WHOLE FOREST HARVEST SCHEDULING IN THE PRESENCE OF THE RISK OF FIRE

by

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ABSTRACT

Several extensions to the procedure for forest-level harvest scheduling in the presence of the risk of fire, developed by Reed and Errico (1985), are considered. They include:- (i) the possibility of multiple timber types and regeneration options, (ii) the possibility of salvage after a fire, (iii) the problem of accessibility, and (iv) the inclusion of recovery costs. In addition, a new method of solving the deterministic scheduling problem by linear programming is considered. This new method is computationally more efficient. Moreover, it allows for the simple determination of the present net worth of hectares of forest in the whole-forest context, and can easily handle the problem of a changing land base. The use of terminal payoffs is shown to result in a considerable reduction in the size of the LP problem.

I. Introduction

In a previous paper (Reed & Errico 1985a), the problem of optimal harvest scheduling at the forest level in the presence of the risk of destruction through fire, was investigated. Since fires occur in a random way the problem had to be formulated as one of stochastic control. While an exact solution to the stochastic control problem was not in general determined, it was shown how an acceptable, approximately optimal solution, could be obtained by applying the solution to a realted deterministic control problem, in a feedback manner. This procedure provided an optimal harvest in any period, given the current state of the forest. Furthermore, the solution to the deterministic control problem provided estimates or predictions of long-run timber supply, under optimal management. The deterministic control problem could be solved by the use of linear programming. It was seen how the presence of even modest fire hazards could result in a considerable reduction in long-run yield and in per-period harvest under optimal management.

The "forest" considered in the earlier paper was very simple, consisting of a collection of many stands with the same volume-age or value-age relationships, but initially at different ages. The control variables were areas cut in the various age classes at different periods. In this latter respect the model is similar to Model II form of FORPLAN (Johnson, Jones and Kent 1980) and MUSYC (Johnson and Jones 1979). In this paper we show how the model can be extended to cover a more complex forest and to cover more complex regeneration options but sill with the risk of destruction through fire present. Spedifically, we include:-

(a) multiple timber types and regeneration options,

(b) the possibility of partial salvage after a fire,

(c) the problem of accessibility, and

(d) the inclusion of costs of recovery dependent on location, terrain, etc.

In addition, a more efficient linear programming formulation of the deterministic control problem is presented. This formulation is useful in that:-

(i) the problem of a changing land base can very easily be incorporated,

(ii) the values of certain dual variables in the optimal solution provide shadow values for timber in various age classes at various times. Thus a true assessment of the present net worth of a stand in the whole forest context can be made.

(iii) The new formulation is computationally more efficient, especially in the case when fire destruction probabilities are age-dependent.

The problem of planning over an infinite time horizon is solved by using a suitably large finite planning horizon with terminal payoff values corresponding to the single-stand net present worth, calculated with the risk of fire present (Reed and Errico 1985(b)). This permits a considerable reduction in the length of the planning horizon needed for solving the problem, and thus a considerable reduction in cost.

Finally, in an appendix, the equivalence of our model when the fire probability is set to zero, with the basic Model II form of MUSYC and FORPLAN is established formally.

II. The Basic Model and Solution

A stochastic dynamic model for the evolution of a forest subject to random destructions through fire is developed in Reed & Errico (1985(a)). The model can be written in matrix form as

$$\underset{\sim}{x}_{t+1} = R_t \underset{\sim}{x}_t - S_t \underset{\sim}{h}_t \tag{1}$$

where: -

$$\underset{\sim}{x}_t = \begin{bmatrix} x_1^t \\ x_2^t \\ \vdots \\ x_k^t \end{bmatrix} \quad \text{and} \quad \underset{\sim}{h}_t = \begin{bmatrix} h_1^t \\ h_2^t \\ \vdots \\ h_k^t \end{bmatrix} \tag{2}$$

represent, respectively, the areas in the forest in the various age-classes (1, 2, ..., k) at the start of period t, and the areas harvested in the various age classes in period t. The matrices R_t and S_t are <u>random</u> matrices of the form

$$R_t = \begin{bmatrix} p_1^t, & p_2^t, & \cdots & p_k^t \\ (1-p_1^t), & 0, & \cdots & 0 \\ 0, & (1-p_2^t), & & 0 \\ \vdots & & \ddots & \\ 0, & & (1-p_{k-1}^t), & (1-p_k^t) \end{bmatrix} \tag{3}$$

and

$$S_t = \begin{bmatrix} (-1+p_1^{\,t}), & (-1+p_2^{\,t}), & \cdots & (-1+p_k^{\,t}) \\ (1-p_1^{\,t}), & 0, & \cdots & 0 \\ 0, & (1-p_2^{\,t}), & & 0 \\ & & \ddots & \\ 0, & & (1-p_{k-1}^{\,t}) & (1-p_k^{\,t}) \end{bmatrix} \tag{4}$$

where $p_i^{\,t}$ $(i = 1,\ldots,k)$ are random variables representing the proportions of the areas in the various age-classes destroyed by fire during period t.

It is assumed that the volume per hectare (or net value per hectare) of timber of age i is v_i $(i = 1,\ldots,k)$. We let

$$\underset{\sim}{v}' = (v_1, v_2, \ldots, v_k) \tag{5}$$

Thus $\underset{\sim}{v}'$ is a volume (or value) at age vector. Corresponding to a harvest (by area) of $\underset{\sim}{h}_t$ in period t, the total volume cut will be $H_t = \underset{\sim}{v}' \underset{\sim}{h}_t$. It is assumed that there are imposed some constraints on the harvests to ensure a fairly even flow of timber from the forest. For example there could be sequential flow constraints of the form

$$(1-\gamma_1)H_{t-1} \le H_t \le (1-\gamma_2)H_{t-1} \qquad t = 2, 3, \ldots \tag{6}$$

where γ_1 is the maximum permitted proportional decrease in volume harvested $(0 \le \gamma_1 \le 1)$ and γ_2 is the maximum permitted proportional increase in volume harvested $(0 \le \gamma_2)$

The problem of maximizing the expected discounted volume (value) of timber from the forest over an infinite time horizon, and subject to the flow constraints is considered in Reed and Errico (1985(a)). It is shown how an approximately optimal solution can be obtained by applying in a feedback manner the first period harvest of the optimal solution to the related <u>deterministic</u> control problem:-

$$\text{maximize} \quad J = \sum_{t=1}^{\infty} \alpha^{t} \, \underset{\sim}{v}' \underset{\sim}{h}_t \tag{7}$$

subject to:-

$$\underset{\sim}{x}_{t+1} = \bar{R}\, \underset{\sim}{x}_t - \bar{S}\, \underset{\sim}{h}_t \tag{8}$$

$$\underset{\sim}{h}_t \le \underset{\sim}{x}_t \tag{9}$$

$$\underset{\sim}{h}_t \ge \underset{\sim}{0} \qquad t = 1, 2, \ldots \tag{10}$$

$$\text{and} \quad (1-\gamma_1)\underset{\sim}{v}'\underset{\sim}{h}_{t-1} \le \underset{\sim}{v}'\underset{\sim}{h}_t \le (1+\gamma_2)\underset{\sim}{v}'\underset{\sim}{h}_{t-1} \qquad t = 2, 3, \ldots \tag{11}$$

where $\bar{R}$ and $\bar{S}$ are the expected values of the (assumed time-homogeneous) random matrices R_t and S_t, and α is the per period discount factor.

In Reed and Errico (1985(a)) the above deterministic control problem was solved by eliminating the $\underset{\sim}{x}_t$ using equation (8) recursively. By doing this it is shown how the constraints (8) and (9) collapse to a single set of constraints of the form

$$\bar{R}^{t-2}\, \bar{S}\, \underset{\sim}{h}_1 - \bar{R}^{t-2}\, \bar{S}\, \underset{\sim}{h}_2 + \ldots + \bar{S}\, \underset{\sim}{h}_{t-1} + \underset{\sim}{h}_t \le \bar{R}^{t-1}\, \underset{\sim}{x}_1 \tag{12}$$

where $\underset{\sim}{x}_1$ is the given initial state of the forest. The maximization problem is thus to maximize (7) subject to (9), (11) and (12). If the planning horizon is set to a large finite value N, then this problem can be solved by linear programming (both the objective (7) and the constraints (9), (11) and (12) are linear in the control variables $\underset{\sim}{h}_t$ $(t = 1,\ldots,N)$. This is the approach used in Reed and Errico (1985(a)). The latter part of the solution trajectory was simply ignored because its behaviour depends on the fact that an optimal solution will leave no timber of value standing at the end of the planning horizon and will thus start moving away from an equilibrium several periods before the end of the planning horizon.

It has been discovered that a more efficient procedure is to include a terminal payoff corresponding to the single stand net present worth (calculated with fire risk present -- see Reed and Errico (1985(b)) of timber standing at the end of the planning horizon. Specifically the objective function used is

$$J = \sum_{t=1}^{N} \alpha^t \underset{\sim}{v}' \underset{\sim}{h}_t + \alpha^{N+1} \underset{\sim}{r}' \underset{\sim}{x}_{N+1} \tag{13}$$

where $\underset{\sim}{r}' = (r_1, r_2, \ldots, r_k)$ is a vector of net present worths of single hectares of timber of ages 1, 2, ..., k. These values can be calculated either analytically or numerically using a policy improvement algorithm (Ross 1970, p. 125 ff). For the case of age-independent fire probabilities the net present worth of a single hectare of forest of age i is

$$r_i = \begin{cases} \dfrac{v_i (\alpha q)^{i_0 - i}}{q^2(1-\alpha)\left(1-(\alpha q)^{i_0-1}\right)} \left\{q(1-\alpha) + p(\alpha q)^i\right\} & i < i_0 \\ \\ v_i + \alpha r_1 & i \geq i_0 \end{cases} \tag{14}$$

where p is the (age-independent) probability of fire, $q = 1 - p$ and i_0 is the optimal single stand cutting age (Reed and Errico 1985(b)).

If the solution to the infinite time horizon problem has reached an equilibrium after N periods, with the harvest flow constraints non-binding, then using the objective function (13) in place of (7), will provide an identical solution. It has been found for the problems that we have considered, that the solution trajectory reaches an equilibrium after 10 to 15 periods. Previously a planning horizon of 35 periods had been used. Thus the use of terminal payoffs has allowed for a reduction in the size of the problem by 50 percent or more.

Furthermore, it has been found to be computationally more efficient <u>not</u> to remove the variables $\underset{\sim}{x}_t$ from the problem. Thus we look at the problem of maximizing (13) subject to (8), (9), (10) and (11).

This problem is linear (in its objective and its constraints) in the variables $\underset{\sim}{x}_t$ and $\underset{\sim}{h}_t$. It can thus be solved by linear programming methods.

Besides leading to increased computational efficiency (which is especially great in the case of age-dependent fire probabilities) this new LP form has several other advantages.

Firstly the problem of a changing land base can easily be handled by a modification to the appropriate dynamic equation in the set (8). For example, if it is anticipated that in period 2, areas $\underset{\sim}{A}' = (A_1, A_2, \ldots, A_k)$ in the various age classes are to be removed from the land base then the appropriate equation is

$$\underset{\sim}{x}_3 = \bar{R}\,\underset{\sim}{x}_2 - \bar{S}\,\underset{\sim}{h}_t - \underset{\sim}{A}$$

All the other equations would remain unchanged. While it is true that such a change in land base could be handled in the earlier formulation, it would be much more complicated, with $\underset{\sim}{A}$ occuring throughout the left-hand side of equation (12). In the case of zero fire probability, the problem of handling a changing land base in MUSYC model II is complicated (essentially it is the same as the case discussed above with the change occurring throughout the left-hand side of (12)). It would be much simpler to use the form presented here with fire probabilities set to zero.

The problem of modelling accessibility can be thought of as one of modelling a changing land base -- extra hectares are added to the land base as new forest areas become accessible. This will be discussed further in Section V.

Another advantage of using the new LP form is that the values of the dual variables corresponding to the equality constraints (8), at the optimum, provide shadow values for land containing timber of various ages, at various times throughout the planning horizon. Thus the per hectare present net worth of standing timber <u>in the whole forest context</u> can be assessed at any time. This may be very different from the single-stand present net worth. It will depend not only on the age (volume) of trees in the given hectare, but also on the age-distribution of stands throughout the forest, and on the harvest history.

In the next sections we extend this model to include salvage, multiple timber types and regeneration options etc. Before proceeding however it is worth noting that if the fire probability parameters are all set to zero the LP model of Reed and Errico (1985(a)) reduces to the basic Model II form of MUSYC. This is formally proved in the Appendix. When there is no risk of fire present, it would seem that our new form here (with the $\underset{\sim}{x}_t$'s kept in the problem) would provide an attractive alternative to MUSYC. While it seems to be computationally slightly less efficient than MUSYC, it has the advantages discussed above of being able easily to handle a changing land base and the problem of accessibility, and to provide the present net worth of land in the whole-forest context.

III. Multiple Timber Types and Regeneration Options

We consider firstly the case of two timber types. We suppose that timber type 1 can be adequately described by k age-classes and a volume (or value) at age vector

$$\underset{\sim}{v}' = (v_1, v_2, \ldots, v_k)$$

i.e. we assume that the volume (or value) at age curve has reached an asymptote after k periods). Similarly we assume that timber type 2 can be described by ℓ age-classes and a volume (or value) at age vector

$$\underset{\sim}{w}' = (w_1, w_2, \ldots, w_\ell)$$

The state of the forest at the beginning of period t will be described by a $(k+\ell)$-dimensional vector

$$\begin{bmatrix} x_1^t \\ \vdots \\ x_k^t \\ \cdots \\ y_k^t \\ \vdots \\ y_\ell^t \end{bmatrix}$$

where x_i^t $(i = 1,\ldots,k)$ denotes the area of timber type 1 at age i and y_i^t $(i = 1,\ldots,\ell)$ denotes the area of timber type 2 at age i.

We shall suppose that areas, $a_1^t, a_2^t, \ldots, a_k^t$ are harvested from timber type 1 in period t, and regenerated as timber type 1 at the start of period $t + 1$, while areas $b_1^t, b_2^t, \ldots, b_k^t$ are harvested from timber type 1 and regenerated as timber type 2. Similarly we shall suppose that areas $c_1^t, c_2^t, \ldots, c_\ell^t$ are harvested from timber type 2 and regenerated as timber type 2, while areas $d_1^t, d_2^t, \ldots, d_\ell^t$ are harvested from timber type 2 and regenerated as timber type 1.

We shall suppose that random proportions $p_1^t, p_2^t, \ldots, p_k^t$ of the areas in timber type 1 are destroyed by fire in period t, while random proportions $q_1^t, q_2^t, \ldots, q_\ell^t$ of the areas in timber type 2 are destroyed by fire in the same period.

The dynamics of the evolution of the forest can be described by the equation (similar to (1))

$$\begin{bmatrix} x_1^{t+1} \\ x_2^{t+1} \\ \vdots \\ x_k^{t-1} \\ \hdashline y_1^{t+1} \\ y_2^{t+1} \\ \vdots \\ y_\ell^{t+1} \end{bmatrix} = \begin{bmatrix} (a_1^t+\ldots a_k^t) + (d_1^t+\ldots d_\ell t) + (e_1^t+\ldots e_k^t) + (h_1^t+\ldots h_\ell^t) \\ (1-p_1^t)(x_1^t-a_1^t-b_1^t) \\ \vdots \\ (1-p_{k-1}^t)(x_{k-1}^t-a_{k-1}^t-b_{k-1}^t) + (1-p_k^t)(x_k^t-a_k^t-b_k^t) \\ \hdashline (c_1^t+\ldots c_\ell^t) + (b_1^t+\ldots b_k^t) + (g_1^t+\ldots g_\ell^t) + (f_1^t+\ldots f_k^t) \\ (1-q_1^t)(y_1^t-c_1^t-d_1^t) \\ \vdots \\ (1-q_{\ell-1}^t)(y_{\ell-1}^t-c_{\ell-1}^t-d_{\ell-1}^t) + (1-q_\ell^t)(y_\ell^t-c_\ell^t-d_\ell^t) \end{bmatrix} \qquad (15)$$

where: -

e_i^t represents area of timber type 1 of age i burnt in period t and regenerated as timber type 1 in period $t + 1$ $(i = 1,\ldots,k)$,

f_i^t represents area of timber type 1 of age i burnt in period t and regenerated as timber type 2 in period $t + 1$ $(i = 1,\ldots,k)$,

g_i^t represents area of timber type 2 of age i burnt in period t and regenerated as timber type 2 in period $t + 1$ $(i = 1,\ldots,\ell)$,

h_i^t represents area of timber type 2 of age i burnt in period t and regenerated as timber type 1 in period $t + 1$ $(i = 1,\ldots,\ell)$.

Clearly if all areas burnt are regenerated we have the extra constraints

$$e_i^t + f_i^t = p_i^t\left(x_i^t - a_i^t - b_i^t\right) \quad (i = 1,\ldots,k) \tag{16}$$

$$g_i^t + h_i^t = q_i^t\left(y_i^t - c_i^t - d_i^t\right) \quad (i = 1,\ldots,\ell) \tag{17}$$

The problem of maximizing expected discounted yield can be expressed as

$$\text{maximize:-} \quad J = \sum_{t=1}^{N} \alpha^t H^t, \tag{18}$$

$$\text{where} \quad H_t = \underset{\sim}{v}'(\underset{\sim}{a}_t+\underset{\sim}{b}_t) + \underset{\sim}{w}'(\underset{\sim}{c}_t+\underset{\sim}{d}_t), \tag{19}$$

subject to the dynamic equation (15), the constraints (16) and (17) and harvest flow constraints

$$(1-\gamma_1)H_{t-1} \le H_t \le (1+\gamma_2)H_t. \tag{20}$$

It should be noted that the control variables are

$$\underset{\sim}{a}_t = \left(a_1^t,\ a_2^t,\ \ldots,\ a_k^t\right)'$$

$$\underset{\sim}{b}_t = \left(b_1^t,\ b_2^t,\ \ldots,\ b_k^t\right)'$$

$$\underset{\sim}{c}_t = \left(c_1^t,\ c_2^t,\ \ldots,\ c_\ell^t\right)'$$

$$\underset{\sim}{d}_t = \left(d_1^t,\ d_2^t,\ \ldots,\ d_\ell^t\right)'$$

As in the single timber type case this is a problem in stochastic control since the p_i^t and q_i^t are random variables. However if we replace them by their expected values $\bar{p}_i$ $(i = 1,\ldots,k)$ and $\bar{q}_i$ $(i = 1,\ldots,\ell)$, the problem becomes one in deterministic control. As before we shall consider the deterministic problem and use the first period harvest of its optimal solution in a feedback manner to obtain an approximately optimal solution to the stochastic problem. The deterministic problem can be solved by linear programming since the objective (18) and the constraints (15), (16) and (17) and (20) are all linear in the variables $\underset{\sim}{x}_t$, $\underset{\sim}{y}_t$, $\underset{\sim}{a}_t$, $\underset{\sim}{b}_t$, $\underset{\sim}{c}_t$ and $\underset{\sim}{d}_t$. However the problem can be simplified considerably by adding the equations with x_1^{t+1} and y_1^{t+1} on the l.h.s. of (15), and defining new variables

$$z_i^t = a_i^t + b_i^t \qquad i = 1, \ldots, k \tag{21}$$

$$u_i^t = c_i^t + d_i^t \qquad i = 1, \ldots, \ell \tag{22}$$

corresponding to total areas harvested in each age class in each timber type in each period.

The objective then becomes

$$J = \sum_{t=1}^{\infty} \alpha^t (\underset{\sim}{v}' \underset{\sim}{z}_t + \underset{\sim}{w}' \underset{\sim}{u}_t) \tag{23}$$

and the constraints corresponding to the dynamic equations (15), and to (16) and (17) become

$$x_1^{t+1} + y_1^{t+1} = \left(z_1^t + \ldots + z_k^t\right) + \left(u_1^t + \ldots + u_\ell^t\right) + \left(\bar{p}_1\left(x_1^t - z_1^t\right) + \ldots + \bar{p}_k\left(x_k^t - z_k^t\right)\right) + \left(\bar{q}_1\left(y_1^t - u_1^t\right) + \ldots + \bar{q}_\ell\left(y_\ell^t - u_\ell^t\right)\right)$$

$$x_2^{t+1} = (1-\bar{p}_1)\left(x_1^t - z_1^t\right)$$

$$\vdots$$

$$x_k^{t+1} = (1-\bar{p}_{k-1})\left(x_{k-1}^t - z_{k-1}^t\right) + (1-\bar{p}_k)\left(x_k^t - z_k^t\right) \tag{25}$$

$$y_2^{t+1} = (1-\bar{q}_1)\left(y_1^t - u_1^t\right)$$

$$\vdots$$

$$y_\ell^{t+1} = (1-\bar{q}_{\ell-1})\left(y_{\ell-1}^t - u_{\ell-1}^t\right) + (1-\bar{q}_\ell)\left(y_\ell^t - u_\ell^t\right)$$

This problem is linear in the variables $\underset{\sim}{u}_t$, $\underset{\sim}{z}_t$, $\underset{\sim}{x}_t$ and $\underset{\sim}{y}_t$ $(t = 1, \ldots)$ both in the objective (23) and in the constraints (20), (25). To solve it we use linear programming after first changing the objective to a finite time horizon and using terminal payoffs corresponding to the net present worth of standing timber at the end of the planning horizon in the manner discussed in Section 2.

The solution will give explicitly only the total harvests from each age class in each timber type $\left(\text{the } z_i^t, \text{ and } u_i^t\right)$ and the state of the forest $\left(\text{the } x_i^t \text{ and } y_i^t\right)$ at the beginning of each period, for the optimal policy. However information as to how the areas cut and burnt are regenerated under optimal management can be recovered from equations (15), (16) and (17).

An example of the method was run using the two volume-age relationships given in Table 1. Timber type 1 corresponds to pure spruce (Picea glauea Moench Voss) on sites of site index 130+M (reference age 100) of medium accessibility to mills in the Fort Nelson Timber supply area of north-eastern

British Columbia. Timber type 2 is a hypothetical type with a volume age curve derived from that of timber type 1, simply by reducing the volume at any given age by fifteen percent. Thus timber type 2 represents a slower growing species. To offset this we have assumed that type 2 is less susceptible to desctruction through fire than type 1. Specifically we have assumed that for all age-classes of type 1 the per annum fire probability is 0.01, while for all age classes of type 2 it is 0.0065. The fire-adjusted volume rotation curves (VRCs) (see Reed and Errico 1985(b)) are shown in Figure 1. It can be seen that for lower rotation ages (below 70 years) the long-run average yield (LRAY) of type 1 is greater than that of type 2, but for higher rotation ages the situation is reversed. The differences, however, are only very slight.

In the example it was assumed that the initial inventory contained only stands of timber type 1. The inventory used corresponds to the current inventory of pure spruce, as described above, in the Fort Nelson Timber Supply Area and is the same as that used in Reed and Errice (1985(a)). The initial inventory is displayed in the top part (period 1) of Figure 2. The remainder of Figure 2 shows how the forest inventory evolves under optimal management using a 5 percent per annum discount rate and sequential flow constraints of ±10 percent of volume per period, and assuming fixed rather than random rates of fire as described above. It can be seen that all volumes cut are regenerated as type 2. Thus under optimal management the forest is eventually converted to a pure type 2 forest. The gain in yield through switching to type 2, although positive is very small, as the fire-adjusted volume-rotation curves of Fig. 2 would suggest. Roughly speaking, in this example, the effect of a reduction in the fire probability from 0.01 per annum to 0.0065 per annum, is equivalent to an increase in the volume growth curve slightly in excess of 15 percent. This would suggest that in fire-prone regions, silvicultural activities could be profitably directed towards reducing the risk of fire as much as toward increasing growth in volume.

The problem of multiple (as opposed to dual) timber types or regeneration options can be handled in essentially the same way. For a problem with m timber types, each described by k age-classes, and with a planning horizon of N periods using terminal payoffs, the resulting linear programming problem, if sequential flow constraints of the form (20) are included, would have $\left((k-1)m+2\right) \times N$ rows (constraints) and km columns (activities). If there are restrictions on regeneration (e.g. that only a certain timber type can be regenerated after a fire) they can be incorporated as extra constraints in the problem.

IV. Salvage

In the model described in Section 2, it is assumed that after a fire a stand is completely destroyed. In practice, very often, some usable timber can be salvaged after a fire. This can fairly easily be incorporated into the model.

Suppose that after a fire in a stand in age-class i in period t a proportion θ_i^t of the usable volume can be salvaged. Given proportions burnt, $p_1^t, \ldots, p_k^t$ in period t, and a harvest $\underset{\sim}{h}_t$, the total usable volume cut (harvest plus salvage) in that period would be

$$\underset{\sim}{v}'\underset{\sim}{h}_t + \sum_{i=1}^{k} v_i \, \theta_i^t \, p_i^t \left(x_i^t - h_i^t\right) \qquad (26)$$

This quantity would be a random variable, since the fire proportions and, quite possibly, the salvage proportions would be random variables. However following the procedure discussed in Section 2, we can replace these random variables by their expected values and consider the corresponding deterministic control problem, and then apply its solution in a feedback way.

Under these assumptions the total discounted volume cut over an infinite time horizon would be

$$J = \sum_{t=1}^{\infty} \alpha^t \left\{ \underset{\sim}{v}'\underset{\sim}{h}_t + \sum_{i=1}^{k} v_i \, \bar{\theta}_i \, \bar{p}_i \left(x_i^t - h_i^t\right) \right\} \qquad (27)$$

where $\bar{\theta}_i = E\left(\theta_i^t\right)$ and $\bar{p}_i = E\left(p_i^t\right)$ for $i = 1, \ldots, k$. We would want to maximize (27) subject to constraints (8), (9), (10) and the harvest flow constraints. We can consider the salvage volume to be either:-

(a) extra to the regular harvest and not included in the flow constraints, or,

(b) included in the flow constraints.

In case (a) the harvest flow constraints would be simply the constraints (11). In case (b) they would be of the form

$$(1-\gamma_1)H_{t-1} \leq H_t \leq (1+\gamma_2)H_{t-1} \tag{28}$$

where H_t would be given by (26).

Using terminal payoffs in the manner discussed earlier (the net present worth of a stand would now include salvage -- see Reed and Errico 1985 (b)), we can reduce the problem to one with a finite time horizon and solve it by linear programming.

As an example we have considered a "forest" comprising pure spruce stands (timber type 1 of the example of Section III) with the initial inventory discussed in that section. The per annum fire rate considered was one percent (age independent) and the discount rate used was 3 percent per annum. Two distinct salvage scenarios were considered:-

(i) expected salvage of 25 percent of volume burnt for ages 70 and older, and zero percent of volume burnt for ages less than 70,

(ii) expected salvage of 75 percent of volume burnt for ages 70 and older and zero percent of volume burnt for ages less than 70.

The predicted harvest trajectories both with salvage included and not included in the flow constraints, along with that for no salvage are shown in Figs. 3 and 4. Figure 3 shows these predictions for the case of 25% salvage (i), while Fig. 4 shows the case of 75% salvage (ii). It can be seen that while the gain through salvage in steady-state yield is relatively small, the gain in the early years of harvesting is greater especially when there is 75% salvage. This is because of the skewed nature of the initial inventory distribution which has much mature timber. The possibility of salvaging losses through fire of this timber, seems to provide fairly substantial benefits. When the salvage volume is outside the flow constraints the initial gains through salvage are higher than when salvage is included in the flow constraints. However in the former case predicted decline in harvest volume is, as one would expect, greater.

It should be noted that an actual harvest trajectory, obtained by applying the first period harvest of the optimal deterministic policy in a feedback manner, will likely deviate more from the predicted harvest trajectory, in the case when salvage is present, than in the case when it is not. The reason for this is that the randomness due to fires enters directly into the total harvest in a given period (see (26)), rather than only indirectly through the state-variable $\underset{\sim}{x}_t$. This additional variance in actual sample paths could, quite possibly, be considerable. When anticipated salvage is included in the flow constraints, it is quite possible that, for an actual sample path, the period to period fluctuation may be greater than the specified flow limits, because of the variation in actual amounts burnt and salvaged.

The question of whether salvage should or should not be included in the harvest flow constraints raises the whole question of the reasons for, and the influence of, such constraints. In whole-forest harvest scheduling models, both with and without the presence of the risk of fire, the influence of harvest flow constraints on the optimal solution, is considerable, and to a large extent seems to override the influence of the discount rate. The cost, in terms of foregone revenue, of imposing flow constraints is fairly easy to assess. The benefits are harder to quantify in simple economic terms. Ultimately the chosen trade-off between evenness of supply and long-run yield will reflect social, political and other preferences, as much as economic criteria. The question of how salvage should be treated, should be addressed in this light.

V. Accessibility

As discussed in Section II, the problem of accessibility can be handled essentially as one of a changing land-base, which, as we have seen, can be easily handled by a modification of the dynamic equation (1).

If we let $\underset{\sim}{x}_t$ denote the areas by age of the accessible forest, then the appropriate dynamic equation is

$$\underset{\sim}{x}_{t+1} = R_t(\underset{\sim}{x}_t + \underset{\sim}{m}_t) - S_t \underset{\sim}{h}_t \tag{29}$$

where $\underset{\sim}{m}_t$ is a vector of areas by age assumed to be newly roaded during period t. To determine optimal cutting strategies one needs estimates of the areas to be roaded, in the future. Long-range plans for road development may be available for this. To estimate the age distribution of areas to be roaded in the future, the effects of fire must be considered.

Suppose that at the beginning of the planning period (in period 1) the total extent of the forest can be partitioned into regions corresponding to those parts currently accessible, those parts to be roaded during period 1, those parts to be roaded during period 2, ..., etc. The areas currently accessible can be described by the vector $\underset{\sim}{x}_1$. In a similar way, let $\underset{\sim}{a}_t$ describe the areas (by current age) which, according to the roading plan, are to be roaded in period t, $t = 1, 2, 3, \ldots$. It follows that the actual areas by age (in period t) which are newly roaded in period t, will be

$$\underset{\sim}{m}_t = R_{t-1} R_{t-2} \cdots R_1 \underset{\sim}{a}_t \tag{30}$$

where $R_1, R_2, \ldots, R_{t-1}$ are random matrices of the form (3). Using the expected values of these matrices, we get estimates of the areas by age to be roaded in period t:

$$\hat{\underset{\sim}{m}}_t = \bar{R}^{t-1} \underset{\sim}{a}_t \tag{31}$$

Substituting this in (29) and again using expected values of R_t and S_t we get a deterministic dynamic equation of the form

$$\underset{\sim}{x}_{t+1} = \bar{R} \underset{\sim}{x}_t - \bar{S} \underset{\sim}{h}_t + \bar{R}^t \underset{\sim}{a}_t \qquad t = 1, 2, \ldots \tag{32}$$

This equation would be used in place of (8) in the LP used to determine the first period harvest. When the second period harvest is determined not only the state $\underset{\sim}{x}_t$ of the roaded regions should be updated, but also the states of the unroaded regions. In addition, any changes in the roading plans should be taken into account.

VI. Recovery Costs

In the previous sections the costs of harvesting timber have been ignored, and we have dealt essentially with the problem of maximizing the expected discounted total volume of timber that can be extracted from the forest. Costs of harvesting timber will depend on many factors (see Williams and Morrison 1985) including location, terrain, piece size etc. Suppose that the per hectare costs of harvesting of a stand can be decomposed into two parts, one corresponding to biological characteristics of the stand (age, size, etc.) and the other corresponding to geographical and topographical characteristics of the site on which the stand is growing. Let

$$\underset{\sim}{w}' = (w_1, w_2, \ldots, w_k)$$

denote, for each age class, the value of timber harvested, net of costs corresponding to biological characteristics. We shall assume that these costs are related only to age, so that $\underset{\sim}{w}'$ is the same for all stands in the forest.

Suppose that the other component of cost (geographic/topographic) can be discretized, so that it assumes one of the values, $c_1, c_2, \ldots, c_m$. We can partition the forest into m regions corresponding to these costs. Suppose the initial inventory can be described by area by age vectors

$$\underset{\sim}{x}_1^{\,1},\ \underset{\sim}{x}_1^{\,2},\ \ldots,\ \underset{\sim}{x}_1^{\,m}$$

(each of form (2)), corresponding to each of the m cost regions. For each cost region there is a dynamic equation of the form (1);

$$\underset{\sim}{x}_{t+1}^{\,j} = R_t^{\,j}\,\underset{\sim}{x}_t^{\,j} - S_t^{\,j}\,\underset{\sim}{h}_t^{\,j} \qquad j = 1, \ldots, m \tag{33}$$

where $\underset{\sim}{h}_t^{\,j}$ is a vector of areas harvested (by age) in period t in the j^{th} cost region, and $\underset{\sim}{x}_t^{\,j}$ is a vector of total areas (by age) at the beginning of period t in the j^{th} cost region.

In period t, the total value net of costs of both types of harvesting areas $\underset{\sim}{h}_t^{\,1}, \underset{\sim}{h}_t^{\,2}, \ldots, \underset{\sim}{h}_t^{\,m}$ will be

$$V_t = \sum_{j=1}^{m} \left\{ \underset{\sim}{w}' \underset{\sim}{h}_t^{\,j} - c_j \, \underset{\sim}{e}' \underset{\sim}{h}_t^{\,j} \right\} \tag{34}$$

where $\underset{\sim}{e}' = (1,1,1,\ldots,1)$, (so that $\underset{\sim}{e}'\underset{\sim}{h}_t^{\,j}$ denotes the total area harvested in cost region j.)

Over an infinite time horizon the expected present net value of a given harvest sequence will be

$$J = \sum_{t=1}^{\infty} \alpha^t \, V_t \tag{35}$$

This is linear in the harvest variables $\underset{\sim}{h}_t^{\,j}$. To maximize expected present value we will have to maximize (35) subject to constraints corresponding to the dynamic equations (33), constraints of the form

$$\underset{\sim}{0} \le \underset{\sim}{h}_t^{\,j} \le \underset{\sim}{x}_t^{\,j} \qquad j = 1, \ldots, M \tag{36}$$

and possibly flow constraints on the <u>volume</u> of timber harvested of the form

$$(1-\gamma_1)H_{t-1} \le H_t \le (1+\gamma_1)H_t \tag{37}$$

where H_t is the total <u>volume</u> harvested in period t and is of the form

$$H_t = \sum_{j=1}^{m} \underset{\sim}{v}' \underset{\sim}{h}_t^{\,j} \tag{38}$$

where $\underset{\sim}{v}' = (v_1,\ldots,v_k)$ is a volume at age vector, presumed the same for all sites in the forest.

This problem is linear in the $\underset{\sim}{h}_t^{\,j}$ and $\underset{\sim}{x}_t^{\,j}$ and so can be solved by the linear programming techniques as described in Section II.

VII. Discussion

In this paper we have described how the technique of optimal harvest scheduling in the presence of the risk of fire of Reed and Errico (1985(a)) can be extended in a number of directions. The following extensions have been considered:-

(a) multiple timber types and regeneration options,

(b) the possibility of partial salvage after a fire,

(c) the problem of accessibility, and

(d) the inclusion of costs dependent on terrain, location, etc.

Each extension has been considered on its own. A fully operational whole-forest scheduling model would incorporate all of these aspects together. While conceptually not difficult, the mathematical description of such a model would be a nightmare of subscripts and superscripts etc., and we have not attempted it here.

In addition to considering the above extensions to the Reed & Errico (1985(a)) model, we have presented a more efficient and revealing method of solving the linear programming problems that emerge from the prescribed technique of determining approximately optimal harvests. This new method allows for a changing land base to be easily incorporated, and provides shadow values for hectares of forest of any age at any period. The use of terminal payoffs greatly reduces the size of the LP problems encountered.

The equivalence of the Reed & Errico (1985(a)) model when the fire probability is set to zero, and the basic Model II form of MUSYC is proved in the Appendix. While current operational forms of MUSYC incorporate many of the extensions discussed in this paper, they cannot handle the problem of catastrophic loss, which appears to have a considerable impact on volume yields. To rewrite MUSYC to allow for catastrophic losses would probably be extremely time-consuming, and the resulting LP problems might be of such dense form that solutions would be extremely expensive, if indeed possible. An alternative is to use the dynamic equation model form discussed in this paper and use the method of solution described in Section II. While there would be a great deal of programming and other work required to develop an operational model including all of the features discussed here, it would likely be a better route to follow than to try to adapt MUSYC or other existing scheduling models.

On the other hand, simulation models such as TREES (Tedder, Schmidt and Gourley 1980) etc., could fairly easily be adapted to allow for random catastrophic losses. The essential dynamic equations for such an adaption are given in this paper.

Table I

Volume-age relationships for two timber types

Age (years)	Volume (m^3/ha) Timber type I	Timber type II
10	0	0
30	0	0
50	16	13.60
70	107	90.95
90	217	184.45
110	275	233.75
130	298	253.30
150	306	260.10

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Figure Captions

Figure 1. Fire-adjusted volume rotation curves (Reed and Errico 1985(b)) for the two timber types with growth curves given in Table I. For timber type I the per annum fire probability is 0.01, and for timber type II it is 0.0065.

Figure 2. Projected age distributions of timber types I and II under optimal management. Note how areas of timber type I are harvested and regenerated with timber type II.

Figure 3. Projected harvest volumes when there is a per annum fire probability of 0.01 and per annum discount rate of 3 percent. The three trajectories correspond to:- (a) no salvage, (b) 25% salvage for stands of age 70 or over which are burnt, with salvaged volumes considered part of the harvest flow, and (c) 25% salvage for stands of age 70 or over which are burnt, with salvaged volumes considered extra to the harvest flow.

Figure 4. Projected harvest volumes when there is a per annum fire probability of 0.01 and per annum discount rate of 3 percent. The three trajectories correspond to:- (a) no salvage, (b) 75% salvage for stands of age 70 or over which are burnt, with salvaged volume considered part of the harvest flow and (c) 75% salvage for stands of age 70 or over which are burnt with salvaged volume considered extra to the harvest flow.

Appendix I

Equivalence with MUSYC Model II of the Reed-Errico model with fire probabilities equal to zero.

The basic Model II form of MUSYC (Johnson and Scheurman, 1977) is

$$\text{maximize} \quad \sum_{j=1}^{N} \sum_{i=-M}^{j-1} D_{ij}\, y_{ij} + \sum_{i=-M}^{N} E_{iN}\, z_{iN}$$

subject to:-

1. area constraints

(a) $$\sum_{j=1}^{N} y_{ij} + z_{iN} = A_i \qquad i = -M, \ldots, 0$$

(b) $$\sum_{k=j+2}^{N} y_{jk} + z_{jN} = \sum_{i=-M}^{j-1} y_{ij} \qquad j = 1, \ldots, N$$

2. harvest flow constraints

$$(1-\gamma_1)H_j \le H_{j+1} \le (1+\gamma_2)H_j \qquad j = 1, \ldots, N-1$$

where

D_{ij} = discounted net revenue per hectare of areas regenerated in period i and harvested in period j (equivalent to $\alpha^j\, v_{j-i}$ in our notation)

E_{iN} = discounted value per hectare of areas regenerated in period i and left as ending inventory at the end of period N (equivalent to $\alpha^{N+1}\, r_{N+1-i}$ in our notation)

y_{ij} = area regenerated in period i and harvested in period j (equivalent to $h_{j-i}^{\,j}$ in our notation)

A_i = area present in period one that was regenerated in period i $(i = -M,\dots,0)$, with M being the age of the oldest timber in the initial (period 1) inventory, (A_{-j} is equivalent to x_1^{j+1} in our notation)

z_{jN} = area regenerated in period j and left as ending inventory after the harvest in period N (equivalent to x_{N-j+1}^{N+1} in our notation)

In general Model II allows for the specification of a minimum cutting age. For the sake of simplicity we have set this equal to one, in establishing the equivalence with our model. To establish a minimum cutting age in our model we would simply have to constrain some of the components of the harvest vectors $\underset{\sim}{h}_t$ to be equal to zero.

Model II, unlike the model of this paper, does not classify all areas with stands above a certain age, into a single age-class. To accommodate for this difference in models we shall have to modify our model slightly by allowing for stands of age up to $M + N + 1$, and modifying the matrices R and S to be the $M + N + 1$ dimensional square matrices

$$R = \begin{bmatrix} 0, & 0, & \cdots & 0 \\ 1, & & & \\ & 1, & & \\ & & \ddots & \\ & & 1, & 0 \end{bmatrix}, \qquad S = \begin{bmatrix} -1, & -1, \dots & -1 \\ 1 & & \\ & 1 & \\ & & \ddots \\ & & 1 & 0 \end{bmatrix}$$

The initial and terminal vectors of our model expressed in terms of Model II parameters are the $M + N + 1$ dimensional vectors

$$\underset{\sim}{x}_1 = \begin{bmatrix} A_0 \\ A_{-1} \\ \vdots \\ A_M \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \underset{\sim}{x}_{N+1} = \begin{bmatrix} z_{NN} \\ z_{N-1,N} \\ \vdots \\ \vdots \\ z_{-M,N} \end{bmatrix} \tag{A1.1}$$

From the recursion relation (24) we have

$$\underset{\sim}{x}_{N+1} = R^N \underset{\sim}{x}_1 - \left(R^{N-1} S \underset{\sim}{h}_1 + R^{N-2} S\underset{\sim}{h}_2 + \ldots + RS \underset{\sim}{h}_{N-1} + S \underset{\sim}{h}_N\right) \tag{A1.2}$$

This equation represents $M + N + 1$ equality constraints. The first N of these are equivalent to the area constraints (b), while the last $M + 1$ are equivalent to the area constraints (a). This can be established by multiplying out the right hand side of (A1.2), and using the equivalences in (A1.1) and the equivalence

$$h_i^{\,j} = x_{j-i,j} \qquad i = 1, \ldots, m, \quad j = 1, \ldots, M + N + 1 \tag{A1.3}$$

To do this for general M and N is quite tedious. For the sake of illustration we do it here for the case $N = 2$, $M = 3$. (A1.2) gives in this case

$$\underset{\sim}{x}_3 = R^2 \underset{\sim}{x}_1 - RS \underset{\sim}{h}_1 - S \underset{\sim}{h}_2$$

i.e. using (A1.1),

$$
\begin{bmatrix} z_{22} \\ z_{12} \\ z_{02} \\ z_{-12} \\ z_{-22} \\ z_{-32} \end{bmatrix}
=
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}
\begin{bmatrix} A_0 \\ A_{-1} \\ A_{-2} \\ A_{-3} \\ 0 \\ 0 \end{bmatrix}
-
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}
\begin{bmatrix} h_1^1 \\ h_2^1 \\ h_3^1 \\ h_4^1 \\ h_5^1 \\ h_6^1 \end{bmatrix}
$$

$$
=
\begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\begin{bmatrix} h_1^2 \\ h_2^2 \\ h_3^2 \\ h_4^2 \\ h_5^2 \\ h_6^2 \end{bmatrix}
$$

Using (A1.3) this gives six equations

$$z_{22} = y_{12} + y_{02} + y_{-12} + y_{-22} + y_{-32} + y_{-42}$$

$$z_{12} = y_{01} + y_{-11} + y_{-21} + y_{-31} + y_{-41} + y_{-51} - y_{12}$$

$$z_{02} = A_0 - y_{01} - y_{02}$$

$$z_{-12} = A_{-1} - y_{-11} - y_{-12}$$

$$z_{-22} = A_{-2} - y_{-21} - y_{-22}$$

$$z_{-32} = A_{-3} - y_{-31} - y_{-32}$$

The last four equations can be written

$$\sum_{j=1}^{2} y_{ij} + z_{iN} = A_i \qquad i = -3, -2, -1, 0$$

which are the area constraints (a) of Model II, while the first two equations are

$$\sum_{k=2}^{2} y_{1k} + z_{12} = \sum_{i=-5}^{0} y_{i1}$$

$$z_{22} = \sum_{i=-4}^{1} y_{i2}$$

which are seen to be the two area constraints (b) once we recognize that

$$y_{-51} = y_{-41} = y_{-42} = 0,$$

since there were no hectares regenerated in periods -5 and -4 available for harvest.

The sequential flow constraints, and the non-negativity constraints are the same in both models. The objective of our model is

$$J = \sum_{t=1}^{N} \alpha^t \underset{\sim}{v}' \underset{\sim}{h}_t + \alpha^{N+1} r' \underset{\sim}{x}_{N+1}$$

The first term can be re-expressed as

$$\sum_{t=1}^{N} \sum_{s=1}^{M+N+1} \alpha^t v_s h_s^{\,t} = \sum_{t=1}^{N} \sum_{s=1}^{M+N+1} \alpha^t v_s y_{t-s,t}$$

By a change of summation variables

$$i = t - s, \quad j = t$$

the sum can be written as

$$\sum_{j=1}^{N} \sum_{i=j-(M+N+1)} \alpha^j w_{j-i} y_{ij} = \sum_{j=1}^{N} \sum_{i=-M}^{j-1} D_{ij} y_{ij},$$

since $i \geq -M$ (i.e. M is age of oldest trees in the initial inventory). This is the first term of the objective function of Model II. Similarly the second term of our model, corresponding to terminal payoffs for areas left standing at the end of the planning horizon, is equivalent to the second term in the objective of Model II.