K_3 (impossible by lemma 3 1 2) or a blue K_3 In the latter case the blue K_3 together with u constitutes a blue K_4 whose vertex set is irredundant in G ■

Lemma 3 2 3 G does not have adjacent vertices of degree 3

Proof Let V = {1, , 8} and 1, 2 be adjacent vertices of degree 3 Since there is no red K_3, N(1) ∩ N(2) = ∅ and we take N(1) = {8, 7} and N(2) = {3, 4} It follows that 34 and 78 are blue, as are all remaining edges joining 1 and 2

<u>Case 1</u> Suppose 56 is red

Vertex 5 is adjacent to a vertex of {3, 4}, otherwise {1, 3, 4, 5} is a blue K_4 We assume 45 is red Similarily, to avoid the blue K_4 {2, 5, 7, 8} we take 57 red Vertex 6 must send a red edge to {3, 4} to avoid the blue K_4 {1, 3, 4, 6} If 46 is red, then{4, 5, 6} forms a red K_3, hence, 46 is blue and 36 is red Similarly 68 is red and 67 is blue To avoid red K_3's, we deduce 38, 35, and 47 are blue

There is now a red 6-cycle C(1, 2, 3, 6, 5, 7) and by lemma 3 1 2 we deduce that 16, 25, and 37 cannot all be blue, otherwise $\overline{IR} \geq 3$ Hence 37 is red In future uses of this 6-cycle structure with lemma 3 1 2, we will abbreviate the previous sentences to

"By C(1, 2, 3, 6, 5, 7), 37 is red" In fact, the next deduction is of this type By C(1, 2, 4, 5, 6, 8), 48 is red Every vertex in G is now saturated (i e has degree 3) and by lemma 3 2 2 no more red edges may be added However, the set X = {1, 2, 3, 7} is irredundant in G The private neighbours are as follows $8 \in I(1,X)$, $4 \in I(2, X)$, $6 \in I(3, X)$, and $7 \in I(7, X)$ This ends case 1

Case 2 Suppose 56 is blue

Each vertex of {3, 4, 7, 8} sends a red edge to either 5 or 6 For example, if 35 and 36 are blue, then {1, 3, 5, 6} is a blue K_4 which implies $IR \geq 4$ Further there is a red edge between {3, 4} and {7, 8}, otherwise these vertices form a blue K_4 We assume without losing generality that 35 and 37 are red and deduce that 57 is blue Since 7 sends a red edge to {5, 6}, the edge 67 is red Now both 3 and 7 have degree 3, therefore, all remaining edges to 3 and 7 are blue The set X = {4, 5, 6, 8} is irredundant in G, since $2 \in I(4, X)$, $3 \in I(5, X)$, $7 \in I(6, X)$, and $1 \in I(8, X)$ Notice no edges added to the graph will destroy these private neighbours, since {1, 2, 3, 7} all have degree three already This completes the proof ■

Lemma 3.2 4 G does not have adjacent vertices of degree two

Proof Suppose the contrary and let 1 and 2 be adjacent in G with degree two Assume 18 and 23 are red It follows that all edges from {1, 2} to {4, 5, 6, 7} and the edges 13 and 28 are blue Let Q = { $v \in V$ | v3 and v1 are blue } Since 3 sends at most two red edges to {4, 5, 6, 7} and 1 sends no red edges to {4, 5, 6, 7}, $|Q| \geq 2$ If there is a blue edge $q_1q_2 \in Q$, then $\{q_1, q_2, 1, 3\}$ is a blue K_4 Hence Q is a red clique By lemma 3 1 2, no red K_3 can exist in G, hence $|Q| \leq 2$ Thus $|Q| = 2$ and we assume 34 and 35 are red, 36 and 37 are blue and deduce that 67 is red Similarly, 8 sends exactly two red edges to {4, 5, 6, 7} and by lemma 3.2.2, edge 83 is blue. If both 48 and 58 are red, then each vertex of Y = {1, 2, 3, 4, 5, 8} either has degree three in G or is adjacent to a vertex of degree three in G. Therefore by lemmas 3.2.2 and 3.2.3, no further red edges may be added to vertices of Y. This implies 6 and 7 have degree one in G, contrary to lemma 3.2.2. The edges 87 and 86 cannot both be red and so we may assume without losing generality that 87 and 85 are red while 86 and 84 are blue. To avoid the blue K_4 {2, 4, 6, 8}, we deduce that 46 is red. By lemma 3.2.3, all remaining edges joining vertices of {4, 5, 6, 7} are blue. By C(3,4, 6, 7, 8, 5), $\overline{IR} \geq 3$. ■

Theorem 3.2.5 s(3,4) = 8.

Proof. Suppose the 8-vertex graph G has IR ≤ 3 and $\overline{IR} \leq 2$. Lemmas 3.2.2, 3.2.3, and 3.2.4 imply that G is bipartite with independent sets U_1 and U_2, the sets of vertices of degree two and degree three respectively. Either U_1 or U_2 has at least four vertices and is irredundant, which shows that IR ≥ 4. Therefore, no such G can exist and s(3, 4) ≤ 8.

The graph C_7 has IR = 3 and $\overline{IR} = 2$, hence s(3,4) > 7. ■

Theorem 3.2.6 s(3,5) = 12.

Proof. As a first step in showing that s(3, 5) ≤ 12, we prove any counterexample, i.e. a 12-vertex graph G with IR < 5 and $\overline{IR} < 3$, is regular of degree four. Let v be a vertex in G. Suppose N(v) ⊇ X = {v_1, v_2, ..., v_5}. Either X is independent, in which case IR ≥ 5, or some v_iv_j is red, in which case vv_iv_j is a red K_3, i.e. $\overline{IR} \geq 3$ by lemma 3.1.2.

If v has degree at most three, then let Y = V - N[v]. Since s(3,4) = 8 and |Y| ≥ 8, it follows that $IR(\overline{G}[Y]) \geq 3$ which implies $\overline{IR} \geq 3$, or IR(G[Y]) ≥ 4. In the latter case v together with an irredundant set of G[Y] forms an irredundant set of G, i e. IR ≥ 5.

Let V = {1, ..., 12} and without losing generality assume that N(1) = {2, 10, 11, 12} and N(2) = {1, 3, 4, 5}. It follows that {2, 10, 11, 12} and {1, 3, 4, 5} are blue K_4's and that all edges joining {1, 2} to {6, 7, 8, 9} are blue. The four vertices {6, 7, 8, 9} cannot be independent (otherwise {1, 6, 7, 8, 9} forms a blue K_5) nor can these vertices contain a red K_3. Further, each vertex of {6, 7, 8, 9} sends at least one red edge to each of the sets {10, 11, 12} and {3, 4, 5} to avoid blue K_5's. If for example, vertex 6 did not send a red edge to {3, 4, 5}, then {1, 3, 4, 5, 6} forms a blue K_5. Therefore, no vertex of G[{6, 7, 8, 9}] has degree three. Hence, the induced (red) subgraph is one of the following five shown in Figure 2. In all five cases, the edges 68 and 79 are blue and 89 is red.

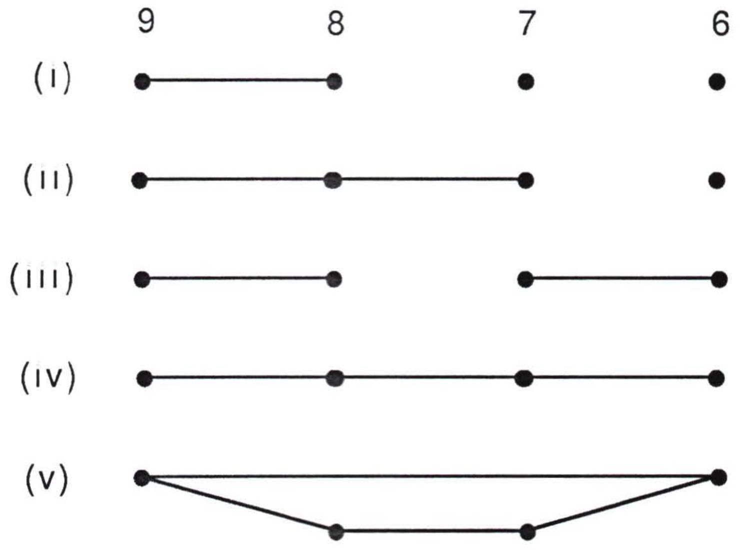

Figure 2. Possibilities for G[{6, 7, 8, 9}]

Vertex 9 sends a red edge to {3, 4, 5} to avoid the blue K_5 { 1, 3, 4, 5, 9}. Hence we assume 59 is red and by a similar argument to avoid the blue K_5 {2, 9, 10, 11, 12}, we may suppose 9 10 is red. The vertex 8 also sends a red edge to both {3, 4, 5} and {10, 11, 12} but cannot have a red neighbour in common with 9, otherwise a red K_3 results. We assume, without losing generality, that 84 and 8 11 are both red and deduce 9 11, 8 10, 5 10, 4 11, 49, 58 are all blue.

We next use C(1, 11, 8, 9, 5, 2) and C(1, 10, 9, 8, 4, 2) to deduce 5 11 and 4 10 are red.

Two cases are now required to show that each of the possibilities of Figure 2 is impossible.

<u>Case 1</u>. Suppose G[{6, 7, 8, 9}] is one of the graphs of Figure 2(i) or (ii).

In G[{6, 7, 8, 9}] we know 89 is red, 78 may be either blue or red and the remaining edges are blue.

If 9 12 were red, then C(1, 12, 9, 8, 4, 2) implies 12 4 is red which gives vertex 4 degree four in G. Then {1, 4, 6, 7, 9} is a blue K_5 and we deduce that 9 12 is blue.

Similarly, if 39 is red, C(2, 3, 9, 8, 11, 1) requires 3 11 red. Now 11 is saturated and {2, 6, 7, 9, 11} is a blue K_5. Therefore 39 is blue.

Hence only the three red edges 59, 10 9, and 89 are incident with 9, contrary to the 4-regular property established above.

<u>Case 2</u>. Suppose G[{6, 7, 8, 9}] is one of the graphs of Figure 2(iii), (iv), or (v). In G[{6, 7, 8, 9}] we assume 76 is red, 78 and 96 may be either blue or red and the remaining edges 68 and 79 are blue. These assumptions cover all the three cases of Figure 2(iii), (iv), and (v).

Vertices 7 and 6 must each send a red edge to {10, 11, 12} to avoid the blue

K_5's {2, 10, 11, 12, 6} and {2, 10, 11, 12, 7}. Further 7 and 6 can not join the same vertex of {10, 11, 12} with red edges, otherwise a red K_3 results. Therefore, there is a red edge from {7,6} to {10, 11} and we assume 7 10 is red. It follows that 6 10 and 47 are blue in order to avoid the red K_3's {6, 7, 10} and {4, 10, 7}.

Vertex 10 now has degree four in G. Hence 3 10 is blue and C(2, 1, 10, 7, 6, 3) implies 63 is blue. Also, C(2, 1, 10, 7, 6, 5) implies that 65 is blue. We deduce that 64 is red to avoid the blue K_5 {1, 3, 4, 5, 6}.

Vertex 4 now has degree four in G and so 4 12 is blue. This requires 7 11 to be blue by C(4, 10, 7, 11, 5, 2) and 7 12 to be blue by C(7, 12, 1, 2, 4, 6).

We now deduce that 9 12 is red to avoid the blue K_5 {2, 7, 9, 11, 12}. But this is impossible by C(4, 2, 1, 12, 9, 8). This completes the proof of Case 2 and hence $s(3,5) \leq 12$.

The lower bound for s(3, 5) was established using a computer search on cyclic graphs with the program in Appendix A. The 11-vertex graph depicted in Figure 3 has $IR = 4$ and $\overline{IR} = 2$, hence $s(3, 5) > 11$. ■

The proof methods for theorems 3.2.5 and 3.2.6 are obviously very similar. If a similar proof is used to calculate s(3,6), the number of cases to examine is much larger. For this reason a new proof technique is used to show $s(3,6) = 15$. This is presented in chapter 4.

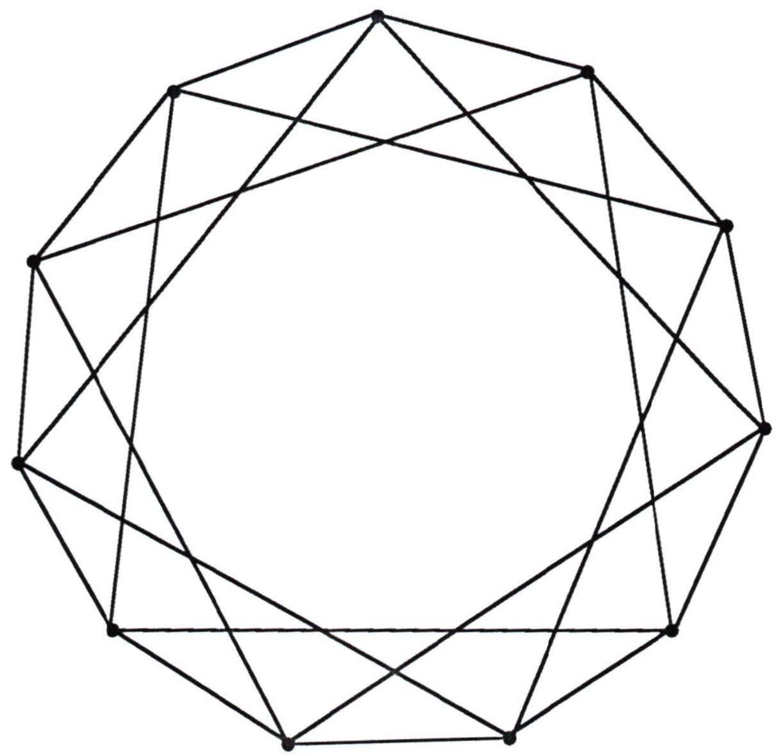

Figure 3 An 11-vertex graph with $IR=4$ and $\overline{IR} = 2$

Chapter 4

Calculation of s(3,6)

In this chapter we show every graph on 15 vertices must have $\overline{IR} \geq 3$ or $IR \geq 6$. In the following G is an assumed counterexample, i.e. a 15-vertex graph with $\overline{IR} < 3$ and $IR < 6$. The calculation is finally completed by the presentation of a 14-vertex graph with $\overline{IR} = 2$ and $IR = 5$.

Lemma 4.1 Each vertex of G must have degree three, four, or five.

Proof Suppose there exists a vertex v in G with degree greater than five. Let $N(v) \supseteq X = \{ v_1, v_2, \ldots, v_6 \}$. If X contains a red edge, then a red K_3 is formed together with v and hence $\overline{IR} \geq 3$. On the other hand, if X contains no red edges, then a blue K_6 exists and we have $IR \geq 6$.

Suppose vertex v has degree two or less. Let $Y = V - N[v]$. Since $s(3,5) = 12$ and $|Y| \geq 12$, we conclude $IR(\overline{G}[Y]) \geq 3$ or $IR(G[Y]) \geq 5$. In the latter case, we can add v to any irredundant set in Y and get $IR(G) \geq 6$. ■

4.1 - Any counterexample contains no vertex of degree five

In this section using a sequence of lemmas we show that G has no vertex of degree five. Suppose to the contrary that G has a vertex v of degree five. Let $Y = N(v) = \{a, b, c, d, e\}$ and let $X = V - N[v] = \{1, 2, 3, \ldots, 9\}$.

Lemma 4.1.1 Suppose vertices v_1, v_2, and v_3 in X form the vertex sequence of a red P_3 and v_1, v_3 both send red edges to Y. Then v_1 and v_3 have a common red neighbour in Y.

Proof Suppose to the contrary that v_1 and v_3 do not have a common red neighbour in Y. Assume v_1s and v_3t are red, where $s,t \in Y$. By C(v, s, v_1, v_2, v_3, t), we conclude $\overline{IR} \geq 3$ since v_1t and v_3s are blue by assumption. ■

Notice Y is a blue K_5, otherwise, a red K_3 is formed together with v. Also, every vertex in X must send a red edge to Y. If a vertex $u \in X$ did not send an edge to Y, then u together with Y forms a blue K_6, contrary to the assumption $IR < 6$. Therefore, every vertex in X sends a red edge to Y. Hence, lemma 4.1.1 is applicable in this case to all v_1, v_2, v_3 which form (in sequence) a red P_3 in X.

We will use lemma 4.1.1 many times in this chapter and for ease of presentation, we will write "By $P_3(v_1, v_2, v_3)$, v_1 and v_3 have a common red neighbour in Y."

Lemma 4.1.2 G[X] contains no 5-cycle.

Proof Assume G[X] contains a 5-cycle. Without loss of generality, let the red 5-cycle have vertex sequence 1, 2, 3, 4, 5. By $P_3(1, 2, 3)$, 1 and 3 must have a common red neighbour in Y. Assume 1a and 3a are red. Similarily, 2 and 4 have a common red neighbour in Y. Notice this vertex must be different from a since 2 cannot join a neighbour of 1, otherwise, a red K_3 is formed. Assume 2b and 4b are red. To avoid red K_3's 1b and 4a are blue. By C(v, a, 1, 5, 4, b), we conclude $\overline{IR} \geq 3$. ■

We now show using similar techniques that G[X] cannot contain a 7-cycle.

Lemma 4.1.3 G[X] contains no 7-cycle.

Proof Assume a red 7-cycle with vertex sequence 1, 2, 3, 4, 5, 6, 7 exists. As above, assume 1a, 3a, 2b, and 4b are red. Also by $P_3(3, 4, 5)$, 3 and 5 have a common red neighbour in Y.

<u>Case 1</u>. Suppose 6a is red.

Edge 3b is blue since 2b is red and 5a is blue since 6a is red, hence, assume 3c and 5c are red without loss of generality. Now, C(v, a, 6, 3, 4, b) exists and 4a must be blue since 3a is red. If 6b is blue, then by lemma 3.1.2, $\overline{IR} \geq 3$ contrary to our choice of G.

Therefore, 6b must be red. By $P_3(7, 1, 2)$, 2 and 7 have a common red neighbour in Y. Both 2a and 2c are blue since 3 joins both a and c. Edge 7b is blue since 6 joins b. Therefore, assume 2d and 7d are red.

Edge 7c must be blue to avoid C(v, c, 7, 1, 2, b). By C(v, c, 5, 6, 7, d), 5d is red. Now, C(v, a, 3, 4, 5, d) exists with 3d blue, since 2d is red and 5a blue, since 6a is red. Therefore, $\overline{IR} \geq 3$ by lemma 3.1.2.

<u>Case 2</u>. Suppose 6a is blue.

By $P_3(1, 7, 6)$, 1 and 6 have a common red neighbour in Y. Since 2b is red, 1b is blue. Assume 1c and 6c are red. Similarily, assume 5d and 7d are red. (Edges 7a and 7c are blue since 1a and 1c are red. Edge 5b is blue since 4b is red.)

By C(v, a, 1, 7, 6, b), 6b must be blue. By C(v, b, 4, 5, 6, c), we conclude 4c is red. This implies 3c is blue. Edge 2d must be blue to avoid C(v, c, 4, 3, 2, d). By C(v, d, 7, 1, 2, b), 7b must be red. Further, 3d must be blue to avoid C(v, c, 1, 2, 3, d). By C(v, a, 3, 4, 5, d), 5a must be red. Hence, C(v, a, 5, 6, 7, b) gives the contradiction $\overline{IR} \geq 3$. ■

Lemma 4.1.4 G[X] contains no 9-cycle.

Proof Suppose to the contrary that 1, 2, . . , 9 is the vertex sequence of a red 9-cycle. As before, assume without loss of generality that 1a, 3a, 4b, and 2b are red.

Case 1. Suppose 8a is red. By $P_3(5, 6, 7)$, 5 and 7 have a common red neighbour in Y. This vertex cannot be a or b since 8a is red, implying 7a is blue, 4b is red, implying 5b is blue. Therefore, assume 5c and 7c are red.

C(1, a, 3, 4, b, 9) and lemma 4.1.3 imply that 9b is blue. Since 1a is red, 2a is blue. Therefore, the common red neighbour of 9 and 2 in Y cannot be a or b. Suppose 2c and 9c are red. Then 3c is blue and by C(v, a, 3, 4, 5, c), 5a is red. Now, C(1, a, 5, 4, b, 2) implies $\overline{IR} \geq 3$. Therefore, it is not the case that both 2c and 9c are red. Hence we may assume that 2d and 9d are red.

By C(1, a, 8, 7, d, 2) and lemma 4.1.3, 7d is blue. By C(v, c, 7, 8, 9, d), 9c must be red and hence 2c must be blue. Now, C(v, b, 2, 1, 9, c) implies $\overline{IR} \geq 3$, contrary to our choice of G.

Case 2. Suppose 8a is blue.

By $P_3(1, 9, 8)$, 1 and 8 have a common red neighbour in Y. Assume 1c and 8c are red.

By C(1, a, 5, 4, b, 2) and lemma 4.1.3, 5a is blue. Edges 5b and 7c are blue to avoid red K_3's. Therefore, the common red neighbour of 7 and 5 is not a, b, or c. Assume 7d and 5d are red. By C(v, a, 3, 4, 5, d), 3d is red.

By C(1, a, 3, 4, b, 9) and lemma 4.1.3, 9b must be blue. Edges 9a, 9c, and 2d are blue to avoid red K_3's. Therefore, the only possible common red neighbour of 2 and 9 in Y is e. By C(v, b, 2, 1, 9, d), 9d is blue. Therefore, 7e is red by C(v, e, 9, 8, 7, d).

Now, C(1, c, 8, 7, e, 2) and lemma 4.1.3 imply $\overline{IR} \geq 3$, contrary to our choice of G. Therefore, G[X] contains no red 9-cycle. ■

Theorem 4.1.5 G contains no vertex of degree five.

Proof. Using lemmas 3.1.2, 4.1.2, 4.1.3, and 4.1.4, we conclude G[X] contains no 3-, 5-, 7-, or 9-cycles. Since |X| = 9 and G[X] contains no odd cycles, G[X] is bipartite. Therefore G[X] must have an independent set of at least size five. This set together with v forms a blue K_6 and hence IR ≥ 6. ■

4.2 - Any counterexample contains no vertex of degree three

At this point we know that the degree of any vertex in a 15-vertex graph G having $\overline{IR} < 3$ and IR < 6 is three or four. We now rule out the former case.

Lemma 4.2.1 If u and v are vertices of degree three in G, then either u and v are adjacent or $N[u] \cap N[v] = \varnothing$.

Proof. Suppose u and v are nonadjacent vertices of G of degree three such that $N[u] \cap N[v] \neq \varnothing$. Since $|N[v]| = |N[u]| = 4$ and the two sets are not disjoint, $|N[u] \cup N[v]| \leq 7$. This implies $|V(G) - (N[u] \cup N[v])| \geq 8$. Recall, s(3,4) = 8 and we chose G such that $\overline{IR}(G) < 3$. Hence $V(G) - (N[u] \cup N[v])$ contains an irredundant set Z of size 4. Then $Z \cup \{u, v\}$ is an irredundant set in G of size 6, contrary to our choice of G. ■

Theorem 4.2.2 G contains no vertex of degree three.

Proof. Suppose $v \in V$ is a vertex of degree three. Let N(v) = {a, b, c} and let X = V(G) - N[v]. Since |X| = 11, $\delta(G[X]) \geq 3$. To see this, suppose $x \in X$ had degree two or less in

G[X] and $T = X - N[x]$. Notice $|T| \geq 8$. Since $s(3,4) = 8$, either $IR(\overline{G}[T]) \geq 3$ or G[T] contains an irredundant set, Q, of size four. In the latter case $Q \cup \{v, x\}$ is an irredundant set of size six in G. Both cases contradict our choice of G.

Note that at most one of {a, b, c} has degree three in G. For if deg(a) = deg(b) = 3, then a and b are nonadjacent vertices of G of degree three with a common neighbour v, contrary to lemma 4.2.1. Further no vertex in X joins more than one vertex in {a, b, c} because $\delta(G[X]) \geq 3$ and $\Delta(G) < 5$. Suppose without loss of generality that $N(a) \cap X = \{1, 2, 3\}$, $N(b) \cap X = \{4, 5, 6\}$, $N(c) \cap X \supseteq \{7, 8\}$.

To avoid an independent set in {1, 2, ..., 6} of size 5, G[{1, 2, ..., 6}] must have two independent edges. If this were not the case, we could remove the one vertex to which all edges were incident. The five remaining vertices would form a blue K_5. This set together with v would give $IR(G) \geq 6$. Without loss of generality say 14 and 25 are red. By C(1, a, 2, 5, b, 4), 15 or 24 is red, assume 15 is red.

If {1, 2, 3, 7, 8} formed a blue K_5, then its union with v is a blue K_6. Hence {1, 2, 3, 7, 8} contains a red edge. Subsets {1, 2, 3} and {7, 8} have no red edges to avoid red K_3's. If 71 is red, then C(v, b, 5, 1, 7, c) exists but 7b and 5c are blue. Similarily 72, 81, and 82 are blue. Without loss of generality assume 83 is red. Similarily, 76 or 86 is red. By C(v, b, 6, 8, 3, a), 86 is blue and hence 76 is red. Neither 7 nor 8 is adjacent in G to any other vertices.

If 3 is adjacent to any $i \in \{4, 5, 6\}$, then C(v, b, i, 3, 8, c) implies $\overline{IR} \geq 3$. Similarily, 6 is nonadjacent to {1, 2, 3}. Hence, {3, 6, 7, 8} all have degree one in G[{1, 2, ..., 8}].

Now if c is adjacent to a vertex $9 \in X$, then to avoid the blue K_5 {4, 5, 7, 8, 9}, 95 or 94 is red. Both cases are impossible by C(v, a, 1, 4, 9, c) and C(v, a, 1, 5, 9, c), hence $deg_G(c) = 3$.

Let X - {1, 2, ..., 8} = {x, y, z}. No vertex in {x, y, z} sends red edges to vertices in more than one of the sets {1, 2, 3}, {4, 5, 6}, {7, 8}. If for example 1x and 6x were red, then C(v, a, 1, x, 6, b) implies $\overline{IR} \geq 3$. However, it now follows that {3, 6, 7, 8} all have degree two in G[X] which is impossible. ■

4.3 - Any counterexample is not 4-regular.

In this section we use a set of lemmas similar to the ones used in section 1. In fact, the proofs are extensions of the proofs already presented. Once again, let G be a 15-vertex graph with $\overline{IR} < 3$ and IR < 6. By sections 4.1 and 4.2, we may now assume G is 4-regular. Let v be a vertex in G with N(v) = Y = {a, b, c, d} and X = V(G) - Y.

Notice that the proof of lemma 4.1.1 is valid for these new sets X and Y, and hence that lemma may also be applied to our current situation.

Lemma 4.3.1 There are at most two vertices in X that do not send an edge to Y. If two such vertices exist, then they are adjacent.

Proof. Since G contains no red K_3's, Y is a blue K_4. If two nonadjacent vertices in X did not send edges to Y, then a blue K_6 is formed with Y. This implies $\beta(G) \geq 6$, from which it follows that $IR(G) \geq 6$, contrary to our choice of G.

Therefore, the set of vertices in X which do not send red edges to Y must form a red clique. The largest red clique allowed in G by lemma 3.1.2 is a K_2. ■

Lemma 4.3.2 G[X] does not contain a 5-cycle.

Proof Suppose X has a 5-cycle labeled 1, 2, 3, 4, 5. Further, assume {1, 2, 3, 4} all send red edges to Y. This implies 1 and 3 have a common red neighbour in Y by $P_3(1, 2, 3)$. Similarily, 2 and 4 have a common red neighbour in Y. Without loss of generality assume 1a, 3a, 2b, and 4b are red. Both 1b and 4a are blue since 2b and 3a are red respectively. Now, C(v, a, 1, 5, 4, b) implies $\overline{IR} \geq 3$.

Therefore, {1, 2, 3, 4, 5} must contain two vertices which do not send edges to Y. Without loss of generality, let 4 and 5 be these vertices. (Recall two such vertices must be adjacent.)

Vertices 4 and 5 have degree four. Also, the 5 cycle {1, 2, 3, 4, 5} can not contain any red chords otherwise a red K_3 is formed. Assume without loss of generality 46, 47, 58, 59 are red.

As before, by $P_3(1, 2, 3)$, 1 and 3 have a common red neighbour in Y. Assume 1a and 3a are red. Now, vertex a has degree 3 and therefore can join one more member of X, but 1 must join a common neighbour of both 8 and 9. (By $P_3(1, 5, 8)$ and $P_3(1, 5, 9)$). Therefore, assume 1b, 8b are red. Similarily, 3 must join a red neighbour in Y of 6 and 7. Assume 3c, 6c are red. (As before a cannot join 6 and 7. Vertex b cannot join 3 plus a vertex from {6, 7}.)

Now both 1 and 3 have degree four. This implies 1c and 3b are blue. By C(v, b, 1, 2, 3, c), we conclude $\overline{IR} \geq 3$ contrary to our choice of G. ■

Lemma 4.3.3 G[X] does not contain a red 7-cycle.

Proof Suppose a red 7-cycle exists with vertex sequence 1, 2, ..., 7. The cycle cannot contain a chord otherwise a red K_3, or C_5 is formed, each of which is impossible.

<u>Case 1</u>. Suppose every vertex in {1, 2, ..., 7} sends an edge to Y.

The proof of this case follows directly from the proof of lemma 4.1.3 and is left to the reader.

<u>Case 2</u>: Exactly one vertex in {1, 2, ..., 7} does not send a red edge to Y.

Without loss of generality let 7 be this vertex. Vertex 7 must join two vertices in {8, 9, 10} with red edges since G is 4-regular. Assume 78 and 79 are red.

As before, assume 1a and 3a are red. At least one of {8, 9} sends an edge to Y. Let 8 be this vertex. By $P_3(6, 7, 8)$, vertices 6 and 8 have a common red neighbour in Y. Assume 6b and 8b are red.

By C(v, a, 1, 7, 6, b) and C(v, a, 1, 7, 8, b), either 1b is red or 6a and 8a are red. In the latter case deg(a)=5 which is impossible. We conclude 1b is red.

Vertices 6 and 4 must have a common red neighbour in Y. Because a has degree 3 and b has degree 4, this common neighbour cannot be a or b. Assume 6c and 4c are red. Both 6 and 1 have degree 4. Therefore, 6a and 1c are blue. By C(v, a, 1, 7, 6, c) we conclude $\overline{IR} \geq 3$, a contradiction.

<u>Case 3</u>: There are two vertices in {1, 2, ..., 7} which do not send red edges to Y.

Without loss of generality assume these vertices are 6 and 7. In order for 7 to have degree four, assume 78 and 79 are red. Vertex 6's only possible remaining red neighbour is 10. Therefore, 6 only has degree 3, contrary to Theorem 4.2.2.

By lemma 4.3.1, the above cases are exhaustive. ■

Lemma 4.3.4 G[X] contains no red 9-cycles.

Proof. Suppose a red 9-cycle exists with vertex sequence 1, 2, ..., 9. We claim every vertex in the 9-cycle sends a red edge to Y. If some vertex u did not, then u must send red edges to two other vertices in G[X] in order for deg(u)=4. If u is on a red chord of the 9 cycle, then a red K_3, C_5, or C_7 is formed. Therefore, only one other vertex in G[X] can join u.

At this point we can use a similar proof to that of lemma 4.1.4. The details are omitted. ■

Theorem 4.3.5 G is not 4-regular.

Proof By lemmas 3.1.2, 4.3.2, 4.3.3, and 4.3.4, G[X] contains no 3-, 5-, 7-, or 9-cycles. Since |X|=10, G[X] is bipartite. This implies $\beta(G[X]) \geq 5$. Since v is nonadjacent to X in G, we have $\beta(G) \geq 6$. This gives IR ≥ 6, contrary to our choice of G. ■

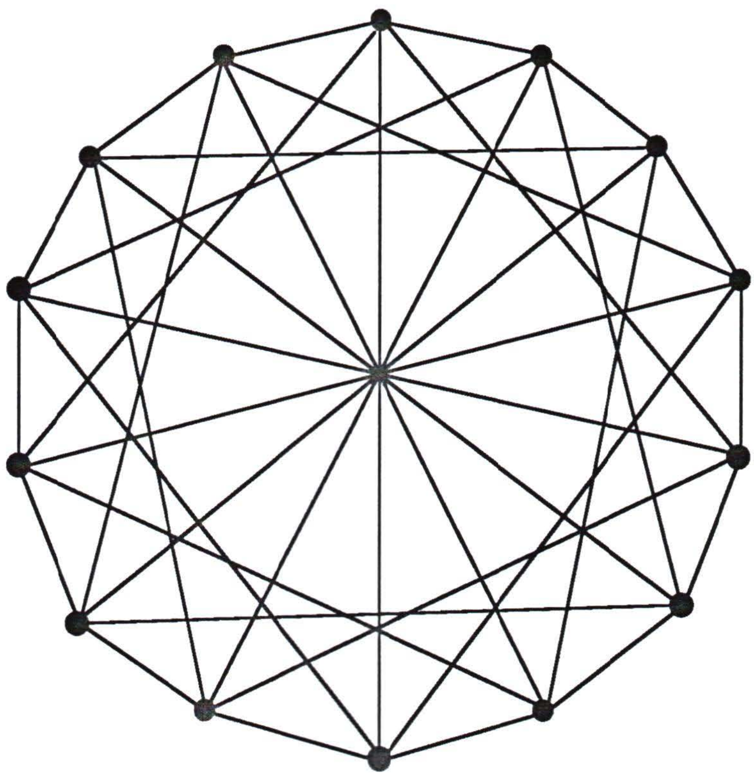

Figure 4. A 14-vertex graph with $\overline{IR} = 2$ and IR = 5.

Theorem 4.3.6 s(3,6) = 15.

Proof Lemma 4.1 and theorems 4.1.5, 4.2.2, 4.3.5 show that s(3, 6) ≤ 15. The graph in figure 4 is an example of a 14-vertex graph with IR = 5 and $\overline{\text{IR}}$ = 2 found with a computer search using the program in Appendix A. ■

4.4 - Further Results and Open Problems

The asymptotic results obtained for classical Ramsey numbers either follow from the recurrence relation in theorem 2.3.1 or depend on the property of independence. For example, the probablistic results by Erdos[18] and Spencer[34] use the independence graph for a set of probabilities, see the Lovász Local Theorem[18]. Since the property of irredundance is not as strong (or as simple) as independence, similar results for irredundance may be hard to derive.

The recurrence relation from theorem 3.1.1 has always produced upper bounds larger than the exact values of irredundant Ramsey numbers calculated so far, unlike the first four classical Ramsey numbers calculated by Greenwood and Gleason[25]. Perhaps this relation can be improved by a constant term.

Conjecture 4.4.1

$$s(m, n) \leq s(m - 1, n) + s(m, n - 1) - c$$

where c is a constant.

A simple lower bound for s(m, n) can be contructed using the following lemma.

Lemma 4.4.2 Let G and H be graphs. IR(G ∪ H) = IR(G) + IR(H). $IR(\overline{G \cup H})$ = max $(IR(\overline{G}), IR(\overline{H}))$.

Proof First we notice, IR(G ∪ H) ≤ IR(G) + IR(H). Let S be an irredundant set in G ∪

H. Since, no edges run between V(H) and V(G), the private neighbours of the vertices of G in S must also be vertices of G, and the private neighbours of the vertices of H in S must be vertices of H. Hence, the vertices of G in S form an irredundant set in G. Similarily, the vertices of H in S form an irredundant set in H. Therefore, S can contain at most IR(G) vertices of G and IR(H) vertices of H. i.e. $|S| \leq IR(G) + IR(H)$. Second, if we let T be an irredundant set of G of size IR(G) and R be an irredundant set of H of size IR(H), $T \cup R$ is an irredundant set in $G \cup H$. Therefore, $IR(G \cup H) = IR(G) + IR(H)$.

Let S be a maximal irredundant set in $\overline{G \cup H}$. Notice all the edges between V(G) and V(H) in $G \cup H$ are blue. Therefore, any set containing one vertex from G and one vertex from H will dominate $G \cup H$ and hence contain a maximal irredundant set. Thus, $|S| \leq 2$, if S contains vertices from both G and H. The result follows whenever $IR(G) \geq 2$ or $IR(H) \geq 2$. If both these are one, then $IR(\overline{G \cup H}) = 1$. ■

Theorem 4.4.4 Let m and n be integers with $m \geq 2$ and $n \geq 2$ then

$$s(m, n) > \max (s(m, i) -1 + s(m, n - i + 1) -1),$$

where the maximum is taken over values of $i = 2, 3, \ldots, n - 1$.

Proof Let G_1 be a graph on $s(m, i) -1$ vertices such that $IR(\overline{G_1}) \leq m - 1$ and $IR(G_1) \leq i - 1$. Let G_2 be a graph on $s(m, n - i + 1) -1$ vertices such that $IR(\overline{G_2}) \leq m - 1$ and $IR(G_2) \leq n - i$.

Consider the graph $G = G_1 \cup G_2$. By the above theorem, G is a graph on $s(m,i) -1 + s(m, n - i + 1) -1$ vertices with $IR(\overline{G}) \leq m -1$ and $IR(G) \leq n - 1$. ■

This theorem gives bounds lower than the exact values for all numbers calculated so far with the exception of s(3, 4). The graph $K_2 \cup C_5$ attains the lower bound for s(3,4).

Finally, we believe that irredundant Ramsey numbers can be calculated for all values for which classical Ramsey numbers are known. Perhaps a generalization to more than two colours is possible. Methods similar to the one used to calculate s(3, 6) may help for s(3, n) when n ≥ 7. It appears however that each new evaluation will be a new and challenging problem requiring further original methods.

References

[1] Ajtai, M , Kolmós, J and Szemerédi, A note on Ramsey numbers, *Journal of Combinatorial Theory Series A* , **29** (1980) 354-360

[2] Allan, R B and Laskar, R , On domination and some related concepts in graph theory, *Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph Theory and Computing, Utilitas Mathematica,* Winnipeg, (1978), pp 43-56

[3] Berge, C Theory of Graphs and its Applications, Methuer, London (1962)

[4] Berge, C Graphs and Hypergraphs, North-Holland, Amsterdam (1973)

[5] Bollobás, B and Cockayne, E J , Graph-theoretic parameters concerning domination, independence, and irredundance, *Journal of Graph Theory* **3** (1979), 241-249

[6] Bondy, J A and Murty, U S R , Graph Theory with Applications, Elsevier Publishing New York, New York (1982)

[7] Burr, S A , Generalized Ramsey theory for graphs -- A Survery, Graphs and Combinatorics (R A Bari and F Harary, eds) Lecture Notes in Mathematics **406**, Springer, (1974), 52-75

[8] Chung, F R K and Grinstead, C M , A survey of bounds for classical Ramsey numbers *Journal of Graph Theory* **7** (1983) 25-38

[9] Chvátal, V , Tree-complete graph Ramsey numbers *Journal of Graph Theory* **1** (1977), 93

[10] Chvátal, V and Harary, F , Generalized Ramsey theory for graphs, *Bulletin of the American Mathematical Society,* **78** (1972), 423-426

[11] Chvátal, V and Harary, F , Generalized Ramsey theory for graphs I, Diagonal Numbers, *Periodica Mathematica Hungarica* **3** (1973), 113-122

[12] Chvátal, V and Harary, F , Generalized Ramsey theory for graphs III Small off-diagonal numbers, *Pacific Journal of Mathematics,* **41** (1972), 335-345

[13] Cockayne, E J Some tree-star Ramsey numbers, *Journal of Combinatorial Theory Series B* , **17** (1974)

[14] Cockayne, E J , Favaron, O , Payan, C , and Thomason, A Contributions to the theory of domination, independence and irredundance in graphs, *Discrete Mathematics* , **33** (1981), 249-258

[15] Cockayne, E J , Hedetniemi, S T , and Miller, D J , Properties of hereditary hypergraphs and middle graphs *Canadian Mathematical Bulletin* **21** (1978), 461-468

[16] Cockayne, E J and Mynhardt, C M On the sequence of upper and lower domination, independence, and irredundance numbers of a graph Submitted

[17] Erdös, P Some remarks on the theory of graphs *Bulletin of the American Mathematical Soceity* **53** (1947) 292-294

[18] Erdös, P , and Lovász, L , Problems and results on 3-chromatic hypergraphs and some related questions, in Infinite and Finite Sets(edited by A Hajnal, R Rado, and V T Sós), North Holland, (1975), pp 609-628

[19] Erdös, P and Szekeres, G A combinatorial problem in geometery *Compositio Mathematica* **2** (1935) 464-470

[20] Faudree, R J and Schelp, R H , All Ramsey numbers for cycles in graphs, *Discrete Mathematics*, **8** (1974), 313-329

[21] Favaron, O Two relations between the parameters of independence and irredundance To appear

[22] Gerencsér, L and Gyárfás, A , On Ramsey-type problems, *Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae Sectio Mathematica* **10** (1967), 167-170

[23] Graham, R L , Rothschild, B L and Spencer, J H , Ramsey Theory, John Wiley & Sons, New York (1980)

[24] Graver, J E and Yackel, J , Some graph theoretic results associated with Ramsey's theorem, *Journal of Combinatorial Theory* **4**, 125-175

[25] Greenwood, R E and Gleason, A M , Combinatorial relations and chromatic graphs, *Canadian Journal of Mathematics* **7**, 1-7

[26] Harary, F , Graph Theory Addison-Wesley, Reading, Mass (1969)

[27] Hedetniemi, S T , Laskar, R and Pfaff, J , Irredundance in graphs -- a survey Private Communication

[28] Kalbfleisch, J G , Construction of special edge-chromatic graphs *Canadian Mathematics Bulletin*, **8** (1965), 575-584

[29] Kalbfleisch, J G , Upper bounds for some Ramsey numbers *Journal of Combinatorial Theory* **2** (1967) 35-42

[30] Parsons, T D Ramsey graph theory Selected Topics in Graph Theory, Editors L W Beineke and R J Wilson, Academic Press, London (1978)

[31] Radziszowski, S P and Kreher, D L , Search algorithm for Ramsey graphs by union of group orbits, *Journal of Graph Theory* **12** (1988), 59-72

[32] Ramsey, F P On a problem of formal logic *Proceedings of the London Mathematical Society* **30** (1930) 264-286

[33] Rosta, V , On a Ramsey type problem of J A Bondy and P Erdös I, *Journal of Combinatorial Theory Series B,* **15** (1973), 94-104

[34] Rosta, V , On a Ramsey type problem of J A Bondy and P Erdös II, *Journal of Combinatorial Theory Series B,* **15** (1973), 105-120

[35] Spencer, J H , Ramsey's theorem -- a new lower bound *Journal of Combinatorial Theory Series A,* **18** (1975) 108-115

```
{*-------------------------------------------------------------*}
{* IR_TEST          Richard Brewster       May 1988           *}
{*                                                             *}
{* Program to compute IR(G) of cyclic graphs.                  *}
{* This program reads the first row of the                     *}
{* adjacency matrix of a cyclic graph. IR is                   *}
{* calculated for both G and its complement using              *}
{* the backtracking procedure Check_IR                         *}
{*-------------------------------------------------------------*}
PROGRAM IR_TEST (input, output);
    USES
        Check_IR;

    VAR
        i, j : integer;
        first_row : STRING;
        subset : set_type;
        cls_nbhd : set_type;
        PN : PN_type;
        go_again : char;

BEGIN

    ShowText;
    go_again := 'y';
    WHILE (go_again = 'y') DO
        BEGIN

    {*--- Read the number of vertices and          ---*}
    {*--- and the first row of the adjacency       ---*}
    {*--- matrix.                                  ---*}
            write(' p : ');
            readln(p);
```

```
        write(' Enter the first row : ');
        readln(first_row);

{*---  Compute the remainder of the adjacency        ---*}
{*--- matrix using cyclic permutations of the         ---*}
{*--- first row.                                      ---*}
        FOR i := 1 TO p DO
            IF first_row[i] = '1' THEN
                A[1, i] := true
            ELSE
                A[1, i] := false;
        A[1, 1] := true;


        FOR i := 2 TO p DO
            BEGIN
                FOR j := 2 TO p DO
                    A[i, j] := A[i - 1, j - 1];
                A[i, 1] := A[i - 1, p];
                A[i, i] := true;
            END;

{*--- Set preliminary values of IR and subset---*}
{*--- for use in Check_IR procedure.          ---*}
        IR := 0;
        FOR i := 1 TO p DO
            subset[i] := false;
```

```
{*--- Send a subset of a single vertex for          ---*}
{*--- for each vertex in the graph                   ---*}
         FOR i = 1 TO p DO
              BEGIN
                   subset[i] = true,
                   FOR j = 1 TO p DO
                        BEGIN
                             cls_nbhd[j] = A[i, j],
                             PN[1, j] = A[i, j],
                        END;

{*--- Calculate IR using Check_IR ---*}
                   Check_IR(subset, 1, i, cls_nbhd, PN),
                   subset[i] = false,
              END,

{*--- write results ---*}
         writeln('IR =', IR  3),
         writeln('Example of set '),
         FOR i = 1 TO p DO
              IF Ex_IR[i] THEN
                   write(i  4),
         writeln,

{*--- take the complement ---*}
         FOR i = 1 TO p DO
              BEGIN
                   FOR j = 1 TO p DO
                        A[i, j] = NOT A[i, j],
                   A[i, i] = true,
              END,
```

```
{*--- Compute IR as above ---*}
        IR := 0;
        FOR i := 1 TO p DO
            subset[i] := false;

        FOR i := 1 TO p DO
            BEGIN
                subset[i] := true;
                FOR j := 1 TO p DO
                    BEGIN
                        cls_nbhd[j] := A[i, j];
                        PN[1, j] := A[i, j];
                    END;

                Check_IR(subset, 1, i, cls_nbhd, PN);
                subset[i] := false;
            END;

        writeln('IR bar =', IR : 3);
        writeln('Example of set ');
        FOR i := 1 TO p DO
            IF Ex_IR[i] THEN
                write(i : 4);
        writeln;

        write('go again ? ');
        readln(go_again);
    END;
END.
```

```
{*---------------------------------------------------------------*}
{* This unit uses a recursive backtracking program          *}
{* to compute the largest irredundant set in a graph        *}
{*---------------------------------------------------------------*}
UNIT Check_IR;
INTERFACE
   TYPE
      set_type = ARRAY[1..20] OF boolean;
      PN_type = ARRAY[1..20, 1..20] OF boolean;

   VAR
      IR : integer;
      Ex_IR : set_type;
      p : integer;
      A : ARRAY[1..20, 1..20] OF boolean;

{*---------------------------------------------------------------*}
{* This procedure receives as input, an irredundant set  *}
{* of a graph. The procedure tries to increase the size of *}
{* the set. If the set is larger than any irredundant set  *}
{* seen so far, the set and its size are stored.           *}
{*---------------------------------------------------------------*}
   PROCEDURE Check_IR (subset : set_type;
                       sub_size : integer;
                       largest_el : integer;
                       cls_nbhd : set_type;
                       PN : PN_type);

IMPLEMENTATION
   PROCEDURE Check_IR;
      VAR
         i, j, k : integer;
         new_sub : set_type;
         new_nbhd : set_type;
```

```
            IR_test : boolean;
            Saves_PN : boolean;
            this_row : boolean;
            New_PN : PN_type;

    BEGIN
{*--- Test to see if we have a largest IR set ---*}
{*---  If yes, save set and its size          ---*}
        IF sub_size > IR THEN
            BEGIN
                IR := sub_size;
                FOR j := 1 TO p DO
                    Ex_IR[j] := subset[j];
            END;

        {*--- Try adding elements to increase the set ---*}
        FOR i := (largest_el + 1) TO p DO
            BEGIN

        {*--- See if the new element has a PN ---*}
                IR_test := false;
                FOR j := 1 TO p DO
                    BEGIN
                        IR_test := IR_test OR (A[i, j] AND NOT (cls_nbhd[j]));
                        New_PN[sub_size + 1, j] := A[i, j] AND NOT
(cls_nbhd[j]);
                    END;
```

```
        {*--- See if the new element destroys any existing PN ---*}
                Saves_PN  = true,
                j  = 1,
                WHILE Saves_PN AND (j <= sub_size) DO
                    BEGIN
                        this_row  = false,
                        FOR k  = 1 TO p DO
                            BEGIN
                            this_row  = this_row OR (NOT A[i, k] AND PN[j, k]),
                            New_PN[j, k]  = PN[j, k] AND NOT A[i, k]
                            END,

                        Saves_PN  = this_row,
                        j  = j + 1,
                    END,

{*--- In the event the new element has a private         ---*}
{*--- neighbour and it does not destroy any other         ---*}
{*--- elements private neighbour, call the procedure      ---*}
{*--- recursively with the larger set                     ---*}
                IF IR_test AND Saves_PN THEN
                    BEGIN
                        FOR j  = 1 TO p DO
                            BEGIN
                            new_sub[j]  = subset[j],
                            new_nbhd[j]  = cls_nbhd[j] OR A[i, j],
                            END,
                        new_sub[i]  = true,
                        Check_IR(new_sub, sub_size + 1, i, new_nbhd,
New_PN),
                    END,
            END,
    END,
END
```

Vita

Surname	Brewster	*Given Names*	Richard Charles
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Title of Thesis

Irredundant Ramsey Numbers

Author [redacted]

Richard Charles Brewster
September 2, 1988